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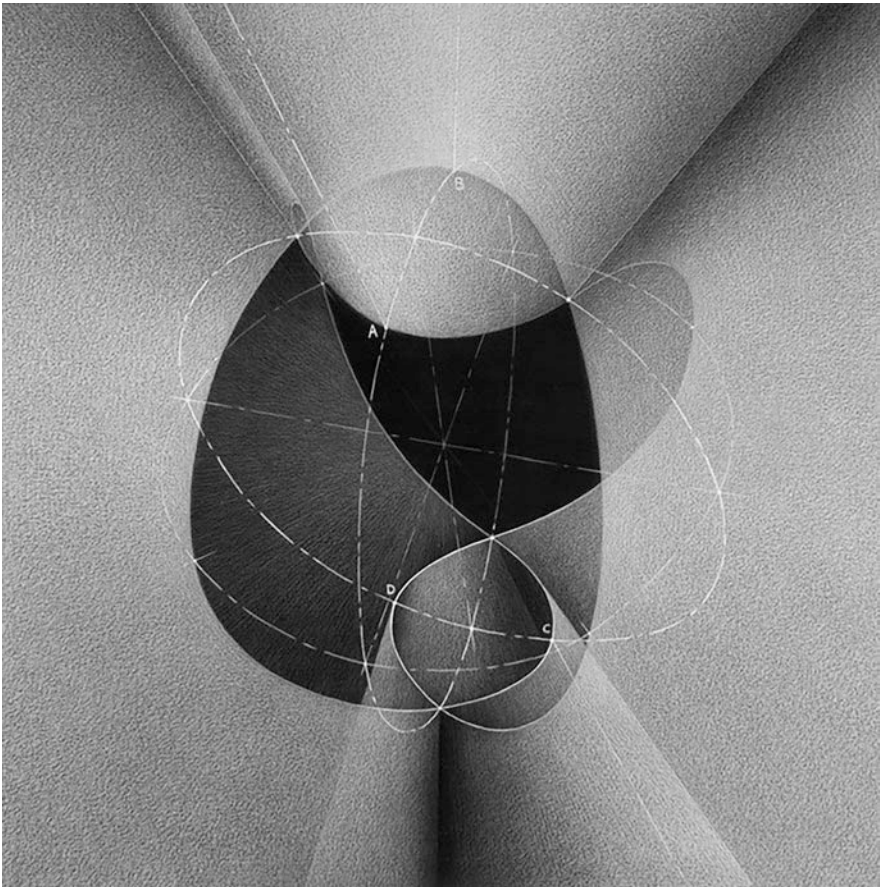

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Introduction

Specialists from various backgrounds, each recognised as references in their field or promising to become so, have been invited in January 2018 to reflect and exchange on a theme that was on the way to becoming a commonplace. This conference was guided by the firm intention of renewing this theme by reactivating all the dimensions that may have been convened in the past, but in scattered order, without avoiding salutary tensions and contradictions.

Such dispersion could not be avoided once again and is, no doubt, due to the nature of the theme itself. One of its poles is the *diagrammatic* and the other is the *question of space* as they intervene in the development of human knowledge in its broadest and most encompassing sense. By indexing these elements under the aegis of a questioning that crosses mathematics, physics, biology, philosophy, literature, and art, that of *form*, we have tried not to drown this heterogeneity under an even broader amphibology, but to turn it towards what resonates as a future or even a promise.

Indeed, diagrams play a fundamental role in the mathematical visualisation and philosophical analysis of forms in space. Some of the most interesting and profound recent developments in contemporary sciences, whether in topology, geometry, dynamic systems theory, quantum field theory, or string theory, have been made possible by the introduction of new types of diagrams, which, in addition to their essential role in the discovery of new classes of spaces and phenomena, have contributed to enriching and clarifying the meaning of the operations, structures, and properties which are at the heart of these spaces and phenomena. This multiplicity of uses covers a certain polysemy that should also be questioned. Diagrams, which are often related to images, drawings, figures, and models, implement a more imaginative and pictorial thinking of scientific and artistic practice, and combine *gesture*, *invention*, and *meaning*. They show that it is possible to elaborate a theory, a model, abstract or concrete, like a thought in movement, which originates in a space that is itself to be reinvented and unfolds in its own time. Developing a diagrammatic thinking thus amounts to trying to understand the dynamics of transformation and the processes of emergence of new properties and qualities of spaces and phenomena.

This volume wants to examine the importance today of diagrams of knots, links, braids, fields, interactions, strings, etc., in topology and geometry, in quantum physics and cosmology, but also in the theories of perception, in the plastic arts and in philosophy. To this end, we propose to study different cases of mathematical and physical theories in which diagrams play an important role, philosophical and phenomenological approaches to space, time, and perception, as well as artistic practices strongly inspired by diagrammatic thinking.

Topological Visualisation, or How to Apprehend the Invisible

The representation of space in topology require a process of "mathematical visualisation" (of idealisation or imagination), which calls upon a new type of intuition, more conceptual and at the same time more pictorial (diagrammatic), and resolutely distant from immediate sensations and empirical intuition. In topology, the figure, the drawing, the diagram, or the graph (in this context, we attribute to these words the same status) are no longer the image of something, of an external object that the image would be in charge of representing, but are themselves the object that represents a universe of relations and "hidden" properties absent from the image. The "semiotization" of the status of the image is even more developed there than in other sciences and has reached a very fine level.

Topology allows another approach to the study of objects which does not restrict itself to quantitative relations of size and visual aspects, but rather considers the form (i.e. the *image* of a deformation, either an embedding or an immersion) as a whole, as well as the spectrum of possible variations (continuous or discrete) of its configurations. It has changed profoundly our scientific thought and culture of the image; it is the domain *par excellence* of images (figures, drawings) in the new and particular sense we have just outlined. It is the most abstract (but, in another sense, the most concrete) part of mathematics, which is defined as the science of transformations of extended objects and more or less abstract spaces by continuous deformation, that is to say, without tearing or gluing. In this field of mathematics, the notions of distance and length no longer play any role, and the concept of homotopy, or more generally of homology proves to be powerful to account for the similarity and intrinsic relations between two objects, two figures, two surfaces, or, more generally, between two abstract spaces. The criterion of similarity by superimposition or by reproduction of sizes, and thus the very notion of visible resemblance, takes all meaning in topology. In no other field of mathematics is the distance from visible reality and the model of figuration as important as it is in topology, where a much subtler concept close to the intrinsic structure of objects plays a fundamental role, namely the qualitative concept (in the topological sense, i.e. where no metric relations intervene) of equivalence or homeomorphism. Thus, in its broad meaning, topology studies the properties of geometrical objects such as manifolds, spaces, knots, and braids that conserve their essential properties when submitted to a certain kind of transformations called homeomorphisms and homotopies.

Feynman Diagrams: A New Way Forward in Theoretical Physics

The Feynman diagrams, and their relationship to knots and links diagrams, are a magnificent example of this point of view, and it is one of the most significant ways of thinking in twentieth century physics. It can be seen as a kind of grammar endowed with the power to create theoretical models and possible interactions between physical but immaterial particles. These theoretical models, symbolised by diagrams generated one after the other thanks to a method of invention, represent a process in the making. Indeed, it allows us to apprehend the meaning of a physical world (that of fields and subatomic particles) invisible and inaccessible to our perception, first through the dynamic construction of a space with a very large number n of parameters (where n is unknown), then by projecting this space onto a simple two-dimensional surface, but which nevertheless conceals hidden possibilities. A fundamental part of physical thought in the twentieth century is closely linked to the invention of the diagrammatic method, largely due to the genius of the physicist Richard Feynman, to whom we owe (together with Tomonaga, Schwinger, and Dyson) the creation in 1948 of Quantum Electrodynamics (QED). It is a quantum field theory that aims to reconcile electromagnetism with quantum mechanics. The concept of field designates a structure that makes it possible to account for the creation or annihilation of particles at any point "in space". Mathematically, it is an Abelian group with a $U(1)$ gauge group of symmetries. The gauge field that intervenes in the interaction between two charges represented by ½ integer spin fields is the electromagnetic field. Physically, this translates into the fact that the charged particles interact through the exchange of photons. Photons, which play a fundamental role in atoms, are like strings that bind electrons to the nucleus. Quantum electrodynamics was the first quantum field theory in which the difficulties in developing a purely quantum formalism allowing the creation and annihilation of particles were satisfactorily resolved, thanks to the so-called renormalisation method, designed to free itself from the infinite undesirable quantities encountered in quantum field theory.

Diagrammatics and Invariants in Knot and Braid Theory

It is a field of research that cuts across several fundamental areas of mathematics, physics, biology, and the philosophy of science. Knots and braids diagrams are among the most fascinating objects of current research at the crossroads of algebraic and geometric topology and quantum field theories. Information about two-dimensional knotted surfaces (i.e. knots that allow for a planar projection) living in the three-dimensional space R^3 or in S^3 results from a structure called a *knot diagram*. For the mathematical study of knots, the fundamental question is whether two knots are equivalent. If they are not, then they can be distinguished, usually by means of numerical, geometrical, or algebraic invariants: one of these invariants is the Jones polynomial. Most of these invariants can be obtained from knots diagrams. If at least two knots are equivalent, then we can define an ambient isotopy that establishes an equivalence relationship between manifolds or subsets of R^n.

In its far-reaching meaning, a knot invariant is a function from the set of all knots to any other set such that the function does not change as the knot is changed (up to isotopy). In other words, a knot invariant always assigns the same value to equivalent knots (although different knots may have the same knot invariant). Standard knot invariants include the fundamental group of the knot complement, numerical knot invariants (such as Vassiliev invariants), polynomial invariants (knot polynomials such as the Alexander polynomial, Jones polynomial, Kauffman polynomial, Homfly-pt polynomial), and crossing, linking, or torsion numbers (i.e. numbers defined in terms of an invariant, and for the torsion number, generated by the finite cyclic covering space of a knot complement). Torsion invariants were introduced by Reidemeister in 1935, and were historically the first non-homotopy invariants for closed three-dimensional Piecewise Linear manifolds.

Starting in the eights the quest for knots' invariants has been one of the most significant lines of research in the field of three-manifold topology. This began with the Jones polynomial invariant for knots in S^3, which was used by Witten to show, with Chern–Simons theory, that the $2+1$ dimensional quantum Yang–Mills theory is exactly soluble. There are combinatorial definitions of the Jones polynomial, basically skein relations that say how the invariant is related before and after changing a crossing in the diagram of a knot or link. These types of relations led to other combinatorial notions of knot invariants and sometimes to 3-manifold invariants. One of the most powerful seems to be the Khovanov homology of knots in S^3. These homology groups should be viewed as a categorification, a kind of enrichment, that is a process consisting in replacing sets point of view by category theory. Like the original Jones polynomial, these invariants are defined starting with a braid presentation of the knot. The invariants of knots and links in S^3 obtained from Chern–Simons theory (a topological three-dimensional quantum field theory) can be used to construct three-manifold invariants. This provides an important tool to study topological properties of three-manifolds.

In the 1990s, it has been uncovered a deep but unexpected relationship between conformal field theories and knot theory. Indeed, quantum field theory can be used to generate new knots polynomials and analytic expressions for them. Knot theory, in turn, is an important tool by which conformal field theories and statistical mechanics can be studied, giving us a topological meaning to quantum groups and to the Yang–Baxter relation.

Philosophical and Scientific Implications

This brings us to the epistemological and philosophical issues. (1) In general, and particularly in the case of knots s and Feynman diagrams in quantum electrodynamics, diagrams have not been used to simply "illustrate" something (objects and events), but rather as symbolic operations or operators of an algebraic and topological nature. Moreover, diagrams make it possible to show and know topological properties of the object, in particular concerning their possible transformations in space and their invariant characteristics. Once known, these topological properties can lead to the discovery of new algebraic invariants of the knot. (2) Diagrams are not limited to offering a picture of the world as it appears to us. A diagram is a powerful symbolic and conceptual construction, a key to reading and rewriting the processes of formation of the "real" world, a kind of open semio-dynamics of his becoming. From this point of view, the diagram translates the form of a way of thinking (of a conceptual strategy) that the physicist geometer is able to give to matter, to the physical world. As a set of plastic qualities of a thought in the process of forming itself as an effective model, as a form, the diagram tends to objectify itself in real processes. Diagrams have three functions at the same time: (a) to elucidate concepts by deploying their articulations within a possible form; (b) to introduce new concepts; (c) to create new properties of the objects that diagrams (graphs, trees) model. (3) Diagrammatic thinking has opened the way to new methods and techniques for visualising objects and phenomena inaccessible to ordinary perception, and also to experiments that can be carried out by even very sophisticated devices. This visualisation has additional explanatory power compared to other classical methods used in the mathematical and physical sciences. Through the creation of mental images of objects and equations, certain visualisation techniques, especially diagrams, are constituted, of which two levels can be distinguished: the first corresponds to models of real objects as they are imagined and not (directly) observed; the second concerns the elements that form the models themselves, i.e. the graphical, denotative and connotative aspects, which give to "see" what is not visible, by a creation of objects and new articulations of meaning.

Diagrammatics and Category Theory

The theory of knots and interlacing gives us to see the stitching of diagram and calculation, implied in each other as in a Feynman diagram. From this interaction between the image and its interpretation, a dialectical pulsation is created between seeing and enunciating in a series of relationships that the diagram imposes and exposes. Explored by Lacan in the meanders of the Borromean knot, the image of these interlaces serves as a support for thought in order to justify and develop the complicated relationships between the real, the symbolic, and the imaginary, and to locate the place of modes of jouissance and meaning between the wedging points of the "bo-knot". It will therefore be a question of studying the theatre of operations of these diagrammatic entanglements, understanding their relationship to the virtual, and giving an account of their creative intuition. But also, to point out the question of indexing, present both in Châtelet's texts and in those of Charles Sanders Peirce, who liked to say that "algebra is nothing other than a kind of diagram".

It is around the algebraic expression of Jones's polynomial that the connections between the low-dimensional topology, of which the theory of knots and links is a part, and the mathematical theory of the categories of Saunders MacLane and Samuel Eilenberg are born. Developed from the middle of the twentieth century onwards, the latter gave rise to many links between various fields of mathematics. The diagram is one of its founding elements and modes of reasoning. It is at work, for example, in Charles Ehresmann's sketch theory and in the theory of locally free diagrams by René Guitart and Christian Lair. These theories deserve to be reinterpreted within the philosophical framework of Alain Badiou's recent writings as set out in his *Logique des mondes*, and of the texts of Gilles Deleuze and Michel Foucault, who saw in Bentham's panoptic a diagram of power, a "political device", organising, according to Deleuze, "a new type of reality". In this confrontation between science and philosophy, notions such as appearance, univocity, duality, and universality will find new dimensions that will extend the reflection on the diagram and the diagrammatic.

A French Singularity in Epistemological Field: Gilles Châtelet

Among the many authors who have devoted all or part of their reflections to this question, Gilles Châtelet is undoubtedly a leading figure to whom this colloquium will pay tribute. Indeed, the mathematical philosopher reminds us that the diagram is distinguished from the figure by the operation that makes it function, by a series of singular points that make it up or that emerge *ab initio* from its own tensions and its own virtualities. For Châtelet, the diagram is "the gestural unfolding of a space" that pushes calculation through "allusive stratagems" that the philosopher in *Les enjeux du mobile* never ceases to hunt down in Argand's texts, Maxwell's electromagnetism, and Grassmann's theories. No less suggestive is Châtelet's renewed look at the history of science and the philosophy of science, as well as at the way in which the phenomenology of space should be resourced with scientific practice. It never goes without a parallel, sometimes implicit and sometimes explicit, with the history of art and painting in particular. It is in this perspective that Châtelet's insistence, following Leibniz, on the "manner of the operation" must be understood. Against a purely operative conception of algebraic and geometric (understood as the provision of a figurative by means of a "simple abstraction" from sensitive space), he rightly insists on the fluidification of space in favour of a *spatium*, understood as the space of virtualities whose constitution "presupposes only a law of coordination of the internal spontaneity of monads".

The resignation that would be the acceptance of a space and a metric already won, attached to the position of a neutral observer, is refused in the name of the requirement of a conquest of space that is a "victory of the 'projective'". Whether it is the conquest of depth in pictorial perspective, the scale of speeds in Oresme's diagrams, Einstein's central intuition of the special relativity that underlies the understanding of Lorentz's contraction, all the experiences of thought or "intuitions" that preside over these "conquests" (or inventions), whether scientific, artistic, philosophical or simply "existential", proceed from a "decision of horizon", where the style of circumspection and the opening of the "thematic" field is decided; which is not surprising, since the horizon "allows us to venture into the turbulent space where science, art and philosophy come close without confusing science, art and philosophy". This decision or "pact" negotiates between two failures: that of geometrical intuition being frozen in "clichés", and that of units of meaning syntactically submitted to the administration of proof. Such pacts correspond to moments of institution and brackets of the "available", and allow for their rediscovery: "a flagrant example is the 'rediscovery' (in Hamilton's case) of imaginary numbers by the introduction of the plane"; "this new encounter of geometry and algebra (and therefore of the visible and calculable) will impose the work of Hamilton and Riemann (and many others), while having an impact on the very notion of the application of mathematics". Such pacts engage us in the wake of Kant, beyond the "classical pact that geometry and algebra had sealed, by tying the images of the first (the "figures") to the literality of the second (the sequences of formulae and calculations of magnitudes)", in this "underbelow" that transcendental aesthetics left in the shadows, without sending it back to an empirical psychology, be it a psychology of the creative imagination or of the mathematical imagination.

The epistemology suggested by Châtelet is a tracking of these gestures and moments of disaffection with regard to clichés, misleading analogies, as well as the domesticated operative, which has become routine. However, this kind of parenthesis has the opposite side to a full and complete subjective involvement in an experience of thought and intropathy, of *Einfühlung*. This point is essential because it provides us with the link between this epistemological attitude induced by the explicit promotion of the diagram to the rank of formal object and writing and the aesthetic attitude instituted by explicitly diagrammatic practices in the plastic arts. Châtelet's study is exemplary in this respect with regard to the *diagrams of intentionality* presented on the occasion of the developments on Grassmann, whose schema and general tendency need only be retained. The reading of these diagrams is pushed by Châtelet to the level of a choreography, depicting the scene of intersubjectivity, resulting in a profound modification of the (classical) subject/object relationship. Intentionality is in turn disturbed in turn, the subject *S* being haunted by the object *O*, haunting or reflecting the impassable abyss between two I's.

Phenomenology of Space (and Time) and Diagrammatic Epistemology

In the wake of these reflections, we are invited to rethink the place and role of a phenomenology of space. By stripping away structures, we discover—underneath the constituted—the trace of these gestures and their rich potential. This is how the diagrams are introduced. It is still through the diagram that one can find and make one's own experiences of radical thought. This is the case with the thought experience of Galileo, the Einstein of General Relativity, or Archimedes. The diagram records and transmits these radical thought experiences. They are a trace of them. In this conception of science, understanding and scientific learning, the diagram plays a decisive role, because diagrammatic communication is based on "intropathy". The appropriation of these experiences therefore presupposes that we find the gesture, the embodied operativity; that we plunge back into the constituent variety of subjective ways of "doing", which appear, after the fact, in the eyes of a normalised science, like a halo or a contingent gangue. But it is first of all constitutive: inventio itself presupposes such a "putting oneself in the place of". What is sought in this way under the diagrams and the techniques of forcing intuition, of provoking an algebraic intuition likely to upset the trivialities of a logically domesticated calculable, are in fact the gestures of constitution and institution. Since the task of phenomenology is to "explicitly renew" these formations of meaning by reactivating them, it must stop naively making use of them to consider them in their formation and in their use. This supposes that, in the description and discourse held on this subject, another form of reading of these "legacies" is practised. This legacy presupposes that they are made available in the form of "writing systems" and a socially constituted and transmitted habitus capable of reading and using what is thus made available. This presupposes the acquisition of new linguistic and graphic "tools" to "describe" the provision itself.

A "transcendental" moment comes to lie at the heart of the institution or the renewal of these practices, each time the hypotheses on which the previous practices were based are awakened and shaken up. And, it is possible to consider as a sample of phenomenology the explicit reflection that accompanies these moments of invention. The attention to the foundations and underpinnings of constructive symbolic activity is phenomenological. Phenomenology digs into the "model cupboard", into the back-kitchen of science. In its exploration of the deepest constitutive levels of constituent subjectivity, it must strive to "re-seize" and "reactivate" the inaugural "gestures", which are not only those of a proto-geometer from an antiquity as mythical as it is remote, but also and first and foremost those of each "inventor". But it is one thing, it will be said, to elucidate the role of the diagrammatic in intellectual activity (categorical synthesis), it is another to use diagrams in the course of this elucidation itself, as Husserl does to study a level of synthesis that is precisely not intellectual, and that intervenes, it seems, in the lowest and most immediate levels of constitution, as is the case with everything that touches on "aesthetic" syntheses: those which are constitutive of our consciousness of space and time.

Towards a "Diagrammatic Critique of Aesthetics"

That this area is also the place of the technicality of reason, that is to say of mathematics and art, is what we should never cease to meditate on; but first and foremost, this deployment takes place through a communicability of gestures, that is to say of hands and manners, "all this talking with hands" "which should perhaps be better called talking with hands" and which should not lead us to conceal the "hands" or the "manners" of speaking, this dimension of the linguistic expression which Kant in the third critique designates as being that of gesture (and gesticulation) always joint, in the spoken word, to word and tone (to articulation and modulation). In the field of the arts, it will therefore be a question of questioning the diagram as a gesture, in the act of writing, photographing, painting, or composing music, but also in its own genesis, in what precedes thought, in the unthought: the diagram as a technical device for indexing, preconceived or not, random or deterministic. In Francis Bacon's interviews with David Sylvester, the act of painting presupposes that there exists on the canvas a set of figurative data, more or less virtual, more or less current, constituting an intermediate place where the play of forces can be exercised. According to these forces, new data will appear, disappear, fade, or stand out according to the artist's desire or will. From a chaotic, primitive, structural, or algorithmic primordial form, the artistic diagram will be born in a middle ground, in a pure becoming. Because etymologically, the word diagram in Greek is the deverbal noun of *diagraphein*, that is to say literally "through the writing", it is posed as a transverse axis to the artist's gesture. The function of a diagram is always to make something explicit, and according to Châtelet, to immobilise "a gesture in order to establish an operation of amplification and intuition". From Kandinsky to Paul Klee, from Pollock to Bacon, or for artists such as Ricardo Basbaum, Wacław Szpakowski, Daniel Sheets Dye, Mark Lombardi who work on lines, their arrangements and relational networks, or for artists such as Edward Tufte, who works on the notion of Feynman's diagram, the diagram does not have here the same meaning or the same status, or particularly for artists such as Jorge Eduardo Eielson and Robert Morris, who worked on different types of knots and their virtual artistic expressions. We will therefore attempt to analyse the aesthetic outline of the diagram, to problematise its links with the work and the artistic process, to situate diagrammatic creativity as a processual whole, that is to say, in short, we will attempt to sketch out a "diagrammatic critique of aesthetics".

The book is divided into eight parts devoted to various mathematical and philosophical themes. The first part of the book is on "Logic, Forms, and Diagrams". The purpose of the paper of L.H. Kauffman is to explore the idea of a sign, using G. Spencer-Brown's work "Laws of Form" as a pivot, a reference, and a place from which Kauffman makes excursions into simplicity and complexity. Julien Bernhard proposes a thorough inquiry on roles of diagrams in logic. Are they merely pedagogical tools, or are they effectively endowed with a demonstrative force and eventually a heuristic potential, at least in the discovery of inferential forms? The paper of Franck Jedrzejewski is a rigorous tentative of giving an interpretation of knot and link diagrams and invariants in terms of theory of category.

The second part of the book is on "Geometrical Spaces and Topological Knots, Old and New". The nice paper of Marco Andreatta is an historical presentation of some ideas and concrete constructions of geometrical surfaces. He briefly considers some recent results in higher dimensional algebraic geometric, which can be summarised in few diagrams. He also points out that some of these ideas and diagrams could be directly connected to biology and life sciences. Alessandro Verra shows that geometry undergone revolutionary shifts of paradigm during all the last century. He addresses the issue of the difference between historical and contemporary geography of algebraic geometry, trying to bring some evidence and concrete examples from the experience of a person working in this field. A major attention is payed to the history of classical and modern rationality problems, for some famous examples of algebraic varieties. In its paper, Luciano Boi shows that knot theory has extensive interactions, not only with different branches of mathematics but also with various and fundamental areas of physics. Knots and links are deeply related to the geometry of 3-manifolds and low-dimensional topology, quantum field theory, and fluid mechanics. The paper surveys some current topics in the mathematical theory of knots and some of their more striking ramifications in physics and biology. This article aims at stressing the importance of considering diagrams in the study of topological and geometrical objects and the key role of knot theory for the understanding of the structure of space and space-time.

The third part of the book is on "Diagrams, Graphs, and Representation". Carlo Petronio explains how planar diagrams, equipped with suited depictions, can be used for describing topological objects of dimension 1, 2, 3, and 4. In that paper, Patrick Popescu-Pampu presents some problems which led to the introduction of special kinds of graphs as tools for studying singular points of algebraic surfaces. He explains how such graphs were first described using words, and how several classification problems made it necessary to draw them, leading to the elaboration of a special kind of calculus with graphs. This non-technical paper is intended to be readable both by mathematicians and philosophers or historians of mathematics.

The fourth part of the book deals with "Diagrams, Physical Forces, and Paths Integrals". In the paper of Sergio Albeverio, Feynman path integrals are first presented for the case of non-relativistic quantum mechanics, both in physical and mathematical terms. Then the case of scalar relativistic and Euclidean quantum fields is discussed. The methods of (constructive) perturbation theory and renormalisation theory in relation to Feynman path integrals are briefly considered, in particular mentioning the visual help provided by Feynman diagrams. The paper ends with mentioning some open problems and presenting some philosophical remarks and reflections on the description of natural phenomena, in particular those of fundamental physics, in mathematical terms. Jean-Jacques Szczeciniarz paper contains some remarks on Penrose diagrams. He explains what a conformal diagram is by expounding the theoretical and geometrical elements which constitute it: complex geometry in several variables, projective geometry, conformal geometry.

Finally, he briefly presents the cosmological and philosophical significance of the synthesis realised by Roger Penrose.

The theme of the fifth part is "Phenomenology in and of Mathematical Diagrams". Frédéric Patras paper is a tentative of giving representations of elementary geometric and combinatorial objects, with particular emphasis on the so-called non-crossing set partitions, in a Husserlian perspective. The paper of Arturo Romero first presents the inner relation between the phenomenological concept of intentionality and space in a general mathematical sense. Then briefly characterises the use of the geometrical concept of manifold (*Mannigfaltigkeit*) in Husserl's work. Next, he presents some examples of the use of the concept in Husserl's analyses of space, time, and intersubjectivity, pointing out some difficulties in his endeavour. Finally, he offers some points of coincidence between phenomenology and category theory suggesting that the latter can work as a formal frame for ontology in the former. The thesis of the paper is that intentionality operates in different levels as a morphism, functor, and natural transformation. Carlos Lobo's paper is a study of the way in which diagrams of time have functioned in Husserlian phenomenology in the most fertile period on the subject, i.e. between the lessons of 1905 and the Manuscripts from 1917-1918. While doing so, he offers a phenomenological clarification on the role of diagrams in science as well as in phenomenology.

The sixth part is on "Diagrams, Gestures, and Subjectivity". The paper of Hye Young Kim presents a topological analysis of space-time consciousness. The paper attempts to explore a possibility to visualise the structure of time-consciousness in a knot shape. By applying Louis Kauffman's knot-logic, the consistency of subjective consciousness, the plurality of now's, and the necessary relationship between subjective and intersubjective consciousness are represented in topological space. Philippe Roy paper is on gestures, diagrams, and subjectivity. According to the author, the subjectivity can be thought through the new articulations of the dualities "Individual/Society" or "Individual unity/Multiplicity of becoming", rather than by the category of substance. The paper of Filipe Varela presents some grounds to further understand what imagination is, what it does, how it possibly works, and where it may be located in our nervous systems. Such an approach follows the imperative to correlate the precious insights coming from philosophy, with those of the sciences that nowadays study our perceptual and cognitive apparatus from a physiological and bio-chemical perspective. Such correlation is vital to a comprehensive understanding of this elusive thing, imagination. The paper of Fabien Ferri is on the diagrammatic language beyond the phenomenological difference. In other words, following the ideas proposed by Bruno Bachimont, the author discusses the possibility of overcoming the traditional difference between scientific language or knowledge whose main function is to calculate, and phenomenological language or knowledge, which can be expressed by natural language and where "saying stands for having a meaning".

The seventh part is about "Diagrams, from Mathematics to Aesthetics". The paper of Charles Alunni shows how *Ars diagrammaticae* discusses the significance of the art of diagram and its philosophical implications through the analysis of some examples took from modern and contemporary mathematical physics. Alunni draws on the concepts and operations at work in contemporary mathematics to question the classical philosophical distinctions between image, figure, and diagram. The paper of Jakub Zdebik shows how, despite its abstract and non-figurative character, Gilles Deleuze's diagram has stimulated a deep methodological renewal in aesthetics of visual arts. Amélie de Beauffort's paper focuses on the idea that drawing allow for a plastic foundation of thought, which is important for creativity and also for living in the world. This viewpoint is motivated by a graphic practice which rests on a plastic morphology of the manipulation of knots and gestures that make and unmake the drawing. Farah Khelil paper deals with the emergence of a reasoning owing to the diagram in its artistic practice and in particular in a recent work entitled *Point d'étape*. She clarifies the singularity of the diagram with respect to the relation of art with philosophy and its relationship with the reality. She also enquires the painting and its relationship to space.

The eighth part of the book is on "Poetics and Politics of Diagrams". Catherine Paoletti focuses on the tentative by Gilles Châtelet, who stressed that the diagram is at the same time tool, object, and locus of thinking, which give rise to new spatial configurations, therefore to potential movements in space and also to new dynamics in the writing. The paper of Tatiana Roque is on diagrams of the possibility, by establishing a link between the phase space (a mathematical object) and the political subject. The function of the diagram is, as stressed by Deleuze, a goal which introduces new effective possibilities in science, as well in social sciences; in other words, it is an operative concept which can change reality. In its paper, Noëlle Batt, after briefly recalling the evolution of the concept of diagram in the philosophical work of Gilles Deleuze, first present the hypothesis that literary and more particularly poetic writing has a diagrammatic dimension. The hypothesis is sustained by a number of resonances between, on the one hand, the features and morpho-dynamic processes which are prominent in the definition of diagram proposed by different disciplines, and, on the other hand, those which characterise the specific reorganisation of signs and infra-semantic elements of the language which constitute the material of the literary text.

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The Semiotics of Laws of Form

Louis H. Kauffman

Abstract The purpose of this essay is to explore the idea of a sign, using G. Spencer-Brown's work "Laws of Form" as a pivot, a reference and a place from which to make excursions into both simplicity and complexity. In order to handle the simplicity of the issues involved in thinking about distinction, Spencer-Brown's introduction of a language that has only one sign is an instrument of great delicacy.

Keywords Semiotic · Diagram · Logic · Proof · Knots

Introduction

The purpose of this essay is to explore the idea of a sign, using G. Spencer-Brown's work "Laws of Form" (Spencer-Brown 1969) as a pivot, a reference and a place from which to make excursions into both simplicity and complexity. In order to handle the simplicity of the issues involved in thinking about distinction, Spencer-Brown's introduction of a language that has only one sign is an instrument of great delicacy.

The Spencer-Brown mark $\urcorner$ is a sign that can represent any sign, and so begins semiotics in both universal and particular modes. The mark is seen to make a distinction in the space in which it is written, and so can be seen, through this distinction, to refer to itself. In the language of Charles Sanders Peirce, the mark is its own representamen and it is also its own interpretant. The sign that the mark produces for somebody is, in its form, the mark itself. By starting with the idea of distinction we find, in the mark, the first sign and the beginning of all possible signs.

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Right, that's how long it takes, not a day less,—Qfwfq said,—once, as I went past, I drew a sign at a point in space, just so I could find it again two hundred million years later, when we went by the next time around. What sort of sign? It's hard to explain because if I say sign to you, you immediately think of a something that can be distinguished from a something else, but nothing could be distinguished from anything there; you immediately think of a sign made with some implement or with your hands, and then when you take the implement or your hands away, the sign remains, but in those days there were no implements or even hands, or teeth, or noses, all things that came along afterwards, a long time afterwards. As to the form a sign should have, you say it's no problem because, whatever form it may be given, a sign only has to serve as a sign, that is, be different or else the same as other signs: here again it's easy for you young ones to talk, but in that period I didn't have any examples to follow, I couldn't say I'll make it the same or I'll make it different, there were no things to copy, nobody knew what a line was, straight or curved, or even a dot, or a protuberance or a cavity. I conceived the idea of making a sign, that's true enough, or rather, I conceived the idea of considering a sign a something that I felt like making, so when, at that point in space and not in another, I made something, meaning to make a sign, it turned out that I really had made a sign, after all.

In other words, considering it was the first sign ever made in the universe, or at least in the circuit of the Milky Way, I must admit it came out very well. Visible? What a question! Who had eyes to see with in those days? Nothing had ever been seen by anything, the question never even arose. Recognizable, yes, beyond any possibility of error: because all the other points in space were the same, indistinguishable, and instead, this one had the sign on it.

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I thought about it day and night; in fact, I couldn't think about anything else; actually, this was the first opportunity I had had to think something; or I should say: to think something had never been possible, first because there were no things to think about, and second because signs to think of them by were lacking, but from the moment there was that sign, it was possible for someone thinking to think of a sign, and therefore that one, in the sense that the sign was the thing you could think about and also the sign of the thing thought, namely, itself.

So the situation was this: the sign served to mark a place but at the same time it meant that in that place there was a sign (something far more important because there were plenty of places but there was only one sign) and also at the same time that sign was mine, the sign of me, because it was the only sign I had ever made and I was the only one who had ever made signs. It was like a name, the name of that point, and also my name that I had signed on that spot; in short, it was the only name available for everything that required a name.

***

In the universe now there was no longer a container and a thing contained, but only a general thickness of signs superimposed and coagulated, occupying the whole volume of space; it was constantly being dotted, minutely, a network of lines and scratches and reliefs and engravings; the universe was scrawled over on all sides, along all its dimensions. There was no longer any way to establish a point of reference: the Galaxy went on turning but I could no longer count the revolutions, any point could be the point of departure, any sign heaped up with the others could be mine, but discovering it would have served no purpose, because it was clear that, independent of signs, space didn't exist and perhaps had never existed. [A Sign in Space, Cosmicomix by Italo Calvino. Copyright © 1965 by Giulio Einaudi Editore, S.p.A. English translation copyright © 1968 by Harcourt Brace & Company and Jonathan Cape Limited]

Finding Distinction

We begin by discussing (the idea of) distinction. If one looks for the definition of distinction in any dictionary the result is circular. Distinction is defined as a difference. Difference is defined as a form of distinction. The meaning of distinction as an indication of outstanding value is also an instance of special difference. Fields of study are founded in the use and examination of certain basic distinctions.

Mathematics is constructed set theoretically by using the concept of a collection. A collection is a distinction of membership. For example the set of prime numbers connotes the distinction between composite and prime among the positive integers. At the level of sets themselves, the empty set, denoted by brackets containing nothing {}, is a distinction between void and an empty container. The very sign for the empty set consists of two brackets (left and right) that together can be interpreted as a container for something that is placed between them. In the case of the empty set, nothing is placed between the brackets. The brackets themselves are shaped as cusps.

$$\{ \quad \}$$

Each cusp can be seen as a process of bifurcation that gives rise to the distinction between the branches of the cusp. The two cusps (brackets) are mirror imaged with respect to one another, and it is this symmetry across an imaginary mirror between them that gives us the possibility to see them together as one container. The brackets are two and yet they are one (via the mirror symmetry).

At this point (in the encounter with the empty set) we reach a semantic divide between the mode of speaking of mathematicians trained in logical formalism and a wider analysis of language that I refer to as semiotic. In speaking of semiotics I am relying for its root meanings as expressed by Charles Sanders Peirce:

> [Semiotics is a] quasi-necessary, or formal doctrine of signs ... which abstracts what must be the characters of all signs used by an intelligence capable of learning by experience, ... and which is philosophical logic pursued in terms of signs and sign processes. [Peirce, C. S., *Collected Papers of Charles Sanders Peirce*, vol. 2, paragraph 227]

> A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object not in all respects, but in reference to a sort of idea which I have sometimes called the ground of the representation. [Peirce—Vol. 2, p. 228]

Peirce goes on to say

> The object of representation can be nothing but a representation of which the first representation is the interpretant. But an endless series of representations, each representing the one behind it, may be conceived to have an absolute object as its limit. The meaning of a representation can be nothing but a representation. In fact, it is nothing but the representation

> itself conceived as stripped of irrelevant clothing. But this clothing never can be completely stripped off; it is only changed for something more diaphanous. So there is an infinite regression here. Finally, the interpretant is nothing but another representation to which the torch of truth is handled along; and as representation, it has its interpretant again. Lo, another infinite series. [Peirce—Vol. 1, p. 339]

Peirce concentrates on the structure of signs and that for him signs either are, or stand for, certain distinctions. To begin with signs is to begin with something apparently definite and yet, as soon as the discussion begins, we find there are only signs (see above) "A sign, or representamen, is something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent sign." Thus what is in the mind of another person is also a sign, albeit a sign that is understood internally by that person. One can look and look for substance that may underlie the sign but the search always leads to more signs. In this expansion of signs related to signs, signs describing signs, signs and interpretant signs, the self becomes yet another sign standing in relation to all the signs that work at the nexus that the person represents. The sign of the self becomes a limit of all the signs that are the life of that self. The distinction of a person is her sign of distinction, her sign of self.

Spencer-Brown in his book "Laws of Form" (here to be abbreviated LOF), makes a new semiotic start, beginning with the idea of distinction. Signs arise as we shall see, but Spencer-Brown begins with the pronouncement:

> We take as given the idea of distinction and the idea of indication, and that we cannot make an indication without drawing a distinction. We take, therefore, the form of distinction for the form. (LOF, page 1).

It is as this point that Peirce and Spencer-Brown come into contact. For in Chapter 12 in the last sentence of Laws of Form, Spencer-Brown writes "We see now that the first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical." Here the mark is the first made sign or indication of a first distinction. The observer can be identified with the interpretant in so much as the interpretant (see the quote from Peirce above) is an equivalent sign created in the mind of somebody, and must for its existence partake of the being of that somebody. At this nexus Spencer-Brown indicates the essential identity of sign, representamen and interpretant. The three coalesce into the form that is the form of distinction.

The form of distinction becomes, in Spencer-Brown, a background for the entire play of signs that is the context of Peircean semiotics. We take the form of distinction for the form. And in this saying "the form" becomes a noun as elusive as it seems to be concrete, just as is the nature of the sign in Peirce. The form of a distinction drawn as a circle in the plane is geometrical form, the circle. But the form of distinction, the form of the idea of distinction, what is this form?

The echo from Peirce is clear as a bell. The form of distinction calls up a sign in the mind of some person. It is an amalgamation or superposition of all the signs for distinction in the history of that mind, that observer or all observers. We come forth in the complexity of experience to a sharp idea of the distinct. We can give instructions for the performance of an act of distinction, while simultaneously

Definition

Distinction is perfect continence.

That is to say, a distinction is drawn by arranging a boundary with separate sides so that a point on one side cannot reach the other side without crossing the boundary. For example, in a plane space a circle draws a distinction.

Once a distinction is drawn, the spaces, states, or contents on each side of the boundary, being distinct, can be indicated.

There can be no distinction without motive, and there can be no motive unless contents are seen to differ in value.

Fig. 1 Definition

understanding that it is a creative act, not bound by any given set of rules or regulations.

The next lines of Laws of Form are shown in Fig. 1. We give these quotes by direct photocopy to show the layout in the text of Laws of Form. After some thought the reader may come to realize that this paragraph is an amalgam of words that all stand for aspects of distinction: *definition, continence, boundary, separate, sides, point, draw a distinction, spaces, states, contents, side of the boundary, being distinct, indicated, motive, differ in value*. The paragraph is not a definition in the mathematical sense of definition: something in terms of previously defined things. There is no possibility to define distinction in terms of previous somethings that are not distinctions. The only possibility is to define distinction in terms of itself. We take the form of distinction for the form.

The paragraph is nevertheless readable. How did this happen? How could readability arise from circularity? The answer is in the injunctive power of language. This same paragraph contains the phrases: "arranging a boundary so that a point on one side cannot reach the other side without crossing the boundary", "a circle draws a distinction", "a distinction is drawn", "spaces, states, or contents … can be indicated", "contents are seen to differ in value". At once the paragraph is an amalgam of synonyms for distinction and it is a catalog of injunctions to arrange a boundary, to draw a circle, to indicate, to see the difference in value. We are invited to take these steps and so enter into a contract of exploring the concept and practice of distinction.

Let us not forget that we have followed already the injunction of the first line: "We take as given the idea of distinction and the idea of indication and that we cannot make an indication without drawing a distinction." It is already given that there is a something called indication that entails the making of a distinction. And implicitly it is called up that a distinction could occur without any indication. We cannot make an indication without drawing a distinction. Can we have a distinction without making an indication? We are falling down the rabbit hole.

> If a content is of value, a name can be taken to indicate this value.
>
> Thus the calling of the name can be identified with the value of the content.
>
> **Axiom 1. The law of calling**
>
> *The value of a call made again is the value of the call.*
>
> That is to say, if a name is called and then is called again, the value indicated by the two calls taken together is the value indicated by one of them.
>
> That is to say, for any name, to recall is to call.

Fig. 2 Calling

But here, we have to look and see. In most circumstances, to draw is to indicate. A notion of privacy is another form of distinction. Can I hide distinctions within the boundary of my privacy distinction? Then I can pretend that there are distinctions that do not have indications. What a tangled web we weave in order to believe. In order to make a distinction without an indication, we are entangled in a web of new distinctions. The very act of drawing is a form of indication, and it must be concealed? Must we search for distinctions that are made without drawing a distinction? I sit before an emptiness. The emptiness is distinct for me. It is empty and I am empty before it. It is possible to have less action not more. And in the limit of acting gently in emptiness, or not at all, there seems to be the possibility of distinction without indication.

Figure 2 illustrates the next few lines of Laws of Form. Take the first sentence shown in Fig. 2: “If a content is of value, then a name can be taken to indicate this value.” Already we have faced the multiplicity of names for a distinction and that making an indication is a special act that cannot happen without the making of a distinction. Nevertheless, it comes as a shock that suddenly a *name* can be called forth. A name can be taken to indicate a value. A distinction can be performed that allows the performance of a distinction. We begin to realize that in this condensed place where there is only creation of distinction, boundary or the crossing of the boundary, the only distinction is at first the distinction between nothing (the unmarked) and the act of creation, and then arises a distinction between name and act. If a state is of value then a name can be taken to indicate this value. If a distinction is a distinction then a distinction of distinctioning can be distinctioned to distinction this distinction. We are down the rabbit hole again. One side makes you smaller. One side makes you larger. Choose a door and pass through it. The act and the name are not different. The indication of a distinction, the crossing of the boundary of the distinction and the distinction “itself” are in the form identical.

We come to the creation of a name and find that this is the same as the creation of a distinction. They are one and the same. And yet a name can be separated from the distinction to which it refers. The name can be taken to be a new distinction that refers to the first distinction. Indeed we can imagine that the original distinction (for example a circle drawn in the plane) is seen (in all quietness) to stand for, to indicate, itself. But in the act of recognizing this possibility that "it" could stand for "itself" we have made a distinction between "it" and "itself". We have allowed a condensation by making the possibility of a separation. The name and the sign are born in that process. The name, the sign, is Peirce's representamen, a sign residing in the mind of somebody. And we conclude that both the sign, the name and the original distinction all reside in the mind of somebody. At the point of condensation, the mind is the sign and the sign is the mind. No mind, no distinction. No distinction, no mind. We take the form as the form of distinction. Form is emptiness. Emptiness is form.

In his "A Note on the Mathematical Approach" Spencer-Brown writes "The act is itself already remembered ... as our first attempt to distinguish different things in a world where, in the first place, the boundaries can be drawn anywhere we please. At this stage the universe cannot be distinguished from how we act upon it, and the world may seem like shifting sand beneath our feet." The act of naming ("If a content is of value, then a name can be taken to indicate this value.") is the key step toward a world of apparent distinctions. It is by naming a distinction that we call it into being. In Fig. 2 we see the first of the Laws of Form, the "Law of Calling: The value of a call made again is the value of the call." It is enough to indicate the name once. For any name, to recall is to call.

In Fig. 3 we find the "Law of Crossing: The value of a crossing made again is not the value of the crossing.". At this point a distinction is made between crossing (the boundary of a distinction) and calling the name of a distinction. For "The value of a call made again is the value of the call." Crossing and calling appear to be given as terms in a similar level of speech, and yet they are declared to be different. We understand that the crossing of the boundary can be the act of naming the distinction. I cross into "riding" when I cross the boundary of balance and actually ride the bicycle. I name riding by actually engaging in the act of riding. If I cease to ride,

Axiom 2. The law of crossing

The value of a crossing made again is not the value of the crossing.

That is to say, if it is intended to cross a boundary and then it is intended to cross it again, the value indicated by the two intentions taken together is the value indicated by none of them.

That is to say, for any boundary, to recross is not to cross.

Fig. 3 Crossing

Construction

Draw a distinction.

Content

Call it the first distinction.

Call the space in which it is drawn the space severed or cloven by the distinction.

Call the parts of the space shaped by the severance or cleft the sides of the distinction or, alternatively, the spaces, states, or contents distinguished by the distinction.

Intent

Let any mark, token, or sign be taken in any way with or with regard to the distinction as a signal.

Call the use of any signal its intent.

Fig. 4 Construction, content and intent

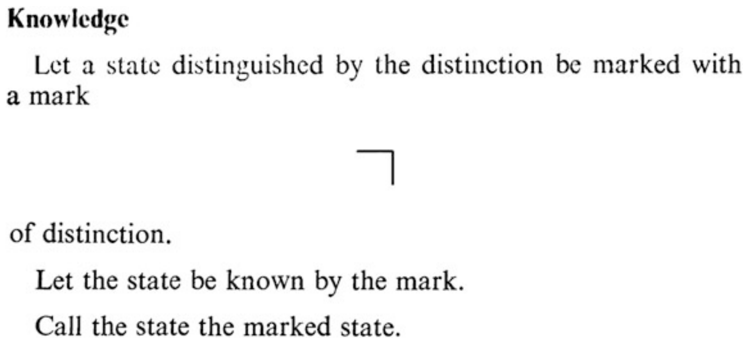
then the value of riding ceases. The distinction of riding is no longer present. And yet it can still be named.

At this point, we have come to the end of Spencer-Brown's discussion of Laws of Form that makes no explicit use of a sign of distinction. The word "sign" has not yet occurred for Spencer-Brown. The first use of the word sign is in the next chapter of the book entitled "Forms Taken Out of the Form" and is shown in Fig. 4.

In this development. The injunctive mode has taken priority. The text tells its reader to "Draw a distinction." and to "Call it the first distinction.". This should sweep away any notion that first distinction is an absolute concept.

The first distinction is the one that is under discussion. The form is the form of the first distinction. And so the form of the first chapter has shifted from the universal to the particular, and the form of distinction is the form of that first distinction. The form is inherent in any act of distinction. Still speaking of Fig. 4, we find that at the point of intent "Let any mark, token, or sign be taken in any way with or with regard to the distinction as a signal. Call the use of any signal its intent.". Here is the entry of the word sign into Spencer-Brown's consideration of distinction and form.

Now we listen again to Peirce. "A sign, or representamen, is something which stands to somebody for something in some respect or capacity." Indeed Spencer-Brown's mark, token or sign is a sign in the Peircean sense. This sign is taken as a signal (in the condensation of Laws of Form, a signal is yet a sign) with regard to or



Knowledge

Let a state distinguished by the distinction be marked with a mark

⏋

of distinction.

Let the state be known by the mark.

Call the state the marked state.

Fig. 5 The introduction of the definition just above (text and symbol)

of the first distinction. Spencer-Brown does not say that this mark is the first sign, but sign it is and with it it is possible to indicate the first distinction.

Finally there enters upon the stage of distinction the mark that will be the pivot for the formalism of Laws of Form. See Fig. 5.

Spencer-Brown writes “Let a state distinguished by the distinction be marked with a mark ⏋ of distinction.” The mark is written upon one side of the first distinction. We shall take the liberty of illustrating this in Fig. 6. This mark is chosen to make and to indicate a distinction in its own form. The mark has (for the observer—our word for Peirce’s somebody) an inside and an outside. Spencer-Brown says, quite explicitly “ Let each token of the mark be seen to cleave the space into which it is copied. That is, each token be a distinction in its own form.” And before this, he gives permissions: “Call the space cloven by any distinction, together with the entire content of the space, the form of the distinction.

Call the form of the first distinction the form. . . . Let there be a form distinct from the form. Let the mark of distinction be copied out of the form into such another form.

Call any such copy of the mark a token of the mark. Let any token of the mark be called as a name of the marked state. Let the name indicate the state.” Now the circle has again closed. Each token of the mark is a sign and a copy of the mark itself. Each token and indeed the mark itself is a distinction in its own form. There is now a plethora of signs, marks and forms. They all indicate the marked side of the first distinction. Only one distinction is being discussed. As many marks as may be needed are available to signal this distinction. We embark upon, not just form, but formalism and the inception of calculation.

As we show in Fig. 6, the mark can indicate the outside of the first distinction, and we take the mark to make a distinction in the outer space of that first distinction. We see then that *we can take the mark itself as the first distinction.*

This move brings the discussion directly in coincidence with C. S. Peirce’s semiotics. If the mark or sign is the first distinction, then it is a sign for itself. It is a sign that makes a distinction and it is a sign that stands for the outside space of that distinction.

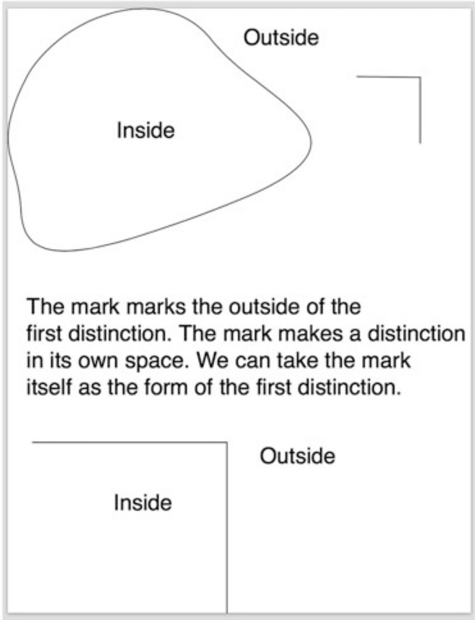


Fig. 6 The mark marks the outside of the first distinction

We are now in a position to summarize the semiotic development of the Spencer-Brown mark

$$\urcorner.$$

The mark is a sign that makes a distinction in the plane within which it is drawn. In that plane there is a distinction between the (bounded) inside and the (unbounded) outside of the mark. The mark is chosen to refer to the outside of the distinction that it makes in the plane. The mark can be seen to refer (via referring to the outside of the distinction that it makes) to itself as the boundary of that distinction. Thus we can write the Law of Calling in the form

$$\urcorner\urcorner = \urcorner.$$

Each mark in the expression on the left is a sign or name for the outside of the distinction made by the other mark in the expression. Each mark is the name of the other mark. The calling of a name made again may be identified with the calling of the name. And so we have the equation as indicated above, condensing the two marks to a single mark.

We take the mark to indicate a crossing from the state indicated on its inside. $\overline{A}\rvert$ denotes the state obtained by crossing from the state indicated by A.

Hence $\overline{\overline{\;}\rvert}\rvert$ indicates the state obtained by crossing from the marked state. Hence $\overline{\overline{\;}\rvert}\rvert$ indicates the unmarked state. In equations, we have the Law of Crossing:

$$\overline{\overline{\;}\rvert}\rvert = \quad .$$

The value of a crossing made again is not the value of the crossing.

We have arrived at a self-referential nexus. The mark, first sign, refers to itself. The first sign is a name and it is identified with the action of crossing from the unmarked state (the state with no sign). We began with the idea of distinction.

The sign of the first distinction acts as the transformation and the boundary between the unmarked state (the state with no sign) and the marked state. The sign of the first distinction is a signal of the emergence of articulated form. The sign of the first distinction is, in the form, identical with the first distinction.

Finding Primary Arithmetic

The formalism that we have arrived at is directly connected with mathematics. Let us recall where we are. We have one sign $\overline{\;}\rvert$ and two laws or rules about that sign:

$$\text{The Law of Crossing: } \overline{\;}\rvert\,\overline{\;}\rvert = \overline{\;}\rvert .$$

$$\text{The Law of Calling: } \overline{\overline{\;}\rvert}\rvert = \quad .$$

At this stage in the development of the sign, these laws are statements about naming and about the crossing of the boundary of an initial distinction. The initial distinction can be the distinction made by the sign itself.

And yet there is another sign. It is the equal sign. And with that sign we enter mathematics.

With the equal sign we formalize condensations of reference and meaning.

It is implicit that we may write expressions such as

$$\overline{\overline{\overline{\overline{\overline{\;}\rvert}\rvert}\rvert}\rvert}\rvert$$

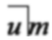
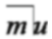
and wonder what this nest of signs can mean. And we find that we have already defined the meaning of this new sign as a five-fold act of crossing-from the previous state, starting from the unmarked state. And being able to count, we know that this means that we will arrive at the marked state after such a process. Thus we have

An infinity of possible equalities of concatenations of signs has opened up before us and since we know how to count, we can evaluate them all and find either the marked state or the unmarked state as an equivalent to each one. Do we need to know how to count to accomplish this task? We do not need to know how to count. We can apply the laws of calling and crossing where we find them. An empty cross with a cross over it can be regarded as an instance of the law of crossing:

The two innermost marks in the left hand nest of marks are an instance of the law of crossing and we can erase them, forming the right hand side with only three marks.

Doing this once more, we find

and the marked value of the nest has been uncovered without the need for counting.

Here is another way. Let u and m stand for the unmarked and marked states respectively. Agree that  has the marked state as its outside value and write  to indicate this state of affairs. Agree that  has the unmarked state as its outside value and let  indicate this state of affairs. Then we can evaluate the nest of marks by marking it with u and m.

Similarly,  by repeated application of the law of calling. And

Here we combine uses of the laws of calling and crossing when they are available.

We see that there is an arithmetic of expressions written in the mark and the equals sign has taken on the crucial role of connecting expressions that indicate the same value.

Oh! You want to know the meaning of $\overline{\overline{\;}|\,\overline{\;}|}|$! It is a multiple action. Think of putting an unmarked signal u at the deepest spaces in the expression and marking it with u and m as we did before.

$$\overline{\overline{\overline{u}|m}|m\;\overline{mu}|\overline{m}|u}|$$

Note that in the space one crossing away from the outside there are two m's.

We take the rules that $mm = m$, $uu = u$, $mu = um = m$. Then any expression can be seen as indicating a multiple process of crossing and re-crossing from the unmarked state of the first distinction. The signals interact with one another and produce the value of the expression as either marked or unmarked. The result is the same as that obtained by using the laws of calling and crossing on the expression. Here is the simplest arithmetic generated by a sign that makes a distinction. Spencer-Brown calls this the primary arithmetic or the calculus of indications.

Finding Logic

The primary arithmetic is a two valued system. Every expression is either marked or unmarked. Remarkably, there is a translation to the two-valued logic of True (T) and False(F). Let $a \vee b$ denote "a or b" (inclusive or—a or b or both a and b),

$a \wedge b$ denote "a and b". Let $\neg a$ denote "not a" and let $a \rightarrow b$ denote "a implies b".

Recall that in symbolic two-valued logic one takes the equivalence $a \rightarrow b = (\neg a) \vee b$.

Now note that if we write algebraically about the primary arithmetic with the variables standing for either the marked or unmarked states, *then ab is marked exactly when a is marked or b is marked.* This suggests that we take the interpretation T for the marked state and F for the unmarked state. Lets write $T = \overline{\;\;}|$ and F = . Then $\overline{T}| = F$ and $\overline{F}| = T$. Thus we can interpret $\overline{a}|$ as $\neg a$.

And then we have

$$a \rightarrow b = (\neg a) \vee b = \overline{a}|b$$

so that implication in logic becomes the operation $\overline{a}|b$ in the algebra of the primary arithmetic. It is then easy to see that and is expressed by the formula

$$a \wedge b = \overline{\overline{a}|\overline{b}|}|$$

since the formula on the right is marked exactly when both a and b are marked.

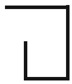
In this way basic logic rests on the primary arithmetic and can be seen as a patterning of its operations and processes. I hope to have convinced the reader that this is a satisfactory entry into logic starting the notions of sign and distinction.

One can explore a great deal from this basis and I will stop here with only a hint of what may come.

One aspect of logic that comes forth at once is the role of paradox. Consider the Liar Paradox in the form $L = \neg L$. Rewriting into primary algebra, we find

$$L = \overline{L}\rceil$$

Since the mark makes a distinction between its inside and its outside, this equation suggests that L must itself have a sign that indicates a form that re-enters its own indicational space. L must have a sign as shown below.



In crossing from the state inside the reentering mark, we arrive again at the inside. The inside is the outside and the outside is the inside. The sign connotes a distinction that contraverts itself and yet it is still a sign in the constellation of all signs and it still distinguishes itself in its own form. Nothing is left but the time of circulation in the oscillation of inside and outside, and beyond this state of time we have returned to void.

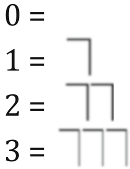
Finding Mathematics

Up to this point we have not actually ventured across a boundary into numerical mathematics. The construction of a sign that can stand for any sign and is self-referential involves no counting, no calculation, no algebra and seemingly no arithmetic of any kind. It would appear that we have arrived at a pivot point where one could begin thinking about the growth of thought and language with no regard to the development of mathematics.

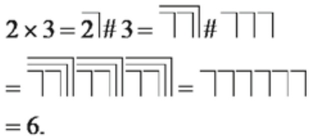
And yet mathematics has symbolic beginnings and is woven into the structure of language. What signs are the least signs needed for number? We might take on the sign | for 1, the sign || for 2 and generally, the sign n = |||...| (with n vertical marks) for the integer n. In this mode we have n + 1 = n|. And n + m = nm, the juxtaposition of the marks for n and the marks for m.

1 = |, 2= ||, 3=|||, 4=||||, 5= ||||| and so on.
3 + 2 = ||| + || = ||||| = 5.

Arithmetic can grow from elemental signs and indeed we can use the Spencer-Brown mark to represent numbers with zero as the unmarked state.

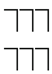


and so on. Note that in order to represent numbers in this way, we must rescind the Law of Calling so that multiplicities of marks stand for different numbers. With the Law of Calling removed, we are no longer working with only one distinction. Each new number is a distinction in its own form. What about the Law of Crossing? It turns out that we can put it to service for defining multiplication as follows: We define $a \times b = \overline{a|}\#b$ where $\overline{a|}\#b$ means the we take each cross in b and insert a copy of $\overline{a|}$ underneath it. Then simplify the resulting expression using the law of crossing. Here is the example of 2 × 3.



Here we have used the algebraic version of the Law of Crossing: $\overline{\overline{a|}|} = a$ for any a, and such an a can be one of our numerals, taking values beyond marked and unmarked.

This is the beginning of arithmetic, the gateway into the depths and beauties of mathematics. This foundation for the theory of numbers will clarify the deep quests of number theory. One can begin by wondering about the prime numbers. Six is not prime as we have just seen. It is a product of 2 and 3. The row of six marks is two rows of three marks and it is three rows of two marks. It seems that numbers want their own distinctions. After conversing with six we see that six prefers to be seen as



or as

but confinement to a single row is just not comfortable for a composite number.

Let us find arithmetic anew by staying close to its origin in the origination of a sign.

The Arctic Essay

This essay on the semiotics of Laws of Form was motivated by the author's discovery of the manuscript shown below. The manuscript was found at the bottom of an abandoned mine shaft in the frozen wastes of the Arctic Circle, abandoned surely in the 1800s. Written on crumbling paper and composed long before the conception of Spencer-Brown's book it is a mystery how the reference to the Spencer-Brown mark could have occurred in this manuscript. I have attempted in this essay to give sufficient background of a semiotic nature that the reader might be able to decipher the manuscript itself. The manuscript was entitled "A Sign in Space", but no author is indicated. I can only speculate that perhaps Spencer-Brown himself saw the manuscript, and yet that would not solve the puzzle of how it came to bear his name. Alas, the original document disintegrated into dust soon after it was found. This essay is all that is left. There is one clue. The document refers, at a crucial point to C. S. Peirce. I suspect that this is a self-reference and that Peirce himself wrote the document. As for Spencer-Brown, Peirce time traveled into the future and took back these notes on Spencer-Brown's work. All that happened before the Russell Singularity that made forward time travel impossible. There can be no other explanation.

A Sign in Space

Let $\overline{\ \ }|$ be the Spencer-Brown mark.

Let there be a distinction with Inside denoted I, and outside denoted O.

Regard the mark as an operator that takes inside to outside and outside to inside.

Then

$$\overline{I}| = O$$
$$\overline{O}| = I.$$

Note that it follows that

$$\overline{\overline{I}|}| = \overline{O}| = I$$

$$\overline{\overline{O}|}| = \overline{I}| = O$$

For any state X we have

$$\overline{\overline{X\,}|}| = X$$

Introduce the unmarked state by letting the Inside be unmarked.
Then I = .
And so

$$\overline{\ \ }| = O$$

$$\overline{O}| =$$

Therefore the value of the outside is identified with the mark and

$$\overline{\overline{\ \ }|}| = \quad .$$

The value of the outside is identified with the result of crossing from the unmarked inside.

$$\overline{\ \ }| = \overline{\ \ }|.$$

This equation can be read on the left as "cross from the inside" and it can be read on the right as "name of the outside".

Once the inside is unmarked, then the mark itself can be seen to be the first distinction.

The language of the mark is self-referential.

$\overline{\ \ }|\overline{\ \ }| = \overline{\ \ }|$ says that either mark names the other.

It is as though I were to wear a name tag that is a picture of myself.

At the level of the form there is no difference between myself and a picture of myself.

A sign can refer to another sign. (Cf. C. Peirce)

The mark is seen as a sign and
as a distinction between the inside of that sign and its outside.
We take the mark, as sign, to refer to its own outside.
In the form, the mark and the observer are identical.
In the form, a thing is identical to what it is not.
Tat Vam Asi.
In this way one arrives at non-duality by abandoning form to void.

Abandon form to void.
Form is emptiness.
Emptiness is form.
The form we take to exist arises from framing nothing.
We take the form of distinction for the form.

Epilogue

In this paper we have examined the use of and development of signs in relation to G. Spencer-Brown's Laws of Form and we have engaged in some wordplay related to a real story "A Sign in Space" by Italo Calvino (giving an extensive quote from it in our introduction) and a fictitious document named "A Sign in Space" that seems to be a precursor to the work of both Charles Sanders Peirce and George Spencer-Brown.

In fact, there is a long history of precursors to the semiotics signs at the base of mathematics, logic and language. Alphabets are historical records and ongoing libraries of signs and the simplest of such forms such as the cuneiform and Sumerian signs. For example, consider the fragment in Fig. 7 below from a Sumerian document, twenty-sixth century BC (https://en.wikipedia.org/wiki/Cuneiform).

There in the document are a nest of left-shaped marks, and since they are nested, the distinction they each make in the plane was clearly part of their use. In modern typography a relative of the Spencer-Brown mark is the square root sign, a connected sign that can be nested and arranged for mathematical purposes.

The language of Laws of Form was discovered, according to Spencer-Brown, in making a descent from Boolean algebra in which he found the notation of the mark, the role of the unmarked state and the double-carry of mark as name and mark as transformation. In Boolean algebra and in symbolic logic the negation sign connotes transformation and it does not stand for a value (True or False). In the calculus of indications, viewed from the stance of symbolic logic, the mark is a coalescence of

$$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}$$

Fig. 7 Sumerian document, twenty-sixth century B.C

the value True and the sign of negation. This comes about because True is what is not False and the False is unmarked in Laws of Form. Hence T = ~F = ~ (using ~ as the sign for negation), but this cannot be said without confusion in symbolic logic since there is no inside to the sign of negation. In Laws of Form we can write $T = \overline{F}| = \overline{\ \ }|$ and the mark as container (as parenthesis) makes it possible for it to take on its double role of value and operator.

Wittgenstein says ((Wittgenstein 1922) Tractatus [97] 4.0621) ". . . the sign '~' corresponds to nothing in reality." And he goes on to say ((Wittgenstein 1922) Tractatus 5.511) "How can all-embracing logic which mirrors the world use such special catches and manipulations? Only because all these are connected into an infinitely fine network, the great mirror." For Wittgenstein in the Tractatus, the negation sign is part of the mirror making it possible for thought to reflect reality through combinations of signs. These remarks of Wittgenstein are part of his early picture theory of the relationship of formalism and the world. In our view, the world and the formalism we use to represent the world are not separate. The observer and the mark are (in the form) identical.

This theme of formalism and the world is given a curious twist by an observation that the mark and its laws of calling and crossing can be regarded as the pattern of interactions of the most elementary of possible quantum particles, the Majorana Fermion (Kauffman 2010, 2012; Buliga and Kauffman 2014). A Majorana Fermion is a hypothetical particle that is its own anti-particle. It can interact with itself to either produce itself or to annihilate itself. In the mark we have these two modes of interaction as calling $\overline{\ \ }|\,\overline{\ \ }| = \overline{\ \ }|$ and crossing $\overline{\overline{\ \ }|}| =$. The curious nature of quantum mechanics is seen not in such simple interactions but in the logic of superposition (a kind of exclusive or) and measurement. Measurement of a quantum state demands the coming into actuality of exactly one of a myriad of possibilities. Thus we may write

$$\overline{\ \ }| * \overline{\ \ }| = \overline{\ \ }|\,\overline{\ \ }| + \overline{\overline{\ \ }|}|$$

to indicate that the quantum state $\overline{\ \ }| * \overline{\ \ }|$ of a self-interacting Majorana Fermion $\overline{\ \ }|$ is a superposition of marked and unmarked states. Upon observation, one or the other (marked or unmarked) will be actual, but before observation the state is neither marked nor is it unmarked. We need a deeper step in semiotics to enter into quantum sensibility. The equation for this interaction can be written in ordinary algebra as PP = P + 1 where P stands for the Majorana Particle and 1 stands for the neutral state of pure radiation. Then we recognize a famous quadratic equation

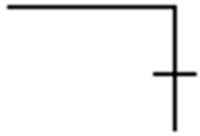


Fig. 8 Peirce's Sign of illation

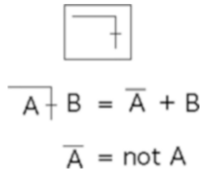


Fig. 9 The Peirce Sign of illation

$P^2 = P + 1$ with solution the Golden Ratio $\left(1+\sqrt{5}\right)/2$ and multifold relationships with the Fibonacci numbers. Indeed this is the legacy of the Majorana Fermion as Fibonacci particle, fundamental entity in the most idealistic and yet soon to be practical searches for quantum computing and understanding of particles as well-known as the electron. Each electron appears to be an amalgam of two Majorana Fermions. It is not the point here to start doing technical physics, but the moving boundary of Sign and Space is changed from the time of Wittgenstein and we should expect to see semiotic insight of a different kind from now on.

Charles Sanders Peirce came very close to inventing the mark ⏋ in his "sign of illation" as shown in Fig. 8.

[C. S. Peirce, "The New Elements of Mathematics", edited by Carolyn Eisele, Volume IV—Mathematical Philosophy, Chapter VI—The Logical Algebra of Boole. pp. 106–115. Mouton Publishers, The Hague—Paris and Humanities Press, Atlantic Highlands, N. J. (1976).]

The Peirce sign of illation is used for logical implication and it is an amalgam of negation as the over-bar and logical or writing as a + sign on the left vertical part.

See Fig. 9 for an illustration of this anatomy of the Peirce sign.

The mark ⏋ goes further since the unmarked state is allowed, and the operation of or is also unmarked and indicated by juxtaposition. Thus we still have the decomposition of *A*⏋*B* as "Not(A) or B" once the mark is understood to operate as negation. The largest difference is semiotic in that the mark can be taken as a universal sign and as a sign for itself. As such it has a conversational domain quite independent of Boolean logic. In this role, the mark can be seen as part of a wider context of distinction that informs and illuminates logic and mathematics.

Peirce spoke of a "Sign of Itself". Here is a key passage from his work.

> But in order that anything should be a Sign it must 'represent', as we say, something else called its Object, although the condition that a Sign must be other than its Object is perhaps arbitrary, since, if we insist upon it we must at least make an exception in the case of a Sign that is part of a Sign. Thus nothing prevents an actor who acts a character in a an historical drama from carrying as a theatrical 'property' the very relic that article is supposed merely to represent, such as the crucifix that Bulwer's Richelieu holds up with such an effort in his defiance. On a map of an island laid down upon the soil of that island there must, under all ordinary circumstances, be some position, some point, marked or not, that represents qua place on the map the very same point qua place on the island...
>
> If a Sign is other than its Object, there must exist, either in thought or in expression, some explanation or argument or other context, showing how—upon what system or for what reason the Sign represents the Object or set of Objects that it does. Now the Sign and the explanation make up another Sign, and since the explanation will be a Sign, it will probably require an additional explanation, which taken together with the already enlarged Sign will make up a still larger Sign; and proceeding in the same way we shall, or should ultimately reach a Sign of itself, containing its own explanation and those of all its significant parts; and according to this explanation each such part has some other part as its Object.
>
> [C. S. Peirce, "Collected Papers—II, p. 2.230—2.231, edited by Charles Hartshorne and Paul Weiss, Harvard University Press, Cambridge (1933).]

There are extraordinary and topological ideas in this passage. There is an implicit reference to the notion of a fixed point so that a map and its image must have a coincidence. There is the notion that Sign and Explanation will undergo recursion until ultimately the Sign, the Explanation and the Object become One. We have begun with a sign $\rceil$ that is a sign for itself in the sense that it represents the distinction that is made by the sign in its coincidence with an Observer. And yet the recursion is always possible. Consider the re-entrant sign that was discussed in Section "Finding Logic". The re-entrant sign can be taken to be a solution to $J = \overline{J}\rceil$ or, in a graphical mode, to be a solution to the re-embedding of J inside a circle as in Fig. 10.

> And yet, the equation $J = \overline{J}\rceil$ asserts the reentry of J into its own indicational space, and it exhibits J as a "part of itself". The equation is the explanation of the nature of J as reentrant and can be taken as a description of the recursive process that generates an infinite nest of circles. It is only *J as an equation* that yields J as a Sign of itself. If we wish to embody the equation in the Sign itself then we need to allow the Sign to indicate its own reentry as we did in the last section with the symbol shown below in Fig. 11. This symbol does "contain its own explanation" in the sense that we interpret the arrow as an instruction to reenter the form inside the circle.

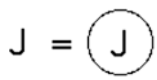


Fig. 10 Reentrant equation

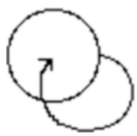


Fig. 11 A reentrant form

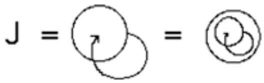


Fig. 12 Equation, indication and reentry

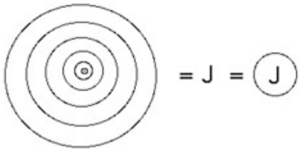


Fig. 13 Infinite regress and fixed point or EigenForm

[L. H. Kauffman, The mathematics of Charles Sanders Peirce, Cybernetics and Human Knowing, Vol. 8, no. 1–2, 2001, pp. 79–110.]

In fact that reentry occurs ad infinitum as indicated in Fig. 12, from which we see that he equational reentry is recaptured from the self-standing form of Fig. 11.

We see that it is a matter of language that fuels the difference between a simple form that stands for itself such as the mark and those reentry forms that partake of infinite regress (as shown in Fig. 13) in order to attain self-reference. This infinite regress is a microcosm of the infinite regress of Peirce and allows us to solve for J as an unending nest of marks.

$$J = \overline{\overline{\overline{\overline{\ldots}\rceil}\rceil}\rceil}\rceil = \overline{J}\rceil$$

It must be mentioned that the work of Church and Curry on the Lambda Calculus (See Kauffman 1994, 2001, 2005, 2009, 2010; Buliga and Kauffman 2014) gives another approach to reentry:

Let $GX = \overline{XX}\rceil$. Then $GG = \overline{GG}\rceil$ and so we can take J = GG to obtain $J = \overline{J}\rceil$ without any infinite regress! How did this happen? At the level of sign and operation, G is a duplicating device. Given an X, G makes two copies of X and places them under a mark. The equals sign means that *GX is replaced by* $\overline{XX}\rceil$, and the operation of G is defined by this replacement. When we replace X by G in the equation, we have put G in the position to act on G. G does act and produces GG with a mark around it. But GG is now ready to act again and so GG moves into the temporal domain and instructs a recursion: $GG = \overline{GG}\rceil = \overline{\overline{GG}\rceil}\rceil = \overline{\overline{\overline{GG}\rceil}\rceil}\rceil = \overline{\overline{\overline{\overline{GG}\rceil}\rceil}\rceil}\rceil = \ldots$ The infinite regress of J has been replaced by the inherent temporality of GG. The Church-Curry idea of recursion is in fact an outgrowth of the Russell Paradox of the Set of all Sets that are not members of themselves. To see how this plays out in the realm of signs *let XY denote that Y is a member of X*. Taking this to heart, we define the Russell Set by the equation

$$RX = \sim XX.$$

As the reader sees immediately, R is now the duplication Gremlin. We have shifted the interpretation of the mark to negation and we use ordered juxtaposition as membership. We find that if RX = ~XX, then RR = ~RR and we now have the self-denial of the Russell Set in regard to its self-membership. This could just as well have been written $RX = \overline{XX}\rceil$ and $RR = \overline{RR}\rceil$. We understand that this need not be a paradox. It is a reentry form and can be taken on its own cybernetic grounds. We have the option to view the Russell set temporally in the Church-Curry recursion. Then Russell oscillates in time between being a member of itself and not being a member of itself. The Russell Pendulum avoids the Russell Singularity.

In our fiction in this paper, we referred to the Russell Singularity as having made time travel into the future for the sake of mathematical and semiotic plagiarism impossible. In fact this was a reference to the weight of the ban (The Theory of Types) on temporal solutions to the Paradox that was presented by Russell and Whitehead in their monumental work Principia Mathematica. With the recursive way out it may be that we have also released the demons of Time Travel once again upon an unsuspecting world.

When representation and explanation are insisted upon, then an infinite regress occurs due to the proliferation of signs that must indicate each stage of explanation. When this "noise" is reduced by the indicational power of an arrow, or the simple recognition of the presence of a distinction, then forms can stand alone and be recognized as being, in form, identical with their creators.

Along with the references quoted directly in the text, I have provided a selection of papers that I have written that are related to the themes of this essay. They are references (Kauffman 1985, 1987, 1994, 2001, 2005, 2009, 2010, 2012, 2016; Buliga and Kauffman 2014). There is much to think about in this domain and we have only just begun.

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Can We "Show" the Correctness of Reasoning? On the Role of Diagrammatic Spatialization in Logical Justification

Julien Bernard

Abstract Our aim is to show that the "analytical diagrams" in the sense of Venn play a role that goes far beyond the simple pedagogical interest. In the first section, we will show that the possibility of using logic diagrams effectively as teaching aids comes from certain cognitive advantages of diagrams. These advantages are worthy of interest in themselves and explain why cognitive sciences and visual communication technicians currently rehabilitate the studies on these diagrams. In the next two sections, our aim will be to show that, beyond the cognitive advantages they bring, diagrams can acquire a truly fundamental function in logic. Two types of question must then be asked in turn. First of all: Can diagrams alone be used to validate all correct forms of reasoning? Or are diagrams destined to be nothing more than illustrations of a valid linguistic demonstration. To answer these first questions, our second section will draw on the results obtained over the last twenty years or so by exporting the methods of demonstration theory to the field of diagrammatic thinking. This section is based on a long appendix which we are publishing in parallel with this article. Then, in the third section, we will test the thesis inspired by Euler's texts: Can (should?) diagrams be given the task of making the form of propositions and the sequence of correct reasoning intelligible? It is no longer simply a question of using diagrams as a support for demonstrative practice, but of using them as a means, dispensable or not (this is what our analysis will attempt to see), of acquiring the logical intuition that legitimizes inferences. Is the possibility of acquiring these logical intuitions from spatial intuitions a sign of an intimate connivance between the domain of space and that of demonstration, or is the rapprochement superficial and contingent? Finally, in a final section, we will return to the origins of logic, recalling the place that perhaps diagrammatic representation already played in Aristotelian logic, and we will articulate our analyses with those of the historians of ancient logic on this point.

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Keywords Philosophy · History of logic · Philosophy of logic · Diagrams · Diagrammatic method · Leonhard Euler · John Venn · Lewis Caroll · Traditional logic · Syllogistic · Boolean algebra · Logic and spatiality

Introduction

The Eulerian Thesis: The Logical Correctness of the Diagrams "Jumps to the Eyes"

In Letter 102, from *Letters to a German Princess, On Different Subjects in Physics and Philosophy*, on the occasion of the invention of his logic diagrams, Leonard Euler proclaims:

> They are a marvellous aid in explaining very clearly what the correctness of reasoning consists of.[1]

He goes on to say that, thanks to diagrams, the nature of propositions and the correctness of syllogistic modes "*jumps to the eyes*".[2]

Euler diagrams, or related diagrams, are often considered as simple teaching tools, serving to lighten the learning of logic by illustrating syllogistic procedures. Moreover, how could a diagram, a simple sensitive representation, endorse on a fundamental role in logic? Isn't logic essentially a matter of articulating discourse, and therefore alien to, or at least indifferent to spatial intuition, which can be dispensed with in the last instance? Does not the fact, put forward by Leibniz, that logic can be practised as a *blind calculation*, confirm this optional character of any visual intuition, or even any spatial intuition in logic, with the exclusive exception of the minimal intuition required to manipulate symbols[3]?

In spite of these presumptions, we aim, in this article, at taking Euler's thesis seriously, by seeking to give a much more important function to logic diagrams than a simple auxiliary pedagogical function. We will test the idea that these diagrams would "show" (Euler says "donne à voir", literally: "give to see"), through intuitions that provide meaning, the correctness of reasoning.

This main goal will lead us to touch on some fundamental questions such as:

- Can we give diagrams the burden of proving validity?

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[1] (Euler 1770, volume II, letter 102, p. 103). Euler's letters were first written in French. I will give some of the original texts in footnotes. Here: « Ils sont d'un secours merveilleux pour expliquer très distinctement en quoi consiste la justesse d'un raisonnement. »

[2] (*ibid.*): "au moyen de ces figures, tout saute d'abord aux yeux".

[3] Of course, in Leibniz's work, it is important that each inference can be justified by giving the symbols back their meaning, as can also be done for calculation in algebra. For Leibniz, therefore, the possibility of practising logical calculation "blindly" is by no means the correlate of a conception of logic as a simple formal game defined by arbitrary rules of calculation. The meaning can always, *de jure*, be given back.

- Can we give diagrams the burden of enlightening or even making each logical inference intelligible?
- What can we learn from the answers to the previous questions about the relationship of logical and discursive thinking to spatial thinking?

Determining the Logical Framework of Our Research

In order to address these fundamental questions, we will remain within the limited framework of traditional logic, in the sense of pre-Fregean logic, that which goes from Syllogistic to Boolean Logic. And, by "diagram", we will mainly mean Euler-Venn type diagrams, i. e. those that John Venn describes as "*analytical*", in the sense that they penetrate into the analysis of the proposition, figuring the relation between the subject and the predicate, as opposed to other older diagrams such as, for example, the trees of Porphyry.[4] Our restriction to traditional logic is justified by the fact that this is the area where the Eulerian thesis of diagrammatic efficiency has reached its maximum validity. In conclusion, we will discuss the limitation of the Eulerian thesis in modern, i. e. post-Fregean logic.

Moreover, our limitation to this field is partly justified by the revival of studies on Euler-Venn type diagrams since the 2000.[5] This renewed interest comes, firstly, from the accelerated development of cognitive sciences and research in artificial intelligence, which are interested in the function of diagrams in human inferences, and their possible implementation on computers; secondly, from the progress of visual representation technologies in science.[6] Finally, the renewed interest in diagrams in geometry, following the important work of K. Manders[7] and the more recent work of J. Mumma,[8] has led to a renewed interest in the role of diagrams in logic as a neighbouring field of mathematics.

[4] In (Venn 1883a, pp. 47–48), Venn gives Euler as the one who popularised the practice of this type of diagram. However, they are considerably older than Euler. Venn reports as the earliest reference found: Ludovicus Vives, *De Censura Veri*, c. 1530.

[5] See the corresponding sections of the international conferences: *Diagrammatic Representation and Inference*: 1998–2000–2002–2004–2006–2008–2010–2012–2014–2016, in particular the introduction to the second edition (2002).

[6] See also (Hammer 1995) for a general reflection about the importance of logical foundations of visual information.

[7] (Manders 1995).

[8] (Mumma 2006, 2010, 2012).

The Late Emergence of "Analytical" Logic Diagrams in a Pedagogical Context

The context in which logic diagrams appeared, those which are "analytical" in the sense of Venn, prompts us to give them no more than a pedagogical and auxiliary function. Indeed, they appeared late in the history of logic, after centuries of demonstration of the effectiveness of non-diagrammatic methods (linguistic or symbolic) for analysing the logical structure of propositions and reasoning. The historical situation is therefore in a reversed direction with respect to the history of Euclidean diagrams, as reported by Vincenzo de Risi.[9]

Geometry and logic do not have the same relationship with space. For this reason, one must be careful when transposing, from one field to another, analyses or conceptual tools about diagrams.

In geometry, the object of study is precisely space (in a post-eighteenth century modern vision) or, in an older version, figures, objects and spatial configurations (triangles, straight lines...). It is therefore obvious that in geometry space has, from the outset, a status that goes beyond that of a simple medium of representation.

In contrast, the correct forms of reasoning studied by formal logic do not immediately present themselves as being of a properly spatial nature. The diagram spatially represents a reasoning, which is not by itself necessarily concerned with spatial issues. The necessity or at least the advantage of introducing space remains to be demonstrated in the field of logic. This is not a given. This difference in relation to spatiality being granted, let us return to the difference between the two histories of diagrammatic thinking, in geometry and logic.

Keneth Manders' work on Euclidean diagrams has shown that they played an indispensable inferential role in the practice of geometry prior to the eighteenth century. These diagrams were indeed used to carry out certain inferential steps that could not be based on Euclidean postulates. Manders identified the properties concerned by these inferential steps as the so-called "coexact" properties, i. e. both topological properties and properties involving metric inequalities. From the eighteenth century onwards, and especially during the nineteenth century, the coexact properties have been expressed propositionally within new postulates, by axioms of incidence, axioms of a topological nature, etc.; until we had reached a point in the history of geometry where diagrams become dispensable as a support of proof. There were no more "holes" in purely propositional demonstrations. This is why, with regard to the burden of proof, Euclidean diagrams have become dispensable from this period in the history of geometry.

If we return to the logic diagrams, history follows a very different path here. In contrast to the field of geometry, very early, the practice of logic has done without diagrams. The use of Eulerian-type diagrams did not really develop until the eighteenth century, superimposing itself on a rather linguistic and symbolic practice

[9] (De Risi 2017, 2019, 2020).

which preceded it. Thus, in the field of logic, the dispensability of diagrams, as elements of proof, was acquired from the origins of this discipline or almost from the very beginning. It is not a result acquired with difficulty through centuries of hard work as in geometry. On the contrary, diagrams have had to (re-[10])demonstrate their usefulness in logic.

In addition to this late arrival, analytical logic diagrams emerge in contexts which are often pedagogical from the outset. They generally do not discriminate differently between correct and incorrect reasoning than earlier non-diagrammatic methods. From this point of view, they seem dispensable. Euler introduced his diagrams to teach syllogistic to Princess Friederike Charlotte of Brandenburg-Schwedt. Charles Dodgson's diagrams appeared in a book entitled *The Game of Logic* (Caroll 1886), which was intended primarily for children and was accompanied by a cardboard diagram with red and white tokens, in order to teach logic as a board game. These diagrams appeared also in *Symbolic logic* (Caroll 1977), a more developed book aimed at a wider audience. However, Dodgson signs this book *Lewis Carroll*, and not by his real name as he usually does for his academic articles. According to L. Gattégno and E. Coumet,[11] before the publication of Volume I of *Symbolic Logic*, Dodgson had leaflets distributed by his publisher which spoke about its content as an exciting intellectual game for children and young people. The book is deliberately entertaining, but with a desire to give an understanding of deep aspects of logic. Gattégno and Coumet add that Caroll compared the symbolic and computational methods accompanied by diagrams to those medicines of our childhood, which our parents skilfully but unsuccessfully hid under a nappy of jam. This explains the choice of the French translation of Caroll's logic texts by *Logique sans peine*. In his preface to the fourth edition, Caroll also develops a whole *economy of pain*. He says he is satisfied that the trouble he has gone through in writing this book is more than ten times compensated by the effort that will be saved by this book for pupils and adults who wish to learn formal logic. The diagrammatic formatting of the logic enables it to be disseminated to a wide audience because it makes it easy to follow the reasoning. And, for Caroll, the ease takes the particular form of the game.[12]

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[10] See our Section "Some Remarks on the Relationship Between the Principles of Logic and Spatiality in Syllogistic (From Aristotle to Hamilton)" to understand the allusion expressed by this parenthesis.

[11] (Carroll 1966, introduction by the editors).

[12] (Carroll 1966, preface to the fourth edition) "In conclusion, let me point out that even those, who are obliged to study Formal Logic, with a view to being able to answer Examination-Papers in that subject, will find the study of Symbolic Logic most helpful for this purpose, in throwing light upon many of the obscurities with which Formal Logic abounds, and in furnishing a delightfully easy method of *testing* the results arrived at by the cumbrous processes which Formal Logic enforces upon its votaries This is, I believe, the very first attempt (with the exception of my own little book, The Game of Logic, published in 1886, a very incomplete performance) that has been made to *popularise* this fascinating subject. It has cost me *years of hard work*: but if it should prove, as I hope it may, to be of real service to the young, and to be taken up, in High Schools and in private families, as a valuable addition to their stock of healthful mental recreations, such a result would *more than repay ten times the labour* that I have expended on it" [emphasis added]. Note the dual

The idea of *pain* or *fatigue* that the mind would feel in following discursive reasoning without the crutch of spatial and diagrammatic intuition was already present in Euler,[13] and in Venn,[14] with nuances that will not be developped here.

First Section: The Cognitive Advantages of the Diagrammatic Method

Beyond its pedagogical interest in teaching academic logic, the economy of effort allowed by Euler-Venn type diagrams is indicative of certain cognitive advantages of diagrammatic representation over linguistic and symbolic representations. These cognitive advantages partially explain the renewal of research on this type of diagram since the 2000s, which we mentioned in the introduction. However, these cognitive advantages were already well seen by Venn as early as his articles of 1880.[15] Among these cognitive advantages are:

A gain in time. Suppose we have to represent a certain Boolean logical configuration involving *n* concepts. The diagrammatic method will require simply drawing *n* potato-shaped figures,[16] rather than symbolically listing the 2^n elementary zones of the underlying Boolean algebra.

Aesthetic pleasure. Venn speaks of the pleasure experienced in drawing a diagram, in the face of the *unpleasant drudgery* of systematic symbolic writing of the elementary logical zones. One can also think of the laborious task of writing a truth table. Let us pass over this aesthetic criterion, but, noting all the same that it is not unrelated to the gain "in intuitiveness" of a diagram. The diagram is all the more "visually speaking" that it is intrinsically elegant.

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role given by Carroll to his diagrams:1) to clarify certain logical ambiguities in everyday language and 2) to verify, by a second method, syllogistic exercises solved by the non-diagrammatic method. Note then the ideas of effort ("labour") and gravity. Carroll gives examples of these "healthful mental recreations" whose logic he wants to bring closer. He thinks of chess, backgammon, etc.

[13] (Euler 1770, beginning of letter 103): "These round figures or rather these spaces (for it does not matter what figure we give them) are very likely to *facilitate* our reflections on this matter, and to discover for us all the mysteries which are boasted of in Logic, and which are demonstrated with great effort, while by means of these signs *everything first jumps to the eye*. [emphasis added]." « Ces figures rondes ou plutôt ces espaces (car il n'importe quelle figure nous leur donnions) sont très propres à faciliter nos réflexions sur cette matière, et à nous découvrir tous les mystères dont on se vante dans la Logique, et qu'on y démontre avec bien de la peine, pendant que par le moyen de ces signes tout saute d'abord aux yeux. ».

[14] In (Venn 1883a), John Venn points out that only analytical diagrams (in the sense given below), in contrast to older diagrams, can be "real aid to the mind in complicated trains of reasoning".

[15] (Venn 1880, 1881, 1883a, 1883b). Dense remarks on this can be found in (Venn 1880, p. 8-sq.).

[16] French logicians sometimes call « potatoes » the shapes that are used in so called « Euler-Venn Diagrams », in order to insist on their irregularity and on the fact that the very precise shape does not really matter in expressing the intended logical relationships. Euler (1770) prefers to use *round* figures, but he is conscious about the fact that the circularity of the shape is arbitrary. See the first line of Euler's text quoted in footnote 12.

Let us simply quote here Venn:

> We have endeavoured above to employ only symmetrical figures, such as. should not merely be an aid to the sense of sight, but should also be to some extent elegant in themselves[17]

Corrective function. The diagrammatic method makes it possible to "make sure against mistake and oversight".[18] Venn quickly passes over this point.

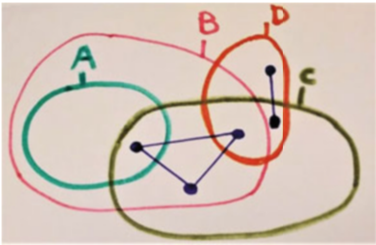
The quasi-synoptic and quasi-instantaneous aspect of the diagrammatic representation. The terms "quasi-synoptic" and "quasi-instantaneous" are mine. Venn simply says that the diagram shows "at a glance" —Venn uses the French expression « d'un coup d'œil »[19] — the logical configuration it represents. It is understandable that this refers to the conciseness and synthetic aspect of the figure, which can allow us to go through all the stages that discursive reasoning can only unfold linearly.

We can therefore see how in Venn—but we would also see it in Euler—the synoptic and instantaneous aspect of the perceptive data of a diagram often slips metaphorically towards the idea that the relations at stake in a logical configuration would themselves be grasped with a single synoptic and instantaneous glance by the mind's eye.

We must of course question the relevance of this metaphorical shift. A phenomenological and epistemological critique is necessary. Undoubtedly, we must not take at face value the instantaneous and synoptic aspect of diagrammatic intuition, when the intuition at stake is not the simple perceptive intuition, but rather the *logical* intuition which grasps the relations between concepts. One can nevertheless give some consistency to this idea, by giving it a weak meaning.

Let's take an example for this.

- All A's are B's.
- Some B's are C's.
- No A is D.
- Some D's are not B.



Here are two representations of the same logical configuration, derived from a set of four premises. On the left side, we have a linguistic representation, which appears as a linear list, one below the other, of propositional statements; on the right

17 (Venn 1880, p. 8). Of course he meant « an aid to reasoning, through the sense of sight ». This is corrected in (Venn 1881).

18 (Venn 1881, p. 107).

19 (Venn 1881, p. 108, footnote 1).

side, the diagrammatic representation (of the "generalised Euler" type, with <*spider graph*>[20]) of the same logical configuration.

The perceptive seizing of the diagram, as a purely visual object, is of course neither more nor less instantaneous than the seizing of the list of statements as pure drawings of the symbols. It is only in the interpretative acts that the difference can take place. For it is obviously not the same thing to have before one's bodily eyes, synthetically, the sequence of symbols that make up a demonstration, and to have before one's mind, synthetically, all the propositions as units of meaning. Each statement, in order to acquire its meaning, requires an act of propositional apprehension. And it seems quite plausible that the pre-propositional act by which the mind grasps the "topological"[21] relations (inclusion, exclusion, overlap) on the diagram is faster than the propositional act of interpreting the utterances. We leave it to phenomenologists, experimental psychologists and psychophysicists to clarify this aspect.

One may even wonder whether the diagrammatic representation would not allow the mind to go further in the possibilities to maintain under the scope of its attention, jointly, a set of premises. Indeed, it seems difficult to keep a plurality of logical relations under our attention, without the help of a spatial support, be it perceived or simply imagined.

It should also be pointed out that the synoptic aspect of the diagram goes much further than simply showing all the premises together. A Eulerian-type diagram is in fact the result of a *synthesis* operation, by which it can ultimately show together not only the collection of all the premises, but also all the propositions logically deducible from them.[22] The diagram is not a disarticulated sum of premises, as the list of statements is, we will come back to this later. The linguistic equivalent of the Eulerian diagram would therefore be rather the fully developed syllogistic demonstration, or even the aggregate of all possible syllogistic demonstrations starting from the same set of premises.

Let us move on to a last important point: the non-linear, instantaneous and synoptic aspects of the diagram are partially weakened when we take into account the operation of the mind by which it grasps the meaning of the diagram and "inhabits" it, so to speak. This operation reintroduces a temporal unfolding and a linearisation of logical thinking.

[20] The *spider graphs* are a mode of notation which makes it possible to generalise Euler's diagrams by using a visible mark in order to identify the zones of the diagram in which one assumes the existence of at least one individual. See (Gil et al. 1999), (Mutton et al. 2004) and (Howse et al. 2001). Instead of spider graphs, one could color the diagram.

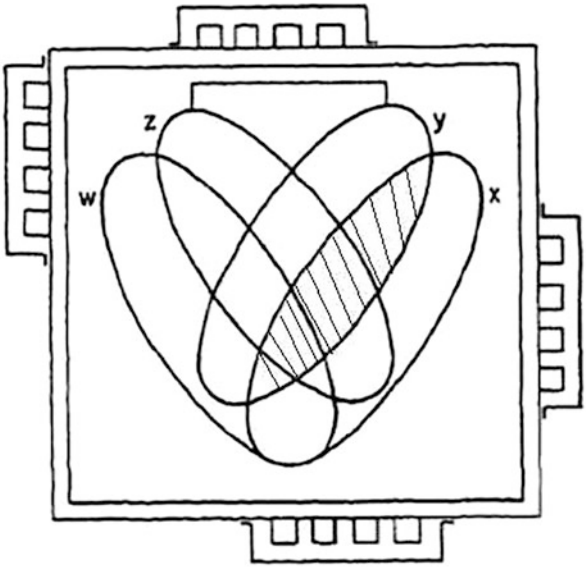
[21] The quotation marks are a reminder that this is not exactly topology in the precise sense of the mathematical discipline that we know, insofar as a diagram is fundamentally a drawing, i. e. a configuration that gives itself *to perception*, with all the vagueness that this implies in terms of the underlying continuum and the exact determination of figures and their relationships. See below, in the next section, our remarks on the problems posed by the exactness of diagrams, when defined mathematically.

[22] See (Bernard 2022, corollaries 10, 16 and theorems 7, 8).

Let us take a new example. We start from a problem borrowed from Venn.[23] It asks us what is the most general possible logical configuration that responds to the following three premises:

- All X's are either (Y and Z) or non-Y.
- If an XY is Z then it is W.
- No WX is YZ.

The Venn diagram below, which summarises these premises, can be given directly. Venn states that this is the diagram, whose hatched ("grey" in the Venn sense) elementary zones correspond to $X \cap Y$.



Venn diagram represents the most general configuration satisfying the three conditions expressed by the premises.

It can be verified that this diagram satisfies each of the three premises. However, it is not clear why it represents the most general logical configuration that corresponds to the premises. Moreover, if a student answers the logical question posed above by giving simply the final synthetic diagram alone, by saying "you see?", the teacher will answer: "OK, 'I see', but how did you get this diagram, and how do you know that it answers the question asked?" In order to understand it, it is necessary to go linearly through the steps of the diagram synthesis by which the diagram was constructed.

[23] (Venn 1880, pp. 10–11).

In order to be able to unfold the construction of his diagram step by step, Venn had designed a "diagram-machine",[24] inspired by Jevons' machines. It was in fact a wooden Venn diagram, with four classes, and for which each of the $2^4 = 16$ elementary zones could be "greyed out" (i. e. considered empty), by removing the corresponding wooden piece. It is a sort of upside-down puzzle, where one starts from the position where all the pieces are nested, and gradually removes pieces. The diagram machine could be reset in an instant, by turning it over.

To understand the solution to the logical problem given above, we start from the initial configuration. Then, we check that the proposition "All X's are either (Y and Z) or not Y" is exactly expressed by the removal of the two pieces of wood corresponding to the zone: $X \cap Y \cap \complement Z$. Next, check that the proposition "If an XY is Z then it is W" is exactly expressed by the removal of the piece corresponding to "$X \cap Y \cap Z \cap \complement W$". Finally, we note that "No WX is YZ" is expressed by the removal of the part "$X \cap Y \cap Z \cap W$".

When one has gone through these three steps, one understands that $X \cap Y$ and only $X \cap Y$ has been emptied.

As this example shows, even if a Euler-Venn-Caroll type diagram does indeed represent synoptically and two-dimensionally all the consequences of a set of premises; on the other hand, the verification and understanding of this fact can only take place linearly and step by step, through an interpretative act by which the mind goes through the diagram and seizes its meaning and legitimacy. It is a matter of the mind going through the diagram, to grasp the stages of its construction. Any path implies the reintroduction of a form of linearity and of temporal and spatial deployment of the stages.

Lewis Carroll saw this clearly when he conceived a diagram not as a fixed figure, but as a dynamic construction game, a succession of gestures, in the case of *The Game of Logic*, the movement of small coloured tokens. And it is by performing the performance of the construction of the diagram that the student or the logician grasps its meaning and understands that it is answering the question.

To conclude this paragraph, the immediacy and instantaneity of the diagrammatic method is partly illusory. The instantaneous and synoptic aspect of the *final diagram* must be nuanced by the dynamic and linear aspect of the *diagrammatic synthesis*. However, the gain in time, and the help to the synthesis of information for the mind, allowed by the diagrammatic method, must not be denied. In the absence of a radical qualitative difference with linguistic or symbolic methods, there is at least a significant difference of degree in some cases, which justifies Dodgson's metaphor on the "economy of pain", and which justifies the use of psycho-physical methods to compare quantitatively the efficiency of the mind in solving logical tasks by the two methods.

[24] (Venn 1880, pp. 16–18).

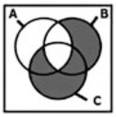
The Plasticity of Diagrammatic Representation

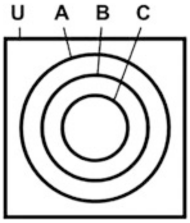
Let us conclude this section by pointing out that the cognitive qualities of the diagrammatic method that we have highlighted are reinforced by another quality shared by many systems of diagrammatic representation: their *plasticity*. To a given set of premises, we do not associate *one* perfectly determined Euler diagram, but a class of diagrams up to continuous deformations. We are not thinking here of diagrams as Euler himself drew them, with “round” figures, but of potatoes-shaped diagrams which rapidly conquered the field of logic after Euler.[25] In such a framework, there is no isomorphism, but only homomorphism, from the space of Euler’s potatoes-shaped diagrams onto the space of possible logical configurations.

This overrepresentation of logical configurations by the diagrams is not a defect, but gives the ingenuity and imagination of the creator of diagrams the power to express themselves; she will then be able to choose the most appropriate diagram, i. e. the one that will exploit at best the cognitive qualities of diagrammatic representation. In particular, if the diagram drawer understands the logical path to be highlighted, she can organise the information graphically to facilitate understanding of this path.

Thus, for example, it seems indisputable that the logical relationships at play in the reasoning of the Barbara modus are more immediately grasped in a Euler diagram than in the corresponding, less plastic, more constrained Venn diagram.

[25] Euler speaks of “circular diagrams” and, in fact, his traced figures are most often approximately circles. See also our footnote 16. In order to express all possible relationships, beyond three classes, the circular form is no longer sufficient. But Euler does not go beyond three round figures per diagram, since he uses them only to check the validity or invalidity of the modes of syllogistic. In the nineteenth century, as the use of diagrams became more widespread, logicians quickly emancipated themselves from the circular form. In (Venn 1883a, p. 52), Venn notes that the only relationships that count are inclusions and exclusions, and that the circular form is fortuitous and has sometimes been replaced by other forms: “Ploucquet used squares”. In this respect, Venn specifies in his text the usages of Kant, De Morgan, Latham, Leechman and Bolzano.





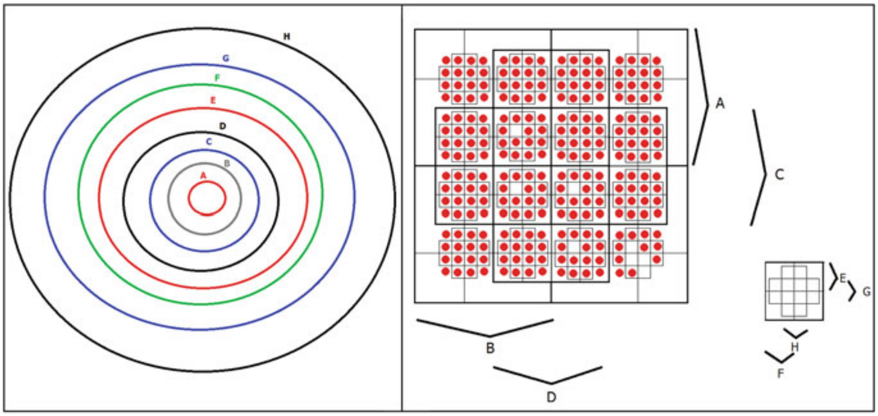
The Euler diagram below shows immediately that one can infer “All C is A’ from the two premises “All C is B” and “All B is A”. Indeed, each of the two premises is represented by the spatial inclusion of a circle within another. The conclusion can be seen directly through the transitivity of the inclusion relationship. It is an obvious principle of our perceptive knowledge of space: “Everything that is in the content is also in the container”. (Euler 1770, letter 104, p. 123).
The Venn diagram above justifies the same scheme of reasoning, but in a much less direct and clear way.
The cognitive inferiority of the Venn diagram over the Euler diagram comes from its rigidity. Whereas the rules of use of Euler diagrams are usually more flexible and can be adapted to the logical configuration under consideration (one represents only the relevant zones, one chooses the shape of the zones, and one arranges them with a certain latitude), in the case of Venn diagrams one is forced to represent all 2^n possible zones of the free Boolean algebra, according to a pre-determined arrangement.

The difference in plasticity between the Euler and Caroll diagrams is even more pronounced. Let us take the reasoning with eight premises:

All A’s are B
All B’s are C
All C’s are D
All D’s are E
All E’s are F
All F’s are G
All G’s are H

(So) All A’s are H

The Eulerian representation of this reasoning is given below on the left.



The conclusion that “All A’s are H’s”, which is derived syllogistically from the succession of six inferences of the modus Barbara, can again be seen through the transitivity of the spatial inclusion relation. The course of the stages of reasoning is certainly indispensable, and the idea that the correctness of the conclusion would be seen immediately is certainly questionable. But the general pattern, the nesting of eight classes, is at least immediately visible and understandable.

If the same set of premises is represented on a Carroll diagram (see diagram on the right), the intuitiveness of the inferential sequence is almost completely lost. One loses the convenient organisation of information, the time saving, in short almost all the cognitive advantages of the Euler diagram. The result no longer “jumps to the eye”, but requires a step-by-step verification, not very intuitive, where one loses the overall meaning of the inferential diagram.

The main reason for this loss of logical intuitiveness comes from the fact that the *zones representing classes E, F, G and H are no longer connected*. The gain in symmetry and automaticity of the operation of construction of the boxes comes at the expense of a loss of the intuitiveness of the predicative relationship. The latter is no longer represented by a simple topological relation of inclusion between two related connected zones. In a way, Carroll went too far in the search for symmetry, which was requested by John Venn to put aesthetics at the service of logical intuitiveness:

> The leading conception of this scheme [i. e. the Venn diagram] is then simple enough; but it involves some consideration in order to decide upon the most effective and symmetrical plan of carrying it out. [...] [Venn speaks of the possibility of replacing Euler’s circles by any closed figure]. The only objection is, that since diagrams are primarily meant to assist the eye and the mind by the intuitive nature of their evidence, any excessive complication entirely frustrates their main object.[26]

The fact that Caroll’s diagrams have lost the connectedness of the regions representing the classes is in fact the same as juxtaposing several diagrams, so that

[26] (Venn 1880, p.6–7).

one loses what Venn called "the advantage of the *coup d'œil* afforded by a single figure".[27] Caroll's diagrams are too constrained. Excessive standards certainly introduce symmetries. But these are accidental symmetries imposed by the mode of representation. Caroll emphasises symmetries that apply only to the free Boolean algebra (i. e. that with 2^n non-empty elementary zones), which corresponds *to* the complete grid, to the detriment of the essential symmetries that apply to the particular configuration that is represented *in* the grid. Highlighting the essential symmetries requires a plasticity of representation, to enable the diagram designer to use his creativity in the service of the particular configuration being studied.

Second Section: Can Diagrams Be Given the Task of Validating Reasoning?

Is the diagrammatic method (through Euler-like diagrams) fallible? Can it lead us to validate reasoning that syllogistic would however consider to be incorrect? Or, on the contrary, would it fail to recognise certain forms of correct inferences? In other words, is it *sound* and *complete* with respect to the usual linguistic method of syllogistic?

We are convinced that, from the time of Euler, users of logic diagrams should have had no doubt about the possibility of using the diagrammatic method to validate or invalidate forms of reasoning, with the same efficiency and rigour as the "meta-linguistic" methods of scholastics.[28] All that is needed for this is a correct control of the rules of diagram synthesis. However, in the teaching of logic, one sometimes still hesitates to consider a diagram as holding for an authentic logical proof, and it is often given only a heuristic role in order to find a demonstration which is written out in words or symbolically.

For at least ten years, we have had mathematical results which establish the soundness and completeness of Euler-Venn type diagrammatic methods. An example can be found in *The Logical Status of Diagrams* and in "Diagrammatic Reasoning System with Euler Circles".[29] In this paragraph, we will outline the

[27] (Venn 1880, p. 8)

[28] We say "meta-linguistics" in the sense that syllogistic is a second degree discourse, a discourse about discourses. Indeed, according to the traditional methodology of syllogistic, the logician starts from a so to speak grammatical analysis (quality of the proposition: affirmative or negative; quantity: universal or particular) of the propositional statements which constitute the reasoning to be studied, then he writes his analysis, and his conclusion as to the validity or logical invalidity of the reasoning studied, in a technical language, that of syllogistic and its classificatory tools. In particulier, such a logician can use mnemotechnical techniques such as those based on the Latin nomenclature (BaRBaRA, DaRi, etc.).

[29] (Mineshima et al. 2008).

principle of such results, of which we publish in appendix a version more adapted to the objectives of this article.[30]

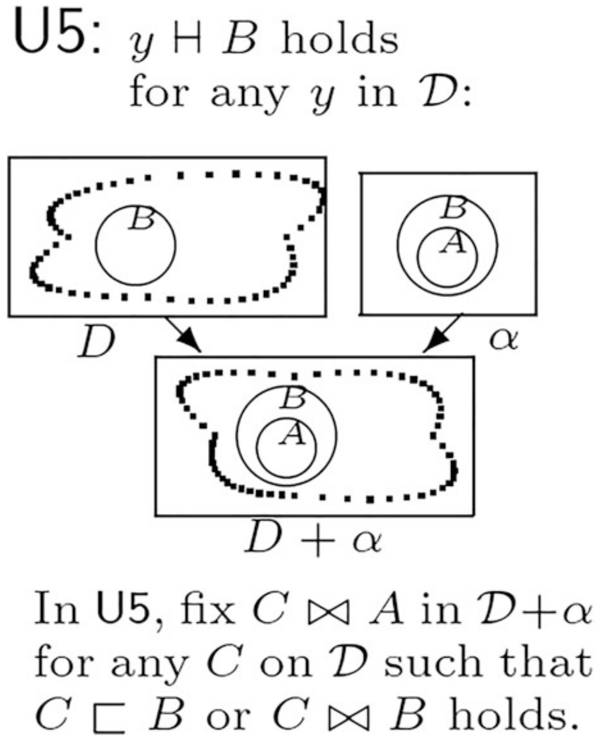
The *diagrammatic space* on which a Euler diagram is drawn can be defined as a mathematical space containing exact figures. Typically, we define the potato-shaped figures of a Euler diagram by means of the continuous loops defined within $\mathbb{R}^2$. By Jordan's theorem, we know that such a curve cuts $\mathbb{R}^2$ into two parts, the inside and the outside of the "potato". We may add constraints of simplicity on the curves and on their mode of composition. And one arranges so that the diagrammatic space responds to a so-called *Algebra of Sets*. That is to say that the possible zones of a diagram must be stable by finite intersections and by complementation.[31]

Once the diagrammatic space has been rigorously defined, diagrammatic synthesis operations are defined which replace the rules of inference usually expressed in a language (symbolic or not). Starting from a diagram which already synthesises (n-1) propositions, and a new proposition to be synthesised, we describe (up to diagram-isomorphisms) how the previous diagram should be modified to integrate the new information. These rules of synthesis must be perfectly explicit, algorithmic. We give in the figure below an example of such rules and refer the reader to complete examples of diagrammatic synthesis games.[32]

[30] (Bernard 2022).

[31] Note in passing that the definition of a concrete Algebra of Sets that is effective for use as a diagrammatic space is a problem that is far from mathematically trivial. Many mathematical obstacles are encountered: what regularity condition should be chosen for potatoes? How can we ensure that the zones cut by the potatoes as they intersect will even be regular? How to define the overall algebra operations so that borders do not play an essential role in the diagram (in particular, the topological distinction between open and closed sets should not play a role for a Eulerian potato, the border must be "neutralised"). Etc. But what is the point of finding an exact mathematical definition of potatoes, when their diagrammatic efficiency is precisely due to the indifference one may have towards their exact contours? Concerning a concrete algorithm to construct Euler diagrams from an abstract description of it, cf. (Flower & Howse, 2002).

[32] Our example is borrowed from the GDS system. See (Mineshima et al. 2008, pp. 195–196).



An example of a diagrammatic synthesis rule:
This rule must be understood as follows:
- Suppose we have already produced a diagram D, containing in particular a potato-shape B (but not potato-shape A), and also an indeterminate number of other potato-shapes which are in some configuration (the dotted shape is a reminder of this, playing a role similar to a syntactic variable).
- Let us suppose then that we want to synthesise diagram D with a simple diagram representing "All A's are B's".
- It is then sufficient to add potato A to diagram D, placing it inside B, and making sure that A is, relative to any potato C that intersects B or is included in B, in the most general relationship possible (A and C intersect without being included one inside the other).

A *diagrammatic demonstration* is then a process of synthesis which makes us navigate from diagram to diagram by explicit rules. The diagrammatic space is then sufficiently well-defined for the mathematical methods of demonstration theory to be applied to it, to obtain results of expressiveness, soundness and completeness of the method.

It should be noted, however, that the use of these methods has a price to pay which is difficult to accept from the point of view of the philosophical analysis of the nature of a diagram and its epistemic function. Indeed, does not considering a diagram as a precisely defined mathematical object distort it from the outset? To make a diagram an exact mathematical object is to forget that a diagram is obviously a *sensitive drawing*, and not a structure cut out of a mathematical space.

However, technically, one can emancipate oneself from the exact shapes chosen to draw the "potatoes", through equivalence classes, by stipulating that, starting from an exact diagram, any diagram obtained by a variation which preserves the topology of the whole, will be another equivalent representation.[33] In a landmark article in the field of geometric diagram research,[34] Manders showed that the inaccuracy of geometric diagrams is harmless and inconsequential. Indeed, the properties for which the diagrams are mobilised are characterised by Manders as being *coexact*.[35] This means that if, starting from a diagram D, one constructs a diagram D' sufficiently close to D, then it expresses the same coexact relations as D. However, the relevant properties in a logical Euler-Venn type diagram (inclusions, exclusions, intersections, presence of markers in certain zones), which could be described as "topological" or "mereological" properties are a subclass of the coexact properties in the sense of Mander.[36] Manders' discourse could be adapted to show that, despite the inaccuracy of the outline of zones in a logic diagram, it is still perfectly effective in expressing the relationships that matter.

To resume, at the price of the distortion of the idea of diagram, which consists in transforming it into an exact object, one obtains precise results on the expressiveness, soundness and completeness of Euler-Venn type diagrammatic methods. Let us state here just a few typical results.

Expressivity of diagrams. *Carroll diagrams can express all configurations of Boolean logic.* This is immediate since, by construction, these diagrams show, in a mixed form between a table and a diagram, the 2^{n} possible elementary zones of a free finite Boolean algebra. However, one can retort that this theorem of expressivity is obtained at little cost, since the spatial and topological constraints on a Carroll diagram are very loose, the zones attached to the notions not even being necessarily connected.

Venn diagrams can also express all configurations of Boolean logic. Since the operation of shading zones does not pose a problem, the only thing to be demonstrated is that a Venn diagram can be drawn consisting of any number n of potato-shapes, cutting out 2^n elementary zones. This theorem of expressivity can be obtained with interesting diagrammatic constraints: connectedness of all

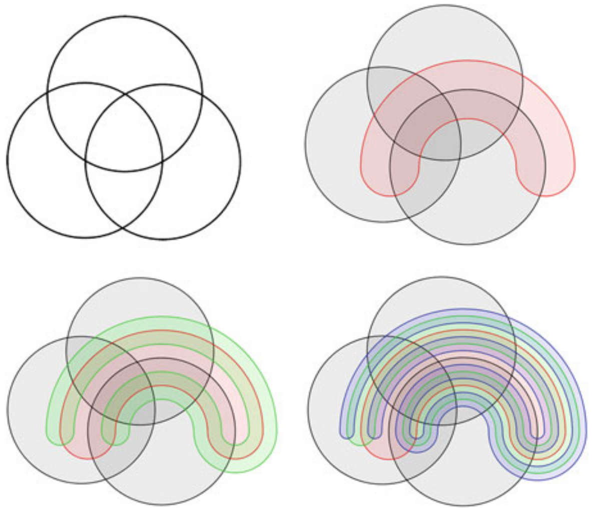
[33] Venn had already seen that the relevant relationships on a Euler or Venn diagram were topological and not metric in nature, (Venn 1883b, p. 59): The compartments yielded by our diagrams must be regarded solely in the light of being bounded by such and such contours, as lying inside or outside such and such lines. We must abstract entirely from all consideration of their relative magnitude, as we do of their actual shape [. . .] and trace no more connection between these facts and the logical extension of the terms which they represent than we do between this logical extension and the size and shape of the letter symbols, A and B and C's".

[34] (Manders 1995). The use of Manders' pioneering work in the field of logic is possible, and can yield fruitful things, provided that it is done carefully, by adapting Manders' ideas correctly to this different framework, and by not forgetting the different status that space has in the field of logic and in that of geometry (see our remarks at the beginning of Section "The Late Emergence of "Analytical" Logic Diagrams in a Pedagogical Context").

[35] (Manders 1995, pp. 91-sq.).

[36] In addition to these properties, Manders' coexact properties include certain metric inequalities.

primitive potato-shapes and elementary zones.[37] John Venn himself had the intuition that it was possible, and sketched the first steps of a construction, generalisable by induction over *n*, of an *n*-potato Venn diagram.



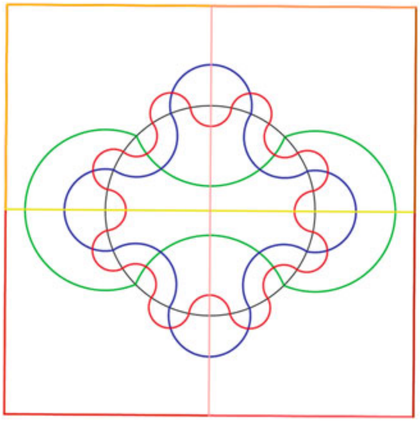
Venn diagrams with respectively 3, 4, 5 and 6 "potatoes".
From the fourth potato added, it is a folding process (with 2^n-1 folds), generalisable by induction, which dictates the shape of each potato.
The process is indicated in (Venn 1880, p. 8–9).
Source of images: Wikipedia, article "Venn diagrams".

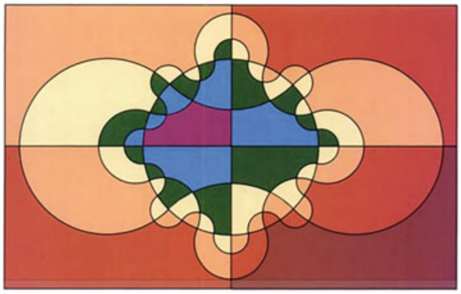
In the second half of the 1980s, on the one hand Mahmoodian, Rezaie and Vatan[38] and on the other Edwards[39] proposed more elegant, more symmetrical constructions. Mahmoodian, Rezaie and Vatan used a trigonometric construction in the plan, adding at each stage arcs of circles of half the radius of the previous stage, and which oscillate twice as fast. The possibility of continuing the construction indefinitely is demonstrated by induction, using trigonometric formulas. Edwards' solution is topologically equivalent to the previous one. It is no longer based on trigonometric formulae, but rather on geometric intuition in space. Edwards starts from a sphere, produces sinusoidal curves oscillating faster and faster around the equator, and then projects them by means of stereographic projection.

[37] On an n-potatoes (P_1, ..., P_n) diagram D, the 2^n elementary zones are those expressed by a set formula of the type ε_1.P1 ∩ ... ∩ ε_n.P_n, where each ε_i is 0 or 1, where "1.P_i" is P_i, and where "0.Pi" is $\complement P_i$.

[38] (Mahmoodian and Vatan 1987).

[39] See (Edwards 2004) for some clues to the history of these diagrams. Edwards seems to have first reported his solution in a poster session (October 1988) at the Royal Society of London's 'Fractals in the Natural Sciences' meeting. The solution of Mahmoodian, Rezai and Vatan thus seems approximately contemporary (Shin 1994), (Mineshima et al. 2008).





Left: the construction of Mahmoodian, Rezai and Vatan. (**Source**: personal production on Geogebra).

On the right: Edwards'. The construction is stopped at 6 curves (the first two "curves" are rectangular in shape and are independent from the rest of the construction). **Source**: (Edwards 2004, p. 58).

Soundness and completeness of the diagrammatic method. Once all the propositions of a set of premises have been synthesized on a diagram, by means of correctly explained diagrammatic rules, then the propositions which are shown on this diagram are *exactly all* the propositions which are deductible from the premises by the usual rules of syllogistic.

See our appendix for details.[40]

Lemma (existence of the diagram). Either α_1, – - - -, α_k a list of propositions of Aristotelian form (i. e., having one of the four forms of the Boethius' Square of Oppositions). The procedure of synthesis of the generalized ("spider") Euler diagram responding to this list of premises is successful if and only if the list is *syllogistically consistent*.

Lemma (uniqueness of the graph). The previously constructed graph is unique (up to a diagram isomorphism) and does not depend on the order in which the premises were synthesized.

Theorem (completeness). If a proposition β (of Aristotelian type) is a consequence of the list α_1, - - - -, α_k by the syllogistic method, then β is *shown* on the diagram which synthesizes α_1, - - -, α_k.

[40] (Bernard 2022).

Theorem (soundness). If a proposition β (of the Aristotelian type) is *shown* on the diagram synthesizing a list α_1, - - - -, α_k of propositions; then β is a consequence of the list by the syllogistic method.

Third Section: Can a Diagram Show ("donner à voir") the Nature of a Proposition or the Correctness of a Reasoning?

Back to Euler

To support Euler's thesis, we will comment on the text where he introduced his diagrams. At the moment when Euler wants to introduce Aristotle-Boethius' logical Square of Oppositions to the Princess of Germany, he writes:

> These four kinds of propositions can also be represented by figures, to express their nature visibly at sight. This is a marvellous help in explaining very clearly what the correctness of a reasoning consists of. As a general concept contains an infinity of individual objects, it is seen as a space in which all these individuals are enclosed. Thus, for the notion of Human, one makes a space[41]

And it is precisely at this moment that Euler drew his first round "figure" which represents the extension of the notion (here of Human).

In the previous section, we saw that in traditional (that is pre-Fregean) logic, one could just as well use diagrams and rules of diagrammatic synthesis, as usual linguistic/symbolic methods, to express all propositional forms and to discriminate between correct and incorrect reasonings. But we see in the text that Euler reserved for diagrams a much superior epistemic function. They are not for him mere elements of proof, but, can "visibly express [...] at sight" (note the pleonasm) the nature of the propositions and the correctness of the reasoning.

Speaking of a "*giving to see*" ("donner à la vue"), Euler suggests a double use of the idea of perception or vision, which I have already mentioned above. On the one hand, the diagram is obviously itself an object of perception. But by saying that it shows (literally: "gives to see") the correctness of a reasoning, we suggest the idea that visual perception[42] is transformed into a vision in a metaphorical sense, a vision of the mind which adequately grasps the form of a proposition, or which intuits the validity of an inference. In order for a diagram to be capable of this, it must be much more than an arbitrary arrangement of visual forms, but rather an "image", one that,

[41] (Euler 1770, Letter 102): "On peut aussi représenter par des figures ces quatre espèces de propositions, pour exprimer visiblement leur nature à la vue. Cela est d'un secours merveilleux, pour expliquer très distinctement en quoi consiste la justesse d'un raisonnement. [. . .]"

[42] In fact, the visual character, although emphasised by Euler, is perhaps not paramount. The intuitions at play in logical diagrams (inclusion and exclusion of zones, presence of markers in certain zones...) lend themselves just as well to a tactile approach. Can a blind person not acquire the intuitions at play through tactile diagrams with differentiated textures for each zone? See (Aldrich 2008) and (Goncu et al. 2010). Spatiality seems essential rather than visuality.

in a certain sense, *resemble* or even be *adequate* to the logical forms it intends to represent. To paraphrase Venn: in an authentic diagram, logical relationships must "explain themselves".[43]

If we place ourselves in Kant's tradition, where intuitions are reserved for the realm of sensibility, as opposed to the realm of understanding and the logical sequence of concepts, then we cannot give meaning to Euler's thesis. In order to give him back his credit, one must instead place oneself in a philosophical tradition, which includes Descartes, Husserl and Weyl, where the domain of logic, like any other sphere of knowledge, is also based ultimately on primitive *intuitions*. The capture of the correctness of a reasoning by the mind is itself, in the last instance, based on an intuition; not a sensitive intuition, but, according to the authors, a *logical*, *formal* or *categorical intuition.*[44] We can then reformulate Euler's sketched opinion in the following way: the visual intuitions underlying diagrammatic practice can be transformed into logical intuitions capable of giving us an adequate view of the logical forms of propositions and the correctness of reasoning.

Showing the Nature of a Proposition

Concerning the possibility for diagrams to show ("donner à la vue") the nature of propositions, Euler's text is very instructive. The Swiss mathematician spontaneously proposes, in his letter 102, a one-to-one association between the four logical forms of the Aristotle-Boethius Square and the following four diagrammatic forms:

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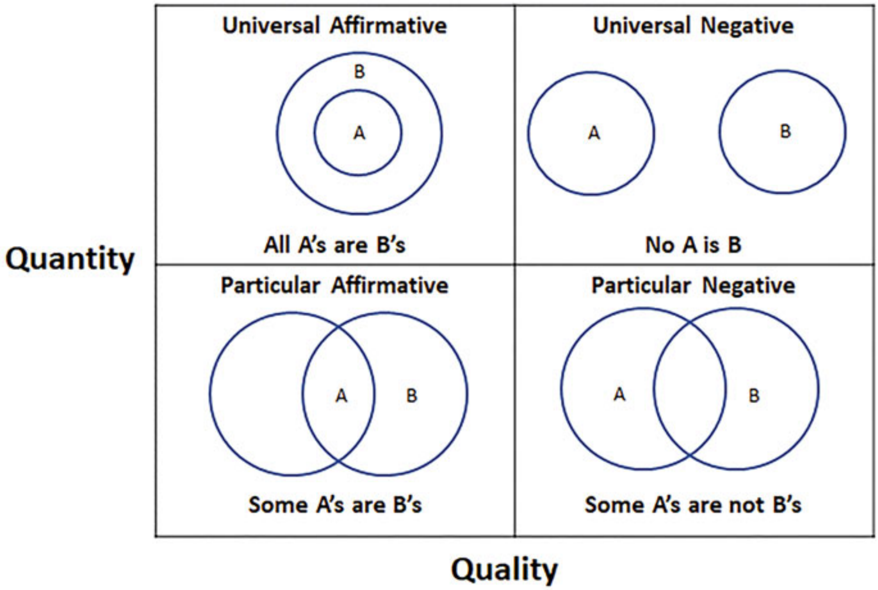
[43] (Venn 1883a, p. 55), cited below.

[44] For example, in *The Continuum*, in a rather Cartesian tradition, Weyl explains that even in the most rigorous mathematical demonstrations, which present themselves as a succession of inferences, "The *'direct* experience of the True' < *Erlebnis der Wahrheit* > is ultimately based on intuitions or direct grasps <*unmittelbarer Einsicht* > of the truth of each inferential step (Weyl 1994, p. 59, note).



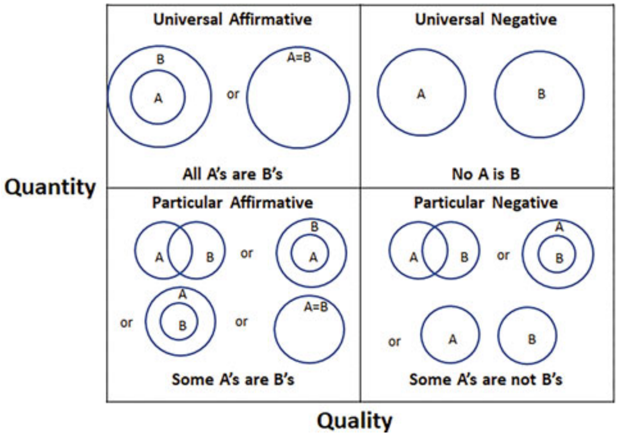
Then he soon realises that it does not work so simply. Firstly, as Venn will explain in 1880, there are in fact five possible extensional relations between two classes A and B:

1. A and B are external to each other,
2. A is strictly included in B,
3. B is strictly included in A,
4. A and B intersect (without one being included inside the other),
5. A and B are coextensive.

There is therefore already a simple arithmetical impossibility to obtain a bijection since there are 4 cells in Aristotle-Boethius' table, which gives 8 cases after possible inversion of *A* and *B*, and only 6 cases if one identifies the equivalent propositions by conversion (Some A is B = Some B is A, None A is B = None B is A).

In fact, to each propositional form corresponds not a single Euler diagram, but a *disjunction of several diagrams*. If one wanted to write a table in the manner of Euler's,[45] but taking into account this disjunction of cases, it would give:

[45] Euler realises in the letter 104 that several diagrams (of the type drawn above) must be *disjunctively* associated with certain of the four cells of Aristotle-Boethius's square. But he shows this only in a concrete example. He does not go back to the logical square in order to rewrite it correctly in its entirety.



The Logical Square, rewritten in disjunctive diagram form.
Such a table *is not* explicitly in Euler's text (see note 45).

The representation of a propositional form by a disjunction of diagrams quickly becomes impracticable, as soon as one wants to use it to articulate several propositions among themselves to test the validity of a somewhat complex reasoning. Indeed, the number of diagrams to be considered disjunctively increases exponentially. The method becomes impracticable in time and space, and the visual intuitivity of the logical form of the propositions is undermined.[46]

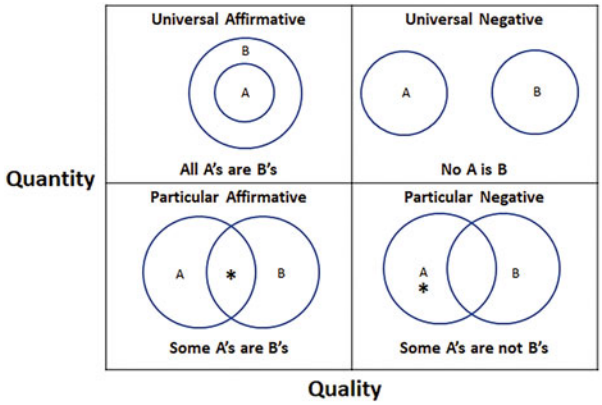
Euler seems to be aware of this problem since he introduces a new notation in letter 105. Indeed, he begins to place small asterisks in the zones of the diagram where it is known that there must necessarily be individuals, as these zones correspond to the subjects of certain particular propositions (i. e. of type I or O in the Latin classification).

Euler does not explicitly state this in his letter, but this introduction of a new notation suggests *a radical change in the rules of the diagrammatic game*. That is to say that, henceforth, contrary to what the diagrams of the letters 102 to 104 suggest, any zone drawn on a diagram may be empty. In this case, the diagram is considered to be committed to the existence of individuals only in the zones marked with an asterisk. This is the ancestor of *spider Euler graphs* or *coloured Euler graphs*. If we stick to these new diagrammatic rules, then *we re-establish a biunivocal association* between the logical forms of propositions and the diagrammatic forms.[47] And the problem of the exponential explosion in the

[46] Cf. also on this subject: (Venn 1880, p. 1–4-sq.).

[47] More precisely, there is a bijection between the *logical forms of proposition* of traditional logic and a correctly constructed set of Eulerian asterisk diagrams (6 possible cases, as explained above). On the other hand, there is no bijection but only an *injection* of the *linguistic* forms of propositions

number of diagrams disappears at the same time. Euler, after having introduced his notation with the asterisk, does not rewrite the logical square. If we do, the result is:



The Logical Square, rewritten with Euler diagrams with asterisks, and with the rule (not explicit in Euler):*a zone without an asterisk is possibly empty*.
This table *does not appear* in Euler, but the notation with the asterisk is present. Euler introduces it in connection with the verification of the correctness of a particular form of syllogism, taken as an example.

John Venn underlined a philosophically crucial point of this story: for a possible adequacy between the diagrammatic representation and the forms of predication, it is essential to express a form of *uncertainty* with regard to the extensional relations between the potato-shapes.[48] Any diagram capable of adequately reflecting predication must therefore show uncertainties as well as certainties. *To be adequate, the diagram must have an irreducibly modal dimension*.

Thus, for example, to transcribe "Some A is B", an asterisk is only put in the central zone of the diagram. The two zones corresponding to $A \setminus B$ and $B \setminus A$ are drawn without an asterisk. It is then necessary to understand that they are *possibly* empty zones. Thus, the four cases of the disjunctive table above have been subsumed under a single diagram of *modal significance*. Euler seems to have seen this confusedly, since he introduces his new notation (the asterisk) by specifying that it is placed where one is "*certain*" (of the presence of individuals), the other zones being zones of uncertainty.[49] Euler had probably not seen the potential of this new notation, however, since he continues to present disjunctive diagrams in his letter 105, even after having introduced his asterisk. This is why Venn does not consider Euler as

into these Eulerian diagrams. Indeed, for example, the two linguistically distinct propositions "Some A is B" and "Some B is A" are mapped to the same diagram. The logical forms bijectively mapped to the diagrams are therefore abstract forms, 'mental' forms as medieval logicians would have said, which should not be confused with their expressions in language.

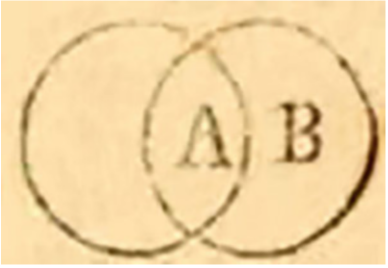
[48] (Venn 1880, p. 4-sq.). Cf. in particular the idea of "partial knowledge".

[49] (Euler 1770, letter 105, pp. 126–127).

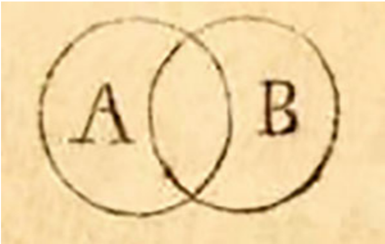
having seen the necessity of the modal dimension of the diagrams,[50] but he clearly sees this idea in: Dr. Thomson, Lambert, then in De Morgan, Jevons and Überweg.[51]

The story of Euler's notational hesitations has another side to it, which is just as interesting for us. Indeed, by changing the rules of the diagrammatic game, under the suggestion of Euler's asterisk, we corrected an inadequacy of his original notation.

Indeed, for the "Some A is B" proposition, Euler had first proposed the diagrammatic representation below.



However, this diagrammatic representation gives non-symmetrical locations for the letters *A* and *B*, even though the proposition in question is logically symmetrical (in the sense of the possible conversion to "Some B is A"). Conversely, Euler had chosen the following spatially symmetrical diagrammatic representation for a proposition "Some A is not B" which is not logically convertible!



Once the notation with the asterisk has been established, the spatial symmetry of the diagram is matched with the logical convertibility of the proposition. From this point of view, the diagrammatic representation is more adequate than the linguistic representation. In fact, the two propositions "Some A is B" and "Some B is A" express one and the same logical relationship between concepts A and B, which one and the same diagram adequately represents. The same would be shown for the negative universal propositions.

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50 (Venn 1883b, p. 49): "The weak point about these [Euler's circle diagrams] consists in the fact that they only illustrate in strictness the actual relations of classes to one another." (Emphasis added)

51 (Venn 1883b, p. 50).

On a diagram, logical symmetry is directly represented by spatial symmetry; where, on the contrary, the usual linguistic representation of the proposition requires the introduction of a conversion rule, to express only *symbolically* and conventionally what the diagram shows by a direct image and by the rules of its use (a diagram is given without any sense of reading[52]).

The existence of a one-to-one relationship between the above asterisk diagram forms and the traditional propositional forms, as well as the natural use of spatial symmetries to represent logical symmetries, are elements which confirm the Eulerian thesis that diagrams show the nature of propositions. It should be noted, however, that Euler does not commit himself precisely to the meaning of this showing ("donner à voir") and does not evoke symmetries in his text.

Showing the Correctness of a Reasoning

Let us move on to the showing of the correctness of a reasoning.

On this point, Euler's indications are clear. His diagrams can "demonstrate the correctness" of the 19 conclusive syllogistic modes.[53] This is possible because they reduce all the logical sequences between propositions to only two principles of a spatial nature (more precisely of a "topological" or "merereological" nature, as we would say today). These two principles make it possible to intuitively and adequately grasp the relations between concepts. Here they are:

- Everything that is in the content is also in the container.
- Everything outside the container is also outside the content.[54]

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[52] In another context, that of the logic of predicates, Hermann Weyl shows that a spatial representation may be more adequate for capturing the logical form of a proposition than its linear symbolic writing. He gives the example of the two different scripts "4 < 5" and "5 > 4" which lead us to believe, wrongly, that they would represent two different relationships (which are called "reciprocal" of each other). Weyl rebels against this idea. There is only one and the same relationship behind these two scriptures. And we would have a clearer representation of this if we replaced the too linear language with a three-dimensional representation, let's say a model that we could walk around, reversing at will the order in which we see the "4" and the "5" in our perception of this relational fact. The example can be found in *The Continuum* (1918). Cf. (Weyl 1994, p. 52 note; p. 129). Weyl's model, like Euler's diagram above, does not introduce any privileged order between the two elements put in relation. Symbolic writing, on the other hand, must artificially introduce an order (which does not appear in the logical relationship to be represented) and then neutralise it by a conventional rule, a rule of conversion, or of passage to the reciprocal relationship.

[53] (Euler 1770, letter 107, p. 139): "Moreover, the correctness of each of these modes is already demonstrated above by means of the spaces I have used to mark the notions." ("Au reste, la justesse de chacun de ces modes est déjà démontrée ci-dessus par le moyen des espaces que j'ai employés pour marquer les notions.") Clearly, Euler places himself in a post-Aristotelian tradition of counting syllogistic modes, where the modes of the 'fourth figure' are counted.

[54] (Euler 1770, letter 104, p. 123).

It is therefore the adequate representation of the relations between the extensions of concepts by the topological relations between potato-shapes, which explains the natural shift from logical intuition to spatial intuition, from the visual to the intellectual gaze. There is a naturality of *diagrammatic* representation, which contrasts it with *symbolic* representation, which presupposes an arbitrary convention.

In a comparable spirit, John Venn tries to distinguish between modes of representation which are properly diagrammatic, and those which are merely "symbolic". Thus, he writes two articles in parallel, on the modes of representation of the usual propositional forms of logic; the first article contains all the notations that take a "symbolic form"[55]; and the other contains all the "sensitive representations" that use a "geometrical diagram".[56] Venn does not delimit the boundary between these two modes of representation by an exact definition. However, he does give some interesting criteria. Symbolic notation presupposes arbitrary conventions. Conversely, a properly diagrammatic representation exploits "geometrical" properties to naturally suggest predicative relationships between concepts. We shall see below, on the occasion of Venn's discussion of Hamilton diagrams, that these geometrical properties include in particular mereological relations of the type used in Eulerian-type diagrams. The latter diagrams are extensional in the sense that they represent the relations between extensions (in the logical sense) by the analogous relations between extensions (in the sense of the spatial extent of the potato-shapes). Thus, says Venn, these diagrams tell their story in a direct way.[57] To our knowledge, no text of Venn's suggests other types of diagrams than extensional diagrams to adequately and directly represent the form of propositions and reasoning.

Furthermore, Venn identifies certain practices, such as Frege's and perhaps also Hamilton's, where one is in a mixed case, mixing symbols and properly diagrammatic representation.[58]

Is the link between space and logic, as we have highlighted in Eulerian-type diagrams, fundamental and irremediable, or is it only an accident, but a happy accident, at the meeting between two spheres of thought?

Moreover, can this link between logic and spatiality be projected back to earlier stages in the history of logic?

55 (Venn 1883a, p. 36).

56 (Venn 1883b, p. 47).

57 (Venn 1883b, p. 53): "[they] therefore tell their tale somewhat directly".

58 (Venn 1883b, p. 43): "Frege's scheme (*Begriffschrift*, 1879) deserves almost as much to be called diagrammatic as symbolic."

Some Remarks on the Relationship Between the Principles of Logic and Spatiality in Syllogistic (From Aristotle to Hamilton)

We recalled from the beginning of our article that *diagrammatic* practice (in the sense that Venn speaks of "analytical diagrams") was not common before Euler's work. But this remark does not end the question. The relation to spatial intuitions could be presented either in the form of diagrams of a different nature, or in forms less immediately obvious than the production of diagrams. Even if the drawing of diagrams is dispensable in a work introducing to the primitive notions of logic, it could turn out that the notions introduced require, in order to be grasped, a type of intuition that would deserve to be called "schematic" or "diagrammatic".

This is why we have carried out a quick historical survey[59] of the lexical fields and metaphors used by Aristotle when he introduced the principles of traditional logic: predication, relationship between genus and species, distinction between particular and universal, etc. The aim was to test the idea of an intuition of a spatial nature, similar to that of Eulerian diagrams, at the basis of primitive logical notions.

At a first level of analysis, it should first be noted that, while the Greek language is propitious for a massive use of spatializing etymologies and metaphors, there is relatively little use on the part of Aristotle of terms of spatial origin, or spatial analogies to introduce the notions and principles of his *logic*. This is particularly the case when Aristotle defines or characterizes the relationships between terms. A first exception is the particle "ἐν" which indicates the fact that one thing is immobile *in* another. Aristotle uses it in particular for the relationship between a substance and one of its accidents (*Categories*, 1b). But it is a much-used particle, and with many deviant meanings. The relation to spatiality is indeed there but remains insignificant. Where Euler based the Barbara modus reasoning (or, if one prefers, the *dictum of omni*) on a spatial intuition, Aristotle uses a legal vocabulary which has nothing spatial about it:

> When one thing is attributed to another, as in the case of its subject, all that can be said of the attribute can also be said of the subject. (Categories, 2a)

It is generally the verb *kategorein*, of legal origin, which means "to accuse", that dominates. When he wants to talk about the way in which terms are brought together to form a proposition, Aristotle uses the vocabulary of the link (*sumploke, sumdesmo*), which often refers to the bringing of bodies into contact in various

[59] To carry out this investigation, not being ourselves Hellenist, we first sought the help of Isabelle Koch to go directly to the Aristotelian texts, with the valuable help of Jonathan Barnes' comments in (Barnes 1991). Koch indicated to us the recent comments of Michel Crubellier (Crubellier 2017) on the diagrammatic interpretation of the texts of the *Prior Analytics* (cf. below). Finally, Venn has led us to the Hamilton texts (Hamilton 1860, 1866) which are important for clearing up the history of the "logical diagrams", in the period of medieval and ancient commentators of Aristotle. See however (Macbeth 2010, 2012).

contexts (kissing, sexual relations, wrestling holds). Aristotle thus describes a statement as a whole composed of parts. In the utterance, the words are linked together (*kata sumploken*), whereas they were previously separated (*kechorismenon*). We can therefore see that Aristotle understands the composition of a proposition from its terms by the idea of a link, a connection that binds the terms together to form a totality. The relation to space is not emphasized, and Aristotle's point of view is more syntactic than semantic. In other words, it is a question of composing statements from words. In the *Prior Analytics* (24a-25a), Aristotle distinguishes the possible quantities of a proposition, i. e. universal or particular, by saying that the attribute is asserted either of the whole thing, or of a part (*kata meros*). There is thus an opposition between the whole and the part, but without a spatial or visual dimension being put forward. On the contrary, it is the vocabulary of accusation/attribution that comes up.

However, at a second level of analysis, if one focuses more particularly, within the *Organon*, on syllogisms, and therefore especially in the *Prior Analytics*, one comes across a technical vocabulary that is largely borrowed from geometry.[60] A proposition is called an "interval" (διάστημα) or a "protension" (πρότασις), i. e. the stretching of a line from one point to another—which could almost be transcribed anachronistically by "vector".[61] Such a proposition links two notions called "terms" (ὅροι) or (ἄκρα). These are terms used to designate geometric points as the ends of a segment. The different ways in which three terms can be arranged together (the middle term appearing twice) to form a reasoning structure are called by Aristotle "scheme" or "figure" (σκῆμα). This arrangement of terms is described by Aristotle in terms of *position* (θῆ θέσει); situating the three terms (major, minor, medium) by prepositions such as: above, below, left, right.[62] Hamilton even warns that the word "syllogism" (συλλογισμὸς) could be borrowed from mathematics, designating something like a calculation.

This mathematical vocabulary, and more precisely geometric vocabulary (except for the word "syllogism") suggests that Aristotle's texts could have been accompanied by diagrams that would unfortunately have been lost through the negligence of the copyists.[63] A certain similarity can be noted here with the fate of the figures accompanying the geometrical texts (especially those of Euclid).[64] In some medieval and Renaissance manuscripts,[65] we find logic diagrams, in particular

60 (Hamilton 1860, p. 469), (Hamilton 1866, footnote p. 663), (Barnes 1991) and (Crubellier 2017).

61 Crubellier sometimes refers to this as the "journey" ("trajet").

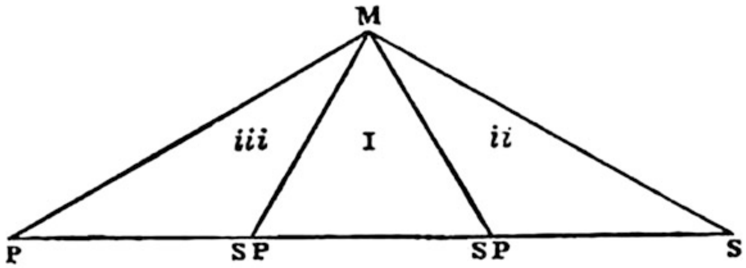
62 See (Hamilton 1866, p. 665).

63 Ammonius expresses his opinion on the very likely existence of "logical diagrams" accompanying Aristotle's texts. Pacius attributes their disappearance to the copyists cf. (Hamilton 1866, footnote p. 663).

64 (De Risi 2017).

65 In (Hamilton 1866, p. 666 footnote), Hamilton tells us that he traced them back to a text of 1533 where there were triangular diagrams related to a certain Faber Stapulensis, who claims that they go back to the Greeks. Crubellier has traced them back to a ninth-century manuscript containing an edition of the *Prior Analytics* containing comparable logical diagrams.

accompanying the editions of the *Prior Analytics*, and it is not known whether they are of Greek origin, or whether they were already reconstructions of Aristotle's supposed logic diagrams, which would have been lost in an earlier period. The schemas in question, since they consist of the articulation of three "intervals" (=propositions) (the two premises, and the conclusion), have a triangular shape. To give the general spirit in which these diagrams are constructed, let us just report here the Hamilton diagrams (see the diagram below). These diagrams represent only the "figures" of the syllogistic, and therefore ignore the quantity and quality of the propositions, simply showing the order between the terms in the premises and the conclusion.



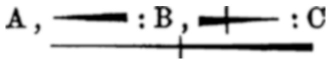
Hamilton's diagram summarizing the three "figures" of Syllogistic. (Hamilton 1866, p. 664)

> Hamilton tries to be faithful to the indications present in Aristotle's text. S is the minor term, P the major term, and M the middle term. All segments should be thought of as vectors or arrows, oriented from left to right. Thus, we see that for the first syllogism figure (central triangle), the major term is attributed to the middle term which is in turn attributed to the minor term. For the second syllogism figure (triangle on the right), the middle term is attributed to the other two terms, and so on.

Hamilton then designed his own diagrams, in which the segments are replaced by small arrows, to express their orientation without having to rely on the convention of reading from left to right. Hamilton adds small signs above or next to the arrows to express the quantity and quality of the propositions.[66] Finally, he often flattens the triangle. Thus, for example, the "Baroco" modus in the second figure is represented by Hamilton[67]:

[66] A small bar perpendicular to the arrow is added to denote negation. The ":" or ";" sign indicates a universal quantization, and "'" or "," indicates a particular quantization. Hamilton advocates quantifying the predicate as well as the subject.

[67] (Hamilton 1860, 457). Hamilton in fact seems to quantify *terms* rather than the *proposition* itself.



Example of Hamilton's diagram representing the "Baroco" syllogistic mode.

Venn considers that this type of representation does not have the same diagrammatic status as his own or Euler's diagrams:

> To my thinking [Hamilton's notation] does not deserve to rank as a diagrammatic scheme at all, though he does class it with the others as "geometric": but is purely symbolical. What was aimed at in the methods above described was something that should explain itself at once, as in the circles of Euler, or need but a hint of explanation, as in the lines of Lambert. But there is clearly nothing in the two ends of a wedge to suggest subjects and predicates, or in a colon and comma to suggest distribution and non-distribution.[68]

Venn's opinion expressed here makes it possible to clarify how he sees the opposition between diagrammatic and symbolic representation. A diagram must "explain itself at once". It shows immediately the relationships it represents. It must "suggest" them without the need for arbitrary conventions. And Venn reminds us that it is the *geometric* properties of a diagram that allow this immediate suggestion of certain logical properties of reasoning. Unfortunately, he does not define exactly what he means by "geometric" here. In any case, it is understandable that, as soon as one accepts the extensional interpretation of logic, universal predication is naturally expressed by an inclusion between two potato-shapes, which must be counted here among the "geometric" properties.

However, we want to qualify Venn's blunt assertion that Hamilton's representations do not deserve the status of a diagram at all. Indeed, even if the symbols added by Hamilton to signify the quantity and quality of propositions do have a symbolic and conventional status, it is not the case for the arrangement of the terms in relation to each other in a "figure" (in Aristotle's technical sense). We are referring here to what Hamilton's diagrams share with the triangular diagrams of Aristotle's commentators.

The triangular representation (with oriented sides or "arrows") represents more adequately the figures of Syllogistic than their symbolic representations can. As we have seen in the case of Eulerian diagrams, triangular figures manage to represent the scheme of reasoning, without having to introduce, when superfluous, arbitrary order between the premises. The perceptive apprehension of two sides of a triangle, unlike the enunciation of two premises, does not need to differentiate between a first and a second element. Thus, for example, Hamilton insists that, in the second syllogistic figure, there is no logical reason to call one of the two terms "major" and the other "minor". Their roles are absolutely symmetrical with respect to the medium term; and this is why many logicians are wrong to distinguish two different syllogistic modes within the second figure (one direct, the other indirect), based on the arbitrary ordering of the premises. In contrast to this attitude, the diagrammatic

68 (Venn 1883a, p. 55).

representation will allow these two modes to be represented by a single triangular diagram.[69]

Generally speaking, an important motivation for Hamilton's work on these triangular diagrams is to seek to capture adequately the logical symmetries in the status of the premises, through the geometrical symmetry of the figures. On a few occasions, Hamilton uses expressions with a tone similar to Euler's, stating that the diagrams, through some of their symmetries, make certain logical symmetries of the problems apparent to the eyes.[70] Hamilton also attributes to Aristotle such a motivation, which could explain, among other things, why Aristotle refuses the syllogistic "fourth figure" that some of his continuators will introduce.[71] We can therefore see how Aristotle manages to extract himself from certain contingencies of language by means of a mathematical type of abstraction, more precisely: geometric. When, at the end of the seventeenth century, in his *De Arte Combinatoria*, Leibniz proposed to re-establish the "fourth figure" of syllogistic as of right, and to admit indirect modes as authentic and conclusive syllogistic modes,[72] he did so in the name of re-establishing certain symmetries. But, if these symmetries had to be re-established, it was perhaps because we had historically moved from a diagrammatic mode of representation to a purely symbolic mode.

Recently, on the occasion of the preparation of his edition of the *Prior Analytics*, Michel Crubellier reopened the question of the diagrammatic interpretation of the *Prior Analytics* (Alristotle 2014) by bringing new arguments from the progress of philology and history of the ancient logic.[73] Crubellier also assumes that Aristotle's text had to be accompanied by drawn diagrams that provided spatial and visual support to the textual discourse, just like the treatises on geometry of the time. Crubellier argues that figuration is a more adequate means of making Aristotelian syllogistic intelligible than "formalisation" in the modern sense, which would consist in producing hypothetical-deductive symbolic systems to encode the rules of syllogistic (he points out, however, that hypothetical-deductive systems are certainly not foreign to Aristotle's thought in the context of science; but logic is precisely not one for Aristotle).

Thus, the diagrammatic interpretation of the *Prior Analytics* allows us to highlight an early use of geometric intuitions to show certain logical properties of our reasoning. However, the diagrams concerned are not extensional diagrams as found in Euler, and they do not manage to represent naturally the complete logical

[69] In the same vein, we can refer in our appendix to the way in which we define, for a set consisting of premises, the 'maximum Eulerian diagram' that satisfies it. It is defined univocally up to a diagram-isomorphism and does not depend on the order in which the premises are considered during its construction.

[70] (Hamilton 1866, p. 656, point 12.): "This is made apparent to the eye."

[71] (Hamilton 1866, p. 655, point 4), the same type of argument is taken up by (Crubellier 2017).

[72] These are those that go through an unnecessary weakening of the quantity of the conclusion or of one of the two premises, such as BarbarI or CelarOnt.

[73] (Crubellier 2017).

structure of our reasoning. This is why this type of diagram required symbolic additions.

The history of logic thus seems to indicate that, as long as logic has been interpreted preferentially in an intensional way, the use of spatial intuitions as a means of acquiring logical intuitions that preside over the acquisition of principles, has seemed to be a path little used, or at least partially used. It is through the shift towards a preference for an extensional interpretation of logic that the link to spatiality seems to have been strengthened.

Conclusion: Summary and Discussion

The history of logic seems to indicate that the relationships between, on the one hand, space and diagrammatic thinking and, on the other hand, the intuitions underlying the principles of formal logic, were not immediately obvious. It is only in Euler's period that logic progressively became preferentially extensional, and that the possibility of a fully efficient diagrammatic method in logic opened up; even if diagrams probably played an important role, already in Aristotle's time, to represent certain symmetries of reasoning, notably in syllogistic "figures".

The use of analytical diagrams, in the sense of Venn, has undeniable cognitive advantages: conciseness, aesthetic advantage, plasticity, quasi-synoptic or quasi-instantaneous character, which epistemology, cognitive sciences, experimental psychology and modern technologies of graphic visualisation nowadays often use.

As recalled in the second section, some contemporary logicians have adopted a completely different approach to Euler/Venn diagrams, consisting in rigorising and exacting them to make them the possible objects of the methods of the theory of demonstration. These works have an undeniable heuristic character. But for them to be harmless, one must be aware of the fundamental denaturation that a diagram undergoes as soon as one sees it as a mere element of formal proof, whereas it is more fundamentally a sensitive drawing that is being sketched out.

Furthermore, is the use of diagrams as mere "elements" of formal proof, in the sense we reported in the second section, sufficient to give the purely diagrammatic method a truly foundational role in logic? This is doubtful.

On the one hand, the soundness and completeness theorems for diagrammatic methods can only be given in *discursive* form. It is hard to imagine writing these theorems themselves using Eulerian potato-shaped diagram. So that, if the burden of proof is attributed to the diagrammatic method (which is always possible), the global understanding of the soundness and completeness of the method seems to rest inescapably on a discursive meta-level which takes care of ensuring the formal properties of the "*object diagrammatic-field*" (by analogy with the usual idea of *object language*).

On the other hand, in order to apply the methods of demonstration theory to diagrams, it is not necessary for the diagrams to be meaningful. One can make "blind algorithmic drawing" based on diagrams, just as one would do blind calculation based on symbols. The diagram then loses its proper diagrammatic function and takes on the same status as symbolic representations, that of a simple encoding of the underlying Boolean structure and the calculation rules that take place within it. To test Euler's idea that diagrams can show the correctness of reasoning, we need to be more demanding about what we call the "justifying" function of diagrams. We are no longer just looking for an algorithmic demonstrative process that confidently leads to an answer (validation or invalidation of the reasoning), but we are looking for the possibility to give an intuition of *why* the validity is there. We must then maintain our attention on the intrinsic meaning of the diagrams.

The opposition which we have just described between two uses or two very different points of view on logic diagrams can also be understood analogously, starting from diagrammatic practice in *geometry*. Thus, one can read an austere treatise on geometry, without a figure, and check the completeness of its inferential sequences. But if we want to understand the meaning of the demonstrations, shouldn't we restore the diagrams, at least at the level of our imagination? Isn't there an obligatory recourse to some form of diagram, whether printed on paper or simply present in the mind? To sum up, it is a characteristic of the practice of rigorous geometry that we no longer *need* to return perpetually to the intuition of the figure in order to continue the demonstration. Intuition can be invoked to justify the axioms, and then is dispensable. This is what has made geometry become an entirely deductive science.[74] However, even if the perpetual recourse to intuition is dispensable, the *possibility* of having recourse to intuition, at any phase of the reasoning, is what makes deduction itself meaningful, and not just a simple deductive game on symbols. In the same way, in the field of logic, one cannot conclude from the dispensability of diagrams for the completion of demonstrations, to the idea that diagrams, at least as a potentiality of the (transcendental?) imagination, would be dispensable to "give meaning" to these demonstrations.

Without the help of cognitive abilities related to spatiality, it is hard to see how one could make sense of Euclid's demonstrations. To put it bluntly, a being endowed with a logical apparatus but devoid of a notion of space (in so far as it can exist) could probably follow the logical path of the "hole-less" demonstrations of the nineteenth century geometers, but could not understand their meaning, acquire an intuition of their truth. They would lack the *referent* of the propositions of geometry, that is Space itself. If Space is necessary to give meaning to the propositions of geometry, one is inclined to think that the passage through the figuration of diagrams is indispensable to the acquisition of the meaning of Euclid's propositions. So that even for the most diagrammatophobic of all geometers, who would practice Geometry as a pure logical activity of inference, without any drawing, his activity would only make sense insofar as he has acquired the primitive intuitions of Euclid's geometry through a certain familiarity with diagrams, at least at the time of his first mathematical training.

[74] See in particular on this point what Weyl says in *Mind and Nature*, p. 120 (from *Mind and nature and other writings*). And (De Risi 2017, 2019).

By analogy with geometry, we wondered whether, despite the dispensability of logic diagrams for discriminating between correct and incorrect forms of reasoning, they might not be necessary, or at least extremely useful, for acquiring the "logical intuitions" which legitimise the form of correct reasoning. Our third section, going back to Euler, showed how, in the field of traditional, pre-Fregean logic, such an idea was defensible. Eulerian type diagrams *show*, the logical form of simple propositions (categorical propositions in the sense of Aristotle), and the correctness of syllogistic inferences. We have analysed some of Euler's wanderings in his text, which gradually led him to recognise the need to give diagrams a modal meaning in order to adequately capture the logical forms of traditional logic. Once this step had been taken, we showed how a certain naturalness emerges from diagrammatic representation, which immediately makes sense, making certain logical symmetries of propositions and reasoning immediately visible, as opposed to conventional symbolic representation.

Returning to the premises of Aristotelian logic, we wondered whether such intuitions, of a spatial or diagrammatic nature, had not played a role in the origin of logic. Taking up analyses dating back at least to Hamilton, and recently renewed by Crubellier, we explained why it is legitimate to think that diagrams should accompany the text of the *Prior Analytics*. These diagrams did not, however, touch on the internal form of the propositions, but only concerned the syllogistic *figures*, i. e. the different configurations according to which two propositions can be articulated by means of a medium term. As far as the development of diagrammatic intuitions for analysing the interior of a proposition is concerned, it would seem that the passage to a preferentially extensional logic played an important role, leading to a moment in the history of logic, that of Euler/Venn diagrams, where the thesis of the power of diagrams to *show* the rules of logic found its maximum acuity.

However, the efficiency of potato-shaped diagrams rests partly, negatively, on the relative poverty of Boolean logic (*a fortiori* of syllogistic), whose configurations can be given synoptically only because they are finite, exhaustible in a synthetic diagram. What about the other sectors of modern logic which have developed subsequently? The Eulerian thesis cannot be defended so easily as soon as one enters the domain of infinite logics, starting with the predicate logic. How could one still, in this field, give a fundamental role to spatial and diagrammatic intuition? Doesn't any representation of an infinite configuration oblige us to introduce an inevitable conventional symbolisation? For sure, diagrammatic thinking is far from being absent or insignificant in the field of infinite logics. But the diagrammatic usages found there (think, for example, of the trees of proofs, Gentzen's sequences, or Jean-Yves Girard's program of geometrisation of proofs) always include mixed usages that associate diagrams and symbols. Venn had already seen in Frege's *Bregiffschrift* such a mixed case of notation. The diagrammatic method cannot be found in infinite logic in the pure form that it could still have in Euler/Venn/Carroll's analytical diagrams, which are like a parenthesis in the history of logic. Nevertheless, this parenthesis has been reopend over the last twenty years, with the revival of research on Boolean logic and syllogistic, which is a sector of it, in the context of cognitive sciences, artificial intelligence or scientific imaging technology. It is therefore important to be aware of the validity of the Eulerian thesis of the power of diagrammatic thinking, even if it concerns, at least in its original form, only a restricted field of logic.

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Catégorification et méthode

Franck Jedrzejewski

Résumé Au cours des trente dernières années, les mathématiques ont connu des bouleversements majeurs dans la façon de poser et de résoudre les problèmes. L'un de ces bouleversements auquel on assiste aujourd'hui est la catégorification. Comme son processus inverse la décatégorification, le mot signifie que l'on transpose un problème posé dans le vocabulaire classique ensembliste en un problème d'un plus haut degré d'abstraction posé dans le monde des catégories. La catégorication a suscité de nouveaux néologismes comme la combinatorialisation (Doron Zeilberger), l'homotopification ou la groupoïdification (John Baez). Si l'on parle de catégorification et non de catégorisation, c'est que l'on veut insister sur le processus de transposition. Il ne s'agit pas seulement de catégoriser, ce qui est une façon de définir des catégories en vue de ranger des objets, mais de catégorifier, ce qui suppose que l'on va, à travers une approche catégorique, donner un plus haut sens à un objet ou à un problème mathématique. Aujourd'hui, seule la théorie des catégories permet de faire ce travail et de donner une plus grande compréhension de certains aspects ou objets mathématiques. En réalité, le mot catégorification a plusieurs sens.

Mots clés Categorification · Philosophy of Categories · TQFT · Khovanov Homology

Mathematics Subject Classification (2010) Primary 1803; Secondary 01A60

En 1984, Vaughan Jones a découvert un nouvel invariant des nœuds orientés, le polynôme de Jones, à valeurs dans l'anneau des polynômes de Laurent à coefficient entiers. Seize années plus tard, en 2000, Mikhail Khovanov construit un nouvel invariant de nature homologique tel que le polynôme de Jones s'exprime comme la caractéristique graduée d'Euler de cette homologie. Il inaugure ainsi une série de résultats de catégorification, une notion introduite quelques années auparavant,

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en 1994, par Louis Crane qui consiste à déplacer un problème ensembliste en un problème catégorique. Depuis, la notion de catégorification s'est développée dans le monde mathématique pour aider à la compréhension et au développement de nombreux domaines dont la théorie quantique des champs. La force de cette notion et la puissance diagrammatique qu'elle véhicule nous incite à nous interroger sur cette nouvelle méthode mathématique qui nous rappelle les théâtres et projets, les raisonnements compliqués qui, des références à la Cabale, de Raymond Lulle à Francis Bacon, de Pierre de la Ramée à Leibniz ont trouvé leur acmé dans la recherche d'un méthode universelle.

Au cours des trente dernières années, les mathématiques ont connu des bouleversements majeurs dans la façon de poser et de résoudre les problèmes. L'un de ces bouleversements auquel on assiste aujourd'hui est la *catégorification*. Comme son processus inverse la *décatégorification*, le mot signifie que l'on transpose un problème posé dans le vocabulaire classique ensembliste en un problème d'un plus haut degré d'abstraction posé dans le monde des catégories. La *catégorification* a suscité de nouveaux néologismes comme la *combinatorialisation* (Doron Zeilberger), l'*homotopification* ou la *groupoïdification* (John Baez). Si l'on parle de *catégorification* et non de *catégorisation*, c'est que l'on veut insister sur le processus de transposition. Il ne s'agit pas seulement de *catégoriser*, ce qui est une façon de définir des catégories en vue de ranger des objets, mais de *catégorifier*, ce qui suppose que l'on va, à travers une approche catégorique, donner un plus haut sens à un objet ou à un problème mathématique. Aujourd'hui, seule la théorie des catégories permet de faire ce travail et de donner une plus grande compréhension de certains aspects ou objets mathématiques.

En réalité, le mot catégorification a plusieurs sens. Une façon simple de considérer le problème est de voir la catégorification comme une mise à jour, un « *upgrade* » de la théorie des ensembles vers la théorie des catégories. Dans ce mouvement, les ensembles sont remplacés par des catégories, les fonctions sont remplacées par des foncteurs, les égalités entre les éléments de l'ensemble sont remplacées par des isomorphismes entre les objets de la catégorie, et les égalités entre fonctions sont remplacées par les transformations naturelles entre foncteurs. Comme une catégorie est définie par une collection d'objets et une collection de morphismes, on voit que catégorifier revient à ajouter des morphismes à un objet ou à un problème. On enrichit ainsi l'objet ou le problème en lui donnant plus de dynamisme ou plus de structure, ce qui fournit en retour plus d'information sur l'objet ou le problème étudié. De ce point de vue, la catégorification est une méthode pour résoudre un problème, de la même manière, mais dans un tout autre domaine, que l'on résout une équation différentielle en prenant sa transformée de Fourier, puis en résolvant l'équation obtenue avant de revenir par transformée inverse à la solution de l'équation de départ. Dans la catégorification, un problème posé dans l'univers ensembliste est généralisé dans le monde catégorique, puis résolu en termes catégoriques avant d'être en retour spécialisé pour fournir une solution dans le monde ensembliste.

Le polynôme de Jones

Les nœuds ont toujours été un sujet d'étonnement. Les Incas les utilisaient pour stocker de l'information dans les *quipus* et ils servaient dans de nombreux contextes d'ornements ou de signes religieux: le *srivatsa* était considéré comme un nœud infini par les bouddhistes tibétains et les juifs décoraient leur vêtement de *tsitsits* car il est dit dans la Thora « Parle aux enfants d'Israël, et tu leur diras de se faire des tsitsits aux coins de leurs vêtements ». Lacan voyait dans le nœud borroméen une représentation de la relation RIS, un entrelacement du Réel, de l'Imaginaire et du Symbolique. Pour le mathématicien, ce que recouvre la théorie des nœuds concerne au moins trois types d'objets : le nœud propement dit, fait d'une seule ficelle fermée et entrelacée, l'entrelacs (*link*) fait de plusieurs ficelles fermées et entrelacées, la tresse qui fait correspondre n points d'un segment de droite en haut de la feuille à n points d'un segment de droite en bas de la feuille et l'enchevêtrement (*tangle*) qui est une tresse dans laquelle on entrelace des ficelles fermées. Les tresses se rapportent aux nœuds par clôture. Alexander a démontré en 1923 que chaque entrelacs orienté peut s'obtenir à isotopie près comme la clôture d'une tresse. Pierre Vogel a montré comment construire le procédé inverse permettant de passer du nœud à une tresse. Pour manipuler ces objets à isotopie près, les mathématiciens emploient les mouvements de Reidemeister qui sont des mouvements locaux déterminant le passage d'un entrelacs à un autre. Pour caractériser ces objets, ils ont trouvé de nombreux invariants dont le plus célèbre est sans doute le polynôme de Jones.

Vaughan Jones est un mathématicien néo-zélandais né en 1952. En 1979, il soutient une thèse à l'Université de Genève sous la direction d'André Haefliger sur les algèbres de von Neumann et la théorie des facteurs – qui sont des algèbres de von Neumann dont le centre est formé de multiples de l'opérateur identité. Ce domaine, qui attira aussi Alain Connes, conduisit Jones sur les terres de la théorie des nœuds. C'est en effet en étudiant des algèbres de von Neumann de dimension finie dont les générateurs vérifiaient des relations proches de celles du groupe des tresses que Jones construisit son polynôme pour tout entrelacs orienté. En 1985, au moment où il publie *A polynomial invariant for knots via von Neumann algebras*, Jones connaissait le groupe des tresses d'Artin, la représentation de Burau, le polynôme d'Alexander, obtenu à partir du déterminant d'une matrice de Burau, et les mouvements de Markov. Il savait aussi que Temperley et Lieb (1971) avaient construit des représentations d'algèbres auxquelles lui-même s'intéressait. Ils avaient appliqué leur construction à des problèmes de mécanique statistique et obtenu une solution analytique exacte pour la percolation. En associant à chaque générateur du groupe des tresses à n brins un générateur de l'algèbre de von Neumann, Jones construisit une représentation qu'il utilisa pour définir son fameux polynôme.

$$J_L(t) = \left(-\frac{(1+t)}{\sqrt{t}}\right)^{n-1} \operatorname{tr}(\pi_L)$$

Il démontra que deux entrelacs isotopes avaient le même polynôme de Jones, et par conséquent définissait un invariant d'entrelacs. En outre, il démontra que ce polynôme vérifiait la relation d'écheveau (*skein relation*) pour trois croisements élémentaires: droit L^+, gauche L^- et parallèle L^0. Ce nouvel invariant a provoqué une moisson de nouveaux résultats et de relations très profondes entre des objets comme les algèbres de Hopf, les groupes quantiques, l'équation de Yang-Baxter et les modules d'écheveau, qui sont aussi appelés *modules de Kauffman*. C'est cette relation d'écheveau qui donnera naissance à toute une multitude de résultats, mettant à jour la richesse structurale des modules de Kauffman.

$$\frac{1}{t} J_{L^-} - t J_{L^+} = \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right) J_{L^0}$$

On s'aperçut très vite que le polynôme de Jones avait plus de potentiel que le polynôme d'Alexander, même après qu'il fut normalisé par Conway. Il tenait sa supériorité du fait qu'il distinguait le nœud de trèfle gauche du nœud de trèfle droit, alors que cela était impossible avec le polynôme d'Alexander ou de Conway. Jones montra que l'invariant de Arf que l'on calcule à partir d'une forme quadratique de la surface de Seifert d'un nœud s'exprime simplement à partir de la valeur du polynôme de Jones prit au point i, (i étant le complexe $\sqrt{-1}$). Cet invariant de Arf (qui vaut 0 ou 1) s'exprime, ainsi que l'a montré Raymond Robertello (1965), comme la somme modulo 2 de certains cœfficients du polynôme d'Alexander.

Le crochet de Kauffman (1987), normalisé à 1 pour le nœud trivial, a lui aussi apporté une moisson de résultats nouveaux. Il se définit simplement à partir d'une désingularisation de croisements élémentaires par les formules:

$$< L \cup O >= (-A^2 - A^{-2}) < L >$$

Ce crochet permet de calculer le polynôme de Jones pour un nœud K de c croisements. On calcule d'abord la vrille $w(K)$, puis le polynôme

$$L(K) = (-A^3)^{-w(K)} < K >$$

et l'on déduit le polynôme de Jones, par un simple changement de variable

$$J_K(t) = L(K)(t^{-1/4})$$

Comme l'expression $(-A^3)^{-w(K)} < K >$ est invariante par les mouvements de Reidemeister, elle définit un invariant d'entrelacs orienté $P_2(K)$ déterminé par la relation de Jones

$$q^2 P_2(L^+) - q^{-2} P_2(L^-) = (q - q^{-1}) P_2(L^0)$$

et la normalisation $P_2(O) = q + q^{-1}$. Le polynôme de Jones $J_K(t)$ s'obtient par le simple changement de variable $q = t^{-1/2}$ et la relation de normalisation $J_O(t) = 1$:

$$J_K(t) = (P_2(K)/P_2(O))_{q=t^{-1/2}}$$

Par la suite, ce polynôme a été généralisé en un autre polynôme qui porte le nom de ses inventeurs: le polynôme *HOMFLY-PT* (acronyme de Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, J. Przytycki et P. Traczyk). C'est un polynôme de Laurent à deux variables de $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ qui vérifie lui aussi pour tout entrelacs orienté L, la relation d'écheveau:

$$v^{-1} P(L^+) - v P(L_-) = z P(L_0)$$

Par des changements de variables, on démontre que le polynôme Homfly permet de retrouver le polynôme d'Alexander

$$\Delta_K(t) = P(i, \theta), \qquad \text{avec } \theta = i(t^{1/2} - t^{-1/2})$$

et le polynôme de Jones, par la formule:

$$J_K(t) = P_K(it^{-1}, -\theta)$$

La catégorification du polynôme de Jones

D'un point de vue algébrique, la catégorification consiste à voir un groupe (resp. un anneau) comme le groupe de Grothendieck (resp. l'anneau de Grothendieck) scindé d'une catégorie abélienne (resp. monoïdale). Ici, ce qui a été fécond finalement en théorie des entrelacs est d'avoir oublier la nature géométrique des entrelacs. Le fait d'avoir considéré les diagrammes de nœuds comme des morphismes de modules sur une algèbre a été la clé de tous les résultats sur les invariants quantiques. Lorsque l'algèbre est une algèbre de Hopf particulière (quasi-triangulaire en ruban), les morphismes sont des invariants des nœuds, c'est-à-dire invariants par les isotopies planaires et les mouvements de Reidemeister. Un invariant comme le polynôme de Laurent à coefficients positifs est vu comme la dimension d'un espace vectoriel gradué. Un nombre entier quelconque s'interprète comme la caractéristique d'Euler d'une homologie. Pour un espace M (vérifiant quelques restrictions), on associera un invariant qui est sa caractéristique d'Euler $\chi(M) \in \mathbb{Z}$. En munissant M d'une triangulation ou d'un complexe cellulaire, on peut calculer de manière combinatoire cet invariant. Par la théorie de l'homologie, l'espace M est associé à un espace vectoriel gradué pour lequel on calcule sa caractéristique d'Euler:

$$\chi(M) = \sum_{i,j\in\mathbb{Z}} (-1)^i \dim(H_i(M;\mathbb{Q}))$$

En ce sens, *l'homologie catégorifie la caractéristique d'Euler*.

Dans le cas d'un polynôme, la graduation permet de coder l'information portée par les monômes de ce polynôme, alors que dans le cas de l'homologie, la graduation code l'information de signe. Pour les polynômes de Laurent, comme le polynôme de Jones, la catégorification nécessite au moins un espace vectoriel bigradué (une graduation pour le signe, l'autre pour le degré).

En 2000, Khovanov (2000) a proposé un nouvel invariant des entrelacs construit entièrement de manière combinatoire. La géométrie de ces entrelacs réapparaît alors par *décatégorification*. L'idée est la suivante. On part d'un diagramme d'entrelacs L et on construit un complexe de chaînes bigradué $C(L)$ associé à L

$$L \stackrel{Khovanov}{\longrightarrow} C(L)$$

c'est-à-dire une suite de groupes abéliens $(H^{i,j})_{i,j\in\mathbb{Z}}$, ou d'objets H_i d'une catégorie abélienne et d'homomorphismes $\partial_i : H_i \to H_{i-1}$ tels que $\partial_i\partial_{i+1} = 0$. Il suffit ensuite de considérer les groupes d'homologie $(Kh(L)^{i,j})_{i,j\in\mathbb{Z}}$ du complexe $C(L)$ pour poser le problème.

$$C(L) \stackrel{Homologie}{\longrightarrow} Kh(L)$$

Khovanov démontre alors que

1. si L' est le diagramme de l'entrelacs L modifié par une suite de mouvements de Reidemeister, alors il existe un isomorphisme entre $Kh(L)$ et $Kh(L')$
2. la caractéristique graduée d'Euler est le polynôme de Jones (non normalisé)

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim(Kh^{i,j}(L)) = \widehat{J}(L)$$

Autrement dit, *l'homologie de Khovanov catégorifie le polynôme de Jones*. À partir de la catégorie des entrelacs $\mathfrak{C}$ dont les objets sont les entrelacs et les morphismes sont les cobordismes (les surfaces orientables compactes) entre deux entrelacs, Khovanov construit un foncteur de cette catégorie vers la catégorie des groupes abéliens, c'est-à-dire définit de manière canonique le complexe $C(L)$ par rapport aux morphismes. L'homologie de Khovanov nous renseigne ainsi sur la structure de l'entrelacs. Elle est plus puissante que le polynôme de Jones, car elle permet de distinguer des entrelacs qui ont même polynôme de Jones. Les nœuds 10_{136} et 11_{n92} (nœud non alterné) ont le même polynôme de Jones mais pas la même homologie. Autre exemple : le nœud de Conway (11_{n34}) a le même polynôme de Jones que le nœud de Kinoshita-Terasaka (11_{n42}) (Les propriétés des nœuds et entrelacs sont décrites sur le site de Dror Bar-Nathan: *katlas.org*)

Sur ce principe qui est d'associer un complexe et d'en calculer l'homologie, les mathématiciens ont cherché à catégorifier d'autres polynômes de nœud en suivant la voie ouverte par Khovanov. En 2008, Khovanov et Rozansky (2008) ont construit une variante de l'homologie de Khovanov et ont catégorifié une sous-famille (pour $a = q^n$) du polynôme HOMFLY-PT. À l'aide de factorisations matricielles, ils ont associé une homologie trigraduée $Khr^{i,j,k}(L)$ à tout entrelacs, invariante par les mouvements de Reidemeister et ont construit l'expression

$$P_L(a,q) = \sum_{i,j,k \in \mathbb{Z}} (-1)^i a^j q^k \dim_{\mathbb{Q}}(Khr^{i,j,k}(L))$$

qui catégorifie le polynôme HOMPY-PT. En 2010, Peter Kronheimer et Thomas Mrowka (2010) ont démontré que l'homologie de Khovanov détecte le nœud trivial. Résultat remarquable qui ponctue des années de recherche.

La méthode de catégorification

Rappelons que la notion de *catégorification* a été introduite par Louis Crane en 1995 (Crane et Frenkel, 1994; Crane, 1995). Elle prend selon les contextes des sens différents et reçoit des définitions variées. Mais toutes les définitions convergent vers un même résultat qui est la construction d'une catégorie. En effet, la catégorification est une méthode de généralisation de problèmes qui nous fait passer d'un domaine ensembliste à un domaine catégoriel. Elle fournit des solutions très générales et permet de comprendre pourquoi certains invariants sont liés entre eux. Comme nous l'avons vu, l'idée de la catégorification est de remplacer les ensembles par des catégories, les fonctions par des foncteurs et les égalités par des isomorphismes. Les éléments de l'ensemble deviennent les objets de la catégorie, les relations entre éléments mutent en morphismes. Les relations entre fonctions deviennent des transformations naturelles entre catégories, c'est-à-dire des transformations entre foncteurs de la catégorie.

Ensembles	sont remplacés par des	**Catégories**
Fonctions	sont remplacées par des	**Foncteurs**
Équations entre éléments	sont remplacées par des	**Isomorphismes**
Équations entre fonctions	sont remplacées par des	**Transf. naturelles**

La méthode de Khovanov pour catégorifier un polynôme a été déterminante dans l'histoire de la catégorification. Dans les années 2000, les mathématiciens ont cherché à catégorifier d'autres structures que les polynômes. En 1990, W. Soergel a utilisé les catégories (Soergel 1990) pour démontrer plusieurs résultats de structure, favorisant une approche algébrique. Deux ans plus tard, il définit ce qu'on appelle aujourd'hui les *bimodules de Soergel* (Soergel, 1992) qui permettront à C. Stroppel de construire une catégorification des algèbres de Temperly-Lieb et des enchevêtrements (Stroppel, 2005). En 2005, Khovanov (2016) définit un mélange d'algèbre homologique et d'algèbre de Hopf qu'il appelle des algèbres *hopfologiques* dans l'espoir de catégorifier des invariants quantiques des 3-variétés.

Les algèbres de Hecke sont des algèbres construites au-dessus du groupe de Weyl comme un $\mathbb{Z}[v, v^{-1}]$-module libre dont la base unique est appelée base de Kazhdan-Lusztig, liée aux polynômes des mêmes auteurs (Kazhdan et Lusztig, 1979). Les bimodules de Soergel servent à construire une 2-categorie qui catégorifie les algèbres de Hecke. En utilisant ces résultats, M. Khovanov, V. Mazorchuk et C. Stroppel ont construit une catégorification des modules de Specht (Khovanov et al., 2008).

En matière de catégorification, Rouquier (2004, 2008) a une approche algébrique alors que Lauda (2008) et Khovanov (2010) ont une approche diagrammatique. Leurs travaux aboutissent à la catégorification des algèbres KLR (acronyme de Khovanov, Lauda, Rouquier) appelées aussi algèbres de Hecke carquois.

En 2006, Rouquier (2006) associe à chaque tresse un complexe de bimodules gradués de sorte que les complexes de tresses isotopes soient équivalents par homotopie. Ainsi Rouquier catégorifie le groupe des tresses. Pour cela, il s'appuie sur des bimodules introduits dans les années 1990 par Wolfgang Soergel en théorie des représentations (Soergel, 1992, 1995). Soergel démontre alors que ces bimodules catégorifie l'algèbre de Hecke. Nous savons que cette algèbre de Hecke est associée au groupe des tresses. En effet, l'algèbre de Hecke $\mathcal{H}_n$ est un quotient de l'algèbre du groupe des tresses $\mathcal{B}_n$ dépendant d'un paramètre q. Elle est engendrée par $n-1$ générateurs $T_1, \ldots, T_{n-1}$, soumis aux relations de tresses

$$T_i T_j = T_j T_i \qquad \text{pour tout } i, j = 1, 2, \ldots, n-1 \text{ avec } |i-j| \geq 2$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{pour tout } i = 1, 2, \ldots, n-2$$

ainsi qu'aux relations

$$T_i^2 = (q^2 - 1)T_i + q^2 \qquad \text{pour tout } i = 1, \ldots, n-1$$

Compte tenu des liens entre l'algèbre de Hecke et le groupe des tresses, Rouquier a cherché une catégorification du groupe de tresses $\mathcal{B}_n$ en utilisant les bimodules de Soergel. À chaque élément générateur b_i de $\mathcal{B}_n$, il associa un complexe de cochaîne $F(b_i)$, et à l'élément b_i^{-1} un complexe de cochaînes $F(b_i^{-1})$. Pour un mot de tresses (un produit d'éléments générateurs b_i, b_i^{-1}), il associa le produit tensoriel au-dessus des bimodules de Soergel des complexes des éléments générateurs du mot. Il démontra que les complexes ne dépendent pas de l'écriture du mot, mais seulement de la tresse considérée. En 2007, Khovanov démontra (Khovanov, 2007) que l'homologie de Khovanov-Rozansky trigraduée peut être retrouvée à partir du complexe de Rouquier $F(K)$ de la fermeture K d'une tresse en utilisant l'homologie de Hochschild.

Ces résultats ont inspiré la catégorification du groupe de tresses singulières (tresses ayant un nombre fini de points doubles) et la catégorification du groupe des tresses virtuelles (tresses ayant une troisième type de croisement, dit *virtuel*), résultats qui ont été obtenus par Volodymyr Mazorchuk et Stroppel (2007) à partir de l'étude d'endofoncteurs de la catégorie de Bernstein-Gelfand introduite dans les années 1975–1976 (Bernstein et al., 1976).

Une autre manière de voir la catégorification est de considérer son processus inverse, car s'il existe plusieurs manières de catégorification, il n'existe qu'une manière de *décatégorifier*. Lors de ce processus, on oublie l'information catégorique. Une façon classique de le mettre en œuvre est de considérer l'*anneau de Grothendieck* d'une catégorie qui se définit de la manière suivante. Pour une catégorie abélienne $\mathfrak{C}$, on considère le groupe libre abélien $F(\mathfrak{C})$ dont une base est formée par les classes d'isomorphie $[X]$ des objets de la catégorie $\mathfrak{C}$ et le sous-groupe $N(\mathfrak{C})$ de $F(\mathfrak{C})$ engendré par les éléments $[X] - [Y] + [Z]$ pour chaque suite exacte courte

$$0 \to X \to Y \to Z \to 0$$

dans $\mathfrak{C}$. Le *groupe de Grothendieck* $K_0(\mathfrak{C})$ est le quotient $F(\mathfrak{C})/N(\mathfrak{C})$. La structure de groupe est donnée par la somme $[X] + [Y] = [X \oplus Y]$ et la structure d'anneau par la multiplication $[X][Y] = [X \otimes Y]$.

La catégorification consiste alors à réaliser un anneau donné A comme anneau de Grothendieck d'une catégorie $\mathfrak{C}$

$$A = K(\mathfrak{C})$$

Lorsque A est catégorifié par une catégorie ou une sous-catégorie $\mathfrak{C}$ de la catégorie des modules d'une certaine algèbre R, la connaissance de R induit des informations sur A. La décatégorification de $\mathfrak{C}$ est l'anneau de Grothendieck $K(\mathfrak{C})$. Le livre de Mazorchuk (2012) donne une présentation plus rigoureuse et plus générale de la catégorification, que celle qui est esquissée ici.

Prenons un exemple simple de décatégorification. La catégorie des ensembles finis $\mathfrak{FinSets}$ peut être interprétée comme une catégorification des entiers naturels $\mathbb{N}$. Dans ce schéma, l'addition de deux entiers est catégorifiée par la réunion de deux ensembles et le produit de deux entiers est le produit cartésien de deux ensembles. Dans la catégorie des ensembles finis, les opérations de commutativité ou d'associativité sont vérifiées à isomorphisme près. L'ensemble des entiers naturels peut être vu comme une 0-catégorie, tandis que la catégorie des ensembles finis est une 1-catégorie. Ce passage d'une dimension à une autre est appelée par Louis Crane l'*échelle dimensionnelle*. La catégorie $\mathfrak{FinVect}$ des espaces vectoriels de dimension finie (sur un même corps) est différente de la catégorie des ensembles finis. Elle a pourtant les mêmes classes d'isomorphie car deux espaces vectoriels de même dimension sont isomorphes. L'ensemble des classes d'isomorphie est donc en bijection avec l'ensemble des entiers naturels. L'ensemble des entiers naturels $\mathbb{N}$ est donc catégorifiable de deux manières différentes. Mais sa décatégorification (l'ensemble des classes d'isomorphie) est unique.

La théorie topologique quantique des champs

Les premières mentions d'une théorie topologique quantique des champs remontent à l'article de Witten (1988), mais la formulation généralisée et axiomatisée à une dimension quelconque est dûe à Atiyah (1989). Depuis l'introduction de l'intégrale de Feynman, la difficulté pour les physiciens est de calculer l'intégrale d'une exponentielle de la fonctionnelle d'action $S(\Phi)$ sur l'ensemble des chemins $\Phi : M \to N$ entre deux variétés riemanniennes

$$Z(F) = \int F(\Phi)e^{-S(\Phi)}\mathcal{D}\Phi$$

L'action n'est autre qu'un lagrangien fonction de Φ et de sa dérivée première. Dans la plupart des cas, l'intégrale de chemins est mal définie, car une mesure naturelle de volume en dimension infinie n'existe pas proprement. Cependant lorsque la fonction de corrélation des observables est indépendante de la métrique, la théorie devient topologique. Le calcul de l'intégrale de chemins $Z(F)$ ne dépend alors que de la variété de départ M et non de sa métrique. Dans ce cas l'intégrale peut donner des invariants topologiques. Un exemple célèbre de théorie topologique des champs est la théorie de Chern-Simons, dans laquelle le champ Φ est remplacé par une connexion A et dont l'action est définie par la formule:

$$S(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

Pour les invariants topologiques du modèle de Chern Simons calculés analytiquement de cette facon, on pourra se reporter à l'article de Lévy et Sengupta (2017) et à celui de (Albeverio et Mitoma, 2009). Pour permettre le calcul de l'intégrale de chemins, on introduit le cobordisme (M, E, F) entre deux variétés E et F. C'est une variété compacte orientée M muni d'un difféomorphisme $\partial M = \overline{E} \sqcup F$. Dans cette notation, le bord de M est ∂M, $\overline{E}$ est la variété E munie de l'orientation inverse et $\sqcup$ est l'union disjointe. La catégorie des cobordismes *n-Cob* en dimension $n + 1$ est la catégorie dont les objets sont les variétés compactes orientées lisses de dimension n et dont les morphismes entre deux variétés M et M' sont les classes d'équivalence de cobordismes de dimension $n + 1$. Deux corbordismes (M, E, F) et (M', E, F) sont dits équivalents s'il existe un difféormorphisme $M \to M'$ compatible avec les difféormorphismes $\partial M \simeq \overline{E} \sqcup F \simeq \partial M'$. La composition entre morphismes est donnée par l'application de recollement. La catégorie n-Cob est munie d'une structure monoïdale donnée par l'union disjointe.

Du point de vue de la théorie des catégories, la théorie topologique quantique des champs (TQFT) est un foncteur de la catégorie des cobordismes *n-Cob* vers la catégorie des $\mathbb{C}$-espaces vectoriels de dimension finie. Ces deux catégories sont des catégories monoïdales symétriques. Le but de la théorie topologique quantique des champs est d'en étudier les représentations algébriques.

C'est en 1989 que Michael Atiyah a introduit la définition axiomatique de la théorie topologique quantique des champs. Du point de vue formel, une théorie topologique quantique des champs (TQFT) de dimension $(n+1)$ au dessus du corps des complexes $\mathbb{C}$ associe à toute variété fermée orientée E de dimension n un espace vectoriel $\mathcal{H}(E)$ de dimension finie sur $\mathbb{C}$ et à tout cobordisme (M, E, F) une application linéaire

$$Z(M) : \mathcal{H}(E) \to \mathcal{H}(F)$$

notée aussi $Z(M, E, F)$ vérifiant les cinq axiomes suivants.

1. *Naturalité*. Tout difféomorphisme entre variétés orientées fermées qui préserve l'orientation $f : E \to F$ induit un difféomorphisme $f_* : \mathcal{H}(E) \to \mathcal{H}(F)$. Pour tout difféomorphisme (h, φ, φ') entre cobordismes (M, E, F) et (M', E', F') on a

$$Z(M') = \varphi'_* \circ Z(M) \circ \varphi_*^{-1}$$

2. *Fonctorialité*. Si un cobordisme (M, E, G) est obtenu par recollement $M = M' \cup_f M''$ le long du difféomorphisme $f : F \to F'$ de deux cobordismes (M', E, F) et (M'', F', G) alors on a

$$Z(M) = Z(M'') \circ f_* \circ Z(M')$$

   En particulier lorsque les variétés coïncident $F = F'$, f est l'identité. Dans ces conditions, $Z(M)$ est la composition des applications $Z(M'')$ et $Z(M')$.
3. *Normalisation*. Pour toute variété M de dimension n, l'application linéaire $Z(M \times [0, 1]) : \mathcal{H}(M) \to \mathcal{H}(M)$ est l'identité.

$$Z(M \times [0, 1], M, M) = \mathrm{Id}_{\mathcal{H}(M)}$$

   Comme Z est un foncteur, cet axiome exprime que Z applique les morphismes unité sur les morphismes unité.
4. *Multiplicativité*. Il existe des isomorphismes fonctoriels

$$\mathcal{H}(E \sqcup F) \simeq \mathcal{H}(E) \otimes \mathcal{H}(F), \quad \mathcal{H}(\varnothing) \simeq \mathbb{C}$$

compatibles avec les isomorphismes

$$\mathcal{H}((E \sqcup F) \sqcup G) \simeq \mathcal{H}(E \sqcup (F \sqcup G)), \quad \mathcal{H}(E \sqcup \varnothing) \simeq \mathcal{H}(E)$$

5. *Symétrie*. L'isomorphisme

$$\mathcal{H}(E \sqcup F) \simeq \mathcal{H}(F \sqcup E)$$

est compatible avec l'isomorphisme

$$\mathcal{H}(E) \otimes \mathcal{H}(F) \simeq \mathcal{H}(F) \otimes \mathcal{H}(E)$$

À la suite des travaux de Witten (1988) et Atiyah (1989) les théoriciens ont cherché à développer les aspects topologiques de la théorie quantique des champs en les croisant avec la théorie conforme des champs de Segal (1988) ou la théorie de Floer (1988) et Donaldson (1995). Baez et Dolan (1998) ont cherché à développer une catégorification dans les n-catégories. Louis Crane et Igor Frenkel ont cherché une TQFT en dimension 4 et ont introduit le concept d'*échelle dimensionnelle* pour construire une théorie algébrique pour passer d'une dimension à l'autre. C'est dans l'article de 1995 *Clock and category : Is Quantum gravity algebraic?* que Louis Crane (1995) a introduit le concept de catégorification, constatant qu'une théorie bidimensionnelle de TQFT a la même structure catégorielle que la théorie tridimensionnelle. Le principe qui permet le passage d'une théorie de dimension $D = n$ à une théorie de dimension $D = n + 1$ est précisément l'*échelle dimensionnelle* que nous avons vue plus haut. Il pose la conjecture suivante : il existe une théorie quadridimensionnelle de TQFT qui est une catégorification de la théorie de Chern-Simons-Witten, factorisable et qui puisse s'écrire en termes d'une somme topologique d'états.

Diagrammes et méthode

On peut dater du milieu du XXe siècle, le tournant diagrammatique des sciences physico-mathématiques. C'est à la conférence de Pocono en Pensylvanie que Feynman présente ses fameux diagrammes: le premier diagramme sera publié dans l'article de 1949 (Feynman, 1949). Du côté des mathématiques, les diagrammes apparaissent en théorie des catégories à peu près à la même époque lorsque Samuel Eilenberg et Saunders McLane publient leur article fondateur en 1945 (Eilenberg et Mac Lane, 1945). Depuis, tant en physique qu'en mathématiques, les diagrammes se sont multipliés, justifiant le développement perturbatif de l'intégrale de Feynman et les relations dans les catégories, topos et autres structures et topologies de petites dimensions. Bien plus, et c'est ici que l'on justifie le tournant diagrammatique, des démonstrations entières se construisent par enchaînements de diagrammes oubliant le formalisme classique des mathématiques.

Pour le philosophe, la catégorification, en ce qu'elle déplace le problème des ensembles vers les catégories pose la question de la vérité. La stricte égalité d'une équation qui ne peut être que vraie ou fausse et ceci de manière unique est remplacée par une égalité d'isomorphismes où plusieurs vérités deviennent possibles. La catégorification introduit donc la multiplicité des êtres et des regards. Deux objets peuvent être égaux de différentes manières. Et cette différence provient des morphismes qui sont impliqués par l'objet. La structure catégorielle d'objets et de morphismes indissociables est donc responsable de cette multiplicité. Deux objets d'un ensemble sont égaux selon une vérité unique, alors que deux objets d'une catégorie ou d'un groupoïde sont égaux par isomorphismes de différentes manières. Certains mathématiciens pensent que ce qui est en jeu est le type d'homotopie. Ils parlent d'*homotopification* bien que leurs discours concernent le plus souvent des catégories d'ordre supérieur (n-catégories ou (n, k) catégories). Ce sont en effet les types d'homotopie qui interviennent dans les nombres de Betti et la formule de calcul de la caractéristique graduée d'Euler que nous avons vue dans la catégorification du polynôme de Jones.

Née de la topologie des petites dimensions et du regard catégoriel que l'on porte sur elle, la catégorification s'est imposée comme une méthode nouvelle des mathématiques, qui est encore en plein essor. Elle s'accompagne des développements diagrammatiques imposés par la théorie des nœuds et les transcriptions graphiques catégorielles dans une mathématique où les démonstrations elles-mêmes ont fait place à un formalisme nouveau. Les méthodes graphiques ont largement démontrées leurs puissances (Khovanov (2010), Khovanov (2014), etc.).

Ce renouveau de la méthode nous rappelle les textes de philosophes des XVI^e et XVII^e siècles où les textes de la méthode comparative de Budé (*De asse*, 1515), de la *Dialectique* (1555) de Pierre de la Ramée, du *Tableau du droit universel* (1568) de Jean Bodin, du *De umbris idearum* (1583) de Giordano Bruno, du *Novum organum* (1620) de Francis Bacon ont précédé le texte de Descartes. Quelques années après le *Discours de la méthode* (1637), Leibniz écrit ce texte d'une brulante actualité.

> Ce n'est pas cette Méthode de bien enregistrer les faits dont je me sois proposé de parler ici principalement, mais plutôt la méthode de diriger la raison pour profiter tant des faits donnés par les sens ou rapport d'autrui que de la lumière naturelle, afin de trouver ou établir des Vérités importantes qui ne sont pas encore assez connues ou assurées, ou au moins qui ne sont pas mises en œuvre comme il faut pour éclairer la raison. Car les vérités qui ont encore besoin d'être bien établies, sont de deux sortes, les unes ne sont connues que confusément et imparfaitement, et les autres ne sont point connues du tout. Pour les premières, il faut employer la méthode de la certitude ou de l'art de démontrer, les autres ont besoin de l'art d'inventer.[1]

[1] G.W. Leibniz, *Discours touchant la méthode de la certitude et l'art d'inventer pour finir les disputes et faire en peu de temps de grands progrès* (1688–1690).

Leibniz ajoute que l'art d'inventer est peu connu en dehors des mathématiques et qu'une longue expérience lui a fait connaître qu'il y a des secrets dans l'art de penser. La catégorification participe aujourd'hui de ces secrets.

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Part II
Geometrical Spaces and Topological Knots, Old and New

Which Came First, the Circle or the Wheel? From Idea *(ιδεα)* to Concrete Construction

Marco Andreatta

Abstract Idea is a Greek world (ιδεα) which means mental representation, rational scheme or mathematical figure; Plato and Galileo said that nature uses the language of mathematics, whose geometrical patterns come from experiences. But, as F. Enriques noted, in the formalization of these ideas "we should take into account not really the experience, but rather the demands of simplification of our mind, in which they are reflected".

Most of our activities, especially in modern time, consists in realizing concretely abstract mathematical concepts and theories, as in the title of this conference: when form becomes substance. Fab Labs (Fabrication Laboratories) are modern workshops in which everybody can use digital fabrication to create real objects from mathematical ideas, especially from those which came from the phenomenology of space.

In the last part I will briefly consider some recent results in higher dimensional algebraic geometric, which can be summarized in few diagrams. I will point out that some of these ideas and diagrams could be directly connected to biology and life sciences.

Keywords Mechanical linkage · Fab Lab · Minimal model program

Introduction

The Greek philosopher Plato in the "Allegory of the Cave" tells that we live in a cave and that we perceive by our senses only shadows coming from another realm, the realm of pure forms, of ideas. This allegory is at the base of our way of doing philosophy and science, in particular mathematics.

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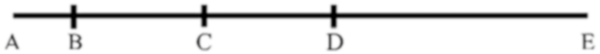
It comes immediately after the "Analogy of the Divided Line" (both contained in the Sixth book of the Republic): this is a nice diagrammatically explanation of how our brain interacts with the external world.

Shortly the analogy proposes to consider the visible things and the intelligible ones disposed on a line divided in two unequal parts, AC and CE. Then divide again each part in the same proportion, as in the figure: thus AB/AC = CD/CE = AC/AE.



The segments represent the following: AB the shadows and the reflections of physical things. BC the physical things themselves. CD mathematical reasoning, where abstract mathematical objects, such as geometric lines, are discussed. DE represents the subjects of philosophy, where we look at Ideas, which are given existence and truth by the Good itself. From the above proportion it follows easily that BC is equal to CD; it is not an accident that physical things corresponds to mathematical reasoning in Plato philosophy.

Idea is a Greek world ($\iota\delta\varepsilon\alpha$) which means scheme, pattern or mathematical figure. Our mind develops a process of knowledge based on this concept: a basic step consists in finding and giving an appropriate description of these ideas. After that, they must be connected to the physical things we perceive from nature, in a coherent and possibly effective way. This is a heavy task for mathematicians or, if one prefers, for philosophers, which is developing since the very beginning of our history.

In this process the use of diagrams, for instance of lines, circles, is fundamental, as Galileo often claimed. Note that the creation of an effective connection with nature comes through what we nowadays call technology.

Concerning the formalization of the ideas I find the following remark of F. Enriques very interesting: "we should take into account not really the experience, but rather the demands of simplification of our mind, in which they are reflected[1] . "In my opinion with this observation he stresses the fact that mathematics is an experimental science: starting from the observation of real things it makes use of the capacity of the brain to construct simplified models of them.

The evolution of human activity in the knowledge process had a critical point during the passage from epos to logos, that is from the oral tradition to the written ones. This passage started with Homer (seventh century b.C.) and was mature with Plato's writings. A big contribution to this transition was given by the works of Euclid, Archimedes, two mathematicians which wrote respectively the Elements and the Methods of Mechanical Theorems. In these books the intuitive power of diagram and the phenomenology of space start to have a written description.

[1] Federico Enriques, entry "Curve" in Enciclopedia Treccani.

I think that nowadays we are through a similar transition, from a written and oral tradition toward a digital tradition, both epos and logos have the chance to become less important. New digital instruments, both conceptual and technological, can, on one hand, easily reflect the natural demands of simplification of our mind and, on the other, make easy connections between the realm of ideas and concrete reality. In a very direct and "user friendly" way.

They are based on the Information Technology and its products, like web, google, wikipedia, robotic and so, ultimately, on mathematics.

Geometric Ideas and their Diagrams

Euclide's Elements (330 b.C), one of the most read and influent book in our history, is a basic step towards a written formalization of the mathematical ideas. In this Summa of the work of many mathematicians, like Thales, Pythagoras, Eudoxus, arithmetics and geometry become logos.

The first lines of the Elements contain some definitions and postulates:

Definitions.

1. A point is that which has no part
2. A line is breadthless length
3. A straight line is a line which lies evenly with the points on itself

...

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another

Postulates.

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any center and radius
4. That all right angles equal one another
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

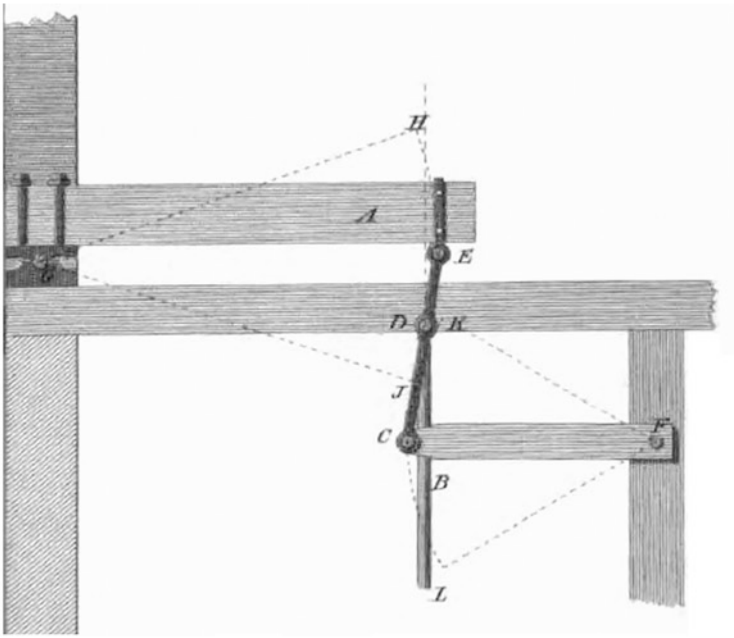
These few sentences give a coherent and ready to use definitions of two basic ideal diagrams, the line and the circle. They are a perfect translation in written words of two fundamental mathematical ideas, which can be find nowadays *verbatim* in any textbook all over the world. They are an "idealization" of real objects, like a wheel, the shape of the moon, the edge of the water at the horizon in a big see: they reflect the demands of simplification of our mind.

The fifth postulate is the famous one; it cannot be derived by the previous and it characterizes Euclidean geometry.

I’d like to concentrate on the postulates 1 and 3 which simply state that we can construct lines and circles. But, is it possible to construct effectively such “diagrams”?

Everybody can draw a circle: take a rope or a piece of an un-extendable material, fix an end of it and move around the other with a pencil attached. With the use of a proper *compass* we can vary the ray of the circle. According to the famous painting “Athen School” by Raffaello Sanzio (see picture),

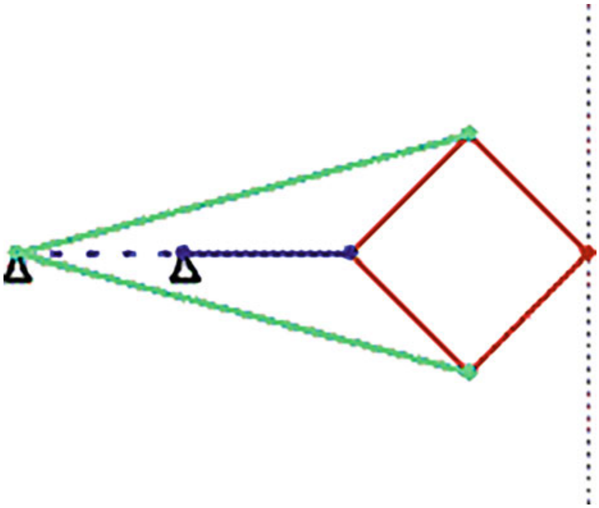
Euclid was using this instrument; there is actually a debate whether the use of compass occurs only later in the work of Arab mathematicians The question whether it is possible to draw a line with a compass, i.e. with a mechanical instrument, is much more difficult. In 1784 James Watt registered its patent specification for the now called Watt steam engine; in it he inserted a mechanism, called Watt’s linkage, with three bars, in which the central moving point of the linkage is constrained to travel on an approximation of a straight line. The following picture is the original design of Watt.



J. Watts wrote: "I have got a glimpse of a method of causing a piston rod to move up and down perpendicularly by only fixing it to a piece of iron upon the beam, without chains or perpendicular guides ... one of the most ingenious simple pieces of mechanics I have invented." (letter to M. Boulton 1784).

This linkage does not generate a true line, and indeed Watt did not claim it. Rather, it traces out a Watt's curve, an eight shaped figure called lemniscate. Nowadays this mechanism is also used in automobile suspensions, allowing the axle of a vehicle to travel vertically while preventing sideways motion.

Charles-Nicolas Peaucellier, a French army officer, constructed in 1864 the first mechanism capable of transforming rotary motion into perfect straight-line motion, called Peaucellier linkage. It consists of seven bars, four and two of the same size, connected as in the figure with two fixed points (marked with a triangle in the figure).



In the history of Mathematics we can find many mechanisms for the construction of curves, from Menaechmus, to Hippia, from Archimedes, to Descartes, Bernoulli, Huyghens and others.

Descartes dedicated much time to the effective construction of new compasses, among them the trisector, which draws a curve trisecting an angle, and the mesolabio, a curve giving the mean proportional of a segment. During these studies he developed a revolutionary theory to deal with curves, based on a new way of postulate the idea of curve which comes from Algebra.

In a letter to the mathematician Beckmann, dated 26.3.1619, he wrote: "So I hope I shall be able to demonstrate that certain problems involving continuous quantities can be solved only by means of straight lines or circles, while others can be solved only by means of curves produced by a single motion, such as the curves that can be drawn with the new compasses (which I think are just as exact and geometrical as those drawn with ordinary compasses), and others still can be solved only by means of curves generated by distinct independent motions which are surely only imaginary, such as the notorious quadratic curve [a curved line discovered by Hippias in the first century BCE; called 'quadratix' because it was used in attempts to square the circle.] With lines such as these available, I think, every imaginable problem can be solved. I'm hoping to demonstrate what sorts of problems can be solved exclusively in this or that way, so that almost nothing in geometry will remain to be discovered. This vast task is hardly suitable for one person; indeed, it's an incredibly ambitious project. But I have glimpsed a ray of light through the confusing darkness of this science, and I think I'll be able with its help to dispel even the thickest obscurities".

The ray of light glimpsed by Descartes can be roughly summarized as follows. Starts with a diagram in a plane consisting of two perpendicular lines meeting in a point, called O; associate to each point in the plane a pair of numbers called coordinates, (x, y). Nowadays we call such construction a Cartesian coordinate system.

A plane curve is given by an equation in two variables, $f(x, y)$; its support is the set of points whose coordinates (x, y) satisfy the equation $f(x, y) = 0$. For instance, a circle centered in O with ray of length $r > 0$ is the set of points whose coordinates satisfies the equation $x^2 + y^2 = r^2$.

Let us read Descartes own words in the book *La Géométrie* (1637): "I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all the points of those curves which we may call "geometric," that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed means of a single equation.*"*[2]

This great idea of Descartes connects geometry to algebra but at that time created some troubles to philosophers: in fact it opposes some theories of Aristotle, who wrote in the book Posterior Analytics: "a proof in one science cannot be simply transferred to another, e.g. geometric truth cannot be proven *arithmetically"*.

This astonishing change of perspective was made possible by the many results in algebra obtained by Italian mathematicians in the previous century. It was in the air and a similar theory was developed independently by the great competitor of Descartes, Pierre Fermat. Nowadays we call this theory Analytic or Algebraic Geometry and it is one of the central area of research in modern Geometry.

Descartes then showed that all classical curves could be described by equations in his new method; he proved for instance that conics are given by polynomials of degree two and that these polynomials give no other curves (a part pairs of straight lines).

Looking at polynomials of higher order he founded new curves, among them for instance the folium, with equation $x^3 + y^3 = 3axy$ (or $x(t) = \frac{3at}{1+t^3}$, $y(t) = \frac{3at^2}{1+t^3}$ where t is a parameter).

This curve has a singular point (a node) at the origin and Descartes was not able to draw it, thinking that it repeats equally in the four quadrants; Christiaan Huygens gave a first exact drawn of it. The problem of finding the tangent to this curve started a famous quarrel between Descartes and Fermat.

In this set up a big difference is to consider curves whose equation is given by a polynomial or those defined by a transcendental or analytic equation. Descartes himself made a distinction between two classes of curves, namely the admissible or geometric ones and the mechanicals or imaginaries; the quadratrix of Hippias mentioned above is among the imaginaries, for example.

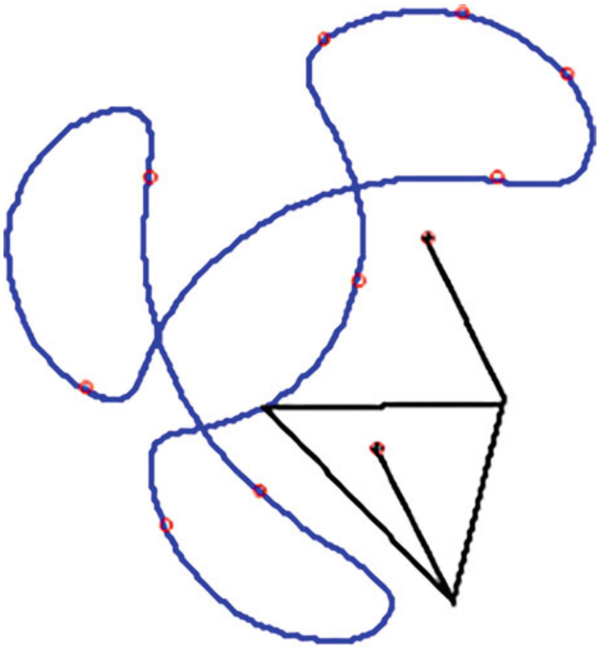
Much later, in 1876, A.B. Kempe[3] proved a fundamental Theorem, following which a plane curve given as zero of a Polynomial of two variables can be drawn by a mechanical linkage.

[2] Descartes *La Géométrie*, (1637).

[3] Kempe (1875), p. 213–216.

This clarifies definitely the problem of when a curve in the plane can be drawn with a “new compass”.

On the other hand, it opens the problem to construct effectively a compass to draw a given curve or a diagram. These problems have been extensively studied by engineers, let me mention for instance the following one: given 9 points in the plane find a Four-bar linkage which draws an algebraic curve passing through the nine points. The existence of the algebraic curve (and of the linkage) is mathematically clear (for more points there could be no solutions). The problem is to find it in a reasonable time; General Motor some years ago asked for a solution in order to create an optimal windscreen wiper. In 1992 the mathematician A. Sommese and others proved that given the nine points one can reduce to 1442 possible linkage; a check in this finite a priori list easily gives the correct ones. More recently other mathematicians reduced the a-priori possibilities to the number 64. The next figure represents a four bars linkage which draws a curve passing from nine given points.

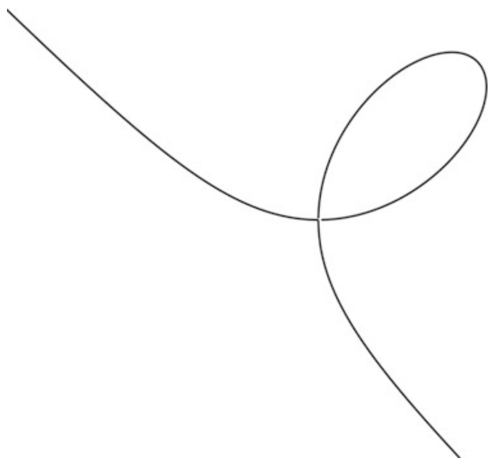


Digital Fabrication

The use of curves or diagrams to describe phenomena in real life and to solve various problems, as indicated by Greek mathematicians and re-proposed by Descartes, is nowadays a general method in all sciences.

Curves are defined *a la Descartes* with the use of equations and they can be drawn with the help of computers. In my opinion this fact is really a big step in the contemporary culture: anybody can easily draw a curve via its equation using a free and user-friendly software, for instance the one denominated GeoGebra.[4]

I still remember the difficulties I found in the high school when the teacher asked to draw a parabol or an ellipse of a given equation. Even Descartes was fouled up by drawing the folium; the following drawn was done in few second simply writing the equation in the low line of a GeoGebra Sheet.

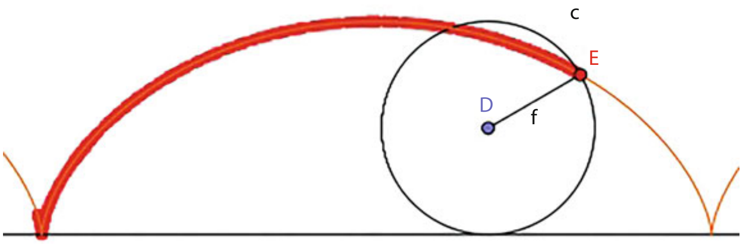


An amusing experience could be to go through the book *La Géométrie*, performing the constructions of Descartes with GeoGebra[3]. This has been actually done recently in a master thesis by Sara Gobbi,[5] a student of the University of Trento. It is surprising to notice, on one hand, how effective is the Descartes language to talk virtually with digital reality; and, on the other hand, how much the digital technology can help in understanding deep ideas of Analytic Geometry. I do not enter here in the question of the approximation, needed by computers to reproduce an abstract curve via numerical analysis.

With this software it is also very easy to construct digitally mechanisms and new compasses which draw curves; the following figure is the drawing of a cycloid made with GeoGebra with a circle rotating on a line.

[4] GeoGebra: https://www.geogebra.org/

[5] S. Gobbi Master Thesis, University of Trento: https://www.geogebra.org/m/azTYkAfn

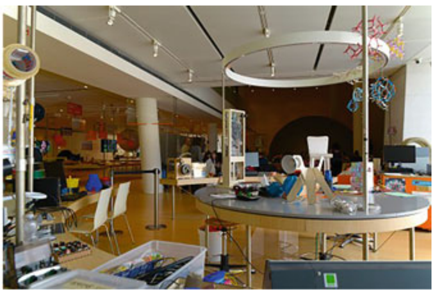


The cycloid is a famous curve which is either a Brachistochrone and a Tautochrone; curves with such properties were searched by Galileo and found by Bernoulli, Huyghens, Leibniz, Newton and others.

Digital visualization of curves may not be such a great novelty for many of us. What is newer is that these digital objects can now “come out of” the computers and become true border shapes. For that one should simply visit a Fab Lab, i.e. a Fabrication Laboratory, a small-scale workshop offering (personal) digital fabrication. These laboratories have many tools which are directly controlled by computers, among others laser cutters (for glass, plastic, metal, wood, . . .) and 3D-printers (printing in plastic, gold, chocolate, . . .).

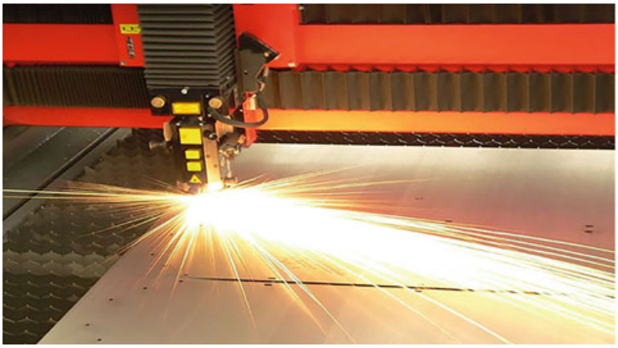
The first Fab Lab seems to be the one created in 2001 at MIT in US. Now, any town of the industrialized world as a Fab Lab and many cultural or scientific institutions host them. On December 2017 about 1200 Fab Labs were officially listed on a web page of the Fab Lab community; on 2017 a Fab Lab based in Grenoble was vandalized and burned by anarchists.

The following are two pictures of the Fab Lab of MUSE, the Science Museum of Trento-Italy.[6]



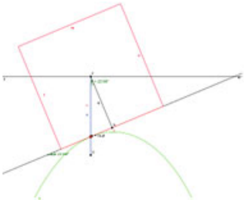
The files constructed with GeoGebra (or other software) can be easily “read” and interpreted by the laser cutters.

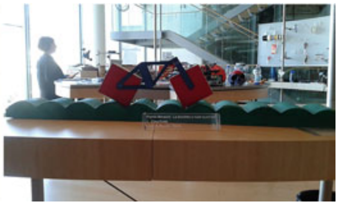
[6] Fabrication Laboratory at MUSE-Trento, Italy: http://fablab.muse.it/




For instance, one can cut a cycloid in the wood (adding a line one produces an exhibit which gives a nice tool to show the brachistochrone property of the curve):

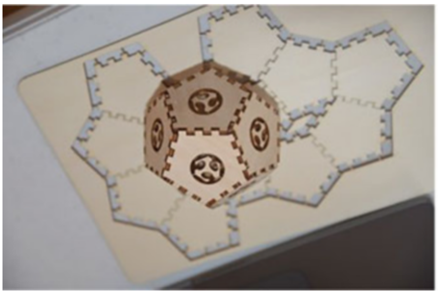
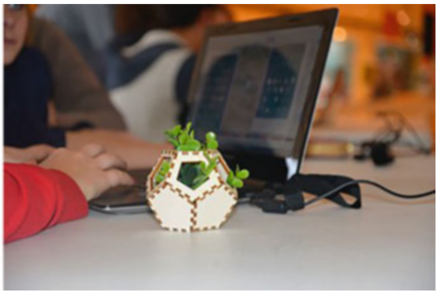
The following pictures represent a project to construct a bicycle with square wheels. In the first one, with GeoGebra, the profile of a suitable ground has been constructed, a catenary. A proof has been made in wood and finally a bike, with the appropriate ground, has been constructed by a school: this project was the winner of a national school competition called Premio Bonaccina in 2017.



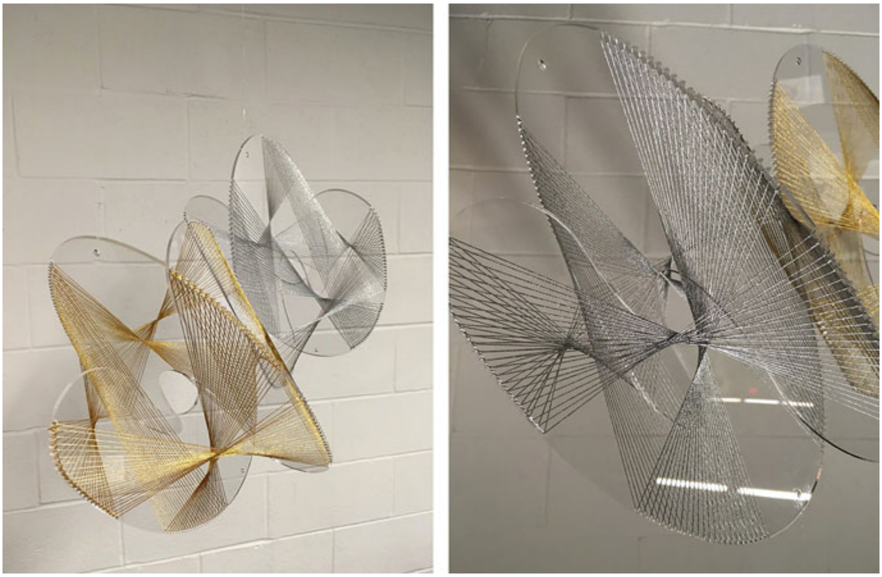


These are simple examples related of the use of a laser cutter to produce objects interesting from the point of view of pure mathematics and its popularization. The principal goal of a Fab Lab is to create economic development for the society in which it acts; the Fab Lab of MUSE hosts everyday public and private enterprises which construct many prototypes of objects they would like to produce.

I like to provide two other examples related to arts: the first is the realization of a flowerpot shaped as a dodecahedron.

The second are two example of "String Art", modeled on the famous Naum Gabo's operas, realized by the artist David Press for the bookshop of the Momath at New York; the plastic support and the holes are cut out with a laser cutter.



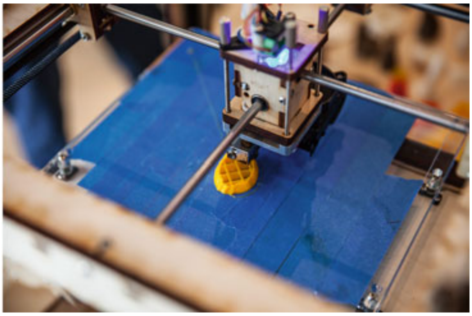
The definition of a curve as the locus of zero of a function in two variables was extended to the case of surfaces in 1700 by A. Parent: for him a surface is the set of points in the space whose coordinates $(x,\ y,\ z)$ have a realization given by an equation $f(x,\ y,\ z) = 0$ where f is a function, for instance a polynomial, in three variables.

Given such a function it is usually rather difficult to figure out what is the shape of the associated surface. Digital visualization on computer has been largely developed in these years, mainly to produce smart video games or animation movies. A very simple and free software which I suggest in order to visualize surfaces associated to algebraic functions is called Surfer, it can be downloaded from the web site imaginary.org.[7]

Here are some examples, the first two represent the hyperbolic hyperboloid $(ax^2 + b\ y^2 - c\ z^2 - 1)$ and the hyperbolic paraboloid $(ax^2 - b\ y^2 - c\ z)$, the two ruled quadrics.

[7] Imaginary Open Mathematics: https://imaginary.org/

The files produced with Surfer (or any other software include the CAD and CAM ones developed by engineers and architects) can be interpreted by a 3D-printers in a FabLab.



In principle one can create any ideal surfaces with many type of material: the possibilities actually depend very much on the quality (and thus on the prize) of the 3D-printer and on the skill of the Fab Lab technician responsible of the printer. Those people face the same problems that Benvenuto Cellini had in producing his famous Saliere and Statues, like for instance the Perseus in Florence. The surface could not usually be printed in one piece but in several ones, which then should be glued together; this job requires good skills and experience.

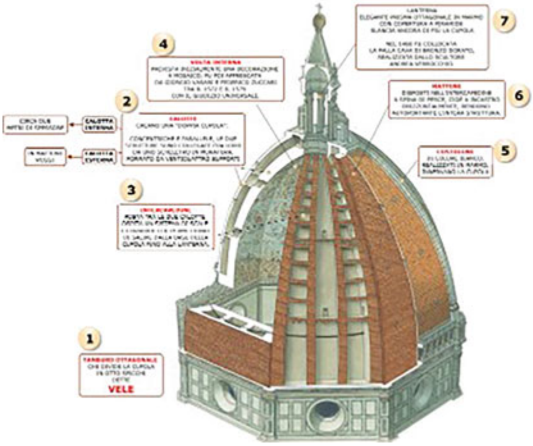
The following pictures give some examples: two famous surfaces, namely the surface of Dini, a surface with negative curvature in any point (a model of non-Euclidean plane geometry), and the surface of Barth, a surface of degree six with the maximum number of singularities. The ready to print files were realized by Oliver Lab and can be downloaded at the above quoted web site: imaginary.org.

The amazing work of Oliver Lab, a "mathematicians with interests in design, programming and several aspects of mathematics", can be explored on his web pages.[8] There one can also buy many surfaces realized with the 3D-printers, artistic objects to display in a room or as jewellery to wear.

In the next three pictures one can see the same cubic surface: the first was constructed in wood by Campedelli in the 1951 and can be found in the collection of Museo della Scienza in Milano. The other two are realized with a 3D-printer by Oliver Lab, the last is a mathematical jewellery.

The famous Brunelleschi Dome of Florence was the biggest dome of the world, still the biggest constructed of masonry. Completed in 1436 with the technique of the "sixth of fifth or fourth of angle", has a catenary as curve section, as discovered by Bernoulli and Huygens in 1690. Together with some of my students of the University of Trento we figure out a digital representation of the Dome which we then produced with the 3D-printer.

[8] Oliver Lab, Mathematical Objects: https://oliverlabs.net/



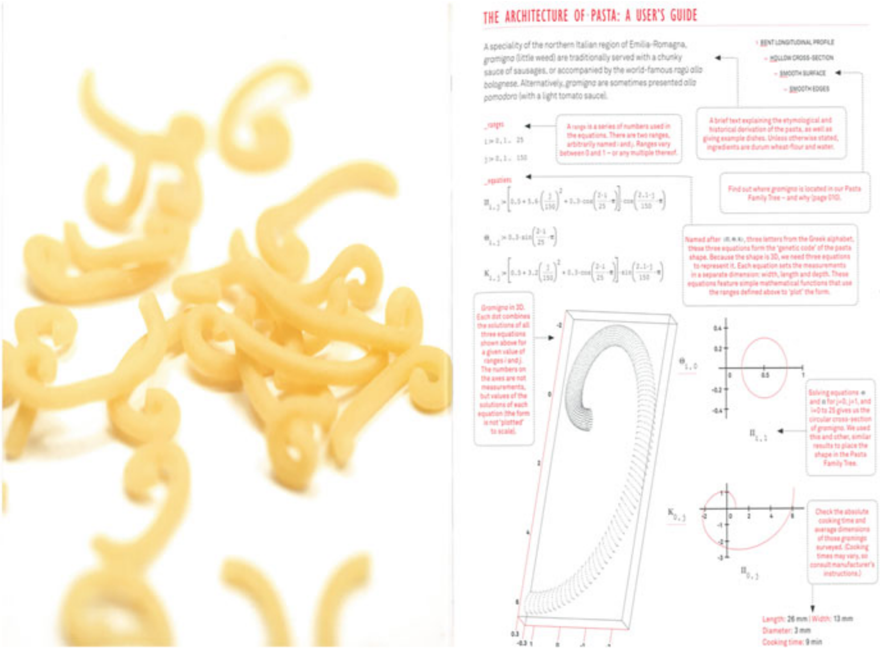
It is possible to construct full houses with appropriate 3D-printers, they can be ordered on the web.

With a 3D-printer one can even try to create surfaces living in higher dimension, like the following Klein's bottle.

The shape of the food is an important feature of its enjoyment; the success of the Pringles chips depends also on their shape, a hyperbolic paraboloid.

Last year the Barilla Company, an Italian food enterprise, open a food contest for the creation of new pasta shapes to be realized through innovative pasta 3D-printer. Interested competitors should, in my opinion, consult the beautiful book Pasta by Design,[9] by architect George Legendre in collaboration with Paola Antonelli, curator of MoMA's Department of Architecture and Design in New York. The book offers a classification of pasta shapes through 92 "canonical models", morphologically different and connected within a phylogenetic tree, described through an equation in three-dimensional space and, finally, associated with a specific sauce; the next figure refers to two pages of the book.

[9] George Legendre et oth. (2011).



Geometry in Higher Dimension

In the XX century Geometry started to explore more general spaces, which are not confined in the usual (standard) 3-dimensional ambient but described by more variables. The great breakthrough was obtained some years before through the work of Bernhard Riemann (1826–1866). In his “Habilitation” thesis”,[10] he overcame the limitations of the classical view and founded modern geometry, introducing n-dimensional varieties, varieties with curvature, the celebrated non- Euclidean geometries and the so-called Riemann surfaces.

Geometry in arbitrary high dimension requires a good capability of abstraction, the objects here are very much ideal. In particular it is hard “to visualize” them and we need new concepts for constructability. The use of diagrams is turning out to be very essential for these purposes, as well as new digital techniques.

In the fall of 2016, I was a curator for an exhibition, at MUSE-Trento-IT, dedicated to mathematics. In one section we considered the question of visualizing higher dimensional objects: for example, to visualize regular 4-dimensional polytopes we described their projections via 3-dimensional diagrams and digital visual construction. The next pictures refer to some of those exhibits dedicated to the polytope denominated “120 cells”.

[10] Bernhard Riemann (1867)

The digital constructions were realized by Gian Marco Todesco and his Digital Video s.p.a.: a sample can be find on YouTube.[11]

Higher dimensional varieties are deeply studied in Algebraic Geometry; the theory was developed in the XXth century by the work of many mathematicians, including the Italians, like Enriques, Castelnuovo, Severi, Segre, Fano. and French ones, like Serre, Weil, Grothendieck.

Alexander Grothendieck in particular introduced the language of Schemes, which clarifies in term of algebra many subtle facts about the reciprocal relations between geometric objects: intersection, multiple or non-reduced structure, ... The use of diagrams to describe algebro-geometric structure reached within this language an acme: examples are the Dynkin diagrams for algebras and singularities, polytopes for Toric Varieties, ...

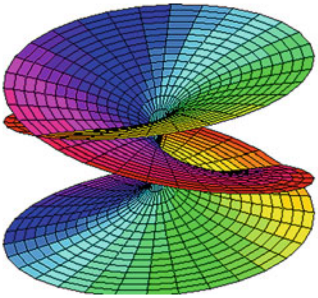
Grothendieck, in his famous proposal for a long-term mathematical research "Equisse d'un Program" (Sketch of a Programme) (1984), introduced also some special diagrams which he called Dessins d'enfants. They are embedded graphs used to study Riemann surfaces and to provide invariants for the action of the absolute Galois group. A dessin d'enfant is the French term for a 'child's drawing' and A. Grothendieck commented his discovery with the following words: "this discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my center of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any childs drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke."[12]

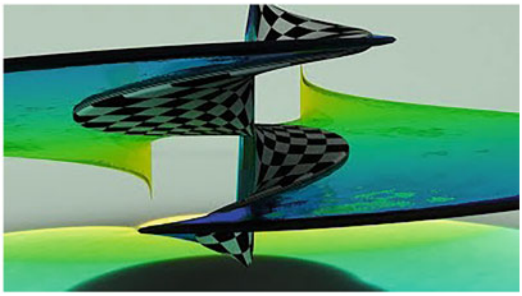
The next picture represents some dessins d'enfants associated to basic Riemann surface.

[11] Gian Marco Todesco video: https://www.youtube.com/watch?v=0uT6q_hrK50

[12] Grothendieck (1984).

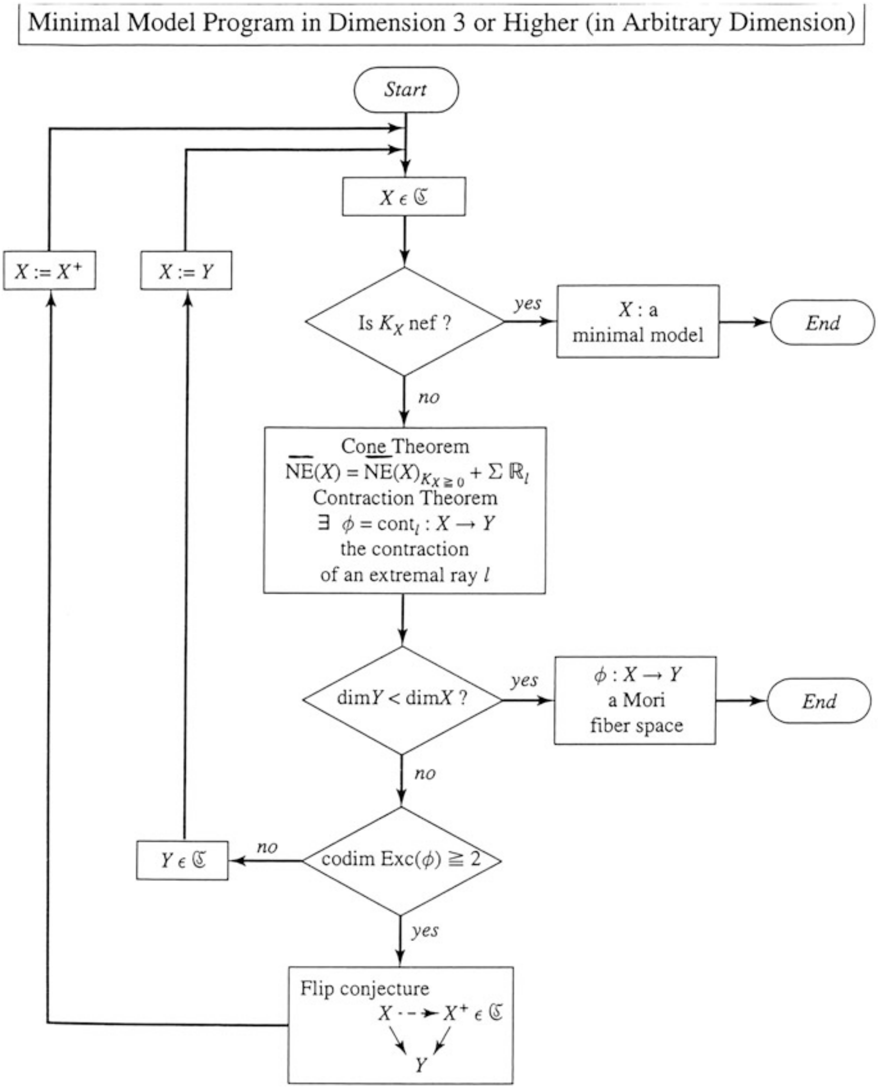
A Riemann surface is actually a 2-dimensional object but it usually lives in a 4-dimensional space; in the next picture you can see a digital representation of a simple Riemann surface and a realization with a 3D-printer.





The classification of Algebraic Varieties embedded in a Projective space is a central topic in modern geometry. In 1980 the Japanese mathematician Shigefumi Mori suggested a Program to classify those varieties in every dimension. This program consists in two main steps: first one has to identify the so-called Special Minimal Models (roughly speaking, varieties that do not admit further projections). Secondly to find a general procedure that, starting from any variety, through projections and projective transformations, leads to a corresponding Minimal Model. The following diagram, taken from the book of K. Matsuki,[13] resumes the full Program and it is well known to specialist.

[13] Matsuki (2002).



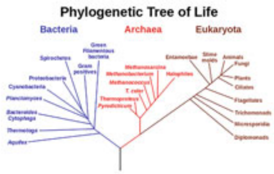
Mori himself has proved the feasibility of the Minimal Model Program in three dimensions; for this he was awarded the Fields Medal in 1990. In 2010, an international team of four mathematicians, Caucher Birkar, Paolo Cascini, Christopher D. Hacon and James McKernan, with a brilliant article in the Journal of the American Mathematical Society,[14] proved the feasibility of the program in any

14 Birkar (2010), n. 2, p. 405–468.

dimension. Now the problem has shifted to find effective algorithms that determine a minimal model for a given variety.

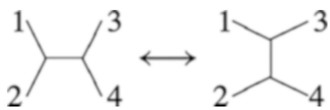
It turned out that the fundamental building blocks of the Program, like the atoms for the ordinary matter, aresomevarietiesintroduced acentury ago by the Italian mathematician Gino Fano.

A group of researchers, including Bernd Sturmfels, proposed an interesting connection between phylogenetic trees, which we find in the theory of biological evolution, and some algebraic projective varieties called Toric and Tropical varieties. In the following picture one can find the original draw of Darwin, a modern example of phylogenetic tree and an exhibit of MUSE on a “universal” phylogenetic tree.



My colleague J. Wisniewski, in this way, associated Fano varieties to trivalent trees. Subsequently, by applying the minimal model program to them, he obtained new examples of Fano varieties. A very nice discover in the field of geometry which starts from a biological diagram. On the other way around, at the moment there is not a good “biological” explanation of the result; I am sure that its importance in life science will be eventually discovered.

A fundamental step in the Minimal Model Program is a mathematical operation called a flop (invented by the Field medalist Sir M.F. Atyah long ago); in the Wisniewski construction on the phylogenetic tree it corresponds to the operation described by the following diagram.



I like to conclude with a nice record of the conference. In that period in Paris I had the opportunity to visit an exhibition on Edgard Degas at the Gare d'Orsay ("Degas Danse Dessin", January 2018). There I could read the following sentence of Paul Valery about Degas's work: "There is a huge difference between seeing something without a pencil in your hand and seeing it while drawing it. Or rather you are seeing two quite different things. Even the most familiar object becomes something else entirely, when you apply yourself to drawing it: you become aware that you did not know it-that you had never truly seen it . . . It dawns on me that I did not know what I knew: my best friend nose.[15]".

I find it very pertinent for the actual occasion: although we are not artists, when we draw a diagram, even digitally, very often we see a different thing, may be something we had never truly seen before.

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[15] Paul Valery (1938).

The Classical Style in Contemporary Geometry: Views from a Person Working in the Field

Alessandro Verra

Abstract As is well known, geometry undergone revolutionary shifts of paradigm during all the last century. This in many cases implies, for instance in the case of algebraic geometry, that history, or at least the history of this discipline, is a discrete, but nevertheless strong and familiar, presence in the work of the persons active in this field. In this perspective the geography of contemporary geometry is silently unified, more than dramatically split as one could think, from the history. The seminar will address this issue, trying to bring some evidence and concrete examples from the experience of a person working in this field. A major attention will be payed to the history of classical and modern rationality problems, for some famous examples of algebraic varieties.

Keywords Diagram · Geometry · Algebraic surface · Unirationality

Introduction

Let me say from the beginning that I do not want, and I cannot pretend, to be a professional on a large part of my contribution here. However this is possibly coherent with the spirit of this conference, where persons from different disciplines join together different points of view, experience and perceptions on the same theme.

A conference on *la pensée diagrammatique* should involve "Diagrammes, Dessins, Esquisses". These are present, implicitely or explicitely, in my exposition. Indeed its language and figures aim to stress that the work of a mathematician in his discipline, and in the surrounding cultural area, is innervated by these ways of thinking and knowing. This is specially true for some case-examples to be considered in the last part of my talk and useful for its goals. These come from my working experience in Algebraic Geometry and are concerned with a sample

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https://doi.org/10.1007/978-3-030-83125-7_5

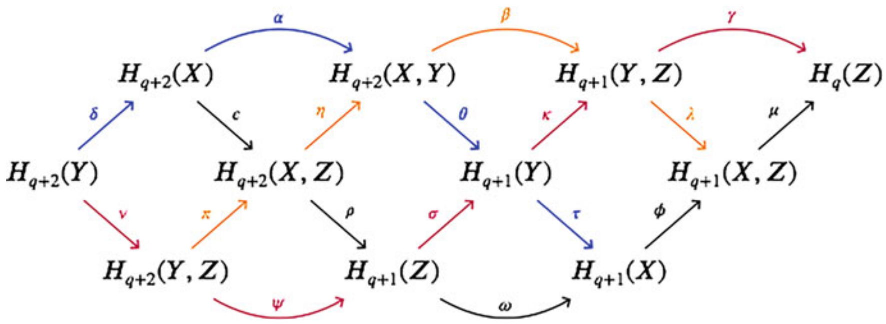

Fig. 1 Suites exactes en Topologie Algébrique. Source: Braid lemma in nLab. https://ncatlab.org/nlab/show/braid+lemma

of classical problems related to the concept of *Rational Space* (more precisely a rational algebraic variety). Indeed its related sequel of *rationality problems* is an example of the complexity and conservativeness of the changes in the time and geography of a mathematical landscape.

Actually, from my working experience, I am used to perceive the word *diagram* in a *colder sense*. In the mind of a contemporary geometer a diagram is, often, a sequence of arrows and letters on a printed book he is reading (Fig. 1).

So this is often conceived as a technical formalism encoded in the last and definitive stage of some mathematical knowledge: *something clean and cleaned*.

On the other hand there is a warmer road, long and windy maybe. Every geometer, every mathematician, writes freely by himself. Diagrams, drawings: on a blackboard, on a scrapbook, on a file in a laptop . . . Let me say that there is a *warmer level of knowledge*, which is certainly tortuous and undetermined for a long while. That road is taken by every geometer when studying geometric objects and the corner of Nature where they are.

Warm often implies alive. In *Tendencious Survey on 3-folds*, Miles Reid (1987, p. 340), a great British geometer of our times, writes that algebraic varieties should be taken as alive beings in their space ambient or Nature:

> A useful image for contemplating the totality of all algebraic varieties is to think of them as being ALIVE; as such they will grow and degenerate and mutate and take on new forms, and will occupy every conceivable nook and cranny in the universe, including some we will find inconceivable.

See also Ulf Persson's paper *Introduction to the geography of algebraic surfaces of general type* (1987) and the book *Life on Earth* by the journalist and naturalist David Attenborough (1979).

I think that this perception of several spaces, each considered as a corner of alive Nature, is very related to that kind of *warm approach to knowledge* which is also passing through explicit, or mental, *diagrammes, dessins and esquisses* in the previous sense. I am not a philosopher, nor an expert *de la pensée diagrammatique*, but the perception of varietes and spaces I have outlined, quoting Miles Reid,

Fig. 2 Man Ray and an Algebraic Surface. Source: The Phillips Collection. https://blog.phillipscollection.org/2015/03/26/man-rays-shakespearean-equations-king-lear/

reminds to me the *sourire de l'être* I have seen in some writings in relation to Gilles Châtelet, you all know very well. So I feel not so alien, as a geometer, in being here and I am honored for having the opportunity of trying to say something (Fig. 2).

Fragments of Basic Algebraic Geometry

At first I want to introduce some elementary or natural constructions so to focus on the geometrical matter to be considered and its shape. My aim is to put in perspective some configurations, in the landscape of XXth century and today Geometry, with its radical revolutions and its persistency of fluent classical forms. It is also due that one would like to intercept relations between ideas and visions in Geometry and the spirit of the times: from classical XIXth century Geometry to present days.

The Affine Space At first let me recall three notions from School Geometry:

1. Space
2. System of polynomial equations
3. Rational parametric equations

The space can be brutally R^3, the set of triples of real numbers. A point is identified to its triple (x, y, z) of coordinates (More precisely we consider an affine space over R. Once a frame is fixed a point is identified to its coordinates).

More in general let k be a field (in the sense of Algebra). Replacing R by k and 3 by n, we obtain the *affine space over k of dimension n*:

$$k^n := \{(t_1, \ldots, t_n) | t_i \in k, i = 1 \ldots n\}$$

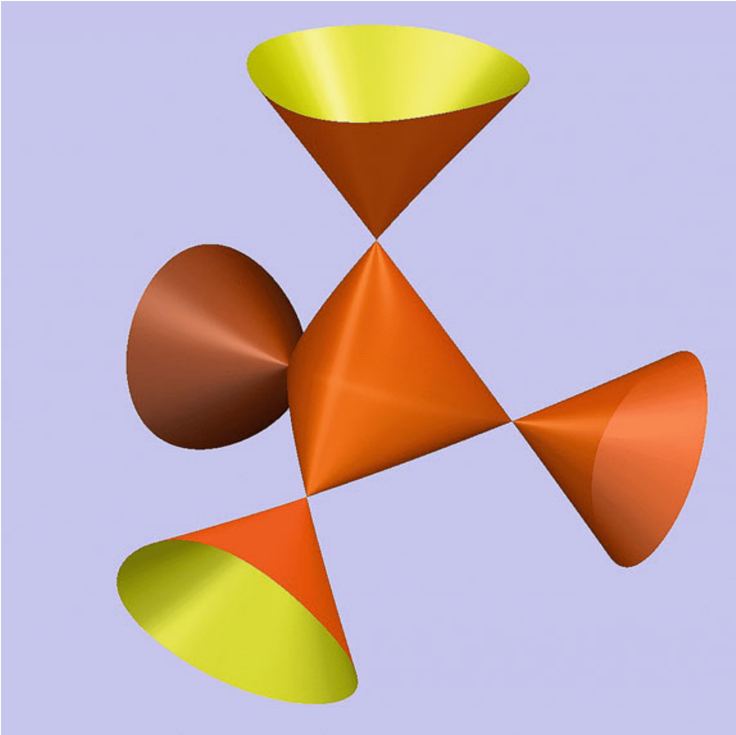

Fig. 3 A Cayley cubic surface: $xy + xz + yz + xyz = 0$. Source: Wikipedia. https://en.wikipedia.org/wiki/Cayley%27s_nodal_cubic_surface#/media/File:CayleyCubic.png

Warning: in what follows the assumption $k = \mathrm{C}$, the field of complex numers, is often necessary. As far as possible we keep $k = \mathrm{R}$ to offer more intuition and vision.

Algebraic Varieties A system of polynomials equations in n variables defines a set V in k^n. V is an *affine algebraic variety*.

The notion of algebraic variety X is more general: the space X is locally, around each of its points, an affine variety. For our goals it will be sufficient to think of V.

The notion of *dimension* of X is defined as usual and corresponds to intuition: point, curve, surface (Fig. 3).

Rational Parametrizations It is a natural experience to search a *rational parametrization* of V, that is rational parametric equations: sometimes with success, though this possibility is restricted to quite special classes of algebraic varieties. A rational parametrization is defined by a n-tuple $f = (f_1, \dots, f_n)$ of rational functions of $t = (t_1, \dots, t_d)$ and it is denoted as

$$f : k^d \to X \subseteq k^n.$$

f works as follows: for a general $t \in k^d, f(t) = (f_1(t), \quad \ldots, f_n(t)) \in X \subseteq k^n$. General means that t belongs to the *domain* D of f, a suitably defined dense open set in the Zariski topology. The outcome is that $f(D) = \{f(t), \quad t \in D\}$ contains a dense open set of X. Notice that f defines the function $f': D \longrightarrow X$ such that $f'(t) = f(t)$. The broken arrow $\dashrightarrow$ suggests that *f not always is a function and that it is not so far from being a function*, since it is a function on D.

Simple Rational Parametrizations V is the unit circle $x^2 + y^2 = 1$ in $\mathbf{R}^2$.
$f : \mathbf{R} \rightarrow V \subset \mathbf{R}^2$:

$$x = \frac{1 - t^2}{1 + t^2}, \; y = \frac{2t}{1 + t^2}$$

V is the quadric cone $x^2 + y^2 = z^2$ in $\mathbf{R}^3$.
$f : \mathbf{R}^2 \rightarrow V \subset \mathbf{R}^3$:

$$x = s\frac{1 - t^2}{1 + t^2}, \mathrm{y} = s\frac{2t}{1 + t^2}, z = s$$

V is the Cayley cubic $xy + xz + yz + xyz = 0$ in $\mathbf{R}^3$.
$f : \mathbf{R}^2 \rightarrow \mathrm{V} \subset \mathbf{R}^3$:

$$z = s + t + 1, y = \frac{s + t + 1}{s}, x = \frac{s + t + 1}{t}.$$

Sterographic Projections in Dimension 1 and 2 (Figs. 4 and 5)

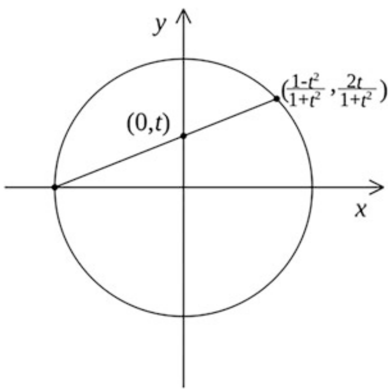

Fig. 4 https://commons.wikimedia.org/wiki/File:Stereoprojzero.svg

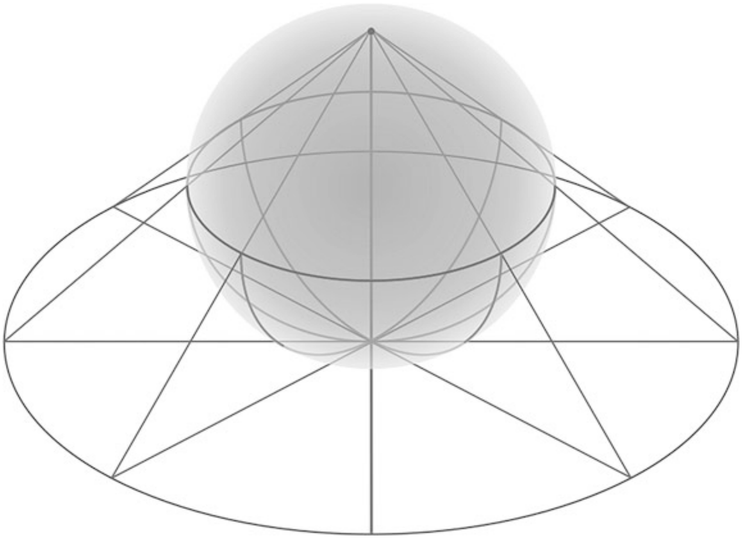

Fig. 5 https://en.wikipedia.org/wiki/Stereographic_projection#/media/File:Stereographic_projection_in_3D.svg

Origins of Birational Geometry and Classification

Rational and Birational Maps The morphisms, in the sense of categories, from an algebraic variety X to an algebraic variety Y are the rational maps $f: X \dashrightarrow Y$.

Let $X \subseteq k^n$, $Y \subseteq k^m$: as in the case of a *rational parametrizion*, f is defined via an n-tuple $(f_1, \ldots, f_n)$ of rational functions as above.

Definition Let D be the domain of f,

1. f is dominant if $f(D)$ contains a dense open set.
2. f is *generically finite* of *degree m* if there exists an open dense set $U \subseteq Y$, such that

   $f^{-1}(y)$ is a set of m points, $\forall y \in U$.

3. f is birational if it is generically finite of degree 1.

Notice that $f: X \dashrightarrow Y$ is birational if and only if it is invertible along a dense open set of Y.

Birational Geometry and Classification From now on A^n is the affine space k^n, (with our usual attitudes: (1) $k = \mathbf{C}$ if necessary, (2) $k^n = \mathbf{R}^n$ if possible, possibly for a better vision). The projective space $\mathbf{P}^n$ is its natural companion: the completion of A^n with *the points at infinity*.

Two algebraic varieties X and Y are said to be *birationally equivalent* if there exists a birational map $f: X \dashrightarrow Y$. The relation of being birationally equivalent is an equivalence relation on the set of all algebraic varieties. Essentially the *birational classification* of algebraic varieties is the description of all classes of birational equivalence classes of this set.

The set $Cr(n)$ or $Bir(n)$ of *all birational maps* f: $\mathbf{A}^n \to \mathbf{A}^n$ is a huge and still very mysterious group with respect to the composition law of rational maps. Replacing A^n by $\mathbf{P}^n$ does not change the group. *Cr(n)* is well known as the *Cremona group* of P^n.

In very concrete and classical terms, part of the birational classification problem is the study of the orbit of $X \subseteq \mathbf{P}^n$ under the action of *Cr(n)*, that is the set {f(X), f $\in$ Cr(n)}. Very roughly speaking, this study is directed towards the quotient set

$$\left\{\text{Algebraic varieties in } \mathrm{P}^n\right\} / Cr(n).$$

For the case n = 2 one can say that Cremona and de Jonquiéres transformations are at the origins of birational plane geometry.

Cremona Transformations (Fig. 6)

Quadratic transformation: a general line transforms in a conic (Fig. 7).

$$f : \mathbf{A}^2 \to \mathbf{A}^2, f(x, y) = \left(\frac{\ell}{x}, \frac{\ell}{y}\right), \ell = x + y + 1$$

de Jonquiéres Transformations (Fig. 8)

de Jonquières transformation: a general line transforms in a curve of degree m with a point of multiplicity $m - 1$ (Fig. 9).

All this brings us to the historical times from where my story on rational algebraic varieties and rationality begins. I am now going to enumerate some classical themes, to be introduced and commented at least partially. The themes I will consider are rooted in that age. It seems quite clear that these themes form like a carsic river in the geography of Algebraic Geometry. This is influencing the landscape even when it seems completely disappeared.

It would be fascinating to see much more deeper, in the folds of history and through the brilliant explosions of novelties, like those in the sixties, the tacit constance and persistence of such a carsic landscape, where time and history slowly rotate.

Fig. 6 Luigi Cremona (1830–1903). Geometer and Garibaldi's follower at the origin of the Italian Algebraic Geometry. Source: Wikipedia. https://it.wikipedia.org/wiki/Luigi_Cremona#/media/File:Luigi_Cremona.jpg

Relying on these themes, I will present some evidence, anecdotes and examples confirming such a landscape and its contrasts. I am not pretending to have clear views on the previous interplay: some remarks seem worth of interest some are possibly obvious.

However it should be not too difficult to see something interesting, since it is not a breaking news that classical problems are back. We live in a post-grothendieckian age, of course having the enormous advantages coming from EGA and its age.

The four themes I am listing are useful to study the so many interplays between classical and contemporary times in the landscape of Algebraic Geometry: evident, hidden, surprising ones.

1. Unirational varieties and Lüroth problem;
2. Plane curves and their birational classification;
3. Geography of algebraic surfaces of general type;
4. Cubic hypersurfaces: the rationality problem.

I will concentrate on a unique historical account, with sepcial regard to (1) and (4).

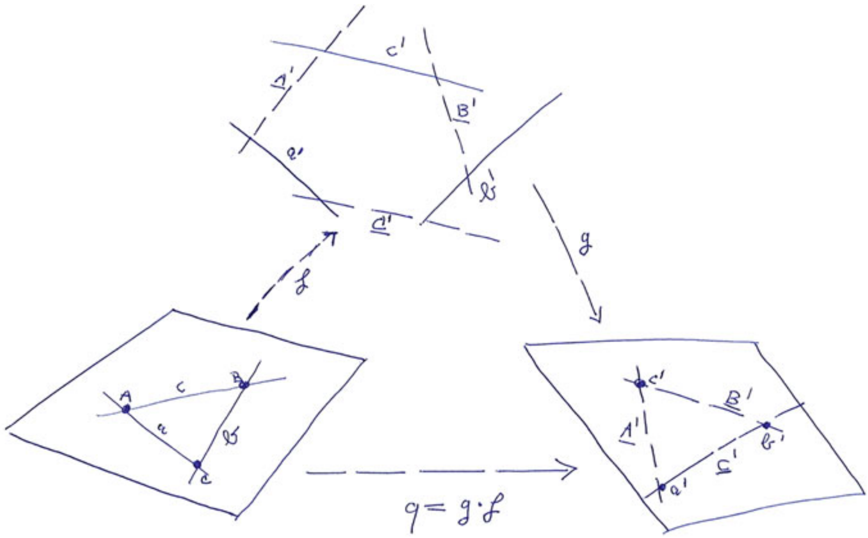

Fig. 7 Transformation quadratique du plan. Source: Personal realization

Fig. 8 Ernest de Jonquières. Sur le navire Gassendi, Terreneuve 1859. Source: Wikipedia. https://it.wikisource.org/wiki/Autore:Ernest_de_Jonqui%C3%A8res#/media/File:Ernest_de_Fauque_de_Jonquieres.jpg; Fonds Paul-Emile Miot. Bibliothèque et Archives Canada, e004156484. MIKAN3529029

Classical Problems and Rational Parametrizations: Curves and Surfaces

Unirational Algebraic Varieties Rational parametrizations are a natural attempt, and an early theme in Algebraic Geometry, since its remote origins to nowadays.

Definition *X* is unirational if admits rational parametric equations.

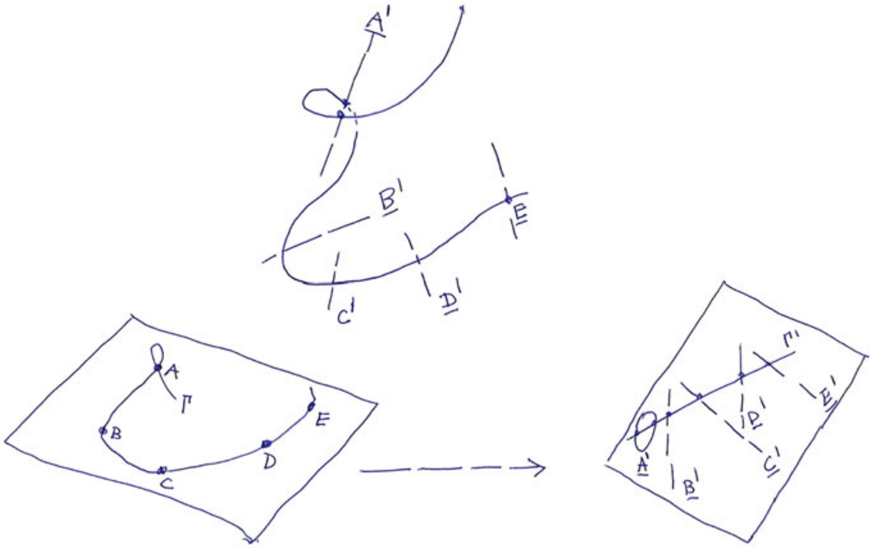

Fig. 9 Transformation de Jonquières, $m = 3$. Source: Personal realization

Starting from a rational parametrization of X it is easy to eliminate parameters so to have a generically finite rational parametrization

$$f : \mathbf{A}^d \to X.$$

So we will assume that the above properties are satisfied by f.

Definition X is rational if admits a rational parametrization of degree one. In other words X and A^d are birational.

Does every X admit rational parametric equations f: $\mathrm{A}^d \dashrightarrow X$? Actually it is well known that A^n has properties which are *innatural* for the structure of most families of algebraic varieties. These are influencing X, if such an f exists, and make X special.

Moreover, as stressed by János Kollár (2001), the most conceivably natural space A^n is somehow unflexible, and not at all simple, in many geometrical senses.

Near to Be Rational Algebraic Varieties For classes of algebraic varieties X strictly related to A^n, (more precisely one should say here: of Kodaira dimension $-\infty$), several notions inducing more flexibility are needed and elaborated. Here is a hierarchy:

1. *Rational*:∃*a birational* f: $\mathbb{A}^d \dashrightarrow X$.
2. *Stably rational*: ∃ *a birational* $f : \mathrm{X} \times \mathbf{A}^m \to \mathbf{A}^{d+m}$.
3. *Unirational*: ∃ *a dominant* f: $\mathrm{A}^n \dashrightarrow X$.

4. *Rationally connected*: *general* $p, q \in X$ *are connected by the image of some* $f\colon A^1 \dashrightarrow X$.
5. *Uniruled: a general* $p \in X$ *belongs to the image of a non constant map* $f\colon \mathbb{A}^1 \dashrightarrow X$.

A^n inherits all these properties, moreover

$$\text{Rational} \Rightarrow \text{Stably Rational} \Rightarrow \text{Unirational} \Rightarrow \text{Rationally connected} \Rightarrow \text{Uniruled}$$

Problems on Rational Parametrizations It is worth to point out that the problem about the existence of a rational parametrization of X is *simple in its formulation*. Like for some deep problems in Number Theory we can discuss it for a while with the background of secondary school. If it stays unsolved, resisting to the time, one says, not only in Mathematics, that it is an *outstanding problem*. To my imagination such a word *outstanding*, more than the word *overwhelming*, suggests *standing outside*.

The themes I am going to outline offered, and offer, a lot of these problems, which are sitting somewhere outside in the landscape, like silent giants. To come to this Conference, they are alive mathematical substance waiting for an appropriate intercepting form. About this it is perhaps appropriate quoting an interview to René Thom:

> [...] êtes-vous un matèrialiste? Thom a dit, je ne pense pas. Je vois la matière dans une perspective aristotèlicienne, une sorte de continu qui peut acquèrir des formes. [...] À mon avis, toute qualitè peut être considèrée, dans une certaine mesure, comme une forme spatiale, une forme rèpandue dans un espace abstrait.

Curves and Unirationality Going back to algebraic geometry and its history let us continue by asking the most elementary questions on unirational varieties: what happens in dimension one that is for unirational curves? The answer can be very well the starting point for a journey in a story whose interest is gone far beyond a reasonable expectation.

Theorem (Lüroth 1876). Every unirational curve is rational.

In particular this is true for a curve defined over any field k. The theorem, essentially a result in fields theory, opens to the most optimistic views in higher dimensions.

Algebraic Surfaces and Unirationality Things become radically complicated already in dimension two. The theorem extends to complex algebraic surfaces, i.e. $k = \mathbf{C}$. However this theorem follows from one of the top result of the Italian School of those times, having as central figures Cremona and, for surfaces in particular, Enriques and Castelnuovo.

Criterion of rationality (Castelnuovo 1894)

$$\text{A surface } X \text{ is rational if } q(X) = P_2(X) = 0.$$

Birational Obstructions to Unirationality *geometric genus* $p_g(X)$, irregularity $q(X)$ and the *plurigenera* $P_m(X)$ are birational invariants of a smooth, projective algebraic variety X.

For complex curves $p_g(X)$ is the genus $g(X)$: the number of holes of the Riemann surface X. Clebsch: X rational $\iff g(X) = 0$.

After Castelnuovo for surfaces and Clebsch for curves: unirational $\iff$ rational $\iff$ some suitable birational invariants of X are zero, assuming X of dimension ≤ 2. Now let us move for a moment to more recent times, in the Fifties of the last century.

Topological Obstructions to Unirationality In the Fifties J. P. Serre was working to build up the new Algebraic Geometry, for instance *Faisceaux Algèbriques Cohèrents*. It is interesting, for the arguments of this paper, that in those years he came to the old theme of unirationality. For an algebraic variety X of any dimension he shows that:

Theorem (Serre 1959). Unirational implies simply connected.

There is a peculiar historical episode related to this theorem, as we will see later. However the subtleness of the (uni)rationality questions already appeared in the work of Enriques and Castelnuovo and it is somehow in relation to the previous theorem.

Obstructions to Unirationality In the correspondence between Castelnuovo and Enriques around 1892, the following equation of a surface in A^3 appears

$$qxyz + x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 = 0,$$

q is a general quadratic polynomial in x, y, z. This is the first appearance of an Enriques surface, in its sextic model passing doubly through the edges of a tetrahedron. It is in some sense very subtly non unirational, and hence not rational, since one has $P_2(X) = 1$ but $p_g(X) = q(X) = 0$. Indeed such a surface deserves a special place in the classification of Algebraic Surfaces over the complex field C and its peculiarity is its non simple connectedness.

The fundamental group of a smooth projective model of an Enriques surfaceis $\pi_1(X) = Z/2Z$. Seeing, mentally or maybe in a picture, an Enriques surface could remind of the *Sourire de l'être* and its hironies (Fig. 10).

Geography of Algebraic Surfaces: K3 Surfaces The universal covering of an Enriques surface is a K3 surface. Topologically it is represented by a quartic surface in the projective space, for instance, in affine coordinates, the Fermat quartic

$$x^4 + y^4 + z^4 = 1.$$

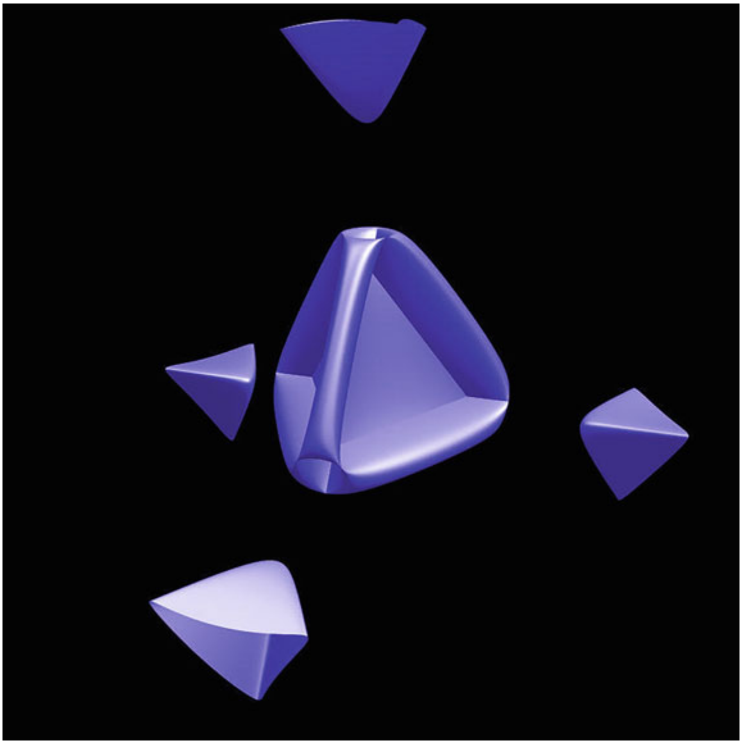

Fig. 10 Surface sextique d'Enriques. Source: Web page on Enriques surfaces. https://www.agtz.mathematik.uni-mainz.de/algebraische-geometrie/

The name K3 reminds of beautiful and hard mountains:

> Dans la seconde partie de mon rapport, il s'agit des variétés Kähleriennes dites K3, ainsi nommées en l'honneur de Kummer, Kodaira, Kähler et de la belle montagne K2 au Cachemire (Weil 1958).

Supersingular K3 Surfaces and Unirationality In the Fifties, as remarked a crucial decade of changes from classical to modern Algebraic Geometry, Oscar Zariski came on unirationality. He constructs *unirational non rational* surfaces over an algebraically closed field k *with char* $k = p > 0$. This is done via *supersingular K3 surfaces*.

For instance let $p = 3$. Then the equations $x = u$, $y = v$, $z = \sum_{i+j\leq 4} u^{\frac{i}{3}} v^{\frac{j}{3}}$ are a rational parametrization of a K3 quartic surface, defined by the equation $x^2 + y^2 - z^3 + x^4 + y^4 = 0$. This is an example, in characteristic 3 of a unirational non rational algebraic variety. Nothing of this belongs to the classical age! In particular Castelnuovo rationality criterion fails in positive characteristic.

Fig. 11 La belle montagne K2 au Cachemire. Source: Wikipedia.https://it.wikipedia.org/wiki/File:K2,_Mount_Godwin_Austen,_Chogori,_Savage_Mountain.jpg

Riposte Armonie Zariski's book *Algebraic Surfaces* (1971), commented by David Mumford and Joseph Lipman, is fundamental to understand more on the relations between classical and modern times. The birational classification of algebraic surfaces over any algebraically closed field k of any characteristic is due to Bombieri and Mumford. This classification is one of the modivations for their Fields medal in 1978. In it non classical Enriques surfaces appear, if k has characteristic 2. About the landscape on Algebraic Surfaces we can close as follows (Fig. 11):

> [...] allora, scherzando sulle difficoltà e le eccezioni che s'incontravano da ogni parte, si soleva dire che, mentre le curve algebriche (già composte in una teoria armonica) sono create da Dio, le superficie invece sono opera del Demonio. Ora si palesa invece che piacque a Dio di creare per le superficie un ordine di armonie più riposte ove rifulge una meravigliosa bellezza (Enriques 1949, pp. 463–464).

Classical Problems and Rational Parametrizations: Cubics

Higher Dimension: The Lüroth Problem Let $f\colon \mathbb{A}^d \dashrightarrow X$ be rational parametric equations of a complex algebraic variety *of higher dimension* $d \geq 3$. The question "is then X rational?" became outstanding in XXth century as the *Lüroth Problem*. It also became an overwhelming discussion: with fake proofs and counterexamples, due to some lack of theoretical foundations and therefore of corresponding tools to obtain rigorous results. In relation to this the classical Italian School, in spite of its glorious results, also stands as an example for such a weakness. Here is some humor about this historical passage:

> algebraic geometry developed by that school was, according to some critics, the only part of mathematics in which a counterexample to a theorem was considered a beautiful addition to it (Halmos 1985, p. 49).

Cubics and the Lüroth Problem In *dimension* ≥ 3 classical schools were not successful in birational geometry, and in the Lüroth problem, as on curves or surfaces. Cubics, that is varieties V defined in A^n by one cubic equation

$$F(X_1, \ \ldots, X_n) = 0,$$

represent since two centuries a crucial example. The rationality problem for a cubic V is indeed a central theme in modern history of Algebraic Geometry, unsolved for dim $V \geq 4$.

The Belle Époque of Cubic Surfaces A very brief historical remind on the rationality problem for V is useful to my purposes. Since a smooth plane cubic is of course not rational, We can start from smooth cubic surfaces in the complex projective.

1. The 27 lines on V and their configuration;
2. Double sixers of lines and rationality;
3. Rationality via double projection;
4. Sylvester pentahedral form of the equation F of V:

$$F = L_1^3 + \cdots + L_5^3,$$

where $L_1 \ldots L_5$ are linear polynomials. In 1911 Archibald Henderson wrote in his book:

> While it is doubtless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year to year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of their discovery of the twenty-seven lines upon the cubic surface (Henderson 1911, p. 1).

Gino Fano and the Cubic Threefold In his 1929 report *The Problem of the Cubic Variety in* $\mathbb{S}_4$, Snyder so writes on a smooth projective complex cubic threefold V, quoting Fano (1915):

> If the sections by $|\mathbb{S}_{n-1}|$ of a manifold M_3 in $\mathbb{S}_n$ are rational surfaces, M_3 is rational except possibly when M_3 is ${V_3}^3$ of $\mathbb{S}_4$, concerning which no conclusion can be drawn (Snyder 1929, p. 620).

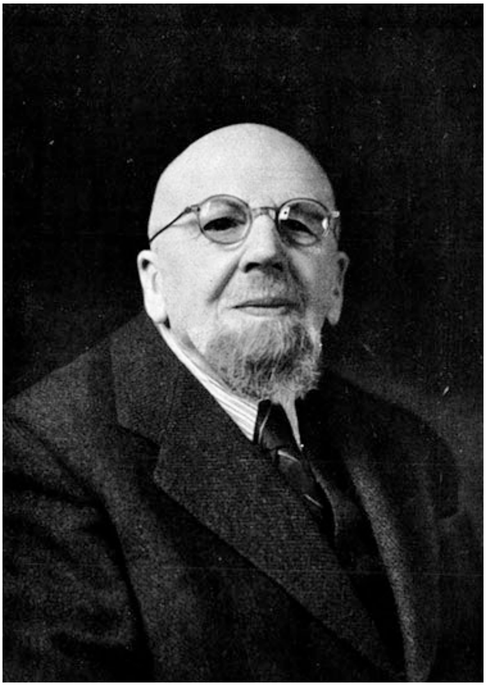

Fig. 12 Gino Fano. Source: Biblioteca digitale italiana di Matematica. http://www.bdim.eu/item?id=GM_Fano

The projective space P^{n} was denoted as $\mathbb{S}_n$ at that time. The unirationality of V, which is attributed to Max Noether, implies that Lüroth problem has a negative answer if V is not rational. Notice also that the unirationality of a cubic hypersurface V of dimension at least 3, defined over any perfect field and smooth over its closure, is true if V has a point over k. See Kollár (2001). Cubic threefolds bring us to the figure of Gino Fano (Fig. 12).

The scientific biography of Gino Fano (1871–1952) is prominent inthe classical XXth century birational classification in dimension 3: Fano threefolds are today part of this classification. The second half of his biography is, taking things wisely, dominated by the rationality problem for cubic threefolds.

In the Fano's archives of Torino's mathematical library one can read, written by hand on a reprint sent to Fano: "Al vincitore della V_3^3" ["To the conqueror of V_3^3 V_3^3"]. The latter denotes the cubic threefold V. Fano's proof of the non rationality of V is developed along some papers and memories. In particular it relies on the realization of a birational model of V which is a different Fano threefold W. Fano addresses the non rationality of W by a well known method, essentially started with him. This is based on proving that, so to say, W cannot contain certain big dimensional families of surfaces which are instead contained A^3. This is an obstruction to the existence of birational maps bewteen A^3 and V. More precisly the surfaces in the families to be considered are birational to K3 surfaces.

Fake Counterexamples Unfortunately his proof is not satisfactory for modern standards. It was criticized, in the early Fifties, by Leonard Roth in *Algebraic Threefolds* (1955), an interesting updating on classical methods.

It is curious that Roth proposes there a new example of *unirational non rational* complex threefold: this should be a hypersurface X, in $\mathbb{A}^4$, the hyperplane sections of which are sextic Enriques surfaces. Roth claims that this unirational X is not rational.

He claims that the non simple connectedness of the Enriques surfaces which are hyperplane sections of X implies that a smooth projective model of X is not simply connected. Hence X would be non rational. However X is unirational. Hence the statement of Roth is contradicted by the mentioned true theorem of Serre, Serre (1959), that *unirational implies simply connected*!

Counterexamples The negative answers to Lüroth problem for an algebraic threefold X came, finally and simultaneously, in the Seventies. This includes the first and original proof that a cubic threefold V is not rational. These negative answers are as follows:

1. *Every smooth cubic threefold is non rational*, Clemens-Griffiths.
2. *Every smooth quartic threefol is non rationald*, Iskovskih-Manin
3. *A suitable quartic double solid is non rational*, Artin-Mumford

The proofs of these negative answers to Lüroth problem are quite different. Nevertheless all represent, in distinct ways, the same extraordinary interplay: between the strength and precision of the new methods and the classical taste for geometry and vision, perceiving algebraic varieties almost like alive inhabitants in the Nature. It is suggestive about this to mention the notion of *general elephant* in the theory of Fano threefolds. It is however not the case of entering in further technical details about this or about the history of these results, which certainly deserves more attention.

For cubics V of dimension ≥ 4 most outstanding problems are still open. These are specially hot today, after a breakthrough due to Claire Vosin, see Voisin (2016), on stably rational threefolds and on hypersurfaces, related to Artin-Mumford example.

A second breakthrough followed, namely Hassett, Tschinkel and Pirutka proved that the property of being rational does not deform in a smooth integral family of integral varieties. This was a longstanding conjecture and it is still open for cubics of dimension ≥ 4.

It is time to conclude this too long focus on the fascinating, modern and classical, landscape of curves, algebraic surfaces and cubics we have considered.

The Classical Turn in Algebraic Geometry

We have seen that classical outstanding problems in some sense are, one could say, invariant: solved or not they mutate their time and shape but something stays unchanged.

Since we are in one of the best schools of France, Lycèe "Henri IV", (like in Rome we have for instance Liceo classico "Tasso" or "Visconti"), let me conclude quoting an interview on *licei classici* by the Italian philosopher Massimo Cacciari. About the meaning of the word *classico* he says: "*Classico non è ciò che appartiene al passato ma ciò che resiste al tempo*." This is true in Geometry as well.

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Knots, Diagrams and Kids' Shoelaces. On Space and their Forms

Luciano Boi

To Vincenzo

La forma universal di questo nodo credo ch'io vidi, perché più di largo, dicendo questo, mi sento ch'í godo.(Dante Alighieri, *La Divina Commedia,* Paradiso, Canto XXXIII, 1316–1321.)

O time! Thou must untangle this, not I; It is too hard for me to untie!
(William Shakespeare, *Twelfth Night,* Act II, Scene 2)

A diagram, so far as it has a general signification, is not a pure icon, but in the middle of our reasonings we forget that abstractness in great measure, and the diagram is for us the very thing. (C.S. Peirce, *Collected Papers,* 1931, 35.)

Abstract Knots and links are mathematical objects living in our three-dimensional space or in other ambient spaces, such as the 3-sphere and the four-dimensional space, and they possess a rich variety of properties and structures, some of them ore quite elementary while some other are very complex and difficult to grasp. Knot theory has extensive interactions, not only with different branches of mathematics, but also with various and fundamental areas of physics. Knots and links are deeply related to the geometry of 3-manifolds and low-dimensional topology, quantum field theory and fluid mechanics. We survey some current topics in the mathematical theory of knots and some of their more striking ramifications in physics and biology. The study of knots is essential to the comprehension of three- and four-dimensional spaces. Knot theory is of central importance in mathematics, as it stands at a crossroad of topology, geometry, combinatoric, algebra and mathematical physics. And it is likewise a key ingredient entering in the comprehension of dynamical systems, macroscopic physics and biochemistry. There are essentially four mathematical approaches to the study of knots and their polynomial invariants, combinatorial, as in the Alexander and Conway case, geometrical, via constructions

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called *Seifert surfaces*, algebraic, by considering the group of the knot, and physical, developed particularly by Witten, in which the Jones polynomial is interpreted and generalized using Chern-Simons theory. This article aims at stressing the importance of considering diagrams in the study of topological and geometrical objects and the key role of knot theory for the understanding of the structure of space and space-time.

Keywords Topology · Geometry · Three-manifolds · Knots · Links · Braids · Diagrams · Moves · Invariants

2010 Mathematical Subject Classification: Primary: 57M25, 57M27; Secondary: 57Q45, 58D10

Introductive Remarks: Shoelaces, Knots, and the Intuition of Space

Trying to tie the shoelaces is for children one of the first and most significant experiences of the intuition of space; in other words, the hand-movements and gestures needed for knotting shoelaces is a way for learning how space is made and build up (Daily-Diamond et al. 2017). We live this experience as well, because we have to show to our child tie shoe laces and pretty soon we teach him tie a square knot, the easiest knot to learn. Indeed, takes two ropes of different color and cross them (red over blue) to form a half not. Cross them a second time (red over blue again) and pull the ends tight to form the square knot. We learn afterwards that the granny knot is a binding knot, used to secure a rope or line around an object. It is considered inferior to the reef knot (or square knot), which it superficially resembles. Neither of these knots should be used as a bend knot for attaching two ropes together. We also learn that the *overhand knot* is the simplest type of knot and is used to make a knob in a rope, string, or cord. It is used for tying packages, to keep rope ends from fraying, and as a first step in making more complex knots such as the surgeon's knot and the square knot. An overhand knot is made by crossing the rope end around the standing part to form a loop, bringing the rope's end through the loop, and pulling the rope taut. A *slip knot* result when, in tying an overhand knot, a loop instead of the ropes ends is slipped through the first loop. And so on . . . Thus, learning to tying knot and understand how they move in our surrounding space is a key process of knowledge and an endless discovering of the stuffs and mysteries of space and time.

Thinking actively about things in our hands and about the movements required to perform an action on them or keep an object steady, is an essential step of our efforts for understanding (see Husserl 1997) the properties and structures of "living" space and thereafter of other more abstract spaces. These two stages are in fact intimately linked and in both geometric intuition not only deals with what you see with your eyes but what you see in the structure of things, at a more fundamental level (see

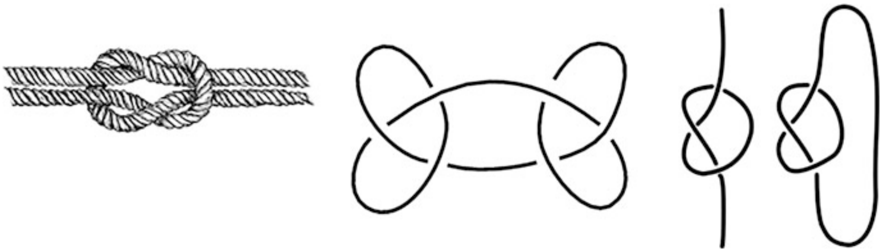


Fig. 1 (Links) A physical square knot. (Middle) A mathematical square knot. (Right) Overhand knot becomes a trefoil knot by joining the ends

Thurston 1998, and Gromov 2010); this powerful geometric intuition consists in taking things like surfaces, manifolds and knots in your hands and putting them in space, just playing with them; this "concrete" geometry deals with simple things, but then we have to learn how to project them to very higher dimensions.

In common usage, knots can be tied in string and rope such that one or more strands are left open on the either side of the knot. To a mathematician, an object is a knot only if its free ends are attached in some way so that the resulting structure consist of a simple looped strand (Adams 1994, and Boi 2005). Physical knots are known and used since longtime (knotted patterns already appeared in Roman floor mosaics in the third and fourth centuries AD, and the artistic use of interlaced knot patterns are found in Byzantine architecture and book Illumination), whereas the study of mathematical knots began about one century and a half ago with few mathematicians and physicists (Fig. 1).

These experience and knowledge have a more general meaning, for lead us to think of mathematics not as an austere and formal subject concerned with complicated and ultimately confusing rules for the manipulation of numbers, symbols equations, rather like the preparation of a complicated income tax return. Good mathematics is quite opposite to this. It is an art of human understanding. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie—or best of all, an integrated combination of several different representations.

The concept of knot is a marvelous example of deeply meaningful mathematics (Zeeman 1966, and Weber 2001). The study of knots and links showed that our three-dimensional space bears a multiplicity of (party correlated) geometries and topologies (Atiyah 1990). From the viewpoint of the geometry and topology conceived as a conceptual and imaginative effort for grasping properties and forms of the mathematical world, many objects and everyday things in nature, of the physical world as well as of the living world, reveal new appeal and complexity (Thurston 1997). Most of these objects and things, especially their shapes, might be mathematically related. Geometry and topology are alive and active things, rather than merely abstract or formal structures, related to our intuition of space and the understanding of the real world (Baez and Muniain 2006). The inner force that drives mathematics isn't to look for immediate applications, it is to understand structure and the inner beauty of mathematics. Mathematical results and findings can be very meaningful and sometimes astonishing. For example, knot theory is a mathematical subject that offer powerful geometric sights of 3- and 4-dimensional worlds on the one hand, and deep spatial visualization and imagination on the other (Bonahon 2002, and Boi 2009, 2014). Both, geometric estimation and spatial visualization are helpful tools allowing for a better adaptation of our organism to the varying environmental conditions. The favourite subject of this paper is the realm of three-dimensional manifold, their inner geometry and topology, and their intricate and multi-layered structures, because we believe that some of the most intriguing and profound mathematical mysteries of our human beings' life are cached by the dimensions three and four of space. And, in this sense, these "low" dimensions could have captured precious information and clues about our intuition, perception and grasping of space (Poincaré 1902, Gromov 2000, 2010, and Donaldson 1983).

The importance of geometry comes also from two other facts. The first is that geometry and especially the geometry of two and three dimensions lies closer to intuition more than all other mathematical subjects. Spatial intuition or spatial perception is an enormously powerful tool and that is why geometry is actually such an influential part of mathematics — not only for thing which are clearly geometrical, but even for those that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool. Geometry is essentially concerned with space, with concepts, ideas, objects, structures, and transformations in it, but also with time somehow, in particular when one thinks of geometry as something evolving in time, that is, a dynamical geometry. Thinking geometrically comes to think about the meaning of "beings", structures, and properties in/of space. This sort of geometrical thinking clearly underlines, for example, homology theory, which starts off traditionally as a branch of topology (Atiyah 1990, Hatcher 2007).

Nevertheless, the possible primacy of space in our perception by no means follows from the Kant's idea of the a priori inevitability of Euclidean geometry (Boi 2016). It often happens that mathematical subjects with strong roots in intuition are actually considerably more difficult than those wherein the structures are more abstract. Thus, for instance, we are at liberty to choose which of the more exotic infinite dimensional vector spaces we want to study; conversely, we have no choice about finite groups or whole numbers or low-dimensional manifolds. Even within topology, the manifolds forced upon us, those of low dimensions, have turned out, in the case of dimensions three and four, to be more intractable than those of higher dimensions that are more nearly of our own creation. This has been well demonstrated by Thurston's geometric program for understanding 3-manifolds (see Thurston 1982), analogous to the uniformization of 2-manifolds but by nature far more complicated and profound. Even in its partial form, this program contains mathematics of great power and significance. If and when it is completed, it will be one of the great achievements of mathematical thought.

Specifically, the comprehension of how to knot flexible objects (like one-dimensional strings and two-dimensional surfaces) in space in our three-dimensional space R^3, or even 3D *objects* in R^n, is perhaps the most primary intuition we can have of space. This accurate intuition allows for grasping the following insights (see Carter 1995, and Hemion 1992).

We cannot see the most the geometrical objects, but only visualize or imagine them in connection with the corresponding mathematical objects such as multi-dimensional space, invariants, connection, homeomorphism, homology, etc. As a first remark, we can say that "three-dimensional manifolds in R^4 are more difficult to visualize than two-dimensional surfaces in R^4, since the ratio of necessary information to visible information is much smaller. To take a very important example in our-day's developments of certain branches of pure mathematics, let's mention the topological knot theory and especially some of its fundamental facts, in order to show that a number of results related to this theory are completely counter-intuitive, although they need, to be grasped, a great deal of inventive intuition and mathematical imagination.

"To 'see' why knotted circles are key in topological understanding of three-dimensions, we need to know three basic mathematical facts about circles, which are completely counter-intuitive in the first sense ("empirical intuition") that we have attributed to the term "intuition". (a) There are no knotted circles in the plane. (b) There are many ways to knot a circle in R^3. (c) There are no knotted circles in R^4. In summary, R^3 has the unique property that in it we can knot circles (1-dimensional spheres). Let's give some other profound and counter-intuitive mathematical properties and results concerning knotted surfaces in four dimensions. (d) It is impossible to knot a sphere in R^3. (e) There are non-trivial knotted spheres in R^4. However, it is impossible to knot a sphere in R^5. Thus, R^4 is the unique dimension for which spheres can knot. (f) An n-dimensional sphere can be non-trivially knotted in a piecewise linear manner in R^m, if and only if $m - 2 = n$ (see Roseman 1997, 68).

Exploring and Visualizing 3-manifols and the Importance of Topology

Let us start with some elementary facts (see Rourke 2006). Topology is the study of spaces under continuous deformation, technically called *homeomorphism*. It arose from Poincaré's work at the end of the nineteenth century (Poincaré 1985, 1904). The 1904s paper, in which Poincaré posed the problem whether any closed, simply connected 3-manifold is homeomorphic to the 3-sphere, marked the founding of topology as an independent discipline within pure mathematics (see Bessières et al. 2010). And its work has been one of the major strands of mathematics through the twentieth century and the beginning of the twenty-first (Prasolov and Sossinsky 1997).

The most important examples of topological spaces are manifolds. A manifold is a space which is *locally* (that is in its infinitesimal regions) like ordinary space of some particular dimension. Thus a 1-manifold is locally like a line, a 2-manifold is locally like a plane, a 3-manifold like ordinary 3-dimensional space, and so on. In order to avoid pathological example, it is important to say that a manifold has some additional structures, which tames it; the principal extra structures which are assumed are: (a) differential (Diff) structure—namely the capability to do analysis on the manifold and hence to apply differential calculus— and (b) piecewise-linear (PL) structure, which is roughly equivalent to assuming that the manifold can be made of straight pieces. For instance, a knot (i.e., an embedding of the curve C^1 into 3-dimensional space R^3) is *tame* if can be formed of a polygonal finite sequence of straight segments (see Rolfsen 1990 and Kauffman 1983, 1991, for an illuminating introduction to knot theory). In low dimensions (2, 3, 4, and in any case << 7), these two seemingly opposed structures are in fact equivalent, and we shall freely assume either of them as necessary. In dimension 3 TOP = PL = DIFF (this is an important result discovered and proved by E. E. Moise 1977), i.e. each topological 3-manifold admits a unique PL/DIFF (that is, polygonal/smooth) structure. This is not true in higher dimensions.

We can explore manifolds by starting in low dimensions and working upwards. Contrarily to what was commonly believed among mathematicians until the late 1960s, dimensions 3 and 4 are more complex, both geometrically and topologically, than the higher dimensions. One reason that high dimensional manifolds are easier to study than ones of dimensions 3 and 4 is that in them there is enough room to move sub-manifolds (e.g., loops and surfaces) around and put them in good position with respect to each other, whereas in lower dimensional manifolds this is not possible. Another reason is that the dimensions 3 and 4 of space are deeply linked with our actual life. On the one hand, with some important constraints grounded on the physiological movements of human body, which combine and interpolate rotations and translations in space (see Gromov 2010). These movements form a rich group of transformations. More precisely, they form a Lie group $SE(3) \times \ldots \times SE(3)$, where $SE(3)$ is the special Euclidean group; each configuration of the human body can be seen as an orbit in the Lie group (see Rohrer 2007, and Zhoce 2018). On the other hand, low dimensions are related with our perception and action, which very possibly enrich of new features the intuitive and mental structures we ascribe to space and time (see Berthoz 1998, and Paillard 1991). Let's return to manifolds and their dimensional particularities. The dimension 1 is very likely the simplest, although not at all trivial. If we assume that our manifold has a PL structure, then it must consist of a number of intervals laid end to end. If we also assume that it is connected (all in one piece) then it either comprises an infinite collection of intervals forming a line or it comprises a finite collection of intervals forming a (closed) circuit. Thus, a connected 1-manifold is homeomorphic either to a line or to circle. Of these only the circle is compact (a technical notion equivalent to being made of a finite number of straight pieces[1]). So, there is just one compact, connected 1-manifold up to homeomorphism.

[1] A more general definition is the following: a space is compact if it is of finite extension and hasn't boundaries.

When working in the category of differentiable manifolds, we will then assume for simplicity that all 2- and 3-manifolds are orientable. (Note that a surface is called *orientable* if each closed curve on it has a well-defined continuous normal field. The sphere and cylinder are examples of orientable surfaces, whereas the Möbius strip and Klein bottle are well-known examples of non-orientable surfaces.) Topologically, one can classify surfaces and higher-dimensional spaces by their *genus*; that is, by the number of holes or handles attached to the surface. Thus, the classification theorem for surfaces says that

Theorem 1 *Any closed, connected orientable surface is topologically equivalent to a sphere with a certain number of holes or handles*; in other words, *it is exactly one of the following surfaces: a sphere, a torus, or a finite number of connected sum of tori*.

It is worth noticing that "the classification theorem is a beautiful example of geometric topology. (...) It is the sort of mathematics that could be thought in schools both for foster geometric intuition, and to counteract the present-day alarming tendency to drop geometry. It is profound, and yet preserves a sense of fun. Examples of surfaces in 3-dimensions are the source of our intuition" (see Zeeman 1960, and Scott 1983).

For the category of differentiable (orientable) manifolds of C^{∞}, the previous definition/theorem can be translated in the following one:

Definition 1 *A geometry is a simply-connected homogeneous unimodular Riemannian manifold X. Unimodularity means that X admits a discrete group of isometries with compact quotient*.

Following Klein's vision (exposed in his *Erlangen Program*),[2] we can also identify geometry with its group of isometries.

Definition 2 *A compact manifold M is called* geometric *if int* $(M) = X/\Gamma$ *has finite volume, where X is a geometry and* Γ *is a discrete group of isometries of X acting freely* (see Thurston 1982, and McMullen 2011).

Let's now stress that one of the most fascinating challenges of mathematics is the understanding of three-dimensional manifolds, by translating topology into geometry on the one hand, and by extracting topological information from geometric structures on the other hand (see Scott 1983, and Thurston 1997, 1998). For example, we can obtain a better understanding of the geometric variation of

[2] In his "Program" (Lecture given at the Erlangen University to obtain a professorship in Mathematics) Felix Klein relates the different geometries known at the time, both Euclidean and non-Euclidean, to the notion of transformation groups and their invariants. Klein's basic idea is that each geometry can be characterized by a group of transformation which preserves elementary properties of the given geometry. It has been the first tentative of giving a *classification* of various geometries through the notion of group, both continuous Lie groups and discrete groups. His unified way of looking at different geometries has had a fundamental impact in contemporary mathematics and physics.

Fig. 2 The simplest Calabi-Yau manifold: a torus with circles passing through its hole as sub-manifolds. (From S.-T. Yau and S. Nadis, *The Shape of Inner Space*, 2010, 176)

the curvature and the metric of a manifold from the classification of its possible deformation (up to equivalence). Conversely, the fact that we may follow the evolution of the metric of a given manifold in a finite time, gives us precious indications about the topological shape (up to deformation) that such a manifold can take. Even though a lot remain to be done, we already have an excellent working understanding of 3 manifolds.

A major purpose of Geometry is to describe and classify geometric structures of interest. We see many such interesting structures on our-to-day life, and other, which we cannot see, exist at different bigger (macroscopic) or smaller (microscopic) scales. For instance, Calabi-Yau manifolds discovered by mathematicians and physicists in the attempt to explain some striking features of physics at the quantum level of space-time, which are supposed to "exist" at the Planck scale, can be imagined but not seen because its extra dimensions are compacted around the usual dimensions, and hence hidden (see Guadagnini et al. 1990, and Witten 1988). However, one can visualize a bi-dimensional section of it; yet to imagine a Calabi-Yau space, which is a kind of complex Riemann surface with a large number of holes, handles and knots, amounts to visualize it mentally or by giving approximate pictures of some sub-spaces contained in it. For instance, it is not too hard to visualize one of the simplest examples of Calabi-Yau manifold, namely a torus with all the loops (circles) passing through its hole (Fig. 2).

Returning for a while to a more philosophical remark, we can say that our spatial imagination, aided (but not substituted) by computer, is a critical tool for the human mind to "see"—better, to visualize—the kinds of geometry that are needed for 3-dimensional topology. The formal construction is only one aspect of the problem. As Thurston has stressed in a thoughtful article (1998), we would indeed improve our understanding about the phenomenology of 3-manifolds addressing the question "What are 3-manifolds like?", rather than "What theorems can currently be proved about 3-manifolds?". We can so further ask: "May there be an overall structure for all 3-manifolds?" If yes, what can be the backbone this structure is made of?

In mathematics, human thinking and understanding doesn't work on a single pathway, like a computer single central processing, because our brains and minds seem to be organized in a variety of distinct hitherto integrated powerful capabilities, which work together loosely, "talking" to each other at high levels rather than at low levels of organization. For mathematical thinking, it is certainly important humane language (for grasping and formulating coherently notions and ideas), logic and deduction (for asserting and developing in a rigorous way our thoughts), process and time,[3] intuition,[4] imagination, analogy, and metaphor. But still more important is vision, spatial sense, and kinaesthetic sense (motion)—as already Poincaré and Husserl pointed out.[5] As argued by Thurston, we have facilities for taking in information visually or kinaesthetically, and thinking with their spatial sense. "We can think spatially at different scales: we can think about little objects in our hands, or we can think of bigger human-sized structures that we scan, or we can think of spatial structures that encompass us and that we move around us" (Thurston 1998, 5). In topology the visual facilities play a very important role for our understanding of the inner and overall properties of two-dimensional surfaces and 3-dimensional spaces. We can visualize the *genus* of a surface, which is the number of holes or handles of the surface. Another related notion we can visualize is the *connected sum* of two surfaces S_1 and S_2, denoted by $S_1 \# S_2$, through which we can construct new (more complex) surfaces connecting S_1 and S_2 along some deleted disk of each surface. Obviously, visualizing the connected sum of 3-manifolds is a harder task (Hilbert and Cohn-Vossen 1999, and Lickorish 1997).

The difference between *homeomorphism* and *diffeomorphism* is pervasive of many fundamental domains of mathematics. Roughly speaking, homeomorphic means "to have the same shape". The idea is that two surfaces in 3-space are homeomorphic if we can deform one into the other continuously, that is without cutting and self-intersections. For instance, topologically the sphere, the surface of a rugby ball and the cube are homeomorphic; the same hold for a doughnut, a pear with a big hole and a coffee cup (see Fig. 3). To be more precise, two surfaces or spaces are homeomorphic if there is a continuous bijection between them having a continuous inverse.

[3] We have a facility for thinking about processes or sequences of actions unfolding in time that can often be used to good effect in mathematical reasoning.

[4] By intuition we "see" (clearly perceive) something without knowing where it come from, something "invisible" or "hidden", we can also imagine surface' and space' which behaviour in an unusual way and whose properties are very different—think of the so-called "pathological mathematical objects", such as Alexander horned sphere and Antoine's necklace (a topological embedding of the Cantor set in 3-dimensional Euclidean space, whose complement is not simply connected.)

[5] See Poincaré (1905), and Husserl (1907). More recently, R. Thom (1992) has emphasized this point, and notably the fact that to "see" or to visualize amount to create geometrical objects or forms which allows to have a mental model of the phenomenon one want understand. This mental form is essentially inspired and built up by intuitions and analogies.

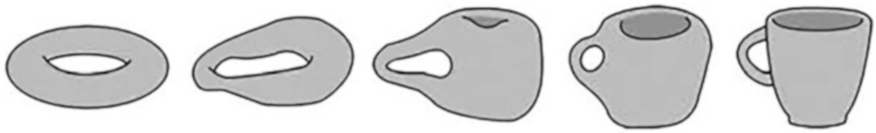


Fig. 3 Deforming a doughnut (a mathematical torus) into a coffee cup (a mathematical surface with one handle)

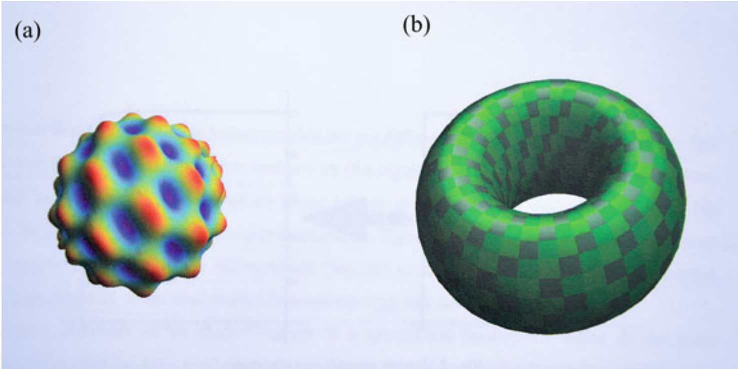


Fig. 4 (**a**) A smooth surface diffeomorphic to the sphere. (**b**) A smooth surface (a single-holed torus) not homeomorphic to the sphere

However, from another point of view the cube and the sphere geometrically do not have the same shape: the cube has edges and corner, whereas the sphere is smooth everywhere. Being smooth is an identifying characteristic of all manifolds; in particular, the cube is not a *smooth* manifold. The rugby ball, on the other hand, is. Thus, the rugby ball and the sphere still have the same shape as smooth manifolds, or, in technical jargon, they are *diffeomorphic* (where "*diffeo*" here comes from "differentiable", to recall that the smoothness can be expressed in terms of derivatives) because we can deform one onto the other without cutting, self-intersecting and in such a way that all the intermediate surfaces in the deformation are still smooth (Fig. 4). Formally the definition is that

Definition 3 *Two smooth surfaces are diffeomorphic if there is a differentiable bijection between them have a differentiable inverse.*

Now it is hard to find a smooth surface homeomorphic but not diffeomorphic to the sphere; and, indeed, it does not exist. In other words, any smooth surface homeomorphic to S^2, in 3-space, is diffeomorphic to it.

It is then natural to conjecture that this holds in any dimension: *any smooth n-dimensional manifold homeomorphic to S^n is diffeomorphic to it*, where n is any positive dimension. This statement is now known as the *smooth Poincaré conjecture*, because similar in nature to the much more famous *topological Poincaré conjecture* stating that

Poincaré Conjecture *Any n-manifold homotopically equivalent to the n-dimensional sphere S^n is homeomorphic to it.*

Roughly speaking, two manifolds are homotopically equivalent if one can be deformed onto the other without cutting but allowing some identification; for instance, a cylinder is homotopically equivalent to a circumference (just squeeze the cylinder onto a base circumference; this operation does not change the number and nature of the holes, which is what homotopy measures.) In 2003, G. Perelman has settled the topological Poincaré conjecture in the positive: *an n-manifold homotopically equivalent to S^n always is homeomorphic to it* (see Boileau 1997).

In the fifth's and sixth's most mathematicians took the smooth Poincaré conjecture for granted. So, J. Milnor discovery (1956) stunned the mathematical community. He showed something very new and profound, precisely, that in dimension 7, *there is a 7-manifold which is homeomorphic (topologically equivalent) but not diffeomorphic (differentiably equivalent) to a sphere: it has creases that cannot be ironed out*.

This 7-manifold is called an exotic sphere: at first glance it looks like a sphere, but becomes positively strange under close examination. The Milnor's result shows that the topological structure does not completely determine the differentiable (smooth) structure; and so, the field of differential topology, i.e. the study of the relationship between topology and differentiability, was born (Thom 1954, and Haefliger 1961).

Equivalence of Images and of Forms: Manifolds, Knots and Diagrams

The most important properties of surfaces and spaces appear during the processes of *embedding* and *immersion*, and can be elucidated thanks to the concepts of *homeomorphism* and *isotopy* (see Papi and Procesi 1999, and Lescop 2012). One very relevant fact, from the topological point of view, is that two objects or surfaces may have the same "form" and nevertheless correspond to (at least) two different diagrammatic representations. In other words, a surface or a space can have different concrete realizations (see Kauffman 2005a, and Boi 2011b). This fact shows first of all that the *equivalence of forms* has a topological meaning much more important than the simple *equivalence of images*. Let's clarify this formal and physical difference with respect to two families of objects or surfaces, the first being knotted and the second unknotted. In fact, the knotted-like form is a property that essentially depends upon the kind of three-dimensional space in which these knotted or unknotted circles or surfaces are imbedded (see Figs. 5, 6, and 7). We think that the study of objects and the spatial environment in which they allow for different types of deformations is deeply correlated with the understanding of the dynamic transformations and the new revealed properties of these objects and spaces.

Mathematically, a knotted manifold is defined to be the image of an embedding of a compact manifold into R^n for some $n \geq 3$ (see Boi 2006a, Burde and Zieschang

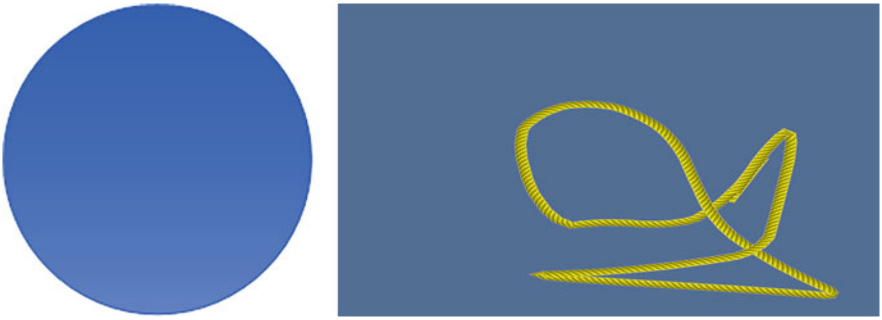


Fig. 5 If we embed a circle (or circumference) C^1 in three-space R^3, then we obtain a knot (like the one pictured above), which is the *image* of the embedding. Here the knot, of course, is not represented as a tame and *symmetric* knot (see later for definitions), rather as a sort of soft ("alive") thing undergoing slight deformations (but we know that a knot with a slight deformation is still a knot)

Fig. 6 A physical knot

1985, Crowell and Fox 1977, for further and more formal definitions). The isotopy equivalence class of a knotted manifold will be called a *knot*. If we have one circle in R^3 this knotted manifold is called a *knotted circle*. If we have several *circle* in R^3 this knotted manifold is called a *link*. It is easy to create examples of knotted circles (or embedded circles) in R^3. A knotted surface is a knotted 2-dimensional manifold, with empty boundary, in R^4. This can be considered as a smooth submanifold or a piecewise-linear manifold. That is, a two-dimensional simplicial complex that approximates a smooth knotted manifold. A knot is called *tame* if it admits a polygonal presentation (Figs. 8 and 9).

Generally, surfaces in R^4 and even in R^3 are difficult to describe and still more to visualize. Knotted circles are an important tool in the study of the topology of R^3. It

Fig. 7 An artistic vision of a mathematical knot (by Jorge Eduardo Eielson)

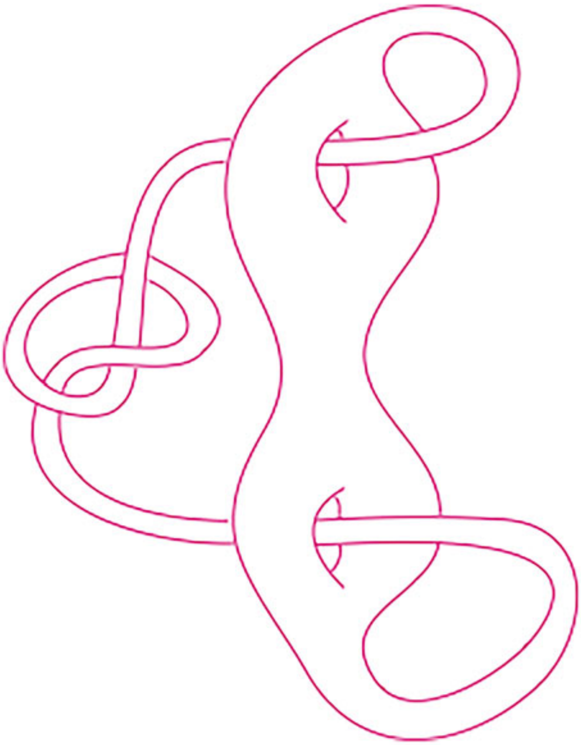


Fig. 8 A knotted Riemann surface of genus 2

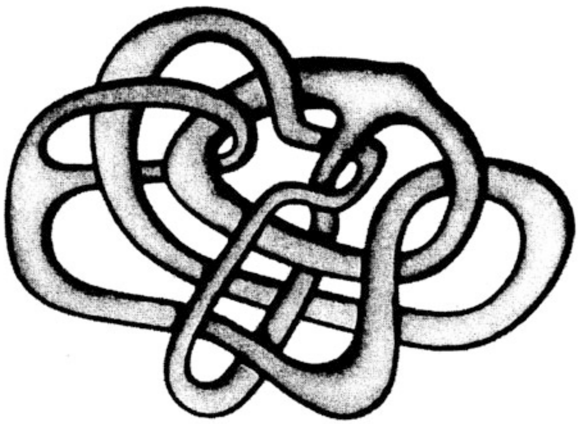


Fig. 9 A self-knotted (tubular) soft 'pathological' manifold of genus 2 (Even if the picture resembles the form of an octopus, a soft-bodied, eight limbed mollusk of the order Octopoda, it would be inappropriate to speak of an "octopus-manifold", precisely because octopuses never seem get tangled up in their own limbs. Differently from the octopus, the hagfishes are able to tie their bodies into complicated three-twists knots to escape tight spots)

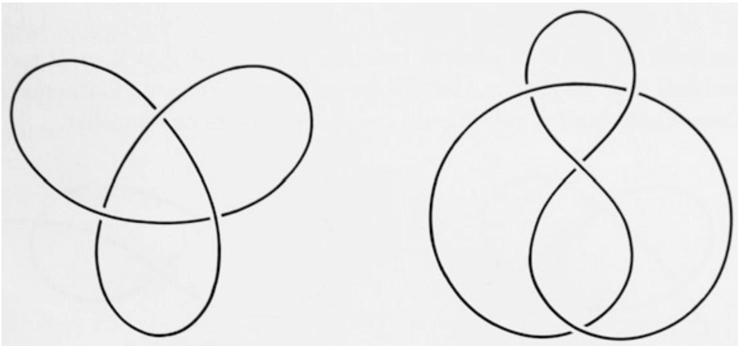


Fig. 10 Mathematical knots: The trefoil knot and the eight-figure knot

is central to the investigation and perception of three-dimensional objects. It is also of interest to chemists and biologists since molecules (including DNA) have been found with a knotted structure (Cozzarelli 1992, and Sumners 1992). Furthermore, properties of these molecules seem to depend on the "knottedness" and its associated numerical and/or topological invariants like the *linking number*, the *writhe number*, and the *twist number* (Kauffman 1991, and Lackenby 2009). In physics, remarkable connections have been found between quantum physics and knots (Witten 1988, Turaev 1994, and Boi 2009), especially to knot diagrams and their representations (Fig. 10).

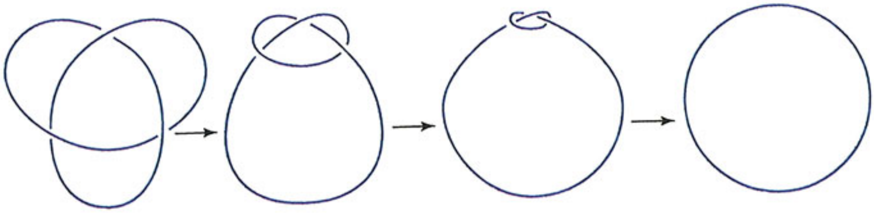


Fig. 11 By a finite sequence of ambient-isotopy deformations, the trefoil knot *K* transforms into the unknotted circumference (the trivial knot)

Informally, a knot is a closed piece of string in space. More formally,

Definition 4 *A knot is a (globally injective) embedding of the circumference S^1 in the Euclidean 3-space R^3 so to bring the first knot exactly onto the second knot (or its mirror image).*

Definition 5 *Two knots are equivalent if there is a homeomorphism of R^3 (i.e., a bijective continuous transformation of the space onto itself with a continuous inverse) transforming the first knot in the second.*

In particular, a knot equivalent to the standard unit circumference in the plane is actually unknotted, and thus considered a trivial knot (Fig. 11).

In some sense, the work of mathematicians is very similar to that of entomologists: they are interested in classifying things in some classes and then to found equivalences and differences between them. For example, one would like to have a list of all possible knots (up to equivalence, of course). The usual way for representing a knot consists in projecting it onto a plane so that the projection crosses itself in a finite number of points, and only two strands of the knot pass through any crossing point. So, one may look for the projection with the least number of self-crossings of a given knot (or, more precisely, of all equivalent forms of a given knot), and tray to organize the knots according to this least number of self-intersections. For instance, the trivial knot clearly admits a representation with no self-crossings: the standard circumference. The first non-trivial knot is the trefoil knot, whose representation has exactly three self-crossings. Here *non-trivial* means that it is not possible to "untie" a trefoil knot in three dimensions without cutting it. Mathematically this means that a trefoil knot is not isotopic to the unknot. In particular, there is no sequence of Reidemeister moves that will untie a trefoil (Reidemeister 1933, and Przytycki 1997). Proving this requires the construction of a knot invariant that distinguishes the trefoil knot from the unknot (Gordon 1978, Coward and Lackenby 2011). The simplest such invariant is *tricolorability*: the trefoil knot is tricolorable, but the unknot is not (Fox 1961, and Cromwell 2004). There are knot polynomials that distinguishes the trefoil knot from an unknot, and other do not distinguishes them (Figs. 12 and 13).

Fig. 12 (Links) A trefoil knot without (visual) three-fold symmetry. (Middle) A left-handed knot with three-fold symmetry. (Right) The trefoil knot is tricolorable

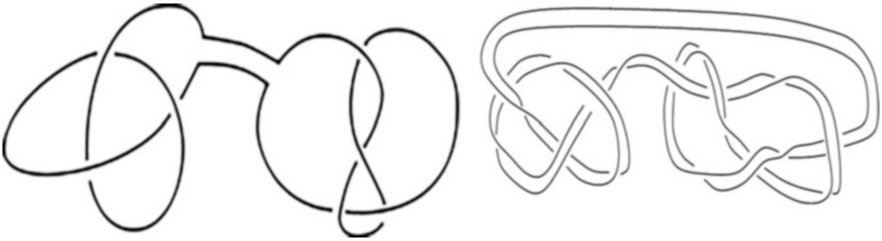


Fig. 13 A collected sum of a trefoil knot and a figure-height knot, and a cable of a trefoil knot

The trefoil knot has many other interesting mathematical properties, in particular (see Fox 1961, Przytycki 1997, Khono 1990): (a) The trefoil knot is the first non-trivial knot, and it is the only knot with crossing number 3 K_1^3. It is a *prime* knot, that is, a non-trivial knot which cannot be written as the knot sum of two non-trivial knots; hence, it is indecomposable. (b) it can be described as the (2, 3)-torus knot. It can also be obtained as the closure of the braid B_1^3. (c) it presents a three-fold symmetry. (d) any continuous deformation of the curve forming a trefoil knot is also a trefoil knot. All these deformations form an isotopy class whose *type* is the trefoil knot. In addition, the mirror image of a trefoil knot is also considered be a trefoil. (e) The trefoil knot is *chiral*; hence it is different from its own mirror image. The two resulting variants are known as the left-handed trefoil and the right-handed trefoil. It is not possible to deform the first continuously into the second, and vice versa; this means that the two trefoil knots are not ambient isotopic. (f) Though chiral, the trefoil knot is also *invertible*, meaning that there is no distinction between a counter-clockwise-oriented and a clockwise-oriented knot. Hence, the chirality of a trefoil knot depends only on the over and under crossings, not on the orientation of the curve. (g) The trefoil knot is an *alternating* knot. However, it is not a *slice knot*, meaning it does not bound a smooth 2-dimensional disk in the 4-dimensional ball. (Recall the definition: *A knot* $K \subset S^3$ *is slice if it bounds a nicely embedded 2-dimensional disk* D *in the 4-ball* B^4). (h) The trefoil is a *fibered knot*, meaning that its complement in S^3 is a fiber bundle over the circle S^1. The exact definition is the

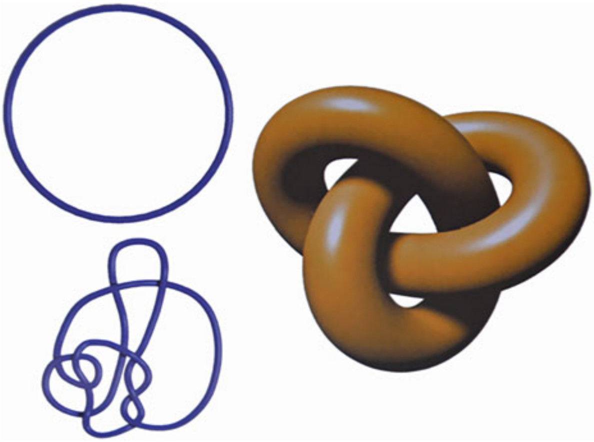


Fig. 14 If the curve is unknotted (links bottom) then it is equivalent to a circle (links top). On the right a tubular trefoil knot obtained from a thick torus (or inner tube) by continuous deformation or homeomorphism

following: A knot or link K in the tree-dimensional sphere S^3 is fibered if there is a one-parameter family F_t of Seifert surfaces (i.e., surfaces whose boundaries are given knots or links) for K, where the parameter t runs through the points of the unit circle S^1, such that if s is not equal to t, then the intersection $F_s \cap F_t$ is exactly K. The trefoil K may be viewed as the set of pairs (z, w) of complex numbers such that $|z|^2 + |w|^2 = 1$ and $z^2 + w^3 = 0$. Then this fiber bundle has the Milnor map $\phi(z, w) = (z^2 + w^3)/|z^2 + w^3|$ as the fiber bundle projection of the knot complement $S^3 \setminus K$ to the circle S^1. (i) The knot group of the trefoil is isomorphic to the braid group with three strands B_3 (see Artin 1947, Cartier 1990) (Fig. 14).

The information about how surfaces knot in three-dimensional space, viewed mathematically, result in a structure called a *knot diagram* (Boileau 1997, Carter 1995, Boi 2011b). In the mathematical study of knotted circles in three-dimensional space, the basic question is: given two knots, are they equivalent? If knots are equivalent, we generally distinguish them by means of topological or algebraic invariants, such as the knot group (Neuwirth 1965, Reidemeister 1927). There are many such invariants, but most are derived from diagrams of a knot. On the other hand, if knots are equivalent, then we need to find an isotopy that shows this equivalence (Calugareanu 1961, Kauffman 1990, Prasolov and Sossinsky 1997). Thus, symbolic representation combined with physical operations, are an important part of mathematics. The role of diagrams notably in geometric topology is hence very important. In most of cases, these diagrams are not used as "illustrations", but as symbolic and concrete operations of a topological and algebraic nature. Furthermore, these algebraic diagrams are able to translate (or to reveal) some striking topological properties of the knot under examination. On the other hand, these topological properties, once known, may lead to the discovery of new algebraic invariants of knots.

Then, a knot diagram isn't simply a "picture of a knot"; it is rather a powerful symbolic and conceptual representation of the knot, useful for further advances into the knowledge of its mathematical properties. As we said, one can use the knot diagram directly to calculate many geometric and algebraic invariants of the knot. The knot group is one such invariant. If M is a sub-manifold of n-dimensional space (or the n-dimensional sphere) the *knot group* of M is defined to be the fundamental group of the complement of M (see Rolfsen 1990, and Boi 2006a). Operations on a given diagram can yield a family of related knots, the so-called *knot skeins*, and may be used to calculate a variety of knot invariants. As L.H. Kauffman so clearly pointed out in (Kauffman 2005a, b): "knot diagrams and geometric manipulations of them provide powerful mathematical results—new topological invariants and new insights into previous ones".

Let's return briefly to the important question of the visualization of geometrical objects. Many results related to topological knot theory are, in a sense, completely counter-intuitive, nevertheless, to be grasped they need a great deal of creative intuition and mathematical imagination. For instance, an important theorem by E. C. Zeeman (1960) states that *spheres are unknotted only in five-dimensions space*. The result generalizes to *unknotting* S^n in k-space, $k > (3/2)(n + 1)$. André Haefliger (1961/62) has shown that *differentiable embedding of the 3-sphere* S^3 *in Euclidean six-dimensions space* E^6 *can be differentiably knotted*. Zeeman proved that any piecewise linear embedding of S^n in the k-dimensional Euclidean space E^k is combinatorially unknotted if $k \geq n + 3$. From these two results it appears that there exists a very significant difference between the *differentiable* and *combinatorial* theories of isotopies. One proves that S^3 can be combinatorially unknotted in E^6. The prof is similar to that of unknotting S^2 in E^5, although it does involve one new idea, that of "severing the connectivity of the near and far sets" (see Zeeman 1960, for details.) Formally the result can be stated as follows:

Theorem 2 *Given a piecewise linear embedding of a combinatorial 3-sphere* S^3 *in Euclidean 6-dimensions space* E^6*, then it is unknotted, i.e. there is a piecewise linear homeomorphism of* E^6 *onto itself, throwing S onto the boundary of a 4-simplex.*

The argument of the prof consists in showing that S^3 is equivalent by cellular moves (precisely three moves) to the boundary of a 4-ball. The definition of a cellular move is as follows:

Definition 6 *If T is another 3-sphere in* E^6*, we say that S is equivalent to T by a cellular move across the 4-ball Q, written S ~ T across Q, if the interior of Q does not meet S, T, and Q is the union of the two 3-balls* $Cl(S - T)$, $Cl(T - S)$.

In recognition of the special nature of geometry, Zeeman observes: "geometry is that part of mathematics that relies on pictures, visual imagination and intuition to suggest the theorems, construct the proofs and inspire the conjectures" (Zeeman 1966, 5). This means that it is worthwhile to think geometrically to visualizing examples of how transformations of surfaces and things dynamically occurs in space, and then to construct mental pictures of such transformations and of the ways in which space itself changes under the action of them (see Carter 1995, Boi 2016, and Kamada 2017). To this purpose, we can use dynamic thought experiments to imagine configurations not only in two and three dimensions, but also in higher dimensions. Experiencing configurations in higher dimensions suggest the intuition that, as the dimensions of spaces increases from a one-dimensional line to a two-dimensional surface to three-dimensional space, we have more and more space to move things and figures around. This enables to visualize and mentally pictures configurations in four dimensions and above in ways that are of great power and significance (Thurston 1998, Kervaire 1965).

Geometrical thinking clearly underlies, for example, homotopy theory and the Ricci flow for compact manifolds, which allow to define the genus (see Seifert 1934, Milnor 1950), which is an invariant, of manifolds, and to classify 3-manifolds by geometric methods and topological surgery (Bing 1983, Haken 1968). The idea of homotopy is a key one. Let us explain the idea in a concise manner (here we follow closely Rourke 2006). The first step was to define the concept of the fundamental group, introduced by Poincaré at the beginning of the last century (Poincaré 2010). We have to consider loops in a manifold. A loop is just a path that starts and ends at a chosen base point. We may add loops, i.e. by making a new path which is the first path followed by the second. The addition makes the set of loops into a group. We need to allow loops to vary continuously; this operation is called *homotopy*. Then the homotopy classes of loops under loop addition defines the fundamental group. Having trivial fundamental group is the same as having that every loop is shrinkable to a point, or homotopic to the constant group.

The next step is to define spheres in any dimension. S^n denotes the n-sphere. The 3-sphere S^3 can be defined using 4-space, i.e. ordinary 3-space with an extra point thrown in. S^3 is really a 3-dimensional object. As we saw, Poincaré conjectured (in Poincaré 1904) that a closed connected 3-manifold whose fundamental group is trivial, $\pi_1(M)$ is homeomorphic to S^3, $M^3 \cong S^3$. For two-manifolds (surfaces) the conjecture makes sense in exactly the same terms: every closed connected 2-manifold whose fundamental group is trivial is homeomorphic to S^2 (Hemion 1979). The other well-known 2-manifolds, as the torus and the projective plane, all have non-trivial group. In dimension 4, it is possible to construct compact connected manifolds with trivial fundamental group which are not the 4-sphere (Freedman 1982, Kamada 2017). One simple example is the complex projective space $\mathbf{C}P^n$.

Embeddings and Isotopies

In order to understand the mathematical signification of a *knot* we first need some basic knowledge about the concept of *embedding*, namely the fact that, X and Y being topological spaces, an *embedding* of X into Y is a one-to-one bi-continuous map of X into Y—that is a homeomorphism of X onto $f(X)$. Further, the fact that, if X and Y are smooth manifolds, an *immersion* of X into Y is a smooth function, f, from X into Y that is locally one-to-one but not necessarily globally one-to-one. Equivalently, we can define an immersion as a smooth map whose differential, df,

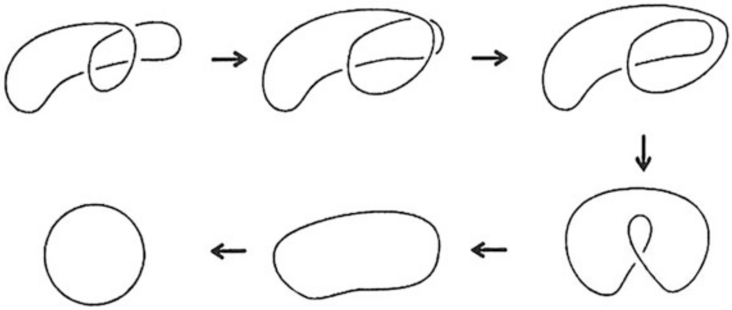


Fig. 15 Steps of the isotopies deforming a trefoil knot into the unknot

is non-singular. An immersed manifold is an image of an immersion. For example, the limaçon, the graph in polar coordinates of $r = 1/2 + \cos\theta$, is an immersed circle in the plane. Another important example, the most familiar image of the Möbius band is a "circular band with a half-twist"— call this subset M_1. This is an image of one embedding of this 2-dimensional manifold into ambient space R^3. Recall the counter-intuitive fact that this manifold is non-orientable.

Let's now introduce the concept of *isotopy*, which is another fundamental concept of knot theory (Birman 1993, Crowell and Fox 1977, and Bonahon 2002). From a topological point of view, it allows to define the equivalence relation of manifolds or subsets of R^n. Two subsets A and A of R^n are isotopic if there is a non-rigid motion in R^n which takes A_1 to A_2. That is, there is an object, A, and a continuous family of embeddings, $f : A \to R^n$ with $f_1(A) = A_1$ and $f_2(A) = A_2$, where $t \in I$, I being the unit interval. Such a map will be called *an isotopy of A*. Two subsets of R^n are equivalent if there is an isotopy between them. Consider a Möbius band in R^3, M (a circular band with a half-twist). Now, let M_2 be a similar example, yet different in that the circular band has three half-twists". We can show that M_1 and M_2 are *not* isotopic. To prove that, we have to consider that, if they were equivalent, then the corresponding boundary sets would be equivalent as circles in R^3. But the boundary of M is a trivial knot and the boundary of M_2 is a non-trivial knot called a trefoil knot (Fig. 15).

Mathematical Propaedeutic for the Understanding of Knots

A systematic study of knots in R^3 was begun in the second half of the nineteenth century by Tait and his followers. They were motivated by Kelvin's theory of atoms modelled on knotted vortex tubes of ether. The theory of knots grew as a subfield of combinatorial and algebraic topology (Boi 2005). Recently new invariants of knots have been discovered and they have led to the solution of long-standing problems in knots theory (Jones 1985, Kauffman 1988, Lickorish and Millet 1987, and Turaev

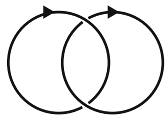


Fig. 16 Hopf link

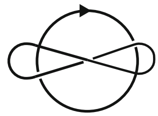


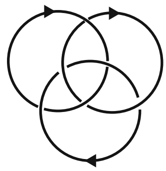
Fig. 17 Whitehead link

1994). Surprising connections between the theory of knots and statistical mechanics, quantum groups and quantum field theory are emerging (Jones 1989). Moreover, knot theory has shown to be intimately connected with many problems in "classical" physics, chemistry and biology (Figs. 16, 17, and 18).

Let M be a closed orientable 3-manifold. A smooth embedding of S^1 in M is called a *knot* in M (Burde and Zieschang 1985, Hemion 1992, Lickorish 1997, and Manturov 2004). A *link* in M is a finite collection of disjoint knots. The number of disjoint knots in a link is called the number of components of the link. Thus, a knot can be considered as a link with one component. Two links L, L' in M are said to be equivalent if there exists a smooth orientation preserving automorphism $\mathrm{f}: M \to M$ such that $\mathrm{f}(L) = L^{'}$. For links with two or more components we require f to preserve a fixed given ordering of the components. Such a function f is called an *ambient isotopy* and L and L' are called ambient isotopic. Here, we shall take M to be $S^3 \cong R^3 \cup \{\infty\}$ and simply write a link instead of a link in S^3. The diagrams of links are drawn as links in R^3. A link diagram of L is a plane projection with crossings marked as over or under.

The simplest combinatorial invariant of a knot K is the *crossing number* $c(K)$ (Alexander 1923, Pohl 1968, Bennequin 1982, Coward 2016, Dehn 1914, and Fox 1970). It is defined as the minimum number of crossings in any projection of the knot K. The classification of knots up to crossing number 17 is now known (Lackenby 2009, Lickorish and Thistlethwaite 1988). The crossing number of some special families of knots is also known; however, the question of finding the crossing number of an arbitrary knot is still unanswered (Coward and Lackenby 2011). Another combinatorial invariant of a knot K that is easy to define is the *unknotting number* $u(K)$. It is defined as the minimum number of crossing changes in any projection of the knot K which makes it into a projection of the unknot. Upper and lower bounds for $u(K)$ are known for any knot K (Hass and Lagarias 2001, Lackenby 2015). An explicit formula for $u(K)$ for a family of knots called torus knots, conjectured by Milnor nearly 40 years ago, has been proved recently by a number of different methods (Ricca and Nipoti 2011). The three manifold $S^3\backslash K$ is called the *knot complement* of K. The fundamental group $\pi_1(S^3\backslash K)$ of the knot complement is an invariant of the knot K. It is called the *fundamental group* of the knot and is denoted by $\pi_1(K)$. Equivalent knots have homeomorphic complements and conversely. However, this result does not extend to links (Boileau 1997, Petronio and Pervova 2009).

Fig. 18 Borromean link



One of the most important results in mathematical knot theory states that knots are determined by their complements (see Gordon and Luecke 1989). Let R^3 be the space in which a knot sits. Then the space "around" the knot, i.e. everything but the knot itself, is denoted by $R^3 - K$ and is called the knot complement of K. If a knot complement is hyperbolic, in the sense that it admits a complete Riemannian metric of constant Gaussian curvature <1, then the metric is unique (Thurston 1986, Gromov 1987, Benedetti and Petronio 1992).

Two (smooth or PL) knots K and K' in S^3 are equivalent if there exists a homeomorphism $h : S^3 \to S^3$ such that $h(K) = K'$. This implies that their complements $S^3 - K$ and $S^3 - K'$ are homeomorphic. Gordon and Luecke proved the converse implication.

Theorem 3 *If two knots have homeomorphic complements then they are equivalent.*

The notion of equivalence of knots can be strengthened by saying that K and K' are isotopic if the above homeomorphism is isotopic to the identity, or, equivalently, orientation-preserving. The analog of Theorem holds in this setting too: *if two knots have complements that are homeomorphic by an orientation-preserving homemomorphism, then they are isotopic.*

Theorem 1 and its orientation-preserving version are easy consequences of the following theorem concerning Dehn surgery (Dehn 1910, Lickorish 1962).

Theorem 4 *Non-trivial Dehn surgery on a non-trivial knot never yields S^3.*

Corollary 4.1 *If two prime knots have isomorphic groups then they are equivalent.*

Let K be a non-trivial knot in S^3, with tubular neighborhood $N(K)$, and let $X = S^3 -$ int $N(K)$ be the exterior of K. Let ρ be a slope on ∂X, that is, the unoriented isotopy class of an essential simple loop on ∂X. Let $K(\rho)$ denote the closed 3-manifold obtained by *ρ-Dehn surgery* on K, in other words, the result of attaching a solid torus T to X so that ρ bounds a disk in T. Let γ be the slope of a meridian of K. Then the *trivial* Dehn surgery yields $K(\gamma) \cong S^3$. Let π be another slope on ∂X, having minimal geometric intersection number $n \geq 1$ with γ.

As we already know, a link invariant is a function defined on links that is invariant under isotopies (Calugareanu 1961, Kauffman 1990). We shall represent links by using their planar diagrams. According to the Reidemeister theorem, in order to prove the invariance of some function on links, it is sufficient to check this invariance under the three Reidemeister moves (Kauffman 1983, Coward and

Lackenby 2014). First, let us consider the simplest integer-valued invariant of two-component links. Let L be a link consisting of two oriented components A and B and let L' be the planar diagram of L. Consider those crossings of the diagram L' where the component A goes over the component B. There are two possible types of such crossings with respect to the orientation. For each positive crossing we assign the number $(+1)$, for each negative crossing we assign the number (-1). Let us summarise these numbers along all crossings where the component A goes over the component B. Thus, we obtain some integer number and, in fact, this number is invariant under Reidemeister moves. The obtained link invariant is called *linking coefficient*. This invariant was first invented by Gauss (Boi 2005, Fiedler 2001, Ricca and Nipoti 2011). He calculated it by means of his famous electromagnetic formula. The linking coefficient can be generalised for the case of p- and q-dimensional manifolds in R^{p+q+1}. The formula for the parameterised curves $\gamma_1(t)$ and $\gamma_2(t)$ with radius-vectors $r_1(t)$, $r_2(t)$ is given by the following formula

$$Lk\,(\gamma_1,\ \ \gamma_2) = 1/4\pi \int_{\gamma_1}\int_{\gamma_2} (r_1 – r_2,\ \ dr_1,\ \ dr_2)^3 / \left| r_1 – r_2 \right|. \tag{1}$$

For one-component link diagrams (knot diagrams), one can define the *self-linking coefficient*. To do this, one should take an oriented knot diagram D and draw a parallel copy D' of it on the plane. After this, one takes the linking coefficient of D and D'. It is easy to check that this is invariant under Ω_2 and Ω_3, but not Ω_1: adding a loop changes the value by ± 1. By changing a link diagram at one crossing we can obtain three diagrams corresponding to links L_+, L_- and L_0 which are identical except for this crossing. In 1928, Alexander gave an algorithm for computing a polynomial invariant $\Delta_K(t)$ (a Laurent polynomial in t) of a knot K, called the *Alexander polynomial*, by using its projection on a plane. He also gave its topological interpretation as an annihilator of a certain cohomology module associated to the knot K. In 1970, Conway defined his polynomial invariant and gave its relation to the Alexander polynomial. This polynomial is called the Alexander-Conway polynomial. The Alexander-Conway polynomial of an oriented link L is denoted by $\nabla_L(z)$ or simply by $\nabla(z)$ when L is fixed. We denote the corresponding polynomials of L_+, L_- and L_0 by ∇_+, ∇_- and ∇_0 respectively. We note that the original Alexander polynomial Δ_L is related to the Alexander-Conway polynomial of an oriented link L by the relation

$$\Delta_L(t) = \nabla_L\left(t^{1/2} – t^{-1/2}\right). \tag{2}$$

In the 1984, Jones discovered his polynomial invariant $V_L(t)$, called the *Jones polynomial*, while studying Von Neuman algebras and gave its interpretation in terms of statistical mechanics. A state model for the Jones polynomial was then given by Kauffman using his bracket polynomial (1987a, b). These new polynomial invariants have led to the proofs of most of the Tait conjectures. The Jones polynomial $V_K(t)$ of K is a Laurent polynomial in t. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$, so that reversing the orientation of all components of L leaves V_L unchanged. In particular, V_K does not depend on the orientation of the knot K. For a fixed link, we denote the Jones polynomial simply by V. Recall that there are three standard ways to change a link diagram at a crossing point. The Jones polynomial is characterized by the following properties (see Jones 1987, Birman 1993, Kawauchi 1996, and Khono 1990):

1. Let L and L' be two oriented links which are ambient isotopic. Then

$$V_{L'}(t) = V_L(t). \tag{3}$$

2. Let O denote the unknot. Then

$$V_O(t) = 1. \tag{4}$$

3. The polynomial satisfies the following skein relation

$$t^{-1}V_+ – tV_- = \left(t^{1/2} – t^{-1/2}\right) V_0. \tag{5}$$

An important property of the Jones polynomial that is not shared by the Alexander-Conway polynomial is its ability to distinguish between a knot and its mirror image. More precisely, we have the following result. Let K_m be the mirror image of the knot K. Then

$$V_{K_m}(t) = V_K\ (t–1)\,. \tag{6}$$

Since the Jones polynomial is not symmetric in t and t^{-1}, it follows that in general

$$V_{K_m}(t) \neq V_K(t). \tag{7}$$

We note that a knot is called *amphicheiral* (*achiral* in biochemistry) if it is equivalent to its mirror image. We shall use the simpler biochemistry term. So, a knot that is not equivalent to its mirror image is called *chiral*. The condition expressed by (7) is sufficient but not necessary for chirality of a knot. The Jones polynomial did not resolve the following conjecture by Tait concerning chirality: *If the crossing number of a knot is odd, then it is chiral*. However, it has been demonstrated recently that a 15-crossing knot provides a counter-example to the chirality conjecture (Lackenby 2016).

There was an interval of nearly 60 years between the discovery of the Alexander polynomial and the Jones polynomial. Since then a number of polynomials and other invariants of knots and links have been found. A particular interesting one is the two-variable polynomial generalizing V, called the *HOMFLY* polynomial (name formed from the initials of authors of the article (Freyd et al. 1985)) and denoted by P. The HOMFLY polynomial $P(\alpha, z)$ satisfies the following skein relation

$$\alpha^{-1} P_{+} - \alpha P_{-} = z P_0. \tag{8}$$

Both the Jones polynomial V and the Alexander-Conway polynomial ∇_L are special cases of the HOMFLY polynomial. The precise relations are given by the following theorem.

Theorem 5 *Let L be an oriented link. Then the polynomials P_L, V_L and ∇_L satisfy the following relations*

$$V_L(t) = P_L\left(t,\ \ t^{1/2} - t^{-1/2}\right) \text{ and } \nabla_L(z) = P_L\left(1,\ \ z\right). \tag{9}$$

Knots and Links: Equivalence, Invariants and the Knot Complement

As we already mentioned, the notion of a knot formalizes that of 1-dimensional strings in ordinary 3-dimensional space. We restrict our study to the case where two ends of the string are brought together, seamlessly. Then we have a loop of string that might or might not contain knots. Knot theory studies the properties of these knotted strings. Which knots are different from other? Conversely, which knots are equivalent? Can we combine knots together or decompose a knot into simpler knots? Can a knot be the boundary of a surface? What kind of three-dimensional space a knot is? A fundamental part of knot theory is the study of knot invariant and the ongoing attempt at classifying them (Hemion 1992, Lescop 2012). This is a still open research problem (proposed by Michael Atiyah in 1988 at the Hermann Weyl symposium), which is to give an intrinsically 3-dimensional definition of the Jones polynomial invariant. The Jones polynomial discovered in 1984 is a new knot invariant which proved to be very powerful at differentiating between different equivalence classes, while at the same time being relatively easy to compute.

We start with the definition of a knot and we will introduce links diagrams, which are essential to the study of knot theory.

Definition 7 A knot K is a continuous embedding of S^1 into S^3 or R^3. Or, equivalently, a knot $K \subset R^3$ is a subset of points homeomorphic to a circle.

We can use the same symbol K for the map $K : S^1 \to R^3$ and for its image $K(S^1)$ and thereby think of a knot either as an embedding or as a subspace of R^3 or S^3.

Topologically, every knot K in R^3 or S^3 is just a copy of S^1, but they are distinguished one from another by the manner in which they are "knotted" by the embedding. There are two notions of knot equivalence we can use for making this distinction mathematically.

1. Two topological knots K_0 and K_1 in R^3 are said to be *equivalent* if there is a homeomorphism $h : R^3 \to R^3$ of R^3 onto itself for which $h(K_0) = K_1$.

Note that such a homeomorphism h extends uniquely to a homeomorphism of S^3 onto itself that carries K_0 onto K_1 so K_0 and K_1 are equivalent if and only if there is a homeomorphism $h : S^3 \to S^3$ of S^3 onto itself for which $h(K_0) = K_1$. This clearly defines an equivalence relation on the set of knots. The corresponding equivalence class of a knot K is called its *knot type*.

2. Two topological knots K_0 and K_1 in R^3 are said to be *isotopic* if there exists a homotopy $H : R^3 \times [0, 1] \to R^3$ for which each $h_t(\cdot) = H(\cdot, t)$, $0 \leq t \leq 1$, is a homeomorphism of R^3 onto itself, $h_0 = \mathrm{id}_{R^3}$, and $h_1(K_0) = K_1$.

Note that isotopic knots are clearly equivalent, but the converse is false. We will refer to the equivalence class of a knot K under this equivalence relation as its *isotopy type*. If K_0 and K_1 are isotopic, there exists an orientation preserving homeomorphism of S^3 onto itself that carries K_0 onto K_1. In fact, one can show that every orientation preserving homeomorphism of S^3 onto itself arises in this way as h_1 from a homotopy $H : S^3 \times [0, 1] \to S^3$ with each h_t, $0 \leq t \leq 1$, a homeomorphism of S^3 onto itself and $h_0 = \mathrm{id}_{S^3}$. Consequently,

Definition 8 *Two topological knots K_0 and K_1 are isotopic if and only if there exists an orientation preserving homeomorphism h: $S^3 \to S^3$ of S^3 onto itself with $h(K_0) = K_1$.* (The difference between *equivalent* and *isotopic* is *orientation*).

A *knot invariant* is a mathematical object (a number, or a group, or a polynomial, etc.) that one can associate with each knot K and with the property that if two knots are equivalent, then their associated objects are the same. Given such a knot invariant one might hope to show that two given knots are *not* equivalent by showing that their invariants are not the same.

One begins the search for such knot invariants by considering the *knot complement* $R^3 - K$. This is an open subspace of R^3 since K is compact. Moreover, two equivalent knots clearly have homeomorphic knot complements so it would seem profitable to look at some of the invariants associated to $R^3 - K$ by algebraic topology (see Przytycki 1997). The homology groups will not do because knot complements all have the same homology. Specifically, for any knot K in R^3,

$$H_p\left(R^3 - K; \mathbf{Z}\right) = \begin{cases} \mathbf{Z}, \text{if } p = 1, \ 2 \\ 0, \text{if } p = 3, \ 4, \ \ldots \end{cases} \tag{10}$$

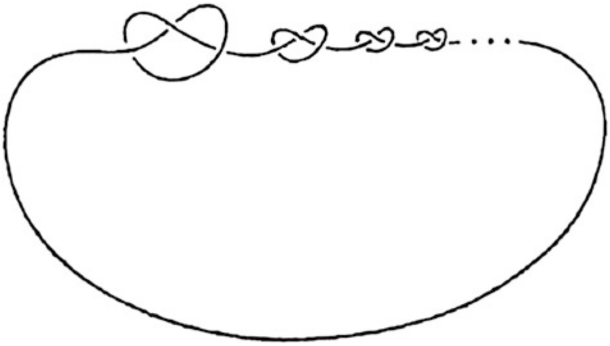


Fig. 19 A Wild Knot

It has been shown by Gordon and Luecke (1989) that if one knows that $R^3 - K_0$ and $R^3 - K_1$ are homeomorphic, then the knots K_0 and K_1 are equivalent and therefore have isomorphic knot groups. Moreover, if there is an orientation preserving homeomorphism of $R^3 - K_0$ onto $R^3 - K_1$, then K_0 and K_1 are isotopic. In this sense the topological type of the knot complement is itself a complete invariant for knots.

A *polygonal knot* is a topological knot that is the union of a finite number of closed straight line segments in R^3. A *smooth knot* is (the image of) a smooth embedding K: $S^1 \to R^3$ of S^1 in R^3 (or S^3). A topological knot is *tame* if it is equivalent to a polygonal knot (or, equivalently, to a smooth knot); otherwise, it is *wild*. Wild knots have rather complicated topological structure (see Fig. 19).

From a regular projection of K one obtains a *knot diagram* of K in the following way (see Kauffman 2005a,b, Papi and Procesi 1999). Each double point $P \in \pi(K)$ is the image under π of two *distinct* points k_1 and k_2 in K. Since π is the orthogonal projection onto P, the distance from k_1 to P is not equal to the distance from k_2 to P. Assuming, without loss of generality, that the former is less than the latter we redraw the crossing at p with the projected arc of K through k_1 *under* the projected arc of K through k_2. This is done by breaking the projected arc through k_1 at p. Repeating the process for each crossing one obtains something like Fig. 20, which is a knot diagram for the so-called *left-handed trefoil knot*. Interchanging the over-crossings and under-crossings in Fig. 17 gives a knot diagram for what is called the *right-handed trefoil knot* shown in Fig. 21.

The "right-handed" and "left-handed" trefoil knots are called *mirror images* of each other and each is the image of the other under an *orientation reversing* homeomorphism of R^3 or S^3 onto itself. In particular, their complements are homeomorphic so, by the result of Gordon and Luecke already cited, the left- and right-handed trefoils are equivalent. They therefore have the same knot group. This group is not Abelian, so, in particular, the trefoil is not equivalent to the trivial knot; it is genuinely "knotted". The left- and right-handed trefoils are, however, *not* isotopic (Figs. 20, 21, 22, and 23).

Fig. 20 Left-handed trefoil knot

Fig. 21 Right-handed trefoil knot

Fig. 22 Regular crossing and irregular crossing

Consider next an oriented knot K and a knot diagram D for it. We would like to use the orientation to associate with each crossing p in D a sign $\text{sgn}(p) = \pm 1$. The procedure is indicated pictorially in Fig. 23 and can be regarded as an application of the so-called "right-hand rule". Choose and fix a normal vector $\boldsymbol{n}$ to the plane in which the diagram D lives and let $\boldsymbol{t}_0$ and $\boldsymbol{t}_u$ be the tangent vectors to the forward over-crossing and forward under-crossing at p, respectively. Then $\text{sgn}(p)$ is $+1$ if curling the right hand from $\boldsymbol{t}_0$ to $\boldsymbol{t}_u$ leaves the thumb pointing in the direction of $\boldsymbol{n}$; otherwise the sign is -1. The sign depends, of course, on the choice of $\boldsymbol{n}$, but once this is fixed the signs are well-defined. Note that these signs are unchanged if the orientation of the knot is reversed. We define the *writhe* $w(D)$ of the knot diagram

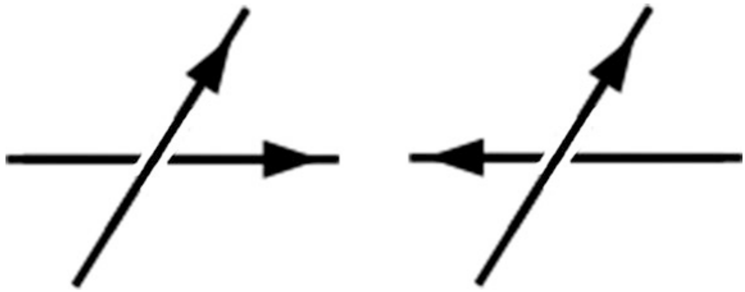


Fig. 23 Crossing points of a diagram: (**a**) left-handed crossing, (**b**) right-handed crossing

to be the sum of these signs over all the crossings in D; it is, by definition, zero if there are no crossings.

$$w(D) = \sum\nolimits_{\text{crossings } p\in D} \text{sgn}(p) \tag{11}$$

Although $w(D)$ is defined from an orientation of K it is, in fact, independent of the orientation and so is a property of the knot diagram itself. It fails to be a knot invariant, however, as one can see by comparing knot diagrams for the trivial knot and the unknot. See Fig. 24. The unknot may have $w(D) \neq 0$.

Next, we consider "linked" families of knots. A *link* in R^3 or S^3 is a disjoint union $L = K_1 \cup \ldots \cup K_m$ of knots $K_1, \ldots, K_m$ each of which is called a *component* of the link. A knot is just a link with one component. The so-called *Whitehead link* is shown in Fig. 25. We will always fix an ordering of the components by $1, \ldots,$ m. The link L is *topological*, *polygonal*, or *smooth* if each of its components K_i, $i = 1, \ldots, m$, is a topological, polygonal, or smooth knot, respectively. An *oriented link* is one for which each of its component knots has been assigned an orientation. Two links $L = K_1 \cup \ldots \cup K_m$ and $L' = K_1' \cup \cdots \cup K_m'$ with the same number of components are *equivalent* (respectively, *isotopic*) if there is a homeomorphism (respectively, orientation preserving homeomorphism) h of R^3 onto itself (or of S^3 onto itself) with $h(K_i) = K'_I$ for each $i = 1, \ldots, m$. In particular, each of the knots K_i in L is equivalent (respectively, isotopic) to the corresponding knot K'_I in L'. The equivalence class of L is called its *link type* (respectively, *isotopy type*). A link is *tame* if it is equivalent to a polygonal (or, equivalently, smooth) link and we will consider only these so that, henceforth, *link* always means *tame link*. A link is *trivial*, or *unlinked*, if it is equivalent to a link that lies entirely in a plane in R^3.

Let P be a plane in R^3 and $\pi\colon R^3 \to P$ the orthogonal projection of R^3 onto P. A point $p \in \pi(L) \subseteq P$ is called a *multiple point* if $\pi^{-1}(p) \cap L$ consists of more than one point. It is a *double point* if $\pi^{-1}(p) \cap L$ consists of two points, a *triple point* if $\pi^{-1}(p) \cap L$ consists of three points, and so on. As in the case of knots we will say that π is a *regular projection* for L if there are only finitely many multiple points, all of which are transverse double points. Every link has regular projections. Indeed, as for knots, there is a sense in which almost every projection is a regular projection for L. In particular, one can find a common regular projection

Fig. 24 (Links) Two simple diagrams. (Middle) Thistlethwaite unknot. (Right) One Occhiai diagram of the unknot

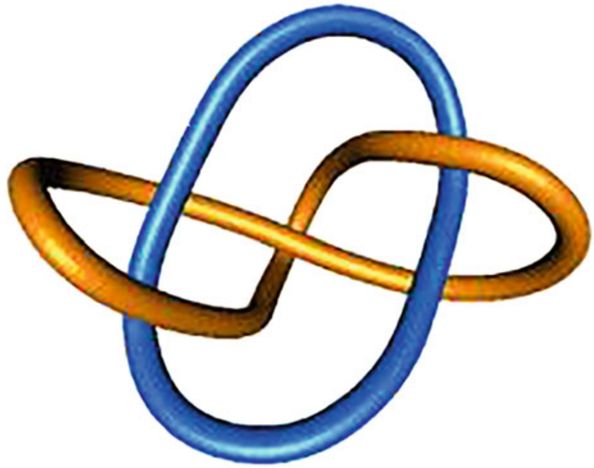


Fig. 25 Whitehead link

for any two links. From any regular projection for L one obtains a *link diagram* from $\pi(L)$ by indicating the over-crossings and under-crossings just as for a knot. Interchanging the over-crossings and under-crossings gives a link diagram for the *mirror image* of L. One possible link diagram for the Whitehead link is shown in Fig. 26. For an oriented link one defines the sign of a crossing in a link diagram and the *writhe* of the diagram in precisely the same way as for a knot diagram. A variant of this is obtained by ignoring all of the "internal" crossings that arise from the component knots themselves and looking only at those that correspond to one component crossing another. More precisely, we consider the oriented link $L = K_1 \cup \ldots \cup K_m$ with link diagram D and fix two distinct components, say, K_i and K_j with $1 \leq i < j \leq m$. Define the *linking number of* K_i *with* K_j, denoted $lk\ (K_i, K_j)$, to be one-half the sum of the signs of the crossings of K_i with K_j. Then the *linking number* of the diagram D, denoted $lk(D)$, is the sum of these over-all distinct pairs of components.

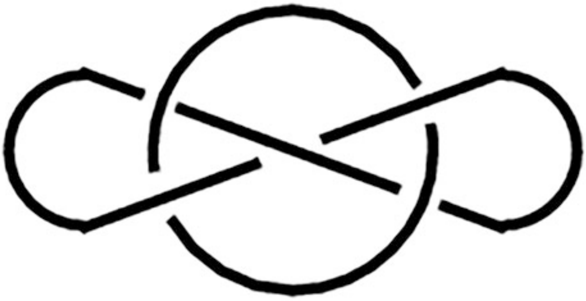


Fig. 26 Whitehead link diagram

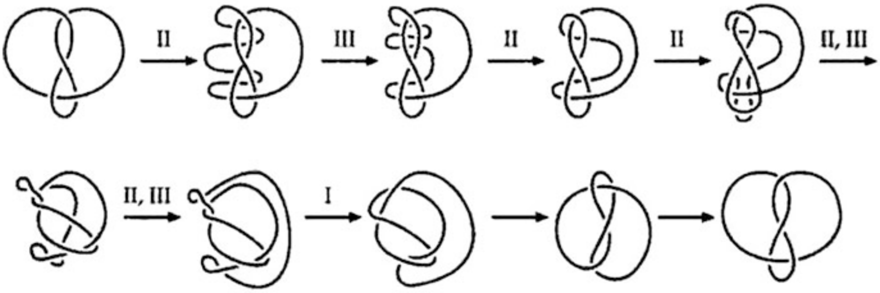


Fig. 27 *Reidemeister moves and planar isotopy*. The figure-eight knot is equivalent to its mirror image, that is, by a finite sequence of Reidemeister moves we can transform the figure-eight knot into its mirror image

$$Lk(D) = \sum\nolimits_{1 \leq i < j \leq m} lk\left(K_i, \quad K_j\right) \tag{12}$$

The definition of the linking number can be formulated in a great variety of equivalent ways, including the famous integral formula of Gauss (see Sect. 4).

Since drawing knots and links in space is rather a tricky business one would like to know if one can tell from their regular projections whether or not two links are "the same" (equivalent or isotopic). An answer to the question was obtained independently by Kurt Reidemeister (1927) and Alexander and Briggs (1926). The procedure involves applying a sequence of local "moves", now called *Reidemeister moves*, which alter a given link diagram in a small region without altering the link type of the link from which the diagram arose. The three basic Reidemeister moves are indicated in Fig. 27 and labeled I, II, and III corresponding to the number of arcs of the projection that are affected by the move. The relevant result proved by Reidemeister is as follows.

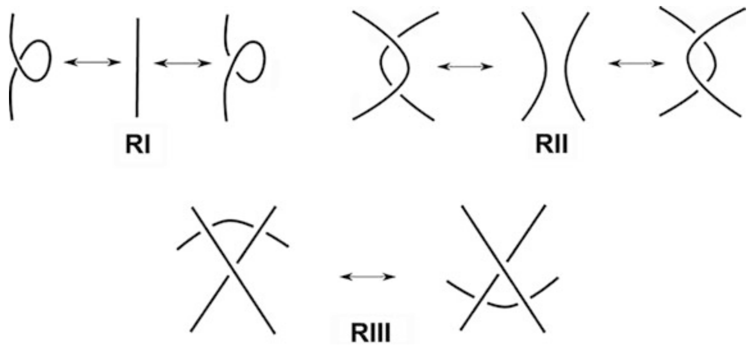


Fig. 28 Reidemeister moves

Theorem 6 *Let L and L′ be two links in* R^3 *and let* $\pi : R^3 \to P$ *be a regular projection for both L and L′. Let D be a link diagram of L in the plane P, and D′ a link diagram of L′ in P. Then L and L′ are isotopic links if and only if D′ can be obtained from D by a sequence of Reidemeister moves and orientation preserving homeomorphisms of P.*

The principal use one makes of the Reidemeister moves is to show that two links are isotopic or that something is a link invariant by proving that it is left unchanged by any of these moves. We include one pictorial application (Fig. 28) of the above theorem which shows that the so-called *figure eight knot* is isotopic to its mirror image.

The Alexander and Jones Polynomials

In 1928 J.W. Alexander introduced an invariant of oriented knots and links that, to a large degree, inaugurated *knot theory* as a branch of topology. The invariant associated to a link L is a Laurent polynomial $\Delta_L(t)$ with integer coefficients in the formal variable $\sqrt{t}$, that is, an integer polynomial in $t^{1/2}$ and $t^{-1/2}$. This is now universally known as the *Alexander polynomial* of L. It is determined only up to multiples of $\pm\, t^{k/2}$ for some positive integer k so one often finds $\Delta_L(t)$ written in a variety of forms.

There are a number of algebraic, combinatorial, and geometric ways to approach the definition of $\Delta_L(t)$. Alexander's original approach via the algebraic topology of the infinite cyclic cover of the knot complement (see Alexander 1923, and Fox 1970). All of these, however, give rise to rather tedious calculations when applied to any particular link or knot. This changed with an observation of Conway (1970) that we will briefly describe. One of the various properties of $\Delta_L(t)$ is a so-called *skein relation* that apparently went unnoticed for quite some time, but has since become a focal point for the definition and application of polynomial invariants.

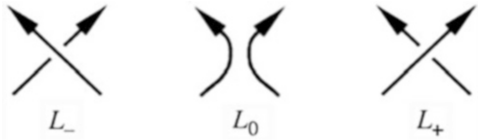


Fig. 29 Skein relation diagrams

This relation can be described as follows. Focus attention on one particular crossing in a diagram D for L. The skein relation compares the Alexander polynomial of three related links, denoted L_-, L_0, and L_+, that are identical to L except perhaps near the crossing in question where, in some neighborhood, their diagrams look like one of the pictures in Fig. 29 or a rotation of one of these. Choosing the normal vector to the plane containing D appropriately, L_- has sign -1 and L_+ has sign $+1$. As we observed earlier this is unaffected by a reversal of the orientation of the link.

The skein relation satisfied by the Alexander polynomial is

$$\Delta_{L+}(t) - \Delta_{L-}(t) = \left(t^{1/2}–t^{-1/2}\right) \Delta L_0(t). \tag{13}$$

Conway showed that this skein relation, together with the normalization

$$\Delta_{\text{unknot}}(t) = 1, \tag{14}$$

completely determines one of the Alexander polynomials (recall that these are determined only up to multiples of $\pm t^{k/2}$). Although one must choose an orientation for L in order to define $\Delta_L(t)$, the polynomial itself is independent of how the orientation is chosen.

With this one can compute the Alexander polynomial of any link recursively. We will illustrate with the example of the right-handed trefoil (see Fig. 31). First, however, a much simpler application of (5) and (6). The first row in Fig. 30 shows three knot diagrams for the unknot. The knots in the second row of Fig. 30 we denote L_+, L_-, and L_0 from left to right. L_0 is the 2-component unlink (two unlinked, unknotted circles). We conclude from (13) and (14) that $1 - 1 = (t^{1/2} - t^{-1/2})\Delta_{L0}(t)$ so $\Delta_{L0}(t) = 0$. The Alexander polynomial for the 2-component unlink is identically zero (Figs. 30 and 31).

Now we turn to the right-handed trefoil. This is the diagram at the top of Fig. 31 and we will denote it $L_+{}^1$, focusing our attention on the circled crossing. The second row in Fig. 30 are, from left to right, $L_-{}^1$ and $L_0{}^1$. The first is the unknot and the second is the so-called *Hopf link*. We conclude that

$$\Delta_{L_+{}^1}(t) - 1 = \left(t^{1/2} - t^{-1/2}\right) \Delta_{L_0{}^1}(t). \tag{15}$$

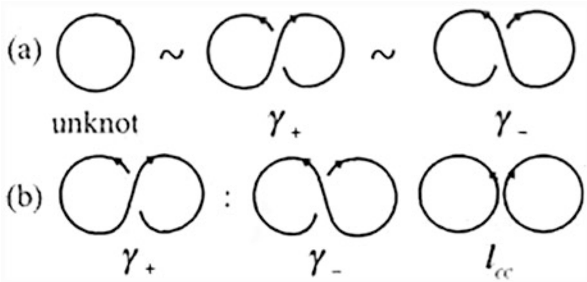


Fig. 30 2-Component Unlink

To compute the Alexander polynomial for the Hopf link we focus our attention on the circled crossing in L_0^1 and rename that link L_+^2. The third row in Fig. 31 is then, from left to right, L_-^2 and L_0^2. The first is the 2-component unlink and the second is the unknot. We conclude that

$$\Delta_{L_+{}^2}(t) - 0 = \left(t^{1/2} - t^{-1/2}\right) 1, \tag{16}$$

so, the Alexander polynomial of the Hopf link is

$$\Delta_{L_+{}^2}(t) = t^{1/2} – t^{-1/2}. \tag{17}$$

Combining these we find that the Alexander polynomial of the right-handed trefoil is $(t^{1/2} - t^{-1/2})(t^{1/2} - t^{-1/2})$, that is,

$$\Delta_{\text{right-handed trefoil}}(t) = t - 1 + t^{-1}. \tag{18}$$

Since the Alexander polynomial is a knot invariant we conclude from this that the right-handed trefoil is genuinely knotted, that is, not equivalent to the unknot. An analogous calculation for the left-handed trefoil gives the same result. Even so, we will see shortly that these two trefoils are not isotopic. The Alexander polynomial cannot, in general, distinguish a knot from its mirror image when these two fails to be isotopic. We will see that the Jones polynomial is more discerning and allow to make that distinction (Fig. 31).

For over 50 years the Alexander polynomial remained the only known polynomial invariant for oriented knots and links. Then, in 1985, Vaughan Jones announced the discovery of another polynomial invariant that is now universally known as the *Jones polynomial* and denoted $V_L(t)$ (Jones 1985, De la Harpe, Kervaire and Weber 1986). The path followed by Jones was somewhat unorthodox in that his research interests centered around von Neumann algebras rather than knot theory. Recall that a von Neumann algebra is a *-algebra of bounded operators on a complex Hilbert space that contains the identity operator and is closed in the weak operator topology. Finite-dimensional von Neumann algebras are all products of matrix algebras for

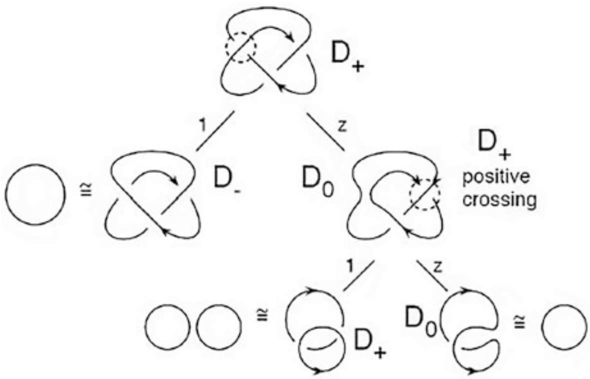


Fig. 31 Skein Tree for the Right-Handed Trefoil

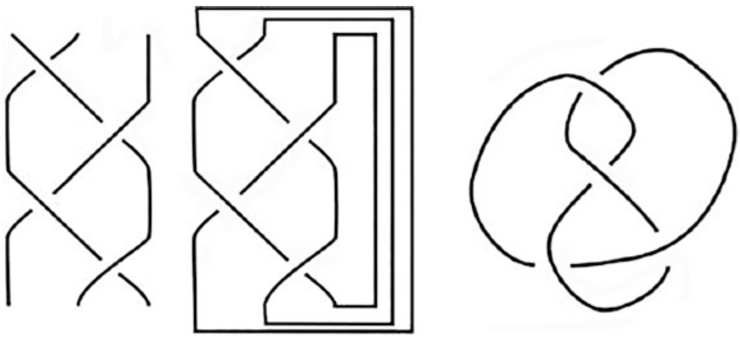


Fig. 32 The figure-height knot as a closed braid

which the *-operation is conjugate transpose. Jones was led to consider a certain family of such finite-dimensional von Neumann algebras J_n (Fig. 32).

The key result upon which the definition of the Jones polynomial is based is obtained by combining the representations R_t (R_t is a representation of the braid group B_n, with $R_t(\sigma_i) = g_i$, $i = 1, 2, \ldots, n-1$) and the traces tr (on each J_n there is defined a trace function tr: $J_n \to \mathbf{C}$)) in the following way. Let ω be an element of the braid group B_n. The closure of the braid ω is an oriented link (or knot) which we will denote ω^-. Furthermore, for each t, $R_t(\omega)$ is an element of J_n. Assuming that t is either a positive real number or a root of unity $R_t(\omega)$ has a trace tr $(R_t\ (\omega)) \in \mathbf{C}$ and Jones proves that the number

$$\left(-(1+t)/\sqrt{t}\right)^{n-1} \text{tr}\ \left(R_t\,(\omega)\right) \tag{19}$$

depends only on the isotopy class of the closed braid $\bar{\omega}$. Now, according to Alexander's Theorem, every link L is $\bar{\omega}$ for some braid ω in some B_n and this gives rise to the following definition. Let L be an oriented link. Then there exists a positive integer n and an element ω in the braid group B_n such that L is isotopic to $\bar{\omega}$. If t is either a positive real number or one of the roots $e^{2\pi i/k}$, $k = 3, 4, 5, \ldots$, of unity we define the *Jones polynomial* of L by

$$V_L(t) = \left(-(1+t)/\sqrt{t}\right)^{n-1} \operatorname{tr}\ (R_t(\omega)). \tag{20}$$

This is an invariant of the isotopy type of L. Just as for the Alexander polynomial, L must be an oriented link in order to define the invariant $V_L(t)$, but the polynomial itself is independent of how the orientation is chosen.

The existence of the $*$-operation on J_n has played no role up to this point, but Jones points out that with it one can extend the definition of $V_L(t)$ to all complex values of t except 0. He then describes the structure of $V_L(t)$ as a function of t.

Theorem 7 *If the link L has an odd number of components (in particular, if L is a knot), then $V_L(t)$ is a Laurent polynomial in t with integer coefficients. If L has an even number of components, then $V_L(t)$ is $\sqrt{t}$ times a Laurent polynomial in t with integer coefficients.*

Jones proves a great many interesting and useful properties of $V_L(t)$, but we will focus on just a few that we will put to use in our examples. The first gives an explicit relationship between the Jones polynomial of an oriented link L and that of its mirror image which we will denote $\hat{L}$. Specifically, we have the following.

$$V_{\hat{L}}(t) = V_L(1/t) \tag{21}$$

Consequently, a link L that is isotopic to its mirror image must have a Jones polynomial that satisfies $V_L(t) = V_L(1/t)$. In particular, if $V_L(t)$ does not have this $t \iff 1/t$ symmetry, then L cannot be isotopic to its mirror image. We will see that this is the case for the trefoil so that the right-handed trefoil and the left-handed trefoil are not the same despite the fact that the Alexander polynomial failed to distinguish them.

Just as in the case of the Alexander polynomial, the definition of the Jones polynomial is computable, but rather unwieldy. One would like to see a skein relation for the Jones polynomial analogous to (13) which reduces the problem of computing $V_L(t)$ to a recursive, combinatorial one. Jones provides this in Theorem 12 of (Jones 1985). Referring again to Fig. 31 the result is

$$(1/t)\, V_{L_+}(t) - t V_{L_-}(t) = \left(t^{1/2} - t^{-1/2}\right) V_{L_0}(t). \tag{22}$$

The skein relation (7) by itself, of course, does not determine the Jones polynomial of a link completely since one must get the recursion started by knowing $V_{\text{unknot}}(t)$.

The usual normalization is the same as in the case of the Alexander polynomial, that is,

$$V_{\text{unknot}}(t) = 1. \tag{23}$$

With this we can compute the Jones polynomial of the right-handed trefoil in precisely the same way we computed its Alexander polynomial earlier. First, referring to Fig. 31, we obtain from the skein relation (22) that

$$(1/t)\,1 - t1 = \left(t^{1/2} - t^{-1/2}\right) V_{\text{2-component unlink}}(t) \tag{24}$$

so

$$V_{\text{2-component unlink}}(t) = -\left(t^{1/2} + t^{-1/2}\right). \tag{25}$$

Now we refer to Fig. 30. The top two rows give

$$(1/t)\,V_{\text{right-handed trefoil}}(t) - t\,V_{\text{unknot}}(t) = \left(t^{1/2} - t^{-1/2}\right) \; V_{\text{Hopf link}}(t) \tag{26}$$

so

$$V_{\text{right-handed trefoil}}(t) = t^2 + \left(t^{3/2} - t^{1/2}\right) \; V_{\text{Hopf link}}(t). \tag{27}$$

From the bottom two rows we obtain

$$(1/t)\,V_{\text{Hopf link}}(t) - t\,V_{\text{2-component unlink}}(t) = \left(t^{1/2} - t^{-1/2}\right) V_{\text{unknot}}(t) \tag{28}$$

so

$$V_{\text{Hopflink}}(t) = -\left(t^{5/2} + t^{1/2}\right) \tag{29}$$

and therefore

$$V_{\text{right-handed trefoil}}(t) = -t^4 + t^3 + t. \tag{30}$$

Since this polynomial is not invariant under $t \to 1/t$, we conclude that the right-handed and left-handed trefoils are *not* isotopic.

Shortly after the introduction of the Jones polynomial, generalizations were discovered that contained both the Alexander and Jones polynomials as special cases. These are referred to as the HOMFLY polynomials and are denoted P_L. The defining relations can be expressed in a variety of ways, but the normalization is always the same as in the case of the Alexander and Jones polynomials, that is, $P_{\text{unknot}} = 1$. One can think of P_L as a homogeneous Laurent polynomial of degree zero in three variables x, y and z satisfying the skein relation

$$xP_{L+}(x, y, z) + yP_{L-}(x, y, z) + zP_{L_0}(x, y, z) = 0. \tag{31}$$

For example, one can proceed exactly as we did for the Alexander and Jones polynomials to show that

$$P_{2\text{-component unlink}}(x, y, z) = -(x + y)/z \tag{32}$$

and

$$P_{\text{right-handed trefoil}}(x, y, z) = y^{-2}z^2 - 2xy^{-1} - x^2y^{-2}. \tag{33}$$

Alternatively, one can think of P_L as a nonhomogeneous Laurent polynomial in two variables. A common way to do this is to define $P_L(l, m) = P_L(l,\ \ l^{-1},\ \ m)$ in which case the skein relation becomes

$$lP_{L+}(l, m) + l^{-1}P_{L-}(l, m) + mP_{L_0}(l, m) = 0. \tag{34}$$

The Alexander polynomial is then recovered by taking $l = i$ and $m = i(t^{1/2} - t^{-1/2})$ while the Jones polynomial corresponds to $l = it^{-1}$ and $m = i(t^{1/2} - t^{-1/2})$. We will make only a few comments about these generalizations in the sequel.

There is one somewhat unsatisfying aspect of the Alexander, Jones and HOMFLY polynomials. Knots and links are intrinsically 3-dimensional objects. There are no knots in R and only the unknot lives in R^2. Moreover, in R^n for $n \geq 4$, one can use the extra dimensions to unknot any embedded circle so that every "knot" is an unknot (see Kamada 2017).

Crossing Changes of Knots

In this section we follow closely M. Lackenby (2016), A. Coward and Lackenby (2014), and M. Long (2005). The so-called recognition problem for knots has been recently proved to have an algorithmic solution, and improved bounds on the complexity of such problems was obtained by A. Coward and M. Lackenby (2014). However, there are many problems in knot and 3-manifold theory which are not known to have an algorithmic solution. Let's mention some of the most important problems in this class, namely those that arise in the study of crossing changes.

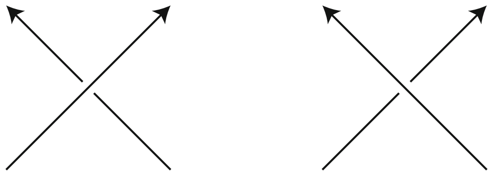


Fig. 33 Positive or right-handed (+1) and negative or left-handed (−1) crossings

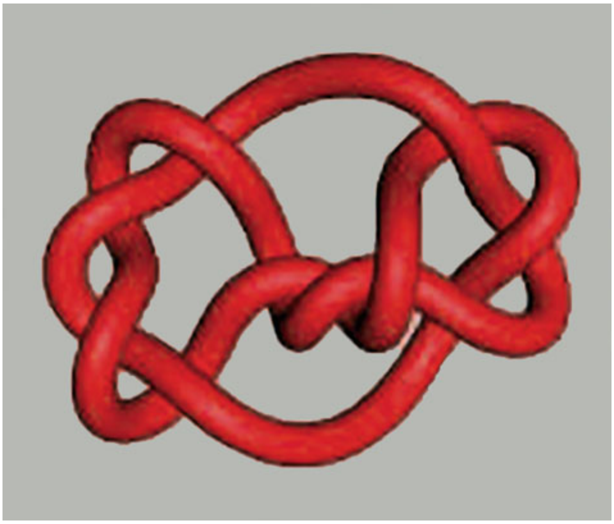


Fig. 34 A knot with unknown unknotting number

One of the most natural ways that a knot can change is via a crossing change, which is the operation shown in Fig. 33. It is a very simple operation whereby one passes the curve out of which the knot is made through itself exactly once, much like the way metal rings appear to pass through each other in the conjuror's ring trick (Figs. 33 and 34).

As M. Lackenby stressed (2016, 23):

> Despite the wealth of knowledge that we have about knots and 3-manifolds, crossing changes are extraordinarily poorly understood and there are many elementary questions that we do not know the answer to. For example, the minimum number of crossing changes required to turn a knot K into the unknot is a measure of the knot's complexity called its unknotting number, denoted $u(K)$. Fig. 34 shows a knot whose unknotting number is not known.

The reason that crossing changes and unknotting numbers are hard to understand is similar to the reason the recognition problem for knots is hard. If one wishes to determine whether two knots are the same, one is really asking if there is a continuous motion of R^3 from one to the other. On the other hand, the question of whether two knots are related by a single crossing change, for example, is also a question of whether there is a continuous motion taking one to the other, but where the continuous motion is allowed to pass the knot through itself exactly once (Lackenby 2016, Coward 2016).

The solution to the recognition problem for knots and links came about because Haken found a way of encoding whether two knots are the same using certain hierarchies of surfaces, and showed how to enumerate possibilities for those hierarchies using normal surface theory (Haken 1968). This raises the question of how may one encode when two knots are related by a crossing change.

The method developed by Lackenby and Coward is the following. A crossing circle for a knot K is a simple closed curve C in the complement of K which bounds a disc that intersects K transversely in two points where K passes through with opposite orientation. A crossing circle describes a crossing change specified by inserting a full turn to the two strands of K near the disk bounded by C. We call this the crossing change specified by C. (Strictly speaking there are two crossing changes specified by C, depending on whether one applies a left-handed or a right-handed turn. Thus, a crossing circle is endowed with a little extra information to make this choice.) The question of deciding whether two knots K and K' are related by a crossing change is best understood as whether there is a crossing circle for K specifying a crossing change that turns it into K' (Kinoshita and Terasaka 1959). This is hard to do because the crossing circle is a curve and not a surface. This means that one cannot use normal surface theory, or any of its adaptations, to search for crossing circles directly.

Crossing circles provide a natural way of saying when two crossing changes on a knot are essentially the same. Two crossing changes for a knot K are said to equivalent if they are specified by crossing circles that are related by a continuous motion of the knot complement, and the handedness associated with the two crossing circles is the same. M. Lackenby and A. Coward used this notion of equivalence in (2014) where they proved the following theorem.

Theorem 8 *Suppose that K is a knot with genus one and unknotting number one. Then, if K is not the figure-eight knot, there is precisely one crossing change that turns K into the unknot, up to equivalence. If K is the figure-eight knot then there are precisely two crossing changes that turn K into the unknot, up to equivalence.*

(Recall that the *genus* of a knot is the minimum genus of any embedded orientable spanning surface for the knot. Thus, genus one knots bound embedded punctured tori (see Sect. 13 for further considerations on this subject)

A. Coward and M. Lackenby proved (2014) that crossing changes can very often be understood in terms of a closely related operation on surfaces called *twisting*. A *twist* on a surface with boundary F is the operation made on F along an arc α (a certain type of self-homeomorphism applied to a surface along an arc) It involves drawing a properly embedded arc α on F, cutting F along α and then regluing F with a single twist inserted. Note that a crossing change is applied to the boundary of F, specified by the curve obtained by pushing α off F a little in all directions.

The problem of determining when two surfaces with boundary are related by a twist seems easier than determining when two knots are related by a crossing change. This is because the later amounts to searching for a curve in a 3-manifold, namely the knot complement, and the former amounts to searching for an arc on a surface. Coward proved that very often a solution to this apparently easier problem implies that one can determine when two knots are related by a crossing change. He then proved more precisely the following theorem:

Theorem 9 *Let K and K′ be oriented knots in S^3, both either hyperbolic or fibered, and with the genus of K being bigger than the genus of K′. Then there are finite lists of oriented spanning surfaces $\{S_1, \ldots, S_n\}$ for K and $\{S'_1, \ldots, S'_n\}$ for K′ such that if K and K′ are related by a single crossing change, then some $S_i \in \{S_1, \ldots, S_n\}$ and some $S'_i \in \{S'_1, \ldots, S'_n\}$ are related by a single twist, up to ambient isotopy of S^3. Furthermore, there is an algorithm that will take diagrams for K and K′ as input, and output such finite lists of spanning surfaces.*

The problem of crossing changes presents three key points which were stressed by Coward in (2016, 16):

> First, crossing changes are about as natural an operation as one could possibly wish to perform on knots, and yet they are very poorly understood, despite all we know about knots and 3-dimensional manifolds. Second, there is now a wealth of technology in knot and 3-manifold theory that one can try to apply to problems like determining when two knots are related by a crossing change, but doing so is often technically difficult because curves such as crossing circles naturally interact poorly with much of this technology, which is generally better at controlling surfaces rather than curves. Third, understanding crossing changes is extremely important because when DNA replicates it untangles itself with the help of enzymes that apply crossing changes to the double helix. So, if we want to understand these most important features of real-world knots we need to vastly improve our understanding of crossing changes.

Back to Classical Invariants of Knots and Links

Deep interests for mathematical properties of knots was initiated by physicists around the 1870s in the tentative to explain the connection between fundamental physics (at the time was the electromagnetic theory developed by Maxwell in its *Treatise* of 1873) and topological features of space. The general idea was very novel and audacious for that time: try to understand the stability of chemical and physical states of matter through the stability of topological objects, in particular through the possible combinations of knots and links. The deeper insight was that a certain stability of matter might be related to (or explained by) the stability of knots and links, that is, to their mathematical invariants. Both stabilities were already recognized to be dynamical, in the sense that the components and states of matter could be held together by the "forces" of topology. For example, the ability of particles to change into new forms of interactions and to acquire other levels of energy might (totally or partially) depend upon the changes occurring in topological structures (for an in-depth study of this issue, see Boi 2006b, 2008). These ideas stimulated, in particular by Tait, the search for classifying knots and links, and hence for discovering invariants of knots or knots types (or, in modern mathematical language, classes of equivalence of knots and links). One of the first feature characterizing knots is the *unknotting number* or Gordian number of a knot K, which is the minimal number of overcrossing-undercrossing changes required to transform K to the unknot. Here one considers obviously *alternating* knots (Menasco and Thistlethwaite 1993).

A knot or link invariant, for example a number or a measure, does not change its value if we apply one of the elementary knots moves. We will see therefore that the search for moves that leave invariant knots and links has been an important task in the development of mathematical knot theory.

A very important knot or link invariant is the *linking number* (already mentioned in Sect. 6) which is, more precisely, an invariant for oriented links. Suppose that D is an oriented regular diagram of a 2-component link $L = \{K_1, K_2\}$. Further, suppose that the crossing points of D at which the projections of K_1 and K_2 intersect are $c_1, c_2, \ldots, c_n$. (Here we are considering only the crossing points of the projections of K_1 and K_2, which are not self-intersections of the knot component.) Then $1/2\{\text{sign}(c_1) + \text{ sign } (c_2) + \ldots + \text{ sign } (c_n)$ is called the *linking number* of K_1 and K_2, which is usually denoted by $Lk(K_1, K_2)$.

This number has some very striking properties, the most important of which is that the linking number $lk(K_1, K_2)$ is an invariant of L, that is, it is the same for two or more diagrams of L. Another important property is that the linking number of the diagram is not changed by Reidemeister type III move. Furthermore, two knots diagrams for the same knot are related by using only type II and type III moves if and only if they have the same writhe and winding number (Trace 1983). Importantly, it has been showed (Manturov 2004) that for every knot type there are a pair of knot diagrams so that every sequence of Reidemeister moves taking one to the other must use all three types of moves. Recently A. Coward showed that for links diagrams representing equivalent links, there is a sequence of moves ordered by type: first type I moves, then type II moves, and then type III. The moves before the type III moves increase crossing number while those after decrease crossing number.

Tait went on to list knots according to their numbers of crossing when drawn on the plane in the most efficient possible way. The *crossing-number* $c(K)$ measures the complexity of a knot, and is one of its basic invariants. If D is a diagram of K, let $c(D)$ be the number of crossing points of D. Then $c(K) = \inf c(D)$. The advantage of this measure is that there are only a finite number of diagrams, hence there are only a finite number of knots to a given crossing-number. Tait called the minimal number of crossing-points achievable for the diagram of a given knotted structure the degree of knottiness of that structure (Fig. 35).

Tait undertook an extensive study and classification of knots (Menasco and Thistelthwaite 1993). He enumerated knots in terms of the crossing number of a plane projection and also made some pragmatic discoveries, which have since been christened "Tait's conjectures". An important class of knots is that of *alternating knots*. A diagram (of a knot) is alternating if

(i) it has at least one crossing;
(ii) the curve goes alternately over and under at successive crossing points;

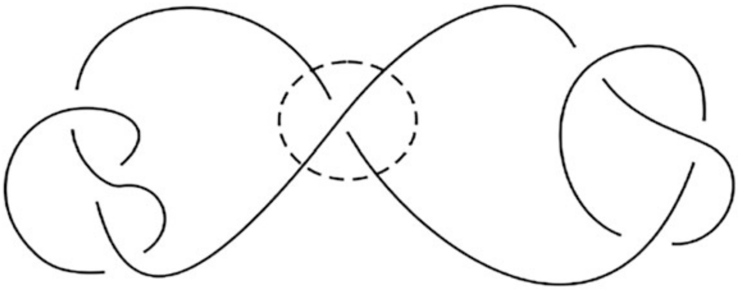


Fig. 35 A nugatory crossing

(iii) the diagram has no 'nugatory' crossing, where the shaded disc represents a portion of knot diagram that may or may not be trivial. Let K be a knot in S^3 and D a diagram of K. The we can ask: In D, which crossing is nugatory? The following definition answered the question: *A crossing in D is nugatory if we have the same knot after the crossing change in it*. (See Torisu 1999).

Thus, although the definition of an alternating knot is a simple matter, it is nevertheless non-trivial. Furthermore, a reduced alternating diagram can readily be understood intuitively, *i.e.*, it is a regular diagram that we cannot change into one with fewer crossings. However, it is only quite recently that this has been mathematically and rigorously proven (Murasugi 1987). The study of alternating knots has a long history, and over this period a panoply of its characteristics had been proven, but the Tait's conjectures (concerning the primary local problems) had haunted researchers but proved unyielding. These were the problems/conjectures that concerned Tait in the nineteenth century and from which a specific type of knot, namely, the alternating knots, emerged. The new invariants, the Jones polynomial and the skein (Kauffman) polynomial have played the crucial role in solving these problems. The three best known of Tait's conjectures can be stated briefly as follows:

First Conjecture A reduced alternating diagram is the minimum diagram of its alternating knot (or link). Moreover, the minimum diagram of a prime alternating knot (or link) can only be an alternating diagram. In other words, a non-alternating diagram can never be the minimum diagram of a prime alternating knot or link. (Here we need to recall what is meant by a *prime* knot. Let K be a knot. When a true (non-trivial) decomposition cannot be found for K, we say that K is a *prime* knot. In this sense, it is equivalent to the way we define a prime number, *i.e.*, a natural number that cannot be decomposed into the product of two natural numbers, neither of which is 1. Thus, a knot K is either a prime knot or can be decomposed into at least two non-trivial knots.

Second Conjecture Suppose that D_1 and D_2 are two reduced alternating diagrams of an alternating knot (or link) K, then the writhe number of D_1 is equal to the writhe

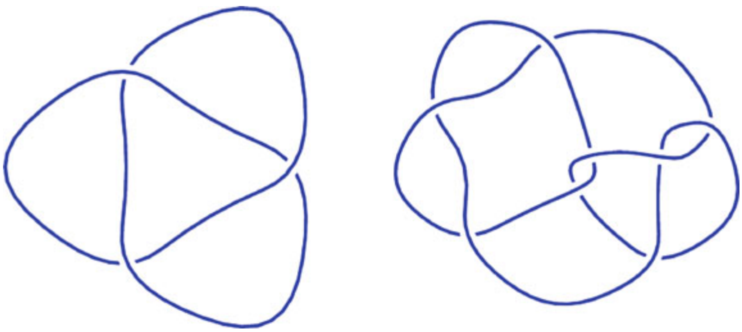


Fig. 36 The diagram for the trefoil knot 3_1 is alternating, whereas the diagram for the knot 8_{19} is non-alternating

number of D_2, that is: $w(D_1) = w(D_2)$. Moreover, when D is a reduced alternating diagram, then this number is an invariant of K. (Recall that if D is an oriented regular diagram of an oriented knot (or link), then the sum $w(D)$ of the signs of all the crossing points of D is said to be the *Tait number* of D, or the *writhe* of D.

Third Conjecture Suppose that D_1 and D_2 are two reduced alternating diagrams of an alternating knot K. Then we can change D_1 into D_2 by performing a finite number of flips. (This conjecture has very recently been shown to be true.)

Despite the great strides made by topologists in the twentieth century the Tait's conjectures resisted all attempts to prove them until the late 1980s. The new Jones invariants turned out to be extremely useful and enabled several of Tait's conjectures to be established. Morwen Thistlethwaite, along with Luis Kauffman and Kunio Murasugi, proved the first two Tait conjectures in 1987, and Thistlethwaire and William Menasco proved the flyping Tait conjecture in 1991.

The new polynomial invariant of links discovered by Vaughan Jones in 1984 is capable of distinguishing many knots from their mirror images, whereas the Alexander polynomial is unable to distinguish between links and their mirror images. So, for example, the trefoil and its trefoil image can be distinguished. What was remarkable was that this new invariant could be defined, as we shall see, using very simple skein relations (Fig. 36).

Historical Note on Knots and their Diagrams

Any non-trivial knot can be represented by one or more diagrams. Two diagrams may correspond to the same knot, although they visually look very different, like in the Perko pairs of knots (Perko 1974). The converse isn't true. Mathematicians are somehow entomologists at heart: they are interested in classifying things (like curves, surface, spaces) in some classes. For instance, one would like to have a list

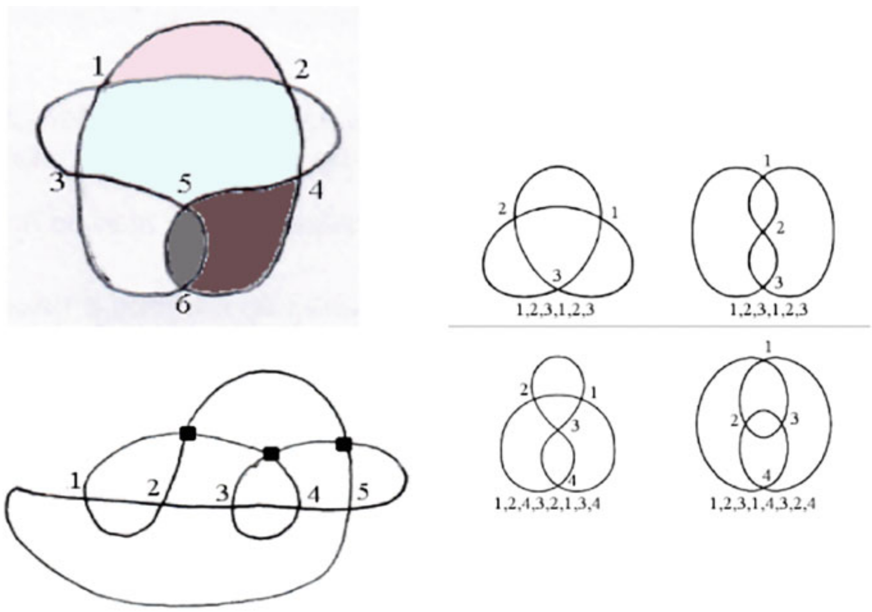


Fig. 37 Gauss's diagrams

of all possible knots, up to equivalence. The more efficient way for representing knots consists in projecting them onto a plane so that the projection crosses itself in a finite number of points, and only two strands of the knot pass through any crossing point. So, one may look for the projection with the least number of self-crossings of a given knot (or, more precisely, of all equivalent forms of a given knot), and tray to classify the knots according to this least number of self-intersections. For instance, the trivial knot (the standard circumference) clearly admits a representation with no self-crossings. The first non-trivial knot is the trefoil knot, whose representation has exactly three self-crossings (Fig. 36).

Gauss's diagrams (1794) is the first mathematical invariant computed from the number of crossings made by any knotted curve. The Gauss diagram is a projection of a closed smooth curve in the plane, and it consists to compute the number of (double-points) crossings which the curve does to form a knot or a link (Fig. 37). On the right of the figure, one sees the Gauss diagram of curves corresponding to the projections of the trefoil knot and the eight-figure knot. On the links (top), one has a curve with six crossings in the Gauss diagram (1, 2, 4, 6, 5, 4, 2, 1, 3, 5, 6, 3). It is possible to complexify the Gauss diagram by adding other crossings.

Precisely, in mathematics Gauss diagram invariants are isotopy invariants of oriented knots in manifolds which are the product of a (not necessarily orientable) surface with an oriented line. These invariants are defined in a combinatorial way using knot diagrams, and they take values in free abelian groups generated by the first homology group of the surface or by the set of free homotopy classes of loops in this surface. There are three main results: 1. The constructions of invariants of finite type for arbitrary knots in non-orientable 3-manifolds. These invariants can distinguish homotopic knots with homeomorphic complements. 2. Specific invariants of degree 3 for knots in the solid torus. These invariants cannot be generated for knots in handle bodies of higher genus, in contrast to invariants coming from the theory of skein modules. 3. T. Fiedler recently introduced a special class of knots called global knots in $F \times \mathbf{R}$ and he constructed new isotopy invariants, called T-invariants, for global knots (Fiedler 2001). Some T-invariants (but not all!) are of finite type but they cannot be extracted from the generalized Kontsevich integral (Kontsevich 1993), which is consequently not the universal invariant of finite type for the restricted class of global knots. Fiedler proved that T-invariants separate all global knots of a certain type.

Equivalence of Knots and Links

The equivalence problem for knots and links asks the most fundamental question in the field: can we decide whether two knots or links are equivalent? This question arises in any branch of mathematics. In topology, there are many well-known negative results. A central result of Markov (1958) states that there is no algorithm to determine whether two closed n-manifolds are homeomorphic, when $n \geq 4$. But in dimension 3, the situation is more tractable. In particular, the equivalence problem for knots and links is soluble.

One original way of interpreting the equivalence problem is in terms of Reidemeister moves. Recall that a Reidemeister move is a local modification to a link diagram as shown in Fig. 28. Reidemeister proved (1927) that any two diagrams of a link differ by a sequence of Reidemeister moves. It is a consequence of Reidemeister's theorem that if the two initial diagrams represent the same link, then this process is guaranteed to produce a sequence of Reidemeister moves taking one diagram to the other.

Of course, this does not provide a solution to the equivalence problem, because if two diagrams represent distinct link types, then the above process does not terminate. But if one knew in advance how many Reidemeister moves were required to take one diagram to other, then one could stop the process when one had tried all possible sequences of Reidemeister moves of this length and if a sequence of Reidemeister moves taking one diagram to the other had not been found, then one could declare that the links are distinct. More specifically, a computable upper bound on the number of Reidemeister moves required to relate two diagrams of the same link provides a solution to the equivalence problem. In fact, it is not hard to show that the converse is also true: if there is a solution to the equivalence problem, then, given natural numbers n_1 and n_2, one can compute an upper bound on the number of Reidemeister moves required to relate two diagrams of a link with n_1 and n_2 crossings. One enumerates all link diagrams with these numbers of crossings, and then one sorts them into their various link types, using the hypothesized algorithm to solve the equivalence problem. Then, for all diagrams of the same link type, one starts applying Reidemeister moves to these diagrams. By Reidemeister's theorem, eventually a sequence of such moves will be found relating any two diagrams of the same link. Hence, one can compute an upper bound on the number of moves that are required.

Reidemeister proved that any two diagrams of a link differ by a sequence of Reidemeister moves. This result has many applications. For example, it is often used to show that an invariant of link diagrams actually leads to an invariant of knots and links, by showing that the invariant is unchanged under each Reidemeister move. It also leads to many natural and interesting questions, including the following.

Problem *Find good upper and lower bounds on the number of Reidemeister moves required to relate two diagrams of a knot or link.*

As already mentioned, there exists a computable upper bound on Reidemeister moves if and only if there exists a solution to the equivalence problem. Moreover, if one just considers diagrams of a fixed link type, then the existence of a computable upper bound on Reidemeister moves is equivalent to the existence of a solution to the recognition problem for that link type. There is, of course, a linear lower bound on the number of Reidemeister moves required to relate two diagrams of a link. More precisely, if two diagrams have n_1 and n_2 crossings, then the number of moves relating them is at least $|n_1 - n_2|/2$, simply because each Reidemeister move changes the crossing number by at most 2.

Hass and Nowik (2010) proved that, in general, at least quadratically many moves may be needed. For each link type, they produced a sequence of diagrams D_n, where the crossing number of D_n grows linearly with n, but where the minimal number of Reidemeister moves relating D_n to D_0 is a quadratic function of n. They did this by finding an invariant of knot diagrams that does not lead to a knot invariant, but that changes in a controlled way when a Reidemeister move is performed. Hence, a 'large' difference in the invariants of two diagrams implies that they necessarily differ by a long sequence of Reidemeister moves.

In contrast to these lower bounds, the known upper bounds on Reidemeister moves are vast. The best known bound in general is due to A. Coward and M. Lackenby (2014): If two diagrams of a link have n and n' crossings respectively, then these diagrams differ by a sequence of at most

$$2^{2^{\cdot^{\cdot^{(n+n')}}}}\Bigg\}\text{ height } c^{(n+n')} \tag{35}$$

Reidemeister moves, where $c = 10^{1000000}$. This is huge, but it was the first known upper bound that is primitive recursive.

A recent theorem of Marc Lackenby (2015) provides a polynomial upper bound on Reidemeister moves for the unknot: *If a diagram of the unknot has n crossings, then there is a sequence of at most $(236n)^{11}$ Reidemeister moves taking it to the trivial diagram (or crossings).*

From the Alexander Polynomial to Seifert Surfaces for Knots

Let's start this section with some important notions and definitions relating to the work of Herbert Seifert. A Seifert surface F for a knot K is an orientable surface S^3 such that $\partial F = K$; i.e., a surface with one boundary component such that the boundary component of the surface is a given knot. Orientable surfaces with a single boundary component are all homeomorphic to genus g surfaces with a single open disk removed. So, if F is a Seifert surface, then we say $g(F) = g(\hat{F})$ where $\hat{F}$ is the surface obtained by capping off F. Also, we can compute that $g(F) = 1 - \chi(F)/2$, since we have $\chi(F) = 2–2g\left(\hat{F}\right)$, $\chi(F) + 1 = 2–2g\left(\hat{F}\right)$, $\chi(F) + 1 = 2 - 2g(F)$. Seifert established an important result in 1927 according to which *there can be different surfaces for isotopic knots*. Stated differently, a given knot or link can have many various spanning surfaces. For example, two isotopic diagrams will have rather different Seifert surfaces. In 1934, Seifert proved the theorem that *such a surface can be constructed for any knot or link*. The process of generating this surface is known as Seifert's algorithm. One has further the corollary that if a knot K has $g(K) = 0$ then it is trivial. If a nontrivial knot has a Seifert surface of genus 1 then $g(K) = 1$. In 1983 David Gabai proved the theorem that *Seifert's algorithm applied to an alternating projection of an alternating knot does yield a Seifert surface of minimal knot genus*. A knot or link L^n in S^{n+2} is said to be fibered if there exists a fibration map $f : S^{n+2} - L + S^1$, and if the fibration is well-behaved near L. That is, each component L_i is to have a neighborhood framed as $S^n \times D^2$, with $L_i \cong S^n \times 0$, in such a way that the restriction of f to $S^n \times (D^2 - 0)$ is the map into S^1 given by $(x, y) \to y/|y|$. It follows that each $f^{-1}(x) \cup L$, $x \in S^1$, is an $(n + 1)$-manifold with boundary L; in fact, a Seifert surface for L. (See Rolfsen 1990, for a formal definition and developments). Then, remind that for a closed Haken manifold M^3, the torus decomposition of Jaco and Shalen (1979) and Johannson (1979) and together with the uniformization Theorem of Thurston (1986; see also Sakuma 2020) say that there is a collection of incompressible tori $W_M \subset M$, unique up to ambient isotopy, which cuts M into Seifert fibered manifolds and hyperbolic manifolds of finite volume. Denote the regular neighborhood of W_M by $W_M \times [-1, 1]$ with $W_M \times \{0\} = W_M$. We write $M \backslash W_M \times (-1, 1) = H_M \cup S_M$, where H_M is the union of the Seifert fibered manifold components (Figs. 38 and 39).

H. Seifert discovered (1934) another important concept in knot theory, which is the *genus* of a knot, classical very important invariant in knot theory. Seifert used a connected, oriented, compact surface that has the knot as its boundary to define the genus of a knot. At first sight, it is surprising that such a surface exists for any knot and link. Seifert showed that such a surface can be generated from a knot diagram using a simple algorithm. It consists in four steps. First of all, assign an orientation to the components of the knot or link. Secondly, eliminate all crossings. At each crossing two strands meet. A crossing is eliminated by cutting the strands, and connecting the incoming strand with the outgoing strand, and vice versa. This gives a set of non-intersecting (topological) circles, called Seifert circles. Thirdly, if circles are nested in each other, offset them in a direction perpendicular to the diagram. Fill in the circles, giving disks. Finally, connect the disks using twisted

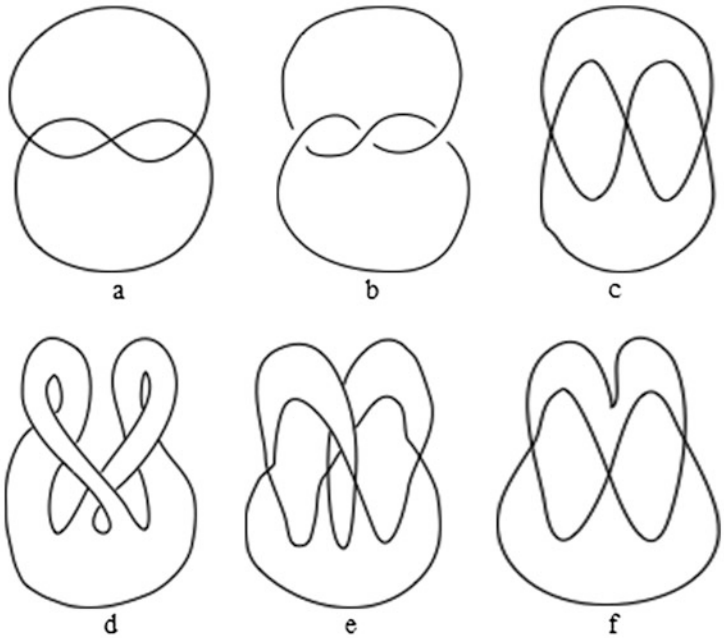


Fig. 38 Knotted surfaces (**d**), (**e**) and (**f**) spanned respectively by figure height-knots (**a**), (**b**) and (**c**)

bands. Each band corresponds to a crossing, and has one twist, with orientation derived from the crossing type. A twist is a rotation over plus (right-hand) or minus (left-hand) 180 degrees. Note that the crossing type does not influence the circles that are generated. The resulting surface satisfies the requirements. Different projection of the knot lead to different surfaces, possibly also with a different genus. The genus of a knot is defined as the minimal genus of all oriented surfaces bounded by the knot. Note that not all surfaces bounded by a knot arise from Seifert's algorithm, and the genus of these other surfaces can be different (Fig. 40).

An important fact is that Seifert surfaces admit a braid representation. To generate Seifert surfaces for arbitrary knots and links, we need an encoding for these knots and links. Many different encodings have been developed, such as the Conway notation and the Dowker-Thistlethwaite notation. The braid representation is also very useful. By means of braids, several different models of surfaces can be generated easily; and also, the braid representation lends itself well to experimentation.

A braid consists of a set of n strings, usually running from a left bar to a right bar. Strings are allowed to cross, and the pattern can be encoded by enumerating the crossings from the left to the right. A crossing is denoted by σ_k^j, which means that strings at the k'th and $k+1'$th row are twisted j times, where $j=1$ denotes a right-hand crossing and $j=-1$a left-hand crossing. The closure of the braids defined by

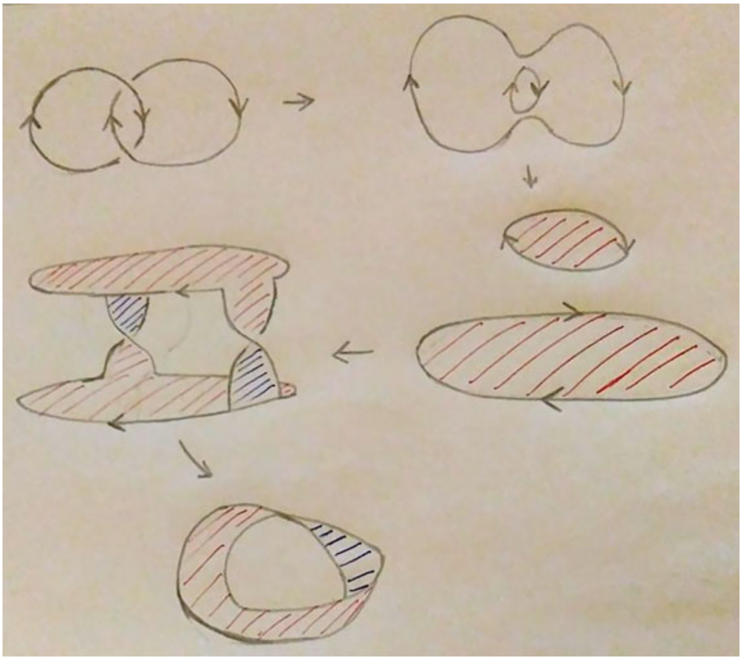


Fig. 39 Seifert surface of a Hopf link

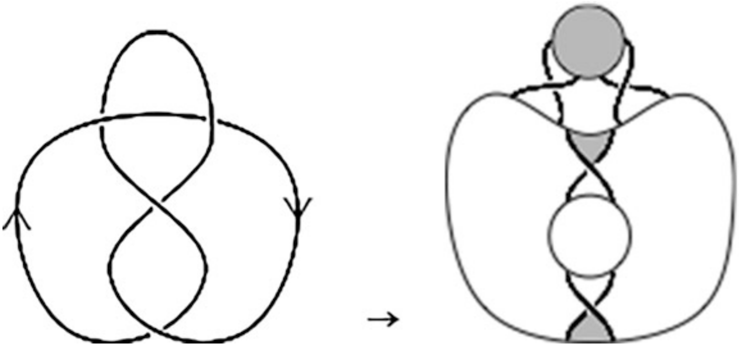


Fig. 40 Seifert algorithm: By smoothing all crossings of a knot diagram, we obtain Seifert circles (mutually disjoint circles in the plane). Construct mutually disjoint disks in R^3 bounded by the Seifert circles, and join them by bands. The resulting surface is a Seifert surface. Steps: (i) assign orientation, (ii) eliminate crossings and (iii) add bands; shown for a knot and a crossing

attaching the left bar to the right bar, such that no further crossings are introduced. In other words, we add n extra strings that connect the beginnings and ends at the same row, without further crossings. Every knot and link can be defined as a braid. A trefoil has the *braid word* $\sigma_1\sigma_1\sigma_1 = \sigma_1^3$, a figure-eight knot can be represented as $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ (Fig. 41). An alternative notation for braids is to use uppercase

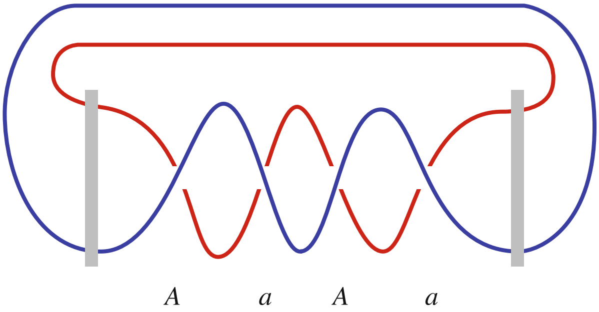


Fig. 41 *AaAa* gives simple boundaries, but a complex topology of the surface

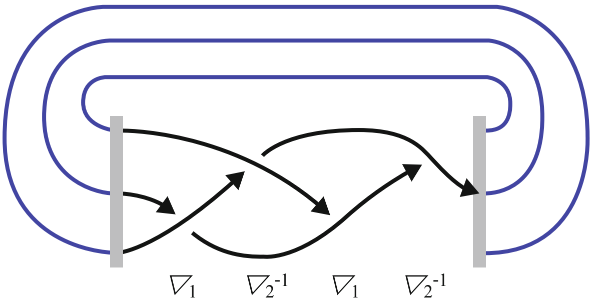


Fig. 42 Braid representation of figure eight-knot

letters for right crossings and lowercase letters for left crossings, and where the character denotes the strings effected, according to alphabetic order. Hence, a trefoil, is encoded by *AAA*, and a figure-eight knot by *AbAb*. Furthermore, every possible braid word defines a knot or a link, which makes this representation well suited for experimentation.

As mentioned, the braid presentation does not always yield a surface with minimal genus. This property can also be used to produce surfaces with a high genus that are bounded by simple knots and links. Consider the knots and links *AaAaAa*... One strand is always on top of the other here (Fig. 41), hence this produces either two loose rings or one unknot, foe an even or odd number L of letters, respectively. The Seifert surface is more complex, and contains $L - 1$ holes. The result of *AaAa* is intriguing. Locally, the shape is simple to understand, but it is hard to form a mental image of the complete shape, like one can imagine a sphere or a torus (Figs. 41, 42, and 43).

The *Alexander polynomial* of a knot or a link, which has become one of the cornerstones of knot theory, is very closely connected with the topological properties of the knot. Besides, of great significance is that there are various methods by which we may calculate the Alexander polynomial; one of these methods is called the *Seifert matrix*. Concisely, the Seifert matrix of a knot is a

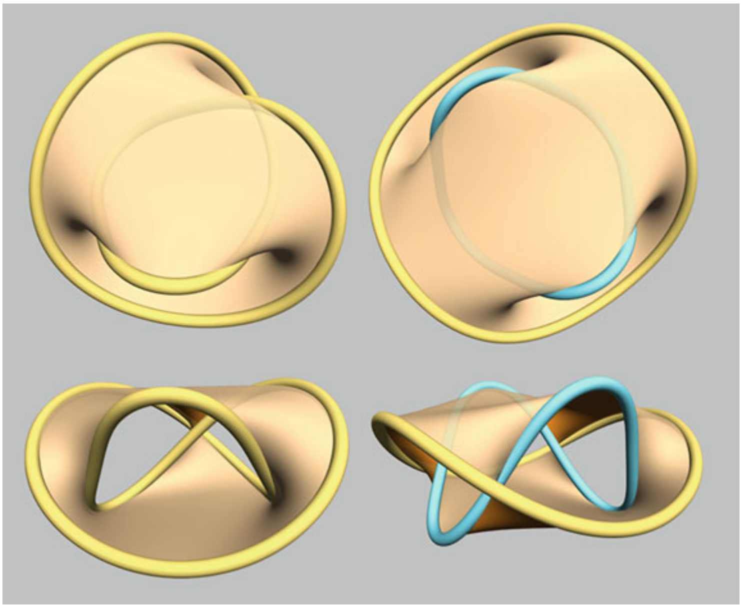


Fig. 43 *AaA* (left) and *AaAa* (right)

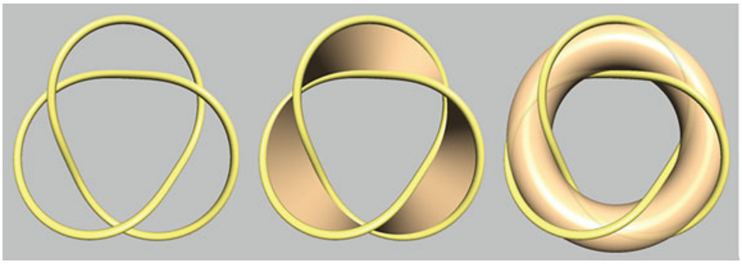


Fig. 44 From links to right, in the first picture we see a trefoil knot. If we twist it three times, and glue the ends together, we obtain a strip which is a kind of Möbius strip. There is an orientable surface that has the trefoil knot as its boundary (picture in the middle). Equivalently, one can embed a trefoil knot in a closed surface. This is called a torus knot (last picture on the right)

very combinatorial topological and very nice tool allowing to show how various surfaces can be constructed from the same knot diagram: it suffices that the linking numbers for all these surfaces be equal or rearranged in such a way that they turn out to be equivalent. Alexander described a method for associating with each knot a polynomial such that if one form of a knot can be topologically transformed into another form, both will have the same associated polynomial. Now, suppose M is the Seifert matrix of a knot (or link) K, then det $(M + MT)$ is an invariant of the knot K. Furthermore, the determinant of a knot (or link) K is completely independent of the orientation assigned to K′′ (see Murasugi 1996, 141–142) (Fig. 44).

It is here important to underline that the Alexander polynomial arises from the homology of the infinite cyclic cover of the complement of a knot. Equivalently it can be derived from considering cohomology of the knot complement with coefficients in a flat line-bundle (Vassiliev 1990). Following Kauffman (1987a; b) and Murasugi (1996), we can now define mathematically, in a summarized way, the Alexander polynomial.

Let $K \subset S^3$ be an oriented knot or link, and $F \subset S^3$ a connected oriented spanning surface for K. Let $\theta : H_1(F) \times H_1(F) \to Z$ be the Seifert pairing.

Definition 9 *Two polynomial f(t),* $g(t) \in \mathbf{Z}(t)$ *are said balanced (written* $f = g$*) if there is a non-negative integer n such that* $\pm t^n f(t) = g(t)$ *or* $\pm t^n g(t) = f(t)$.

Definition 10 *Let K, F, θ be as above. The Alexander Polynomial,* $\Delta_K(t)$*, is the balance class of the polynomial* $\Delta_K(t) = D(\theta - t\theta')$*. It follows from S-equivalence (due to Seifert) that this determinant is well defined on isotopy classes of knots and links up to multiplication by factors of the form* $\pm\, t^n$*"* (see Kauffman 1987a, b, 229).

Reidemeister Moves and Classical Knot Invariants

We already stressed the fact that *Reidemeister moves* have played a very important role in the development of knot theory.

A *link* is an embedding of a collection of circles into the oriented three-dimensional sphere or, equivalently, into three-dimensional Euclidean space. A link of n circles is said to have n components, and a 1-component link is called a *knot* if it cannot be continuously deformed into a standard (unknotted) circle.

A *knot (or link) invariant*, by its very definition, does not change its value if we apply one of the elementary knots moves. It is often useful to project the knot onto the plane, and then study the knot (or link) via its regular diagram. We must now ask ourselves what happens to, what is the effect on, the regular diagram if we perform a single elementary knot move on it? This question has a beautiful answer. Two diagrams are equivalent (or, mathematically stated, represent the same isotopy class) if and only if one can get from the first to second by a sequence of (local) deformations known as the *Reidemeister moves* (from the name of the German mathematician who wrote the first book on topological knot theory) (Jones 1985).

Each of these moves, usually called I, II, and III, consists in modifying a small portion of a link diagram while keeping the rest fixed. We will call a small portion of a component of a link, as it appears in a diagram, a *strand*. Move I consist in modifying a diagram in a neighborhood containing only a single strand by putting a "twist" in the strand. Move II consists in "cancelling" a pair of crossings, as shown below. Move III consists in sliding a strand under a crossing as shown below. (Note that we can also interpret this move as sliding a strand *over* a crossing.) (Fig. 45)

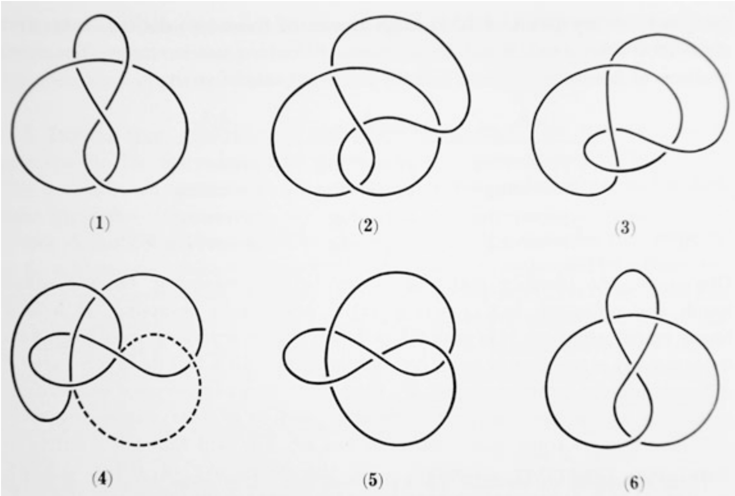


Fig. 45 The figure-height knot (1), which is antichiral, transforms into its mirror image (6) by a sequence of planar isotopies (or Reidemeister moves)

In fact, these Reidemeister moves are sufficient to get between any two diagrams representing the same isotopy class of knots (or links). In other words, he proved that if we have two distinct projections of the same knot, we can get from the one projection to the other by a series of Reidemeister moves and planar isotopies. Therefore, according to Reidemeister, there is a series of Reidemeister moves that takes us from the first projection to the second. The idea of the proof is simple; in performing an isotopy of a link, its projection onto the plane will occasionally encounter certain "catastrophes" where it does something non-generic; if one looks at the link diagram before and after one of these catastrophes, it will be seen that a Reidemeister move has occurred.

One can now define an equivalence (isotopic deformation) between two regular diagrams D and D' of knots K and K'.

Definition 11 *If we can change a regular diagram D to another D' by performing, a finite number of times, the operations (moves) I, II, III and/or their inverses, then D and D' are said to be equivalent: $D \approx D'$. Suppose that D and D' are regular diagrams of two knots (or links) K and K', respectively. Then $K \approx K' \iff D \approx D'$.*

Therefore, a knot (or link) invariant may be thought of as a property (quality) that remains unchanged when we apply any one of the above Reidemeister moves to a regular diagram. It is interesting to play around with the Reidemeister moves and discover that some knots are isotopic to their mirror images, while other are not. The trefoil is not isotopic to its mirror image, so there are really two forms of trefoil, the right-handed and left-handed trefoil. On the other hand, the figure-eight knot is isotopic to its mirror image; such knots are said to be amphicheiral.

Reidemeister moves and Alexander polynomials are among the most important invariants in topological knot theory. The Alexander polynomial makes it possible to distinguish between knots that otherwise look very similar, or, conversely, to detect new invariants which permit to recognise when two or more seemingly different knots are in fact equivalent. The polynomial, written $\Delta_K(t)$, is constructed according to the numbers of different kinds of crossing in a knot diagram. A simple trefoil knot, for example, has $\Delta_K(t) = t - 1 - 1/t$. Differently deformed versions of the same knot have the same Alexander polynomial; knots with different polynomials are different. Yet two knots with the same Δ are not necessarily equivalent. The Alexander does not distinguish, for example, between the square knot and the granny knot. Reidemeister moves simplify the analysis of knots. Two knots are the same if and only if their diagrams can be made identical by some combination of (two-dimensional) moves. A knot is not perceptively changed if we apply only one elementary knot move. However, if we repeat the process at different places, several times, then the resulting knots *seem* to be completely different.

Let us take, for example, *Perko's pair* of knots (see below). In appearance, Perko's two knots look completely different from each other. In fact, for the better part of a hundred years, nobody thought otherwise. However, it is possible to change the one into the other by performing the elementary knot moves a significant number of times, as showed in 1970 by the American lawyer K.A. Perko. More precisely, we say that two knots K_1 and K_2 are equivalent if there exists an orientation-preserving homeomorphism of R^3 that maps K_1 to K_2. Moreover, if two knots K_1 and K_2 that lie in S^3 are equivalent, then their complements $S^3 - K_1$ and $S^3 - K_2$ are homeomorphic (Fig. 46).

Knot invariants are functions which remain constant when we apply some small changes to knots; in other words, they are mathematical expressions that describe certain properties of knots. They derive their name from the fact that they do not

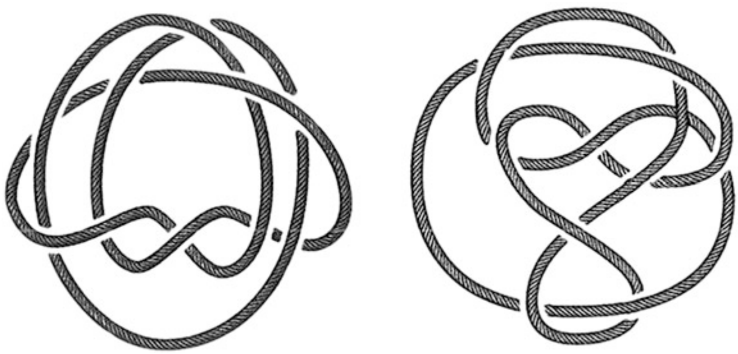


Fig. 46 *Perko knots*. These two diagrammatic representations correspond to the same knot. They are visually different, but topologically equivalent, meaning that they can be deformed one into the other by a finite sequence of Reidemeister moves, i.e., of planar isotopies. Perko knots are antichiral knots. *Antichiral* means that each knot is the mirror image of the other; otherwise stated, the two knots are invariant by reflection or mirror symmetry, although their images or planar representations looks very different

change according to the way the underlying knot is pushed, pulled or twisted. The path from a loop of string to an equation in powers of t is a complex one, however. How do we determine an invariant, for example, the Alexander or Jones polynomial, of a given knot? The simplest method is the skein relation invented by British mathematician John Conway. Another technique, the Kauffman "states model," is notable for its elegance and profound connection to statistical mechanics. A knot is a line drawn in three-dimensional space, which begins and ends at the same point, and which is never broken. The nature of a knot depends neither on its form nor on its size, nor on the position of its loops. The primary task of the knot theory is to prove that such knots are clearly differentiated. However, there are knots which though seems apparently different, they are actually intrinsically equivalent or isotopic, as for example Perko has shown, or as that is the case for Kinoshita's and Conway's non-trivial knots, which have $\Delta K(t) = 1$ for Alexander's polynomial.

Dehn Surgery of Knots and his Work on Knot Theory and 3-dimensional Manifolds

We need here to recall another important result in knot theory, due to Max Dehn (Dehn 1914). In short, it states that *a left-hand trefoil knot cannot be deformed into a right-hand trefoil knot*. Dehn's proof is based on the observation that a deformation of one trefoil knot into the other would induce an automorphism of the knot group which associates certain geometrically significant elements, namely "latitude" and "longitude" curves on the neighborhood torus of the knot, which determine an orientation of the ambient space. The trefoil knot is the simplest knot, i.e. the simplest interwoven space curve which cannot be continuously transformed in space into a circle. It is the only knot with a plane projection having only three double points. In space one can distinguish two different kinds of trefoil knot, which result from each other by reflection, say, in a plane. The problem is then to find a way of distinguishing the two kinds of trefoil knot in space topologically, i.e. to show that the left knot is not convertible into the right by a continuous space transformation. This enables to point out a very remarkable phenomenon. The space with a left trefoil knot and the space with a right trefoil knot are homeomorphic, i.e. so composed that the knots occupy corresponding regions in the two spaces. Despite this the two knots in the same space are not convertible into each other by a continuous transformation of space. Thus, we have before us a kind of topological symmetry, called "mirror symmetry", which plays nowadays a fundamental role in the attempts to explain the essential features of space and matter.

The above isn't the only contribution of Dehn to knot theory. An even major contribution came from him and concerns the topology of 3-manifolds. In 1910 he published a paper where he showed that taking two non-trivial knot complements (the complements of a solid-torus neighborhoods of two non-trivial knots in the 3-sphere) and sewing them together so that the meridian of one matches the longitude of the other and vice-versa produces a 3- manifold with the homology of S^3. Dehn also claimed that these manifolds were not diffeomorphic to S^3 since they contain a torus that does not bound a solid torus. But it was not until 1924 that Alexander established that every torus in the 3-sphere bounds a solid torus. On the other hand, it is not too difficult to see (using Van Kampen's theorem which was formulated later but more or less understood even in Poincaré's day) that the fundamental groups of the manifolds that Dehn constructed in this way are non-trivial, so that these manifolds are indeed distinct from S^3. This produces a plethora of homology 3-spheres (manifolds with the homology of the 3-sphere) (see Bonahon 2002, Thurston 1997, and McMullen 2011).

Dehn also introduced a notion that has proven to be of central importance in the theory of 3-manifolds; namely, the notion of Dehn surgery. Given a 3-manifold M and a knot $K \subset M$, one removes a solid torus neighborhood of K from M and sews this solid torus back into the resulting complement using some self-diffeomorphism of the boundary 2-torus. These self-diffeomorphisms up to isotopy are identified with $SL(2, \mathbf{Z})$ via the action of the diffeomorphism on the first homology of the two-torus. Dehn then constructed manifolds by this method, starting with a (2, q)-torus knot in S^3—these are knots lying on the surface of a standard torus and wrapping that torus linearly twice in one direction and q times in the other (q must be odd). Dehn identified which of the non-identity Dehn surgeries on (2, q)-torus knots produce manifolds with the homology of the 3-sphere and showed that they all have non-trivial fundamental groups (usually infinite). When the knot is the (2, 3)-torus knot there is a surgery that produces a manifold with the same group as Poincaré's example (later proved to be diffeomorphic to Poincaré's example).

The examples of Dehn surgery on (2, q)-torus knots were much better understood after the work of Seifert (1933) and Seifert and Threlfall (1934). They considered 3-manifolds that admit locally free circle actions (now called Seifert fibrations). They showed that all of the examples coming from Dehn surgery on (2, q)-torus knots were such manifolds and they showed how to compute the fundamental group of these manifolds. In particular, Dehn's exceptional example was shown to have the same fundamental group as Poincaré's original example—it is the pre-image in $SU(2)$ of the symmetries of the regular icosahedron (Dehn had earlier considered this fundamental group but slightly miscomputed it).

The Fundamental Group of Knots and Links

Groups apply to the study of knots in a fundamental way (see Birman 1974, and Boi 2006c). Instead of assigning a polynomial to each knot, one assigns a group, that is an algebraic structure. These structures forms invariants in that whenever two knots have the same type, there exists an isomorphism between their groups (Neuwirth 1965, Burde and Zieschang 1985).

Informally a group is a collection of *elements* with a single *operation*. The operation is typically addition or multiplication, but transformations of geometric objects can also form a group such as the group of rotations and reflections of a regular polygon. A group G must also obey certain constraints on its elements and its operations, it is associative, it has an identity element, and it must have an inverse. In the case of abstracts groups, we do not require the elements of the group to represent "things" in the real world (such as numbers, rotations, functions, etc.) and instead simply use formal symbols. The operation on the symbols is concatenation to form *words*. For example, the combination of symbols $a \cdot b \cdot a^{-1} \cdot c$ forms the word $aba^{-1}c$. To specify an abstract group, one gives a *presentation* of it. A presentation of a group G comprises a set S of *generators*—so that every element of the group can be written as a product of powers of some of these generators—and a set R of *relations* among these elements. For example, the group presentation $\langle a, b, c \,|a^2, b^2, c^2, abc\rangle$ represents a group with three generators. All elements of the group are words in a, b, c and their inverses and $a^2 = b^2 = c^2 = abc = 1$.

In order to find a group presentation associated to a knot, we need to find a way of encoding the geometric properties of a knot in an algebraic structure. We do this by considering loops in the space $R^3 - K$: the complement of the knot. Fix a base point x_0 somewhere in $R^3 - K$ and consider the collection of all paths that begin and end at that point. We can define a composition of two paths by travelling down the first path and then down the second. We consider two paths to be equivalent if one may be continuously deformed into the other within $R^3 - K$ (i.e. without passing through the knot). In the diagram, p_1 and p_1' are equivalent are p_2 and p_2'. This means that any path that does not pass around an arc of the knot can be shrunk down to the base point (a constant path). In the diagram, loops p_2 and p_2' are both equivalent to the constant path. Formally, the continuous function which maps one path onto an equivalent path is called a *homotopy* of paths. Call the equivalence class of all loops equivalent to a particular loop p a *loop class*. The composition of two loop classes is well-defined and is taken to be the class of the composition of the loops (Figs. 47 and 48).

If we consider the collection of all loop classes in our space the we see that there arises a natural inverse for each class. If a composite loop is formed by travelling down a loop in one direction and then returning to the base point down the same path but in the other direction then we may continuously deform the composite path to the base point. If the original loop was p, call the same loop with the opposite orientation $\bar{p}$. Then the opposite loop $\bar{p}$ plies in the loop class of the constant path at the base point. Thus, we have the

Theorem 10 *The collection of all loop classes in* $\mathbb{R}^3 - K$ *forms a group with composition of loop classes as the group operation.*

To summarize the previous description: each loop class has an inverse; the class of the constant path at the base point can be taken to be the group identity. To make these concepts precise it is necessary to define explicit homotopies between paths in the space. Using these homotopies it is straightforward to show that composition of loop classes is an associative operation.

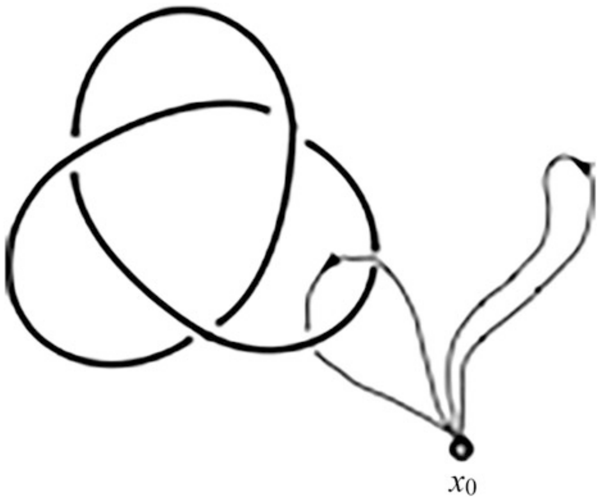


Fig. 47 Paths in $R^3 - K$

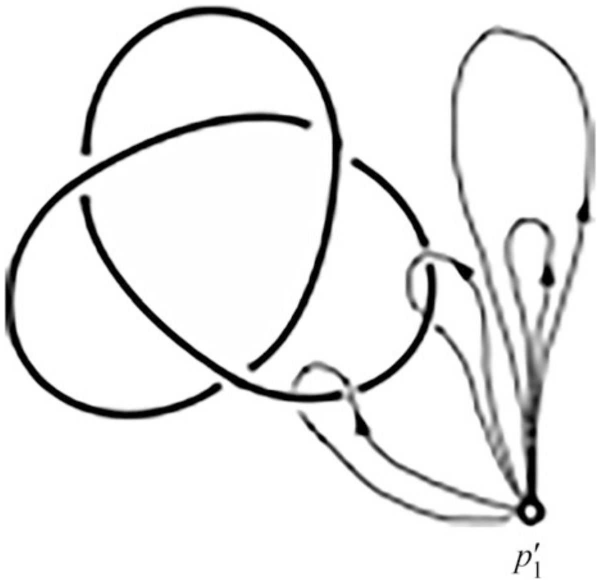


Fig. 48 Equivalent paths

We can read off a set of *generators* for the knot group from the diagram of the knot. Indeed, each region corresponds to a generator of the group. By convention, we place the base point for the loops in the outside region r_0 and so this element is the group identity. We define the loops corresponding to each region as starting from some point in r_0, passing 'above' the knot diagram, through some region r_i

and back 'underneath' the diagram. Fo simplicity, we represent the group element by the same symbol as the label of the region. But what is a loop that passes through a number of regions? Can we verify that it is equivalent to a word in the generators $r_0, \ldots, r_{v+1}$? This is easy to show. If a loop r_* passes through region r_1 from top to bottom, then through region r_2 from bottom to top and finally through region r_3 from top to bottom before returning to the base point, we can imagine continuously deforming the loop so that it visits the base point again in between each region (showed by the dotted lines in the diagram). Hence r_* is in fact equivalent to the word $r_1 r_2^{-1} r^3$. Clearly, this argument can be applied to any possible loop around the knot.

We also need to define a number of relations to give a full presentation of the knot group. Firstly, we must denote the outside region as the identity of the group, since any loop staying in that region can be shrunk to the base point. To show this we place the relation r_0 in the presentation of the knot group. Now consider a crossing point of the diagram with surrounding regions r_j, r_k, r_l, r_m in some precise cyclic order. For each such crossing point, we also obtain the identity $r_j r_k^{-1} r_l r_m^{-1} = 1$, and so we add the relation $r_j r_k^{-1} r_l r_m^{-1}$ to the group presentation. It must be stressed here that the knot group is non-commutative (Fig. 49).

We see the reasoning for this relation by attempting to draw the loop it represents. The loop passes below the overpass at the crossing and above the underpass. Hence the entire loop can be pulled free to lie outside the knot. So compound loops of this type are all equal to the identity. Denote these identities by $c_i\,(r)$ and we have a group presentation for a knot K with v crossing points and $v + 1$ regions: $G(K) = \langle r_0, r_1, \ldots, r_{v+1} | r_0, c_1(r), \ldots, c_v(r) \rangle$.

$$r_j \;\; r_k^{-1} r_l r_m^{-1}$$

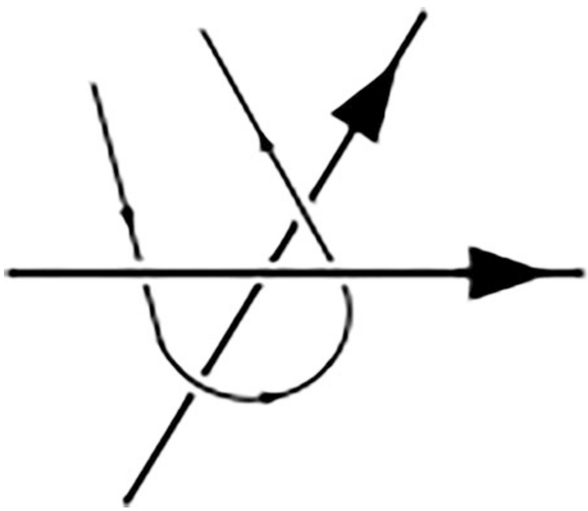


Fig. 49 A loop that can be pulled free from the knot

One of the fundamental problems in knot theory is determining when two knots are equivalent. In general, it is much simpler to show that two knots are equivalent than to show that they are not. All one needs to show equivalence is to provide an ambient isotopy. In the case of two knots given explicitly by diagrams, this can be done easily (though indirectly), through what are called "Reidemeister moves".

The most common method of distinguishing knots is by finding "knot invariants," which are properties that are the same for any two knots that are equivalent. Showing that two knots have different values of a knot invariant then proves that they are not equivalent. It follows directly from the definition of equivalence that for any two equivalent knots, the complements of the images of the knots in S^3—i.e. their *knot complements*—are homeomorphic. Many knot invariants, including the *knot group*, work by using this fact, distinguishing nonequivalent knots by distinguishing their knot complements. Even the knot complement itself could be considered a knot invariant, albeit a very useless one on its own.

Definition 12 *The knot group of a knot K with a base point* $p \in S^3 - \text{Im}(K)$ *is the fundamental group of the knot complement of K, with p as the base point.*

The knot groups are isomorphic for any pair of equivalent knots and base points, since equivalent knots have homeomorphic complements and homeomorphic spaces have isomorphic fundamental groups. Just like how we use a knot type to refer to an equivalence class of knots, we can also use a type of group to refer to equivalence classes of isomorphic groups. This allows us to talk about *the* knot group of a knot, without reference to a base point.

Unfortunately, the knot group is not always enough to show nonequivalence. For example, the right- and left-handed trefoil knot, as mirror images of each other, have the same knot group, but are not equivalent. Nonetheless, the knot group can be used to show two knots are distinct.

Example 2 The simplest knot group to calculate is, of course, that of the unknot. However, this knot group is not trivial.

Proposition 1 *The knot group of the unknot is the infinite cyclic group C.*

The general idea of the proof is that the knot group of the unknot is the fundamental group of the circle, which is the infinite cyclic group. Formally, the fundamental group of the circle $\pi_1(S^1)$ is $\cong \mathbb{Z}$. A similar example is that of the Hopf link.

Proposition 2 *The knot group of the Hopf links the free abelian group with two generators,* $C \times C$.

The fundamental group of the figure-eight knot is the free group on two letters. To prove this choose as base point the point where the two circles meet (dotted in black in the figure); any loop γ can be decomposed as $= a^{n+1}b^{n+1} \ldots \; a^{nk}b^{mk}$, where a and b are the two loops winding around each half of the figure as depicted, and the exponents $n_1, \ldots, n_k, m_1, \ldots, m_k$ are integers. Unlike $\pi_1(S^1)$, the fundamental group of the figure-eight knot is not abelian: the two ways of composing a and b are not homotopic to each other: [a] · [b] ≠ [b] · [a]. The knot group of the trefoil knot can be defined as the braid group B_3, which gives another example of a non-abelian fundamental group (Fig. 50).

A general method for finding the knot group of any tame knot was given by Wilhelm Wirtinger around the beginning of the twentieth century (see Rolfsen 1990, and Kawauchi 1996). It has the advantage of being intuitively simple and easy to compute. Constructing the Wirtinger presentation starts by considering the oriented knot diagram of a knot K. It is viewed entirely in the xy-plane in $\mathbb{R}^3$, except for the lower part of each crossing, which dips down below to avoid intersection with the above segment. Remember that a knot diagram of a tame knot consists of finitely many arcs in the plane, with finitely many crossings at the ends were one arc bridge under another. At each crossing, we consider the arc that passes over to be unbroken, so each side is part of the same arc. Meanwhile, the piece that passes under is broken, so the two sides are ends of two different arcs (or in some cases, the two ends of the same arc). We can let n be the number of arcs in the knot diagram, and we can number the arcs $a_0, a_1, \ldots, a_{n-1}$. If K is a true knot, then we can assign the numbers such that a_{i+1} is the arc that comes after a_i with the given orientation, with addition in $\mathbb{Z}/n\mathbb{Z}$. Since the Wirtinger presentation can be used for strict links as well as for true (tame) knots, we will in general use a_{i+1} to refer to the arc that follows a_i. Here "+" is no longer an operation; it is a function mapping the set of arcs to itself (Fig. 51).

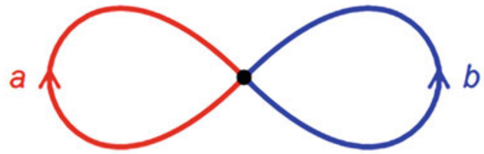


Fig. 50 The fundamental group of the figure-eight knot is the free group on two generators a and b

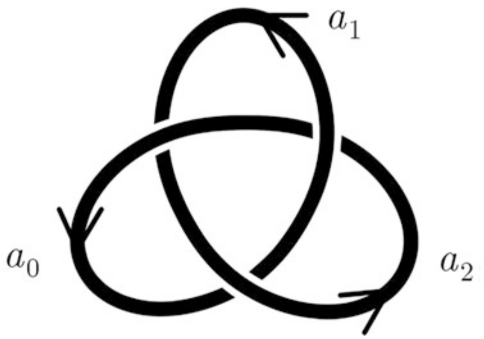


Fig. 51 A trefoil knot with labelled oriented arcs

Invariants of 3-manifolds and Hyperbolic Knots

Knots and links provide a glimpse of the full complexity of 3-dimensional topology. Thurston's results on hyperbolic 3-manifolds, and their impact on knot theory has been one the most significant advances in low-dimensional geometry and topology (see Thurston 1997, Sakuma 2020).

A hyperbolic knot is one whose complement admits a complete Riemannian metric with constant curvature -1. William Thurston famously showed that every knot, with the exception of certain specific types of knot called torus knots and satellite knots, is hyperbolic (Fig. 52).

Hyperbolic knots are an important class among knots. A knot is a smooth embedding of a circle S^1 in S^3. The union of finitely many disjoint knots is a link. By removing a thickened link in S^3 (a union of solid tori) and gluing it back in with a twist, one obtains a new 3-manifold M.

W.B.R. Lickorish showed (1962) that all orientable 3-manifolds can be obtained by surgery on links in S^3. This provides an important tool to study topological properties of three-manifolds. Furthermore, these result has been an important step in the construction of invariants of knots and links in S^3 from the link invariants of $SU(2)$ Chern-Simons theory (see Guadagnini et al. 1990, Kaul 1999, and Turaev 1994). The result states precisely that

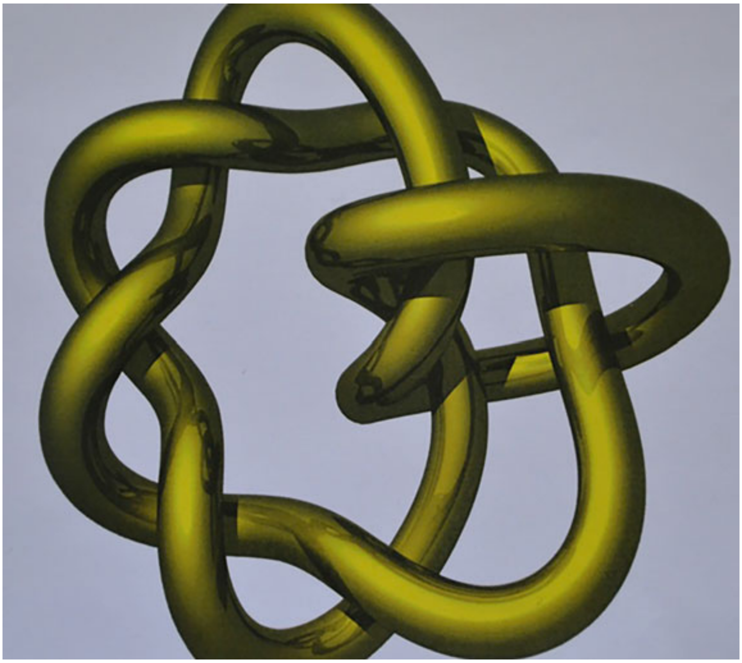


Fig. 52 A hyperbolic knot, i.e. a knot which is the boundary of a surface with negative curvature

Theorem 11 (Lickorish-Wallace). *Every closed, orientable, connected three-manifold, M^3, can be obtained by surgery on an unoriented framed knot or link $\{L, f\}$ in S^3.*

The framing f of a link L is defined by associating with every component knot K_s of the link an accompanying closed curve K_s parallel to the knot and winding $n(s)$ times in the right-handed direction. That is the linking number lk (K_s, K_{sf}) of the component knot and its frame is $n(s)$. A particular framing is the so-called vertical framing where the frame is thought to be just vertically above the two-dimensional projection of the knot as shown below. We may indicate this sometimes by putting $n(s)$ writhes in the strand making the knot or even by just simply writing the integer $n(s)$ next to the knot. Next the surgery on a framed link $\{L, f\}$ made of component knots $K_1, K_2, \ldots, \mathrm{K}_r$ with framing $f = (n(1), n(2), \ldots, n(\mathrm{r}))$ in S^3 is performed in the following manner. Remove a small open solid torus neighborhood N_s of each of each component knot K_s disjoint from all other such open tubular neighborhoods associated with other component knots. In the manifold left behind $S^3 - (N_1 \cup N_2 \cup \ldots. \cup N_r)$, there are r toral boundaries. On each such boundary, consider a simple closed curve (the frame) going $n(s)$ times along the meridian and once along the longitude of the associated knot K_s. Now do a modular transformation on such a toral boundary such that the framing curve bounds a disc. Glue back the solid tori into the gaps. This yield a new manifold M^3. The theorem of Likorish-Wallace assures us that every closed, orientable, connected three-manifold can be constructed in this way (Wallace 1960, and Lickorish 1962).

Move I For a number of unlinked strands belonging to the component knots K_s with framing $n(s)$ going through an unknotted circle C with framing $+1$, the unknotted circle can be removed after making a complete clockwise twist from below in the disc enclosed by the circle C. In the process, in addition to introducing new crossing, the framing of the various resultant component knots, $K_s{}'$ to which the affected strand belongs, change from $n(s)$ to $n^{'}(s^{'}) = n(s) + (lk(K_s, C))^2$. In the process, in addition to introducing new crossing, the framing of the various resultant component knots, $K_s{}'$ to which the affected strand belongs, change from $n(s)$ to $n^{'}(s^{'}) = n(s) + (lk(K_s,\ \ C))^2$.

Move II A disjoint unknotted circle with framing $+1$ can be dropped without affecting the rest of the link.

Thus, Likorish-Wallace theorem and equivalence of surgery under Kirby moves reduces the theory of closed, orientable, connected three-manifolds to the theory of framed unoriented links via a one-to-one correspondence: Framed links in S^3 modulo equivalence under Kirby moves $\Longleftrightarrow$ Closed, orientable, connected three-manifolds modulo homeomorphisms (Kirby 1978).

This consequently allows us to characterize three-manifolds by the invariants of the associated unoriented framed knots and links obtained from the Chern-Simons theory in S^3. This can be done by constructing an appropriate combination of invariants of the framed links which is unchanged under Kirby moves: Invariants of a framed unoriented link which do not change under Kirby moves $\Longleftrightarrow$ Invariants of associated 3-manifolds (Witten 1989, Kaul 1999).

The Geometrization conjecture introduced by Thurston in the early heights says roughly that every prime 3-manifold can be cut open along incompressible tori into pieces that admit complete homogeneous geometries of finite volume. A direct and immediate consequence of this conjecture is the Poincaré conjecture according to which any 3-manifold has to be homeomorphic or homotopically equivalent to the three-sphere. There are eight possibilities for the type of geometries with the most plentiful and the most interesting being hyperbolic geometry. This reduces the problem of classifying closed 3-manifolds to the problem of classifying discrete, torsion-free subgroups of $SL(2, \mathbf{C})$ of finite co-volume up to conjugacy. It gives a completely satisfying conceptual picture of the nature of all 3-manifolds, and it shows how closely related these topological objects are to homogeneous one. All closed 3-manifolds are made from homogeneous geometric ones by two simple operations: gluing along incompressible tori and connected sum. As a sidelight, this resolves in the affirmative the Poincaré conjecture.

Thurston established several major cases of the geometrization conjecture, including the following unexpected results:

1. Almost all knots are hyperbolic;
2. Almost all surgeries of S^3 along hyperbolic knots and links yield hyperbolic manifolds; and
3. The result M of *gluing together* two hyperbolic 3-manifolds is hyperbolic, unless $\pi_1(M)$ contains a copy of $\mathbf{Z}^2$.

Here a knot or link L is *hyperbolic* if $S^3 - L$ is homeomorphic to a finite volume hyperbolic manifold $\boldsymbol{H}^3/\Gamma$. In the first statement, just torus knots and satellite knots must be avoided; in the second, finitely many surgeries must be excluded on each component of the link. The third statement is the key to proving the first two. All three results make precise, in various ways, the statement that most 3-manifolds are hyperbolic.

Recall that, in 1982, William Thurston proved that universal links exist for manifolds of dimension three. That is, he showed there is a six-component link L_T in S^3 such that every closed orientable 3-manifold, M^3, is a branched covering space of S^3, $p : M^3 \to S^3$, with branch set the link L_T. Thurston then asked whether universal knots exist. Hugh M. Hilden (1985) showed that in fact they exist. The main result he announced is:

Theorem 12 *There is a knot K in S^3 such that every closed orientable 3-manifold M^3 is a branched covering space of S^3, $p : M^3 \to S^3$, with branch set the knot K.*

Thurston geometrization conjecture entails two important processes for deforming a topological 3-manifold towards its optimal geometric shape: conformal iteration and Ricci flow. This allow for grasping the deep philosophical idea that geometric structures can evolve. The first important notion we need to introduce is that of *Haken manifold*. Let's begin with reminding some notions. Let M be a compact orientable 3-manifold, possibly with boundary. A connected orientable surface $S \subset M^3$ is incompressible if $S \neq S^2$ and $\pi_1(S)$ maps injectively into $\pi_1(M)$. A 3-manifold is *Haken* if it can be built up, starting from 3-balls, by successively gluing along incompressible submanifolds of the boundary. Any knot or link complement is Haken, as is any irreducible 3-manifold with boundary (see Thom 1954, Wallace 1960, Haefliger 1961/62 and Waldhausen 1968). Thus, most of the results stated in previous paragraphs for knots are consequences of:

Theorem 13 (Thurston). *The geometrization conjecture holds for Haken 3- manifolds.*

Next needed notion is that of *iteration on Teichmüller space*. Since the Thurston seven simpler geometries are understood for Haken manifolds, the main point in the proof of theorem is to treat the hyperbolic case. At the critical inductive step, one has an open hyperbolic 3-manifold M with incompressible boundary, and a gluing involution $\tau : \partial M \to \partial M$. The task is to produce a hyperbolic metric on the closed manifold M/τ.

A generalization of Mostow rigidity shows that hyperbolic structures on the interior of M correspond to conformal structures on ∂M. They are therefore parameterized by Teichmüller space, a finite-dimensional complex manifold homeomorphic to a ball. Thurston showed that a solution to the gluing problem corresponds to a fixed point for a topologically-defined holomorphic map

$$\sigma^{\circ}\tau : \mathrm{Teich}\,(\partial M) \to \mathrm{Teich}\,(\partial M)\,. \qquad (36)$$

By iterating this map, we obtain an evolving sequence of hyperbolic structures on M. If the sequence converges, then M/τ is hyperbolic.

One obstruction to convergence comes from $\pi_1(M/\tau)$: the fundamental group of a closed, negatively curved manifold never contains a copy of $\mathbf{Z}^2$. In fact, as Thurston showed, this is the only obstruction.

Theorem 14 (Thurston) M/τ is hyperbolic $\iff$ $\pi_1(M/\tau)$ does not contain $\mathbf{Z}^2$.

By Thurston's theorem 2, any 3-manifold contains a knot such that $M^3 - K$ is hyperbolic. One can then try to increase the cone angle along the knot from 0 to 360°, to obtain a geometric structure on M. This cone-manifold approach to geometrization works well for constructing orbifolds, but it runs into difficulties, still unresolved, when the strands of the knot collide.

The Ricci flow, on the other hand, smooths out such conical singularities, diffusing the knot so it can freely pass through itself. There are two main obstacles to long-term evolution under the Ricci flow: singularities may develop, which rapidly pinch off and break the manifold into pieces; and the manifold may collapse: it may become filled with short loops, even though its curvature remains bounded.

Perelman's work addresses both of these obstacles, and indeed turns them into the cornerstones of a successful proof of the geometrization conjecture. In brief, he shows that in dimension three:

1. Singularities of the Ricci flow always occur along shrinking 2-spheres, which split M into a connected sum of smaller pieces. These singularities can be sidestepped by an explicit surgery operation.

2. Curvature evolution with surgery defines a flow which continues for all time.
3. In the limit as $t \to \infty$, a geometric structure on the pieces of M becomes visible, either through convergence to a metric of constant curvature or through collapsing.

As a consequence, we have:

Theorem 15 (Perelman). *Both the Poincaré conjecture and the geometrization conjecture are true.*

There has been much other work in 3- manifold topology independent of the Geometrization conjecture. As we already have seen, this began with the Jones polynomial invariant for knots in S^3 (Jones 1985), which was generalized by Witten (1989; see also the important work by Kovanhov 2000) using a physical theory to invariants of all 3-manifolds. There are combinatorial definitions of the Jones polynomial, basically skein relations that say how the invariant is related before and after changing a crossing on the knot and doing surgery at the crossing. These types of relations led to other combinatorial notions of knot invariants and sometimes to 3-manifold invariants. One of the most powerful seems to be the Khovanov homology of knots in S^3 (2000). These homology groups should be viewed as a categorification of, i.e., enrichment of, the Jones polynomial (Kronheimer and Mrowka 2011). Like the original Jones polynomial, these invariants are defined starting with a braid presentation of the knot. There is now a proposal due to Witten, coming from physics, for how to extend Khovanov theory for knots to give 3-manifold invariants but no mathematical treatment yet exists.

The Importance of the Linking Number in Molecular Biology

As is now quite clearly understood, the information content of a DNA molecule is embodied in its sequence of paired nucleotide bases. Yet, because DNA is a topological living entity, it is therefore independent of how the molecule is twisted, tangled or knotted (see Boi 2011a,b, Dean et al., 1985, and Sumners 1992). Nonetheless, twisting, coiling and knotting operations are able to enhance (increase) or to decrease (reduce) the structural and physiological functions of the genome and the cell nucleus. In the past decade, it has become clear that the topological form of a DNA molecule, the structural modifications of the chromatin and the spatial architecture of the chromosome exert an important influence on the way in which DNA acts within the cell. Moreover, these three levels of organisation of the most fundamental nuclear components seem to be deeply related. Also, their functions are controlled by the action of different complexes of regulatory factors and co-factors, which may affect locally and globally the metabolism and physiology of cells. Among these different families of proteins' regulatory complexes, the remodellers of chromatin structure play a fundamental role in replication and repair of DNA sequences and in the transcriptional activities of the entire genome.

Let us consider, particularly, the basic level of DNA structure and coiling. Enzymes topoisomerases (Berger et al. 1996), which convert DNA from one topological form to another, appear to have a profound role in the central genetic events of DNA replication, transcription and recombination (Wang 1996, and Boi 2011a). It is a long-standing problem in biology to understand the mechanisms responsible for the knotting and unknotting of DNA molecules. Large amounts of DNA are wound up and packed into the average cell. DNA molecule is an incredibly long polymer whereas the cell's nucleus has a very thin spatial volume. This obviously means that the embedding of the DNA into chromatin within the cell core is exceedingly complicated; therefore, many complex structural modifications, topological deformations and regulatory networks interactions must work together in order to perform the proper packing of DNA into several folding-levels of chromatin, as well as to ensure the stability of the genome.

It is worth of noticing that differential geometry and knot theory can be used to describe and explain the 3-dimensional structure of DNA and protein-DNA complexes. Biologists devise experiments on circular DNA, which elucidate 3-dimensional molecular conformation (helical twist, supercoiling, etc.) and the action of various important life-sustaining enzymes (topoisomerases and recombinases). These experiments are often performed on circular DNA molecules, in which changes in the geometric (curvature, writhing, twisting and supercoiling) or topological (knotting and linking) state of DNA can be directly observed.

The link between the structure of the DNA double-helix and some differential geometrical concepts appear very highlighting in the 'White's formula' (J. White 1989) relating the linking, twisting and writing properties of a space curve. In order to make clear the meaning of this fruitful relationship between geometry and biology, let's start with a rigorous formulation of the 'Jordan Curve Theorem', which constitutes a mathematical prerequisite of White's formula (for further mathematical details, see Massey 1991). It is well-known that a simple, closed, continuous (or if you like smooth, or piecewise smooth, or even piecewise linear) curve separates the plane R^2 into two parts with the property that it is impossible to get from one part to the other by means of a continuous path avoiding the given curve. The same conclusion (as for a simple, closed, continuous curve) holds for any 'complete' curve in R^2, i.e. a simple, continuous, unboundedly extended, non-closed curve both of those ends go off to infinity, without nontrivial limit points in the finite plane. This principle generalizes in the obvious way to n-dimensional space: a closed hypersurface in R^n separates it into two parts.

There is however another less obvious generalization of this principle, having its most familiar manifestation in 3-dimensional space R^3. Consider two continuous (or smooth) simple closed curves (loops) in R^3 which do not intersect:

$$\begin{aligned}&\gamma_1(t) = (x_{11}(t),\ \ x_{21}(t),\ \ x_{31}(t))\,, \gamma_1\,(t+2\pi) = \gamma_1(t);\\&\gamma_2\,(\tau) = (x_{12}(t),\ \ x_{22}(t),\ \ x_{33}(t))\,, \gamma_2\,(t+2\pi) = \gamma_2(t).\end{aligned} \tag{37}$$

Consider a 'singular disc' D_i bounded by the curve γ_i, i.e. a continuous map of the unit disc into $R^3 : x_i^{\alpha}(r, \alpha)$, $i = 1, 2$, $\alpha = 1, 2, 3$, where $0 \leq r \leq 1$, $0 \leq \varphi \leq 2\pi$, sending the boundary of the unit disc onto γ_i:

$$x_i^{\alpha}(r, \varphi)\big|_{r=1} = x_i^{\alpha}(\varphi), \alpha = 1, \; 2, \; 3 \tag{38}$$

where $\varphi = t$ for $i = 1$, and $\varphi = \tau$ for $i = 2$. So, we have the following

Definition 12 *Two curves γ_1 and γ_2 in R^3 are said to be nontrivially linked if the curve γ_2 meets every singular disc D_1 with boundary γ_1 (or, equivalently, if the curve γ_1 meets every singular disc D_2 with boundary γ_2).*

In n-dimensional space R^n certain pairs of closed surfaces may be linked, namely sub-manifolds of dimensions p and q where $p + q = n - 1$. In particular a closed curve in R^2 may be linked with a pair of points (a "zero-dimensional surface")—this is just the original principle that a simple closed curve separates the plane (Kervaire 1965).

Gauss introduced an invariant of a link consisting of two simple closed curves γ_1, γ_2 in R^3, namely the signed number of turns of one of the curves around the other, *the linking coefficient or linking number* $\{\gamma_1, \gamma_2\}$ of the link. His formula for this is

$$N = \{\gamma_1, \gamma_2\} = 1/4\pi \int_{\gamma 1}\int_{\gamma 2} ([d\gamma_1(t), d\gamma_2(t)], \gamma_1 - \gamma_2) / |\gamma_1(t) - \gamma_2(t)|^3, \tag{39}$$

where [,] denotes the vector (or cross) product of vectors in R^3 and (,) the Euclidean scalar product. Thus, this integral always has an integer value N. If we take one of the curves to be the z-axis in R^3 and the other to lie in the (x, y)-plane, then the previous formula (40) gives the net number of turns of the plane curve around the z-axis. It is interesting to note that the linking coefficient N may be zero even though the curves are nontrivially linked. Thus, his having non-zero value represents only a sufficient condition for nontrivial linkage of the loops.

Let's now return to the White's theorem, which states that:

$$Lk = Tw + Wr. \tag{40}$$

This equation is fully valid for differentiable curves in three-dimensional space. By combining two quantities, twist and writhe, that depend upon metric considerations, we obtain the linking number—a topological invariant of a pair of entangled curves.

To summarize the previous considerations, let's say that the linking number is a mathematical quantity existing in two, three and also higher dimensions, topologically invariant by deformations, which tells us a great deal about the structural properties and qualitative behaviours of DNA during the cell cycle. First, it is closely related to the number of time that the two sugar-phosphate chains of DNA wrap around, or are 'linked with', one another. Here take DNA in its stress-free, relaxed state as the reference point for counting *Lk*, where hence $Lk = 0$. Now consider the simple model of a circular DNA with the values: $Tw = +3$, $Wr = 0$, $Lk = +3$. Thus, $Lk = +3$ tells us that the DNA has three more double-helical turns than it would have in a relaxed, open-circular form. In general, *Lk* measures the total excess or deficit of double-helix turns in the molecule. Note, in particular, that *Lk* can only be an integer, because the DNA can only join to itself by some integral number of turns.

Geometrical and Topological Properties of the Double Helix and Supercoiling

Every physical, chemical and biological property of DNA—its transcription, hydrodynamic behavior, energetics, enzymology and so on—are essentially affected by closed circularity and the deformations associated with supercoiling. Understanding the mechanism of supercoiling and the consequences of this structural feature of DNA, however, presents problems of considerable mathematical complexity. Fortunately, there are two branches of mathematics that offer substantial help in this effort: topology, which studies the properties of objects that remain unchanged when these objects are deformed, and differential geometry, which applies the methods of the differential calculus to the study of curves and surfaces. In what follows we shall first describe a mathematical model of closed circular DNA and then discuss the implications of the model for the real DNA (see Boi 2005, 2011a, b for more details).

For our purposes it is preferable to choose a model whose axis coincides with that of the double helix. In addition, we specify that the ribbon must always lie perpendicular to the pseudo-dyads, or twofold axes of rotation, which are distributed along the double helix. This ribbon model follows the axis of DNA double helix and twists as the two chains of the molecule twist around that axis. In addition, because the sequences of atoms in the two-polynucleotide chains run in opposite directions the edges of the ribbon will be assigned opposite orientations. This model can be analyzed mathematically in a number of different ways.

One way consists in studying the relation between the oppositely directed edges of the ribbon. When the ends of a ribbon are joined, each edge describes a closed curve in three-dimensional space. Furthermore, when the ribbon represents a closed circular molecule of DNA, a number of 360-degree twists are introduced before the ends of the ribbon are joined, and so the two curves described by its edges are linked. In other words, it is impossible to separate the curves without "cutting" one of them. If each loop in a linked pair represents a covalently bounded molecule, as is the case with the two-polynucleotide chains of the double helix, the two are said to be joined by a topological bond. This is a peculiar type of bond in that although no part of one molecule is covalently joined to any part of the other, it is nonetheless necessary to break a covalent bond in order to separate the two (Fig. 53).

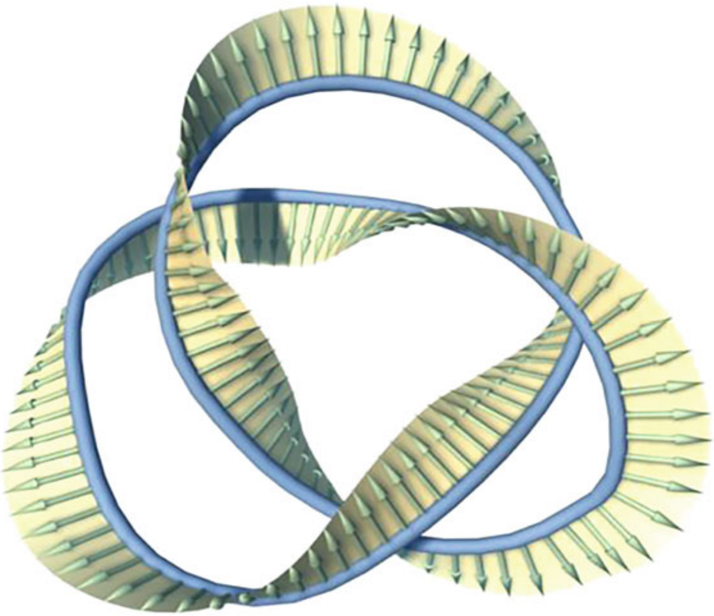


Fig. 53 Pictured here is a framed trefoil knot. You can see how the framing can be thought of as both a vector field or as a knot tied out of a ribbon. You can see the twists in the ribbon and you can imagine a trefoil with more or fewer twists, corresponding to a different framing

In mathematical terms the linking number of two closed curves is a topological property: no matter how the curves are deformed (pulled, twisted and so on), as long as neither one is broken they will remain linked in exactly the same way. As we have seen in the previous section, the linking number *Lk* is defined as a signed integer that describes a property of two closed curves in space. To separate a pair of curves without actually cutting them the value of *Lk* must be 0 (although the converse is not always true). If the curves in question are the edges of a closed ribbon with N turns in it, their linking number will remain unchanged when the ribbon is deformed. Notice that although the edge of the ribbon model of DNA were no chosen to coincide with the sugar-phosphate backbones of the double helix, the linking number of the ribbon will be exactly that of the backbones.

In mathematical terms, the *twist of a ribbon* measures how much it twists around its axis and is defined as the integral of the incremental twist around the ribbon. A formula for the twist is given by

$$\mathrm{Tw}(K) = 1/2\pi \int_K ds\ \varepsilon_{\mu\nu\alpha}\ dx^{\mu}/ds\ n^{\nu} dn^{\alpha}/ds, \tag{41}$$

Where K is parametrized by $x^{\mu}(s)$ for $0 \leq s \leq L$ along the length of the knot by parameter s, and the frame K_f associated with K is $y^{\mu} = x^{\mu}(s) + \varepsilon \;\; n^{\mu}(s)$, where ε is a small parameter and $n^{\mu}(s)$ is a unit vector field normal to the curve at s. Letting Lk be the linking number of the two components of the ribbon, Tw be the twist, and Wr, be the writhe, then the Calugareanu theorem (1959) states that

$$Lk(R) = \mathrm{Tw}(R) = Wr(r). \tag{42}$$

Another way to analyze the ribbon model of DNA is by looking not at the relation between its edges but at the way the ribbon twists. For a ribbon whose axis follows a straight line the idea of a numerical value expressing twist is intuitively obvious. Here we shall adopt the convention that a right-handed twist of 360 degrees has a value of $+1$ and a left-handed twist has a value of -1. The definition of twist is less obvious, however, for a ribbon whose axis is not straight. Perhaps the best way to understand this concept is to imagine a small arrow placed perpendicular to the axis of the ribbon, pointing to one of its edges. As the arrow is moved along the twisting ribbon it rotates about the axis, and the twist of the ribbon can be defined as the integral of the arrow's angular rate of rotation with respect to the arc length of the axis curve. In the special case where the axis of the ribbon is confined to a plane this value can be measured simply as the number of rotations the arrow completes about the axis as it is moved along the ribbon. For example, when the ribbon models is a closed circular piece of DNA 5.000 base pairs long that is relaxed (that is, its axis lies in a plane), the arrow makes one complete rotation for every turn of the double helix, and so the total twist Tw equals $+500$, with the plus sign arising one again because the double helix is right-handed. For a relaxed circular piece of DNA 5.000 base pairs long, then, both the linking number and the twist are equal to $+500$. From this example one might well assume that linking number is just another way of expressing twist, but that is not the case. Indeed, it is particularly important to understand the distinction between these two quantities. To begin with, linking is a topological property, whereas twist is geometrical: if a ribbon is deformed, its twist may be altered. Moreover, to compute the linking number (which is always an integer) the ribbon must be considered as a whole. On the other hand, twist (which may not be an integer) can be considered locally, and the twist values of individual sections can be summed to obtain the total twist for the ribbon.

The realization that linking and twisting are distinct properties raises another question. Is there a geometrical significance to the difference between these properties, that is, to the difference between the linking number of a ribbon and its total twist? In 1968, it has been proved by J.H. White that the linking number of a ribbon and its total twist differ by a quantity that depends exclusively on the curve of the axis of the ribbon. This quantity is well known to mathematicians as the Gauss integral of the axis curve. In other words, assume that the axes of two closed ribbons follow the same curve in three-dimensional space; then even if the ribbons themselves turn and twist in entirely dissimilar ways, their values of linking number and total twist will differ by exactly the same amount. At about the same time, it was suggested the name *writhing number* for the quantity by which the two differ. Thus, for a closed ribbon in three-dimensional space the writhing number *Wr* equals the difference between the linking number *Lk* and the total twist *Tw*, or $Wr = Lk - Tw$. The writhing number of a ribbon is a powerful quantity whose value generally changes if the axis of the ribbon is deformed in space. Hence writhing, like twisting, is not a topological property of the ribbon but a geometrical one.

The writhing number can be obtained by computing the Gauss integral, but is generally far easier to calculate it by evaluating the linking number and the total twist of the ribbon in question and then taking their difference. It is only in certain special cases that it is convenient to compute *Wr* directly. For example, if the axis of a ribbon lies entirely in a plane or entirely on the surface of a sphere, then it can be shown that *Wr* is zero. Substituting this value into the equation $Wr = Lk - Tw$ gives $Lk = Tw$, which explains why in the example of the relaxed closed circular molecule of DNA both *Tw* and *Lk* were found to be +500. Now consider what happens if the axis of this DNA molecule is made to writhe in such a way that its writhing number is no longer zero. When the writhing number of the molecule is made to change, the linking number remains the same (it can be altered only if one of the backbones of the double helix is broken) and so the twist must change. It is this relation that underlies the phenomenon of supercoiling.

A ribbon's linking number, total twist and writhing number do not depend on the ribbon's location or orientation in space. They are also independent of scale, but if one axis of space is inverted (as it is the case when the ribbon is reflected in a mirror) or three axes are inverted (as is the case when the ribbon is inverted through a point), then the sign of all three quantities is changed. On the other hand, if any two axes are inverted, as happens if one looks into an ordinary microscope (that is, not a dissecting one), their sign are unchanged. In fact, any operation that turns a right-handed screw into a left-handed one without introducing other distortions will change the sign of the linking number, the total twist and the writhing number. It is also been shown that there is one other special mathematical operation that changes the sign of these quantities but leaves their magnitude unaltered: inverting the ribbon through a sphere. This result explains why the writhing number of a ribbon whose axis lies on the surface of a sphere is zero. Under this operation the closed curve described by the axis is transformed into itself.

Although the equation $Wr = Lk - Tw$ demonstrates that linking and twist are mathematically distinct, the physical difference between quantities may not yet be evident. It may be helpful, then, to consider what happens when a mathematical ribbon is wound around a cylinder in such a way that its surface is always flat against the cylinder. We shall call this *pitch angle* of the helix described by this ribbon ε. In other words, ε is the angle at which each turn of the helix inclines away from the horizontal, so that when ε is small, the helix is shallow, and when $\langle$ is large, the helix is steep. Now assume the ribbon is wrapped around the cylinder *N* times before its ends are joined in the most straightforward way. Then if the effects are ignored, it can be demonstrated that the linking number of the ribbon *Lk* equals $N\sin\varepsilon$. Therefore, when the helix is stretched out so that the pitch angle ε increases, the number of turns and thus the linking number remain the same, but the twist goes from a small value to a large one, clearly demonstrating the difference between linking and twist. Moreover, since a ribbon's writhing number is defined as the difference between its linking number and its total twist, the value of *Wr* for this ribbon is $N - N\sin\varepsilon$, or $N(1 - \sin\varepsilon)$. As this formula indicates, when ε is small and the twist is small, the writhing is substantial, but when ε is large and the twist is large, the writhing is minimal. The relation can be easily observed in a coiled telephone wire: when such a wire is unstressed, it assumes a highly writhed form with little twist; when its ends are pulled out, a highly twisted form that writhes only slightly is obtained.

Knots, Links, and Topological Quantum Field Theories: An Overview

Topological quantum field theories (TQFTs) can be used as a powerful tool to probe geometry and topology in low dimensions (Atiyah 1988). Chern–Simons theories, which are examples of such field theories, provide a field theoretic framework for the study of knots and links in three dimensions (Atiyah 1990, Kaul 1999). These are rare examples of QFTs which can be exactly (nonperturbatively) and explicitly solved. Abelian Chern–Simons theory provides a field theoretic interpretation of the linking and self-linking numbers of a link (i.e. the union of a finite number of disjoint knots). In non-Abelian theories, vacuum expectation values of Wilson link operators yield a class of polynomial link invariants; the simplest of them is the well-known Jones polynomial (Guadagnini et al. 1990, Witten 1989). Powerful methods for complete analytical and nonperturbative *computation* of these knot and link invariants have been developed. From these invariants for unoriented and framed links in S^3, an invariant for any three-manifold can be easily constructed by exploiting the Lickorish-Wallace surgery presentation of three-manifolds. This invariant up to a normalization is the partition function of the Chern–Simons field theory. Even perturbative analysis of Chern–Simons theories are rich in their mathematical structure; these provide a field theoretic interpretation of Vassiliev knot invariants (Vassiliev 1990, Vogel 1992-93). In Donaldson–Witten theory perturbative methods have proved their relations to Donaldson invariants. Nonperturbative methods have been applied after the work by Seiberg and Witten on $N = 2$ supersymmetric Yang–Mills theory. The outcome of this application is a totally unexpected relation between Donaldson invariants and a new set of topological invariants called Seiberg–Witten invariants (Donaldson 1983, and Floer 1998). Not only in mathematics, Chern–Simons theories find important applications in three- and four-dimensional quantum gravity also. Work on TQFT suggests that a quantum gravity theory can be formulated in three-dimensional space–time (Witten 1988, Atiyah 1989). Attempts have been made in the last years to formulate a theory of quantum gravity in four-dimensional space–time using "spin networks" and "spin foams".

There are several examples of gauge invariant functions. For example, primary characteristic classes evaluated on suitable homology cycles give an important family of gauge invariant functions. The instanton number k of $P\,(M, G)$ belongs to this family, as it corresponds to the second Chern class evaluated on the fundamental cycle of M representing the fundamental class $[M]$. The point-wise norm $|F_\omega|_x$ of the gauge field at $x \in M$, the absolute value $|k|$ of the instanton number k and the Yang–Mills action are also gauge invariant functions. Another important example of a quantum observable is given by the Wilson loop functional $W_{\rho,\alpha}$ associated to the representation ρ and the loop α. The definition of the Wilson loop functional and a given equation expressing a gauge transformation which changes the holonomy by conjugation by an element of the gauge group G, implies that the Wilson loop functional is gauge invariant and hence defines a quantum observable. Regarding a knot κ as a loop we get a quantum observable $W_{\rho,\kappa}$ associated to the knot.

The idea of a topological field theory was introduced by Witten (1988) as a rudimentary structure to which, in principle, any QFT reduces at very long distances and low energies. An axiomatic formulation of TQFTs has also been proposed by Michael Atiyah (1989). There are a number of examples of topological field theories which are very relevant in geometric topology. One of them provides a unified point of view on the knot invariants discovered by V. Jones, and the associated invariants of three- manifolds. Another encodes the Donaldson invariants of four-manifolds (1983), and the Floer cohomology groups of three-manifolds (1998; see also Ozsváth and Szabó 2005).

TQFTs are independent of the metric of curved manifold on which these are defined; the expectation value of the energy–momentum tensor is zero, $\langle T_{\mu\nu}\rangle = 0$. These possess no local propagating degrees of freedom; only degrees of freedom are topological. Operators of interest in such a theory are also metric independent. The mathematics of topological field theories is closely linked with string theory as a theory of gravitation and elementary particles. One possible way to look at it, from the mathematical point of view, is to say that it replaces the finite-dimensional space–time manifolds of conventional QFT by a completely new kind of "stringy" manifold. To give a conventional manifold is the same as to give the commutative algebra of smooth functions on it. A "stringy" manifold is described not by a commutative algebra but by a more sophisticated algebraic structure which is a fairly natural generalization of a commutative algebra.

To illustrate how ideas of QFT can be used to study topology, we shall focus our attentions here on recent important developments in Chern–Simons gauge field theory as a TQFT on a three-manifold. This theory provides a field theoretic framework for the study of knots and links in a given three-manifold. The new famous Jones polynomial is intimately related to Chern–Simon theory. Witten set up a general field theoretic framework to study knots and links. Since then enormous effort has gone into developing an exact and explicit nonperturbative solution of this field theory. The interplay between QFT and knot theory is a very fascinating topic and a very promising area of research both for geometry and theoretical physics (Kontsevich 1994 and Kreimer 2000). Many of the open problems in knot theory have found answers in the process.

Wilson loop operators are the topological operators of the Chern–Simons gauge field theory. Their vacuum expectation values are the topological invariants for knots and links (see Cho 2007), which do not depend on the exact shape, location or form of the knot and links but reflect only their topological properties. The power of this framework is so deep that it allows us to study these invariants not only on simple manifold such as three-sphere but also on any arbitrary three-manifold. The knot and link invariants obtained from these theories are also intimately related to the integrable vertex models in two dimensions. These invariants have also been approached in different mathematical frameworks. A quantum group approach to these polynomial invariants has been developed (Turaev 1994). Last decade or so has seen enormous activity in these directions in algebraic topology. A mathematical important development is that these link invariants provide a method of obtaining a specific topological invariant for three-manifolds in terms of invariants for framed unoriented links in S^3.

Now, let us consider, more specifically, the general case of a non-Abelian Chern–Simons theory as a description of knots and links. A non-Abelian Chern–Simons theory, instead of being a gauge theory of one vector field, involves, say for gauge group $SU(2)$, three such fields, $A_\mu^a (a = 1, 2, 3)$. These three are collectively written as a matrix valued vector field $A_\mu = A_\mu^a \ \sigma^a/2i$, where anti-Hermitian matrices are the generators of the group $SU(2)$. Action functional defined in a three-manifold, say S^3, is given by:

$$kS = k/4\pi \int_{M_3} tr\left(AdA + 2/3A^3\right) \tag{43}$$

Like Abelian Chern–Simons theory, this action has no metric dependence. Besides a gauge invariance, it is also invariant under general coordinate transformations. The topological operators are the Wilson loop (knot) operators defined as

$$W_j\,[K] = tr_j P \exp \int_K dx^\mu A_\mu^a T_j^a \tag{44}$$

for an oriented knot K carrying spin j representation reflected by the associated representation matrices T_j^a $(a = 1,\ 2,\ 3)$. The symbol P stands for path ordering of the exponential. This is done by breaking the length of the knot K into infinitesimal intervals of size dx_m^μ around the points labelled by the coordinates x_m^μ along the knot. Then path-ordered exponential is:

$$P \exp \int_K dx^\mu A_\mu^a T_j^a = \prod^m \left[1 + dx_\mu^m \ \ A_\mu^a\,(x_m)\,T_j^a\right]. \tag{45}$$

For a link L made up of oriented components knots $K_1, K_2, \ldots, K_s$ carrying spin j_1, $j_2, \ldots, j_s$ representations, respectively, we have the Wilson link operator defined as

$$W_{j_1 j_2 \ldots j_r}\,[L] = \prod_{1=1}^{s} \ W_{j_l}\,[K_l]\,. \tag{46}$$

Knots and Dynamics Systems

One of most interesting illustrations of the essential role of knot theory in mathematical physics is the relation between knots and chaotic dynamical systems. After the classical and fundamental work of V. Arnold and others on the connection between topological invariants and dynamical systems, Etienne Ghys showed (2007) that the class of Lorenz knots, pertaining to the theory of chaotic dynamical systems and ordinary differential equations, and the class of modular knots, pertaining to the theory of 2-dimensional lattices and number theory, coincide.

Before to explain this kind of correspondence, let's give the following definition:

Definition 13 *A torus knot of type (p, q) is a curve in R^3 that in cylindrical coordinates r, z, θ, is given by the equations $r = 2 + \cos t$, $z = \sin \theta$, $\theta = pt/q$, where $t \in [0, 2\pi q]$. Here p and q are coprime natural numbers. The torus knot lies in the surface of the unknotted torus $(r - 2)2 + z^2 = 1$, intersecting the meridians of the torus at p points and the parallels at q points. The torus knots of types (p, 1) and (1, q) are trivial.*

The simplest non-trivial torus knot is the trefoil, which is of type (2, 3). The group G of the torus knot T of type (p, q) has a presentation $\langle a, b\colon a^p = b^q \rangle$, and the Alexander polynomial is given by $(t^{pq} - 1)(t - 1)/(t^p - 1)(t^q - 1)$. All torus knots are Neuwirth knots. The genus of a torus knot is $(p - 1)\,(q - 1)/2$. Recall that a *Neuwirth knot* is a polynomial knot (S^3, K^1) whose group has a finitely-generated commutator subgroup. The complement $S^3 \backslash K^1$ of a Neuwirth knot is a fiber space over a circle and the fiber F is a connected surface whose genus is that of the knot.

Further, recall that a *satellite knot* is a knot that contains an incompressible, non-boundary-parallel torus in its complement (Schubert 1953). Every knot is either hyperbolic, a torus or a satellite knot. The class of satellite knots include composite knots, cable knots, and Whitehead doubles. A satellite knot is one that orbits a companion knot K in the sense that it lies inside a regular neighborhood of the companion. A satellite knot K can be described as follows: start by taking a non-trivial knot K' lying inside an unknotted solid torus T. Here "non-trivial" means that the knot K' is not allowed to sit inside of a 3-ball in T and K' is not allowed to be isotopic to the central core curve of the solid torus. Then tie up the solid torus into a non-trivial knot. When $K \subset \partial T$ is a torus knot, then K is called a *cable knot*. If K' is a non-trivial knot in S^3 and a compressing disc for T intersects K' in precisely one point, then K is called a *connected sum* of K and K' (Fig. 54).

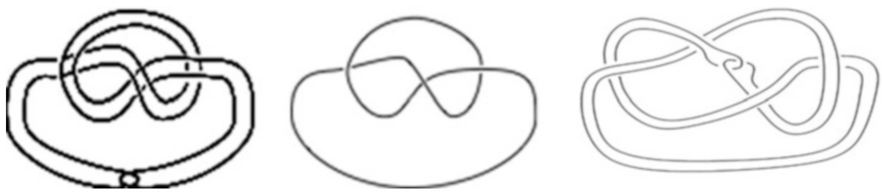


Fig. 54 A satellite knot. A companion knot. The Whitehead double of the figure-height knot

Mostly chaotic knots are torus knots. Recall first that a *torus* is a doughnut-shaped surface, that is the Cartesian product of two circumferences. A *torus knot* is a knot on a torus. In other words, in a torus knot the string on the surface of a torus contains the representation of the simplest torus knot.

Lorenz knots appear in the first, and still most famous, example of chaotic dynamical system, introduced by Edward N. Lorenz in 1963 as a simplified model for convection in the atmosphere. This model consists of three (mildly non-linear) ordinary differential equations:

$$x' = 10\,(y\text{–}x)\,,\; y' = x\,(28\text{–}z)\,\text{–}y,\; z' = xy\text{–}\,(8/3)\,z. \tag{47}$$

Lorenz has discovered one of the most distinctive characteristics of chaotic dynamical systems: *sensitive dependence on initial conditions*. It may happen that the slightest change in the initial state can cause a completely different result, the so-called butterfly effect. But in his model Lorenz has discovered another butterfly, which is more relevant to our discussion. There exists a Lorenz model of a chaotic dynamical system. Let's explain the general idea a little more in details. The Lorenz model, as any system of ordinary differential equations in three variables, prescribes at each point in space a velocity vector, we can then start from any point in space, and move according to the speed and direction given by these velocity vectors. The itinerary we follow is an *orbit* in the model. Lorenz noticed that almost all orbits tended to accumulate onto a peculiar and approximately butterfly-shaped set, having a very intricate geometric structure (later on it was proved that it is a fractal set of dimension slightly larger than two). This set, the *Lorenz attractor*, was the first example of strange attractor for a chaotic dynamical system. Most orbits go around wildly getting closer and closer to the Lorenz attractor; but a few special ones actually live in the Lorenz attractor itself. These are *periodic* orbits: orbits that after a finite amount of time come back to their starting point. Periodic orbits are thus (never self-intersecting) closed curves in Euclidean space, that is, they are knots. And . . . the *Lorenz knots* are exactly the periodic orbits of the Lorenz model.

It turns out that *Lorenz knots* fill out (they are *dense*, another typical feature of chaotic dynamical systems is the coexistence of periodic behavior with very wild behavior) the Lorenz attractor, and so understanding them might give important information on the structure of the Lorenz attractor. In the Eighties, Joan Birman and Bob William (1983) started studying Lorenz knots, trying to understand and classify them. They showed that *all torus knots are Lorenz knots*; and recently (2009) Birman and Ilya Kofman have proved that *every Lorenz knot is a twisted torus knot*, a knot that can be obtained from a torus knot by a simple procedure (amounting to cutting the knot in several carefully chosen places, twisting the strands according to specific rules, and then gluing the strands together.) Twisted torus links are given by twisting a subset of strands on a closed braid representative of a torus link. T-links are a natural generalization, given by repeated positive twisting. The authors established a one-to-one correspondence between positive braid representative of Lorenz links and T-links, so Lorenz links and T-links coincide. They used this correspondence to identify over half of the simplest hyperbolic knot as Lorenz knots. They further showed that both hyperbolic volume and the Mahler measure of Jones polynomial (in terms of which the Jones polynomial behaves like hyperbolic volume under Dehn surgery) are bounded to infinite collections of hyperbolic Lorenz links. The correspondence provides new unexpected symmetries for both Lorenz links and T-links.

To explain now what is a *modular knot* we must first explain what a *lattice* is. Roughly speaking, a lattice is a discrete family of points (in a line, a plane, a space ...) uniformly distributed. The easiest example of lattice is the set of integers numbers in the real line. From a geometrical point of view, a lattice in the line is obtained by covering the line with infinitely many copies of the same basic block, an interval (of length one if the lattice is normalized). In the plane, the situation in considerably more complex. As building block for a lattice we can use a parallelogram; but even assuming (as we may up to a translation) that one of the vertices of the parallelogram is the origin, we still have infinitely many distinct cases to consider.

To describe a normalized lattice, we need four real numbers (the coordinates of two vertices of the basic parallelogram) satisfying one condition (area equal to 1); this means that we can identify the space of all normalized lattice with a suitable subset of the Euclidean 3-space. It turns out that this subset is exactly the complement of a trefoil knot—the first but not last appearance of knots in this setting.

There is another way of describing the space of normalized lattices. Instead of considering the two vertices separately, we can put their coordinates in a 2×2 matrices; the normalization condition then amounts to saying that the determinant of this matrix is 1. If we multiply a matrix with determinant 1 by another matrix with determinant 1 we still get a matrix with determinant 1, that is another normalized lattice. In particular, this holds if we multiply by the diagonal matrix having e^t and e^{-t} as diagonal elements, where t is any real number. Letting t vary in the real numbers, we then get a whole family of normalized lattices that can be thought of as a curve in the complement of the trefoil knot, an orbit of the *modular flow*.

The most interesting aspect of all that is that the modular flow has periodic orbits, forming knots contained in the complement of the trefoil knot; these periodic orbits are called *modular knots*. It turns out that they are in one-to-one correspondence with (similarity classes of) 2×2 matrices with integer coefficients, determinant 1 and absolute value of the trace (the sum of the diagonal elements) greater than 2; these matrices are *hyperbolic* elements of the *modular group* (the group of 2×2 matrices with integer coefficients and determinant 1).

Modular knots have been studied for a long time; however, Ghys found a new way of looking at them, giving unexpected results. Ghys' surprising discovery is that

Result *A knot can be realized as a Lorenz knot if and only if it can be realized as a modular knot*. In other words, *the class of Lorenz knots coincides with the class of modular knots*.

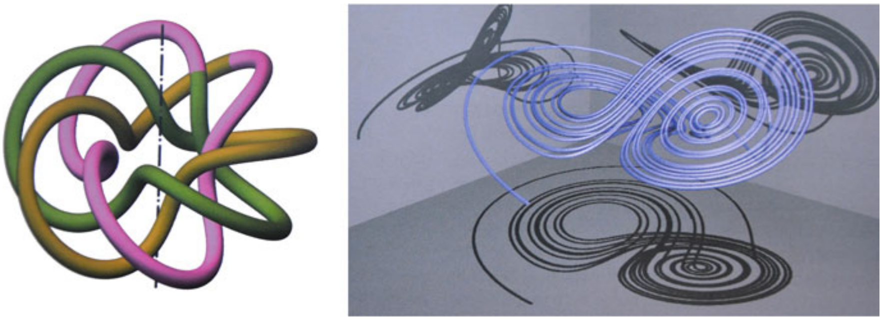


Fig. 55 On the right, a Lorenz strange attractor and their periodic orbits; on the links, a torus knot

To prove this, Ghys gave a way to pass from a Lorenz knot to a modular knot and conversely, based on the idea of Lorenz template previously introduced by Birman and Williams (1983). The Lorenz template is a figure-eight-shaped surface, similar to—and thus still sort of butterfly-like—but much simpler than the Lorenz attractor, with the very useful property that every Lorenz knot can be continuously pushed onto the Lorenz template (remaining equivalent to the original knot). This discovery has already had profound consequences in the theory of modular flow, the topology of knots and dynamical systems, and in celeste mechanics (Fig. 55).

The Energy of Knots

The energy E of a knot K is a real valued functional, which is well-behaved for embedded circles in the 3-dimenasional Euclidean space, and which blows up for curves with self-intersections. Let $E_{\lambda B}$ be the sum of the energy E and the total squared curvature functional. It has been showed (O'Hara 1991) that for any real number x, there are only finitely many ambient isotopy classes of embeddings (i.e. knotted types) with the value of $E_{\lambda B}$ not greater than x. Other functionals defined for the space of closed curves in R^3 with suitable conditions, and especially for knots, such as the total curvature, the total squared curvature, and the Gauss integral of the linking number for a simple curve, do not have the above property. They do not blow up for curves with self-intersections, and we cannot in general show the finiteness of knot types by them, though we can distinguish the trivial knot from non-trivial knots by the total curvature, and hence, by the total squared curvature.

Freedman et al. (1994) introduced an energy $E(\Gamma)$ for a simple closed curve $\Gamma \subset R^3$. The functional E is continuous on each isotopy class of curves, and tend to infinity as Γ nears self-intersection. Moreover, E is "proper" on the set of all isotopy classes, in the sense that there are only finitely many knot types below a given energy level.

A useful geometric property of E is Möbius invariance: if μ is a Möbius transformation of $R^3 \cup \infty$ and $\mu(\Gamma) \subset R^3$, then $E(\mu(\Gamma)) = E(\Gamma)$. This can be used to prove that (Freedman et al. 1994) each *prime* knot class has an energy-minimizing representative (of differentiability class $C^{1,1}$), and the round circle is the unique energy minimizer among all curves, with $E = 4$. It is also a non-trivial result by the authors that, at least for curves $C^{1,1}$, the functional E is sufficiently smooth to have a "gradient" dE. Thus, it becomes an interesting problem to find E-critical curves, that is, solutions of $dE = 0$. Kim and Kusner (1993), constructed the first explicit examples of knotted curves that are critical for E. Their basic observation is that the Möbius-invariant energy E extend naturally to simple curved curves $\Gamma \in R^m$. In particular, if Γ is a simple closed curve in $S^3 \subset R^4$ and $\sigma : S^3 \to R^3 \cup \infty$ is its stereographic projection, then $E(\sigma(\Gamma)) = E(\Gamma)$ provided $\sigma(\Gamma) \subset R^3$. They then used the principle of symmetric criticality to show that for each relatively prime pair of integers (p, q) there is a (p, q)-torus knot $\Gamma_{p,\,q} \subset S^3$ critical for E. This curve $\Gamma_{p,q}$ is a principal orbit of an isometric action of S^1 on S^3.

The principle of symmetric criticality states in brief that critical symmetric points are symmetric critical points. In more detail, let M be a smooth (i.e., C^∞ manifold) in which a group G acts by diffeomorphism (a smooth G-manifold) and let $f : M \to R$ be a smooth G-invariant function on M (that is, f is constant on the orbits of G). Then a *critical point* of f is a point p of M where df_p, the differential of f at p, vanishes. And a symmetric point of M is an element of the set $\sum = \{p \in M | gp = p$ for all $g \in G\}$ of points fixed under the action of G. The principle states that *in order for a symmetric point p to be a critical point it suffices that it be a critical point of* $f/\sum$, the restriction of f to $\sum$; in other words, if the directional derivatives $df_p(X)$ vanish for all directions X at p tangent to $\sum$, then the principle claims that directional derivatives in directions transverse to $\sum$ also vanish. In particular, for example, an isolated point of $\sum$ (where there are n directions tangent to $\sum$) should automatically be a critical point of f.

The mathematical notion of energy functional for classical knots is based on the main idea of defining a real-valued functional on the space of knots so that gradient descent along the values of the functional does not change the ambient isotopy class of the knot and leads to a well-defined minimum, which may be regarded as the normal form of the knot (corresponding to the given functional); if this gradient descent leads to the same normal form starting from any two knots in the same ambient isotopy class, we would have a solution of the knot classification problem. Knots would be entirely classified by their normal forms. However, none of the functionals studied so far have achieved this ambitious goal. Nevertheless, the functionals considered, in particular the Möbius energy functional devised by J. O'Hara (1991), possess some important properties, for example, any knot diagram of the knot is taken to the normal circle via gradient descent along the values of the Möbius energy functional, but the claim that the unknot has a unique normal form (the circle) has not been proved.

The functionals studied by Karponkov and Sossinsky (2011) differ from those mentioned above in that they are defined on knot diagrams (rather than knots in 3-space); the gradient descent for them is not invariant with respect to the Reidemeister

I move, but is invariant with respect to II and III Reidemeister moves, and the diagrams themselves are not curved lines: they are like vert thin solid tori lying in the plane.

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Part III
Diagrams, Graphs and Representation

Diagrammes planaires qui représentent des objets de dimension 1, 2, 3 et 4

Carlo Petronio

Résumé On explique comment des diagrammes planaires avec des décorations appropriées peuvent être employées pour décrire des objets topologiques de dimension 1, 2, 3 et 4.

Problème de classification

De façon très générale, on peut décrire un problème de classification de mathématique comme suit. On se donne

$$\mathcal{M} \quad \text{la classe d'objets qu'on veut classifier}$$
$$\sim \quad \text{l'équivalence avec laquelle on veut classifier } \mathcal{M}$$

on essaye ensuite de décrire le quotient $\mathcal{M}/\!\sim$. Les outils principaux qui peuvent aider à cette fin sont:

- Des *méthodes constructives* pour (essayer de) vérifier que $M_1 \sim M_2$ pour $M_1, M_2 \in \mathcal{M}$;
- Des *invariants* $I : \mathcal{M} \to V$, c'est à dire des fonctions telles que $M_1 \sim M_2$ implique $I(M_1) = I(M_2)$; et donc $I(M_1) \neq I(M_2)$ implique $M_1 \not\sim M_2$.

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Approche combinatoire

Souvent, un problème de classification comme celui mentionné ci-dessus peut être attaqué en se donnant:

$$\begin{aligned} \mathcal{C} &\quad \text{une classe d'objets combinatoires;} \\ r : \mathcal{C} \to \mathcal{M} &\quad \text{une fonction de reconstruction surjective;} \\ \mu_1, \dots, \mu_k &\quad \text{des mouvements locaux tels que } r(C_1) \sim r(C_2) \\ &\quad \text{implique que } C_1 \text{ et } C_2 \text{ sont liés par une suite} \\ &\quad \text{de mouvements de types } \mu_1, \dots, \mu_k. \end{aligned}$$

Pour vérifier que $M_1 \sim M_2$ avec $M_1, M_2 \in \mathcal{M}$ on peut donc:

- Trouver $C_1, C_2 \in \mathcal{C}$ tels que $M_j = r(C_j)$;
- Chercher une suite de mouvements de types $\mu_1, \dots, \mu_k$ qui permet de passer de C_1 à C_2.

De même, on peut essayer de vérifier que $M_1 \not\sim M_2$ de la façon suivante:

- Construire une fonction $J : \mathcal{C} \to V$ telle que si C_1 et C_2 sont liés par n'importe quel mouvement μ_i, alors $J(C_1) = J(C_2)$; l'idée est que la valeur de J sur $C \in \mathcal{C}$ soit calculable de façon combinatoire;
- Trouver $C_1, C_2 \in \mathcal{C}$ tels que $M_j = r(C_j)$, calculer les valeurs de $J(C_1)$ et de $J(C_2)$, et vérifier qu'elles sont différentes.

Noeuds (dimension 1)

Le problème de classification des nœuds dans l'espace se ramène au schéma précédent en choisissant:

$$\begin{aligned} \mathcal{M} &\quad \text{un ensemble des plongements raisonnables} \\ &\quad \text{du cercle } S^1 \text{ dans l'espace } \mathbb{R}^3 \\ \sim &\quad \text{une déformation sans rupture.} \end{aligned}$$

Un approche combinatoire au problème est réalisée Rolfsen (1990) en définissant $\mathcal{C}$ comme l'ensemble des diagrammes planaires, voir Fig. 1, et μ_1, μ_2, μ_3 comme les trois mouvements de Reidemeister, voir Fig. 2.

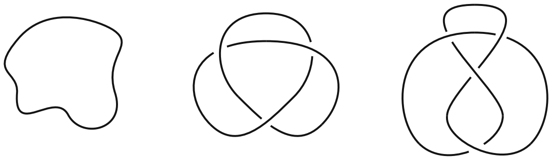

Fig. 1 Diagrammes de nœuds

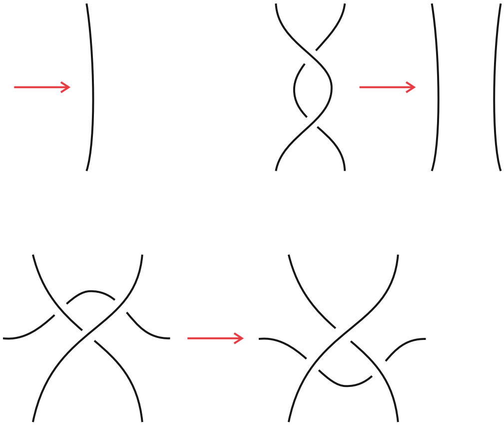

Fig. 2 Les mouvements de Reidemeister

Fig. 3 Apparence locale d'une surface autour d'un de ses points. Les points du deuxième type sont les points de bord

Surfaces (dimension 2)

Une *surface* Σ (à bord) est un espace connexe qui apparaît localement comme en Fig. 3. On dit que Σ est *orientable* si le voisinage régulier de chaque lacet (plongement de S^1) de Σ qui évite le bord, est toujours un anneau. On dit que Σ est non-orientable s'il existe un lacet qui a comme voisinage régulier une bande de Möbius, voir Fig. 4. Un exemple de surface orientable est donné en Fig. 5.

Si on note par $\mathcal{M}$ l'ensemble des surfaces compactes et par $\sim$ la relation d'homéomorphisme, le quotient $\mathcal{M}/_{\sim}$ est très bien connu (Massey, 1967), et classifié au moyen du schéma général décrit ci-dessus. On utilise comme $\mathcal{C}$ l'ensemble des surfaces triangulées, comme mouvements certaines transformations élémentaires d'une triangulation, et comme invariant l'homologie.

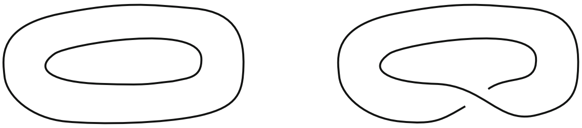

Fig. 4 Un anneau et une bande de Möbius: les seuls possibilités pour le voisinage régulier d'un lacet dans une surface

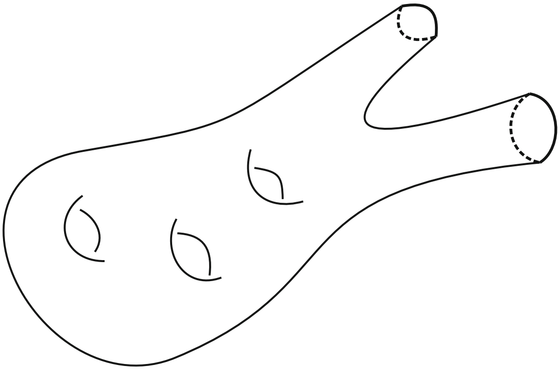

Fig. 5 Une surface orientable

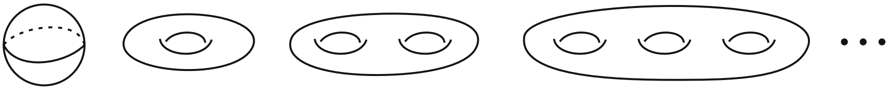

Fig. 6 Les surfaces sans bord orientables de genre $g = 0, 1, 2, 3 \ldots$

Une surface Σ compacte est déterminée par le nombre de ses composantes de bord et par la surface $\widehat{\Sigma}$ sans bord, obtenue en collant un disque à chaque composante de bord de Σ. Les surfaces sans bord orientables forment une suite indexée par un entier $g \in \mathbb{N}$ qui s'appelle *genre*, voir Fig. 6. De la même façon, les surfaces sans bord non-orientables forment une suite indexée par un entier $g \in \mathbb{N} \setminus \{0\}$.

Codage planaire des surfaces

Une surface qui possède un bord non vide peut être codée par un graphe planaire avec des "vrais sommets" de valences arbitraires et des "faux sommets" de valence 4 (les croisements d'un plongement générique dans le plan d'un graphe abstrait), comme suggéré en Fig. 7. Si on note par $\Sigma(G)$ la surface associée à un graphe G, on peut bien sûr définir la surface sans bord $\widehat{\Sigma}(G)$. En plus, on peut inclure dans ce

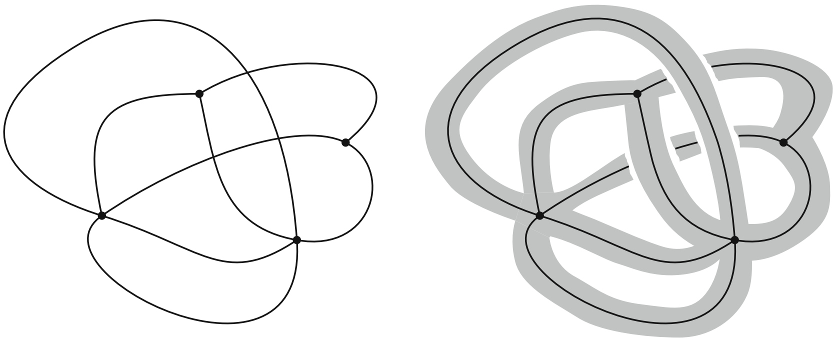

Fig. 7 Codage en diagramme d'une surface orientable avec bord non vide

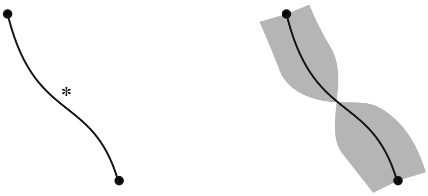

Fig. 8 Codage d'une surface non-orientable

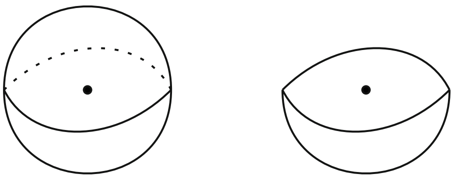

Fig. 9 Apparence locale d'une variété de dimension 3. Les points du deuxième type sont ceux de bord

cadre les surfaces non-orientables en ajoutant une décoration sur certaines arêtes, avec un effet comme en Fig. 8.

Pour laisser plus d'espace aux objets de dimension 3 et 4, on ne va pas décrire de façon explicite ici les mouvements sur les graphes planaire qui engendrent la relation d'équivalence correspondante à l'homéomorphisme des surfaces associées.

Variétés de dimension 3

Une *variété de dimension* 3 est un espace connexe M dans lequel chaque point a un voisinage homéomorphe soit à un disque dans l'espace $\mathbb{R}^3$ soit à un demi-disque, voir Fig. 9.

On dit que M est *orientable* s'ils n'existent pas dans M un anneau et une bande de Möbius (voir Fig. 4) ayant comme intersection leur cœur commun. Cette condition topologique peut s'exprimer de plusieurs manières différentes. Par exemple, on peut appeler *orientation locale* sur un petit morceau de M un choix cohérent de sens de rotation dans le morceau considéré, soit droitier, soit gaucher; une *orientation* de M est alors un tel choix global qui soit localement cohérent. On voit donc que M est orientable si et seulement si elle admet une orientation, et dans ce cas elle en admet exactement deux (rappelons que M est connexe).

On va noter par $\mathcal{M}_{\partial\neq\emptyset}$ l'ensemble des variétés de dimension 3 compactes et avec bord non vide, et par $\mathcal{M}_{\partial=\emptyset}$ celui des variétés de dimension 3 compactes et avec bord vide. Si on rajoute l'hypothèse que M est orientée, on définit les ensembles $\mathcal{M}^{\text{ori}}_{\partial\neq\emptyset}$ et $\mathcal{M}^{\text{ori}}_{\partial=\emptyset}$. Sur tous ces ensembles on va considérer la relation d'équivalence d'homéomorphisme (qui respecte l'orientation s'il y en a une). Dans les pages suivantes on va donner un codage graphique planaire des quotients

$$\mathcal{M}_{\partial\neq\emptyset}/\sim \qquad \mathcal{M}_{\partial=\emptyset}/\sim \qquad \mathcal{M}^{\text{ori}}_{\partial\neq\emptyset}/\sim \qquad \mathcal{M}^{\text{ori}}_{\partial=\emptyset}/\sim.$$

Si $M \in \mathcal{M}_{\partial\neq\emptyset}$ alors le bord de M a un nombre fini de composantes, chacune d'entre elles est une surface compacte sans bord. Si en plus M est orientée, ses composantes de bord sont des surfaces orientables, comme celles qui apparaissent en Fig. 6. À chacune des composantes du bord de M qui est une sphère, on peut coller de façon canonique un disque de dimension 3, on obtient alors une variété $\widehat{M}$ bien définie. En particulier, si le bord de M ne contient que des sphères (peut-être une seule) alors $\widehat{M} \in \mathcal{M}_{\partial=\emptyset}$. De plus, si $M \in \mathcal{M}_{\partial=\emptyset}$ on peut enlever de M et de façon canonique un disque ouvert. On obtient alors une variété dans l'ensemble $\mathcal{M}_{\partial=S^2}$ de celles dont le bord est composé d'une seule sphère.

Les considérations précédentes impliquent que la fonction $M \mapsto \widehat{M}$ donne une bijection naturelle entre $\mathcal{M}_{\partial=S^2}$ et $\mathcal{M}_{\partial=\emptyset}$, qui respecte la relation d'homéomorphisme. Bien sûr, ça reste vrai dans un contexte orienté, donnant une correspondance entre $\mathcal{M}^{\text{ori}}_{\partial=S^2}$ et $\mathcal{M}^{\text{ori}}_{\partial=\emptyset}$.

Polyèdres spéciaux

Pour tout ce qui concerne le codage des variétés de dimension 3 décrit dans la suite, le lecteur est invité à voir Benedetti and Petronio (1995) et les références qui y sont citées. Un espace P compact est appelé *polyèdre presque spécial* si chaque point de P a un voisinage comme dans la Fig. 10. Les points du premier type sont appelés *non-singuliers*, ceux du deuxième et du troisième type sont appelés *singuliers*, et ceux du troisième type sont appelés *sommets*. Chaque composante connexe R de l'ensemble des points non-singuliers de P est une surface avec une compactification naturelle $\overline{R}$ (dont le bord a une projection naturelle localement injective, mais peut-être pas globalement injective, sur le lieu singulier de P).

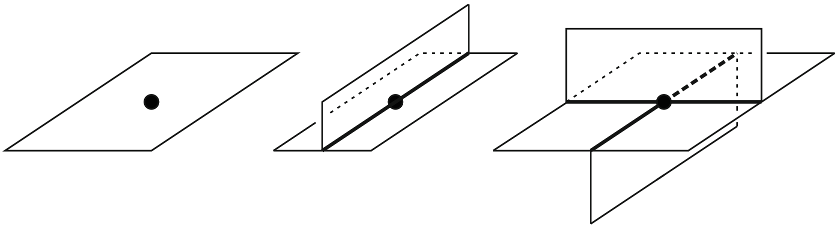

Fig. 10 Voisinages des points dans un polyhèdre presque spécial

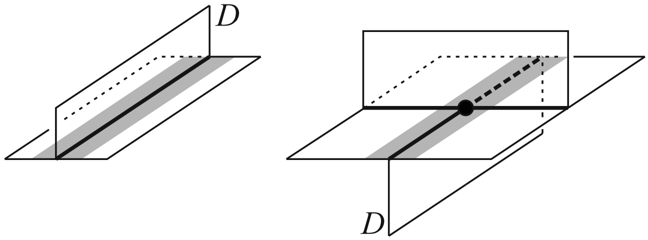

Fig. 11 Fibré en intervalles sur le bord de la compactification $\overline{D}$ d'un disque non-singulier D d'un polyèdre spécial P. La fermeture de D dans P peut bien avoir un passage multiple à travers un sommet, mais ça ne donne aucune singularité au bord de $\overline{D}$ ou au fibré en intervalles sur le bord de $\overline{D}$

On dira que P est *spécial* s'il est presque spécial et que chacune des composantes R de l'ensemble des points non-singuliers de P est un disque ouvert. Sa compactification abstraite $\overline{R}$ est alors un disque fermé.

Polyèdres spéciaux épaississable

Si P est un polyèdre presque spécial et D est un disque non-singulier de P, on peut définir de façon naturelle un fibré en intervalles sur le cercle de bord de $\overline{D}$, voir Fig. 11. On dira que P est *épaississable* si pour chaque disque D non-singulier de P, le fibré en intervalles sur le bord de $\overline{D}$ est trivial, c'est à dire si son espace total est un anneau (et non une bande de Möbius).

Reconstruction

Si P est un polyèdre spécial épaississable, on peut associer à P une variété compacte $M(P)$ de dimension 3 avec bord non vide, comme suggéré en Fig. 12. En plus, $M(P)$ est bien définie à homéomorphisme près.

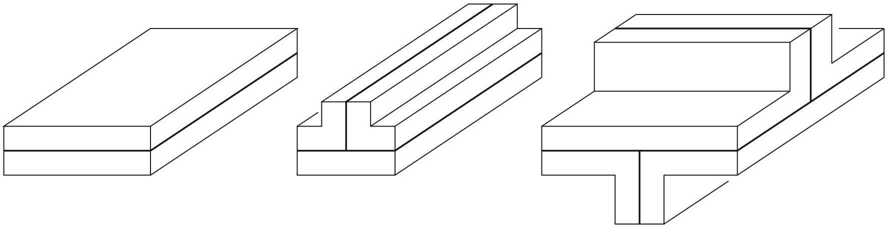

Fig. 12 Les morceaux du polyèdre spécial de Fig. 10 sont épaissis comme montré ici. La construction est globalement possible seulement grâce à l'hypothèse que chaque fibré défini comme en Fig. 11 soit trivial

On va maintenant noter par $\mathcal{P}_{\partial\neq\emptyset}$ l'ensemble des polyhèdres spéciaux épaississables et par $\mathcal{P}_{\partial=S^2}$ le sous-ensemble de ces P tels que $M(P) \in \mathcal{M}_{\partial=S^2}$. Il faut noter que S^2 est la seule surface de caractéristique d'Euler-Poincaré égale à 2, et donc cette condition est très facile à vérifier de façon combinatoire. Sans donner tous les détails, nous remarquons qu'il existe une notion combinatoire très simple d'*orientation* pour un polyédre spécial P, qui correspond à celle d'orientation pour la variété $M(P)$. Donc on a des ensembles $\mathcal{P}^{\text{ori}}_{\partial\neq\emptyset}$ et $\mathcal{P}^{\text{ori}}_{\partial=S^2}$. Sur tous ces ensembles on va considérer la relation d'équivalence d'homéomorphisme (qui respecte l'orientation s'il y en a une).

Grâce à l'idée d'épaissement qu'on a donnée, on a des fonctions de reconstruction

$$\begin{aligned}
\mathcal{P}_{\partial\neq\emptyset} &\ni P \mapsto M(P) \in \mathcal{M}_{\partial\neq\emptyset}\\
\mathcal{P}_{\partial=S^2} &\ni P \mapsto \widehat{M}(P) \in \mathcal{M}_{\partial=\emptyset}\\
\mathcal{P}^{\text{ori}}_{\partial\neq\emptyset} &\ni P \mapsto M(P) \in \mathcal{M}^{\text{ori}}_{\partial\neq\emptyset}\\
\mathcal{P}^{\text{ori}}_{\partial=S^2} &\ni P \mapsto \widehat{M}(P) \in \mathcal{M}^{\text{ori}}_{\partial=\emptyset}
\end{aligned}$$

et le résultat profond est qu'elles sont toutes des surjections. En plus, sans perte de généralité et sans changement de notation, on peut toujours se limiter aux polyèdres spéciaux qui ont au moins deux sommets.

Codage graphique

On va maintenant expliquer comment coder un polyèdre spécial P par un graphe planaire. Supposons par exemple que P possède trois sommets, et plongeons leur voisinages de façon disjointe dans l'espace comme dans Fig. 13 en haut. Si on se place au point à $+\infty$ sur la verticale et que l'on regarde ces voisinages, on voit leurs bords comme en Fig. 13 au milieu, et on peut décider de les coder avec les morceaux de graph planaire montrés en bas dans la même figure, qui représentent

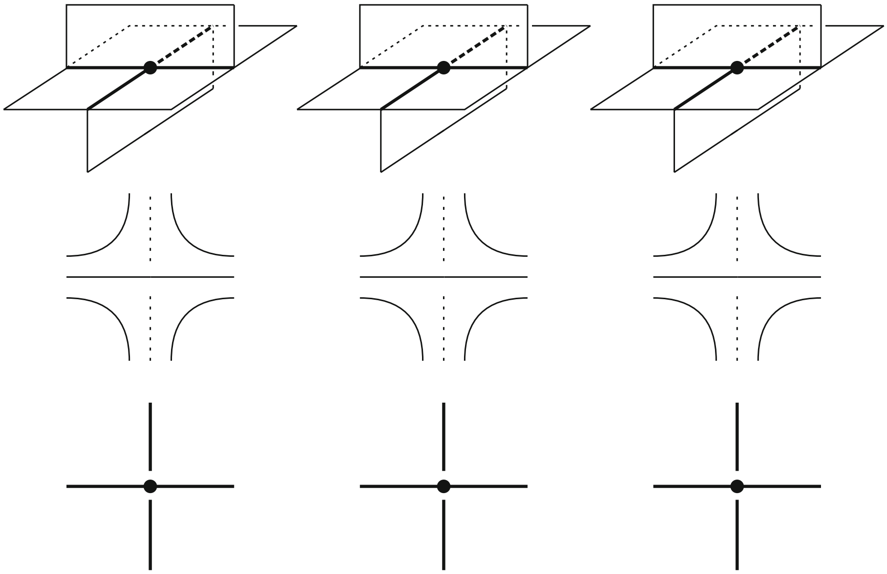

Fig. 13 Codage d'un voisinage des sommets

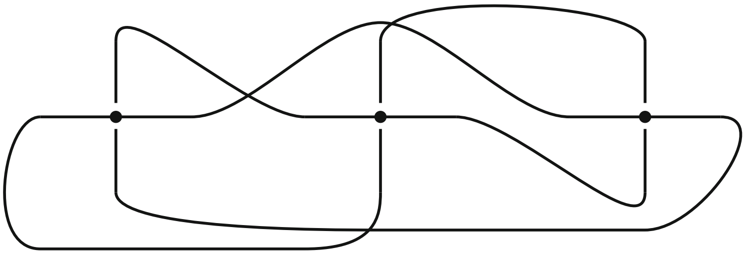

Fig. 14 Immersion planaire de l'ensemble singulier

essentiellement un plongement dans le plan d'un voisinage des sommets dans l'ensemble singulier de P, avec juste un peu plus d'information nécessaire pour reconstruire les voisinages dans P. On peut maintenant étendre ce plongement à une immersion générique dans le plan de tout l'ensemble singulier, comme en Fig. 14.

Grâce à la condition que les composantes non-singulières de P soient des disques, on voit que P est déterminé par un voisinage de son ensemble singulier. Donc pour reconstruire P à partir d'un graphe comme celui en Fig. 14 il suffit de rajouter sur les arêtes une décoration qui puisse expliquer, dans des situations comme en Fig. 15, comment coller les trois ailes d'une côté aux trois ailes de l'autre. C'est alors assez simple de voir qu'une telle décoration est donnée par le choix d'une direction sur l'arête et d'un élément de $\mathfrak{S}_3$. En plus, on peut montrer que dans le cadre orienté, la direction n'est même pas nécessaire, et qu'on se restreint au sous-groupe de permutations paires dans $\mathfrak{S}_3$, qu'on peut identifier à $\mathbb{Z}/3$.

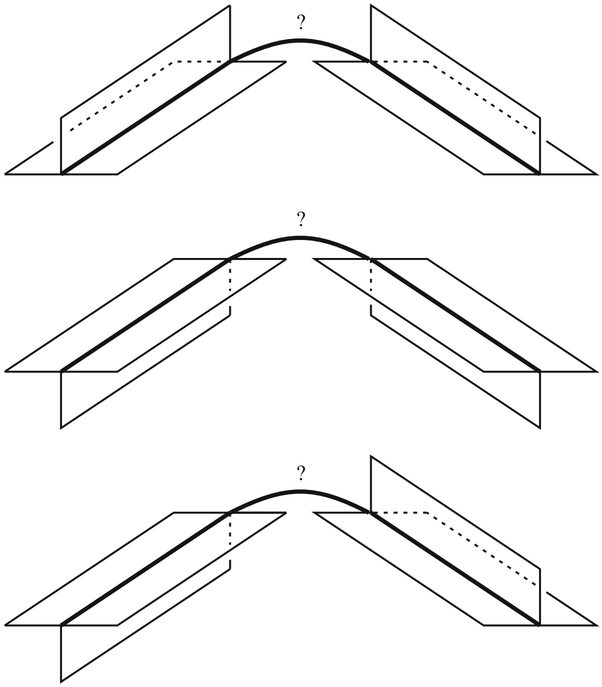

Fig. 15 Reconstruction d'un voisinage de l'ensemble singulier

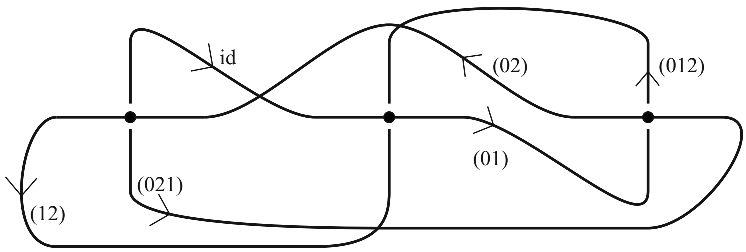

Fig. 16 Un graphe qui représente une variété compacte non orientée avec bord non vide

On définit maintenant $\mathcal{G}_{\partial\neq\emptyset}$ comme l'ensemble des graphes planaires G avec des vrais sommets quadrivalents, décorés comme les diagrammes des nœuds, des faux sommets quadrivalents, un direction et un élément de $\mathfrak{S}_3$ attaché à chaque vraie arête, voir Fig. 16 pour un exemple. Et on a alors une surjection naturelle

$$\mathcal{G}_{\partial\neq\emptyset} \ni G \mapsto M(G) \in \mathcal{M}_{\partial\neq\emptyset}.$$

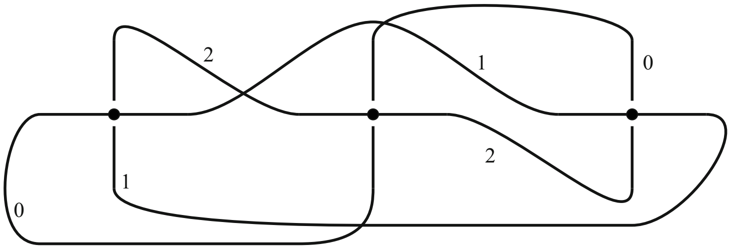

Fig. 17 Un graphe qui représente une variété compacte orientée avec bord non vide

De même, si $\mathcal{G}_{\partial=S^2}$ est le sous-ensemble de $\mathcal{G}_{\partial\neq\emptyset}$ des G tels que le bord de $M(G)$ soit la sphère, on a alors une surjection naturelle

$$\mathcal{G}_{\partial=S^2} \ni G \mapsto \widehat{M}(G) \in \mathcal{M}_{\partial=\emptyset}.$$

Dans le contexte orienté, on définit $\mathcal{G}^{\text{ori}}_{\partial\neq\emptyset}$ comme l'ensemble des graphes planaires G avec des vrais sommets quadrivalents, décorés comme les diagrammes des nœuds, des faux sommets quadrivalents, et un élément de $\mathbb{Z}/3$ attaché à chaque vraie arête, voir Fig. 17 pour un exemple. Dont une surjection naturelle

$$\mathcal{G}^{\text{ori}}_{\partial\neq\emptyset} \ni G \mapsto M(G) \in \mathcal{M}^{\text{ori}}_{\partial\neq\emptyset},$$

qui donne lieu, avec une notation évidente, à une autre surjection naturelle

$$\mathcal{G}^{\text{ori}}_{\partial=S^2} \ni G \mapsto \widehat{M}(G) \in \mathcal{M}^{\text{ori}}_{\partial=\emptyset}.$$

Dans tous les cas, sans perte de généralité et sans changer de notation, on peut supposer que les graphes ont au moins deux vrais sommets.

Mouvements locaux

Un célèbre théorème affirme que pour chacun des ensembles

$$\mathcal{P}_{\partial\neq\emptyset} \qquad \mathcal{P}_{\partial=S^2} \qquad \mathcal{P}^{\text{ori}}_{\partial\neq\emptyset} \qquad \mathcal{P}^{\text{ori}}_{\partial=S^2}$$

la relation d'équivalence induite par l'homéomorphisme (orienté) de la variété correspondante en

$$\mathcal{M}_{\partial\neq\emptyset} \qquad \mathcal{M}_{\partial=\emptyset} \qquad \mathcal{M}^{\text{ori}}_{\partial\neq\emptyset} \qquad \mathcal{M}^{\text{ori}}_{\partial=\emptyset}$$

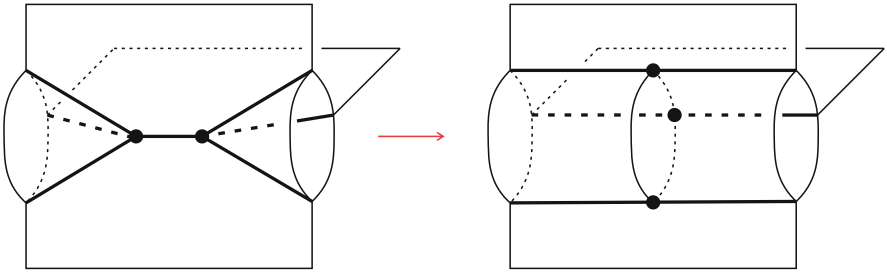

Fig. 18 Le mouvement de Matveev-Piergallini

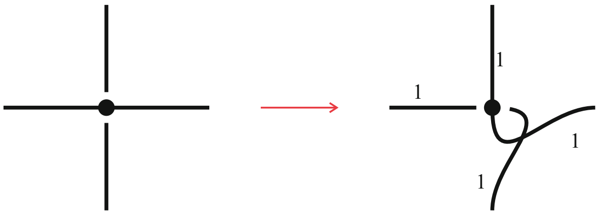

Fig. 19 Premier mouvement sur un graphe qui ne change pas la variété associée

est engendré par l'homéomorphisme (orienté, défini de façon naturelle), et par un seul mouvement local, dit de Matveev-Piergallini et illustré en Fig. 18.

Calcul graphique

On va ici se restreindre au cas orientable. Grâce au résultat topologique sur les polyèdres spéciaux qui représente la même variété, on peut vérifier que la relation d'équivalence sur les graphes de $\mathcal{G}^{\text{ori}}_{\partial\neq\emptyset}$ ou de $\mathcal{G}^{\text{ori}}_{\partial=S^2}$ qui correspond à l'homéomorphisme dans $\mathcal{M}^{\text{ori}}_{\partial\neq\emptyset}$ ou dans $\mathcal{G}^{\text{ori}}_{\partial=\emptyset}$ est engendrée par:

- L'isotopie planaire;
- Des mouvements naturels semblables aux mouvements de Reidemeister, qui traduisent le fait que les faux sommets sont en effet faux, donc la seule chose qui compte pour les arêtes sont leurs deux extrémités;
- Le mouvement de Fig. 19, ce qui traduit la nature arbitraire du plongement dans l'espace du voisinage d'un sommet; la convention ici et ci-dessous est que les étiquettes multiples sur les arêtes doivent être additionnées dans $\mathbb{Z}/_3$;
- Le mouvement de Fig. 20, qui traduit le mouvement de Matveev-Piergallini.

Le codage des variétés de dimension 3 par les polyèdres spéciaux, et sa version graphique, ont eu plusieurs applications. En particulier, ils ont donné des méthodes très efficaces, à l'aide d'un ordinateur, pour reconnaître l'homéomorphisme des

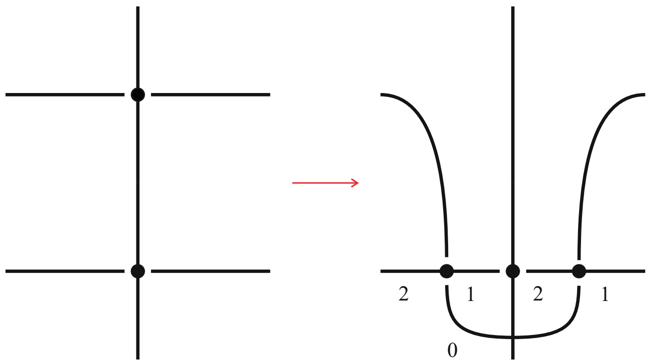

Fig. 20 Deuxième mouvement sur un graphe qui ne change pas la variété associée

variétés, et ils ont été utilisés pour définir plusieurs invariants, notamment ceux de Turaev-Viro.

Variétés de dimension 4

La définition de variété qu'on a donnée ci-dessus en dimension 2 (surface) et 3, a une extension évidente en dimension quelconque. Ici on va parler de variétés de dimension 4 compactes et orientables (sans donner un définition explicite de cette dernière notion). Pour construire des telles variétés on va employer la méthode des ombres, voir Costantino (2005) et les références citées y dedans.

L'idée des ombres est d'épaissir un polyèdre spécial en 4 dimensions plutôt qu'en 3 seulement. On part donc d'un P qui n'est pas nécessairement épaississable dans le sens tridimensionnel. On s'occupe d'abord des sommets et des arêtes de P, près desquels on multiplie tout simplement par l'intervalle $[0, 1]$ l'épaississement tridimensionnel montré dans la Fig. 12, on n'a donc aucun choix à faire. Sur un disque D non-singulier de P, quand même, l'épaississement n'est pas unique, étant classifié par un demi-entier $g \in \frac{1}{2}\mathbb{Z}$ tel que $2g$ modulo 2 est l'obstruction à l'existence d'un épaississement tridimensionnel sur D. C'est à dire que g est entier si le fibré en intervalles sur le bord de $\overline{D}$ décrit ci-dessus est trivial, demi-entier autrement. Le demi-entier g est en effet défini comme la moitié du nombre algébrique d'intersection entre D et D', oú D' est un déplacement générique de D dans son épaississement de dimension 4. Un exemple en dimension moitié de cette construction est donné en Fig. 21.

On appelle donc *ombre* un polyèdre spécial P avec des demi-entiers $g(D)$ attachés aux disques non-singuliers D de P, tels que $2g$ modulo 2 soit l'obstruction à l'existence d'un épaississement tridimensionel sur D. À P on peut donc associer une variété $X(P)$ compacte orientable de dimension 4 avec bord non-vide. La

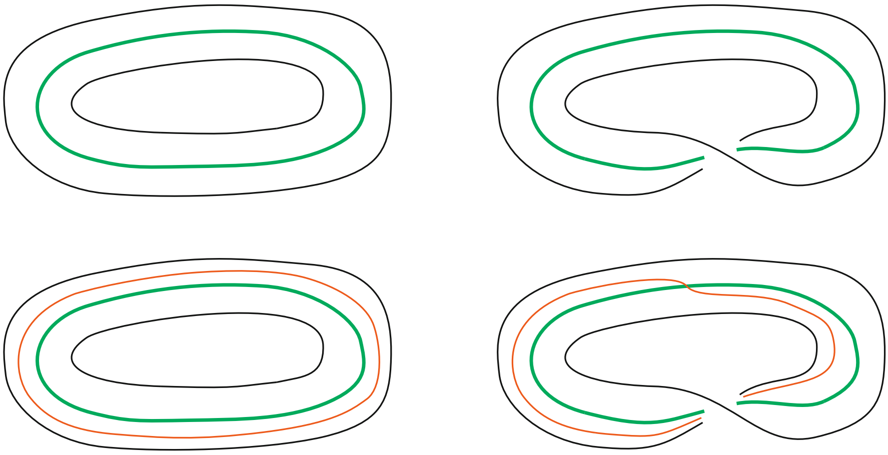

Fig. 21 Les deux épaississements en dimension 2 d'un lacet α, et l'intersection entre α et un déplacement générique α' de α dans l'épaississement

situation en 4 dimensions est quand-même plus compliquée qu'en 3 dimensions; en effet, on sait que:

- Si Y est une variété compacte orientable de dimension 4 telle qu'il existe P avec $Y = X(P)$, alors Y admet une décomposition en anses d'indices au plus 2;
- À une variété Y de dimension 4 orientable avec composantes de bord qui soient S^3 ou des sommes connexes de $S^1 \times S^2$ on peut uniquement associer une variété sans bord $\widehat{Y}$;
- Si Y est une variété de dimension 4 compacte orientable sans bord et on considère une décompositions de Y en anses, alors il existe P tel que $X(P)$ soit Y moins ses anses d'indices 3 et 4.

La théorie des mouvements pour les ombres des 4-variétés est trop compliquée pour la décrire ici, mais elle a eu des applications remarquables.

Codage graphique des ombres

On peut facilement rajouter aux graphes par lesquels on décrit les polyèdres spéciaux, une décoration capable de coder un demi-entier associé à chaque composante non-singulière, et donc de définir une ombre. La règle près d'un sommet est celle de Fig. 22, et un exemple globale est montré en Fig. 23

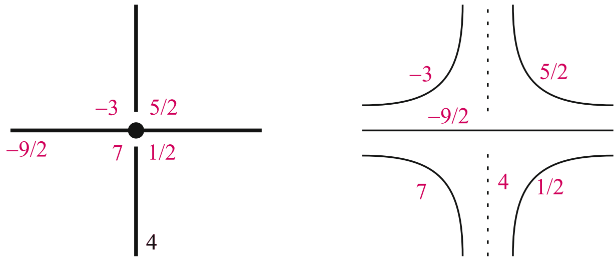

Fig. 22 Demi-entiers près d'un sommet

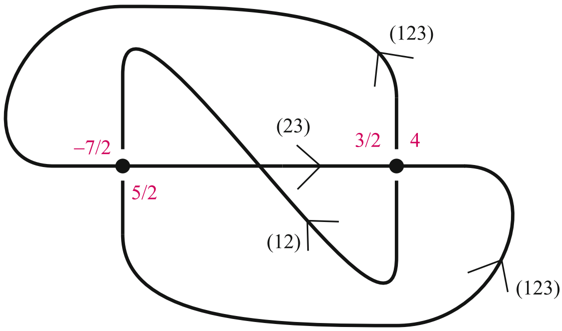

Fig. 23 Un graphe qui définit une ombre, donc une 4-variété orientable

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From Singularities to Graphs

Patrick Popescu-Pampu

Abstract In this text I present some problems which led to the introduction of special kinds of graphs as tools for studying singular points of algebraic surfaces. I explain how such graphs were first described using words, and how several classification problems made it necessary to draw them, leading to the elaboration of a special kind of calculus with graphs. This non-technical paper is intended to be readable both by mathematicians and philosophers or historians of mathematics.

Keywords Coxeter diagrams · Dual graphs · Du Val singularities · Graph manifolds · Models of surfaces · Plumbing calculus · Resolution of singularities · Singularity links · Surface singularities

2010 Mathematics Subject Classification 14B05 (primary), 32S25, 32S45, 32S50, 57M15, 01A60

Introduction

Nowadays, graphs are common tools in singularity theory. They mainly serve to represent morphological aspects of algebraic surfaces in the neighborhoods of their singular points. Three examples of such graphs may be seen in Figs. 1, 2, and 3. They are extracted from the papers Némethi and Szilárd (2000), Neumann and Wahl (2005) and Chung et al. (2009), respectively.

Comparing those figures, one may see that the vertices are diversely depicted by small stars or by little circles, which are either full or empty. These drawing conventions are not important. What matters is that all vertices are decorated with numbers. I will explain their meaning later.

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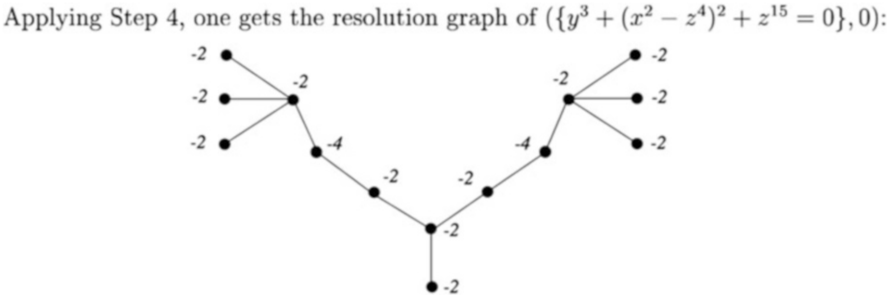


Fig. 1 An example from a 2000 paper of Némethi and Szilárd

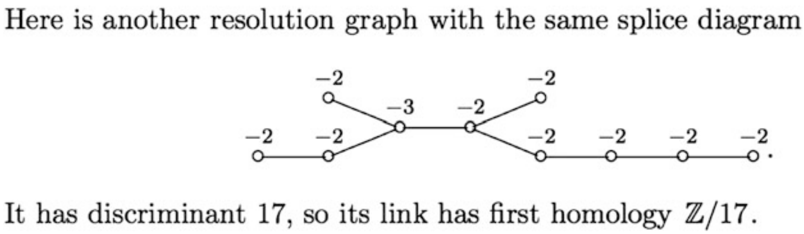


Fig. 2 An example from a 2005 paper of Neumann and Wahl

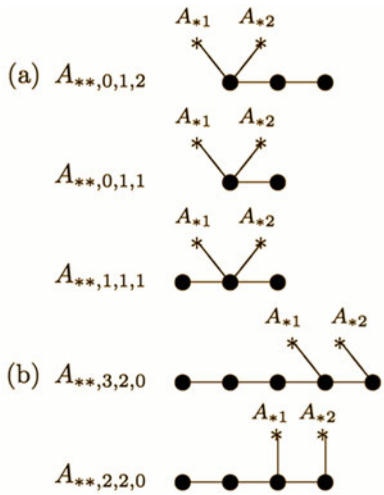


Fig. 3 An example from a 2009 paper of Chung, Xu and Yau

My aim in this paper is to understand which kinds of problems forced mathematicians to associate graphs to surface singularities. I will show how the initial idea, appeared in the 1930s, was only described in words, without any visual representation. Then I will suggest two causes that made the drawing of such graphs unavoidable, starting from the beginning of the 1960s. One of them was a topological reinterpretation of those graphs. The other one was the growing interest in problems of classification of special types of surface singularities.

Let me describe briefly the structure of the paper. In section "What Is the Meaning of Such Graphs?", I explain the meaning of such graphs: they represent configurations of curves which appear by resolving surface singularities. I continue in section "What Does it Mean to Resolve the Singularities of an Algebraic Surface?" by describing what it means to "resolve" a singularity. In section "Representations of Surface Singularities Around 1900" I present several models of surface singularities made around 1900 and I discuss one of the oldest configurations of curves, perhaps the most famous of them all: the 27 lines lying on a smooth cubic surface. In section "Du Val's Singularities, Coxeter's Diagrams and the Birth of Dual Graphs", I present excerpts of Du Val's 1934 paper in which he described a way of thinking about a special class of surface singularities in terms of graphs. In the same paper, he made an analogy between his configurations of curves and the facets of special spherical simplices analyzed by Coxeter in his 1931 study of finite groups generated by reflections. It is perhaps the fact that Coxeter had described a way to associate a graph to such a simplex which, through this analogy, gave birth to Du Val's idea of speaking about graphs of curves. In section "Mumford's Paper on the Links of Surface Singularities", I jump to the years 1960s, because until then Du Val's idea of associating graphs to singularities had almost never been used. I show how things changed with a 1961 paper of Mumford, in which he reinterpreted those graphs in the realm of 3-dimensional topology. Hirzebruch's 1963 Bourbaki Seminar talk about this work of Mumford seems to be the first place in which graphs representing arbitrary configurations of curves were explicitly defined. I begin section "Waldhausen's Graph Manifolds and Neumann's Calculus with Graphs" by discussing a 1967 paper of Waldhausen, in which he built a subtle theory of the 3-dimensional manifolds associated to graphs as explained in Mumford's paper. I finish it with a discussion of a 1981 paper of Neumann, which turned Waldhausen's work into a concrete "calculus" for deciding whether two graphs represent the same 3-dimensional manifold. In section "Conclusion", I conclude by mentioning several recent directions of research concerning graphs associated to singularities of algebraic varieties, and by summarizing this paper.

What Is the Meaning of Such Graphs?

Let me begin by explaining the meaning of the graphs associated to singularities of surfaces. In fact, the construction is not specific to singularities, one may perform it whenever is given a configuration of curves on a surface. The rule is very simple:

- each curve of the configuration is represented by a vertex;
- two vertices are joined by an edge whenever the corresponding curves intersect.

A variant of the construction introduces as many edges between two vertices as there are points in common between the corresponding curves.

Note that this construction reverses the dimensions of the input objects. Indeed, the curves, which have dimension one, are represented by vertices of the graph, which have dimension zero. Conversely, the intersection points of two curves, which have dimension zero, are represented by edges of the graph, which have dimension one. Remark also that an intersection point lies on a curve of the configuration if and only if its associated edge of the graph contains the vertex representing the curve. It is customary nowadays in mathematics to speak about “duality” whenever one has such a dimension-reversing and inclusion-reversing correspondence between parts of two geometric configurations. For this reason, one speaks here about the “*dual graph*” of the curve configuration, a habit which became common at the end of the 1960s.

An example of this construction is represented in Fig. 4, which combines drawings from Michael Artin’s 1962 and 1966 papers (Artin, 1962, 1966). In the upper half, one sees sketches of curve configurations, each curve being depicted as a segment. This representation is schematic, as it does not respect completely the topology of the initial curve configuration, which consists of curves without boundary points. But it represents faithfully the intersections between the curves of the configuration: two of its curves intersect if and only if the associated segments do. The corresponding “dual graphs” are depicted in the lower part of the figure. For instance, the vertex which is joined to three other vertices in the graph of the lower right corner represents the horizontal segment of the curve configuration labeled (v), on the right of the second row.

Note that the representation of the curve configurations as dual graphs emphasizes better visually its overall connectivity pattern than the representation as a

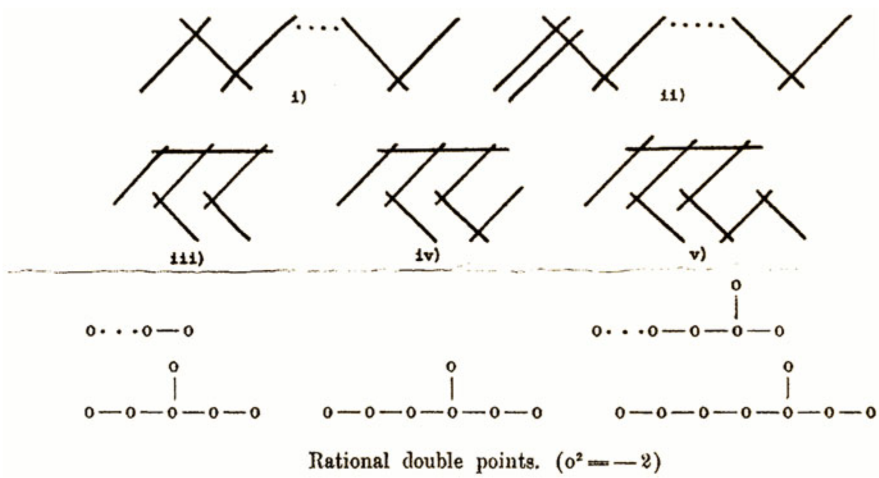


Fig. 4 Artin’s depictions of curve configurations and associated dual graphs

configuration of segments. This is probably one of the reasons which led Artin to pass from drawings of configurations of segments in his 1962 paper to drawings of dual graphs in his 1966 paper.

More generally, dual graphs may be introduced whenever one is interested in the mutual intersections of several subsets of a given set. It is not important that the given sets consist of the points of several curves lying on surfaces, they may for instance be arbitrary subsets of manifolds of any dimension or, less geometrically, the sets of members of various associations of persons. Then, one represents each set by a vertex and one joins two such vertices by an edge if and only if the corresponding sets intersect.

As a general rule, one represents any object of study by a vertex, whenever one is not interested in its internal structure, but in its "sociology", that is, in its relations or interactions with other objects. A basic way to depict these relations is to join two such vertices whenever the objects represented by them interact in the way under scrutiny. In our context, one considers that two curves interact if and only if they have common points.

Let us come back to the configurations of curves and their associated dual graphs from Fig. 4. Those drawings represent the classification of a special type of surface singularities, namely the "*rational double points*", in the terminology of Artin's 1966 paper (Artin, 1966). That list was not new, it had already appeared in the solution of a different classification problem—leading nevertheless to the same objects—in Du Val's 1934 paper in which he had informally introduced the idea of dual graph. We will further discuss that paper in section "Du Val's Singularities, Coxeter's Diagrams and the Birth of Dual Graphs".

Before passing to the next section, let me mention that Artin's paper (Artin, 1966) contained also a new classification, that of the dual graphs associated to the "*rational triple points*" (see Fig. 5). Being more abundant than those of the lower part of Fig. 4,

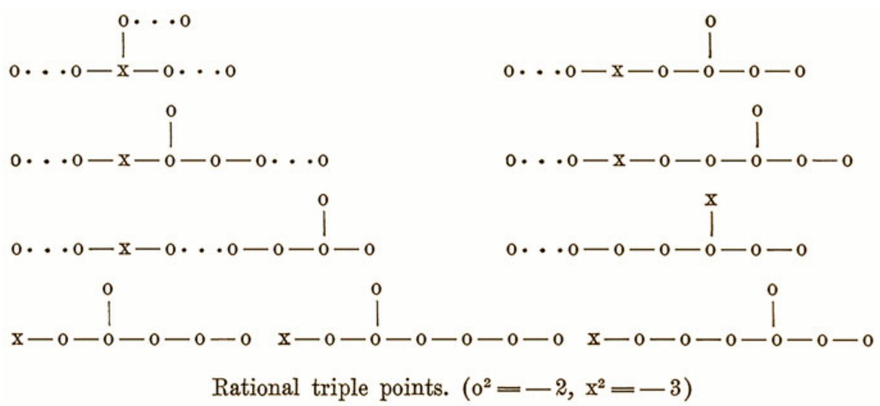


Fig. 5 Artin's classification of dual graphs of rational triple points

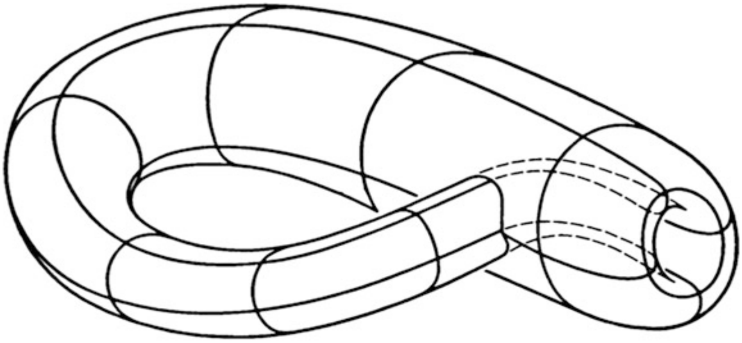


Fig. 6 A Klein bottle

it becomes apparent that it is also more economical for printing to draw such graphs rather than configurations of segments.

Now that we understood how configurations of curves lead to dual graphs, let us see in which way singularities of surfaces may lead to configurations of curves. This is the object of the next section.

What Does it Mean to Resolve the Singularities of an Algebraic Surface?

What is a *singularity of an algebraic surface*? It is a special point, at which the surface is not smooth. For instance, a sphere does not have singular points, but a double cone, idealization of the boundary of a nighty region illuminated by a lighthouse, has a singular point at its vertex. In this case, the singular point is isolated, but other surfaces may have whole curves of singularities. Such curves may be either self-intersections of the surface, as shown in Fig. 6,[1] or they may exhibit more complicated behaviour, as shown in Fig. 7.[2]

In this last figure, a polynomial equation in three variables is written next to each surface. The reason is that each of those surfaces is a portion of the locus of points which satisfy the associated equation in the 3-dimensional cartesian space of coordinates (x, y, z). As polynomials are algebraic objects, such a locus

[1] This illustration of an immersion of a Klein bottle in three-dimensional cartesian space comes from the wonderful 1932 book (Hilbert and Cohn-Vossen, 1952) of David Hilbert and Stefan Cohn-Vossen. Note that in addition to a circle of self-intersection, this illustration represents also curves of apparent contours and several closed curves drawn on the surface.

[2] This figure comes from https://homepage.univie.ac.at/herwig.hauser/gallery.html, January 2018. Copyright: Herwig Hauser, University of Vienna, www.hh.hauser.cc.

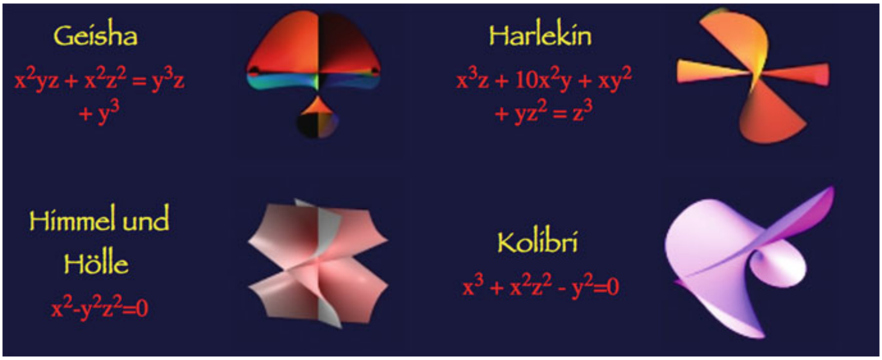


Fig. 7 Several real surface singularities from Hauser's gallery

is called "*algebraic*". There are also algebraic surfaces in cartesian spaces of higher dimensions, defined by systems of polynomial equations in more than three variables. In fact, all surfaces considered in the papers discussed here are algebraic. One advantage of working with such surfaces is that one may consider not only the real solutions of those equations, but also the complex ones. In this way, one expects in general to make the correspondence between the algebraic properties of the defining equations and the morphological properties of the associated surface easier to understand.

A prototype of this expectation is the fact that a polynomial equation in one variable has as many complex roots as its degree, provided that the roots are counted with suitable "multiplicities" (this is the so-called "fundamental theorem of algebra", but it is rather a fundamental theorem of the *correspondence* between algebra and topology). If one considers instead only its real roots, then their number is not determined by the degree, but there are several possibilities. In fact, as I will briefly explain at the beginning of section "Representations of Surface Singularities Around 1900", whenever one considers families with three parameters of polynomials, these possibilities may be distinguished using "*discriminant surfaces*", which have in general non-empty singular loci.

Because of this expected relative simplicity of complex algebraic geometry versus real algebraic geometry, it became customary in the nineteenth century to study the sets of *complex* solutions of polynomial equations in three variables. One gets in this way "*complex algebraic surfaces*". Nevertheless, in order to build an intuition of their properties, it may be useful to practice with concrete *models* of the associated *real* surfaces. Around the end of the nineteenth century, such models were either drawn or manufactured using for instance wood, plaster, cardboard, wires and string. Nowadays they are also built using 3D-printers or, more commonly, simulated using techniques of computer visualization. This is for instance the case of Hauser's images of Fig. 7.

Why is it important to study singular surfaces? Because, in general, surfaces do not appear alone, but rather in families depending on parameters (which, in physical contexts, may be for instance temperatures or intensities of external fields), and that for some special values of these parameters one gets surfaces with singularities. Understanding the singular members of a family is many times essential for understanding also subtle aspects of its non-singular members. For instance, one may understand part of the structure of a non-singular member by looking at its portions which "vanish" when one converges to a singular member.

The techniques of differential or algebraic geometry used in the study of smooth algebraic surfaces may be extended to singular surfaces using three basic procedures:

- by decomposing a singular surface into smooth "*strata*", which are either isolated points, smooth portions of curves or smooth pieces of surfaces; this is similar to the decomposition of the surface of a convex polyhedron into vertices, edges and faces;
- by seeing a singular surface as a limit of smooth ones; when this is possible, one says that the surface was "*smoothed*"; such a process is not always possible, and even if it is possible, it can be usually done in various ways;
- by seeing a singular surface as a projection of a smooth one, living in a higher dimensional ambient space; if such a projection leaves the smooth part of the initial singular surface unchanged, then it is called a "*resolution of singularities*"; resolutions of singularities always exist, but are not unique.

Intuitively speaking, resolving the singularities of a surface means to remove its singular locus and to replace it algebraically by another configuration of points and curves, so that the resulting surface is smooth. In the special case of an *isolated* singular point of algebraic surface, one replaces that singular point by a configuration of curves, called the "*exceptional divisor*" of the resolution. For instance, all the graphs appearing in Figs. 1, 2, 3, 4, and 5 are dual graphs of exceptional divisors of resolutions of isolated singularities of complex surfaces.

In simple examples, one may resolve the singularities of an algebraic surface by performing finitely many times the elementary operation called "*blowing up a point*", which is a mathematical way to look through a microscope at the neighborhood of the chosen point. This operation builds new cartesian spaces starting from the space which contains the initial singular surface. Each of those spaces contains a new surface, which projects onto a part of the initial one. If one of those surfaces is smooth, then one keeps it untouched. Otherwise, one blows up again its isolated singular points. It may happen that finitely many such operations lead to a family of *smooth* surfaces, each one of them projecting onto a portion of the initial singular surface. Those surfaces may be glued, together with their projections, into a global smooth surface which "*resolves the singularities*" of the initial one.

Let us consider for instance the surface with equation $x^2 - y^2 + z^7 = 0$, illustrated in Fig. 8. It has an isolated singularity (called "of type A_6") at the origin.[3] If one

[3] This figure was taken from the web-page https://www.krueger-berg.de/anne/aufl-bilder/A6.html of Anne Frühbis-Krüger in September 2020. This is also the case of Figs. 9 and 10.

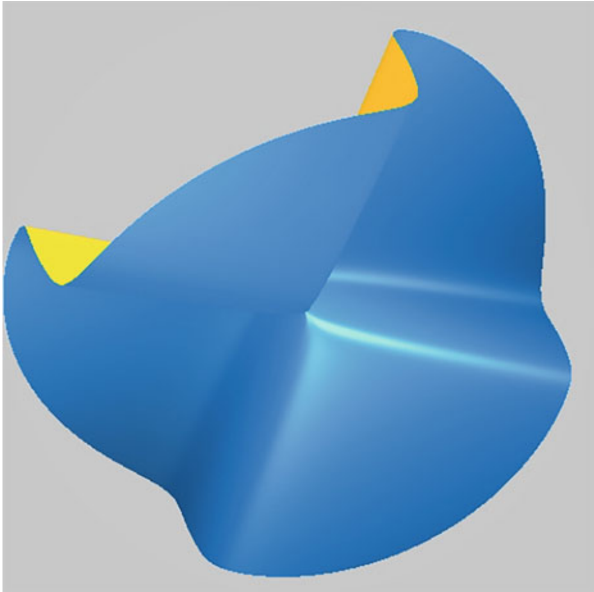


Fig. 8 An isolated A_6 surface singularity

performs the previous iterative process of blowing up the singular points of the intermediate surfaces, one gets a "tree" of surfaces, represented diagrammatically on Fig. 9. The initial surface is indicated in the top-most rectangle, and each edge of the diagram represents a blow-up operation.

The final smooth surfaces produced by the process are represented in Fig. 10. Each of them contains one or more highlighted lines. Those lines glue into a configuration of curves on the total smooth surface which resolves the initial singular one. This configuration is the exceptional divisor of this resolution of the starting isolated singularity. By looking carefully at the way the gluing is performed, one may show that its associated dual graph is a chain of five segments. This means that it is of the type shown on the left of the third row in Fig. 4.

For more complicated singularities, it may not be enough to blow up points, as previous blow-ups may create whole curves of singularities. Other operations which allow to modify the singular locus were introduced in order to deal with this problem. One may learn about them in Kollár's book (Kollár, 2007), which explains various techniques of resolution of singularities in any dimension. The reader more interested in gaining intuition about resolutions of surfaces may consult Faber and Hauser's promenade (Faber and Hauser, 2010) through a garden of examples of resolutions or my introduction (Popescu-Pampu, 2011b) to one of the oldest methods of resolution, originating in Jung's method for parametrizing algebraic surfaces locally.

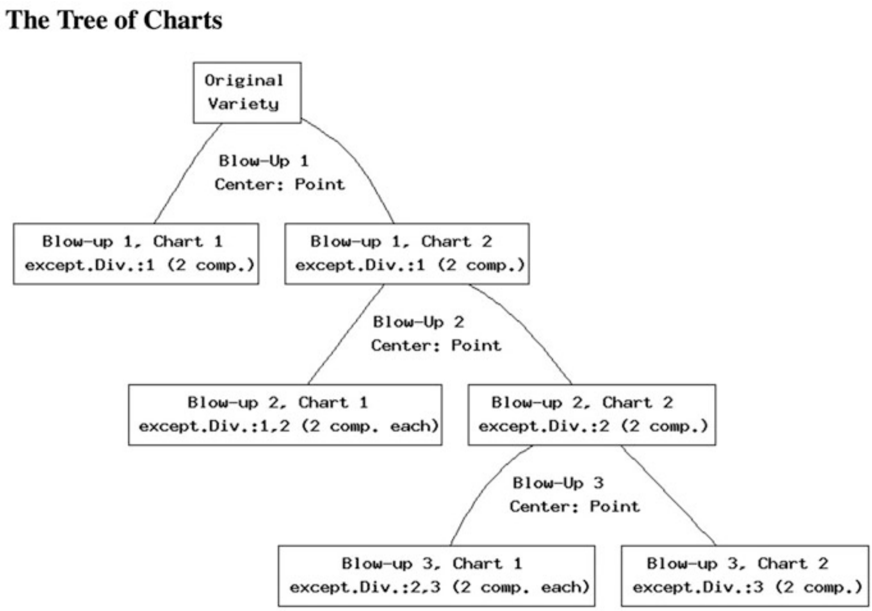


Fig. 9 Frühbis-Krüger's representation of a resolution process

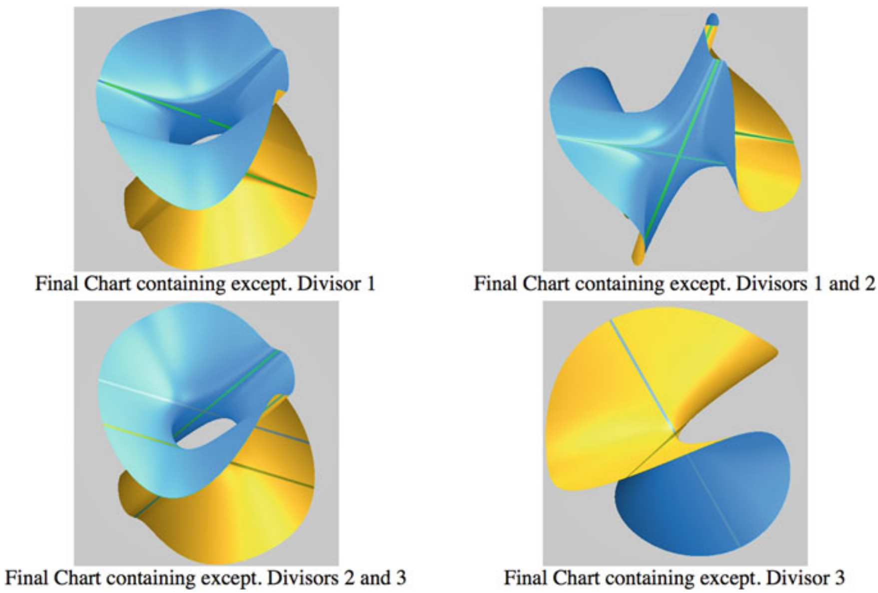


Fig. 10 The final stages of the previous resolution process

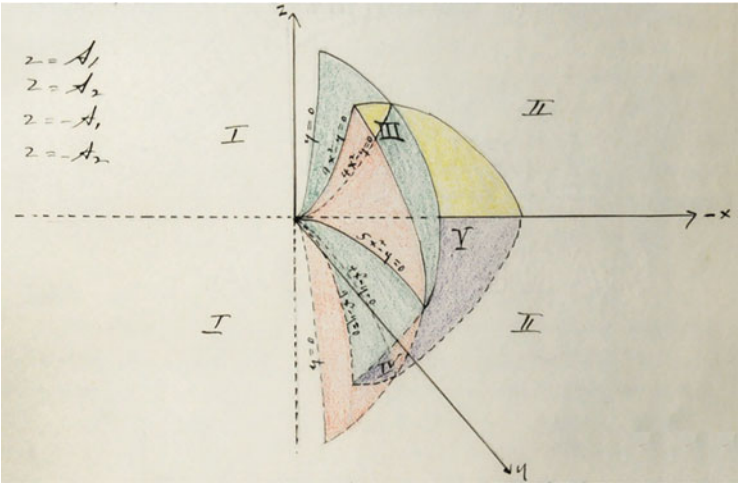


Fig. 11 Sinclair's representation of a discriminant surface

Representations of Surface Singularities Around 1900

In the previous sections we saw contemporary representations of surface singularities, obtained using computer visualization techniques. Let us turn now to older representations, dating back to the beginning of the twentieth century. This will allow us to present one of the oldest sources of surfaces with singularities—the study of discriminant surfaces—and one of the most famous configurations of curves—consisting of the lines contained in a smooth cubic surface.

Figure 11 shows[4] a hand-drawn "*discriminant surface*", which has whole curves of singular points, as was the case in the examples of Figs. 6 and 7. It reproduces a drawing done by Mary Emily Sinclair in her 1903 thesis. Let me discuss this surface a little bit, as it emphasizes another source of interest on the structure of singular surfaces. As explained in its caption from the paper (Top and Weitenberg, 2011), Sinclair was studying the family with three parameters (x, y, z) of polynomials of the form $t^5 + xt^3 + yt + z$. One may associate to it an algebraic family of sets of points, namely the sets of roots of the polynomial in the variable t obtained for fixed values of the parameters. The "discriminant surface" is the subset of the cartesian space of coordinates (x, y, z) for which the associated polynomial has at least one multiple complex root.

More generally, consider any family of points, curves, surfaces or higher-dimensional algebraic objects, depending algebraically on some parameters. If there are exactly three parameters, then the set of singular objects of the family is usually

4 This drawing and the text immediately below it were extracted from the paper (Top and Weitenberg, 2011) of Jaap Top and Erik Weitenberg.

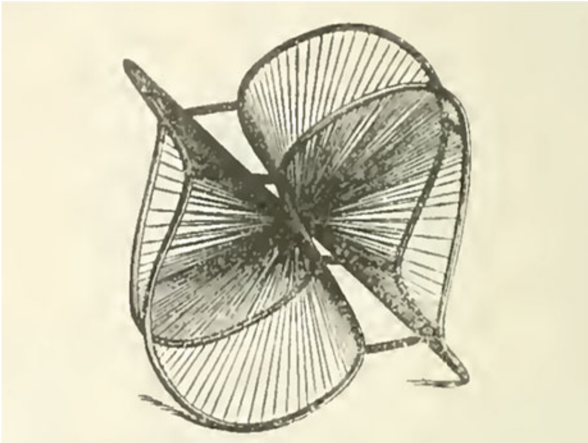


Fig. 12 An image from Schilling's catalog of mathematical models

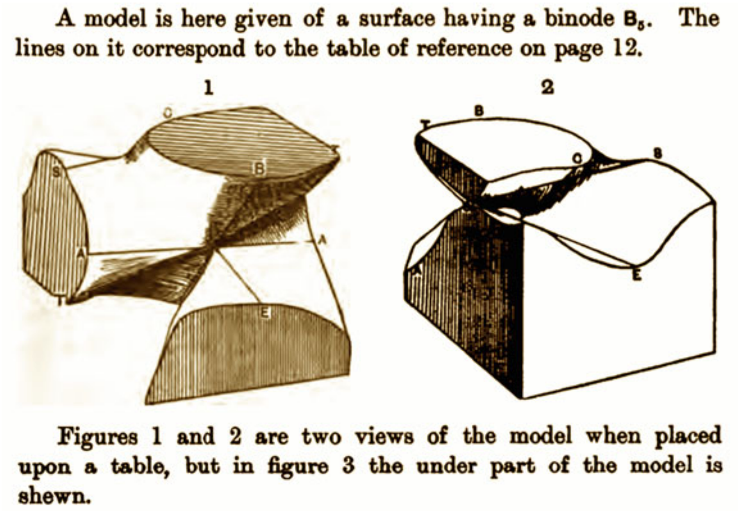
a surface in the space of parameters. All the surfaces obtained in this way are called "*discriminant surfaces*", because they allow to discriminate the possible aspects of the objects in the family, according to the position of the corresponding point in the space of parameters, relatively to the surface. For instance, by determining in which region of the complement of the surface of Fig. 11 lies the point with coordinates (x, y, z), one may see if the set of real roots of the polynomial has 1, 3 or 5 elements—those being the only possibilities for a quintic polynomial equation, because the non-real roots come in pairs of complex conjugate numbers. The reader interested in the analogous study of quartic polynomial equations may read Michel Coste's paper (Coste, 2010).

Let us pass now to material models of surfaces with singularities. Figure 12 reproduces an engraving[5] from the 1911 catalog of mathematical models of Martin Schilling's enterprise. It depicts a cone over a smooth cubic curve, that is, a smooth curve contained in the projective plane and defined by the vanishing of a homogeneous polynomial in 3 variables. This cone has therefore only one singular point, its vertex. Figure 13 shows a reproduction from the 1905 book (Blythe, 1905) of William Henry Blythe. It depicts two plaster models of cubic surfaces with singularities.

One of the most famous discoveries of the nineteenth century regarding the properties of algebraic surfaces is that all smooth complex algebraic cubic surfaces situated in the projective space of dimension three contain exactly 27 lines. This discovery was done in 1849, during a correspondence between Arthur Cayley and George Salmon, and it triggered a lot of research.[6] Starting from around 1870,

[5] It may be found on page 123, part II.3.b of the catalog (Catalog mathematischer Modelle, 1911).

[6] For instance, in the historical summary of his 1915 thesis (Henderson, 1915), Henderson mentions that "in a bibliography on curves and surfaces compiled by J. E. Hill [in 1897] [...] the section on cubic surfaces contained two hundred and five titles. The Royal Society of London


A model is here given of a surface having a binode B_5. The lines on it correspond to the table of reference on page 12.

1 2

Figures 1 and 2 are two views of the model when placed upon a table, but in figure 3 the under part of the model is shewn.

Fig. 13 An image from Blythe's book on models of cubic surfaces

material models of parts of real cubic surfaces with all 27 lines visible on them started to be built. One may see such a model in Fig. 14. It represents a portion of "*Clebsch's diagonal surface*".[7] This surface does not contain singular points, but it is interesting in our context because it exhibits a highly sophisticated configuration of curves, composed of its 27 lines.

Much more details about the building of models of algebraic surfaces around 1900 may be found in the books (Mathematical models, 2017; Institut Henri Poincaré, 2017). As illustrated by Fig. 13, plaster models were built not only of smooth cubic surfaces, but also of singular ones. The manufacturing process was based on Rodenberg's 1878 work (Rodenberg, 1878). The complete classification of the topological types of real cubic surfaces was achieved by Knörrer and Miller in their 1987 paper (Knörrer and Miller, 1987). Other historical details about the study of the configurations of 27 lines lying on smooth cubic surfaces may be found in Polo Blanco' and Lê's theses (Lê, 2013; Polo-Blanco, 2007), as well as in Lê's paper (Lê, 2013) and Labs' paper (Labs, 2017).

Catalogue of Scientific Papers, 1800–1900, volume for *Pure Mathematics* (1908), contains very many more."

[7] Clebsch's diagonal surface is usually defined by the pair of homogeneous equations $x_0 + \cdots + x_4 = 0$ and $x_0^3 + \cdots + x_4^3 = 0$ inside the projective space of dimension 4 whose homogeneous coordinates are denoted $[x_0 : \cdots : x_4]$. This photograph of a model belonging to the University of Göttingen was taken by Zausig in 2012. It comes from Wikimedia Commons: https://commons.wikimedia.org/wiki/File:Modell_der_Diagonalfl che_von_Clebsch_-Schilling_VII,_1_-_44-.jpg.

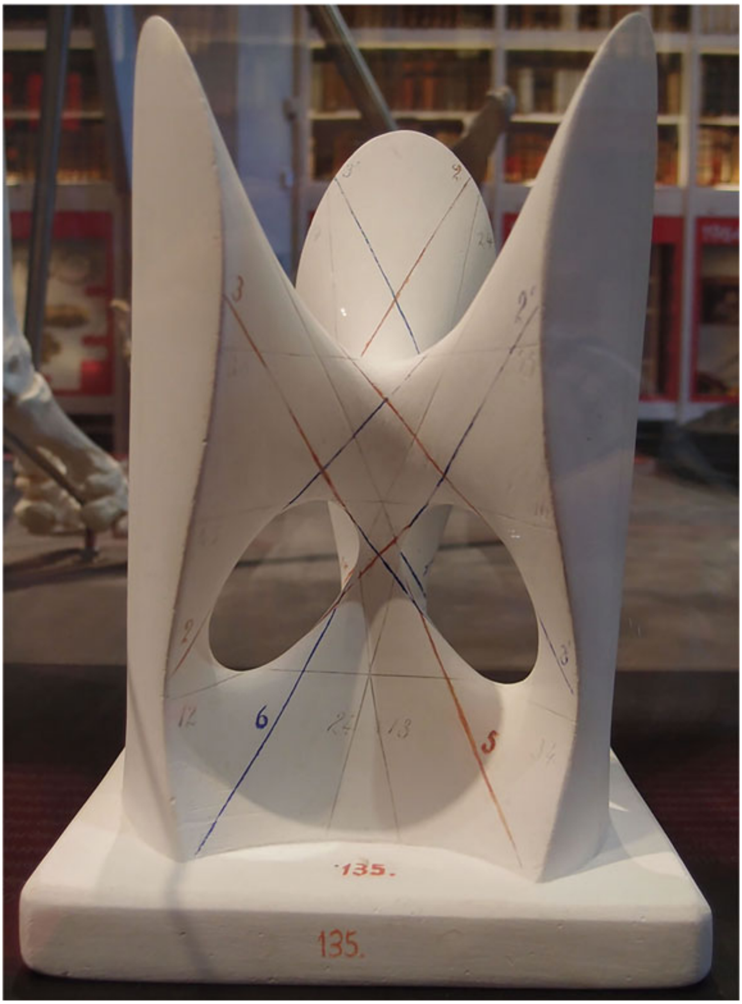


Fig. 14 A model of Clebsch's diagonal surface and of its 27 lines

Note that at the beginning of the twentieth century, some artists from the Constructivist and Surrealist movements were inspired by material models of possibly singular surfaces, as explained in Vierling-Claassen's article (Vierling-Claassen, 2017). It would be interesting to know in which measure computer models as those of Fig. 7 inspire nowadays other artists.

We are thus led to consider a double point having further double points in its neighbourhoods, all of which are rational. If these are transformed into curves we obtain a "tree" of rational curves, each of which meets enough of the others for the tree to be connected, and each of which (as arising from the neighbourhood of a double point) has grade -2. Not every such tree however is capable of representing the whole neighbourhood of a multiple point on a surface; since if a system $|f|$ represents the

Fig. 15 Du Val's introduction of dual graphs

Du Val's Singularities, Coxeter's Diagrams and the Birth of Dual Graphs

Let us discuss now the 1934 paper (Du Val, 1935) in which Patrick Du Val considered, seemingly for the first time, the idea of *dual graph* of an exceptional divisor of resolution of surface singularity.

As indicated by the title of his paper, Du Val's problem was to classify the "*isolated singularities of surfaces which do not affect the conditions of adjunction*". Given a possibly singular algebraic surface contained in a complex projective space of dimension three, its "*adjoint surfaces*" are other algebraic surfaces contained in the same projective space and defined in terms of double integrals. I will not give here their precise definition, which is rather technical.[8] Let me only mention that the adjoint surfaces must contain all curves consisting of singularities of the given surface. By contrast, it does not necessarily contain its *isolated* singular points. Those through which the adjoint surfaces are not forced to pass are precisely the singularities "which do not affect the conditions of adjunction".

Du Val analyzed such singularities by looking at their resolutions. It is in this context that he wrote that for each one of those singularities, there is a resolution whose associated exceptional divisor is a "*"tree" of rational curves*" with supplementary properties (see Fig. 15). For instance, each curve in this "tree" has necessarily self-intersection -2 in its ambient smooth surface (this is the meaning of the syntagm "*has grade* -2"). Du Val continued by giving a list of constraints verified by such "trees", if they were to correspond to singularities which do not affect the conditions of adjunction (see Fig. 16). Using those constraints, he arrived exactly at the list of configurations of curves depicted in Fig. 4. But, in contrast with Artin's papers (Artin, 1962, 1966) from 1960s, his article does not contain any schematic drawing of a configuration of curves, or of an associated dual graph.

It is not even clear whether Du Val really thought about dual graphs. Perhaps he drew for himself some diagrams resembling those of the upper part of Fig. 4, and he

[8] The interested reader may find it in Merle and Teissier's paper (Merle and Teissier, 1977). The whole volume containing that paper is dedicated to a modern study of the singularities analyzed by Du Val.

point. We are thus able to restrict the trees which need to be considered by the elimination of the following:

(i) All trees containing a cycle of curves, each of which meets its two neighbours; since the least sum of positive multiples of the curves of such a cycle which has non-positive intersection number with each of them is just the sum of them all, and this has zero intersection with each (the grade of each being -2, and each meeting two others).

(ii) All trees containing a curve which meets four others; for twice this curve plus the sum of the other four has zero intersections with each of the five.

(iii) All trees containing more than one curve which meets three others; for if there are two such curves, since the tree is connected there is a chain (or sequence in which each curve meets its predecessor and successor) joining them, and this chain, with the two given curves, forms a total curve of grade -2 meeting four others.

Fig. 16 Du Val's restricted class of graphs

It may be noted that the "trees" of curves which we have had to consider bear a strict formal resemblance to the spherical simplexes whose angles are submultiples of π, considered by Coxeter†. If in fact we let the r curves which form a tree correspond to the bounding primes of a simplex in $[r-1]$, making intersecting curves correspond to primes inclined at an angle $\pi/3$, and non-intersecting curves to mutually perpendicular primes, then comparison of the results obtained above with Coxeter's indicates that to a simplex which can be constructed in spherical space corresponds a tree of curves which can be fundamental to a linear system, i.e. which can actually represent the neighbourhood of a singular point; while a simplex which can be constructed

Fig. 17 Du Val's analogy with Coxeter's spherical simplices

saw an analogy with some "trees" considered by other mathematicians. Note that it is possible that for Du Val the term "tree" meant what we call "graph". Indeed, one sees him stating in the excerpt of Fig. 16 that the "trees" under scrutiny should not contain "a cycle of curves", a formulation which allows some "trees of curves" to contain such cycles.

At the end of his article, Du Val mentioned an analogy with results of Coxeter[9] regarding finite groups generated by reflections (see Fig. 17). In order to understand

[9] Du Val cites the paper (Coxeter, 1932), but Coxeter already considered this problem one year earlier, in Coxeter (1931). Note that Du Val and Coxeter were friends and that they discussed regularly about their research. One may learn a few details about their friendship and discussions in Roberts' book (Roberts, 2007), especially on pages 71–72.

this analogy, we have to know that Coxeter started from a finite set of hyperplanes passing through the origin in a real Euclidean vector space of arbitrary finite dimension. He assumed that they were spanned by the facets of a simplicial cone emanating from the origin, and he looked at the spherical simplex obtained by intersecting the cone with the unit sphere centered at the origin. Coxeter's problem was to classify those spherical simplices for which the group generated by the orthogonal reflections in the given hyperplanes is *finite*.

Du Val realized that his classification of isolated singularities which do not affect the conditions of adjunction corresponds to a part of Coxeter's classification of spherical simplices giving rise to finite groups of reflections. In order to make this correspondence visible, he associated to each curve of a given exceptional divisor a facet of the simplex, two curves being disjoint if and only if the corresponding facets are orthogonal, and having one point of intersection if and only if the facets meet at an angle of $\pi/3$.

Exactly in the same way in which Du Val introduced in 1934 his dual graphs verbally, without drawing them, Coxeter had verbally introduced in 1931 "*diagrams of dots and links*" in order to describe the shapes of his spherical simplices (see Fig. 18). It is only in his 1934 paper (Coxeter, 1934) that he published drawings of such graphs (see Fig. 19), which were to be called later "*Coxeter diagrams*", or "*Coxeter-Dynkin diagrams*", in reference to their reappearance in a slightly different form in Dynkin's 1946 work (Dynkin, 1946) about the structure of Lie groups and Lie algebras.

Much later, Coxeter explained in his 1991 paper (Coxeter, 1991) that analogous diagrams had already been introduced by Rodenberg in 1904, in his description (Rodenberg, 1904) of the plaster models of singular cubic surfaces from Schilling's catalog. Rodenberg's intepretation was different, not related to reflections, but to special subsets of the configuration of 27 lines on a generic smooth cubic surface (see Fig. 20, containing an extract from Coxeter (1991)). More details about Rodenberg's convention, based on his older paper (Rodenberg, 1878), may be found in Barth and Knörrer's text (Barth and Knörrer, 2017).

It is therefore desirable to enumerate all spherical simplexes whose dihedral angles are submultiples of π. Since these angles must not be too small, only a few of them can be acute; the rest are all right angles. It is useful to represent such a spherical simplex by a diagram of dots and links. Every prime is represented by a dot; and if $(r\ s) = \pi/k$ $(k > 2)$, the dots r and s are joined by a link marked "k". The dots representing perpendicular primes are not joined at all. We can suppose the diagram to consist of a connected chain, since otherwise the corresponding group is merely the direct product of the groups which correspond to the various disconnected portions.

Fig. 18 Coxeter's introduction of his diagrams

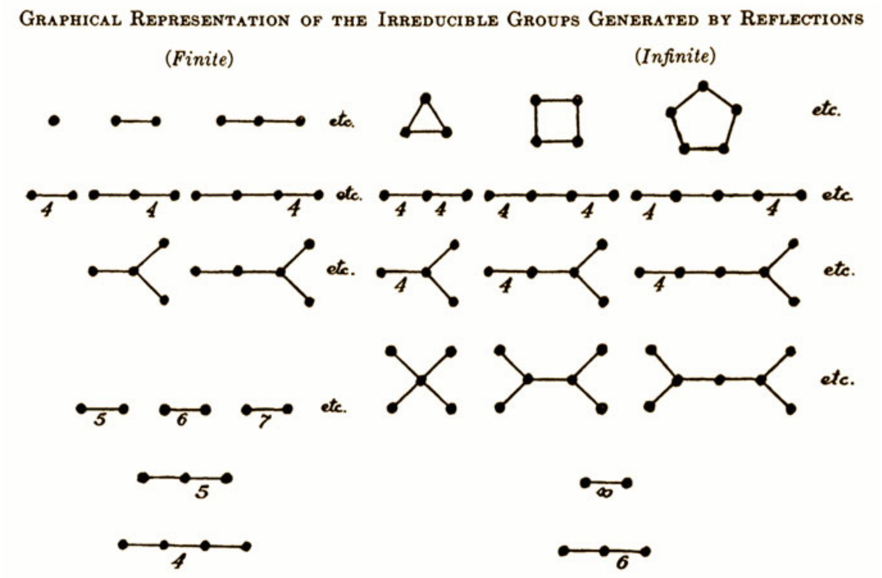


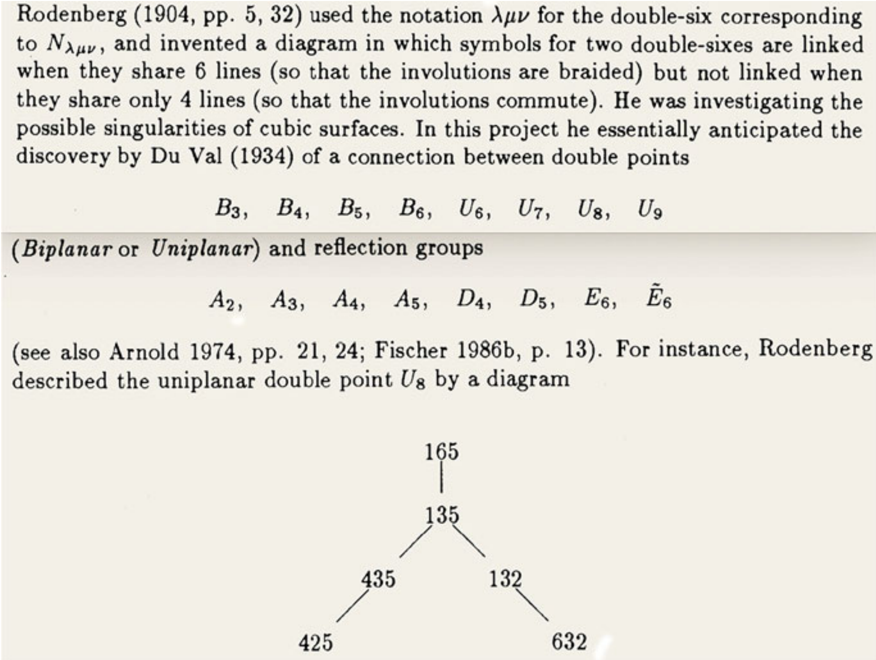
Fig. 19 Coxeter's pictures of his diagrams

Mumford's Paper on the Links of Surface Singularities

One could believe that the combination of Du Val's analogy between his "trees of curves" and Coxeter's spherical simplices on one side, and Coxeter's diagrams on another side, would trigger research on the possible dual graphs of isolated surface singularities. Such an interest indeed developed starting from a 1961 paper of David Mumford. In this section I explain the aim of Mumford's paper and how it led to the first explicit formulation of the notion of dual graph of a configuration of algebraic curves contained in a smooth algebraic surface.

I could find only one article published between 1934 and 1961 which contained a drawing of dual graph of resolution of isolated surface singularity.[10] It is Hirzebruch's 1953 paper (Hirzebruch, 1953), in which he proved that one could not only resolve the singularities of complex algebraic surfaces, but also of the more general *complex analytic* ones. That article contains a single illustration (see Fig. 21), which depicts the general shape of possible dual graphs of resolutions for a class of singularities which is crucial for his method.[11] Unlike the case of Du

[10] One may think, from a rapid glance, that Du Val's paper (Du Val, 1944) is an exception. But the graphs of that paper, which I rediscovered with a different interpretation in Popescu-Pampu (2011a), are not dual graphs of configurations of curves. They indicate relations between infinitely near points of a given smooth point of an algebraic surface, being variants of Enriques' diagrams introduced in Enriques and Chisini (1917).


Rodenberg (1904, pp. 5, 32) used the notation $\lambda\mu\nu$ for the double-six corresponding to $N_{\lambda\mu\nu}$, and invented a diagram in which symbols for two double-sixes are linked when they share 6 lines (so that the involutions are braided) but not linked when they share only 4 lines (so that the involutions commute). He was investigating the possible singularities of cubic surfaces. In this project he essentially anticipated the discovery by Du Val (1934) of a connection between double points

$$B_3,\quad B_4,\quad B_5,\quad B_6,\quad U_6,\quad U_7,\quad U_8,\quad U_9$$

(*Biplanar* or *Uniplanar*) and reflection groups

$$A_2,\quad A_3,\quad A_4,\quad A_5,\quad D_4,\quad D_5,\quad E_6,\quad \tilde{E}_6$$

(see also Arnold 1974, pp. 21, 24; Fischer 1986b, p. 13). For instance, Rodenberg described the uniplanar double point U_8 by a diagram

Fig. 20 One of Rodenberg's diagrams

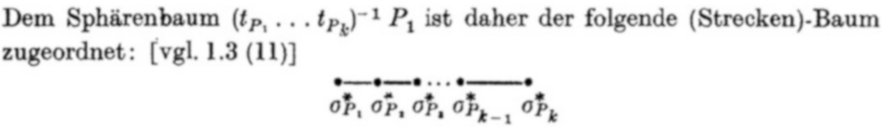

Dem Sphärenbaum $(t_{P_1}\ldots t_{P_k})^{-1}\,P_1$ ist daher der folgende (Strecken)-Baum zugeordnet: [vgl. 1.3 (11)]

Fig. 21 The dual graph of Hirzebruch's 1953 resolution paper

Val's singularities which do not affect the conditions of adjunction, which have resolutions for which all the curves composing the exceptional divisor have self-intersection -2, here the self-intersections can be arbitrary negative integers.[12] But a common feature of both cases is that all those curves are smooth and

[11] Such singularities are variously called nowadays "*Hirzebruch-Jung singularities*", "*cyclic quotient singularities*" or "*toric surface singularities*". The first name alludes to their importance for Hirzebruch's method or resolution of arbitrary complex analytic surfaces (explained in Popescu-Pampu, 2011b), inspired by the ideas of Jung's method of local parametrization of algebraic surfaces mentioned at the end of section "What Does it Mean to Resolve the Singularities of an Algebraic Surface?".

[12] The "normal" surface singularities which admit resolutions with such a dual graph, its composing curves being moreover smooth, rational and pairwise transversal, are exactly the Hirzebruch-Jung singularities alluded to in Footnote 11. Their "*links*", in the terminology explained later in this section, are the so-called *lens spaces*. One may consult Weber's recent paper (Weber,

rational. This means that from a topological viewpoint they are 2-dimensional spheres.

Hirzebruch used the expression "*Sphärenbaum*", that is, "tree of spheres" for the configurations schematically represented in Fig. 21. He explained that this terminology had been introduced by Heinz Hopf in his 1951 paper (Hopf, 1951) for the configurations of 2-dimensional spheres which are created by successive blow-ups of points, starting from a point on a smooth complex algebraic surface. Hopf chose that name because those spheres intersect in the shape of a tree.[13]

Let us look again at the models of algebraic surfaces from Figs. 7, 8 and 12. In each case, one has a representation of only part of the surface, as the whole surface is unbounded. The chosen part is obtained by considering the intersection of the entire surface with a ball centered at the singular point under scrutiny. By this procedure, one obtains a portion of the surface possessing a boundary curve. When the ball's radius is small enough, one gets a curve whose qualitative shape (number of connected components, number of singular points on each component, etc.) does not depend on the radius. It is called nowadays the "*link*" of the singular point.

One may perform an analogous construction starting from a point of a *complex* algebraic surface. When the point is taken on a "*normal*" complex surface,[14] then its associated link is a 3-dimensional manifold. In the late 1950s, Abhyankar conjectured that it was impossible to obtain a counterexample to the Poincaré conjecture following this procedure. In other words, that it was impossible to find a point on a normal complex surface, whose link is simply connected and different from the 3-dimensional sphere. The aim of Mumford's paper (Mumford, 1961) was to prove this conjecture, as may be seen in Fig. 22, which reproduces the end of its introduction.

2018) for details about the history of the study of lens spaces and their relation with Hirzebruch-Jung singularities.

[13] In Section 4 of Hopf (1951), Hopf wrote: "*[...] die σ_i lassen sich den Eckpunkten eines Baumes—d. h. eines zusammenhängenden Streckencomplexes, der keinen geschlossenen Streckenzug enthält—so zuordnen, dass zwei σ_i dann und nur dann einen gemeinsamen Punkt haben, wenn die entsprechenden Eckpunkte eine Strecke begrenzen.*". This is to be compared with Hirzebruch's explanation given in section (11) of Hirzebruch (1953): "*K_n wird von H. Hopf als Sphärenbaum bezeichnet, da sich die Sphären σ_i^* eineindeutig den Eckpunkten eines Baumes zuordnen lassen: Zwei σ_i^* haben genau dann einen Schnittpunkt, wenn die zugeordneten Eckpunkte im Baum eine Strecke begrenzen.*"

[14] Normality is a technical condition which implies that all the singular points of the surface are isolated. One may find details about it in Laufer's book (Laufer, 1971) or in Mumford's book (Mumford, 1999).

From the standpoint of the theory of algebraic surfaces, the really interesting case is that of a singular point on a *normal* algebraic surface, and *m* arbitrary. M is then by no means generally S^3 and consequently its own topology reflects the singularity P! In this paper, we shall consider this case, first giving a partial construction of $\pi_1(M)$ in terms of a resolution of the singular point P; secondly we shall sketch the connexion between $H_1(M)$ and the algebraic nature of P. Finally and principally, we shall demonstrate the following theorem, conjectured by Abhyankar:

Theorem. — $\pi_1(M) = (e)$ if and only if P is a simple point of F (a locally normal surface); and F topologically a manifold at P implies $\pi_1(M) = (e)$.

Fig. 22 Mumford's motivation

are connected ([1]). In order not to be lost in a morass of confusion, we shall now restrict ourselves to computing only H_1 in general, and π_1 only if $\pi_1(\cup E_i) = (e)$. Note thats this last is equivalent to (*a*) E_i connected together as a tree (i.e. it never happens $E_1 \cap E_2 \neq \emptyset$, $E_2 \cap E_3 \neq \emptyset$, $\ldots$, $E_{k-1} \cap E_k \neq \emptyset$, $E_k \cap E_1 \neq \emptyset$ and $k > 2$ for some ordering of the E_i's), (*b*) all E_i are rational curves.

Fig. 23 Mumford's curves "connected together as a tree"

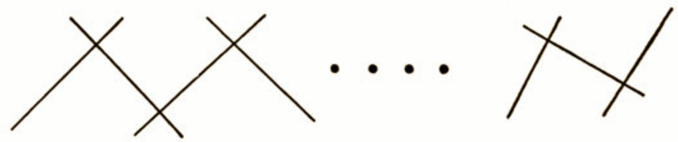
Mumford's proof started from a resolution of the given point, assuming—which is always possible—that its exceptional divisor has "*normal crossings*".[15] It proceeded along the following steps:[16]

1. Show that the link M of the given point p of a normal complex surface is determined by the exceptional divisor and by the self-intersection numbers of its composing curves.
2. Show that if M is simply connected, then the curves of the exceptional divisor are "*connected together as a tree*" and are all rational (see Fig. 23).
3. Under this hypothesis, write a presentation of the fundamental group $\pi_1(M)$ of M in terms of the configuration of curves of the exceptional divisor and of their self-intersection numbers.
4. Deduce from this presentation that if $\pi_1(M)$ is trivial, then one may contract algebraically one of the curves of the exceptional divisor to a point and obtain again a resolution whose exceptional divisor has normal crossings.
5. Iterating such contractions, show that the given point p is a smooth point of the starting surface, which implies that its link M is the 3-dimensional sphere.

In what concerns step (1), Mumford proved in fact that the link M is determined by the dual graph of the exceptional divisor, decorated by the genera and the

[15] This means that the exceptional divisor is composed of smooth curves intersecting pairwise transversally, and that three such curves do not have common points.

[16] One needs to know a certain amount of algebraic topology in order to understand this proof completely. The reader with a strong taste for visualization may learn the needed notions from the website (Saint Gervais, 2017).



Case 2. — It remains to consider the case where no E_i intersects more than two others. Then the E_i are arranged as follows:

Fig. 3

Fig. 24 Mumford's diagram

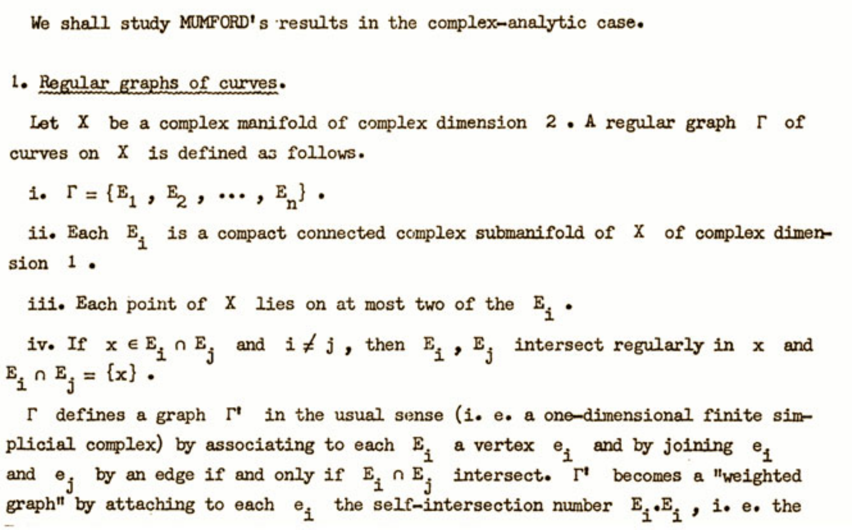
self-intersection numbers of the associated curves. This formulation is slightly anachronistic, because he still did not formally introduce this dual graph. He said only that the curves of the exceptional divisor were "*connected together as a tree*", which is similar to Du Val's terminology of his 1934 paper discussed in section "Du Val's Singularities, Coxeter's Diagrams and the Birth of Dual Graphs" (see again Fig. 15). Unlike Hirzebruch in his 1953 article, he did not even draw a dual graph, but only a schematic representation of the same type of configuration of curves as that of Hirzebruch's paper (Hirzebruch, 1953) (see Fig. 24). One may notice that the same drawing convention was to be followed by Michael Artin in his 1962 paper (Artin, 1962) (see the upper half of Fig. 4), before his switch to dual graphs in the 1966 paper (Artin, 1966).

It seems that the notion of dual graph of an arbitrary curve configuration, not necessarily formed of smooth rational curves, was formally defined for the first time in Hirzebruch's 1963 Bourbaki Seminar talk (Hirzebruch, 1963) discussing the previous results of Mumford. Hirzebruch associated to any "regular graph of curves" a "graph in the usual sense" (see Fig. 25), without using the terminology "dual graph", which seems to have appeared later in the 1960s. Then, he stated the result of step (3) formulated above as the fact that the dual graph determines an explicit presentation of the fundamental group of the link M—of course, under Mumford's hypothesis that the exceptional divisor has normal crossings, that all its components are rational curves and that the graph is a tree.

Less than 10 years later, appeared the first books explaining—among other things—algorithms for the computation of dual graphs of resolutions of singularities of normal surfaces: Hirzebruch, Neumann and Koh's book (Hirzebruch et al., 1971) and Laufer's book (Laufer, 1971). All those algorithms followed Hirzebruch's method of his 1953 paper from which Fig. 21 was extracted.

Before passing to the next section, let me quote an e-mail received on 9 January 2018, in which Mumford answered my questions about the evolution of the notion of dual graph:

> Perhaps the following is useful. In much of the twentieth century, math papers never had any figures. As a geometer, I always found this absurd and frustrating. In my "red book" intro to AG [Algebraic Geometry, Mumford (1999)], I drew suggestive pictures of various schemes, trying to break through this prejudice. On the other hand, I listened to many lectures by Oscar Zariski and, on rare occasions, we, his students, noticed him making a small drawing



We shall study MUMFORD's results in the complex-analytic case.

1. Regular graphs of curves.

Let X be a complex manifold of complex dimension 2. A regular graph Γ of curves on X is defined as follows.

i. $\Gamma = \{E_1, E_2, \dots, E_n\}$.

ii. Each E_i is a compact connected complex submanifold of X of complex dimension 1.

iii. Each point of X lies on at most two of the E_i.

iv. If $x \in E_i \cap E_j$ and $i \neq j$, then E_i, E_j intersect regularly in x and $E_i \cap E_j = \{x\}$.

Γ defines a graph Γ' in the usual sense (i. e. a one-dimensional finite simplicial complex) by associating to each E_i a vertex e_i and by joining e_i and e_j by an edge if and only if $E_i \cap E_j$ intersect. Γ' becomes a "weighted graph" by attaching to each e_i the self-intersection number $E_i . E_i$, i. e. the

Fig. 25 Hirzebruch's switch from "graphs of curves" to graphs "in the usual sense"

> on the corner of the blackboard. You see, the Italian school had always in mind actual pictures of the real points on varieties. Pictures of real plane curves and plaster casts of surfaces given by the real points were widespread. If you want to go for firsts, check out Isaac Newton's paper classifying plane cubics. So we were trained to "see" the resolution as a set of curves meeting in various ways. The old Italian theory of "infinitely near points" was, I think, always drawn that way. Of course, this worked out well for compactifying moduli space with stable curves. I'm not clear who first changed this to the dual graphs. Maybe it was Fritz [Hirzebruch].

Waldhausen's Graph Manifolds and Neumann's Calculus with Graphs

Mumford's theorem stating that the link of a complex normal surface singularity is determined by the dual graph of any of its resolutions whose exceptional divisor has normal crossings raised the question whether, conversely, it was possible to recover the dual graph from the structure of the link.

Formulated in this way, the problem cannot be solved, because resolutions are not unique. Indeed, given a resolution whose exceptional divisor has normal crossings, one can get another one by blowing up any point of the exceptional divisor. The new resolution has a different dual graph, with one additional vertex. Is there perhaps a minimal resolution, from which all other resolutions are obtained by sequences of blow ups of points? Such a resolution indeed exists,[17] and one may ask instead whether its dual graph is determined by the corresponding link.

This second question was answered affirmatively in the 1981 paper (Neumann, 1981) of Walter Neumann, building on a 1967 paper (Waldhausen, 1967) of Friedhelm Waldhausen. Let me describe successively the two papers, after a supplementary discussion of Mumford's article (Mumford, 1961).

Mumford looked at the link M of a singular point of a normal complex algebraic surface as the boundary of a suitable "tubular neighborhood" of the exceptional divisor of the chosen resolution. One of his crucial insights was that the assumption that this divisor has normal crossings[18] implies that the 3-dimensional manifold M may be described only in terms of real curves and surfaces, which are objects of smaller dimension. Namely, the link M may be obtained by suitably cutting and pasting continuous families of circles—called *circle bundles*—parametrized by the points of the curves of the exceptional divisor. Let me explain why.

In the simplest case where the exceptional divisor is a single smooth algebraic curve—topologically a *real surface*, because it is a *complex curve*—such a tubular neighborhood has a structure of disk bundle over the curve: one may fill it by discs transversal to the curve. Being its boundary, the link has therefore a structure of circle bundle over this surface.

In general, the exceptional divisor has several components. Then the boundary of one of its tubular neighborhoods has again a structure of circle bundle far from the intersection points of those components, but one has to make a careful analysis near such points. Mumford worked in special neighborhoods of them, which he called "*plumbing fixtures*". These allow to see in which way one passes from the circle bundle over a curve of the exceptional divisor to that over a second such curve, intersecting the first one at the chosen point. In terms of the dual graph, the description is very simple: to every edge of it one can assign a "plumbing fixture". In the link, which is identified with the boundary of the tubular neighborhood, it gives rise to a torus. By moving inside the link M and crossing this torus, one passes from the first circle fibration to the second one. In order to understand precisely in which way the transition is made, one has to look at the relative positions of the circles of both fibrations on the separating torus. These intersect transversally at exactly one point.

This phenomenon gave rise to the notion of "*plumbed* 3-*manifold*". It is a 3-dimensional manifold constructed from a decorated graph by performing a "*plumbing*" operation for each edge of the graph, similar to that described by Mumford in his paper. Each vertex comes equipped with two numbers, one representing the genus of a surface and the second its self-intersection number in an associated disk-bundle of dimension four. There is a subtlety related to orientations, which obliges one to decorate the edges with signs.

[17] The figures in sections "Introduction" and "What Is the Meaning of Such Graphs?" present in fact dual graphs of *minimal* resolutions of the corresponding surface singularities.

[18] Recall that this notion was explained in footnote 15.

One witnesses here a metamorphosis of the interpretation of the weighted dual graphs. If they started by representing the configurations of curves obtained as exceptional divisors of resolutions of singularities, they became blueprints for building certain 3-manifolds. It was Waldhausen who developed a subtle theory of those manifolds in Waldhausen (1967). He called them “*Graphenmannigfaltigkeiten*”—that is, “*graph-manifolds*” and not “plumbed manifolds”, in order to emphasize the idea that they are defined by graphs. In fact, he considered slightly more general graphs, whose edges are also decorated with pairs of numbers (see Fig. 26). This convention allowed the transitions from one circle fibration to another one across a torus to be performed by letting the fibers from both sides intersect in any way, not necessarily transversally at a single point. One of his main theorems states that any graph-manifold has a unique minimal graph-presentation, except for an explicit list of ambiguous manifolds.

In his 1981 paper (Neumann, 1981), Neumann turned this theorem into an algorithm, allowing to determine whether two weighted graphs in the original sense of the plumbing operations determined the same oriented 3-dimensional manifold. Roughly speaking, this algorithm consists in applying successively the rules of a “*plumbing calculus*”—some of them being represented in Fig. 27—in the direction which diminishes the number of vertices of the graph. Two graphs determine

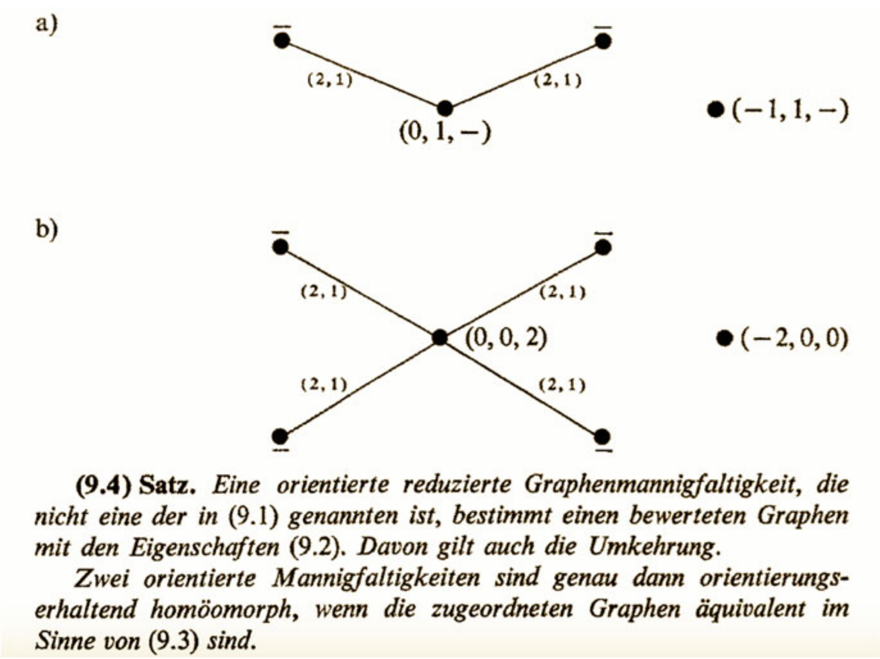


Fig. 26 Two of Waldhausen’s “graph manifolds”

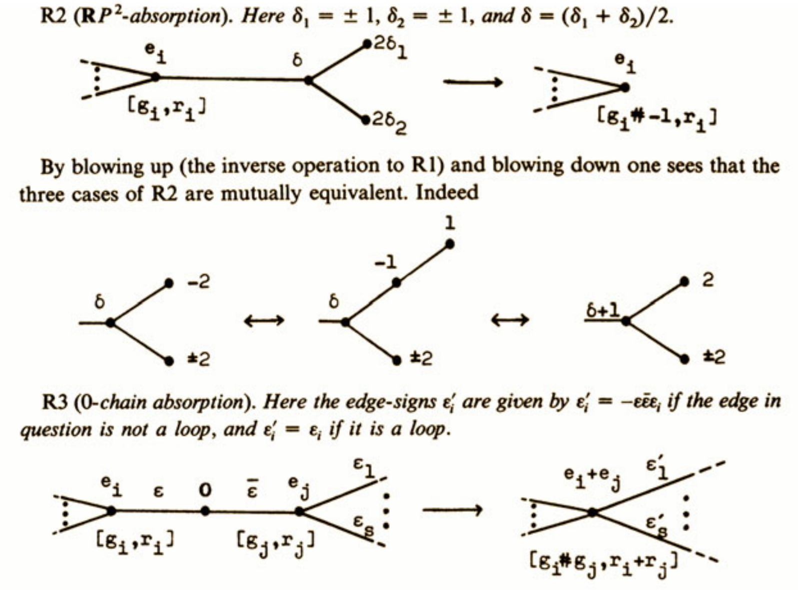


Fig. 27 Part of Neumann's "plumbing calculus"

the same 3-dimensional manifold if and only if the associated "minimal" graphs coincide—again, up to a little ambiguity related to the signs on the edges.

The algorithm allowed Neumann to prove important topological properties of normal surface singularities and of families of smooth complex curves degenerating to singular ones. For instance, he showed that the decorated dual graph of the minimal resolution with normal crossings is determined by the *oriented* link of the singularity. This unified the two viewpoints on the graphs associated to surface singularities discussed in this paper (as dual graphs of their resolutions, and as blueprints for building their links).

Conclusion

We could continue this presentation of the interaction between graphs and singularities in several directions:

- By examining other classification problems of singularity theory which led to lists of dual graphs, between Mumford's paper (Mumford, 1961) and Neumann's paper (Neumann, 1981) (for instance Brieskorn's paper (Brieskorn, 1968), Wagreich's paper (Wagreich, 1972) and Laufer's papers (Laufer, 1973, 1977)). We saw that it was such a classification problem which led Du Val to his consideration of "graphs of curves", and which led to Artin's lists of dual graphs shown in Figs. 4 and 5. Another such problem led to the more recent paper of Chung, Xu and Yau from which Fig. 3 is extracted. Note that in Laufer (1973), Laufer classified all "*taut*" normal surface singularities, that is, those which are determined up to complex analytic isomorphisms by the dual graphs of their minimal resolutions. He showed in particular that all rational double and triple points of Figs. 4 and 5 are taut. This result had already been proved by Brieskorn in Brieskorn (1968) for rational double points. As a consequence, this class of singularities coincides with Du Val's singularities which do not affect the condition of adjunction. This result is much stronger than the fact that their minimal resolutions have the same dual graphs.

- By examining the applications and developments of Neumann's "plumbing calculus". One could analyze its variant developed by Eisenbud and Neumann in Eisenbud and Neumann (1985) for the study of certain links (that is, disjoint unions of knots) in integral homology spheres which are graph-manifolds, its applications initiated by Neumann (1989) to the study of complex plane curves at infinity, or those initiated by Némethi and Szilárd (2012) and continued by Curmi (2020) to the study of boundaries of "*Milnor fibers*" of non-isolated surface singularities.
- By discussing generalizations of dual graphs to higher dimensions. In general, when one has a configuration of algebraic varieties, one may represent them by points, and fill any subset of the total set of such points by a simplex, whenever the corresponding varieties have a non-empty intersection. One gets in this way the so-called "*dual complex*" of the configuration of varieties. In the same way as there was a substantial lapse of time since the idea of dual graph emerged till it became an active object of study, an analogous phenomenon occurred with this more general notion. It seems to have appeared independently in the 1970s, in Danilov's paper (Danilov, 1975)—whose results were rediscovered with a completely different proof by Stepanov in his 2006 article (Stepanov, 2006)—in Kulikov's paper (Kulikov, 1977) and in Persson's book (Persson, 1977). Information about recent works on dual complexes may be found in Payne's paper (Payne, 2013), in Kollár's paper (Kollár, 2013) and in the paper (de Fernex et al., 2017) by de Fernex, Kollár and Xu. One may use Nicaise's paper (Nicaise, 2016) as an introduction to the relations between dual complexes and "*non-Archimedean analytifications in the sense of Berkovich*".
- By presenting the notion of "*fan*" of the divisor at infinity of a toroidal variety, introduced by Kempf, Knudson, Mumford and Saint-Donat in the 1973 book (Kempf et al., 1973). It is a complex of cones associated to special kinds of configurations of hypersurfaces in complex algebraic varieties. When the configuration has normal crossings, the projectivisation of the fan is in fact the dual complex of the configuration. Fans had been introduced before by Demazure for "*toric varieties*" in the 1970 paper (Demazure, 1970), and since then they were mainly used in "*toric geometry*". Following this direction, we could arrive at the notion of "*geometric tropicalization*", which expresses "*tropicalizations*" of subvarieties of algebraic tori in terms of the dual complexes of the divisors at infinity of convenient compactifications (see Hacking et al., 2009 and Maclagan

and Sturmfels, 2015, Theorem 6.5.15). Note that Berkovich's analytification (alluded to at the end of the previous item) and tropicalization are intimately related, as explained by Payne in Payne (2009, 2015). Note also that the paper Barroso et al. (2019) studies dual graphs of resolutions of normal surface singularities in the same spirit.

- By discussing how Waldhausen's theory of graph-manifolds led to Jaco-Shalen-Johannson's theory of canonical decompositions of arbitrary orientable and closed 3-manifolds into elementary pieces, by cutting them along spheres and tori (see Jaco and Shalen's book (Jaco and Shalen, 1979) and Johansson's book (Johannson, 1979)). This is turn gave rise to Thurston's geometrization conjecture of Thurston (1982), proved partially by Thurston, and which was finally completely settled by Perelman's work (Perelman, 2003). For details on Perelman's strategy, one may consult the monographs Bessières et al. (2010) and Morgan and Tian (2014).
- By speaking about the second, more recent, main source of graphs in singularity theory: the dual graphs of configurations of "*vanishing cycles*" in Milnor fibers of isolated hypersurface singularities. Such dual graphs, called sometimes "*Dynkin diagrams*", began to be described and drawn after 1970 for special classes of singularities in A'Campo (1975a,b), Gabrielov (1973, 1974) and Gusein-Zade (1974). One may consult Arnold's papers (Arnold, 1973, 1975), Gusein-Zade's survey (Gusein-Zade, 1977) and Brieskorn's papers (Brieskorn, 1981, 1983) for a description of the context leading to those researches on Dynkin diagrams and of their relations with other invariants of hypersurface singularities. Du Val's singularities possess configurations of vanishing cycles isomorphic to the dual graphs of their minimal resolutions indicated in Fig. 4. One may consult Brieskorn's paper (Brieskorn, 2000) for a description of the way he proved this theorem instigated by a question of Hirzebruch. In fact, this property characterizes Du Val's singularities (see Durfee's survey (Durfee, 1979) of many other characterizations of those singularities). In general, the relation between the two types of dual graphs is still mysterious. Note that Arnold described in Arnold (1975) a "*strange duality*" inside a set of 14 "*exceptional unimodular singularities*", relating the two types of dual graphs. This duality was explained in Pinkham (1977) on one side and Dolgachev and Nikulin (1977) on another side (see Dolgachev's Bourbaki seminar presentation (Dolgachev, 1983)). Later, Dolgachev related it in Dolgachev (1996) to the very recent phenomenon—at the time—of "*mirror symmetry*", but this seems to be only the tip of an iceberg.

I will not proceed in such directions, because this would be very difficult to do while remaining reasonably non-technical. I made nevertheless the previous list in order to show that dual graphs and their generalizations to higher dimensions are nowadays common tools in singularity theory, in algebraic geometry and in geometric topology. It is for this reason that I found interesting to examine their births and their early uses.

We saw that dual graphs of surface singularities were first used mainly verbally, in expressions like "tree of curves", "Sphärenbaum". Drawing them became important for stating results of various problems of classification. This made their verbal description first too cumbersome, then completely inadequate for the description of the wealth of morphologies under scrutiny. Then, their reinterpretation as blueprints for building graph-manifolds led to the development of a "plumbing calculus", which transformed them into objects of algebra. The necessity to develop an analogous "calculus" appears every time one gets many different encodings of the structure of an object, leading to the problem of deciding which encodings correspond to the same object (another instance of this phenomenon is Kirby's calculus of Kirby, 1978). In other situations—for instance, that of finite presentations of discrete groups—it is known that the problem is undecidable. But for plumbing graphs it is solvable, as shown by the works of Waldhausen and Neumann alluded to before.

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Part IV
Diagrams, Physical Forces and Path Integrals

Mathematical Aspects of Feynman Path Integrals, Divergences, Quantum Fields and Diagrams, and Some More General Reflections

Sergio Albeverio

Abstract Feynman path integrals are first presented for the case of non-relativistic quantum mechanics, both in physical and mathematical terms. Then the case of scalar relativistic and Euclidean quantum fields is discussed, with a particular consideration of the mathematical problems arising when discussing interactions as non-linear functionals of the fields. The methods of (constructive) perturbation theory and renormalization theory in relation to Feynman path integrals are briefly discussed, in particular mentioning the visual help provided by Feynman diagrams. The paper ends with mentioning some open problems and presenting some philosophical remarks and reflections on the description of natural phenomena, in particular those of fundamental physics, in mathematical terms.

Keywords Quantum Feynman diagrams · Path integrals · Quantum field · Divergence · Renormalization

Introduction

In work by R. Feynman (starting in 1942) (see, e.g., Feynman and Hibbs, 2010; Feynman, 1948, 2005) a new approach to quantum mechanics including quantum field theory was presented. To give a simple example, let us consider first the case of non-relativistic quantum mechanics for a single particle of mass $m > 0$ moving in a σ-dimensional Euclidean space $\mathbb{R}^\sigma$ under the influence of a (scalar) potential V. The solution of the time dependent Schrödinger equation at time t having as initial

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condition at time 0 a square integrable complex-valued function ψ_0 on $\mathbb{R}^d$ is given by

$$\psi(t,x) = \mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H}\psi_0(x), t \geq 0, x \in \mathbb{R}^\sigma,$$

where $\hbar$ is a positive constant (reduced Planck's constant) and H is the self-adjoint operator (Hamiltonian or energy operator), acting in the space $L^2(\mathbb{R}^\sigma)$ of complex-valued square integrable functions. $\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H}$ is the corresponding one parameter strongly continuous unitary group.

For V bounded and continuous, we have, e.g. on the domain of smooth quickly decaying functions $H = H_0 + V$, with $H_0 := -\frac{\hbar}{2m}\Delta$, Δ being the Laplacian in $\mathbb{R}^\sigma$. For $V \equiv 0$, we have, e.g. for ψ_0 smooth

$$\psi(t,x) = \int_{\mathbb{R}^\sigma} K_0(t, x-y)\psi_0(y)\mathrm{d}y, \tag{1}$$

with

$$K_0(t, x-y) := \frac{\mathrm{e}^{\mathrm{i}m\frac{|x-y|^2}{2t}}}{\left(2\pi\mathrm{i}\frac{\hbar}{m}t\right)^{\frac{\sigma}{2}}},$$

the fundamental solution to the Schrödinger equation with $V \equiv 0$ ("free evolution") (i.e. the kernel of $\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H_0}$).

Already in 1932 P. Dirac had discovered that, at least heuristically, $\mathrm{e}^{-\mathrm{i}\frac{t}{\hbar}H}\psi_0$ can be thought of as the limit for $N \to \infty$ of

$$\left(\mathrm{e}^{-\mathrm{i}\frac{\Delta t}{\hbar}H_0}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t V}\right)^N, \tag{2}$$

with $\Delta t := \frac{t}{N}$ (when H, H_0, V are replaced by symmetric finite-dimensional matrices this is so, as a consequence of a formula derived first by S. Lie in connection with his considerations on groups of geometric transformations).

Noticing that the action of $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t H_0}$ on nice functions can be expressed by (1), and combining this with

$$\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t V}\psi_0\right)(y) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t V(y)}\psi_0(y),$$

we obtain the above (2) for $N = 1$ as applied to ψ_0; iterating this we easily arrive at the formula

$$\left(\left(\mathrm{e}^{-\mathrm{i}\frac{\Delta t}{\hbar}H_0}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta t V}\right)^N\psi_0\right)(x) = \int_{R^{\sigma N}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_N(x,x_1,\ldots,x_N;t)}\psi_0(x_N)A_N\mathrm{d}x_1\ldots\mathrm{d}x_N,$$

with

$$A_N := \left(2\pi \mathrm{i}\frac{\hbar}{m}\Delta t\right)^{-\frac{N\sigma}{2}},$$

and

$$S_N(x, x_1, \ldots, x_N; t) := \sum_{k=1}^{N} \Delta t \left\{ \frac{m}{2} \left| \frac{x_k - x_{k-1}}{\Delta t} \right|^2 - V(x_k) \right\}.$$

The above integral over $\mathbb{R}^{\sigma N}$ looks heuristically as an approximation of the following "idealized (symbolic) infinite-dimensional integral":

$$\int_{\Phi} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\varphi)} \psi_0(\varphi(t)) \mathrm{d}\varphi, \tag{3}$$

with

$$S(\varphi) = \int_0^t \left[\frac{m}{2} |\dot{\varphi}(s)|^2 - V(\varphi(s)) \right] \mathrm{d}s. \tag{4}$$

$\mathrm{d}\varphi$ stands heuristically for the limit for $N \to \infty$ of $A_N \mathrm{d}x_1 \ldots \mathrm{d}x_N$ and the "integral" is over the variable φ which belongs to the space Φ of paths φ from $[0, t]$ into $\mathbb{R}^{\sigma}$ with $\varphi(0) = x$ (in classical mechanics, S is the "action function").

The above "integral" is the instance of a (heuristic) Feynman path integral. Let us note that if Φ is replaced by the space of polygonal maps $\varphi^N \in \Phi_N$ (with φ^N starting at time zero in x, being at time $k\Delta t$ in x_k, $k = 1, \ldots, N$, we have $S(\varphi) = S(\varphi^N)$).

It was the merit of Feynman to look at the formula as containing more than an approximation, but rather as a new way to formulate the dynamics in quantum mechanics. From here it was quite tempting to extend it to other situations, including quantum fields. In fact, this approach is nowadays probably the one which is most used in the physics of quantum phenomena, in particular those involving geometry, e.g. gauge theory, see, e.g. Albeverio et al. (2008), Mazzucchi (2009), Atiyah (1979), Roepstorff (1994), Kolokoltsov (2000), Albeverio (1997), Albeverio and Sengupta (1997), Albeverio et al. (2003).

Feynman stressed a number of points which I summarize here:

1. This approach stems mainly from a Lagrangian point of view, complementary to a Hamiltonian one (which incidentally *a posteriori* can be incorporated in a Feynman formalism, see, e.g., Albeverio et al., 2015; Albeverio and Mazzucchi, 2016; Albeverio et al., 2017; Daubechies et al., 1987; Feynman, 2005).
2. It permits to see the "classical world" emerging from the "quantum world" via a heuristic stationary phase method, as seen in the above example by using that $\hbar$ has a small value (in natural units) and replacing the integral

by its main contribution in $\hbar$ as given by the stationary (critical) points of the phase function $S(\varphi)$ (these critical points are known to correspond to orbits of the underlying classical system, for a Newton particle moving in the potential V)

Mathematical realizations of the Feynman path integrals in non-relativistic quantum mechanics and the above considerations have been worked out rigorously in many approaches. Here, we only recall briefly one of them that has been particularly used not only in non-relativistic quantum mechanics, but also in other areas, at least as a starting point, including quantum field theory, string theory, and quantum gravity. This is the "analytic continuation approach" which in the non-relativistic case amounts to replacing the Schrödinger equation by a corresponding heat equation, in which time t is replaced by purely imaginary time $\tau \geq 0$ (in the physical literature this is sometimes called "Wick rotation"). The analogue of Feynman's solution formula for such a heat equation with potential V is the Feynman-Kac formula solving the heat equation

$$\frac{\partial}{\partial \tau} u_\tau = \left(\frac{\Delta}{2} - V \right) u_\tau$$

on $\mathbb{R}^\sigma$, $\tau \geq 0$, by an integral with respect to a measure μ_W^x (conditional Wiener measure) on a space of paths

$$u_\tau(x) = \int_\Phi \mathrm{e}^{-\int_0^\tau V(x+\varphi(s))\mathrm{d}s} u_0(x + \varphi(\tau)) \mu_W^x(\mathrm{d}\varphi)$$

$\mu_W^x(\mathrm{d}\varphi)$ is the translate by the constant path $\varphi(s) = x$ for all $s \in [0, \tau]$ of Wiener measure $\mu_W^0(\mathrm{d}s)$, heuristically given by $\mathrm{e}^{-S_0(\varphi)}\mathrm{d}\varphi$, where $S_0(\varphi) := S(\varphi)_{V\equiv 0}$, $S(\varphi)$ corresponding to the one we used for the Schrödinger equation. $\mu_W^0(\mathrm{d}\varphi)$ can be realized as a probability measure on the space $C_{(0)}([0, \tau]; \mathbb{R}^\sigma)$ of continuous paths φ in $\mathbb{R}^\sigma$, in time $[0, \tau]$, vanishing for $\tau = 0$. Indeed, μ_W^0 is a Gaussian probability measure determined by its mean 0 and its covariance function

$$\int \varphi(s)\varphi(t)\mu_W^0(\mathrm{d}\varphi) = s \wedge t,$$

which is the fundamental solution at $s, t \in [0, \tau]$ of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ in $L^2[0, \tau]$ with boundary conditions given by $\dot{\varphi}(\tau) = 1$, $\varphi(0) = 0$.

The typical paths in Φ are only continuous (as reflected by the fact that $s \wedge t$ is only continuous, not differentiable on the diagonal of $[0, \tau] \times [0, \tau]$). Putting together the integral involving V and the action functional S_0 entering μ_W^0 we see the expression under the integral sign in the Feynman-Kac formula heuristically, for $x = 0$, as

$$\mathrm{e}^{-\int_0^\tau V(\varphi(s))\mathrm{d}s} \mu_W^0(\mathrm{d}\varphi) u_0(\varphi(\tau)) = \mathrm{e}^{-S_E(\varphi)} u_0(\varphi(\tau)) d\varphi$$

with

$$S_E(\varphi) := \frac{1}{2}\int_0^\tau \left[|\dot{\varphi}(s)|^2 + V(\varphi(s))\right] \mathrm{d}s \tag{5}$$

(an analogous formula holds for general initial value x).

If V and the initial conditions are analytic, by analytic continuation in τ one goes from solutions of the heat equation to solutions of the Schrödinger equation, and the Feynman path integral is realized in this way as analytic continuation of a Wiener integral. There is a large literature on this, starting from Cameron in the 40s, E. Nelson in the 60s, see, e.g. Albeverio (1997), Albeverio et al. (2017), Albeverio et al. (2015), Albeverio et al. (2017), Doss (1980), Nelson (1964), Thaler (2005).

Remark As shown by Cameron (1960), the "Feynman measure $e^{\frac{i}{\hbar}S(\varphi)}d\varphi$" is not σ-additive, but the integral itself has been shown, at least for a certain class of continuous potentials, to be a continuous linear functional on a nice space (which is enough for many considerations, and constitutes an infinite-dimensional extension of the mathematical theory of oscillatory integrals Fujiwara, 2017; Hörmander, 2003, see, e.g. Albeverio et al., 1982, 1996, 2020; Albeverio and Høegh-Krohn, 1977; Albeverio and Mazzucchi, 2005, 2006, 2009; Albeverio et al., 2017; Cartier and DeWitt-Morette, 2006; Elworthy and Truman, 1984; Johnson and Lapidus, 2000)

There exist several other approaches, some of them presenting also different physical applications, see, e.g., the white noise calculus approach Hida et al. (1993) and Grothaus and Riemann (2017), and for the non-standard analysis approach (Albeverio et al., 1986; Herzberg, 2013), see also e.g. the references in Albeverio et al. (2017), Albeverio et al. (2015).

One common class of potentials which can be covered by all these approaches are functions V that belong to the space $\mathcal{F}(\mathbb{R}^\sigma)$ (called Fresnel space) of complex-valued functions which can be written as Fourier transforms of complex measures of bounded total variation. By the analytic approach, some classes of polynomial or exponential functions can also be covered (Albeverio and Mazzucchi, 2016; Grothaus et al., 2012). Some additional results can are obtained for special V, particularly via the Lie-Kato-Trotter formula, see, e.g. Chatterjee (In preparation).

Remark A rigorous method of stationary phase for performing a study of the emergence of classical orbits from the path representation of solutions in quantum mechanics has been worked out in Albeverio and Høegh-Krohn (1977), Albeverio et al. (1982), Albeverio and Arede (1985), Albeverio et al. (1996)

Remark A main use in quantum physics of path integrals is for performing computations, see below in connection with our discussion of the case of quantum fields.

The Case of Quantum Fields

A source for the concept of classical fields was the mechanics of fluids, describing the evolution of the velocity $u(t,x)$ of the fluid at time t and at position x (to these belong the classical fields described by the Euler resp. Navier-Stokes equations). Also Maxwell's classical electrodynamics and field (which arose from work of Faraday and Maxwell) $A_{\mathrm{cl}}(t,\vec{x})$, $t \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^{\sigma}$ are well known (see, e.g. for the story of the discovery of this concept Albeverio, 1995; Jost, 1995).

To illustrate how quantization of classical fields works it is simpler to consider the case of classical fields of the scalar type (i.e. real-valued fields) satisfying a non-linear wave equation (called non-linear Klein-Gordon equation)

$$\Box\varphi_{\mathrm{cl}} = -V'(\varphi_{\mathrm{cl}}),$$

with $\Box := \frac{\partial^2}{\partial t^2} - \Delta$, where Δ is the Laplacian with respect to the space variable $\vec{x} \in \mathbb{R}^{\sigma}$. V is a real-valued non-linear function on the real line, e.g. of the form

$$V(y) = m^2 y^2 + \lambda y^4, m^2 > 0, \lambda \geq 0, y \in \mathbb{R},$$

m resp. λ are a mass resp. coupling constant.

Because of the presence of the d'Alembert (wave-)operator $\Box$ and the local form of the non-linear term the equation is an example of a relativistic invariant equation (in the sense of Einstein's (1905) special relativity). The corresponding classical action functional (which can be "guessed" from the analogy with the case of classical mechanics of a non-relativistic particle) is

$$S(\varphi) = \int \left\{ \frac{1}{2}\left[|\dot{\varphi}(s,\vec{x})|^2 - |\nabla\varphi(s,\vec{x})|^2 \right] - m^2|\varphi(s,\vec{x})|^2 - \lambda|\varphi(s,\vec{x})|^4 \right\} \mathrm{d}s\mathrm{d}\vec{x}$$

The space Φ of φ's is now a space of maps from space-time $\mathbb{R}^{\sigma+1}$ to $\mathbb{R}$, and, from a variational principle, one gets from S the Klein-Gordon equation as Euler's equation. Similarly as for non-relativistic mechanics, following Feynman one expects to perform the quantization building up a path integral with a heuristic measure of the form "$\mathrm{e}^{\mathrm{i}S(\varphi)}\mathrm{d}\varphi$" (taking units so that $\hbar = 1$).

In the case of fields, it turns out that the basic quantities are of the heuristic form

$$\int_{\Phi} \varphi(t_1,\vec{x}_1)\ldots\varphi(t_n,\vec{x}_n)\mathrm{e}^{\mathrm{i}S(\varphi)}\mathrm{d}\varphi, \tag{6}$$

expressing "correlations" $\langle\Phi(t_1,\vec{x}_1)\ldots\Phi(t_n,\vec{x}_n)\rangle$ between the quantum fields $\Phi(t_i,\vec{x}_i)$ as values (in the vacuum state) at time-space points $(t_i,\vec{x}_i)$, $t_i \in \mathbb{R}$, $\vec{x}_i \in \mathbb{R}^{\sigma}$, $i = 1,\ldots,n$, for any $n \in \mathbb{N}$ (in the usual Minkowski space-time, we have $\sigma = 3$).

It turned out, especially since the ’60s, that it is mathematically less difficult to look at the corresponding “Euclidean correlation functions” $\langle\varphi_E(\tau_1, \vec{x}_1)\ldots\varphi_E(\tau_n, \vec{x}_n)\rangle$ given by

$$\int_{\Phi} \varphi(\tau_1, \vec{x}_1)\ldots\varphi(\tau_n, \vec{x}_n)\mathrm{e}^{-S_E(\varphi))}\mathrm{d}\varphi, \tag{7}$$

with S_E defined as S but with the subtraction signs on the right side replaced by addition signs, and

$$\tau_j := \mathrm{i}t_j, \qquad j = 1, \ldots, n.$$

Note that S_E is positive.

A role analogous to Wiener’s measure is here played by Nelson’s free field (Euclidean) measure

$$\mathrm{e}^{-S_{E,0}(\varphi)}\mathrm{d}\varphi,$$

where $S_{E,0}$ is the value of S_E for $\lambda = 0$. This is rigorously defined as the mean zero Gaussian measure μ_0 with covariance given by $(-\Delta + m^2)^{-1}$, where now Δ stands for the Euclidean Laplacian on $\mathbb{R}^{\sigma+1}$. We have then for the correlation functions of Euclidean fields (for $\lambda = 0$):

$$\langle\varphi_E(\tau_1, \vec{x}_1)\varphi_E(\tau_2, \vec{x}_2)\rangle = \int \varphi(\tau_1, \vec{x}_1)\varphi(\tau_2, \vec{x}_2)\mu_0(\mathrm{d}\varphi) = (-\Delta + m^2)^{-1}\big((\tau_1, \vec{x}_1); (\tau_2, \vec{x}_2)\big)$$

Note that these expressions are invariant under the action of the Euclidean group on $\mathbb{R}^{\sigma+1}$ and the so obtained quantization is for Euclidean fields, recovering however the expressions of correlations for time ordered relativistic fields by heuristic analytic continuation in the time parameters.

Remark Since $(-\Delta + m^2)^{-1}$ is singular on the diagonal $(\tau_1, \vec{x}_1) = (\tau_2, \vec{x}_2)$ one can expect the integration space Φ in (7) to consist not in continuous functions, but rather in functions which are distribution-like in the space variables. This gives then difficulties in interpreting the non-linear term in $S_E(\varphi)$, which is an expression of the well known problem of mastering divergences in quantum field theory. We shall briefly discuss this problem in the next section, setting up also the connection with Feynman’s diagrammatic approach.

Divergences and Diagrams

Heuristically, one can hope to be able to compute an integral of the form (7) by expanding in powers of λ under the integral. Doing so and interchanging the integral with the sum in the expansion we obtain a sum (from $N = 0$ to ∞) of terms of the form

$$\frac{(-\lambda)^N}{N!} \int_{\Phi} \varphi(\tau_1, \vec{x}_1) \dots \varphi(\tau_n, \vec{x}_n) \left(\int_{\mathbb{R}^{\sigma+1}} |\varphi(\tau_0, \vec{x}_0)|^4 \mathrm{d}\tau_0 \mathrm{d}\vec{x}_0 \right)^N \mathrm{e}^{-S_{E,0}(\varphi)} \mathrm{d}\varphi.$$

Again, exchanging heuristically the integral over Φ with the N ones over $\mathbb{R}^{\sigma+1}$ and noticing that the latter are equal to

$$\prod_{j=1}^{N} \int_{\mathbb{R}^{\sigma+1}} \left| \varphi\left(\tau_0^{(j)}, \vec{x}_0^{(j)} \right) \right|^4 \mathrm{d}\tau_0^{(j)} \mathrm{d}\vec{x}_0^{(j)},$$

we get to look at

$$\frac{(-\lambda)^N}{N!} \int_{\mathbb{R}^{\sigma+1}} \left(\int_{\Phi} \varphi(\tau_1, \vec{x}_1) \dots \varphi(\tau_n, \vec{x}_n) \prod_{j=1}^{N} \left| \varphi\left(\tau_0^{(j)}, \vec{x}_0^{(j)} \right) \right|^4 \mathrm{e}^{-S_{E,0}(\varphi)} \mathrm{d}\varphi \right) \prod_{j=1}^{N} \mathrm{d}\tau_0^{(j)} \mathrm{d}\vec{x}_0^{(j)}$$

An analogy with a finite dimensional situation helps to get further. $\mathrm{e}^{-S_{E,0}(\varphi)}\mathrm{d}\varphi$ is then replaced by a mean zero Gaussian measure $\mu_0(\mathrm{d}y)$ on a Euclidean space $\mathbb{R}^M$ of the form

$$\left(\det\left(2\pi A^{-\frac{1}{2}} \right) \right)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}(y, Ay)} \mathrm{d}y.$$

μ_0 is determined by its covariance $C := A^{-1}$ such that

$$\int_{\mathbb{R}^M} (e_i, y)(e_j, y) \mu_0(\mathrm{d}y) = C_{i,j}, \qquad i, j = 1, \dots, M,$$

where $\{e_i\}$ is an orthonormal base in $\mathbb{R}^M$, and (,) denotes here the scalar product in $\mathbb{R}^M$.

Simple combinatorics show that the higher moments of μ_0,

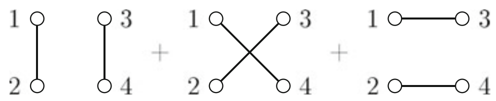
$$\int_{\mathbb{R}^M} (e_{i_1}, y) \dots (e_{i_{2k}}, y) \mu_0(\mathrm{d}y),$$

can be expressed as

$$\sum_{\pi} C_{i_1, j_1} \dots C_{i_k, j_k},$$

where the sum is over all partitions of $\{1, \dots, 2k\}$ into different pairs in $\{1, \dots, 2k\}$, e.g., for $k = 2$ we have $C_{1,2}C_{3,4} + C_{1,4}C_{2,3} + C_{1,3}C_{2,4}$.

A diagrammatic representation is



Similarly we can have a diagrammatic expression of the contributions for any k. See, e.g., references in Witten (1999d), Witten (1999c), Witten (1999b), Witten (1999a) and Ebrahimi-Fard et al. (1984).

If we replace μ_0 similarly as in the Euclidean field formulation—where we had

$$\mathrm{e}^{-S_E(\varphi)}\mathrm{d}\varphi = \mathrm{e}^{-\lambda\int_{\mathbb{R}^{\sigma+1}}|\varphi(s,\vec{x})|^4\mathrm{d}s\mathrm{d}\vec{x}}\mu_0(\mathrm{d}\varphi),$$

by a measure of the form

$$\mathrm{e}^{-\lambda\sum_{j=1}^{M}(e_j,y)^4}\mu_0(\mathrm{d}y)$$

and try to compute the corresponding moments

$$\int_{\mathbb{R}^M}(e_{i_1},y)\dots(e_{i_{2k}},y)\mathrm{e}^{-\lambda\sum_{j=1}^{M}(e_j,y)^4}\mu_0(\mathrm{d}y),$$

again exchanging expansion and integration, we get heuristically to consider a sum over $j=1,\dots,M$ of terms of the form

$$\frac{(-\lambda)^N}{N!}\int_{\mathbb{R}^M}(e_{i_1},y)\dots(e_{i_{2k}},y)(e_{j_1},y)^4\dots(e_{j_M},y)^4\mu_0(\mathrm{d}y).$$

Applying the diagrammatic technique we have to distinguish diagonal terms in the C's. Such terms of the form $C_{i,j}$, will diagrammatically be indicated by a loop with start point j. Remembering that in the continuum Euclidean case $(-\Delta+m^2)^{-1}$ corresponds to C, and the Green function

$$G(y,y') := (-\Delta+m^2)^{-1}(y,y'),$$

$y, y' \in \mathbb{R}^{\sigma+1}$, is actually singular for $y=y'$ we have to take special care of the terms involving loops.

Feynman diagrams in quantum field theory permit to visualize all terms, especially those which are divergent; this is also useful to sort out relative combinatorial problems, see, e.g. Ebrahimi-Fard et al. (1984), Baez and Dolan (2001).

The method of renormalization theory as developed by physicists in various versions can be summarized in the formalism of path integrals and, at least for "super-renormalizable models" (which for our scalar fields amount to considering only $\sigma \leq 2$), amounts in replacing the terms containing $\int_{\mathbb{R}^{\sigma+1}} V(\varphi(x))\mathrm{d}x$ in the action functional by a doubly regularized expression

$$\int_{\mathbb{R}^{\sigma+1}} V_\varepsilon(\varphi_\varepsilon(x))\chi_\Lambda(x)\mathrm{d}x,$$

where heuristically $\varphi_\varepsilon(x) \to \varphi(x)$ as $\varepsilon \downarrow 0$ (removal of the "ultraviolet cut-off" ε) and with χ_Λ (e.g. smooth) of compact support such that $\chi_\Lambda \to 1$ as the bounded region Λ expands to the whole space $\mathbb{R}^{\sigma+1}$ (removal of the "infrared cut-off"). V_ε is obtained from V by subtracting suitable "counterterms", e.g. if $V(y) = y^4$, and $\sigma = 1$,

$$V_\varepsilon(\varphi_\varepsilon) =: \varphi_\varepsilon^4 :,$$

with : denoting Wick ordering with respect to $\mu_{E,0}$ and

$$: \varphi_\varepsilon^4 : (x) := \varphi_\varepsilon^4(x) - 6G_\varepsilon(x,x)\varphi_\varepsilon^2 + \frac{3}{2}(G_\varepsilon(x,x))^2,$$

with

$$G_\varepsilon(x,x) := \int \varphi_\varepsilon(x)\varphi_\varepsilon(x)\mu_{E,0}(\mathrm{d}\varphi).$$

Calling $\mu_{\Lambda,\varepsilon}$ the so modified Euclidean measure μ_E, it has been shown for $\sigma = 1$ that $\lim_{\Lambda\uparrow\mathbb{R}^d} \lim_{\varepsilon\downarrow 0} \mu_{\Lambda,\varepsilon}$ exists (in the sense of weak convergence of measures, e.g. on $\mathcal{S}'(\mathbb{R}^{\sigma+1})$), see, e.g., Simon (1974), Glimm and Jaffe (1987), Da Prato and Tubaro (2018), Da Prato and Debussche (2003). For $\sigma = 2$, similar results were obtained by replacing : φ_ε^4 : by suitably modifying coefficients involving the second and zeroth power of φ_ε.

These constructions, however, cease to work for the space dimension $\sigma = 3$ of the $\sigma + 1$-dimensional Minkowski space-time.

Remark In recent years, a theory of singular stochastic partial differential equations with space-time Gaussian white noise has been developed that provided also alternative constructions of the measures (for $\sigma = 1, 2$) which are relevant for quantum field theory (Albeverio and Kusuoka, 2020; Gubinelli et al., 2015, 2018; Gubinelli and Hofmanov, 2021; Hairer, 2014). However, the construction for the case $\sigma = 3$ still constitutes an open problem. Other examples of (regularized) quantum fields and related more geometrical objects (quantum electrodynamics, quantum gravity in various forms such as loop quantum gravity, string theory, discrete networks, ...) have been discussed, but also there the most interesting critical dimension (corresponding to $\sigma = 3$) has not yet been mastered, see, e.g., Albeverio et al. (1997), Boi (2011), Rovelli (2004), Bahns et al. (2015), Segal (1976).

Some Conclusions, Philosophical Remarks and Reflections

The suggestive power of symbols, ideograms, diagrams for scientific research, in particular in the area of quantum fields and other theories of contemporary physics, is great.

It is the personal view of the author that in these areas only somewhat abstract structural insights have been achieved, but the question of existence (in the mathematical sense) of models is not solved. That means the objects under discussion have not yet a mathematical existence on their own, except for an idealized existence as approximations of a possible non-existing intended entity. The very essence of physics is presently under discussion, it suffices to mention discussions about experimental verifiability of phenomena involving very high energies, like in string theory, see, e.g., Witten (1999a). Possibly this will require a necessary rethinking of the basis of theoretical and mathematical physics, cf. Jaffe and Quinn (1993), Jaffe and Quinn (1994), Atiyah et al. (1994).

We are reminded to the origins of Husserlian phenomenology in the studies of the crisis in foundations (in mathematics and natural sciences, at the turn of the XIX. century), see, e.g. Boi et al. (2007) and Mehrtens (1990).

- The intrinsic difficulties of relativistic quantum field theory are undoubtedly connected with the continuum structure of the Minkowski space-time and the intended quantum character of the theory. Replacement of a continuum space-time by discrete models of it assures mathematical existence of all quantities of interest, but the theoretical justification for such a replacement is problematic. Let us note that some attempts of developing space-time models with co-existing discrete and continuum components have also been undertaken (Dragovich, 2004; Schmidt, 2008).
- The very nature of mathematics and its role in describing "extra-mathematical objects" (of nature or society) has been questioned at different levels, see, e.g. Albeverio et al. (2006), Linnebo (2017), Lolli (2002), Wilder (1981), Jacquette (2002), Heintz (2000), Triplett (1986), Albeverio and Tetens (2004).
- The problems of interpretation of the quantum world raise, as is well known, many epistemological problems, see, e.g. Blanchard et al. (2016), Dürr and Teufel (2009), Dürr et al. (2013), Nelson (1988).

  In recent years, its possible relation with the philosophical problems of the mind-body type has been strongly advocated, see Penrose (2005) and criticized (see references in Penrose, 2005). This is also connected with the more general problem of limits of (scientific) knowledge (Albeverio, 2018; Albeverio and Tetens, 2004).
- For a fascinating discussion of ontological aspects of quantum field theory (in its classical, quantum and stochastic aspects), see Boi (2011).
- The diagrams in present day physics, as shortly hinted to above, have a rather instrumental function. But there are attempts to attribute to them a more fundamental role, according to a position where a "diagram language" might be able to better capture aspects of the outer world as investigated by scientists than

the "linear language" prevalently still used in mathematics (and, e.g., linguistics). Semiotics has underlined possibilities of languages beyond the traditional linear ones. Linguistics has justly stressed the difference between syntactical and semantical aspects, and has also influenced newer approaches to foundational issues in mathematics (Nelson, 2006), with possible implications also for physics (Nelson, 1986).

Quantum gravity seems to provide an arena for discussions of this type, since both its physical and mathematical foundations are still largely missing.

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Some Remarks on Penrose Diagrams

Jean-Jacques Szczeciniarz

Abstract We give an explanation of what a conformal diagram is by expounding the theoretical and geometrical elements which constitute it: complex geometry in several variables, projective geometry, conformal geometry. We specify (partially) the cosmological and philosophical significance of the synthesis realized by Roger Penrose, a synthesis deposited in his diagrams.

Keywords Conformal · Complex · Projective · Geometry · Penrose diagrams

Introduction

The notion of a diagram could be presented as an expression of any solution of a classical—even if derived—philosophical question. Any abstract form, idea or concept, ontologically considered as intelligible being, is related to sensible representation. Thought must be spatialized, Hegel considers that even writing is spatialization of thought.To become a concept, as a tool of knowledge, abstraction must be realized in this sensitive representation. This *presentatio in concreto* is always subtle and complicated.

All geometrical figures are sensible realizations of abstract ideas. Plato wanted to distinguish the triangle in itself from the sensitive triangle, a copy of the first one. But it should be noticed that any figure is endowed with an internal "abstraction ability". There is also an ascent of this force of abstraction from the sensible to the intelligible. I must explain myself about the use of the expression "abstraction of sensible or in the sensible". I mean by this that the geometrical picture possesses an abstract support. And this abstract support as such is not visible but it organizes the picture in space.The picture is visible, its support is invisible.

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So that the figure is worked in its form for itself. What does this substance of the figure, object of the geometer's work, hold?

The strength of the figure comes from the fact that "it is structured at its level", it produces a new form of intelligibility. It goes back to the intelligible that it contains and the mixed product develops the reflection. One must deal with a horizontal Platonism. The imagination or power of production of images gives its own rules to the concept, because these rules are already conceptual as sketches.

But a diagram is not only a geometric figure or a set of figures ; it is also a symbolic form. The symbol is essentially characterized by the fact that the representation bears an analogy to what is represented and what represents. This analogy is based on a schema/concept that indicates a property. Moreover there is a discrepancy between the represented and the representative. Because the representative possesses a conceptual meaning. In our explanation, the representative will depend upon a geometry, namely a conformal geometry. As you will see, the objective of such a diagram will be to represent infinity and to deal with the concept of infinity in order to establish cosmological thoughts.

In this introduction I would like to localize a kind of abstract topography in order to explain what a diagram consists of. The diagram is a "conceptual image" that unfolds horizontally, capturing the strength of the image and of the concept that structures it.

Reflexions on the Different Notions of Dimensions

In geometry we are currently working in dimensions greater than 4. The question is how to represent a geometric figure in these dimensions.This is where the concept of diagram comes into play, as a reworking of the notion of geometrical figure, provided with an additional abstraction consubstantial with the figure represented. I will briefly explain how we represent complex n-dimensional figures. From there I will give a preliminary definition of "diagram".

I will now present what domains in the complex geometry in several variables consists of.

We are in $\mathbf{C}^n$ which means complex space with n dimensions. Recall that $\mathbb{C}$ is a complex plane that corresponds to real dimension 2 thus $\mathbf{R}^2$, $\mathbb{C} \simeq \mathbf{R}^2$. $\mathbb{C}$ has one complex dimension. If we want to work in this space we need to find some means of geometrically representing any object of this space. We dispose of domains of $\mathbf{C}^n$.

In complex geometry a domain is an open set (it contains each point with its neighborhoods) and connected which means that it is one piece.

A ball with radius r and center $a \in \mathbf{C}^n$ is defined as the set of points

$$B(a, r) := \{z \in \mathbf{C}^n; |z - a| < r\}$$

It is the usually Euclidean ball whose frontier B is the sphere with $2n - 1$ real dimensions

$$S^{2n-1} := \{z \in \mathbf{C}^n ; |z - a| = r\}.$$

There is a discrepancy between the represented and the representative.

These objects are sketched, since what we represent is a ball with two dimensions in 3-dimensional space $\mathbf{R}^3$.It should be noted that we draw in space -in the 3-dimensional space $\mathbf{R}^3$, a ball, (the sphere S^2) which comes out of the plane, its points have three coordinates. We are limited by our 3-dimensional perception. We sketch the other dimensions (in order to represent the object in $\mathbb{C}^2$) rather as a projection in dimension 3 from higher dimensions. It is difficult to yield a representation of this projection. We carry out a construction that allows us to capture the frontier or the edge of any domain of $\mathbb{C}^2$.

A polydisk or polycylinder with radius r and center $a \in \mathbf{C}^n$ is the set of points

$$U(a, r) := \{z \in \mathbf{C}^n : ||z - a|| = r\}.$$

It is a ball with a as center and the polycircular metric ρ. This metric is given by the distance

$$||z - w|| = \max_{\nu} |z_\nu - w_\nu|.$$

The frontier ∂U of any polydisk is the set of all points of which at least one coordinate belongs to the frontier of the νth disk that constitutes U and the others $z_\iota, \iota \neq \nu$ vary arbitrarily in the closed disc. This frontier decomposes naturally in n sets

$$\Gamma^\nu = \{z : |z_\nu - a_\nu| = r_\nu, |z_\mu - a_\mu \leq r_\mu, \mu| \neq \nu\}.$$

each of them with $2n - 1$ dimensions, (since the $2n$ coordinates of one point z are related to each other by only one real relation $|z_\nu - a_\nu| = r_\nu$). Then the frontier of the polydisk $\partial U = \cup_{\nu=1}^n \Gamma^\nu$ has $2n - 1$ dimensions. The sets Γ^ν cut each other according to a set

$$\{z; |z_\nu - a_\nu| = r_\nu, \nu = 1, \cdots, n\}.$$

This set, product of n circles, is called the *skeleton of the polydisk*.

We use a diagram to represent (as you see, the meaning of this representing process is complicated), a geometric figure, and here one represents a domain of space $\mathbb{C}^2$, *-of real dimension greater than 3, thus unrepresentable in dimension 2 or 3*. The dimension 2 is that of the sheet of paper, and dimension 3 is the one that can be reproduced on the sheet of paper by simulation of a representation in space (in the case of a sphere its part visible and covering the other part that is made invisible, i.e. the back part of the sphere).

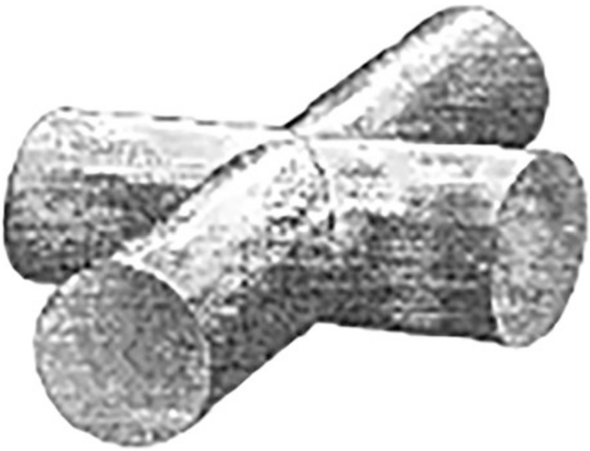
The situation is different in dimension 4. The diagram has one visible and one imaginary side, and allows us to conceptualize the non-visible side. The first side is in dimension 3 but this one is in the visible side, incomplete and points towards the higher dimension. I leave scattered pieces which I know to gather in higher dimension. The diagram ensures the presence of larger dimensions. By presence I mean the fact that it makes it conceivable that in the inaccessible dimension the figure bears a resemblance to the one I see. But to conceive a resemblance is not to perceive the objects that are similar.

Let us describe in detail the bidisk with radius 1 and with center a at the origin.

$$U(a, r) := \{z \in \mathbf{C}^2 : |z_1| < 1, |z_2| < 1\}.$$

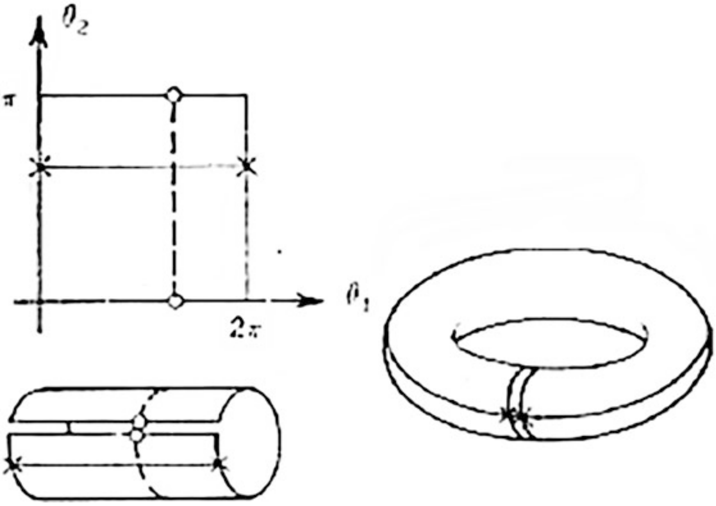
It is a body with four dimensions which is the intersection of two cylinders.

$x_1^2 + x_3^2 < 1$ and $x_2^2 + x_4^2 < 1$.



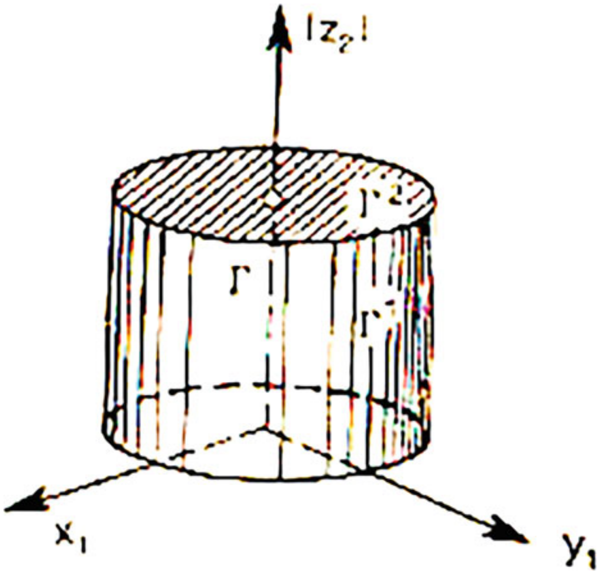
Its frontier is a body with three dimensions $\partial U = \Gamma^1 \cup \Gamma^2$ with $\Gamma^1 = \{|z_1| = 1$ $|z_2| \leq 1\}$ is a body with three dimensions which is decomposed into a family with one parameter of disks $\Gamma^1 = \cup_{\theta=0}^{\theta=2\pi}\{z_1 = e^{i\theta}, |z_2| \leq 1\}$. The skeleton of the bidisk $\Gamma = \Gamma^1 \cap \Gamma^2$ has two dimensions.

It is a torus $\Gamma = \{|z_1|, |z_2|\}$. Indeed, $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}$, is a homeomorphic transformation of the square $\{0 \leq \theta_1 \leq 2\pi\}$, $\{0 \leq \theta_2 \leq 2\pi\}$ on Γ where the opposite sides coincide because $(e^{i\theta_\nu + 2\pi} = e^{i\theta_\nu})$ and this yields a torus.



This figure shows a representative of each family. This torus is the intersection of two cylinders with three dimensions $\{x_1^2 + x_2^2 = 1\}$ and , $\{x_3^2 + x_4^2 = 1\}$ which is contained on the 3-dimensional sphere $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2\}$ of the space $\mathbf{R}^4$.

One can represent geometrically a bidisk in the following way. On the sphere with three dimensions of $\mathbf{C}^2$ $\{|z| = \sqrt{2}\}$, take the torus $\Gamma = \{|z_1| = 1, |z_2| = 1\}$. Spread out on it the 3-dimensional bodies $\Gamma^1 = \{|z_1| = 1, |z_2| \leq 1\}$ and $\Gamma^2 = \{|z_2| = 1, |z_1| \leq 1\}$ lying on the spherical layer $\{1 \leq |z| \leq \sqrt{2}\}$. Their union $\Gamma^1 \cup \Gamma^2$ will be the frontier of a bidisk.



Note
The diagram comes from : B. Chabat Introduction to Complex Analysis ed. Mir 1985

We can distinguish on this picture the pieces with 3 dimensions of the frontiers of Γ^1 and Γ^2 and the skeleton Γ of the bidisk.

Comment
It is easy to distinguish two features of such a diagram. First there is the simple, usual diagram : A representation of a sphere, a circle, a torus. In order to conceive and to perceive the torus I have to represent for myself two identifications that are effective. The perception corresponds to a real object, which can be not only constructed, but also mentally given and exactly drawn. Secondly, as for the bidisk, it is not the case that our perception corresponds to a real object. One only has pieces of the figure we are trying to represent. And each piece in our case is 3 dimensional (3-dimensional sphere, 2-dimensional torus,) but the whole body is 4-dimensional.

A diagram in this second meaning is a *device* where by the result, as a synthesis, is obtained. One only obtains here the 3-dimensional frontier of a bidisk. A diagram is such a construction that makes visible the frontier of a body, which is invisible.

I should complete the concept of a diagram after analyzing this kind of diagram. I call it a device for constructing a relation between the 2 or 3 dimensional visible objects and n-dimensional invisible objects,

Let us try to describe this ideality. This is a new type of abstraction. It is necessary to continue the geometry at a second level: the diagram with represented figures points towards figures of the same type that are not representable. It is, however, the representation as an image that surpasses itself. One might think that it is the engine of any diagrammatic thought. And this is the condition for continuing to think.

The Notion of a Function

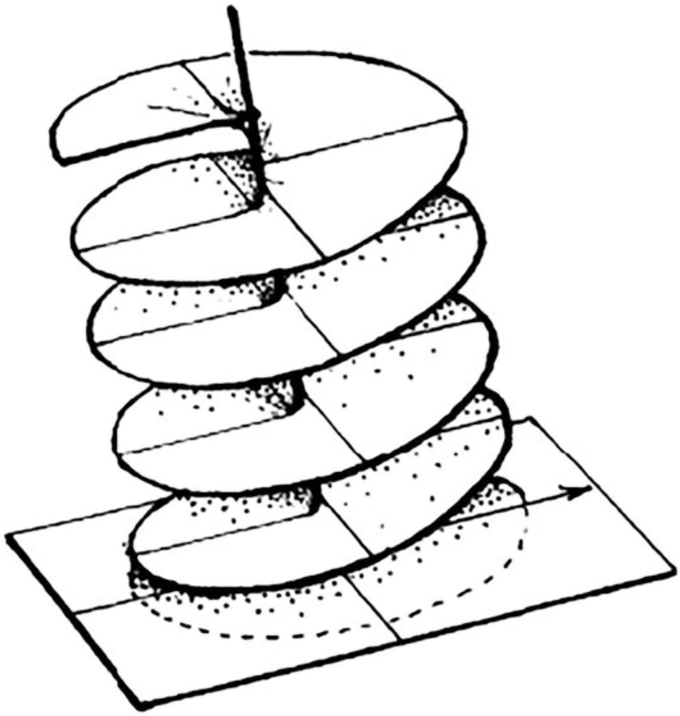
This notion has been transformed by Bernhard Riemann, who renewed the face of the mathematical body on our planet. When trying to define a complex variable function, we must look for a domain of definition that is not merely a subdomain of the complex plane.[1] Indeed, Riemann's idea was to think of such functions as being defined on a domain which is not simply a domain of the complex plane, but a many-sheeted region. In the case of $log z$ we can picture this as a kind of spiral ramp flattened vertically to the complex plane. It would be defined on a region with several layers. in the case of $log z$ it is a spiral ramp projected vertically on the

[1] Penrose Roger, *The Road to Reality* , p. 164.

complex plane. On this multi-leafed version of the complex plane the logarithmic function is univalued because, at each complete turn of the origin which leads to the addition of 2π we find ourselves on another sheet of sound.[2]

Mathematicians have been at odds as to how to treat these so called "many-valued" functions that would be cut in some arbitrary way by a line out from the origin to the infinity. To my way of thinking this a brutal mutilation of a sublime mathematical structure.[3]

Penrose's analysis is more conceptual, and offers to us another deep kind of diagram. It transforms the notion of a function and at the same time furnishes another meaning of "domain". As we have seen, with analytic continuation, a holomorphic function "has a mind of its own," and decides itself what its domain should be.[4] While we may regard the function's domain to be represented by the Riemann surface associated to it, the domain is the explicit form of the function itself that tells us which Riemann surface the domain actually is.



[2] The figures are borrowed, with his kind agreement to the two books of Professor Roger Penrose, *The Road to Reality A Complete Guide to the Laws of the Universe*, Jonathan Cape 2004, *Cycles of Time, An Extraordinary New View of the Universe*, Vintage Books, in French Translation Odile Jacob resp. 2007, 2013.

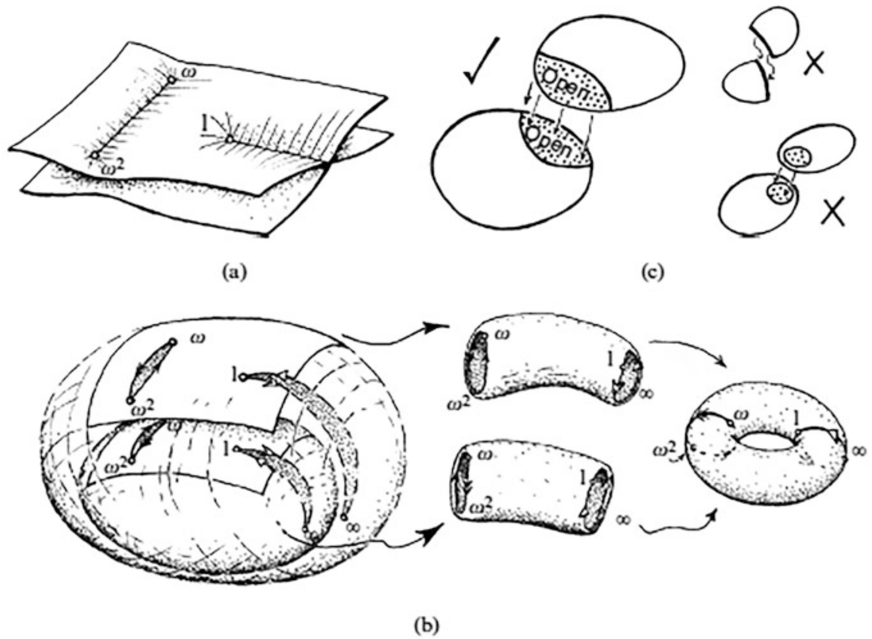
[3] Penrose, *ibid. p. 165*.

[4] *ibid.*

This notion of a function gives the diagram a role in the construction of mathematical knowledge. Since the logarithm function is undefined, Riemann shows us that we must pass over a stacked leaflet at the top of each course of a round. The function itself becomes an agent in itself of our knowledge. The function z^a is perhaps marginally more interesting than $logz$ with regard to its Riemann surface (with a rational). The expression of the lowest term is $a = m/n$. And in this case the spiraling sheets join back together again after n turns. The origin $z = 0$ in all these cases is called a *branch point*. The spatially drawn figure explains the nature of the function that prescribed the drawing. Riemann explains why we must leave the plane.

It is a new exploration of the represented space, and as such it is a diagram.It is a symbolism that uses the form of spatiality to express a geometric property. Unlike Kantian schematism but also in its continuity with it, the realization *in concreto* is done in the spatial construction. Its meaning is spatial.

Let us consider the function $(1 - z)^{1/2}$. This function has three branch points, at $z = 1$, $z = \omega$, $z = \omega^2$ where $\omega = e^{i\pi/3}$. Here as we circle in one complete turn around each individual branch point, staying in its immediate neighborhood ('immediate' means near the point), we find that the function changes sign, after another turn, the function goes back to its original value. This means that the branch points have order 2.



(a) (c)

(b)

As Penrose recalled, this surface actually has the topology of a torus, which is topologically the surface of a bagel with four tiny holes in it corresponding to the branch points themselves. To see that the Riemann surface is topologically a torus, imagine the planes of (a) as two Riemann spheres with slits cut from ω to ω^2 and from 1 to ∞, identified along matching arrows.These are topological cylinders glued correspondingly, giving a torus. This cutting work, according to the points of branching is a work in the representation of thought which is prescribed by the properties of the surface. The figure has become a diagram and thus propagates an analogy by constructing the space of its support.

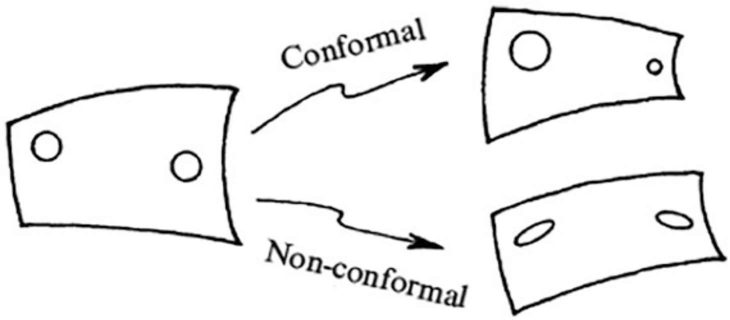
In this way the concept absorbs its own conditions of construction, expressing them from its own point of view. The Riemann surface concept is an extension of this theoretical movement. And it is always in this line and this movement that we must understand the concept of a manifold.

A Second Kind of Diagram

In assembling the various parts of a manifold care must be taken to preserve its local structure between each part and the next. Most often we manipulate real manifolds, and their different parts are pieces of Euclidean space (of fixed dimension) where the various open regions meet. The structure to preserve in the union of one part to the next is continuity or differentiability.

In the case of complex surfaces it is the complex differentiability that must be preserved. This is a question that finds its solution thanks to concepts with a remarkable power. This essential concept is that of conformal geometry.

Roughly speaking, in conformal geometry, we are interested in shape but not in size, as we are referring to shape on the infinitesimal scale. In a conformal map from one (open) region of the plane to another, shapes of finite size are generally distorted, but infinitesimal shapes are preserved. We can think of this applying to small (infinitesimal) circles drawn on the plane. In a conformal map, these little circles can be expanded or contracted, but they are not distorted into little ellipses. Infinitesimal shapes would be completely unaltered. There is a slightly different characterization of conformal mapping: *angles* between curves are unaltered by conformal transformation.



What is smoothness for a complex function? Let us consider a complex function $w = f(z)$ providing a mapping of a certain region of z's-complex plane into w's-complex plane. We would apply the idea of differentiability in the real case to the map from the z-plane to the w-plane. To examine the immediate neighborhood of the point, we imagine (as Penrose does) magnifying the immediate neighborhood of z by a huge factor and the corresponding neighborhood of w by the same huge factor. In the limit, the map so expanded will simply be a linear transformation of the plane, but if it is to be holomorphic, this must be, in the general case, a transformation from z's neighborhood to w's neighborhood that combines a rotation with a uniform expansion (or contraction). That is to say that the small shapes or angles are preserved. They are preserved without reflection; the map is indeed conformal and non-reflective. The small shapes preserved by the transformation are not '"turned over"; rather, the orientation is preserved.

There exists indeed a kind of idea of conformity, like this one of holomorphicity. This idea makes possible another kind of diagram. Let us consider the general case of the combined (non homogeneous-linear) transformation

$$w = az + b.$$

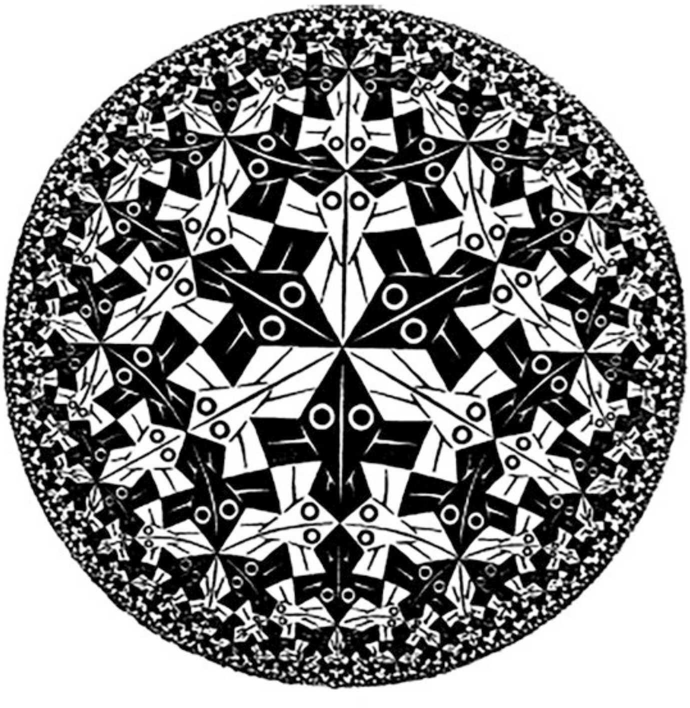
Such transformations provide the Euclidean motions of the plane (without reflection) combined with uniform expansion (or contractions). As Penrose emphasizes, they are the only conformal maps of the entire complex z-plane to the entire w-plane. Let us consider the transformation

$$w = \frac{az + b}{cz + d}.$$

This Möbius transformation actually maps the entire complex plane with the point $-\frac{d}{c}$ removed to the entire complex plane with $\frac{a}{c}$ removed. The point removed from the z-plane is that value ($z = \frac{d}{c}$ which would give $w = \infty$ and correspondingly the point removed from the w-plane is that value $w = \frac{a}{c}$ which would be achieved by $z = \infty$). The whole transformation would make more global sense if we were to incorporate a quantity '∞' into both the domain and the target. As you know, this is one way of thinking about the simplest (compact) Riemann surface of all, the *Riemann sphere*. This is also one way to realize a conformal transformation. This kind of diagram aims to construct the concept of a conformal transformation. The essential feature of the conformity will be the preservation of angles and of orientation, which means also the preservation of shapes. This concept promotes another concept of space, a space without consideration of metrics or distance. The diagram as such realizes these preservations. It is a kind of formula where the symbolism lies essentially in the manner of the integrating the represented space.

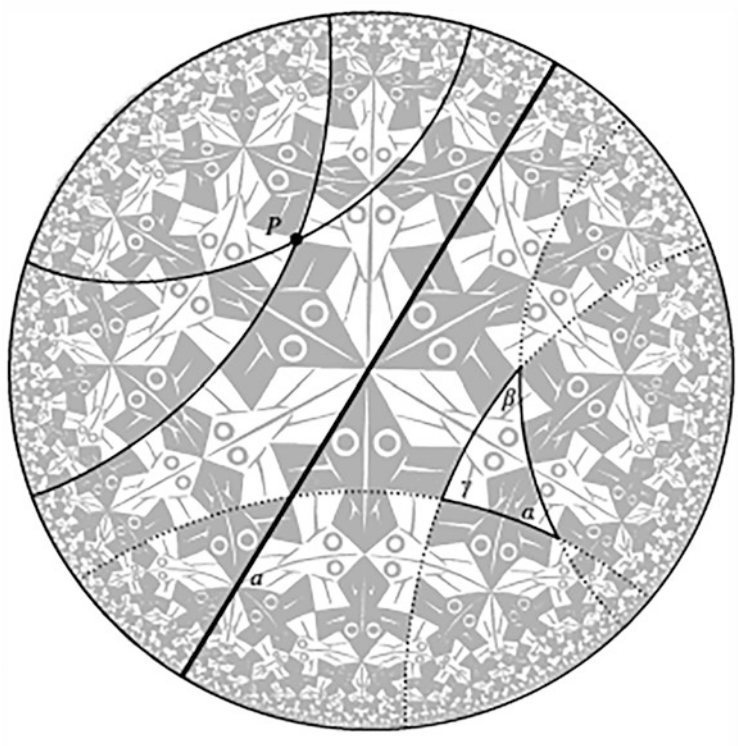
The Introduction of Infinity

We want to make a theoretical gesture. This gesture comes from 17th century-mathematics. We simply add an extra point called "∞" to the complex plane. We must make clear that the required seamless structure holds in the neighborhood of infinity, the same as everywhere else. This was the case of the initial projective geometry of Desargues. Let us regard the sphere to be constructed from two 'coordinate patches', one of which is the z-plane and the other the w'plane. I follow Penrose here, all but two points of the sphere are assigned both a z- coordinate and a w- coordinate (related by the Möbius transformation above). But one point has only a z-coordinate (where w would be "infinity") and another has only a w-coordinate (where z would be "infinity"). We use either w or z or both in order to define the needed conformal structure. We obtain the same conformal structure using either because the relation between the two coordinates (change of the map) is holomorphic.



I would like to add some precisions on hyperbolic geometry and the corresponding diagrams. I consider the famous reproduction of one of the M. C. Escher's woodcuts (see above), called "*Circle Limit I*". It provides us with a very accurate presentation of a kind of geometry - called *hyperbolic, (or some-*

times "Lobatchevskian" geometry- in which the parallel postulate is false, the Pythagorean theorem fails to hold, and the angles of a triangle do not add up to π. And very importantly, as Penrose recalls, for a shape of a given size, there does not in general exist a similar shape of larger size. The bounding circle represents 'infinity' for this hyperbolic universe. We can see that, in Escher's picture, the fish appear to get very crowded as they get close to this bounding circle. Penrose explains that we must think of this as an illusion. Suppose you are one of the fishes. Whether you are situated close to the rim of Escher's picture or close to its centre, the entire (hyperbolic) universe will look the same to you. From the "hyperbolic" perspective of the white or of the black fish themselves, the fish near the bounding circle think that they are exactly the same size and shape as those near the center, even if from our Euclidean perspective the fish near the bounding circle appear to us to be getting very tiny. In more mathematical terms, the set of points lying in the interior of a Euclidean circle represents the set of points in the entire hyperbolic plane. Straight lines according to hyperbolic geometry are to be represented as segments of Euclidean circles which meet the bounding circle orthogonally. This is a conformal representation. Escher uses a representation of hyperbolic geometry that is called the*conformal model* of the hyperbolic plane.

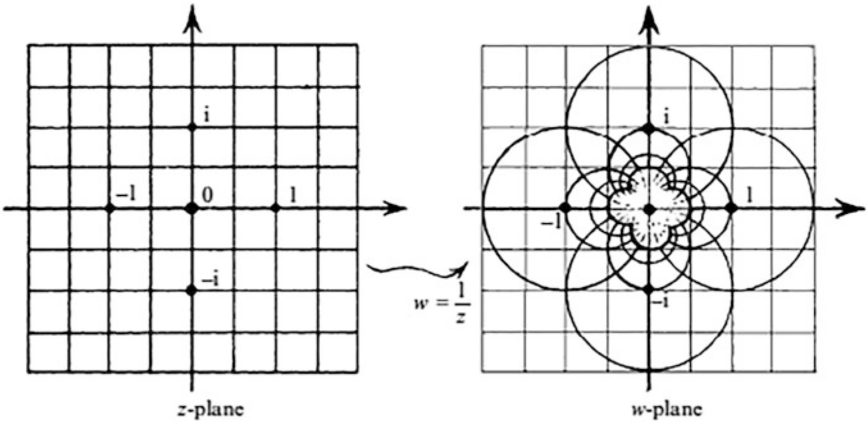


In the often described picture above, we dispose of more than a diagram; the geometric shape is enriched with different motives. They show the pictorial feature of a diagram.

In Escher's picture you have hyperbolic straight lines (A Euclidean circle of lines meeting the bounding circle orthogonally) and a hyperbolic triangle. The hyperbolic angles agree with the Euclidean ones.

We come back now to the Möbius transformation. The simpler form we dispose of is

$$w = \frac{1}{z}, z = \frac{1}{w}$$



This figure shows how to patch the Riemann sphere from the complex $z-$ and w- planes via $w = 1/z$ and $z = 1/w$. the z grid lines are shown also in the w-plane. The overlap regions exclude $z = 0$ and $w = 0$ giving '∞' in the opposite patch.[5]

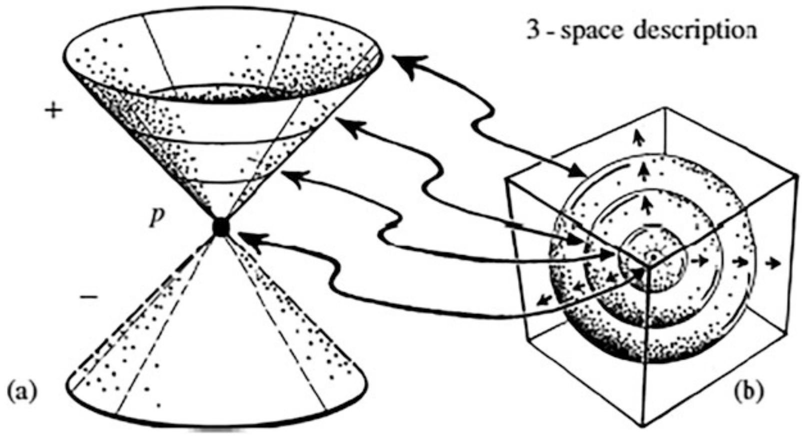
Some Complements to Conformal Geometry and General Relativity

The term 'conformal', as used by Penrose refers to the fact that angles in this geometry are correctly represented in the Euclidean plane in which they have been depicted. The sizes, according to the background Euclidean measure, are represented as tinier the closer to the circular boundary we examine them, but the representation of *angles* or infinitesimal *shapes* remain true, as close to the boundary as we care to examine them. The circular boundary itself represents *infinity* for this

[5] Penrose, *ibid.* p. 172.

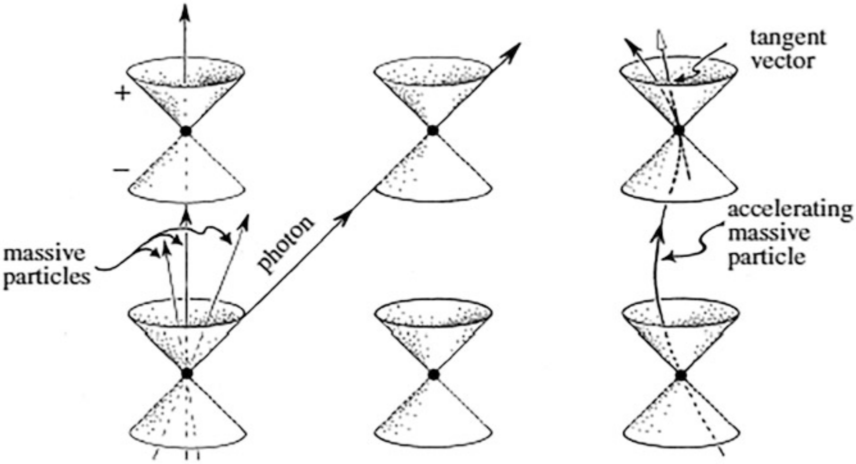
geometry, which should be emphasized for the diagram as we will explain. Penrose tells us that it is this *conformal representation of infinity* as a smooth finite *boundary* that will play a central role in the ideas to which he will turn later.

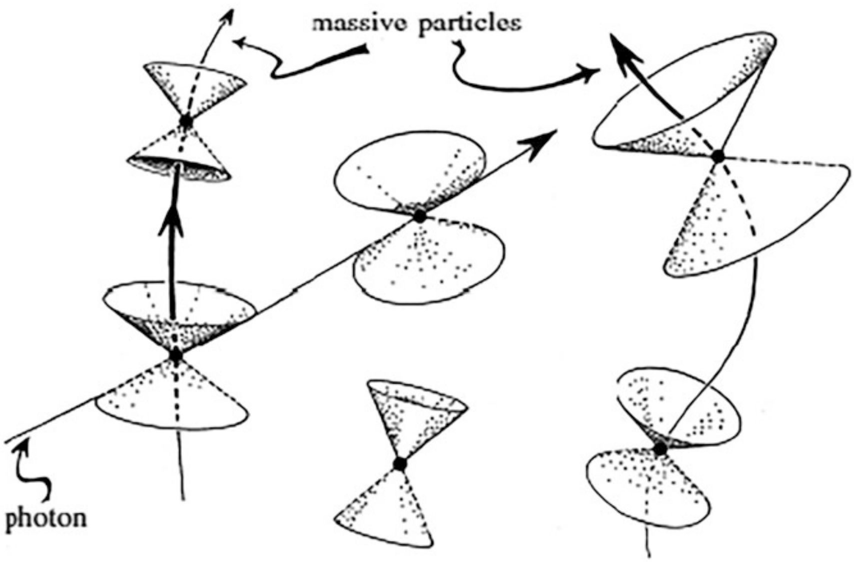
Space-time theory is achieved by Minkowski who presents an objective geometry that is not dependent on any arbitrary observer. Penrose needs to use a kind of *structure* for the 4-space to replace the idea of a temporal succession of 3-spaces. He uses the letter $\mathbb{M}$ to denote Minkowski's 4-space. The basic geometrical structure Minkowski assigned to $\mathbb{M}$ is the notion of a *null cone*. It describes how light propagates at any particular event in $\mathbb{M}$.The intuitive picture of a null cone is provided by flash of light initially focusing itself inward towards the event (past null cone) and immediately afterwards spreading itself out from p (future null cone) like the flash of an explosion.



This diagram—the famous light cone—is a concept that provide us with Special Relativity theory. It should be referred to the argesian geometry for which the observer who perceives the space is the summit of the cone. he observer has been transferred to the top of a cone that becomes itself an observer. It contains the light that emanates from him. The diagram as a cone of light is a conceptualization of observant subjectivity and its spatial representation.This work of installing in the figure of the observer itself, which allows it to become an observer, is what is particular to the new diagrams.

Einstein theory tells us that the speed of any massive particle must always be less than that of light.That means in the theory, the *world-line* must be directed *within* the null cone at each of its events. A particle may have a motion that is accelerated at some place along its world-line. This acceleration is expressed, in space-time terms, as a *curvature* of the world-line.





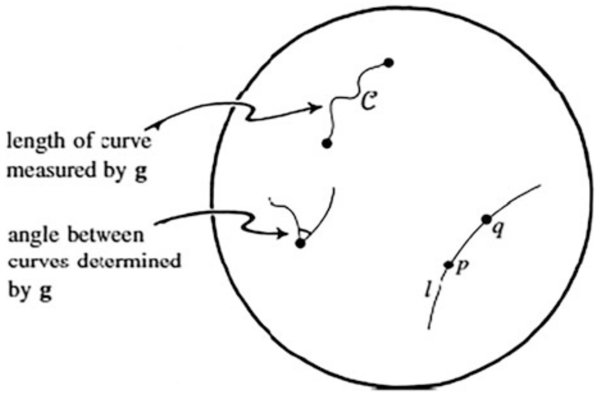
Penrose has labelled a past component with a ‘−’ sign and a future component with a’+’. sign. The past null is distinguished in Penrose’s drawings by the use of broken lines. The causal influences proceeds from past-to future direction. When we pass to Einstein’s general relativity, the generality of the previous world-lines is generally lost. We again have a continuous assignment of time-oriented null cones. And again it is true that any massive particle has a world-line whose tangent vectors all lie along null cones. The following diagram of Penrose depicts the kind of situation that occurs in general relativity where the null cones are not now arranged in a uniform fashion. One should think of these cones as being drawn on some kind of rubber sheet. Any deformation of these sheets is done in a smooth way. Our null cones determine the ‘causality structure’ between events and this is not altered by any such deformation. Penrose mentions that we can imagine Escher’s picture printed on such an ideal rubber sheet. We might choose one of the devils that appear to close to the boundary and move it by such a deformation of the sheet so that it comes into the location previously occupied by one near the center. Such a motion would describe a symmetry of the hyperbolic geometry illustrated by Escher’s picture. The possibility of carrying out such ‘rubber sheet’ deformations is very much in line with Einstein’s general relativity theory, these being referred to as diffeomorphisms or (‘general coordinate transformations’). The principle of ‘general covariance’ which is a cornerstone of Einstein’s general relativity is that we formulate physical laws in such a way that such ‘rubber sheet’ deformation (diffeomorphisms) do not alter the physically meaningful properties of the space and its content.[6]

We are not specifically interested in the theory itself but in the diagram as such.The diagram expresses exactly the essential properties of the theory.The fact that the (theoretical) observer is located inside the diagram provides it with its mathematical and physical properties. This is the case for cones and rubber sheets.The Light cone is a concept/image. And as such this concept/image is a guide for theoretical exploration. In a Kantian framework it must be said that in a diagram the imagination gives its rules to the concept that follows it. The situation is close to that of artistic creation or even of the judgement of beauty, with this essential difference that we produce an ‘effect of knowledge’.

I should add that the only kind of geometry that concerns our space might be something merely of the nature of its topology, indeed sometimes referred to as ‘rubber-sheet geometry’ in which the surface of a teacup will be identical to that of a ring. The term ‘manifold’ is frequently used for such a space of some definite finite number of dimensions. Any deformation of the rubber sheet would carry with it any curve connecting p and q and the length of the segment of $\mathcal{C}$ joining p to q assigned by $\mathbf{g}$ is deemed to be unaffected by this deformation. One says that $\mathbf{g}$ is ‘carried out’ by the deformation.

From the notion of length we pass to the notion of geodesic, such a line l being characterized by the fact that for any two points p and q on l, not too far apart, the shortest curve (in the sense of the length provided by $\mathbf{g}$ from p to q) is pq.

[6] Penrose, *Time Cycles* p. 99.



The ordinary notions of geometry are available to us once **g** has been fixed. I will not concentrate on hyperbolic geometry but on the conformal structure we can assign to it.That is, the structure that provides a measure to the angles between two smooth curves wherever they meet. But (recall) a notion of distance or length is not specified.The concept of angle is actually determined by **g** but **g** itself is *not* fixed by the angle notion. While the conformal structure does not fix the length measure, it does fix the*ratio* of the length measure in different directions at any point. So it determines the infinitesimal *shape*. **g** and Ω^2**g** give us the same conformal structure whatever positive Ω we choose, but a different metric structure.The reason for Ω appearing in squared form in the expression Ω^2 is that the expression for the measure of the (spatial or temporal) separation must take a square root.

Returning to Escher's figure, we find that the conformal structure of the hyperbolic plane is actually identical to that of the Euclidean space interior to the bounding circle.

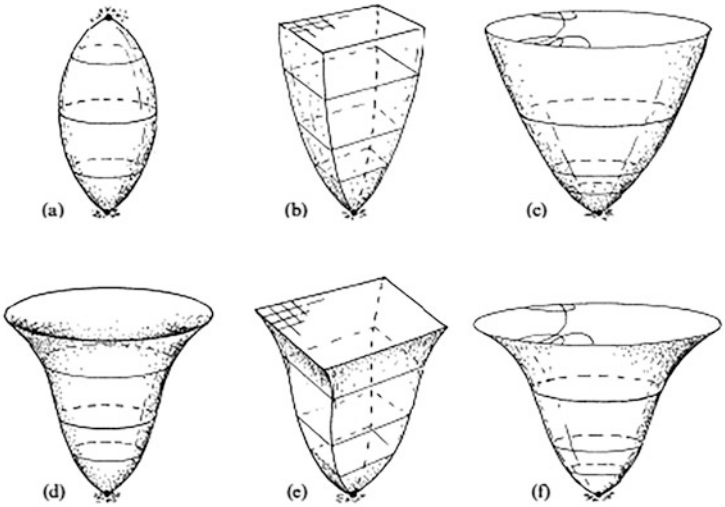
In order to understand the significance ot conformal diagrams, let us consider some characteristics of the new geometry. Minkowski introduced a twist' in[7] the ideas of Euclidean geometry. This 'twist' is what mathematicians refer to as a change signature of the metrics. By changing a few + signs to − signs, it basically tells us how many of a set of n mutually orthogonal directions are to be considered as time-like (within the null cone) and how many space-like (outside the null cone). In Euclidean geometry and in its curved version known as Riemannian geometry we think of all directions as being spacelike. The usual idea of 'space-time' involves only one of the directions as time-like in such an orthogonal set, the rest being space-like. We call it Minkowskian if it is flat, Lorentzian if it is curved. In n dimensional space the signature 1+ 3 separating 4 mutually orthogonal direction into 1 time-like direction and 3 space-like ones. Orthogonality between spacelike directions (and between time like directions when we have more than 1 of them) means simply 'at a right angle' and orthogonality between a space-like direction and a time-like direction means symmetrically related to the null cone between them. We are ready to understand and analyze what a conformal diagram consists of.

[7] Penrose *ibid.* p. 89.

Cosmology and Conformal Diagrams

I present the models in standard form which are called the Standards Cosmological Models of cosmology.

The cosmological models of the general class that Friedmann studied (sometimes with a different type of matter source than Friedmann's "dust") are now referred to as Friedmann-Lemaître-Robertson-Walker (FLRW) models, owing to later contributions, clarifications, and generalization from these others.



In this figure, Penrose has depicted the time -evolution of the universe, according to Friedmann's analysis of the Einstein equation forthe different choices o spatial curvature. The unverse starts from the Big Bang where spacetime curvarures become infinite and then it expands rapidly outwards. In a- we have $K > 0$ the expansion eventually reverses and the universe returns to a singularity (it is the Big Crunch). If $K = 0$ b) the expansion manages to hang on and there is no collapse phase. If $K < 0$, c) there si no "prospect of collapse" (Penrose's words) as the expansion approaches a limiting rate which does not slow down like $K < 0$.

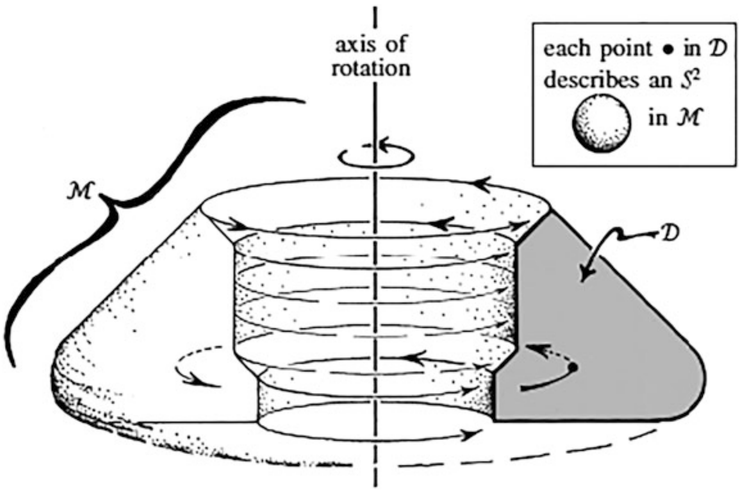
This figure is interesting because its conceptual significance is completely borne by the drawings.

An FLRW model is completely homogeneous and isotropic. Isotropic means that the universe is the same in all directions. Spatially homogeneous means that the universe looks the same at each point of the space. This pair of assumptions is in good accord with observation of matter distribution on a very large scale. Spatial isotropy is found directly to be a very good approximation (from observation of very distant sources). It seems that the homogeneous and isotropic cosmologies-the FLRW models- are excellent approximations of the structure of the actual universe, at least out to the limits of the observable universe which extends to a distance that includes around 10^{11} galaxies, containing 10^{80} baryons.[8] Geometrically we have to do with 3-dimensional 'constant time' spatial section $\mathcal{T}_t$. The three essentially different possibilities for the 3-geometry depend upon whether the (constant) spatial curvature is positive ($K > 0$), zero ($K = 0$), or negative ($K < 0$).

What is important is that we have a diagram that relates to the cosmos. It is a diagram of the "whole" with its properties. It therefore has a finality, and an object of representation and exploration. He contributes to this real science which relates to the whole as such.

As Penrose explains, there is a convenient way of representing space-time models in their entirety, especially in the case of models possessing spherical symmetry. One uses*conformal diagrams*. One distinguishes two types of conformal diagrams: the *strict* and the *schematic*.

Let us start with the strict conformal diagram. It is used to represent space-time with exact spherical symmetry Space time is denoted by $\mathcal{M}$.The diagram would be a region $\mathcal{D}$ of the plane, and each point in the interior of $\mathcal{D}$ would represent a whole sphere's worth (i.e. an S^2's worth). To get something of a picture of what is going on, we lose one spatial dimension, and imagine rotating the region $\mathcal{D}$ around some vertical line off to the left: this being referred to as an axis of rotation. Then each point of $\mathcal{D}$ will trace out a circle (S^1). This is good enough for our visual imagination.[9] But for the full 4-dimensional picture of our space $\mathcal{M}$ we would need a 2-dimensional rotation, so each interior point of $\mathcal{D}$ has to trace out a sphere S^2 in $\mathcal{M}$. We cannot see this sphere S^2.



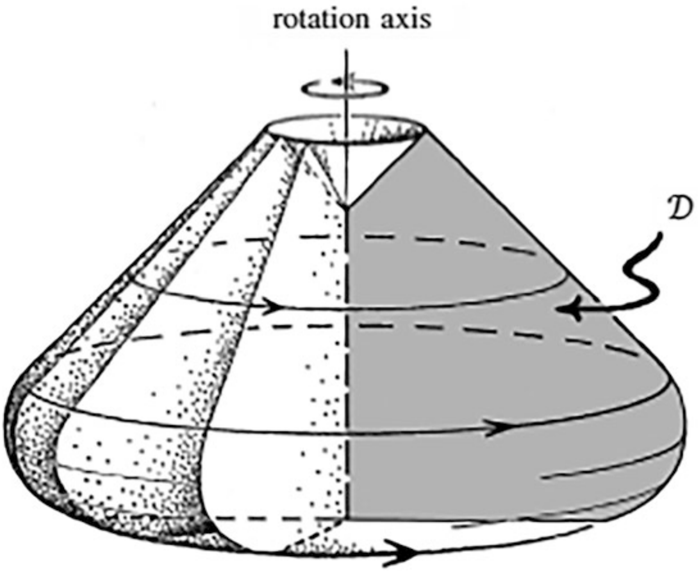
[8] Penrose *A complte Guide to the Laws of the Universe p. 705 Jonathan Cape 2004 in French translation. Odile Jacob 2007* p. 747.

[9] Penrose, *Times Cycles*. p. 106 in French translatation.

Let us consider the process of visualizing. We have space-time in 2 dimensions (actually 4-dimensional) with exact spherical symmetry. A 2-dimensional sphere with its axis of rotation rotates the 2-dimensional region $\mathcal{D}$ of the plane. Each point • in $\mathcal{D}$ describes an S^2 in space-time. We can imagine a 2-dimensional rotation in order to realize the 4-dimensional $\mathcal{M}$.

This diagram, as a device to see in 4 dimensions, is more than intuitive, *intuitio in concreto*, as Kant calls it. Our 3-dimensional intuition is carried beyond the three dimensions by conceptualizing the fourth dimension which we obtain locally. Imagine that you combine two 1-dimensional rotations. The forms of intuition as usually grasped are not sufficient. The intuition cannot be a receptive power. Or rather if it remains this power, because I want to *see*, we want to exercise this receptive power, it is pushed beyond itself: I see the edge, which is provided with 3 dimensions of an object which has 4 dimensions. I cannot do any more.

In Penrose's strict conformal diagrams, we find that we have an axis of rotation which is part of the *boundary* of the region $\mathcal{D}$. Then the boundary points on the axis,-represented in the diagram as *broken* line, would each represent a single *point* (rather than an S^2) in four-dimensional space-time, so that the entire broken line would also represent a single line in $\mathcal{M}$.

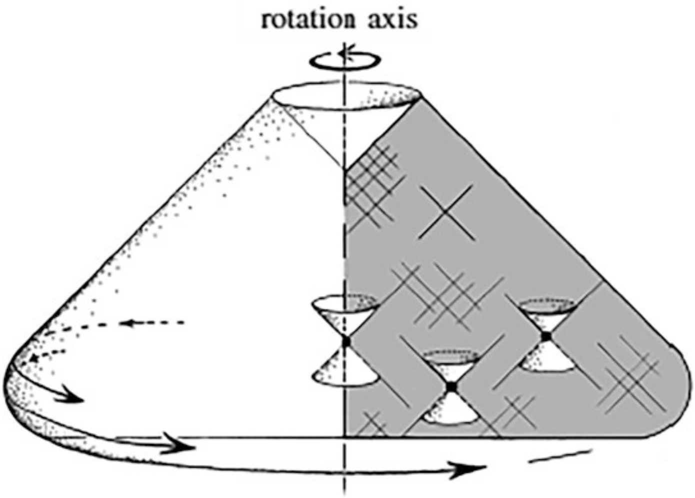


It is true [10] that this diagram gives us an impression of how the whole of space time $\mathcal{M}$ is constituted as a family of 2-dimensional spaces identical to $\mathcal{D}$ in rotation about the broken-line axis. Each point of the broken line represents a single space -time point.

[10] Penrose *Time Cycle.*

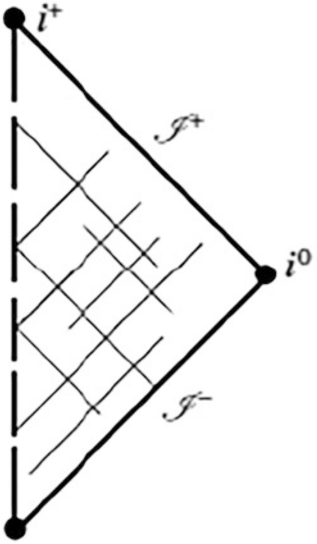
We are going to think of $\mathcal{M}$, following Penrose, as a *conformal* space-time, and not worry too much about the particular scaling that gives $\mathcal{M}$ its full metric **g**. It is important to see that the metric is not important because of the conformal structure.

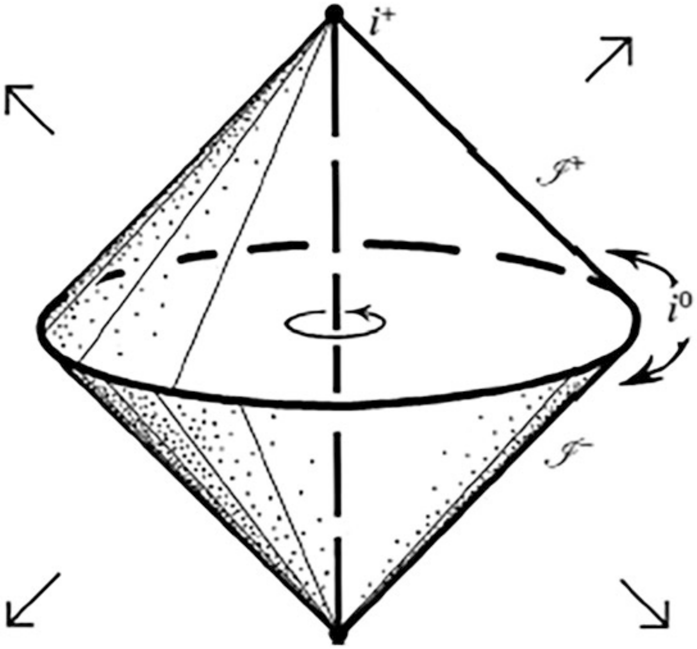
$\mathcal{M}$ is provided with a full family of (time -oriented) null cones. $\mathcal{D}$, it self being a 2-dimensional subspace of $\mathcal{M}$, inherits from it a 2-dimensional space-time structure, and has its own 'time-oriented' null cones. These are the intersections of the planes defining the copies of $\mathcal{D}$ with the future null cones of $\mathcal{M}$.



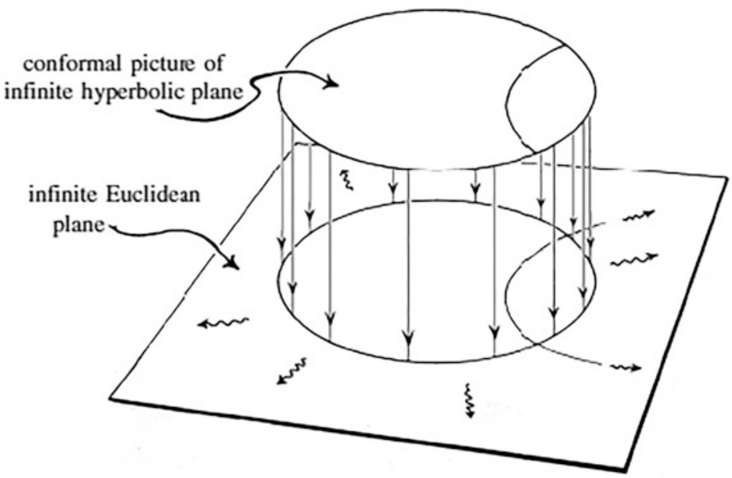
Null cones in $\mathcal{D}$ are angled at 45° to the vertical. Penrose has drawn a conformal diagram for the entirety of Minkowski's space-time $\mathbb{M}$, the radial null lines being drawn at 45° to the upward vertical. We see below that the figure exhibits an important feature of conformal diagrams: the picture is of a merely *finite* (right-angled) triangle, despite the entire infinite space-time $\mathbb{M}$ being encompassed by the diagram.

Here we have the most important feature of conformal diagrams: they enable the infinite region of 'the space time' to be squashed down so as to be encompassed by a finite picture. Here is the theoretical objective of such a diagram: to represent the infinite in a finite way, but in such a manner that the infinite retains it conceptualized content. Infinite itself is also represented in the diagram. The two bold sloping boundary lines represent past null infinity and future null infinity where every null geodesic (null straight lines) in $\mathbb{M}$ acquires a past end-point on $\mathfrak{I}^-$ and a future end-point on $\mathfrak{I}^+$. There are also three points i^-, i^0 i^+ respectively representing past time-like infinity, space-like infinity and future time-like infinity, where every time-like geodesic in $\mathbb{M}$ acquires the past endpoint i^- and the future acquires the past endpoint i^- and the future endpoint i^+, and every space-like geodesic closes into a loop via the point i^0.





We might return to Escher's picture providing a conformal of the entire hyperbolic plane.The bounding circle represents its infinity in a conformally finite way, in an essentially similar manner to the way in which $\mathfrak{I}^-$, $\mathfrak{I}^+$, i^-, i^+, i^0 together represent infinity for $\mathbb{M}$. Penrose develops a very powerful geometrical reasoning. Just as we can extend the hyperbolic plane as a smooth conformal manifold, beyond its conformal boundary, to the Euclidean plane inside which it is represented, we may also extend $\mathbb{M}$, smoothly, beyond its boundary to a larger conformal manifold.[11] The conceptualization of the geometry is here founded on the analogy of two concepts of geometry, the one is hyperbolic geometry, the other is conformal geometry. In both cases we dispose a way to deal with the infinity. The infinity is represented by a circle—in a similar way we represent a line at infinity in projective geometry. When we proceed to a compactification of $\mathbb{C}^4$: we glue a set of points at infinity of $\mathbb{C}^4$, after the inclusion of the manifold in a Grassmannian (a projective complex generalized space). The important thing is that we have at our disposal a representation of infinity and of its role. $\mathbb{M}$ is conformally identical to a portion of the space-time model known as the *Einstein universe* $\mathcal{E}$ or the 'Einstein cylinder'. This is a cosmological model which is spatially a 3-sphere S^3 and completely static.

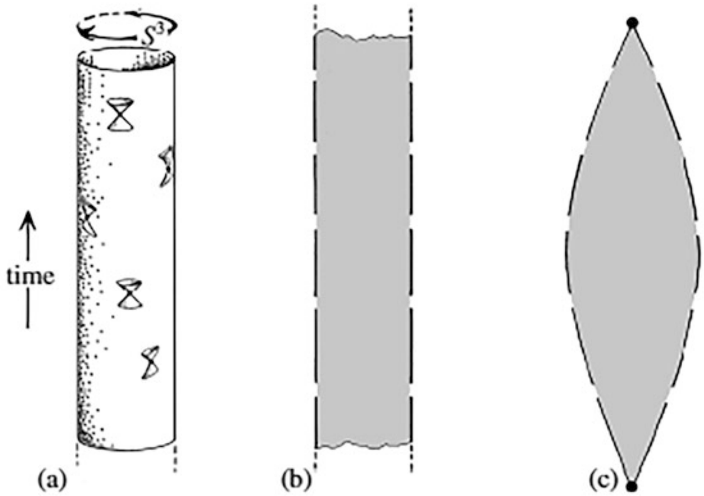


This picture represents the extention of the hyperbolic plane as a smooth conformal manifold beyond its conformal boundary to the Euclidean plane inside which it is represented.

We also have an intuitive picture of the model below, the one with which Einstein introduced originally his cosmological constant Λ in order to achieve. It is interesting to give a strict conformal diagram representing it. in this diagram

[11] Penrose, *Time cycles* p. 123.

there are two separate 'axes of rotation ' represented by two broken lines. this is completely consistent ; we simply think of the radius of the S^2, which each point in the interior of the diagram represents, as shrinking down to zero as a broken line approaches.

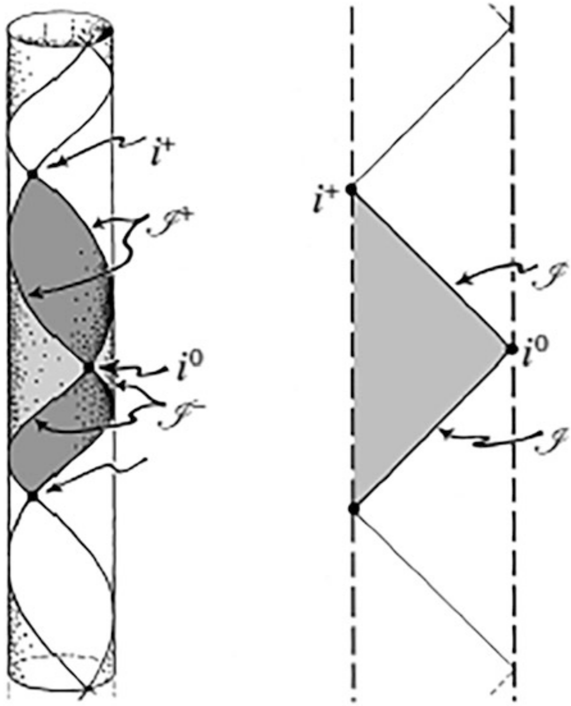


This serves to explain the fact that spatial infinity for $\mathbb{M}$ is conformally just the *single* point i^0 for the radius of S^2 that it would seem to have represented (expression by Penrose) has shrunk down to zero. The spatial S^3 cross sections of the space time $\mathcal{E}$ arise from this procedure.

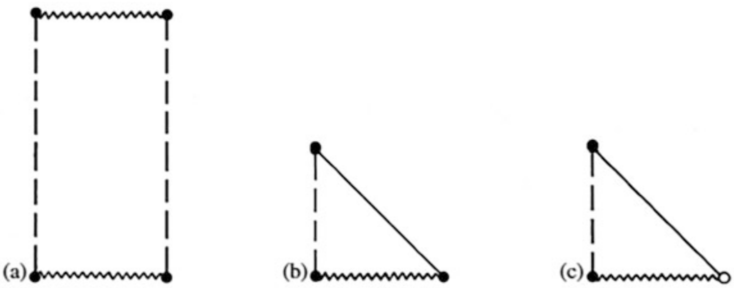
When we compare the two diagrams (the intuitive picture and the strict conformal diagram) we might say that we dispose complementary geometrical thinking of on space time's reality. The passage from the first one to the second one is carried away by a reduction -the radius of S^2 are shrinking to zero - points- and the introduction of a concept of infinity is different.

The spatial infinity for $\mathbb{M}$ is conformally a single point i^0. $\mathbb{M}$ arise as a conformal subregion of the manifold $\mathcal{E}$. It can be considered to be made up, conformally, of an infinite succession of spaces $\mathbb{M}$. And the second (strict conformal) diagram shows this in its frame. In this succession the $\mathfrak{I}^+$ of each space $\mathbb{M}$ is joined on to $\mathfrak{I}^-$ of the next . This is done in terms of strict conformal diagrams.

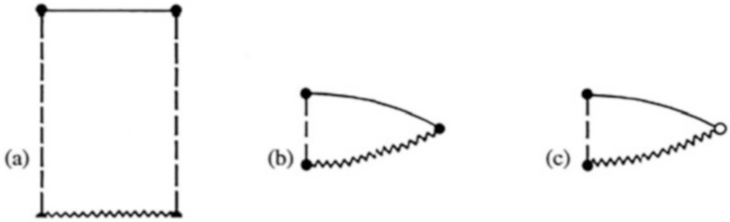
It is explained by the following diagram how this is done in terms of conformal diagrams.



We see how the two infinities (towards the future and towards the past) are connected. Consider now the three cases (a, b, c).



A white dot ∘ on the boundary represents an entire sphere S^2, whereas the black dots •(which already occurred in $\mathbb{M}$) represent single points. These white dots actually represent the boundary spheres on hyperbolic space boundary, in the conformal representation that Escher used in the 2-dimensional case. The corresponding cases for the cosmological constant $\Lambda > 0$ where in case $K > 0$ Penrose assumes that the spatial curvature is not large enough to overcome Λ and produce an ultimate re-collapse. An important feature of these diagrams must be pointed out: the future infinity $\mathfrak{I}^+$ is *space-like* as indicated by the final bold boundary line, and this is always more horizontal than 45°, in contrast with the future infinity that occurs when $\Lambda = 0$ (in the cases illustrated above where the boundary is at 45° so $\mathfrak{I}^+$ is then a *null* hypersurface.

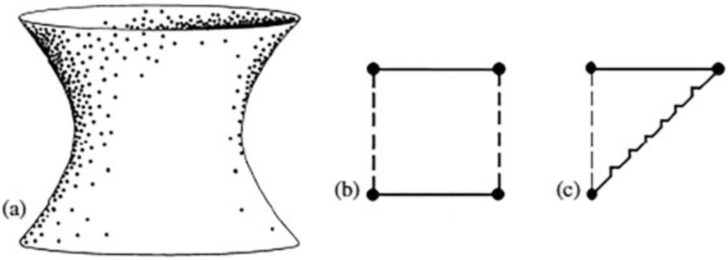


Let us consider these two diagrams. The first one with cosmological constant $\Lambda = 0$ the second one with the cosmological constant $\Lambda > 0$. The first one with vertical figures, the second one with horizontal figures. Both values for curvature are $K = 0$, $K > 0$, $K < 0$.

Let us recall that a white dot '∘' on the boundary represents an entire sphere S^2 an d the black dot '•' represents single points. These white dots represent the boundary spheres of hyperbolic spaces in the conformal representation that Escher used. For the second diagram the future infinity $\mathfrak{I}^+$ is spacelike like we mentioned it, the final bold boundary line being always more horizontal than 45°, for the first diagram with the future infinity that occurs when $\Lambda = 0$, when the boundary is at 45°. So $\mathfrak{I}^+$ is the *null* hypersurface. Penrose mentions that it is a general feature between the geometrical nature of $\mathfrak{I}^+$ and the value of the cosmological constant Λ. I remind that, he cosmological constant is a constant that Einstein added to his equations and then removed as the biggest mistake of his life and which now has significant significance in cosmology.

In this presentation we make use of a "coherence's argument". These Friedmann model with $\Lambda > 0$ all have a behavior in their remote future (i.e near $\mathfrak{I}^+$) which closely approaches de Sitter space-time $\mathbb{D}$, a model universe that is completely empty of matter and is extremely symmetrical (being a Minkowskian analogue of a 4-dimensional sphere). Penrose has sketched a 2-dimensional version of $\mathbb{D}$, with only one spatial dimension represented (where the full de Sitter 4-space $\mathbb{D}$ would be a hypersurface in Minkowski 5 -space. Penrose gave a strict conformal diagram for it b). The *steady-state* model is just one half of $\mathbb{D}$ c).

A cut through $\mathbb{D}$ is required and we have jagged boundary. And thanks to the schema we can understand what Penrose calls "nocomplete in past direction". Indeed, Penrose continues to explain the concepts of the model. There are ordinary timelike geodesics whose time measure does not extend to earlier values than some finite values. Here Penrose is in a situation where he has to work on the model itself to make its meaning more consistent. The finitude conditions might have been regarded as a worrying flaw in the model, if it applied to future direction, since it could be apply to the future of some particle or space traveller,[12] "but here we can simply say that such particle motions were never present." In a note Penrose explains that in a time reversed steady-state model an astronaut, in free motion, following such an orbit would encounter the inward motion of ambient material passing at greater and greater velocity until it reaches light speed with infinite momentum impacts in a finite experienced time.[13]



Philosophical Comments

I have presented a diagrammatic thought of. Let us try to summarize its features. The first step of the thought is the construction of the hyperbolic geometrical situation. The essential point is to represent a 4-dimensional space, the Minkowski space. As I have shown it, this is a construction that needs complex conformal geometry. We gave an example of complex domain in $\mathbb{C}^2$. As I mentioned before, the remarkable feature of this diagram is the ability to manipulate infinity. What does it mean to manipulate infinity? What is the purpose of this construction?

The basic idea of projective geometry is to complete Euclidean geometry by adding a point to infinity which makes it possible to treat the point at infinity thus glued as an ordinary point. This theoretical process must be understood as what mathematicians call *compactification*. Penrose has thought a lot about this construction and has generalized it, in the case of several variables and several lines or even planes or more to infinity. More precisely he has generalized the use of compactification. This brings the questions of infinity back to finite distances. In the same line of reflection, he analyzed the meaning of hyperbolic geometry, which gives us a way of going beyond Euclidean geometry, and he joined to this reflection a concept that concerns another essential concept that guides his thought: the concept of conformity.

The concept of conformity (conformal geometry) implies a very different point of view on space. It is mainly interested in the preservation of angles under the transformations he promotes. At the same time, it provides complex geometry with a

[12] Penrose, it op.cit. p.114.

[13] Penrose *op. cit.* note 2.45.

conceptual framework (a complex holomorphic function is seen in Riemann's work and probably before as a conformal application of one plane in another).

It must be understood that these elements, thus synthesized, allow the development of a true cosmological thought. Penrose can thus give us cosmological models of reflection that are at the same time geometrically structured. We must indeed note that this theoretical reflection is directly inscribed in a geometry: the geometry of conformal diagrams.

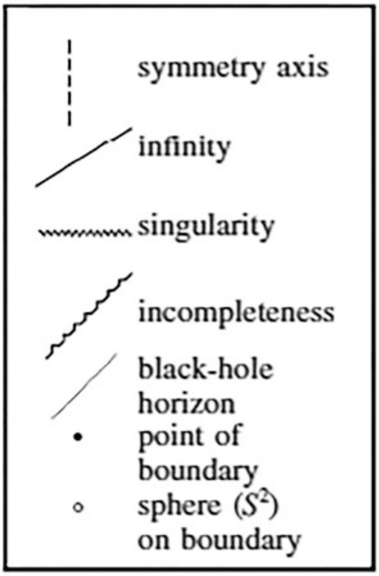
We have highlighted the synthesis made by these diagrams between the geometric representation and the conceptual construction. This synthesis is borrowed from the fact that the representation must provide a geometric and necessarily theoretical access to perceptually inaccessible dimensions.

The case of the infinite is from this point of view, quite peculiar. It is well represented in this compactification (point or straight line or plane). It is placed in a position to be manipulated, like the other concepts, but it retains its "extension". This ceases to be an obstacle to thought.

Geometry has found a way to extract from this concept what makes it thinkable. And since this was a limit for theoretical reflection, this new construction gives this reflection a real theoretical framework.

Final Complements

Let us consider tools that Penrose built up;



Key for strict conformal diagrams.

By analyzing these concepts/images, we could say that Penrose has achieved by means of a work on the representation of geometric objects and dimensions to put in first place the concept of edge, of border to operate with a concept of infinity. Infinity is represented by a line of the diagram. Concerning singularities, I can only give some indications. In fact this is a very deep theory. There are two tensors which determine the curvature of the universe in different models. for example rays passing through the Sun are focused just by Ricci curvature, resulting in magnification, outside the Sul's rim we get essentially purely astigmatic Weyl distortions, so a small circular pattern in the star field would appear elliptical.

Weyl curvature finally diverges to infinity as the block-hole singularities. are reached. For a universe to start out closely FLRW, we expect the Weyl curvature to be extremely small, as compared with Ricci curvature the later actually diverging at the Big Bang. The geometric difference between the initial singularity- of exceedingly low entropy-and the generic black-hole singularities, of very high entropy.[14]

The Weyl curvature vanishes at the initial singularity, and is unconstrained, no doubt diverging widely, at final singularities. We should add the notion of a naked singularity.. A naked singularity would be a space-time singularity resulting from a gravitational collapse, which is visible to outside observers, so it is not ‘clothed’ by an event horizon. A naked singularity is timelike, in the sense that signals can both enter and leave singularity.

I would like to conclude with these final remarks. There exists a notion of cosmic censorship. This is basically a mathematical conjecture concerning general solutions of the Einstein equation. if we assume this conjecture then physical space-time singularities have to be spacelike (or perhaps null) and never timelike. There are two kinds of spacelike (or null) singularities namely ‘initial’ or ‘final’ ones, depending upon whether timelike curves can escape from the singularity into the future or enter it from the past. Penrose refers to what he calls *the Weyl hypothesis conjecture*. This conjecture asserts that the Weyl curvature is constrained to be zero (or at least very small) at *initial* singularities in the actual physical universe.

These terminal claims can shed some light on the place of singularities in the conformal diagrams. They are represented as an edge of the triangular diagram like the infinity, because of the conceptual and geometrical links between both concepts.

Let us insist on this point. The diagrams take precedence over the discursive elements in the argumentation. In Penrose's work it is the diagram that leads. On the one hand, we go beyond a geometric representation in three dimensions. But even more, on the other hand, the processes like the compactification, the conceptual dynamics—as well topological as geometric—of the infinite is supported by the diagram. Cosmology gives its rules to geometry, as much as the schematism of the conceptual imagination gives its rules to cosmological construction.

[14] Penrose R. *The Road to Reality*. p. 176.

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Part V
Phenomenology in and of Mathematical Diagrams

Phénoménologie, représentations, combinatoire

Frédéric Patras

Résumé La phénoménologie husserlienne a largement ses origines, ou du moins ses proto-fondations, dans les mathématiques et la logique. Pour autant, on trouve finalement chez Husserl assez peu de descriptions phénoménologiques détaillées des moments clés de la pensée mathématique, comme si tout l'éventail de méthodes développées par exemple dans les Ideen achoppait à parler vraiment des contenus scientifiques et de leurs modes d'appréhension. On s'intéressera ici, dans une perspective husserlienne, aux représentations d'objets géométriques et combinatoires élémentaires, avec une insistance particulière sur les partitions d'ensembles dites non croisées.

Mots clés phénoménologie · diagrammes · partitions d'ensembles · représentations

La phénoménologie husserlienne a largement ses origines, ou du moins ses proto-fondations, dans les mathématiques et la logique. La *Philosophie de l'arithmétique* (Husserl, 1972) en témoigne amplement, où se trouvent déjà en germe de nombreux ingrédients de la pensée phénoménologique en dépit de références à la psychologie qui seront ensuite abandonnées par Husserl. Les nombres, puis la géométrie, ont donc joué un rôle essentiel dans le processus de constitution de la méthode phénoménologique, et les mathématiques sont au cœur de textes clés de la maturité husserlienne comme *Logique formelle et logique transcendantale* (Husserl, 1960).

Pour autant, on trouve finalement chez Husserl assez peu de descriptions phénoménologiques détaillées des moments clés de la pensée mathématique, comme si tout l'éventail de méthodes développées par exemple dans les *Ideen* (Husserl, 1963) achoppait à parler vraiment des contenus scientifiques et de leurs modes d'appréhension. Il s'agit-là d'un état de fait regrettable, qui n'a été qu'en partie corrigé par des auteurs comme Desanti (Desanti, 1968), Vuillemin (Vuillemin,

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1962) ou Rota (Gandon, 2016; Patras, 2017; Rota et al., 2005) qui, sans employer toutefois la méthode phénoménologique dans toute son étendue, ont su s'en inspirer à des moments clés de leur œuvre.

L'objectif de cet article sera d'aller dans cette même direction en suivant deux approches complémentaires. La première situera la phénoménologie par rapport à d'autres approches des mathématiques à l'aide d'un exemple classique: les objets géométriques élémentaires. Le cas du triangle est étudié plus en détail, mais surtout pour des raisons historiques puisque les arguments avancés vaudraient pour tout objet élémentaire de la géométrie euclidienne, non euclidienne, différentielle, projective..., mais également pour tout diagramme, toute construction illustrant un concept géométrique. La seconde approche envisagera ensuite des exemples plus complexes, toujours liés à des représentations, mais où se jouent d'autres phénomènes, comme les liens entre algèbre et géométrie.

Représentations géométriques: une approche classique

Considérons d'abord un des gestes inauguraux de la géométrie: le tracé, à main levée, d'un triangle. Un problème philosophique des plus classiques tient au rapport entre le triangle dessiné et le concept correspondant. Ce problème peut être décliné selon des modes multiples qui fourniraient à eux seuls un chapitre de la philosophie mathématique : rapport concept/objet ; notion de présentation et de représentation ; esthétique comme théorie de la sensibilité..., il sera abordé ici dans la seule perspective de son possible traitement phénoménologique.

Que dit classiquement la philosophie du rapport entre représentation et objet ? Le triangle dessiné ne coïncide évidemment pas avec le triangle général, abstrait. Au mieux en est-il une instanciation imparfaite, une représentation (lorsque, connaissant le concept de triangle on cherche à en donner une construction dans la sensibilité et, en l'occurrence, de façon matérielle) ou une présentation (lorsqu'on cherche plutôt à construire le concept dans l'intuition - la mienne ou celle d'autrui).

Ce qui se joue avec le problème du triangle est donc finalement le rapport entre concepts et représentations et la possibilité même d'intuitionner des concepts. Dans une tradition classique, kantienne par exemple, la question est abordée au prisme d'une analyse globale du fonctionnement de ce rapport avec, d'un côté, la théorie de l'imagination transcendantale et, de l'autre, la théorie du schématisme des concepts de l'entendement. A grands traits, l'imagination transcendantale est la faculté d'abstraire, d'aller vers des concepts en partant de donnés intuitifs qui disparaissent dans le processus pour laisser place aux idées, aux concepts. C'est elle qui œuvre dans une démonstration au tableau, lorsque les élèves abstraient de la matérialité des figures les principes organiques sous-jacents. Le schème, quant à lui, est la structure sous-jacente à la faculté de reconnaître les instances d'un concept - autre moment constitutif d'une démonstration de géométrie au tableau. Là encore, le mécanisme est clair, correspond de fait à une modalité de fonctionnement de la connaissance, mais, comme dans le cas de l'imagination transcendantale, ses ressorts restent cachés.

Le problème laissé ouvert par ce type de théories, qu'elles soient descriptives ou normatives, c'est que, donnant un cadre pour penser les modalités du rapport entre théorie et intuitions, elles n'interrogent pas les processus cognitifs à l'œuvre dans le détail de leur fonctionnement, détails pourtant constitutifs.

Représentations géométriques: l'approche mathématique

Que pourraient dire les mathématiques sur ces questions ? Lorsque nous dessinons une figure, quelle qu'elle soit, elle a souvent une dimension générique. Nous ne raisonnons alors pas sur une instance particulière d'objet, mais sur sa classe d'équivalence modulo une certaine famille de transformations. Selon une autre modalité, que nous n'explorerons pas ici car sa codification mathématique est moins évidente, cette instance a parfois une valeur universelle du fait que le processus de sa construction même a des traits universels, si bien que le même geste qui nous fait tracer ce triangle puis en déduire telle ou telle propriété pourrait être répété indépendamment sur d'autres instances. Ces deux modalités sont de toutes façons intimement liées, et l'analyse mathématique de la seconde, si on essayait de la pousser jusqu'au bout, reconduirait sans doute à utiliser des actions de groupes de transformations.

Quoiqu'il en soit, il s'agit de questions aussi fondamentales que délicates qui nécessiteraient des développements poussés, mais notre propos est limité ici à donner quelques indications sur des points de vue naturels, conceptuellement et historiquement, afin de pouvoir les utiliser comme étalons de la méthode phénoménologique et de ses spécificités.

Les mécanismes cognitifs auxquels il vient d'être fait allusion sont implicites et ne deviennent conscients qu'à des niveaux assez élevés de la pratique mathématique. Pourtant, ce sont bien eux qui garantissent, mathématiquement, la validité générale du raisonnement effectué sur le triangle dessiné au tableau. Ainsi, qui fait de la géométrie euclidienne va travailler modulo le groupe des translations, celui des transformations orthogonales, et souvent aussi celui des homothéties. Qui fait de la géométrie affine, modulo le groupe affine. Effectuer une transformation globale de toutes les constructions effectuées ne changerait rien à tous les arguments qui peuvent être déployés à partir de figures géométriques tracées sur un tableau. Et c'est le fait de travailler ainsi modulo un groupe de transformations qui garantit la portée générale d'un argument, même s'il se déploie sur un cas particulier, une figure dessinée.

Articuler cette approche mathématique à celle philosophique décrite précédemment ne présenterait guère de difficultés de principe : c'est que l'imagination transcendantale tout comme le schématisme des concepts ont eux-mêmes des structures implicites que l'analyse mathématique permet de mettre en partie au jour. Pour autant, l'analyse philosophique classique n'en dit rien, et ne peut rien en dire, compte-tenu de ses méthodes et de son extension. Quand aux mathématiques, ce n'est évidemment pas leur propos que de s'intéresser à la théorie de la connaissance pour elle-même, tout au plus leurs idées peuvent-elles permettre d'individuer des moments constitutifs, des structures fondamentales des processus cognitifs. C'est donc à la phénoménologie, ou d'autres théories qui pourraient être développées selon des principes directeurs analogues, d'accomplir ce type d'analyses et de descriptions.

Représentations géométriques: l'approche phénoménologique

Que dirait la phénoménologie ? D'abord, que nous n'avons jamais d'intuition propre, parfaite, des concepts mathématiques, ne serait-ce que pour des raisons de principe et d'inhomogénéité entre ce qui serait le contenu, l'essence d'un concept, et sa saisie intuitive.

Aussi la phénoménologie double-t-elle la question classique, qui est celle de la légitimité des raisonnements géométriques sur une figure dessinée, de la question, d'une nature différente, de l'adéquation de notre intuition, de notre perception de la figure, au concept, à l'idée, à la notion correspondante. En termes concrets, la question de la structure et des propriétés des objets considérés et des modalités de leur appréhension se double de la question des structures immanentes à cette appréhension et de leurs légalités.

Les questions nouvelles qui en résultent peuvent se déployer selon des directions multiples. Husserl, dans ses années de formation intellectuelle (Husserl, 1975), s'interroge par exemple sur la nature même de l'intuition. Il remarque que le rapport que nous avons aux objets mathématiques est toujours accompagné d'une intention de signification. C'est le remplissement intuitif de cette intention qui va donner un sens à l'objet et lui permettre de s'insérer dans un horizon cognitif et/ou théorique.

Dans le cas du triangle, le fait de raisonner implicitement modulo l'action du groupe affine ou du groupe des déplacements correspond à une modalité de notre rapport aux objets mathématiques: la possibilité de leur faire subir certaines transformations sans altérer la structure même de ce rapport. Se dégage ainsi l'idée que nos représentations d'objets mathématiques s'accompagnent d'intentions de signification, de visées théoriques dont la philosophie des mathématiques peut et doit interroger la structure implicite.

Il se pourrait bien que ce qui est appelé classiquement la structure des objets et des théories mathématiques ait ainsi, sinon une implémentation (ce qui constituerait une thèse platonicienne forte), du moins des répondants dans les structures de notre intentionnalité. C'est là, me semble-t-il, la thèse majeure, pour les mathématiques, que l'on puisse inférer de la phénoménologie. C'est en tout cas une interprétation possible de textes comme *Logique formelle et logique transcendantale*. La seconde partie de cet article visera à illustrer ces idées sur des exemples concrets tirés de situations ou d'expériences vécues.

Partitions non croisées

Les partitions sont des objets fondamentaux en combinatoire, que l'on rencontre dès les tout premiers pas dans la théorie. Les partitions non croisées (Kreweras, 1972; Simion, 2000) en forment une sous-classe, qui a des applications en particulier en probabilités (dans le domaine des probabilités libres, associées à la théorie des matrices aléatoires), mais également en théorie quantique des champs (dans les théories dites planaires) (Ebrahimi-Fard et Patras, 2016).

Rappelons qu'une partition L de l'ensemble $[n] := \{1, \ldots, n\}$ consiste en la donnée d'une collection de sous-ensembles non vides $L = \{L_1, \ldots, L_b\}$ de $[n]$, appelés blocs de la partition, mutuellement disjoints ($L_i \cap L_j = \emptyset$ pour tout $i \neq j$), et dont l'union vaut $[n]$ ($\cup_{i=1}^{b} L_i = [n]$). Pour $p, q \in [n]$ on écrit $p \sim_L q$ lorsque p et q sont dans le même bloc (et $p \nsim_L q$ sinon). Les partitions d'un ensemble ont une structure d'ensemble (partiellement) ordonné par raffinement: $L \leq K$ si L est une partition plus fine que K, c'est à dire si ses blocs sont des sous-ensembles des blocs de K. La partition $\hat{1}_n = \{L_1\}$ qui est formée d'un bloc unique, $|L_1| = n$, est l'élément maximal de cet ordre. La partition $\hat{0}_n = \{L_1, \ldots, L_n\}$ qui a n blocs formés de singletons est l'élément minimal.

Une partition L de $[n]$ est dite non croisée si pour $p_1, p_2, q_1, q_2 \in [n]$ on n'a jamais

$$1 \leq p_1 < q_1 < p_2 < q_2 \leq n$$

et

$$p_1 \sim_L p_2 \nsim_L q_1 \sim_L q_2.$$

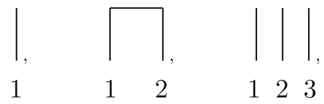
Par exemple, la partition $\{\{1, 3\}, \{2, 4\}\}$ n'est pas non croisée. Les partitions non croisées d'ensembles totalement ordonnés finis se définissent de façon analogue.

L'ordre par raffinement se restreint aux partitions non croisées. Quelques propriétés résultent immédiatement de leur définition, par exemple si L est une partition non croisée de $[n]$, sa restriction à un sous-ensemble arbitraire S de $[n]$ (obtenu en intersectant les blocs de L avec S) définit une partition non croisée de S.

Représentations graphiques

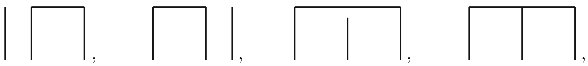
La définition qui vient d'être donnée des partitions non croisées, en dépit de sa simplicité, est peu parlante si l'on ne dispose pas d'une compréhension intuitive de son contenu. Aussi, lors par exemple de l'utilisation de ces partitions dans le contexte d'un exposé ou d'un cours, est-il naturel de donner, plutôt que la définition elle-même, un exemple caractéristique dont il soit facile d'abstraire cette définition. Cet exemple suffit, en général, aux mathématiciens exercés, pour effectuer automatiquement ce travail d'abstraction et il est donc le plus souvent inutile, pour peu que l'on prenne quelques précautions oratoires, d'énoncer la définition abstraite.

Pour ce faire, l'usage le plus courant est de représenter les partitions sous forme diagrammatique. Un ensemble totalement ordonné fini est toujours isomorphe à un sous-ensemble initial des entiers, $\{1, \dots, n\}$, et peut toujours être représenté par une suite de points alignés, et ses sous-ensembles en reliant graphiquement les points correspondants. Ainsi,

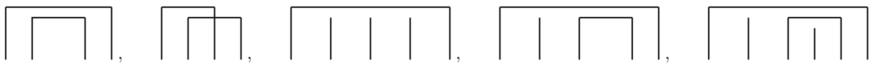


le premier diagramme représentant le singleton $\hat{0}_1 = \hat{1}_1 = \{1\}$. Le second est l'élément maximal $\hat{1}_2 = \{1, 2\}$ dans le cas d'un ensemble totalement ordonné à deux éléments. On trouve ensuite l'élément minimal associé à un ensemble totalement ordonné à trois éléments, $\hat{0}_3 = \{\{1\}, \{2\}, \{3\}\}$. Dans la suite, pour alléger les représentations on omettra de numéroter les éléments dans les représentations diagrammatiques.

Dans le cas de l'ensemble totalement ordonné à trois éléments on a encore

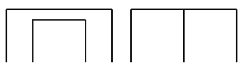


qui représentent les partitions $\{\{1\}, \{2, 3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$ et $\hat{1}_3 = \{1, 2, 3\}$, respectivement. Avec plus d'éléments on trouve par exemple

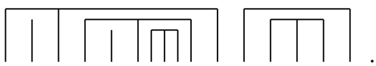


pour les partitions, $\{\{1, 4\}, \{2, 3\}\}$, $\{\{1, 3\}, \{2, 4\}\}$; $\{\{1, 5\}, \{2\}, \{3\}, \{4\}\}$, $\{\{1, 5\}, \{2\}, \{3, 4\}\}$ et $\{\{1, 6\}, \{2\}, \{3, 5\}, \{4\}\}$.

Graphiquement, une partition est non croisée lorsqu'on peut la représenter sans croisements. Ainsi, la partition $\{\{1, 4\}, \{2, 3\}, \{5, 6, 7\}\}$



est non croisée. Elle est toutefois un peu trop simple pour qu'on y devine toute la structure des partitions non croisées, aussi convient-il, si on doit donner un exemple le plus générique possible, de choisir une partition à la combinatoire plus riche, par exemple

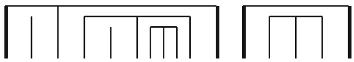


De l'usage des représentations diagrammatiques.

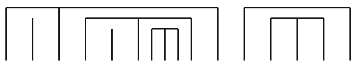
Quel avantage y a-t-il à travailler avec une telle représentation typique plutôt qu'avec une définition formalisée? Il y a en plusieurs. Tout d'abord, la représentation diagrammatique instaure une tension, une visée cognitive et intentionnelle. Celui qui la découvre au cours d'un séminaire, par exemple, va d'abord chercher à trouver la logique sous-jacente à la représentation. S'il est un peu entraîné, il aboutira rapidement à la définition abstraite ; s'il a des doutes, il posera une question qui permettra à l'orateur de préciser les contours de la notion, soit au travers d'autres exemples, soit en énonçant la définition formelle. Il pourra aussi, plutôt que de poser une question, garder en mémoire ses incertitudes et chercher à les lever au cours des développements ultérieurs de l'exposé : loin d'obérer sa compréhension, elles pourront même servir alors de fil conducteur dans l'analyse de certains de ses développements.

L'essentiel n'est pourtant pas là: l'existence d'une visée intentionnelle (en l'occurrence le fait de chercher à dégager des significations d'une intuition, d'une représentation) fait très vite apparaître des structures additionnelles qui restent cachées si l'on s'en tient à la définition formelle des partitions non croisées. Ainsi:

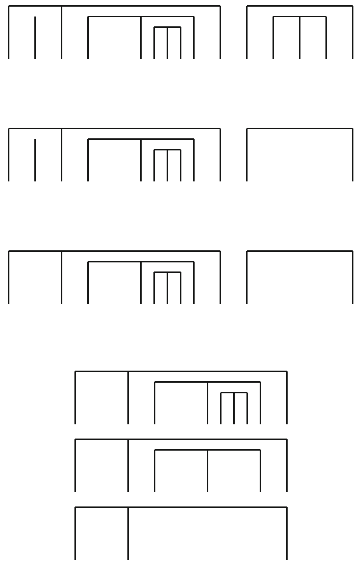
1. La représentation diagrammatique fait naturellement apparaître une décomposition des partitions non croisées en groupes de blocs: dans le cas de la partition $\{1, 3, 11\}, \{2\}, \{4, 6, 10\}, \{5\}, \{7, 8, 9\}, \{12, 16\}, \{13, 14, 15\}$

   on voit que les jambes verticales associées aux éléments 1, 11 et 12, 16 (tracées en gras) déterminent deux "blocs de blocs" selon une structure qui existe de façon générale sur les partitions non croisées et pourrait être formalisée.[1]
2. Appelons intervalle d'une partition non croisée tout bloc formé d'une suite d'éléments consécutifs. Dans l'exemple considéré les intervalles de la partition de départ sont $\{2\}, \{5\}, \{7, 8, 9\}, \{13, 14, 15\}$. Comme dans le jeu de Mikado, on peut déconstruire pas à pas une partition non croisée en enlevant itérativement des intervalles. Par exemple:

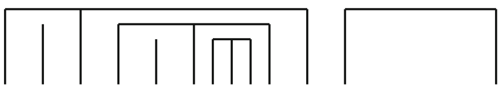
   

[1] Par exemple de façon récursive en observant que l'élément minimal (1) et maximal (11) du bloc contenant 1 déterminent le premier de ces blocs de blocs selon un schéma qui peut être poursuivi de façon itérative (en remarquant qu'une fois enlevé l'ensemble de ces blocs on obtient une partition non croisée de l'ensemble totalement ordonné complémentaire: dans l'exemple considéré, de $\{12, 13, 14, 15, 16\}$).

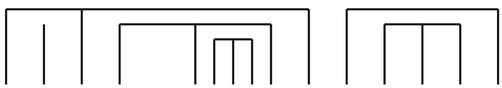


$\emptyset$

On notera que la déconstruction n'est pas unique: on aurait très bien pu choisir, par exemple à la première étape, la partition

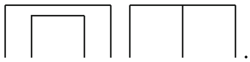


au lieu de

.

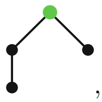
3. L'ambiguïté du processus de décomposition qui vient d'être décrit est intéressante en elle-même et intervient à divers moments de la théorie des partitions non croisées et de leurs applications, par exemple dans la définition de "partitions monotones" (Muraki, 2002). Une autre façon de coder mathématiquement et graphiquement ces idées est par l'intermédiaire de structures arborescentes, un autre type de diagrammes omniprésents en combinatoire et ses applications.

   Pour associer un arbre à une partition non croisée, on remarque d'abord que deux blocs d'une partition non croisée sont soit totalement ordonnés (au sens où tous les éléments de l'un sont strictement à gauche de ceux de l'autre dans la représentation graphique de la partition) ou bien imbriqués (un bloc est "sous l'autre" dans la représentation graphique). Considérons par exemple $\{\{1,4\},\{2,3\},\{5,6,7\}\}$



Le bloc {2, 3} est “sous” le bloc {1, 4}. Le bloc {5, 6, 7} est par contre à droite de {1, 4} et graphiquement indépendant (au sens où il n’y a pas de relation de subordination verticale entre eux). En d’autres termes, la relation “être sous” définit un ordre partiel sur les blocs et permet d’associer un arbre (planaire) à chaque partition non croisée, arbre qui code la hiérarchie des blocs de la partition. Dans l’exemple ci dessus tiré de (Ebrahimi-Fard et Patras, 2016),

est associé à l’arbre

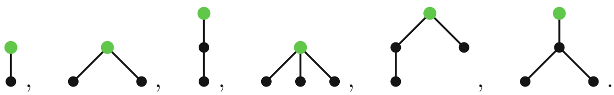


la racine (en vert) étant introduite en sus au dessus de tous les sommets représentant les blocs. On a adopté ici la convention anglo-saxonne qui représente les arbres à l’envers, avec la racine en haut.

Concrètement, on représente chaque bloc par un sommet et on connecte les sommets conformément à la relation d’ordre (partiel) associée au fait qu’un bloc soit sous un autre dans la représentation graphique. Cette association (il en existe d’autres, tout aussi importantes) entre partitions non croisées et arbres n’est pas bijective. Par exemple,

est associée au même arbre que la partition étudiée précédemment. Les partitions

ont comme arbres associés respectifs



De la portée des représentations diagrammatiques

Tous ces exemples de visées accompagnant les partitions non croisées ont la particularité d'être intuitivement évidents, au sens où leur simple énonciation suffit en général à convaincre de la validité du processus général associé. On pourrait multiplier ainsi les exemples d'usage de représentations et du jeu consistant à aller d'une représentation à une autre pour illustrer tel ou tel phénomène caractéristique associé à une notion mathématique.

Que visaient donc à illustrer ces exemples ? Le fait que la visée intentionnelle associée à la représentation intuitive d'un objet mathématique est souvent très riche car elle met en évidence des renvois, des significations, des structures que le formalisme ignore. De façon plus importante encore, c'est cette richesse de la visée intentionnelle qui nourrit la pensée et le travail du mathématicien. C'est à toutes ces intuitions que celui-ci va en appeler pour démontrer des propriétés, des théorèmes, faire des conjectures. Lorsque la structure de la visée intentionnelle est thématisée (prise elle-même comme objet d'étude), on peut faire émerger de nouveaux objets formels (comme les arbres). Cette formalisation n'est toutefois pas toujours nécessaire, et il arrive souvent que l'on se contente d'en appeler à ces structures de façon seulement implicite, au travers d'une construction ou d'une preuve, par exemple.

Le lecteur trouvera une illustration de ces thèses dans les travaux de R. Speicher. C'est à lui que l'on doit d'avoir développé systématiquement l'utilisation des partitions non croisées en probabilités. Ses intuitions et constructions ont ensuite été utilisées et développées dans différents contextes, y compris pour d'autres types de partitions que celles non croisées. Détailler comment les différentes intuitions structurelles qui ont été décrites précédemment interviennent dans le détail de son travail nous entraînerait trop loin ici, mais le lecteur familier avec l'algèbre et la combinatoire s'en convaincra facilement en lisant ses travaux fondateurs et la littérature qui en est issue (Nica et Speicher, 2006; Speicher, 1994).[2]

L'exemple des partitions non croisées conclura cette tentative de confrontation à la pratique mathématique, dans le contexte des représentations, en particulier diagrammatiques, de la méthode phénoménologique, revisitée à la lumière de développements mathématiques récents qui permettent d'en actualiser les contenus. Il conviendrait de multiplier de telles analyses, chaque domaine des mathématiques se prêtant à autant d'illustrations et de développements permettant de dégager des modalités différentes du fonctionnement de l'intentionnalité et des structures sous-jacentes.

[2] L'auteur de cet article ne peut évidemment pas mesurer quel rôle a joué exactement la faculté d'opérer au travers de représentations diagrammatiques chez R. Speicher ou d'autres auteurs du domaine. Ayant cependant travaillé lui-même sur le sujet, il peut certifier que, chez lui, cette faculté a joué un rôle décisif, aussi bien dans la compréhension des phénomènes que lorsqu'il s'est agit d'en découvrir de nouveaux ou de les démontrer, souvent au fil de discussions et échanges étayés par des dessins au tableau.

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Husserl, Intentionality and Mathematics: Geometry and Category Theory

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> *Foucault quotes a text from Borges: [. . .] où il est écrit que « les animaux se divisent en a) appartenant à l'Empereur, b) embaumés, c) apprivoisés, d) cochons de lait, e) sirènes, f) fabuleux, g) chiens en liberté, h) inclus dans la présente classification, i) qui s'agitent comme des fous, j) innombrables, k) dessinés avec un pinceau très fin en poils de chameau, 1) etcetera, m) qui viennent de casser la cruche, n) qui de loin semblent des mouches ». [. . .] La monstruosité [. . .] consiste [. . .] en ceci que l'espace commun des rencontres s'y trouve lui-même ruiné. Ce qui est impossible, ce n'est pas le voisinage des choses, c'est* ***le site lui-même où elles pourraient voisiner***
> *Foucault. Les paroles et les choses.*
>
> ***What pattern connects*** *the crab to the lobster and the orchid to the primrose and all the four of them to me? And me to you? And all the six of us to the amoeba* ***in one direction and the backward*** *schizophrenic in another?*
> *Bateson. Mind and Nature.*
>
> *Un ensemble variant est globalement f: $X_0 \to X_1$. Et sous-jacente à cela, comme un réel dont on ne parle pas, il y a la flèche $0 \to 1$, la flèche du temps, ou de mouvement, ou de changement pur.* ***Au commencement était la flèche****.*
> *René Lavendhomme. Lieux du sujet.*

Abstract The following text is divided in four parts. The first presents the inner relation between the phenomenological concept of intentionality and space in a general mathematical sense. Following this train of though the second part briefly characterizes the use of the geometrical concept of manifold (*Mannigfaltigkeit*) in Husserl's work. In the third part we present some examples of the use of the concept in Husserl's analyses of space, time and intersubjectivity, pointing out some difficulties in his endeavor. In the fourth and final part we offer some

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points of coincidence between phenomenology and category theory suggesting that the latter can work as a formal frame for ontology in the former. Our thesis is that intentionality operates in different levels as a morphism, functor and natural transformation.

Keywords Intentionality · Category theory · Phenomenology · Geometry

Intentionality and Space

Intentionality is the core concept of phenomenology, it describes the essential mode of being of consciousness, its mode of experiencing: "*Die Grundeigenschaft der Bewußtseinsweisen, in denen ich als Ich lebe, ist die sogenannte Intentionalität*, ist jeweiliges Bewußthaben von etwas. Zu diesem Was des Bewußtseins gehören auch die Seinsmodi wie daseiend, vermutlich seiend, nichtig seiend, aber auch die Modi des Schein-seiend, gut-, wert-seiend usw." (Husserl 1950, p.13). This is an early definition of intentionality but has already the ingredients for all further phenomenological investigations. First, one identifies two poles: the subjective (consciousness) and the objective (something). Second, it is clear that these poles are impossible to dissociate, they constitute the fundamental being of consciousness. The I *lives as* being conscious of something. Third, being conscious of something is a condition for all modes of being (*Seinsmodi*), but also other modalities of existence, like presumption, appearance and even not-being (given that every not-being is stated *through* some positive "something").

Intentionality operates as the "absolute stage" in which experience takes place. It is a "space" in a wide sense. Phenomenological investigation can be effectuated only in a particular region called by Husserl my *own sphere* or originary region (*Urregion*, Husserl 1977, p. 158). We gain *access* (*Zugang*) to this absolute region of phenomenological investigation though a suspension of judgement on the existence of the world (i.e., its naïve transcendence) through the so-called *epoché* (Husserl 1977, p. 68). The relevant issue at stake here is the *nature* of this "sphere", of this "region", where the I lives, and where the theater of cogitations takes place and being presents itself as sense (*Sinn*). Concepts like world (*Welt*), world of life (*Lebenswelt*) or horizon (*Horizont*), which Husserl developed later in his oeuvre, will hint also at an ever-growing sphere of *being*, structured throughout levels or strata of foundation and concurrence of different ontological regions.

But intentionality is not only a space or region, but also the very *mode* how things *appear* or *show themselves*. Being modalizes itself not *in* but also *as* the correlation *noesis-noema* (corresponding to the subjective and the objective poles). It would not be correct of speaking of a neutral space "where" cogitations would occur, and different objects present a parade. On the contrary, it is impossible to separate the formal space (or formal ontology, which for the Husserl of the *Logical Investigations* is a *mereology*), the matter (*hyle*) of the experience, and the mode of appearance (the how).

We claim in this paper that intentionally *can* be read in mathematical, specifically, geometrical terms. Husserl will constantly refer to the mathematical concept of manifold (*Mannigfaltigkeit*). But even the very concept of intentionality already includes a spatial dimension. Brentano rescues the concept from its use in the Middle Ages and writes:

> Jedes psychische Phänomen ist durch das charakterisiert, was die Scholastiker des Mittelalters die intentionale (auch wohl mentale) Inexistenz eines Gegenstandes genannt haben [. . .] die **Beziehung** auf einen Inhalt, die **Richtung auf ein Objekt** (worunter hier nicht eine Realität zu verstehen ist) oder die immanente Gegenständlichkeit [. . .] Jedes enthält etwas als Objekt in sich, obwohl nicht jedes in gleicher Weise. In der Vorstellung ist etwas vorgestellt, in dem Urteile ist etwas anerkannt oder verworfen, in der Liebe geliebt, in dem Hasse gehaßt, in dem Begehren begehrt usw. (Brentano 1874, 124–125).

Brentano claims that every mental phenomenon includes its object *immanently* as we can see in Fig. 1. However, immanence does not mean indifference or indistinguishability, since here is a *tension*, a pointing *towards*, a sort of *arrow* between two "subregions". In Fig. 1 we see the unity of a mental phenomenon and its corresponding poles, a mental act and its content, as well as the arrow corresponding to the particular mode of their *relationship*.

Important here is not that there *is* a correlation between acts and their objects, but the *nature* of such correlation. To show the spatial character of intentionality and of consciousness, we will allow ourselves a historical reference to Augustine of Hippo insofar as he anticipates the problematic relationship between mind, space and time in phenomenology. It suffices to remember that Husserl's lessons on the *Phenomenology of the Consciousness of Internal Time* (*PCIT*) begin with a quote of the Book XI the *Confessions*, acknowledging how he discovered fundamental paradoxes on the nature of time. Augustin exposes in this book the perplexity of time: the past is not anymore, the future is not yet, and present seems to be an ever-evanescent point. How is internal time possible? Augustin will argue that we could not have the slightest experience if our soul would not last *in itself*. He thus speaks of a *distention of soul* (*distentio animi*), which inevitably leads us to spatialization of the soul. I do not know myself because I am an extended place which has to be traversed in order to find God in me.

Magnavacca highlights in her dictionary of Medieval philosophy the etymological and conceptual proximity of space and soul in the concept of *distentio*:

> This notion, especially important in Augustine stems from the verb *distendo* meaning to extend or spread [*estirar*], [. . .] both to put in tension and to distract [. . .] Speaking

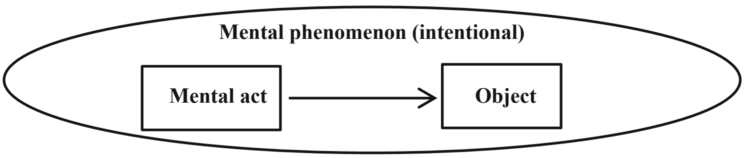

Fig. 1 Diagram of intentionality

> of temporality [. . .] Augustine characterized -not defined- *distentio* as *distentio animi* in Conf. XI, 26, 33. In this context [. . .] we may translate the expression, for example, as 'extension' or 'distension'. In this regard it is interesting to appreciate the crossings between distention of soul [*alma*, *anima*], *extensio* as the realm of spatiality and *intentio*, (that trait of consciousness which will be interpreted in Medieval philosophy as consciousness-of-something). (Magnavacca 2005, p.230).

As we will see, there is also in Husserl's concept of intentionality, a need of "spatial" or "geometrical" approach to explain the "distention" (and differentiated *structure*) of time. The three Latin terms contain the root *tendere*: i.e. to stretch, with three different prefixes: *dis*, *ex*, and *in*. Such prefixes, as most prepositions, add a "spatial" dimension to words. The term *distentio* implies thus a sort of stretching *away* or apart, but without tearing. The term *intentio* shows a tension and a *direction* of soul as it points *towards* an object (we recognize a *Meinung*, in phenomenological terms) and, eventually, a presence (*Erfüllung*, also in the language of phenomenology). In the case of Augustine, this tension points towards a transcendence *of* myself (God) but in myself (interiority).

For Augustine the soul has no form, it does not belong to the world of objects and yet, because it is extended, one must ascribe to it some spatial traits, like continuity, indivisibility, wholeness, etc. Also, soul is not *in* time, it is the very condition of lived time or temporality. In this sense, the soul is *stricto sensu* neither spatial nor timely, but the *origin* of *lived* time and space (temporality and spatiality). Now, if there is some self-constitution, a temporal soul must *return* to itself through memory to assure some *simultaneity*, only possible in *space*.

The Idea of a *Mannigfaltigkeitslehre*

Let us now return to Husserl and evaluate to which extent consciousness but, above all, intentionality, has a spatial or a geometric dimension (intertwined with the temporal) in phenomenology. Concerned with ideal objects, like those of mathematics and with logical necessity (entailment in judgements) Husserl considers, unlike Brentano, impossible to ground them adopting a psychological stance. Logic and mathematics correspond to sciences of *principles*, while psychology (and history) corresponds to *fact* sciences. The empirical is contingent and, as Hume taught, it delivers no necessity, leading to skepticism. Kant's solution grounds causality also in human mind, but it ascribes to it a transcendental value, capable of assuring its inner necessity.

Husserl will demand an *access* to a pure ego capable of granting necessity and universality, but at the same time with no other ground than experience itself. The intentional ego must become the *absolute space of presentation of being* which will take the form of objective apprehensions. To gain access to such a pure *region*, Husserl will call for the necessity of a fundamental "bracketing" (*Einklammerung*), called the *epoché* (suspension of judgment on the transcendence of the world), to move from that world considered as transcendent and empirically existent, to a position in which it appears merely in its essential traits. The world of pure ego-immanence, governed by intentionality, presents essences through reflection on the structure of concrete experiences (*Erlebnisse*).

The concept of intentionality will change along Husserl's work. We will remain for the moment in the early formulations. In *Ideas I*, he describes the whole "spatio-temporal" world of realities as merely intentional being [*bloßes intentionales Sein*] as the absolute region, where all possible *manifolds of appearance* [*Erscheinungs-mannigfaltigkeiten*] can show themselves and be determined (Husserl 1976, p. 106).

What calls the attention in the given quote is the concept of manifold as *what is given* in general, and the concept of intentionality as the *common element or ground* for the appearance of the former. Husserl introduces the concept of manifold in phenomenological sense as early as his *Logical Investigations* (Husserl, 1901). With the focus on the *grounding* of logic, the first meaning ascribed to the concept is oriented towards the construction of *theories*. Different theories correspond to different *regions of experience*, but the emphasis is given to explicit knowledge. He defines thus a *Mannigfaltigkeitslehre* (science of manifolds) as the science of theories or a theory of theories, capable of providing *a priori* the conditions of possibility of *theories* (I stress the *plural* form) in general. This enterprise can be read as Husserl's project of a *mathesis universalis* (Husserl 1901, p.221). The aim here it to depart from purely categorial concepts to obtain multiple concepts of possible theories, to construct pure forms of theories and their reciprocal connections (*Beziehungen*). More precisely Husserl seeks the:

> [. . .] möglichen Formen zu construiren, ihre gesetzlichen Zusammenhänge zu überschauen, also auch die Einen durch Variation bestimmender Grundfactoren in die Anderen überzuführen vermögen [. . .] Es wird, wenn auch nicht überhaupt, so doch für Theorienformen bestimmt definirter Gattungen, allgemeine Sätze geben, welche in dem abgesteckten Umfange die gesetzmäßige Auseinanderentwicklung, Verknüpfung und Umwandlung der Formen beherrschen. (Husserl 1901, p. 247).

As we can see, the *Mannifaltigkeitslehre* pursues a science of science (i.e. a *Wissenschaftslehre*) capable of showing the form of different theories and their interconnections, both dynamical (reciprocal development) and static (connectivity and transfers, or *connections* and *translations* among forms, including ideal variation or *deformation*). But why equate science in general with the study of manifolds? Husserl writes:

> Das gegenständliche Correlat des Begriffes der möglichen, nur der Form nach bestimmten, Theorie ist der Begriff eines möglichen, durch eine Theorie solcher Form zu beherrschenden Erkenntnisgebietes überhaupt. Ein solches Gebiet nennt aber der Mathematiker (in seinem Kreise) eine Mannigfaltigkeit. Es ist also ein Gebiet, welches einzig und allein dadurch bestimmt ist, daß es einer Theorie solcher Form untersteht, d. h. daß für seine Objecte gewisse Verknüpfungen möglich sind, die unter gewissen Grundgesetzen der und der bestimmten Form (hier das einzig Bestimmende) stehen. (Husserl 1901, p. 249).

As we can appreciate, a general theory of manifolds should deliver, first, the objects and connections of different areas of knowledge (which are always referred to areas of experience) and, later, in a more general theory, the connections between different manifolds, taken now as objects to find connections of a higher order.

In his own words: "*Die allgemeinste Idee einer Mannigfaltigkeitslehre* ist es, eine Wissenschaft zu sein, welche die wesentlichen Typen möglicher Theorien bestimmt ausgestaltet und ihre gesetzmäßigen Beziehungen zu einander erforscht" (Husserl 1901, p. 249). Husserl takes the concept of *Mannigfaltigkeit* to be the highest achievement of modern mathematics. But his definition surprises if one considers his own words about the sources[1] of the concept:

> Wenn ich oben von Mannigfaltigkeitslehren spreche, die aus Verallgemeinerungen der geometrischen Theorie erwachsen sind, so meine ich natürlich die Lehre von den *n*-dimensionalen, sei es Euklidischen, sei es nicht - Euklidischen Mannigfaltigkeiten, ferner Grassmann's Ausdehnungslehre und die verwandten von allem Geometrischen leicht abzulösenden Theorien eines W. R. Hamilton u. A. Auch Lies Lehre von den Transformationsgruppen, G. Cantor's Forschungen über Zahlen und Mannigfaltigkeiten gehören, neben vielen Anderen, hieher. (Husserl 1901, p. 250).

It is important to note that Husserl was influenced in his early conceptions by Cantor's set theory. However, Riemann was a constant figure along Husserl's oeuvre and provided phenomenology with a more "geometrical" approach, where relationships among elements are not "imposed" from the outside on a more primitive notion of set, but the belong to the very mathematical object at stake.[2] There isn't either any need to accept *a priori* the implicit ontology of set-theory, like the existence of points as the ultimate constituents of varieties. Another important element to note is the wide mathematical spectrum included by Husserl. Next to set-theory we find n-dimensional manifolds (Euclidian and non-Euclidian), but also Lie groups, approaching thus to abstract algebra.[3]

In the second volume of the *Logical Investigations*, we find another approach to the pure form of objects, i.e., a general theory of something in general (*etwas überhaup*). This objectivity concerns the most elementary forms of objects in general and how they are grounded (i.e. if they are concrete, abstract, grounded or grounding). In the *third logical investigation* Husserl lays the ideas for a general theory of objects (*Gegenstände*) considering different classes of relationships like

[1] Ortiz Hill (2002) writes about Husserl's sources on the concept: "Cantor used the terms 'Menge', 'Mannigfaltigkeit' and 'Inbegriff' interchangeably [. . .] Although Husserl did use the various terms for set interchangeably in the late 1880s, [. . .] he only began to use more frequently in posthumous writings of the 1890s, when he particularly studied geometrical manifolds [. . .] (p. 80)", "by Mannigfaltigkeit Cantor merely meant an aggregate of any elements combined into a whole [by a law] [. . .] [but for Husserl] a Mannigfaltigkeit is an aggregate of elements that are not just combined into a whole, but are ordered and continuously interdependent" (p. 86).

[2] Some contest the importance of the usual references of Husserl's early work, like Frege or Cantor, arguing how Riemann played a key role in all his works. See, for example: (Rosado 2017).

[3] It is indeed possible to think phenomenological ideal *variation* of abstract objects through continuous maps (in topological sense), through functions among sets or even as group homomorphisms. A concrete determination of correspondences between Husserl's mathematical and phenomenological concepts is a hard task and would depend to great extent on speculation. One can try, however, to follow the spirit of his own words extracting the proper consequences. See, for example: (Tieszen 2005), who claims that essences can be thought of as invariants through a set of transformations.

those of parts and wholes, (*a mereology*), subject and constitution (*Beschaffenheit*), individuum and species (*Spezies*), genus and species (*Art*), relation and correlation, unity (*Einheit*), number (*Anzahl*), series (*Reihe*), ordinal number, size (*Größe*), etc. (Husserl 1913, p. 225). Husserl will privilege, however, the relationship part-whole *because it allows to define relations of foundation*, replacing Stump's use of dependent and independent contents.[4] Husserl's theory should then be considered a *formal ontology* or the presentation of the fundamental *formal ontological categories*. But how should we understand Husserl's mereology, i.e., his formal ontology?

In the §10 and §15 of the third investigation he uses again the concept of manifold. Objects in general *are also manifolds* of some kind, included the relationships of wholes and parts. So, one may ask, what is the difference between object-manifolds and theory-manifolds? To answer these questions, we need to introduce a fundamental distinction in Husserl between ground (*Begründung*) and foundation (*Fundierung*). *Begründung* can be translated as the operation of offering a principle from which concrete sentences can be derived or proved (like a theorem). The inverse of this type of foundation or grounding is instantiation. To ground means to offer a general concept or principle capable of *subsuming* particular cases. *Fundierung* means the ontological or logical relationship of *dependence* of some element on another. Husserl writes: "Kann wesensgesetzlich ein α als solches nur existieren in einer umfassenden Einheit, die es mit einem μ verknüpft, so sagen wir, e*s bedürfe ein α als solches der Fundierung durch ein μ*" (Husserl 1913, p. 261). Husserl's formal ontology aims at defining every possible form of objectivity whatsoever. Such objects are of very general nature and are given by mathematics, considering that he uses the term of manifold. There are relationships of order or, more generally, of structure in such objects. But then, what is the link between objects and theories? *The essential question here is, clearly, what does it mean to ground* (*fundieren*) and what is the *Grund*: meaning in German both fundament and reason, i.e. if there is something like *a last instance* a *bottom of being*. It wouldn't be precipitate to affirm that the whole understanding and development of phenomenology revolves around this issue. We should ask to which extent the concept implies a dual (clear-cut opposition ground vs grounded and without one term ever one passing onto the other), vertical (the ground is absolutely first, the grounded is derived) and unidirectional (the arrow flows from the ground to the grounded, but never in the opposite direction). But, as we will see, this concept of grounding, close to how set-theory intended to ground the whole of mathematics, will show insufficient in Husserl, offering in its place a structure with horizontal relationships (among regions) and irreducible layers (i.e. it is not the same to be

[4] This approach could have also been expressed in terms of set-theory by the relationship of membership and the usual operators. Yet, it may be *speculated* that Husserl preferred mereology, since it requires some structure from the outset. Mere aggregates do not constitute meaning or phenomenological experience. A set-theoretical approach would face the problem of introducing form (or structure) from outside and its ultimate elements, ideal points, would be indifferent to their organization in further structures.

a part of some wider instance, than belonging to a new level of organization, like when going from perception of objects, to categorical intuition), being the concept of *totality* displaced by those of *connectivity* and *translation* and making "unity" non-simple and distributed in different regions from the outset. But we will come back to this later.

In the first volume of the *Logical Investigations*, it becomes clear that a theory in eminent sense must constitute a *systematic unity of truths* (considered as contents of thought: *Denkinhalte*). Now, such unity is based on laws and principles, which assure a basic relationship of *Fundierung*. Husserl offers an image and characterizes theories as a "systematic tissue of foundations" (*systematisches Gewebe von Begründungen,*[5] Husserl 1913, p. 25). *Objects* already show relationships of foundation, but a *theory* offers a *universe* of *objects together with their reciprocal relationships*. Considering the *wide* range of mathematical *relationships* offered by Husserl (parts and wholes, subject and constitution, individuum and species, genus and species, relation and correlation, unity, number, series, ordinal number, size) the operation of *Fundierung* is hard to define in unitary fashion. A field of mathematics is rather founded in a structure given by elements and relationships, which constitute "concrete universes", even if they are infinite. We recognize a myriad of relationships among objects, *not necessarily* (or not only) of foundation, which we could equate with the mathematical concept of *morphism*.[6] We could say that intentionality at this level is always a morphism. All this objects and relationships would then "live" or "sit" in a particular space of phenomenological region. Finally, the task of a *Wissenschaftlehre* would consist in providing the *highest (or lowest) level of foundation*, taking theories as its objects and their connections as their fundamental relations. As we see, there are *progressive levels of foundation* which should end in the highest unity. And this unity, who would grant it but the ego itself? If the ego operates always in an intentional structure, this structure should give us the *ultimate space* in which all theories literally *take place* (though not in a psychological sense) relating to each other. This would be the intentionality of higher-order and could be compared in category theory with functors or natural transformations.[7] Jocelyn Benoist (2007) has called the attention to the fact that Husserl did not separate intuition and *categorical* form, as it is stated

[5] Husserl is not always consistent in the use of the words *Begründung* and *Fundierung*. Important is the meaning implied. In this section both meanings could be correct.

[6] The concept of morphism is very abstract and general. In category theory it is defined as a "structure preserving mapping". It is a very general name for "relationship" though *specified* in some context of objects, a category. Morphisms are presented by arrows. Arrows may indicate "static" relationships among collections of objects or dynamic relationships, like transformations or sequences of states.

[7] Functors are morphisms of second order, when categories operate as their objects. Natural transformations repeat the same operation but at an even higher level. MacLane claims to have taken the term "functor" from Carnap (Krömmer 2007, p. 69), where it means a *grammatical function*, a mapping between categorematic expressions. But it might be interesting to consider Hjemslev's, a knower of Husserl, concept of *functor* as an organizer of relations of dependency and *form* as the composition of such dependencies.

in the *Logical Investigations* in the concept of categorial intuition. Benoist tries to ground Husserl's views on mathematics in an anticipation of category theory. However, if we are to take his claim seriously, then category theory should structure all categorial apprehension and in a pre-linguistic way. Or rather, language would be legible only as a structure in the categorial sense.

We took here the *Logical Investigations* as the main framework to understand Husserl's use of the concept of *Mannigfaltigkeit*. A *Mannigfaltigkeitslehre* should provide, as we have seen, the general forms of theories and their connections. But as Husserl advanced in his phenomenological investigations, we appreciate a move from the general aim of founding sciences through an analysis of logic to a general theory of *experience*. In this sense, manifolds will not involve theories, but their *ideal*-lived correlates. In the late text of the *Krisis* (§9), Husserl resorts to the concept of manifold but in reference to the world:

> 'Mannigfaltigkeiten' sind also in sich *kompossib1e Allheiten von Gegenständen überhaupt*, die nur in leerformaler Allgemeinheit als 'gewisse', und zwar als durch bestimmte Modalitäten des Etwas-überhaupt definierte gedacht sind. [...] mit der, wie man sagen kann, die *formal-logische Idee einer 'Welt überhaupt' konstruiert wird*. [...] (Husserl 1976, p. 45).

Here Husserl is not concerned anymore with theories, but with the form of the experienced world. The latter appears to us through totalities of objects, or general forms of some thing. It has been alleged that Husserl privileged the objective and thematic side of experience, as if science would grant the most originary access to the world. But as he moved backwards (through *reduction* and the insistent *Rückfrage*) from general scientific theories to fundamental experience, he displaced his focus of analysis to a more and more indeterminate "background" out of which objective experience would emerge (in the sense of a relationship of foundation). But this dark, passive, non-thematic ground, never fell into absolute philosophical silence. There is, indeed, a tendency in phenomenology to pursue reduction of every positive instance, in order to conduce it to its genesis. Genesis of form itself cannot have any form, and we move towards an indeterminate ground, such as the concept of horizon (a central concept of transcendental phenomenology), over which concrete objects stand out. The very idea of founding requires, to avoid an infinite regress, a last instance, a bottom line, from which everything else is in relation of *some* last dependency.

But Husserl never gave up on the possibility of penetrating phenomenologically this dark ground, resorting to mathematics rather than poetry. The first examples of mathematical areas given by Husserl certainly point towards very determinate forms of objectivity, namely, fields of objects with several axioms to be fulfilled. But mathematics moved from the nineteenth century onwards towards generalization and brought with it the discovery of very abstract and conceptual fields. From the phenomenological point of view a transcendental grounding of mathematics requires a theory of subjectivity involving a locus (the ego) and a set of operations (intentional acts). But it is also true that to describe the structure of the ego requires to speak *of* and *through* some structure or form, mathematics being the most general and powerful language in this respect. Its abstraction becomes its weakness and strength at the same time. There is a certain circularity that did not escape Husserl. At stake is a subject (a space) grounding itself non-thematically but already sitting in some structured "space" (or structure). In this way, the most fundamental space of subjectivity is not a night for thought, but a *minimally* structured space. With this space we mean *almost* a void (some presuppositionless point of departure), *but* with some *restrictions*, which assure a *minimal structure*. But at the same time, such a minimum would not suffice to apprehend the world, but to move across different spaces or modes of being. The world would be such a space of spaces, a multiplicity of variously interlaced multiplicities. This is precisely what mathematical generalization achieves, to think the most general *forms* of being. But we may now ask, aren't these forms or collections of forms closed domains, already constituted universes, derived forms to be brought back to a more originary base? Is there one last instance, the bottom line of all possible constitution? And if there is, does it have the form of a simple unity or does it have "parts"? Or, are there many different "spaces" whose interconnection constitute the common world?

Space and Time in Phenomenology

We devoted the pages above to show how Husserl understood objectivity in general. Now, Husserl gave preeminence to perception over other forms of objective apprehension. Space and time become thus fundamental to grasp objectivity. We will not dwell in the details of his analyses of time and space. Regarding space we want to show that a) constitution of spatiality introduces the mathematical notion of transformation (and thus variation); b) objects are not "lose" things, but they appear in a "region", which provides a priori not only its possible objects, but also the operations (and more generally: morphisms) on and among them (i.e. operations on objects and operations to combine objects). Regarding time, we want to point out that: a) time consciousness has a "form" or an inner relationship to space; b) time "appears" both as a continuous stream and as a discrete structure. We will thus content with providing the examples of the key issues we want to highlight.

In *Ideen I* we find a first approximation to the region of spatial things (*Raumdinge*):

> [...] zu ihrem Wesen gehört die ideale Möglichkeit, in bestimmt geordnete **kontinuierliche Wahrnehmungsmannigfaltigkeiten** überzugehen, die immer wieder fortsetzbar, also nie abgeschlossen sind. Im Wesensbau dieser **Mannigfaltigkeiten** liegt es dann, daß sie Einheit eines einstimmig gebenden Bewußtseins herstellen, und zwar von dem einen, immer vollkommener, von immer neuen Seiten, nach immer reicheren Bestimmungen erscheinenden Wahrnehmungsdinge. Andererseits ist Raumding nichts anderes als eine intentionale Einheit [...] (Husserl 1977, p. 89).

Spatial things appear in a continuum. Further, each one appears only in perspectives (*Abschattungen*, or adumbrations), but they "add up" to constitute a unity. It is clear that this quality belongs not to this or that object, but to all three-dimensional objects. This creates a *class* of possible objects. As I vary the position of my body respect to the object, there is covariation of its appearing perspectives, but I can also rotate it ideally in my mind (or see an object from one perspective and complete it though ideal continuation-deformation in my mind). Mathematically said (see Boi 2004), I can *continuously deform* an object without altering its structure (conserving its unity). This rather simple example points already at the general notion of a category: a collection of abstract objects and morphisms (relationships or transformations), respecting some additional axioms (like associativity, identity and composition).[8]

Spatiality does not only deliver objects and their relative visible perspectives. It is rather a *system of relationships* that allows to identify an object through its variations, to create classes or subclasses of objects (according to dimension, for example), but also to translate one system of relations to another.[9] For Husserl *there is only one world*, not despite, but *thanks* to its different perspectives. It belongs to the experience of a spatial world (but not only) to be given in adumbrations,[10] i.e., partially, incomplete, in the midst of an indeterminate yet infinitely determinable horizon. In this sense, there are not only different possible perspectives of an object for me, but many *actual and different but coexisting* perspectives of the same world, exhibiting various *degrees* of matching. Without *individual subjects* perceiving it, the world would be homogenous. There is an experience of multiplicity and experience as multiplicity. Now, *intersubjective experience* is possible because I can move (i.e. transform) different positions in a system of reference of space thanks to idealization:

> Jedes Subjekt hat seinen ‘Orientierungsraum‘, sein ‘Hier‘ und sein mögliches ‘Dort‘, dieses Dort sich bestimmend nach dem Richtungssystem des Rechts-Links, Oben-Unten, Vorn-

[8] We won’t offer definitions here. For excellent introductory presentations of the subject we remit the reader to Marquis (2009), Goldblatt (1984), Awodey (2010) and, with a psychoanalytic emphasis (Lavendhomme (2001).

[9] For example, I have to translate into and “glue” together my visual to my bodily experience of an object to create a unitary experience. When I *see* the lines of a railroad intersecting at infinity, I operate under a projective geometry framework (non-Euclidian). But when I *walk* along it, I move in the familiar three-dimensional Euclidian space. It is not only possible, but a constant activity, to translate and merge information from two different types of spaces simultaneously lived. We glue information from different sources or “spaces”, like sound, kinesthesias, flavors, odors of the same object or complex situation (*komplexe Sachverhalte*). A further point in this train of thought can be illustrated by the famous art piece “One and three chairs” from Joseph Kosuth. It presents a chair of wood next to a real-sized photograph of it next to a cardboard with the word “chair” and its dictionary definition written on it. A chair is not any of the three objects isolated, but the “knot” tying the three different registers (real, image and language in this case) and the bundle of intentional rays involved in each register. This multiplicity of spaces does not lead to a more fundamental and unitary “ground”, on the contrary, *it constitutes the ground itself*.

[10] As early as in the *Logical Investigations*, non-plenitude was part of experience. Significative intentions are *in certain way prior* to intuitions, for the later fulfil the former. On another sense, intuitions are first, because every possible act is related to some intuition. We can say that experience is always a mixture of empty mentions intentions and actual intuitions, but also that *firstness is not univocally given*.

> Hinten. Aber so ist die Grundform aller Identifizierung von intersubjektiven Gegebenheiten sinnlichen Gehalts, daß sie notwendig einem und demselben *Ortssystem* angehören [. . .], das sich nicht sinnlich sehen läßt, aber verstehbar, in einer höheren Anschauungsart, gegründet auf Ortswechsel und Einfühlung, 'erschaubar' ist. (Husserl 1952, p.83)

I can transform my "here" in a "there", which in turn is your "here", exactly as I do with the pronouns "me" and "you", or the temporal indexes "now" and "then". All relative spatial and temporal (deictic terms) but also grammatical indexes can be reversed, interchanged, or translated into different positions.[11] This already complex conception of space and spatial things is expanded by Husserl to the most general notion of a (single) *world*:

> Ich kann meine Aufmerksamkeit wandern lassen [. . .] zu all den [abwesenden] Objekten, von denen ich gerade 'weiß', [. . .] ein Wissen, das nichts vom begrifflichen Denken hat und sich erst mit der Zuwendung der Aufmerksamkeit [. . .] nur partiell und meist sehr unvollkommen in ein klares Anschauen verwandelt [. . .] Aber auch nicht mit dem Bereiche dieses anschaulich klar oder dunkel [. . .] erschöpft sich die Welt [. . .] Sie reicht vielmehr in einer festen Seinsordnung ins Unbegrenzte [. . .] umgeben von einem dunkel bewußten Horizont unbestimmter Wirklichkeit. Ich kann **Strahlen des aufhellenden Blickes der Aufmerksamkeit in ihn hineinsenden** [. . .] der Kreis der Bestimmtheit erweitert sich immer mehr und ev. so weit, daß der **Zusammenhang** mit dem aktuellen Wahrnehmungsfelde, als der zentralen Umgebung, hergestellt ist. Im allgemeinen ist der Erfolg aber ein anderer: ein leerer Nebel der dunkeln Unbestimmtheit bevölkert sich mit anschaulichen Möglichkeiten oder Vermutlichkeiten, und nur die 'Form' der Welt, eben als 'Welt', ist vorgezeichnet. Die unbestimmte Umgebung ist im übrigen unendlich. Der nebelhafte und nie voll zu bestimmende Horizont ist notwendig da. (Husserl 1977, p. 57-58).

As we stated above, Husserl is ambivalent regarding the world as a totality. On the one hand, it seems to be already contained in the a priori forms of objectivity. Its infinity is really a "bad infinity", just a multiplication of already available (*vorhanden*) objectivities. We should ask: what does it mean to be something? As we saw, it is not the objects that matter, but the "space of possibilities" in which they sit. Space and time can be thought of as infinite manifolds, but it is something different to take infinity *as a form of objectivity or a quality of some region*. There are limit-concepts (totality, nothing, unity, multiplicity, infinity, differential, incompleteness,

[11] Husserl is however ambiguous here. He considers often that objectivity, subjective and intersubjective constitution actually render the *same* objectivity. The question is if "same" means "identic", which would render intersubjectivity superfluous. This issue can be expanded to animals. Husserl ascribed mental activity to "superior" animals, i.e. capacity to constitute a world. We confront here a problem raised by Konraz Lorenz (1941) between the empirical and the transcendental. In phenomenology, my *experience* of things is linked to my bodily constitution, but phenomenological *essences* cannot be determined by my biological constrains. Even though different animals have different apprehensions of space, we could not speak of that diversity if we could not count with a more abstract concept of space in which we can *translate* such different space-constitutions. At the same time, I do not only constitute the world, but I am *part*, a *member* (*Mitglied*) of it (Husserl 1977, p. 58) and thus a part of nature and its history. When considering animals, we must accept that the *transcendental possibility of constitution*, even if it does not depend *directly* from bodily constitutions, it does evolve from nature itself. This would enlarge the very idea of intersubjectivity.

incommensurability, etc.) that are neither empirical concepts, nor categories, but that present the very *mode* of appearing of somethingness. Take for instance the concept of cardinality coined by Cantor. After him infinity cannot be taken as a form of indeterminacy, an "excess" (regarding natural numbers, for the latter is countable and the former is not), but as a positive property of some sets. Even if we cannot experience cardinalities higher than $\mathbb{R}^3$, it belongs to our ideal world the possibility achieving it. In this sense, we can think different "types" of infinity, or "indetermination" of "openness". The idea of an obscure horizon could for example admit different interpretations: as a finite space with fuzzy boundaries, as a non-compact or as a non-simply connected space.

The "world" actually is not directly defined by Husserl. It is more a limit-idea. From the outset Husserl insisted in thinking manifolds as something more than mere aggregates of things. The world should integrate not only time-space manifolds, but also complex states of affairs, aesthetic objects, values, etc. different perspectives, both from different persons (belonging or not to the same tradition) and from animals, in both historical and natural[12] historical perspective. It is obvious that a concept of the world as a single, simply and univocally connected space is impossible. But here we should ask ourselves if there is still place for *intentionality* within the idea of a horizon. "Horizon" is a term than points towards potentiality, openness and indeterminacy, something very different to constituted, present objects. However, as Husserl states, a horizon operates like intentional empty intentions, characterized by limit-concepts like the stated above. A horizon is like a bundle of arrows pointing to potential areas of objectivity. The idea of intentionality as pointing-towards becomes thus more flexible and richer. Even if we can determine objectivities *locally* or by region, the form in which they are *combined* (i.e., the global form of the world) and *arranged* remains open.[13] In the quote provided above about the world there is an interesting concept that plays a key role in his lessons on time (*PCIT*) (Husserl 1969): the "intentional rays".

In *PCIT* we come across the concept of intentional rays as *constituting* the fundamental stream of time-consciousness. We find again a structure of empty and fulfilled, which together constitute a "space" in its own right. Husserl presents the idea of an absolute stream of consciousness, self-constituting and non-thematic (i.e.

[12] For a study of topological forms in nature, both biology and physics, see: (Boi 2005).

[13] Here we can remember Jakobson's claim that language as a structure has *different levels of order* in which the degree of freedom for the speaker grows together with complexity. We are determined at the level of phonemes, but we are freer at the level of sentences. It is not potentiality and indetermination, but differentiation and accumulated orders of relative rigidity what grants more degrees of freedom. This is also true for higher orders of experience: for example, from perception of objects to categorial intuition. Phenomenological reduction does not lead to a last instance, but to the original modes of givenness with their corresponding objectivities and degrees of subjective freedom. But, precisely, because there is no last instance, we should consider a *back and forth* movement between ground and determination. The individual, monadic ego is not a beginning but a *result*, embedded in the intersubjective tissue. But at the same time, it becomes a *new beginning*, *subtracting* or *separating* himself from the common, so that intersubjectivity takes distance from itself.

passive). But here Husserl will rely again on mathematical ideas. First, Husserl claims that time-consciousness, not *time-objects*, is a manifold. As he writes, time has the "Charakter einer einseitig begrenzten orthoiden Mannigfaltigkeit" (Husserl 1969, p.99). This was, of course, not Husserl's last stance on the form of time-consciousness, but it reveals the extensive use of the concept of *Mannigfaltigkeit* when grasping abstract and ideal forms, even the alleged *basis* of all experience. Husserl's analyses of inner time-consciousness begin with *PCIT* but were continued in other important unpublished writings, like the *C-Manuscripts* and the *Bernau Manuscripts*. His aim, as with the phenomenology of space, was to characterize the *form of appearance of time-objects*. But it became clear that what is really lasting along objects is *consciousness itself*. Once we have reduced every content of time-consciousness we end up with its pure flow apprehending itself non-thematically. As we know, Husserl will identify a basic structure of *retention, present and protention*, which constitute a series of moments or phases (*Zeitphasen*). We cannot *separate* this flow in parts, but we can, however, *distinguish* different moments or phases (past, present and future). Time is a continuous manifold but with some additional rules (sequence and irreversibility, while space is simultaneous and path-capable). Husserl says that, if every moment were to definitively pass, without retention, we would forget it, and it would be nothing for us; we would have no continuity of experience. On the other hand, if things would absolutely remain in us, we would listen a "cluster chord": all notes at the same time. Time must pass by and be retained at the same time. Time must include the emptiness of the "already gone" and its virtual presence in memory. Husserl will distinguish between primary and secondary memory. Primary memory is not a voluntary act of bringing into consciousness past events. On the contrary it belongs to presence (it is the relative absence in presence) like a "comet tail". Present could be thought of as a point and memory as a line. But we can't build the continuum by adding points. There is a constant and continuous transit (*Übergang*) and penetration (*Ineinandergehen*) from one moment into the other. In mathematical terms we could say that lived present is *like* an *open set* around an ideal point called pointwise-present, including the neighborhoods of past and present. Now, to distinguish the three main times within a fundamental flow, Husserl *maps* the continuous structure of time with a discrete structure of phases.[14] It is true that Husserl shifts his analysis of time constantly between a discrete and a continuous framework, but this should not be read as an inconsistency, but as a complementary mode of viewing. Time is both continuous and discrete, depending on the perspective assumed. There is no need of trying to ground one mode in the other.

[14] It is true that Husserl conceived of time as essentially continuous. However, he also wanted to stress its inner separation needed so that time actually is "cut" into past, present and future. Time must be continuous and introduce qualitative cuts. We claim he proposes a non-trivial continuity of time. Derida (2010) also showed in his early engagement with Husserl's theory of time, some important homologies between it and Saussure's ordering of signs in a syntagmatic chain.

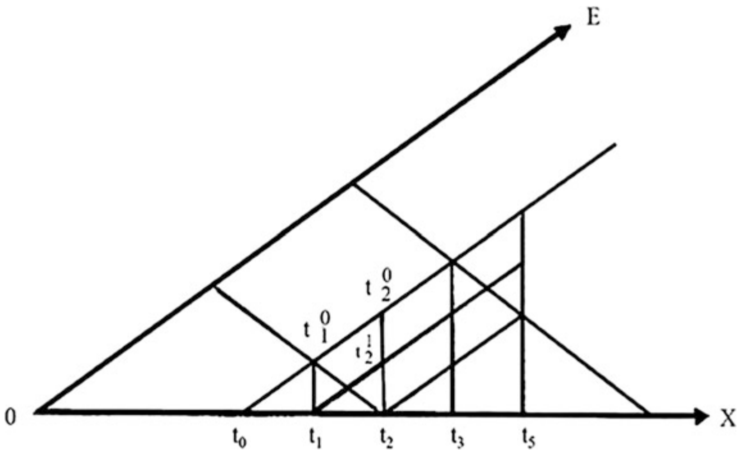

Fig. 2 Diagram of time-consciousness showing memory. Nach: (Husserl 1966, p. 330)

These diagrams drawn by Husserl show that time flows at least in *two directions*: forwards, leaving events behind in a straight line, and downwards (or upwards, depending on perspective), sinking into memory.

In Fig. 2 we appreciate the time series (the horizontal) flowing from t_0 to t_5 ... t_n. But we have also the retention of those past moments in memory: in t_1 we remember (the vertical) t_0; in t_2 we remember both t_0 and t_1, etc. But we do not only remember time, we also have *expectations* of it. The future or anticipation flows first into present and then to past, as we see in Fig. 3. Husserl should have added a third axis to draw future. But he didn't. One can speculate that this decision would have complicated the diagram. But the *product* of the two lines of past and present can deliver a *surface*, an idea that Husserl incorporated in a late diagram called "edge-consciousness" (*Kantenbewusstsein*) (Fig. 4).

This is the surface-model of time-consciousness, but as we already said, Husserl described time also in terms of intentional rays (Husserl 1969, p. 29) in interlacement or entanglement (*Verflechtung* Husserl 1969, p. 83) forming a sort of braid or even a knot.[15] But how is this possible? Husserl explains that future is an empty mention that is fulfilled in the present, just to be emptied again as it sinks into the past (Husserl 1969, p. 83). But it is also true that the past conditions expectations just as expectations condition the past, and that all three phases obey a structure or reciprocal remissions: there are intentional rays going from every time to every other time. Following the image of time as a ray, we could give both a continuous (Fig. 5) and a discrete (Fig. 6) interpretation, see: Romero Contreras (2018, p.199–203). In Fig. 5 we see a single curved line (a *lemniscate*) integrating past, present and future in a *single* flow. In Fig. 6 we appreciate the three time-phases and arrows

[15] See the paper of Hye Young Kim in this volume: *A topological analysis of space-time consciousness: self, self-self. Self-other*.

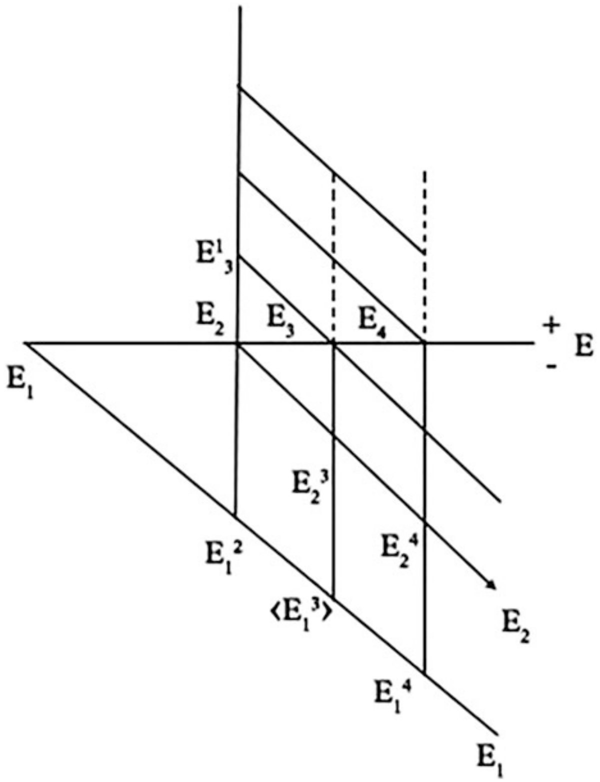

Fig. 3 Diagram of time-consciousness showing memory and expectation. Nach (Husserl 2001, p. 22)

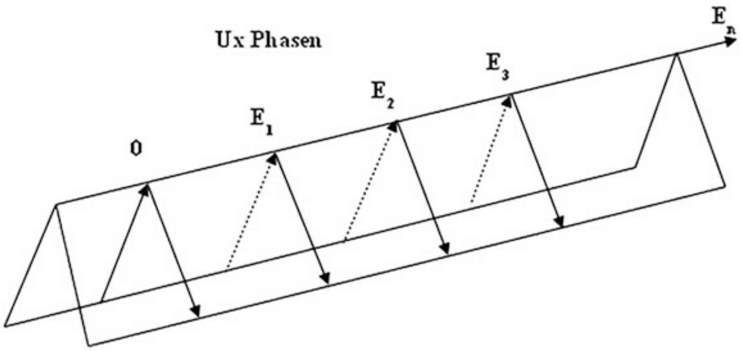

Fig. 4 Three-dimensional diagram of time-consciousness. Nach (Husserl 2001, p. 34–35). The diagram is only described in Husserl's text

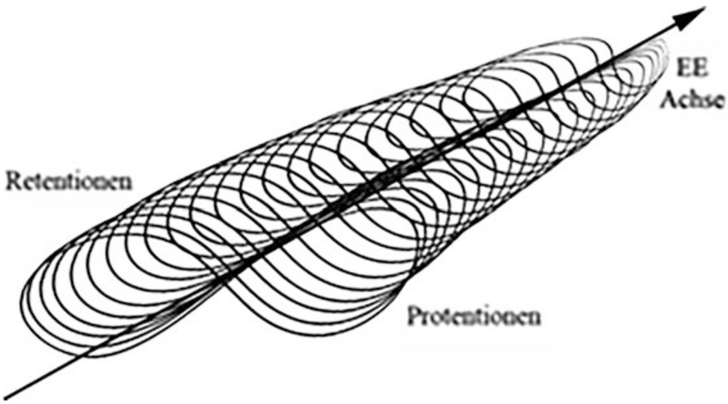

Fig. 5 Our own diagram of time-consciousness as a continuum

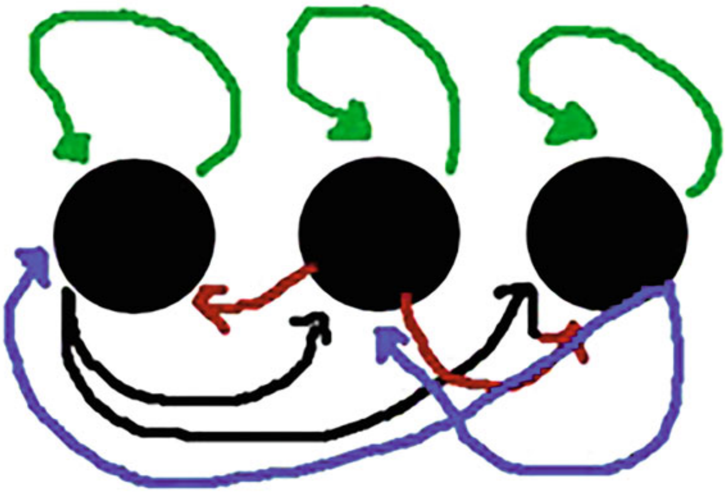

Fig. 6 Our own diagram of time-consciousness as a transit between different times

representing intentional rays going from every time, to every other, included self-reference arrows.

The diagram provided by Husserl where time constitutes a surface has the problem of not being able to show the strict *continuity* of the arrows. We have an infinite number of arrows "climbing" from future, crossing the axis of present (E_X) and sinking into past. But this diagram cannot show how the end of an arrow connects with the beginning of another. In Fig. 5 lines are connected in a lemniscate moving along the arrow of flowing present. A diagram is, of course, only a *projection* of a more complex structure. But we can analyze such a structure *choosing* which trait to show. We can also consider either a topological or an algebraic structure, a continuous or a discrete one to describe time, and *map* *one with the other* instead of choosing one over the other. There is some *hinted* but inconsistent use of an “algebraic geometry” where some types of structures are mapped by others, like when assigning discrete patterns to varieties. But in a second step it becomes possible to establish transits between very different universes of interpretation, (topology, set-theory, logic, groups, etc.). This looks not only close but also very promising regarding what Husserl understood as a *Mannigfaltigkeitslehre*, which now could be interpreted in a phenomenological-categorical approach (see: Peruzzi 1989 and 2006).

Issues of Foundation in Phenomenology through Category Theory

The central concept of all phenomenology is that of *intentionality*, later conceived of as the *a priori of correlation*. We have seen that the side of objectivity always points at *mathematical* concepts.[16] But the ego exhibits also a mathematical as we saw in the constitution of inner time-consciousness. Generally speaking, a subject is only the “who” of experience, the one *living* objectivities (and through its own relative objectivity) in the originary stream of consciousness, whose main *function* is to *ground* the series of objects and relationships in an absolute time-flow. But this isolated ego shares the world with other egos in an intersubjective tissue (there are several perspectives, as we recognized in space constitution and a shared time), and this intersubjective humanity is distributed in different traditions, which are subject to a transcendental history of humanity, which also shares (co-constitutes) the earth with other animals, both belonging to a natural history. At this point, we obtain a complicated intentional structure, where it is not easy to set a last *instance* or a simple structure of layers of experience, one on top of the other, not even a clear directed hierarchy of foundation, not to speak about the absolute relationship of parthood presented in the third logical investigation. We should now ask how to think this *common space of things* and the *imbrication* of *several spaces* of experience.

Husserl identified in this *Logical Investigations* historicism and *psychologism* with *empiricism* (as sciences of *facts*, not of *principles*) and both with skepticism. He also understood the universality of laws of logic, which in turn could gain nothing with temporality. But the “natural” progress of phenomenology demanded

[16] But such concepts do not transform phenomenology into mathematics. It provides only the most abstract and general consideration for objects and relations. Mathematics as a discipline requires phenomenological foundation, but the structure of foundation is philosophico-mathematical. At this point phenomenology seems very abstract, far from every-day life, our body experience, or our emotions. It is not grounded in sensibility or language and does not choose art to express itself. Yet, this abstraction is also a *liberation from* particularity. But we do not impose instead an anonymous generality. There is no such encompassing point of view, but a combination of abstract regions and *modes to transform and connect them*.

to conduce the static to the dynamic (self)constituting ego, this to intersubjectivity and to historical intersubjectivity, and human intersubjectivity came to be embedded in the wide concept of earth. Husserl's late concept of earth (*Erde*, Husserl 1940) should be read as the last expansion of the concept of "world" (with its *horizon*), but also as a general space grounded in nature. Earth does not move, says Husserl. It is, instead, an absolute *point of reference*. In this sense, the earth founds every possible material body relationship that can appear to us. Earth is both an instance that founds natural bodies (place, time and movement) and a ground four our own living body (*Leib*). But it is not clear anymore what does it mean to ground, because this earth only has *sense* as it *appears to us* in its *form*, and yet, we, as humans with our bodies and cogitations, stem from this very earth. This does not mean that we should naturalize phenomenology, but to trace the transcendental at play and its genesis in nature. Grounding becomes a "multidirected" arrow.

We see ourselves naturally confronted with different meanings of grounding (always in the sense of *Fundierung*) and thus with different types of arrows. Heidegger's relationship between being and humans was defined in his late work as "belonging to-" (*Zugehörigkeit*), but there is, despite everything, a main arrow starting in being and flowing in the direction of beings: B $\rightarrow$ b. Coming into being, "desocultation" (*Entbergen*, *Offenbaren*), his translation of the Greek's *aletheia*, all these terms imply this hierarchic and one-directional arrow. This is beginning itself, being *speaking* to humans and only to them. Despite everything, Heidegger seems to be caught in a *simple* relationship of ground-grounded, even if the first means an abyss (*Abgrund*). Heidegger speaks in *Being and Time* about the shared experience of the world coining the term being-with (*Mitdasein*), but this mention is not enough to display the whole problematic of intersubjectivity, where continuities and discontinuities (obstructions) give rise to a complex intentional structure in Husserl. This is why Heidegger insisted in the return to the source (*Quelle*) of phenomena. Like him, many phenomenologists devoted their efforts to bring the constituted to the constituting, to go from beings to being, from consciousness to being or to nature or to some primordial abyssal origin. However, as Fink[17] insisted, this is only the "half" of phenomenology, because the true enigma is

[17] This is precisely what concerned Eugen Fink in his early writings. In his *sixth Cartesian Meditation* (Fink 1988) he addresses what he considers to be the last non-questioned supposition of phenomenology: the ego. But to reduce the ego introduces a shortcut, since it is the ego the only figure entitled to bracket the existence of the world. If this happens, argues Fink, we are not led to being (as Heidegger thinks), nor to a more fundamental space, like the world, but to nothingness. Phenomenology leads to a *meontology* (from the Greek *mé ón*: non-being). This is the real transcendental subject. But, to avoid radical silence, the *task* of phenomenology consists in inventing concepts capable of bringing nothingness back to being. These are the *Entnichtungsbegriffe* (concepts that revert nothingness). Now, such concepts necessarily *ontify* being, offer *forms*. Now, these new concepts are not regular categories. They belong to a *logos hamártikos* or logos of "failure". But failure means here creating limit-concepts, capable of unveiling unthought complexities but always from some perspective. *Entnichtungsbegriffe* are thus paradoxical, self-referenced, ironic, complex or multi-layered (See Fink: 1988 and 2006). This is precisely what contemporary mathematics can offer to phenomenology.

not that things stem from an obscure ground of possibilities, but first, *how*, and second, how do they *last*, not alone, but *in a complex structure of remissions*. How do forms emerge (*morphogenesis*), how they last (*structural stability*),[18] how they transform (*metamorphosis*), how they merge with each other (gluing or patching different spaces), how being distributes in different spaces (types, modes, layers, centers of experience), how these spaces evolve into, connect with and translate into each other. In this sense there is no being in general (B), but several spaces of being (B_1, B_2, B_3 . . .) or universes,[19] which link to each other through different types of arrows (some indicate transformations in time, other functions, other deformations, i.e. morphisms).

We ask now, how should we understand then the main concept of phenomenology, namely intentionality in the light of the former discussion? Intentionality is only the name for an arrow called *correlation*. It is essential to understand that the link between noesis and noema is not that of a fixed presence. There is *variation from both sides*, which assures the richness of the object and a richness of modes of access to it. The shift in Husserl from a static to a dynamic phenomenology revealed that objects are constituted in their becoming, but also that consciousness is constantly shifting or moving through different acts (but also through different regions, modes of being and strata of constitution). There is some type of *abstract* "function" leading from noesis to noema, or at least some *covariation*, in which some invariance is stated. Intentionality is constantly actualized in the correlation noesis-noema and recognized in the §48 of the *Krisis* as the fundamental *a priori of correlation*:

> sobald wir nur anfangen, das Wie des Aussehens eines Dinges in seinem wirklichen und möglichen Wandel genauer zu verfolgen und konsequent auf die in ihm selbst liegende ***Korrelation von Aussehen und Aussehendem*** als solchen zu achten, sowie wir dabei den Wandel auch als **Geltungswandel** der **in den Ichsubjekten und in ihrer Vergemeinschaftung** verlaufenden **Intentionalität** betrachten, drängt sich uns eine feste, sich immer mehr **verzweigende** Typik auf [. . .] [durch] **Weisen der Selbstgegebenheit** [. . .] [und] **Weisen der Intention in Modis der Geltung**, [. . .] in ihren **Synthesen** der **Einstimmigkeit** und **Unstimmigkeit**, **einzelsubjektiver und intersubjektiver** [. . .] [man erkennt] [ein] gewaltiges System neuartiger und höchst erstaunlicher apriorischer Wahrheiten [. . .] [Das Subjekt gilt als] Index seiner **systematischen Mannigfaltigkeiten**. (pp-168-169) [Aber] Impliziert ist [. . .] ein ganzer 'Horizont' **nichtaktueller** und doch

[18] We borrow these concepts from René Thom (1975).

[19] There is of course, the problem that we could simply be dealing with different senses of being. But if being is distributed in spaces, then, a *conditio sine qua non* of a world would be that they "touch" each other, i.e. if there is no totality, no ultimate ground but several interlaced spaces, we need a principle of non-trivial connectivity. Now, the point of thinking in terms of spaces in the mathematical sense of the term is that being does not dissolve in its meaning and in Dasein's apprehension. Being always implies some sort of structure. Givenness is always formed. There is no duration outside form in general. Now the structure of remissions in the world of the Dasein could couple (or be inserted) with (or in) other structures. Husserl recognized that animals are capable of intentional relationships and, as Uexküll and some Gestalt psychologists like Köhler showed it, they may also have an *Umwelt*, which touches the human world in several points and not only through human meaning.

> **mitfungierender** Erscheinungsweisen und Geltungssynthesen [. . .] bald stehen wir auch vor den **Schwierigkeiten** einer **konkreten Entfaltung dieses Korrelationsapriori**. Es kann nur in einer **Relativität** aufgewiesen werden, [. . .] daß unbeachtete Beschränkungen, **manche nicht fühlbar gewordene Horizonte zur Befragung neuer Korrelationen hindrängen**, die mit den schon aufgewiesenen **untrennbar zusammenhängen**. (p. 162) (Husserl 1976).

This quotation confirms the *correlative* character of intentionality, i.e., the fact that all being must be *given* to me (as sense) in some way. Things, even if they cannot be exhausted, must be given in some sense to consciousness, and the subject, even if he is opaque (unconscious) to himself, must also be relative accessible to itself to allow self-constitution (the most basic mode of self-reference or self-relationship). Intentionality does not exhaust any of its poles, but recognises that there are no things at all if they are not for someone, in some sense and manner. There is no what (*Was*) without a who (*Wer*), and both must be brought together *in a context* to which it corresponds a mode of givennes (*Wie*). *What*, *who* and *how* constitute the most elementary set of elements related to each other making sense possible, but not taking it for being. Next to change or transformation (*Wandel*) of things and experiences, Husserl acknowledges the necessary relationship between presence (or actuality) and non-presence (or non-actuallity), but always claiming that there *must be some* presence, givennes.[20]

The personal ego does not exhaust the vast intersubjective experience. Simple presence does not exhaust the complex constitution of time concisousness, which includes the non-actuallity of past and future (retentional and protentional *original* consciousness). Language includes empty intentions and present signs, such that sense is not exhausted by peception. But signs work in abstract systems and are not exhausted (fulfilled) in intuitions. Since animals also constitute their world, humanity does not exhaust being. The seen face of a three-dimensional object does not exhaust the object. There is always some non-presence, some surplus in presence and presentation, but this means, that there is *also* always a relationship to presence in otherness. The task of phenomnology is not to describe pure presence, but *the general space of being through different types of presence* together with the *implied forms of non-presence or non-actuality*. This is the enigma of the link operating in intentionality. As Barbaras points out, the apriori of correlation should not be interpreted in an idealist wake.[21] It is also clear that realism is not refutated,

[20] The same is valid for intersubjective experience. If there is some common, it also must appear in some way (direct or indirect, even as sign or symptom) in my *personal* experience. I am not *everything*, but a point of contact, a site for appearance or hearing (in the legal sense), a point of intersection of rays stemming from relationships that start before/outside the ego and continue after/outside it. But once something appears, it gains a life for itself in the “mental” space, where it intertwines with objects in bundles of relationships.

[21] This is clearly a contentious issue in phenomenology. Speculative realism has objected phenomenology in this point. But we should here remain attentive to Husserl’s own path to avoid both naïve idealism and realism. Transcendental idealism constitutes Husserl’s position. The transcendental approach is needed to avoid a fall in empiricism and to confuse a *quid juri* with a *quid facti*. What remains open if there is something like a transcendental realism,

but *mediated* and *complicated* by subjective givenness.[22] The relationship subject-object is not the absolute space of being, its main locus. It is, rather, *a* place, where intentional rays, surfaces, groups of abstract objects *touch* the subject (that means givenness).[23] Husserl himself spoke of time as a *braid* of intentional rays (*Verflechtung intentionaler Strahlen*). Objects are structures, like rays or surfaces or groups, and the subject is the place in which they "knot" in "bundles" and "braids" or other structures of relationships, but not their absolute origin. Barbaras writes:

> En termes husserliens, la corrélation entre l'étant transcendant et ses modes subjectifs de donnée est un a priori universel [. . .] on ne préjuge en rien du statut exact du sujet de l'apparaître ou, plutôt, on souligne que **la référence de l'apparaître à un sujet ne compromet pas l'autonomie de cet apparaître**. Que l'apparaître soit destiné à un sujet ne signifie pas encore que **ce qui apparaît soit constitué *au sein* de ce suje**t et que sa teneur d'être propre soit finalement celle de la conscience et de ses vécus. (Barbaras 2012, p. 49).

This means that some *degrees of freedom* exist *both* in the subjective *and* in the objective side of the correlation, i.e. the subject is "more" than the object because it can apprehend it in different forms, or ideally variate in several ways, allowing constructions of higher order (and also of creative re-ordering); but the object is also "more" than the subject because it never gives itself in totality (it belongs to my experience the inexhaustibility of my horizon of meaning). Intentionality is thus a correlation (some sort of partial or local interaction), or more precisely, an abstract function relating subjective acts (including its "spaces" or regions, its temporal character and the variation and association of different acts) to appearances of actualities (modes of objectivity) in particular spaces (regions). There is nothing "ontological" said about the nature of the subject or of the object, for being as sense is always only a *how* of one relating to the other. We do not affirm that there is nothing but representations, but that every being "testifies" through some (re)presentation, and this means an *encounter*. Subjective *constitution* means *a place* in which different intentional relationships *gather* in a particular context or space. But again, intentionality does not exhaust either what a subject (or an object) "is" or *can* be. It is not that subject and object are something in themselves, absolutely separate from correlation, but rather that there is a *multiplicity of objects (modes of "somethingness") and types of subjective acts, a multiplicity of correlations* (between objects, between subjects, between objects, within subjects and within

as Schelling characterized his later philosophy, supplementing transcendental philosophy with *Naturphilosophy*.

[22] *Speculative realism* has accused that the *a priori of correlation* constitutes a circle that prohibits an access to the real. However, as Graham Harman (2002) has pointed out, a realist should not *eliminate* the correlation, but *extend it to all* beings to include object-object relationships but affirming, as Barbaras, that no correlation (no encounter or relationship) can exhaust things. And we would add . . . nor the subject.

[23] In this sense we could we establish a parallel between Heidegger's and Husserl's notion of givenness as inseparable form otherness. The "es gibt" could be precisely represented by an arrow of unilateral giving.

objects) and a *multiplicity of spaces, which in term may relate to each other through a multiplicity of relations*.

Following some key reflections on Husserl's phenomenology, we have intuitively arrived at core ideas of category theory. We have made scattered references to mathematical concepts throughout the text. We made special emphasis in the notion of manifold in Husserl, involving not only Riemann's geometrical interpretation, but also set and group theory, as he himself acknowledged. It was clear from the very beginning that Husserl tried to bring together different branches of mathematics to offer a general concept of objectivity in general. But now, *the aimed unity of objectivity cannot be granted without the unity of mathematics in which it rests*. This leads us to the historical context in which Husserl begun his phenomenological project, namely the crisis on foundations of mathematics, which mutatis mutandis entailed a crisis in the foundations of science in general. A ground should grant firstness (difference ground-grounded), unity (difference unity-multiplicity) and a structure of foundation of multiplicities (an order of being). Husserl advanced the ego as a transcendental solution to the problem of foundation but, as we saw, the very idea of grounding (*Fundierung*) depends of ideal forms which should assure objectivity. In every effort to surmount constituted (i.e. scientific) objectivity, as it is the case of the constitution of inner time-consciousness, Husserl resorts to the concept of manifold. The reason is not that science should grant the main access to being, but that being in general is always given in some *form*, without which there would be no phenomenology, but only its shadow. The central misunderstanding here lies in the concept of objectivity. It is normally understood as a figure, capable of being manipulated and positively presented. Against this, phenomenology directed its efforts to unravel the realm of the non- or pre-objective. But in the end objectivity is nothing but the form of presentation, even if this form is vague, without clear borders and indeterminate, it is the structure without which matter (*hyle*) would sink in darkness and silence. Giving up on objectivity means to give up on presentation, intersubjectivity and above all, *form*. Category theory is the most ambitious enterprise to expand the concept of structure in mathematics. It does not deliver an ontology, i.e. a set of objects and relationships, but a multiplicity of them and modes to relate each other.

Husserl distanced from Kant in a very precise point: he would not accept a set of constituted ideal forms pertaining to mind or understanding that would apply to sensible matter (*hyle*) from the *outside*. However, Husserl seems to claim that material ontologies only deliver the places of instantiation of pure forms, pertaining to formal ontology. This means that although experience is always material, because it takes place in a particular region, with its own rules and modes of givenness, it can always eventually be *subsumed* in ideal forms of objectivity regardless of the region involved. This would render materiality phenomenologically irrelevant and it might even reintroduce the classical dualism form-matter. In the same line of thought, intersubjectivity seems to pose an important problem for objective constitution, since it involves different perspectives, mediation of signs and transmission along history. Husserl acknowledges the problem of otherness and the impossibility of constituting the alter ego originally, such that intersubjectivity has the need of *mediation*. But he claims also that every ego constitutes the world in *identical* ideal manner and that changes of position and perspective of observers all belong to a single system, so that objectivity and mutual agreement suffer no risk. This is already contentious, but a fundamental issue considered in the *Logical Investigations*, signs, would hardly be explained by this reasoning, and, as we move from temporal and spatial beings to other layers of constitution involving values and opinions, arriving to what Husserl considers more important: ethics, it seems impossible to hold the same objectivity claims, making multiplicity superfluous.

In mathematics, it was Hilbert's axiomatic view that imposed as an answer to the crisis of foundations. Later, it would be set-theory the mathematical language in charge of grounding the whole field of mathematics. Here, founding meant choosing a universe of objects (sets), relationships (functions) and a small number of axioms, with which all mathematics could be *derived*. We don't find here any strict ontological definitions (although there are ontological *implications*) of the primitives, like "set"; signs are devoid of all meaning outside a formal system; and there is no claim that axioms are "evident" or "universal" but just a convention. Husserl reacted against this reasoning and reintroduced a transcendental subject to grounding mathematics in experience but retaining from the logical school, its critique of psychologism and historicism. We showed at the beginning of this article how set-theory could not serve the purpose of phenomenology for it is devoid of all meaningful content for a *subject*. But the idea of grounding was put in peril by the discoveries of logic itself. Already Cantor had discovered the paradoxes of infinity, which Russell only found to be operating in (naïve) set-theory appearing in the case of self-reference (i.e. sets counting as members of themselves). The last chapter in this history is to be found in Gödel's theorem of incompleteness of arithmetic, where he proved axiomatic mathematical systems to be either inconsistent or incomplete.[24] Next to the inherent problems of set-theory inherited by its axiomatic constitution, it also proved to be insufficient to *encompass* all the mathematical universe. Category theory appeared in the mid-twentieth century to contest the centrality of set-theory in the foundation of mathematics. But this is possible because *the*

[24] We may remember here Alain Badiou, who also draws on mathematical concepts to ground his ontology. However, he seems to be fixed to a *particular region* of mathematics, precisely set-theory *and* to classical logic. He claims that ontology must assert multiplicity as its most fundamental concept (a formless void). But if we start with a formless being, a "pure multiple" as he states in *Being and Event* (Badiou 2005), not being grasped by any unity whatsoever ("the one"), we have to explain how concrete beings emerge, how determination takes place from the outside. Remaining faithful to Heidegger's ontological difference, being must be pre-objective and pre-subjective, indeterminate but determinable in different manners (in concrete beings). But in this view concrete beings have no subsistence at all outside the contingent filed of interpretation. Moreover, Badiou's ontology (re)introduces "the one" at many levels. It ties ontology to *one* single logic and to *a single* field of objects (those from set theory). It *unifies* ontology in *one* formal system. It establishes the ideal points of set theory as the *ultimate single* constituents, etc. Husserl's ontology, on the contrary, retains multiplicity on another level, namely as a *plurality of regions and levels of constitution*, a plurality of spaces, instead of a plurality of points in a set. See Plotnisky (2012) for a related discussion.

meaning of foundation changes when moving from one theory to the other. Set-theory allows to define notions precisely once the axioms are *decided*. There is no ground for deciding axioms, only their *usefulness* in a mathematical field. We then seek to express different regions of mathematics in the common language of sets and functions. But category theory operates on another level of abstraction and constructions may sometimes be arduous and complicated. We already said that we won't explain the main concepts of category theory, we will rather characterize it in a philosophical fashion to show how phenomenology sometimes approached to it and how sometimes it *could* and *should*.

Category theory does not rely on an (quasi)ontology as set-theory does. The simplest category counts only with a collection of objects and a collection of morphisms between them (respecting the very general axioms of associativity, composition and identity), without having to state *what* that objects and morphisms *are*. Objects are defined by what and only insofar as they are *for another* (or many other) object(s). This being-for-another is specified by a particular *morphism* or a collection of *several* morphisms in a single category. In this manner, "what there is in the world" remains open, but always related to some encounter or correlation. Categories are like universes, in which certain objects populate a space, obeying certain rules.[25] This variety of categories or spaces and not the sheer and unstructured "multiple" of set-theory seems suitable to do justice to the multiplicity of being. Category theory states that at first glance very different domains actually obey the same categorical structure. This makes the afore mentioned domains to be *relatively commensurable* applying the same categorical structure, but at another level, called functors. The development of category theory owes much to algebraic geometry, while trying to associate algebraic structures to topological spaces in order to extract information. Phenomenology must fulfill the *double* task of finding *common essences* of objectivity across different regions *and grounding* essences in particular regions. Category theory would provide phenomenology with a general frame to think objectivity along very different concrete domains.

We stated above that in category theory objects are not defined from the inside, it is not important how they are "internally constituted". An object is only "revealed" through its relationships (morphisms) with other objects. This seems a rather precise definition of manifestation or appearing. However, objects retain possibilities form them, not always expressed in a category, but to appear in correlation with other objects. This possibility expresses in other terms Barbara's concern of avoiding a reification of consciousness. An object may appear in several ways and establish several relationships with other objects. By the same token, a "subject" is always revealed not by its abstract "possibilities" but by the objects and subjects he relates to and by the concrete relationships established in the concrete fields that serve as

[25] To explain category theory, one usually gives some example: in the category of sets objects are sets and morphisms are functions; in the category of topological spaces objects are topological spaces and morphisms continuous functions, in the category of groups objects are groups and morphisms are group homomorphisms, etc.

"stages" of experience. Phenomenology may thus be compatible both with realism (an object is always more that the concrete correlation in which it appears), with a procedure (it investigates the varied modes of givenness and their correlations, without further assumptions) or with ontological pluralism (modes of appearance are modes of reality and a multiplicity of the former implies a multiplicity of the latter).

We can now advance the thesis that a category *expresses* the most general form of intentionality (and it can be complicated or scaled in complexity *ad libitum*). To achieve this, one must step out of the exact mathematical definition and extend the idea to a conceptual correlation, i.e. Categories show universes of objectivity (regions) and seem to obscure the underlying subject. But a category, with its objects and relationships is more the *result* of the interplay of subjective acts and modes of being, which is continued in higher levels (functors and natural transformations). Objectivity does not mean simple objects, but a set of possible transformations of the object and between objects. Since categories imply relationships between objects (which may be spaces, sets, groups, etc.) we always count with *structured universes of objects*. This means also that not every universe of objects is *reduced* to its categorical structure. Every region remains singular. This is precisely what phenomenology looked after: a general theory of objectivity capable of respecting the peculiar modes of appearance within the limits of a certain region or "universe". The idea of a structure-preserving map played already a role in Husserl's analysis of space and intersubjectivity: objects vary according to perspective, but this change preserves the structure of the object, both in me, when I change position respect to the object, and among persons, who despite never being in exactly the same place, ideally variate positions in a system to make common experience possible.

As we find in the first volume of Husserl's *Logical Investigations*, a *Mannigfaltigkeitslehre* should deal with theories, not with direct objects. And since theories are linked to (directly or indirectly, but always grounded in) regions of experience, a phenomenological theory of objectivity should analyze the region of regions, the space of spaces. Now, this raises the question about the *possibility* of such a *unity*. If we said that the unity of mathematics cannot be granted by set-theory, what does category theory has to offer in this respect? Category theory does not offer a super-theory of mathematical objects, capable of encompassing every thinkable domain, nor does it seek a theory of ultimate elements. It is an abstract theory that offers a *common language* for the most distant domains of mathematics to establish equivalences that not only allow to identify common features, but also to solve problems of one domain by recourse of another. The unity of mathematics is not achieved neither top-down (a theory capable of subsuming all others as mere cases) nor bottom-up (defining absolute elementary constituents), but "horizontally". Making abstraction of some features, we may render two mathematical (or objectivity) universes comparable to perform different operations that would be impossible from the outset. In the first formulations of phenomenology the ego, or better, the relationship noesis-noema in its constituted ideality was meant to be the absolute space of manifestation. Later intersubjectivity pointed at a *distributed* (and nor reiterated or repeated) subjectivity. Also, genetic phenomenology had to accept not only constituted forms, but also their *process* of constitution (emergence, genesis) (and not only of objects, but of forms of objectivity) and, eventually, a form of time and a form of genesis or change. Correlation, translation, gluing, transformation, etc., may be conceived of as morphisms. Locality, perspective, incompleteness, and multiplicity are part of the world, but they are not unsurmountable differences. Phenomenology traces the "graph" of a back and forth movement from equivalence and obstruction, identity and difference.

Of course, one thing is to be locally bounded and another to have the *concept* of boundedness, to be incapable of uniting a multiplicity and to have the concepts of unity, multiplicity and impossibility. Now, phenomenology must do *both*: it must explain what multiplicity is in terms of objectivity and show how this is *experienced* by synthetic acts. In this sense, phenomenology not only needs ways of transforming objects both in space and time preserving their structure, but also connect different regions of experience to effectuate translations and equivalences. Husserl insisted on identity and on how it assures objects to be identical for everyone. But mathematical objects as well as values depend heavily on the category or *context* they operate. If we move from one domain or region to another, or from one mode of givenness to another, equivalences are not always possible. This is another important feature of category theory derived to great extent from topology and function-theory: there are degrees of likeness. Just as functions may be bijective, injective or surjective, we may find in categories different types of relationships (morphisms) between classes or objects, like monomorphisms, epimorphisms, bimorphisms, retractions, sections, isomorphisms, endomorphisms, automorphisms. The idea is to formalize the *degree* to which two objects or categories are "alike" or how "similar" is their structure and under which criteria. Instead of the metaphysical idea of unity (of the cosmos, God or the subject), phenomenology could investigate the local sites of manifestation and then show how they relate to each other through laws of *equivalence* and not through a simple all-encompassing unity. Identity and difference appear as the two poles of isomorphism and radical obstruction. This mode of approaching equivalence allow to address two phenomenological subjects: analogy (Romero Contreras 2016) (for example between bodies in intersubjectivity) and translation (from one domain to another, but also from one language to another). Let's not forget that objectivities include not only isolated objects, but their combinations in complex situations, i.e. a "grammar". We may very well share the "objects" we are speaking about (a river, for example), but all the connections of that river with my beliefs, my expectations, my social class, etc., are part of complex objectivity, which can never coincide exactly with that of others. We understand partially each other not because we share some objects underlying our discourse, but because we can achieve or produce partial overlappings of our universes of discourse and experience. The so-called "fusion of horizons" from which Gadamer speaks can be better understood in category theory, especially through the concept of sheaves.

Sheaves are an abstract concept in mathematics, developed (and expanded into topos theory) in the language of category theory by Serres and Grothendieck. Sheaves are method to obtain global from local information associating some rich objects to the open sets of a topological space. We may, for example, associate rings to the open sets of a topological space to apply the properties of the former, to better know the latter. We have a sheaf when we apply an arrow (an inverse) going from, in this case, the open sets of an algebraic structure like rings to points of the topological space. In this way we can introduce, for example, order relationships in the latter. A point ceases to be a dimensionless object without structure. Being associated to an open set of another structure, our space becomes richer. We combine geometry (relations of proximity and distribution of points in space) and algebraic structures (richer structure, operations). We can thus construct a space with arrows.

One of the important uses of sheaves in philosophical sense relies on the possibility of obtaining global from local information. After we associate an algebraic structure to a topological space, we can glue isolated parts to produce a global, smoother section. We are not *subsuming* a complicated space into a simpler one, nor we reduce information to force global coherence. The procedure, intuitively speaking relies on associating richer structures to local parts of a space and the gluing that parts in a complex global and richer space. Fernando Zalamea defines sheaves in a philosophical wake as follows:

> The ancient philosophical question "how to move from the multiple to the one'" [...] (phenomenological transit) becomes the mathematical question "how to move from the local to the global?" (technical transit), which subdivides in turn into the questions: a) "how to register differentially the global?" and b) "how to integrate globally these registers". When addressing question *a)* **analytically** we obtain the natural mathematical concepts of neighborhood, covering, coherence and gluing, while, when addressing **synthetically** question *b)* we obtain the natural mathematical concepts of restriction, projections, preservations and sections. Presheaves (term coined by Grothendieck) cover the combinatory of **discrete** links neighborhood/restriction and covering/projection, while sheaves cover the **continuous** combinatory linked to the pairs coherence-preservation and gluing-section. (Zalamea 2009, p. 161).

It seems possible, for example, to glue together different first-person perspectives though association with other structures like language, thanks to local overlappings, possible by an ideal system of space. We do not exchange science directly through evidences but by linguistically structured reports. But at the same time, we cannot aim at global concepts, without being faithful to things themselves. Zalamea proposes a back and forth movement between difference and synthesis, which could be complemented by Finks back and forth movement between reduction (following *ontological difference*) and expression (following a movement of *ontification*).

We arrive finally to the difficult concept of *topos of Grothendieck*. As noted above, this is not the place to mathematically consider the subject. We want, however, to highlight a big possibility reserved for phenomenology in this regard. We referred to the notion of sheaf. A topos generalizes the ideas behind sheaves and pre-sheaves, making possible to associate not only algebraic structures to topological spaces, but to relate via functors virtually any object of mathematics to any other. One of the most surprising results of Grothendieck was the possibility to associate mathematical structures to different logics. A topos is equivalent, for example, to intuitionistic logic. If we further generalize the concept of topos, we can also generate structures with different sub-object classifiers allowing thus other types of logics. While Husserl aimed at grounding "logic" in general, as it follows from his attempts in the *Logical Investigations*, the question of genesis also affected the *contents* of both logic and mathematics. This means that even if "the formal" operates in intuition from the outset, concrete mathematical and logical structures are historically grounded and may also experiment further modifications. However, it is not clear whether "genesis" meant also a multiplicity of valid coexistent frameworks. His reflections were constantly led by *one* single logic, as geometry had been led by one single geometry. Just like geometry experimented a revolution in the hands of Gauss and Riemann or Lobachevsky by suspending (not adding or inventing) an axiom (the one referring to the parallels) it was considered the possibility of expanding the domain of the reasonable by suspending in a regulated manner the principles of non-contradiction and excluded middle. And just as non-Euclidian geometry supposed any space to be *locally* Euclidian, emerging from "patches" of it, it is natural to defend the hypothesis that classical logic may be *locally valid* (i.e., bounded to some abstract space or domain), while some phenomena may exhibit more complicated forms. Phenomenology was constantly caught between two poles: on the one hand, the need of find universal forms capable of assuring the univocity of the world at least in its formal aspect; and, in the other, the need of grounding even the most general laws in concrete experience and this in concrete spaces and moments of history. Topos theory allows to do justice to this double exigency (see: Caramello, 2016). Topos theory bounds logic to a "context" and the material constrains and possibilities of a particular region, but at the same time, thanks to its degree of abstraction, it allows to effectuate *transits* between regions and logics. But Husserl's ideas of some "protogeometry" operating in perception may well demand a "protologic" to explain categorial intuition, both operating like a "formal landscape" of possible determinations.

When one takes a closer look to the history of phenomenology, it is easy to find a clear tendency against all forms of objectivity and formal thought. It was tacitly accepted that the task of phenomenology consisted in moving "back" towards an open origin which could deliver us from the closed and exhausted forms of scientific and objectifying thought in general. It became common sense to claim that the "constituted" forms of thought constrain a more fluid and indeterminate state, which could be seen as "constituting" and of more originary nature. For this reason, time became the model for openness, indeterminacy, and possibility, but also of subjectivity, while space was interpreted as the realm of exteriority, quantification and rigid structures. But this tendency only shows how philosophy distanced from the most powerful insights if contemporary mathematics. Already topology and its associated flexibility to conceive space made it a suitable tool to study dynamical systems and change. If we think of a meagre definition of space, like that of topology, fluidity, indeterminacy and openness can be formally addressed and articulated. But this is possible thanks to *formal and exact approaches*. The fluid becomes thinkable through formal inventions of thought. It is rather the constant movement from fluidity to structure and back what provides thought its power. For the same token we should not give more privilege to the possible and amorphous (or poorly structured) than to the actual and formed, but rather to the *transits* between those poles. Rigidity makes more complex systems possible, but less structure also allows freer transformations, deformations and types of motion. The actual carries its own possibilities not despite, but *thanks to its constraints*. To change form, a constraint must be left behind to find another. In phenomenology, mathematical tools opened deep possibilities of analysis. This choice turned later into an obstacle for its development. Now category theory and topology offer new means to reconsider Husserl's ambitious task.

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Diagrams of Time and Syntaxes of Consciousness: A Contribution to the Phenomenology of Visualization

Carlos Lobo

Abstract The linear representation of time became undoubtedly a dominant representation of what is considered as a fundamental dimension of any natural "external" phenomena as well as lived experiences. It enters into a considerable number of graphic representations and, under the form of the coordinate system, it has provided physics with the indispensable tool for a fine mathematisation of physical processes in their rich diversity and still seems necessary to understand its applicability.

In what follows, we will address three problems: 1. The historical and scientific significance of phenomenology from the point of view of the diagram of time. 2.Phenomenological elucidation of the use of diagrams in science, because there is a phenomenology of diagrams of knowledge, the most radical form of which will consist in a self-elucidation of the use of diagrams in phenomenology. 3.Elucidation of why and how symbolisation and diagrammatisation are required and call each other in the phenomenology of time in particular, description in reflection, 4.Wewill thus be able to approach the specific study of the way in which diagrams of time have functioned in phenomenology between the most fertile period on the subject, i.e. between the lessons of 1905 known as the Lessons on the Intimate Consciousness of Time and the said Bernau Manuscripts of the years 1917–1918.

Keywords Diagrams · Time · Consciousness · Intentionality · Symbolization · Writing

In science, at least from the perspective of modern times, the linear representation of time became undoubtedly a dominant representation of what is considered as a fundamental dimension of any natural "external" phenomena as well as lived experiences. It enters into a considerable number of graphic representations and, under the form of the coordinate system, it has provided physics with the indispensable tool for a fine mathematisation of physical processes in their rich

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diversity and still seems necessary to understand its applicability. The role of this diagrammatisation of time remains nonetheless ambiguous, an ambiguity which modern philosophy has never ceased to confront, but without managing to master it.

This is exemplarily the case of the schematisation proposed by Kant, with whom this representation finds both one of its deepest justification and its most problematic and inadequate character. Indeed, as the ultimate source of the *schematism* of pure concepts and as a condition of the construction of mathematical concepts in the intuition of space, time is the *a priori* form of sensibility and as such conditions the applicability of any concept to external and internal experience. Any epistemological problem, be it physical, chemical, psychic, biological, economical, historical, etc. depends each time on the form and the scope given to this schematism. Whether to promote it or to criticise it (Heidegger or Bergson), there seems to be a general agreement that this schematism is a *condition of possibility* for the constitution of science. Everyone seems to agree that this is the *primitive and ultimate schematism* of time. However, it seems to be unable to support any schematisation other than that of a continuous, unidimensional, unidirectional, homogenous line, etc. which is something spatial, and lets escape almost all the phenomenological characteristics of time (its "flow" and "flight", its irreversibility, a certain unpredictability, the asymmetry of its so-called "dimensions" which are the present, the past and the future, etc.). It is certainly possible to complete this linear representation by adding determinations (arrows, additional axes, etc.). But the diagram will be all the more improper and inadequate, as it is attested by Kant's position on the impossibility of a mathematization of lived experience and thus of proposing, even indirectly, a "geometrization of the soul", because of its alleged one-dimensionality.[1]

However, modern science and philosophy have inherited an historical decision that Weyl sums up perfectly in one of his key epistemological formulas: the invention of the coordinate system, which is so decisive in the enterprise of mathematising phenomena that Galileo initiated, is the ineliminable residue of the elimination of the ego.[2] It is like the cipher of the central enigma of modernity and modern science, concerning the relationship between subjectivity and objectivity. This cipher concentrates especially on the so-called line or arrow of time. This essential tool is, as Weyl phrases it, the ambivalent symbol of an *erasure of subjectivity* and at the same time the *ineliminable residue* of that same subjectivity, *within* the mathematical symbolical construction. In the following, I have tried, in the footsteps of Husserl and Weyl, to unfold this enigma and formulate it as a philosophical problem, as I did in a previous work for the problem of space. Yet, the problem of time cannot be posed in symmetrical or parallel terms, even if, as every

[1] Preface to the *First Metaphysical Principles of Natural Science*.

[2] We won't enter here into the debate whether we could push mathematization a step further, by eliminating any coordinate system, and whether this does not evacuate by the same token the understanding and motivation of the "applicability" of mathematical models "to reality", or less naively stated, if it does not contribute to render even more unreasonable their so-called efficiency. On this particular point, "Space and Time coordinates: Residuum of the Annihilation of the Ego", Ryckman (2005, 128–135).

physicist knows, a physical coordinate system, insofar as it expresses an aspect of physical processes, necessarily includes in an explicit or implicit way at least one temporal axis (considered most of the time as a schema of the real line). Whether we see in this schema the most solid expression of what temporality is for a subject or for the *Dasein* or, on the contrary, the symptom of the ignorance, we cannot but be struck by its permanence. This schematism of time has undergone in recent times a few metamorphoses and enrichments, according to the mathematical elaborations of linearity (infinitesimal, vectorial, computational, etc.), but without motivating, with very rare exceptions, serious questioning of its presuppositions.

This schema expresses certain commonly accepted properties which apparently faithfully describe what is at the core of a phenomenology of time. So much so that, following Heidegger, Husserl's lessons of 1905, in which this diagrammatic finds its justification and support, through the deployment of the famous diagrams of retentions, could be considered as the last word of transcendental phenomenology and what allows one to consider it a the last "classical" metaphysical conception of time. Convinced, moreover, that the diagrams only played a secondary and illustrative role, because of the intuitionism proclaimed by transcendental phenomenology, insufficient attention was paid not only to subsequent developments in later works of Husserl but also to the analyses that implemented them and reflected on their functioning, as well.

However, these analyses and diagrams have been extensively commented on, without their function and, above all, their scope being the subject of serious study, with a few exceptions. Starting from a seemingly phenomenological division between *description* and *mathematical construction*, and forgetting the phenomenology *of* the latter one, the use of diagrams in general and in phenomenology has not been seriously reflected on, by commentators. Most of the time, diagrams have been regarded as a *mere illustrative* and *pedagogical procedure* which inadequately presents the results of descriptive analyses, supposed to be exclusively accessible to phenomenological intuition and reflection. Since this inadequacy have not been *thematized*, phenomenological researches on the subject have been led to obscuring or underestimating the *heuristic function* and *epistemological potential* of these diagrams, as well as their fecundity, for phenomenology itself, especially in the investigation of unsuspected dimensions of time. As a result, this neglect has contributed to covering them up and lead to a common misunderstanding of the nature of categorial intuition or, which amounts to the same, it has impeded the full understanding of *formal or material essences*, and of their constitutive relations. This explains why despite their banality, as Husserl insists, the intuitive essences continued to appear as mystical and inaccessible. Ignoring the role that writing and drawing can play in it leads to misunderstand categorial intuition or eidetic intuition. What made *essences* or *ideas* so enigmatic and mysterious is nothing else than more or less spontaneous theories of empirical abstraction, of mental images, etc. that one throws back on the acts of understanding, and which are incapable of unfolding the intentionality at work in the experience of intellectual evidence (*Einsich*). The latter, one can at least hope so, must happen sometime a day in the millions of research institutions that cover the planet. And this does not correspond to a late evolution, to a "second" or "third Husserl". For, as early as *Logical Investigations*, Husserl was sufficiently clear about the role of *imagination* and diagrammatic (drawing, writing, etc.) in categorial knowledge, that is, in the foundation of acts of categorial intuitions.

Among exceptions, we must certainly count Weyl and Francisco Varela.[3] Both tackle the question of time in relation to this phenomenological approach and in its relation to the diagram. This occurs as early as 1918, especially in *The Continuum*, and continues to inform Weyl's later philosophical meditation, in 1949. On the other hand, Varela deals with Husserl's diagrams in the scope of the so-called naturalisation of phenomenology and welcomes the progress over all preceding history represented by the Husserlian analyses and diagrammatisation of time.

In what follows, we will address three problems:

1. The historical and scientific significance of phenomenology from the point of view of the diagram of time.
2. Phenomenological elucidation of the use of diagrams in science, because there is a phenomenology of diagrams of knowledge, the most radical form of which will consist in a self-elucidation of the use of diagrams in phenomenology.
3. Elucidation of why and how symbolisation and diagrammatisation are required and call each other in the phenomenology of time in particular, i.e. a description in reflection,
4. We will thus be able to approach the specific study of the way in which diagrams of time have functioned in phenomenology between the most fertile period on the subject, i.e. between the lessons of 1905 known as the *Lessons on the Intimate Consciousness of Time* and the said *Bernau Manuscripts* of the years 1917–1918.

The Time Diagram Has Been Touched Upon (Varela, Weyl)

Some authors, among which Weyl and Varela, noticed that phenomenology did not leave the diagram of time untouched. Their analyses are of particular and complementary interest to us in the present perspective. Weyl, because he proposes a phenomenological "reconstitution" of the classical linear diagram of objective time and time-measurement, where he indicates with a true phenomenological finesse the levels of logical and mathematical construction. Varela's approach is significant too, when he points out, in Husserl's researches, motives for a neuropsychology concerned with proposing a non-reductionist modelling of lived time, and breaks

[3] However, we should add to this a number of works, such as those by Longo, Bailly and Montévil (see Longo and Montévil 2011; Montévil 2012) which, following the example of Weyl, and ignoring certain prejudices, in particular regarding the role of philosophy in scientific theorising, draw from Husserl's analyses the resources of another mathematisation of biological time. Or in another direction, Jean-Marc Chouvel (2018, 357–388), who, following in Husserl's footsteps, extends Nunes' aesthetic and musical deepening of phenomenology of time.

with several classical postulates concerning time, in particular that of linearity, which, beyond the classical dilemma discrete vs continuous, conditions the thesis of the *one-dimensionality of time*.

Phenomenological Elucidation of the Subjective Resources of the Mathematical Construction of Linear Time

Herman Weyl, without knowing the details of Husserl's contemporary analyses on the time consciousness, if not through texts published before 1918, including *Ideen I*, develops an original reflection within a major treatise on mathematics in the history of mathematics, situated in a mathematical crisis between logicism (Russell), formalism (Hilbert) and intuitionism (Brouwer). Weyl's reflections on this occasion led, rightly or wrongly, to describe him as a semi-intuitionist. On the other hand, it is indisputable that they are explicitly placed under the patronage of Husserl and a phenomenological question, even though Weyl would only later become aware of the scope of phenomenological analyses for the interpretation of the theory of relativity and non-Euclidean geometries, through the intermediary of one of Husserl's students, Oskar Becker.[4] Weyl's approach is a remarkable example of the reciprocal interpenetration of philosophy and mathematics which he liked, since it takes up the central problem of knowledge for Husserl, which is the relationship between *intention and the fulfilment of intention*, or if one wants *opinion* (Meinung) *as taking-for-true* and its *intuitive validation*, while at the same time he tackles the fundamental problem for mathematics of the construction of the line of reals.

The elucidation proposed by Weyl is explicitly placed under the sign of phenomenology. The text published under the title *Das Kontinuum* is much more than an attempt to elucidate the foundations of topology, but a remarkable sample of the mathematical and philosophical "interpenetration" so dear to Weyl, since he traces the genesis and stratification of the levels of symbolic construction characteristic of mathematics, themselves considered from a phenomenological point of view, i.e., of the *system of acts,* producing (constituting) such units of meaning.[5] This investigation holds as well as an exploration of the phenomenological resources of the schematisation of time, which is at stake in the construction of coordinate systems so important in differential geometry and its application to physics.

The understanding of the process of the mathematisation of time led Weyl to identify, within the logical or arithmetical framework, the place where *subjectivity is erased* and consequently *remains in its symbolical surrogate*.[6] While identifying

[4] Becker (1923).

[5] "Our examination of the problem of the continuum," writes Weyl, "is a contribution to the question of the relationship between *the immediate (intuitive) given* and the formal concepts (of the mathematical sphere), concepts with the help of which we try, in geometry and physics, to construct this given." Weyl (1987, 2) and Weyl (1918, iv).

[6] Order relation of anterior and posterior, equality between instants and time intervals, measurement of intervals, addition and multiplication of intervals, group structure.

and formulating the primitive conditions of such a mathematisation, which subsequently condition the construction of clocks in the abstract sense and by marking the levels of increasing complexity (logical, arithmetic, algebraic), Weyl reminds us at each stage of the mode of intervention of pure subjectivity in the constitution of categorial objectivities—or, what amounts to the same thing, the steps and forms of its withdrawal. It is particularly important for our purpose, i.e. to grasp the part played by the mathematician and the philosopher in Weyl's work, to identify the articulation of different orders of conditions of possibility, and to retrace these stages.

We should do this on the basis of three major texts where this is at work: *Philosophy of Mathematics and Natural Sciences*, *Space. Time. Matter* and *The Continuum*.[7] The philosophical problem at stake is that of the relations between intuition and concept, and through them the distinction between two types of levels and two modes of solving the metaphysical problem of individuation.

1. The possibility of a mathematical concept of time depends on an ideal noetic possibility, not relative to a determined subjectivity or to empirical psychological conditions: "the ideal possibility of the *position* in time of a now rigorously punctual". Without losing its noetic and ideal status, so as not to remain a vague, sterile, ideal possibility, this presupposes in turn the equally ideal possibility of a presentation of rigorously punctual moments. This presentation is necessarily ideal, because it contradicts the concrete practice of the absolutely individualising effective "ostension",[8] whose model and source is the word 'I'[9] or the deictic "this", and all the expressions endowed with an essentially incomplete and fleeting significations[10] (*you*, *here*, *now*, *there*, etc.)
2. Similarly, the establishment of a relation of order and the equality of the intervals of time displaced in time, for the purpose of comparison, are based on the noetic possibility of evacuating the content of the experience that fills the form of the flow of experiences.
3. The measurement of time intervals, and *a fortiori* of duration, presupposes a homogeneous time. This presupposition is once again of a noetic nature. The fact remains that subjectivity can only be erased and forgotten in its productions.

As Weyl puts it: "Time is homogeneous, that is to say that a singular instant can only be given in an individual presentation; there is no founding property in the general essence of time that would affect one instant and not another." This *indistinguishability* of moments which, nevertheless, require an individual act of presentation, is at the basis of the "logical" principle of homogeneity, and of the set of relations which derive from it, and as a result of the numeration of time. There is

[7] Respectively: Weyl (1949, 1918, 1921).

[8] *PMNS*, Weyl (1949, 101).

[9] Weyl (1949, 124).

[10] Such is Husserl's characterization of those expressions, in contrast to accidentally incomplete expressions.

therefore a difference between individual and individualistic intuitive exhibition and the conceptual construction of singularities, a fruitful difference in that it conditions the possibility of measurement.

> Numbers allow us to identify, in the continuum of time, distinct instants in relation to an *OE* unit of measurement, by means of a conceptual process that is objective and precise. But the objectivity of things obtained by the exclusion of the self [*Ausschaltung des Ich*] and of its intuitive immediate life is not without leaving a residue: the system of co-ordinates, which it is necessary to present (and this in an only approximate way) by means of an individual action [*durch individual Handlung*], remains as the necessary residue of the self-elimination of the ego.[11]

Between the passage that begins the final considerations on the group structure of transformations from one coordinate system to another and the one devoted to the numbering of time, there is, in the fourth edition of R.Z.M. Weyl (1952), instead of the sentence ("*Die Logik wird hier zur Arithmetik*"), a development that partially takes up a formula from his writing from 1918 on the continuum, the impact of which is indisputable, but which should be related to the moment of arithmetization in the process of objectification of time. The transition to the arithmetization of time is indeed the main object of *Das Kontinuum* (in § 6 and 7). Starting from a comparison between the *amorphous intuitive continuum* and the mathematical constructed continuum[12] and marking their "incommensurability," Weyl proposes a parallel construction of the line of reals and the time axis (which any physical theory will have to make use of), thus rigorously establishing the analogy between *real numbers* and *instants*, and more generally the parallel between pure theory of time and pure theory of numbers. Pursuing his objective of clarifying and sanitizing the foundations of analysis, against Russell's claim, one can no more rely on the notion of logical "function" than on that of analytical "function".

In this context, the analogy of real numbers with the intuition of time is essential, an analogy that refers to other analogies. "It is possible that this function represents, for example, the position of a mass point as a function of time."[13] Conceptually: the "continuity of the function" means that "for all real values of its argument belonging to a certain interval, the function itself only takes on values belonging to certain play spaces (*Spielräume*)." Intuitively: this corresponds, for example, to the "lapse of time" during which I perceive an object lying there in front of me at rest: e.g. this pen on the desk.

In an underground dialogue with Bergson's philosophical intuitionism and Russell's logicism,[14] Weyl admits that one expresses by means of "objectification",

[11] *RZM.* p. 8. Weyl (1921, 8).

[12] The "complete induction" is presented by Weyl in one of his epistemological theorems as a "projection of the given following a fix procedure onto the background of the possible".

[13] Weyl (1918, 66).

[14] *Our Knowledge of the External World.* And *Monist* for 1914–5; *Introduction to Mathematical Philosophy,* 1919, p. 106.

"idealisation" and "schematisation",[15] is just as well-founded as the assertion by the other (Bergson) based on present perception of "duration". Something is probably lost in this "translation". The rights on the sides of the intuition or the logical side are equally legitimate.[16] It is then that, settling in this "divorce" (*Diskrepanz*) between phenomenal time and the concept of number, Weyl simultaneously proposes an *epistemological (and phenomenological) analysis of the relationship between intuition and concept*, and *an elucidation of the relationship between intuitive continuum and the concept of number*, on which depends the objectivity of the analysis as used in mathematical physics.[17] In doing so, Weyl will deploy a stratum which remains implicit in the usual mathematical construction and which provides a starting point for the construction of the temporal axis and a presentation of the first principles of a pure (mathematical) theory of time[18]. The fact that it takes place in parallel with the construction of the theory of real numbers does not lock us into a vicious circle but illustrates, on the contrary, the source of Weyl's thought, that is, the way a (voluntary or unvoluntary) phenomenological reflexion is fruitful for mathematical thought.[19]

The possibility of such a mathematisation of time depends on the possibility of identifying "instants", understood as "points in time". This supposes the abandonment of the *absolute but vague individuation by ostension*, characteristic of the relation of consciousness to the world, with its qualitatively individuated and elusive "instants" in their fluent individuality as they are lived, in favour of an exact (parametric) indication of interchangeable and individuated "instants" by means of an absolutely fixed system of assignment of temporal "places".[20] Such a system of points with an order relationship makes it possible to define a time segment or "interval" (*Zeitstrecke*).

The introduction of the 4-ary relation of equality between time-segments provides the basis for the possibility of measurement, that is the comparison of time segments. This phase of the construction of objective time presupposes an evacuation of what "fills" each instant, which is one of the constituent moments of formalisation. On the subject's side, it supposes the capacity to abstract oneself from the lived content that fills this time in order to retain only one moment of this lived experience: its duration ("a *certain* duration"). However, this act presupposes two others.

[15] *Das Kontinuum*, p. 66). Weyl (1918, 66).

[16] Ibid.

[17] "In order to be able to establish a relationship with the world of mathematical concepts, it is necessary to be able to fix a strictly punctual 'now' within this duration, and to discern moments (Zeitpunkte): points in time."

[18] The one he takes up in a more embryonic way in *Space. Time. Matter*, Weyl (1952) of which an essential counterpoint can be found in the heuristic and introductory reflections on the theory of Quaternions by Hamilton, 1853, Preface, 3 sq.

[19] See. Rota & Kac (1986) and Rota (1992) and our commentary on the subject, Lobo (2017b, 2018a, b).

[20] Weyl (1918, 67).

1. The definition of the segment as an infinitely divisible interval, which implies an intuitively driven act of "partitioning" duration (*Zertsückung*). I observe, for example, that during the time that I perceived this pen, there was "room" for three bells ringing in the background (more or less regularly and rhythmically), or that I can project imaginatively into the background.
2. A second act is presupposed, that of a "stretching" of a segment of time. For two unequal segments $OE < OL$, it must be possible to multiply ("*stretch*") the OE segment by a rational λ that allows them to relate to each other exactly and to pose: $OL = \lambda \bullet OE$.

> 1. The immediate expression of the intuitive observation that during a certain time I saw the pencil resting in front of me should be glossed over so as to replace the term *'during a certain time"* by '*in each of the instants that fall within the OE time segment'*—which no longer makes it intuitive, but it will have been restored if the division into instants is justified.
> 2. This would have to be true: P being an instant, the domain of rational numbers to which λ exactly belongs, if there is an instant L prior to P such that $OL = \lambda \bullet OE$, is also arithmetically constructible in pure number theory from our principles of definition, and therefore is a real number in our sense. Moreover: in this style, by positing the time segment OE as a unit, not only is associated to each point P a real number as an 'abscissa', but also inversely to each real number is associated a given instant.[21]

This is followed by a development that draws explicitly on Husserl's phenomenological analysis with reference to *Ideen I*, §§ 81–82. These epistemological possibilities justify, against Frege's critiques, the logical application of the principle of *creative definitions*, and correlatively, they ground the "arithmetic constructability" of time "in pure number theory".[22] To ask for more, *i. e.* a point-by-point concordance between conceptual construction and intuition, would be *abusive and absurd*. Moreover, from the very first step, we have replaced the intuitively given by something else. All that would be achieved on the way to a deployment of time consciousness is a system of *approximate* temporal location,—never something like a measure ("a clock"). In concrete time, there are no strict instants and no exact measurement. It is the introduction of the number that provides the real basis for a pure theory of time, the intuition of time can only provide support, an intuitive underpinning (*Stütze*).[23] Mathematisation presupposes a leap which consciousness alone is incapable of ("incapable that it is of jumping over its shadow"), and which, as abstraction, is the mark and work of reason.

The analysis of the foundations of such an abstraction would presuppose a phenomenology of reason, which Weyl here merely points out. This abstraction is indeed rational and "reasonable". Among the motives invoked by contemporary philosophers ("economy of thought" or "schematising violence"), Weyl sticks to the epistemological fundamental requirement of *objectivity*. These motives are

[21] Weyl (1987, 90) and Weyl (1918, 68).

[22] A larger picture of Frege's theory of definition and its impact in his discussion with Husserl's was given in Lobo (2002).

[23] Weyl (1987, 90) and Weyl (1918, 68).

reasonable from the perspective of knowledge ("understanding the world") and its shift in the modern age, and of what this understanding means to seek "the 'truly 'objective' physical world' in the background ("*zu der 'hinter' ihr steckenden 'wahrhaft objektiven' Welt*"). The deepest justification that can be given for the analysis of the concepts of real numbers and continuous functions, therefore, lies in their *physical applicability* (the justification of what they are "able to grasp", of "what 'movement' means in the world of objectivity of physics"). This justification comes from the fact that, with modern science, reason has decided to submit phenomena to the standard of objectivity, under the lead of the principle of relativity. The repercussion of such a decision, as far as subjectivity is concerned, is its expulsion from the universe constructed by mathematics, precisely because consciousness has been simultaneously discovered by Descartes as absolute.[24] The result of this crossroads is the system of coordinates—whose temporal axis we have just studied—*poor residue and reduction of the ego, against which subjectivity may well protest*, and which is the price to pay for objectivity. This system, as we have seen, is "extracted" (*herausschält*) from the given and inserted (or more precisely projected) into a background of possibility constituting the proto-mathematical substratum of any geometric and physical universe, in order to submit the "sensible world" to the norm of objectivity ("*welche die Vernunft aus dem Gegebenen unter der Norm der 'Objektivität' herausschält*"). The residual character consists in the fact that we have instead of the true and "absolute" individuation (which is abandoned) a mere and poor symbolic and schematic substitute. Weyl then goes on to describe what he calls this "simple and little informative substitute" (*ein wenig aufschlußreiches Surrogat*) for what should be a "true philosophy of continuity". In the balance, between the need for a theory of knowledge and the requirements of mathematics, it is the urgency of the latter that will have prevailed.

A pure constructive theory of space and time (i. e. axiomatic and mathematical) is only possible on the basis of:

1. The position of sets of points (instants or spatial points) in their dependence on a chosen coordinate system.
2. The axiom of continuity.
3. The introduction of real numbers as a fundamental numerical category, and the only one likely to allow a transition to mathematical *analysis*. With the introduction of the system of coordinates at this elementary level, we also understand what is the historically imposed bridge between geometry and arithmetic, as well as the turn that analysis has taken with Riemann: "the true geometry of continuity can only be treated analytically, that is to say by developing Analysis as part of

[24] For this very reason, any positing of an absolute (that of space, time, etc.) is condemn as a metaphysical unsustainable stance; such is the philosophical meaning of the relativity theory (Weyl 1927). Same analysis in Weyl (1949).

the pure theory of numbers, and by applying the theorems geometrically with the help of the principle of transfer contained in the concept of coordinates".[25]

In order to achieve a fully analytical treatment (with differentials and integrals) of time, it is therefore necessary to have "*quantities*" and "*measure-numbers*". Even if the construction of the pure theory of time can no longer, at this stage, be dissociated from that of space, Weyl continues its construction by concentrating the pure theory of quantities on a category ("time segments"), and the introduction of the Λ relation such that to any pair of points O and E, where O is prior to E, is associated to one and only one point P, satisfying the relation $\Lambda(OEP)$, and making possible a one-to-one correspondence between (real) numbers and time (and space). Yet, the theory of the measurement of time is only embryonic at this stage. The only reals we have are the "time points" located on the "real line of time". Thanks to the relation and the principle of constructive existence (principle 6 in § 1, of *Das Kontinuum*), it will be able to proceed to the ultimate stage in the construction of a pure theory of time: that of algebraization.

Then, come the well-known formal laws governing time segments (analogously: real numbers) once (1) the equality "$\mathbf{a} = \mathbf{b}$" has been introduced (among other proprieties, transitivity, commutativity); (2) the addition $\mathbf{a} + \mathbf{b} = \mathbf{c}$ (associative and commutative). Then, moving to the formal-logical level in terms of content, "we replace each proportion [a relationship between two segments **a** and **b**] with the corresponding *two-dimensional* set of segments: we call this set the *number-measure* of the proportion. The fact that **a** and **b** constitute a system of elements of this *measure-number* will be expressed by the formula $\boldsymbol{b} = \boldsymbol{\Lambda} \,.\, \boldsymbol{a}$; in other words: **b** is to **a** in the ratio $\boldsymbol{\Lambda}$. We have: if two segments are to each other in the ratio $\boldsymbol{\Lambda}$ and the ratio $\boldsymbol{\Lambda}$*, then $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}$* coincide. Otherwise the segments **a** for which $\boldsymbol{a\Lambda} = \boldsymbol{\alpha\Lambda}$* would *form a one-dimensional set of segments distinct from the empty set and the full set*. The number-measures *multiply* and *add up*. [Weyl's emphasis] The definition of these operations is contained in the equations: $(\boldsymbol{a\Lambda})\,\mathrm{M} = \boldsymbol{a}\,(\boldsymbol{\Lambda} \bullet \mathrm{M})$; $(\boldsymbol{a}\,\boldsymbol{\Lambda}) + (\boldsymbol{a}\mathrm{M}) = \mathbf{a}\,(\boldsymbol{\Lambda} + \mathrm{M})$."[26]

After a development marking the differences between these measured numbers and the numbers of pure number theory developed in the preceding paragraphs of Chapter II, Weyl concludes that this "addition and multiplication of the measured numbers corresponding to fractions take place in complete parallelism with the addition and multiplication of these fractions themselves."[27]

[25] Weyl writes: « *eigentliche Kontinuitäts-Geometrie laßt sich immer nur analytisch behandeln, d. h.* indem man die Analysis als einen Teil der reinen Zahlenlehre entwickelt und ihre Sätze hernach mit Hilfe des im Koordinatenbegrif enthaltenen Übertragungsprinzips geometrisch wendet », *Das Kontinuum*, p. 115). Weyl (1918, 115).

[26] Weyl (1918, 118).

[27] Weyl (1987, 100).

The Specious Time of Neurophenomenology

In another yet complementary direction, some neuro-psychological analyses of lived experiences have recourse to the resources of phenomenology. Criticising the latest avatars of the linear representation of time provided, among other, by information theory, in the form of discrete but oriented (like an arrow) linear time, Francisco Varela engages in a dynamic and complex modelling of time which draws explicitly on Husserlian analyses.[28]

Critically, Varela observes that the linear representation of time imported into neurology came from Turing's idea of the Turing machine, and aimed to propose a refined model in terms of information theory: "the writing head of a Turing machine writes symbols one by one on an infinite string, giving rise to time as a current-sequence, exactly as in classical mechanics".[29] Despite its inadequacy, this model has found its way into the cognitive sciences. It leads in a more or less refined way to seek to reduce the time of consciousness to computer time, and by this means to explain the variability of temporal experiences, what is called lived time, by using an "internal clock" that emits transmitted pulses that translate into behaviour. In this way, time is reduced to the measure.[30] However, Varela challenges this claim. After an analysis, he proposes to abandon the computational scheme.[31]

On another front, against the interpretation hastily adopted by a whole section of the phenomenological tradition which sees in the time analysed by Husserl a variant of the time-measurement of tradition, Varela objects that even a reduction as summary as those proposed by Augustine or James in their reflections on lived time is sufficient to establish the extent to which it differs from the time measured by a clock. This is followed by a series of distinctive features:

1. Lived time is not only linear, but also has a *complex texture* that dominates our existence to a significant degree, with a centre (the 'now' moment), which is not a purely empty form, but always has an "intentional content on which it is focused".
2. "This centre is bounded by a horizon or a margin that has already passed". Hence there is a multiplicity of "moving horizons" *relative to their now*, forming a system of modifications: the "moment" of present and actual coming moment, which "has just passed", and continuously transforms the present into the "immediately past present".

[28] As they had been rigorously analysed by Bernard Besnier (1993).

[29] Varela (1999, 267).

[30] Varela refers here to Church Broadbent, "Alternative representations of time, number, and rate", 1990.

[31] "Strict adherence to the computational scheme will in fact be one of the research frameworks that will be abandoned at the end of the neuro-phenomenological analysis proposed here"

3. This shift is a *darkening* at the edge of which the "moment" escapes my sight. The keeping-hold of this fleeing moment corresponds to an additional dimension of time, a depth, so to speak (that of retentions), and accordingly, that of protention, if we take into account the ever-new moment coming into presence. This tripartite famous structure of phenomenological time represents, according to Valera, "one of the most remarkable results of Husserl's research using the technique of phenomenological reduction".[32]

This preliminary analysis of the consciousness of time leads Varela to distinguish three levels of temporality which he designates, 1. as the level of temporal objects and events in the world: 2. that of the acts of consciousness which constitute these temporal events and objects, which presupposes the method of phenomenological reduction, and finally 3. the level of absolute time constituting the flow of consciousness.

Keeping those two important testimonies, let us turn to the diagrammatic analysis of time proposed by Husserl, and its evolution.

Phenomenology of the Use of Diagrams in Science Including Phenomenology

But first of all, we must question the very use of diagrams in phenomenology. From the outset phenomenology has been linked with this analytical deconstruction of the traditional scheme of time. At the same time, because of its eidetic and reflexive character, it is obliged to engage in a deepening of the close relations between diagrammatic and consciousness of time, as well as a certain formalisation of *the syntaxes of temporal modifications*—namely the specific synthetic level, underlying the traditional logical level, and which cannot however be referred to the informal nor to the indeterminate and empty formal, the so-called formalist position having the monopoly neither of the term nor of what engages under the title of formalisation. Although it constitutes, for essential and well-known reasons, one of the most debated questions in phenomenology, the diagrammatic work of in the phenomenology of time consciousness has not been thematically studied and even less thought about from this double point of view. In spite of numerous and rigorous studies of the Bernau Manuscripts, not to mention current studies of the latest research on time, the potential of the diagrammatic representation of Husserlian time, both for the exploration of subjectivity and for the revision of the presuppositions that guide scientific work, has not yet been realised — at least to my knowledge.

[32] Varela (1999, 268) and Lobo (2010).

Diagrammatic Underpinnings of Scientific Knowledge

In his analysis of the "crisis of European science", Husserl evidences two mediations that have contributed to the spread of what he diagnoses as a "fateful naturalism", whose unquestionable performance is not a "tragedy of culture", or a blindness to the world of culture, but an obscuring, a covering up of the "ground" on which this naturalism and its avatars have grown up. The "ground" on which the "naive position of reality" rests is called the "ground of presuppositions", and it can be called: subjectivity, the world of life, but also Earth and humanity.

But the very first name for this ground is the "*Naturalthesis*" or "thesis of the world" which produces an immediate identification between "nature" and "world", between two levels of structuring. In other words, the nature of this soil has to do with "transcendental subjectivity" understood in a renewed sense, as a *constitutive subjectivity*, actively and passively, that is to say also as a subjectivity "involved" ("committed", body and soul) in this vast collective and transgenerational activity of objectification that is science. It is also necessary to have a clear notion of the levels of such an activity and its precise "modalities": starting from the most elementary levels and going up from there to the most complex levels, without forgetting the difficult problems of the passage from an activity of production and/or individual appropriation to a regime of co-production, co-validation and intersubjective communication.

Another trait: although *a priori* constitutive, this position is not only stratified, but open to a "transcendental history", which, although a priori, is *not* written in advance, but by its very nature exposed to "adventures", "crises", which are unthinkable without the phenomenon of sedimentation of meaning, of which, writing and the mutations of writing systems are an integral part. Now one of the roots of the crises lies in a strange movement of covering this ground, that of the natural thesis, and incorporating into it "protheses", which belong to the technological fundament of scientific knowledge. This "props" or "crutches" are "cultural objects", and in particular systems of symbolisation that have been incorporated into scientific thought, so much so that their status and the "things" they express have been forgotten. More specifically, the transcendental historical analytic proposed by Husserl is an "discrimination" of the ins and outs of this major historical event, which is the advent of mathematical physics. This event is, in modal terms, a decision in reality, a position, or "superposition", which claims to confer on a certain regime of writing a fundamental and definitive role in the constitution of all objectivity. Far from contesting the legitimacy of this decision, Husserl's analytics aims first of all at manifesting the *ambivalence of this gesture* (discovering and covering) that is the "Galilean reduction". Like any reduction, the latter has converted a pre-given field (the sensitive world with its qualitative differences) into an index (a system of indices) for a field that remains to be constructed, and which is postulated to be constructible *a priori*.

To achieve such a reduction, modern objectivism draws on a system of available mediations, which are both technical and cultural. These intermediations are of two kinds: the praxis of measurement, on the one hand, and, on the other, the phenomenon of consignment (*Dokumentierung*) or "incorporation" (*Verkörperung*)—whose main concretisations and modalities Husserl lists: speech (*Sprache*), writing (*Schrift*), graphics or models (*Zeichnungen, Modelle*), these cultural objects which are called "instruments" or "tools" (*Werkzeuge*). These two operations are of course at work in a "pre-Galilean vision of the world" and have contributed to the birth of this fundamental idea for all knowledge, that of a *substitutability*, of a restricted *exemplarity*.

The idealisation at stake in the Galilean reduction operates a hyperbolic generalisation (a passage to the limit) of this idea at the end of which: nothing will henceforth be presumed *objectively real* that does not allow itself to be subjected to such a principle of "transcription". It is thus that the "subjectivity" at work has become opaque and is itself (within the frame of mathematical physics) retroactively incompletely eliminated and reduced to a residue, namely a "system of coordinates", i.e. to the regulated play of straight lines (of one, two or three axes) which progressively transcribe increasingly richer and richer parts of the experience of this subject, and reduce, correlatively, the infinite diversity of observable movements and changes, to variables in equations—"differential equations", partial differential equations, etc. This reduction, which is both an "opening" (a discovery) is at the same time a "closing" (a covering), in that it forecloses the precise sequence of mediations, which have to be transmitted and re-effected from generation to generation, of which the tools, symbols, graphs, etc. are just an index.

The most direct way to disclose this forgotten and hidden history still consists in going back from this system of coordinates to the activities that presided over its "constitution"[33]; in characterising the choices and elucidating the motivations that determined them. In this case, it will be a question of explaining why one first chooses a particular type of coordinates (orthogonal and cartesian and the Galilean group of transformations); to show that this initial choice is underpinned by restrictions imposed on a potential of much richer forms. Finally, the transcendental history establishes that the "progress of mathematical physics" corresponds, from the constitutive point of view, to the lifting[34] of some tacit restrictions, to the exhumation of such potentiality motivated partly by the historical scientific context, motivated, for the rest, by the rational skill, the theoretical ingenuity of certain major authors (Galileo, Descartes, Hamilton, Galois, Riemann, Einstein).

Following the *Krisis*, the mathematization in sciences went through different technical mediations, which are related to subjective performance. As it was the case in the stemming of geometry, the process of idealization is extended to new spheres of reality, "which also encompasses the dimension of time". In such a

[33] As Husserl did in the *Ding und Raum* Lessons (Husserl 1973a, 175; 315, 319–321), and in his investigations on intersubjectivity (Husserl 1973b, 116–117). And my comment in Lobo (2019d: 185 sq.)

[34] As Rota, G-C., Kac, M., Schwartz, J. T. (1992, 168, 173–175) has clearly pointed out, each "lifting" corresponds to a crisis, i.e. an *épokhè* in a functional sense, see Lobo (2018a, 181–183).

way, is constituted for us, “mathematicians of ‘pure’ form”, a universal “idealized spatiotemporal form”. The idealization itself and its practical underpinnings, the means and the modes of transmission, and finally the whole sphere of culture and history undergoes a radical change of meaning and functioning.

> Instead of real practice—be it action or empirical possibilities, which concerns real and real-possible empirical bodies—we now have an *ideal practice of ‘pure thinking’,* which remains exclusively in the realm of a limit form. Thanks to the oldest method of idealisation and construction in history, to be practised within the framework of intersubjective communalisation, these have become habitual acquisitions with which something new can always be elaborated: an infinite and yet autonomous world of ideal objects as a field of work. Like all cultural acquisitions that result from human labour, they remain objectively recognisable and available, even without the need for repeated explicit renewal of meaning; they are simply captured in an apperceptive manner and operatively manipulated through their sensitive embodiment, for example through language and writing. Likewise, the function of sensitive ‘models’, to which paper drawings constantly used during work belong, to learn to read the drawings printed in textbooks and the like. The same applies to other cultural objects (pliers, drills, etc.).[35]

Since its task is, in this regard, to “explicitly renew” these formations of meaning by reactivating them, in order to consider them in their formation and use, phenomenology must stop *simply* making use of them. This supposes that, in the description and discourse held on this subject, another form of reading of these “legacies” is practised. Since this legacy presupposes a provision in the form of “writing systems” and a socially constituted and transmitted habitus ready to read and use what is thus made available (double availability: *habituell-verfügbaren*), it presupposes that new linguistic and graphic “tools” are acquired which make it possible to “describe” this provision itself. Is this possible? Yes, in as much as there is, as we have suggested above, a ‘phenomenological-transcendental” *moment* each time these “practices” are instituted and/or renewed; and even a sample of “spontaneous” or “wild” phenomenology, when new models or systems of transcriptions are “invented” or each time this moment is accompanied by explicit reflection.

This is why it should come as no surprise that phenomenology goes digging in the model cabinet, searching the back-kitchen of science, and, why, in its exploration of the deepest levels of constituent subjectivity, phenomenology strives to “regain” and “reactivate” the inaugural “gestures”, which are not only those of a proto-geometer from an antiquity as mythical as it is remote, but those of *each* “discoverer”, *each time* the hypotheses on which previous practices were based are reawakened. This attention to the foundation and underpinnings of constructive symbolic activity is constant in Husserl. And the first mistake he warns us against is that of forgetting the “intellectual work” that produced and motivated them and justifies their use: the activity of thinking that he calls “*categorial*”. In *Ideas I,* Husserl warned against this peculiar misunderstanding of the nature and technical conditions of categorial intuition, which holds for phenomenological eidetic intuition itself.

[35] *Krisis*, Husserl (1950b, 22).

> There is a not insignificant circumstance that influences these misinterpretations: *the lack of sensitive intuitiveness that is peculiar to all categorial units produced by thought* (and it is particularly striking, of course, in the case of those formed at a very high level of mediation) *as well as the inclination*, in the practice of knowledge, *to attach sensitive images, 'models', to these units, are misinterpreted*. In this way, that which is not the possible object of a sensitive intuition is *understood as a symbolic representative of something hidden*, which could become the object of a simple sensitive intuition if one had a more favourable intellectual organisation, and *the models are understood as serving as intuitive schematic images, in place of this hidden reality, endowed*, as a result, with a function comparable to that of the hypothetical drawings of extinct existing beings which the palaeontologist draws on the basis of meagre data. *Not enough attention is paid to the true meaning of these constructive units produced by thought*, and the fact that here the hypothetical is restricted to the sphere of the synthesis of knowledge is neglected. Even divine physics cannot limit itself to producing intuitive determinations from these categorial determinations of realities that are produced by thought, any more than divine omnipotence can make someone draw elliptical functions or play them on the violin. [36]

In other words, they are *only accessible through* the hypotyposes which are the necessary sensible and technical underpinnings of categorial intuition. These considerations echo and deepen the argumentation and analyses of the Sixth Logical Investigation. They summarise Husserl's position against the theory of *intuitus originarius* which underlies the Kantian refutation of all intellectual intuition and the answer to the objection that most of the general objects that categorial intuition is supposed to make accessible escape all perception, at least adequate perception. To go a step further: If one wants to maintain the parallel between individual perception and eidetic perception, another parallel is missing, that between imagination as a modification of the positionality of perception. Husserl refutation underlines both the *role of models* in the grasp of general objects and he establishes that categorial intuition can either be positional or non-positional, and as inadequate, as the sensitive perception can be.

Let us note, at the outset, two examples taken from the field of technology and mathematical analysis, highlighting the role of graphs and drawings among the many resources of the analogising imagination at the basis of categorial intuition.

> However, it must be pointed out that the examples we invoked were precisely the kind of adequate perception of the general. It is on the basis of particular cases that truly correspond to it that the general was, in this case, actually apprehended and given. Where this is the case, it seems that in fact we lack a parallel imagination with the same intuitive content—as in every case of adequate perception. How could a content, even in the individual domain, propose itself as an analogue of itself, since taken in itself it is impossible for it to be at the same time aimed at as an analogue of itself? And how could the *positional* character be lacking where the content is aimed at is precisely the one that is experienced and given? It is different when, for example, by means of mathematical analysis we have indirectly conceived the idea of a certain kind of curves of the third degree without any such curve ever having been intuitively given to us. An intuitive figure, for example, a particular case known to us, of curves of the third degree, *no matter whether we have actually drawn it or simply imagined it,* can in this case serve us as an intuitive image, as an analogue have generality aimed at, which is to say that the consciousness of generality, as an intuitive but analogical

[36] Hua 03/1: 129. (Emphasis mine).

> consciousness, is constituted on the basis of an individual intuition. And doesn't the primary drawing, roughly traced, already establish a relationship of analogy with the ideal figure, contributing to condition the imaginative character of the general representation? In the same way, we have an intuition of the idea of the steam engine starting from a model of a steam engine and, in this case, there can of course be no question of adequate abstraction or design. In these examples, we are *not dealing with simple meanings*, but with *general representations that are representative by analogy*, i.e. *general imaginations* [emphasis mine]. But when this awareness of a simple analogy is not given, which can happen, for example, in the context of the intuition of a model, then we are precisely in the presence of a case of perception.[37]

It is therefore the explicitly analogical use of diagrams that preserves from confusion. The formalist reduction of mathematical objects to the writing system understood as a system of traces commits such a mistake and constitutes an inadequate grasp of idealities, which reduces the categorial to the meanings of play (Spielbedeutungen). Conversely, the expressly analogical use of figures, tracings, symbols, etc., while avoiding this misunderstanding, does not necessarily suspend the position of these general objects, but leaves open the possibility of two regimes: positional and non-positional, and this despite the fact that it passes, as is most often the case, through a founding analogical imagination.

> In the same way, we now find the differences that were lacking earlier, between positional generality consciousness and neutral generality consciousness. When we conceive only by analogy, imaginatively, a general object, we can aim at it in the positional mode, and this act can, like any positional intention, be confirmed or refuted by an adequate subsequent perception. The first case occurs when the general intuition is filled in an adequate perception, i.e. in a new consciousness of generality, which is constituted on the basis of an 'effective' abstraction of the corresponding particular case. Then the general object is not simply represented and posed but given itself. We can also represent the general in the analogical mode, but *without posing it*. *We conceive it but leave it in suspense*. The intention of the general, built on an intuitive basis, does not therefore decide the 'being' or the 'non-being', but on the other hand makes it possible to decide whether the general and its given being are *possible or not* on the basis of an adequate abstraction. [38]

Categorial intuitions based on imagination can be positional or non-positional.The latter case sheds light on the way in which mathematical and formal thinking decides on the mathematical existence and, consequently, on the possible validity and coherence of a model.

The Use of Diagrams in Phenomenology

But it is one thing, one might say, to elucidate the role of the diagrammatic in intellectual activity (categorial synthesis) in general, it is another to use the diagram in the course of this elucidation itself, as Husserl does to study a level of synthesis

[37] Husserl (1984, 163).

[38] Husserl (1984, 164). Emphasis mine.

which is precisely not intellectual, and which intervenes, it seems, in the lowest and most immediate levels of constitution, as is the case with everything that touches on transcendental "aesthetic" syntheses: those which are constitutive of our consciousness of space and time.

However, we feel that this distinction of levels should not be interpreted in a naive and static way, just as we must get rid of the classical architectural image of the construction of knowledge or the progress of science, which would not result in a deepening that reason or intelligence has of itself, and of its resources. If we follow the suggestions given above, some, if not all, advances in science coincide with a "moment of épochè", when presuppositions are touched on the ground, as in the case of the reform to which Descartes subjected mathematics, Riemann's, Grassmann's and Hamilton's reforms. More precisely, they touch on a new stratum of transcendental aesthetics. It is not surprising that Husserl, who had practised these and many other authors, tried to elevate these ingenious forays to the level of a methodical approach.

One remark at this point: this functioning of the diagrams amounts, in a nutshell, to contesting the summary opposition between philosophy of the concept and philosophy of consciousness (of reflection), as Cavaillès puts it. This opposition actually stems from a limited knowledge of the Husserlian corpus, in particular with regard to the "conditions of possibility of objectivity" (and, correlatively, of objectification), it is nourished by prejudices of Kantian origin, on what intuition or intuitive method means, and on an assimilation of categorial intuition to the intuition of essences, or intellectual intuition in the traditional sense, rightly proscribed by Kant—at least if we put under this term what he understands and if we place as limits to sensitive intuition those assigned to it by transcendental aesthetics. Now, as we have seen, this assimilation is unfounded if one sticks to what Husserl's doctrine is. It is sterile and sterilising from the scientific and epistemological point of view, since in so doing one renounces understanding the resources of the development of modern mathematics, for which, it is true, self-interpretations are no more a sufficient guarantee. Not only because of their diversity and multiple contradictions. This would be a sceptical argument of limited scope. But also, for structural reasons related to the inevitable misunderstandings about the foundations of categorial activity, for any reflection in the natural attitude. It is only in pure reflection, i.e. under transcendental reduction, that the "kingdom of positionality" and its *forms* opens up.[39]

Conversely, it is remarkable how phenomenological reflections always support the conditions for a phenomenological attestation with the critical examination and constructive use of diagrams. As Gilles Châtelet[40] wanted, there are many ways of inhabiting a diagram, of living in it *phenomenologically*, in order to re-enact the "constituting acts" which led to their tracing and allowed to give meaning to the marks affixed along these tracings, to transform them into symbols.

[39] Hua 03/1: 353.

[40] See Châtelet (1993). My commentary in Lobo (2018b).

The simple one-dimensional (and unidirectional) line to which time seems to be reduced is thus underpinned by a multiplicity of correlative forms and modes of operation, which are as many "ways of living", I would almost say, after Spinoza, as *mores*, so much so that the rules we set and in which we freeze this line in an axis, determine a way of being together and communicating, a whole economy of circulation of "signs". The exploration of these acts and this "silent language", through signs and gestures, comes back to a phenomenology attentive to forms (forms of experience and experience of forms).

In this, phenomenology is also constitutive. But constitution is an open process and the form of time is not fixed once and for all, but it is each time a form which is given and in which the genesis finds its greatest point of stability. It is indeed such a point of stability that we reach with the so-called classical objective form of time, a form that aggregates the properties that we attribute to time since classical times (uniform, continuous, homogeneous, unidirectional, unidimensional, infinite . . .) and that we symbolise by the line of the real.

A Second Example: Intersubjective Constitution and Relativisation of the Original Coordinate System

Alongside numbers, the coordinate system plays a crucial role in this overall system of culture, and in particular in the "idealising culture" of modern science. For phenomenology, it is the index and entry point par excellence for a transcendental aesthetic, which is itself renewed. Renewed and deepened, because if we do not arbitrarily limit the activity of "constituting subjectivity" to rudimentary operations, but consent to see it everywhere at work, then, any formation of meaning (whatever its level of abstraction, of refinement) only "holds" insofar as it is subjectively and intersubjectively appropriable. This is why it is important to find behind the products, the gestures of production, behind the administered and standardised, automated operations, the modes of operating, and even behind the "given", the giving and the modes of giving, etc.

The self ego is, so to speak, the "zero point of the coordinate system", from which all things in the world, whether known or not, are observed and known. In the lessons of 1907, Husserl called this point, from which the space of orientation unfolds, *coordinate system*[41] or "point of orientation", "zero point" (*Orientierungspunkt*, *Nullpunkt*),[42] and the unfolding (Enthüllung)—of the constitution—was itself equated with a linear transformation of coordinates.[43] The expression "coordinate system zero point" was corrected by Husserl in 1924 or 1924 with the

[41] Hua 16: 175.

[42] Hua 16: 302.

[43] Hua 16: 302.

expression: “it is the original coordinate system from which all other coordinate systems receive their meaning”.[44] “Every ego is the centre, the zero point of the coordinate system, so to speak, from which it looks, orders and recognises all things in the world, already recognised or not. But everyone grasps this centre as something relative, he changes, for example, his physical place in space, and if he constantly says ‘here’, he knows that the ‘here’ is a different place in each case. Everyone distinguishes objective space as a system of objective spatial places from spatial phenomena such as the way in which space appears with ‘here and there’, with ‘in front and behind’, ‘to the right and left’. And the same applies to time.”[45]

The intersubjective constitution must therefore, at the very least, account for the “coordination” of its coordinate systems. This corresponds to a connexion (a *Zusammenhang*), which is one of the fundamental axes of transcendental monadology.[46] To do this, it is necessary to guarantee a “system of real communicability” that is not a mere empty possibility, rather a “real possibility”. Therefore, it is necessary to make possible the constitution by individuals of a “common coordinate system” through the exchange of observations and knowledge. This is why, beyond any mathematical construction of a group of transformations, it is necessary for *real observers* to agree in a univocal way on a common spatial reference point, the earth, for example, or the sun etc., and a point of time. Every empirical determination therefore contains a reference to a *This* which is common for a human group.[47]

An eidetic variation will make it possible to ensure the a priori nature of this condition, which may seem factual or contingent.

> For example, there is no *real possibility* of establishing an empathic relationship between people on Earth and any ‘people’ on a fixed satellite millions of light years away. But it may be something contingent. If they were people like us, an empathic relationship would be conceivable, for example, to be established in the progress of physics. But how would this be possible, if we and these people had completely different senses and if these basic conditions for the possibility of empathy were therefore not fulfilled? The *conditions for the*

[44] « *es ist das Urkoordinatensystem, durch das alle Koordinatensysteme Sinn erbalten* ».

[45] Hua 13: 116–117. (Translation mine). Compare with Weyl (1949, 80–84, 97), who addresses these questions in a discussion of Kant, and who “anticipates” and points in the direction of Husserl’s contemporary research, of which, he had only indirect knowledge, probably, through Becker and Heidegger. On the treatment of this residue in its relation and distance to Husserl’s phenomenology see Lobo (2019a, 77–93).

[46] Hua 13: §§ 39–41.

[47] “All empirical knowledge of each human being is *linked to his body and therefore to his environment*, which has its own specificity, and for each individual the body of the other belongs to his environment and vice versa. *By ‘exchanging’ their knowledge and their cognitive relationships, the different individuals can constitute a common coordinate system*, a point on the earth, for example, or the sun, etc. Every empirical determination therefore contains a relationship with a ‘this’, which is at best a common relationship for a group of people, for example for all people on earth. In principle, only this condition is valid: any group of people or group of intelligent beings to be found in the unity of nature, who are in a relationship of empathy, constitutes an intersubjective cognition. Each intersubjective experiential cognition is linked to a real or possible group of intelligent beings who are in a possible empathic relationship. However, this possibility means that it is *actually possible*.” Hua 13: 218–219. (Emphasis mine).

> *possibility of an identification of the experiences of different individuals must be fulfilled* and with them the *main conditions for the possibility of mutual understanding*. Ideally, people's experiences hundreds of thousands of years ago have intersubjective validity, even in relation to us, although any factual connection is cut off. But it is conceivable in principle. However, empty possibilities are not enough; there must always be *real possibilities*.[48]

As for the coordination of two consciousnesses and what the true mode of connection (*Zusammenhang*) is: two consciousnesses are then phenomenologically coordinated, connected to each other, even if they are not given phenomenologically as being effectively and continuously unified.

> The question here is not either about the misunderstanding of those who are connected. Two consciousnesses are phenomenologically coordinated with each other, are linked to each other, but are not given phenomenologically as continuously united, and this as if we find in one with the given content of one's own consciousness also the given foreign content, that is, we find the content of the two consciousnesses themselves [in one] and can now see their connected unity and form as founded in the essence of connected consciousnesses. It is only in *one consciousness* that there is a direct, authentic seeing, therefore also connections of essence as connections of unity (*Wesensszusammenhängen als Einheitszusammenhangen*), of a founding itself, of an ensuing etc.[49]

Conversely, the possibility of solipsistic subjectivity is never ruled out, since it is presupposed by the intersubjective constitution: "A current of consciousness restricted to itself would be conceivable, i.e. it would be conceivable that every 'other' would be suppressed."[50] The theory of empathy must be taken up in this perspective. The *Einfühlung* is indeed an 'analogical apperception", but not a reasoning by analogy.[51] It is also important to situate this analogical understanding, because there are direct and indirect, intuitive and imaginative analogies.[52] The essential point is to understand how the transition from an absolute, original coordinate system to its relativisation takes place. This happens, phenomenologically, in the intersubjective constitution, conceived as an "intersubjective group of transformation".[53] This group is a "complex group" which matches or pairs imaginary possibilities of being other[54] (imaginary, in a mathematical sense, since they are incompatible with my effective being) with the given of an effective being. It is in this context that space and time are constituted as a form of the phenomena of nature, as "intersubjective structure by transformation, transfer".[55]

The research of the 1917s comes back to one of the constitutive elements of the system of coordinates: "a *hic et nunc* presupposes a self with its living present and its living body (a zero point of orientation) and it is a general and pure possibility, the

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48 Hua 13: 218.

49 Hua 13: 222.

50 Hua 13: 222.

51 Hua 13: 316–318.

52 Hua 13: 188 & 223.

53 Hua 13: 266–267.

54 Hua 14: 154. Lobo (2014).

55 Hua 13: 374–375.

hypothesis of an absolute self (or of a plurality of absolute selves) and of a 'nature' in relation to it". Above it is written, with regard to individuation which underlies the use of time as a principle of individuation: "several such moments have an identical specific birth which is repeated in them, but which is diversely individuated. Thus, by the way we identically pose what individualises, the moments are also identical, [are] identical in the change of individuation." What, applied to nature, provides the following: "the space-time form, the general form of the *tode ti*, the singularisation is an individuation by the *hic et nunc*, which is not a geometrical difference insofar as geometry does not speak of individually determined points of space-time, but only, at the level of a general statement: of possible and 'certain' determined space-time."[56]

It is to this singular colloquium between phenomenology and science (physics and mathematics) that Weyl tirelessly refers, by his insistence on the phenomenological significance of the coordinate system, a poor symbol, he says, an ineliminable residue of the eviction of (constituent) subjectivity. It marks the necessary place of insertion of subjectivity within this universe of pure objectivity that science seeks to be. Tracing this process of elimination is therefore the task of a genetic phenomenology, among other things a genesis of the processes of objectification leading to the constitution of an objective time.

Such stabilization marks a moment in the development of mathesis, where the "constituent multiplicity" produces a definable, i.e. axiomatizable "constituted multiplicity". But beyond this form, there are others, less determined and parameterised, and this is what Husserl calls 'time itself'. Phenomenology does not stop at the "form of the consciousness of time". It is the form of time itself, time "as an objective form", that constitutes the term of its exploration. The time constituted in consciousness is itself an objective form where "every temporal point is a point (a limit) of an objective form".[57] Phenomenological time is therefore "a form of transcendental objectivity", which must be distinguished from "objective time in the sense of the form of nature", which the latter presupposes[58] and which corresponds to what is traditionally called a principle of individuation.

Is this form even susceptible to mathematisation and axiomatisation? Moreover, this is what is explicitly proposed in the texts we are aiming at here. The form of the time in question is a "system of forms", about which Husserl deploys a strange arithmetic[59] that I cannot engage in here.

It is neither surprising nor scandalous that this exploration should lead to diagrams of time that make one-dimensionality the result of a fold of two surfaces, or that, contrary to popular belief, time has several dimensions, or constitutes a system of forms that is not necessarily rigid.

[56] Husserl (2001, 300). This has to be related again to what Weyl writes about it in PMNS as well as Becker (1923). Lobo (2008).

[57] Husserl, (2001, 181).

[58] Husserl, (2001, 184).

[59] Hua 33: 239. About "multiplicities of individualities" see: "Das Formensystem des Urlebensstroms und seine stetigen Modifikationen im Diagramm)!" Husserl (2001, 239).

Symbolisation and Formal Writing in Phenomenology

In the same way as writing, but following different and complementary modalities, the diagram provides a support for a continued eidetic intuition. We will not be able to start again from the whole of the constitutive analyses of *Ideen I* and the way in which it is possible to transcribe them, but we will gradually introduce the elements indispensable to the understanding of Husserlian reflections on time and diagrams of time.

Symbolisation and Diagrammatisation of Intentional Analysis

In order to do so, let us start from a pivotal text[60] which gives us an overview of the problem concerning both the difference between phenomenological time and objective time as well as the reasons for the transition from phenomenology in 1905 to that of the years after 1918 which introduce important considerations on all aspects of the problem: time as an objective form, time as a principle of individuation, the difference between objective and subjective time, as well as the limits and aporias with which a phenomenology of time is confronted etc.

Time itself (with its temporal filling or *Erfüllung*, its events and its "substrates" of events) differs from the modes of data of time (its time-points and event-points, its temporal-objects that last, its filled time intervals, etc.). The system of these modes of givenness nevertheless constitutes a *non-transcendent* "objective form", whose components are time-points, as the limits of the objective form at stake. At first reading, the whole passage appears rather obscure; but it becomes clearer as we reintroduce, at this level too (and as we must do for all passive syntheses in general), the noetic-noematic correlation.

All lived experience is made up of real (*reelle*) or hyletic and intentional or unreal (*ireelle*) components. Furthermore, "a distinction must be made in lived experience between a *subjectively oriented side* (subjektiv-orientierte) and *an objectively oriented side* (objektiv-orientierte)". And Husserl adds: "this way of expressing oneself must not be misinterpreted, as if we were teaching that the possible 'object' of the lived experience was something analogous to the pure self"[61]— i. e. *what* is temporally *identified*. The latter are two-sided units, so to speak. In order to fix the distinctions and laws introduced by Husserl, we shall symbolise them. We are therefore authorised to introduce, as expressing double sidedness of the c_i, a separation bar (*Trennung*) between the face turned towards the subject and the face turned towards the "correlate" ("objective" with inverted commas).

$$c_i = \mathrm{c}\left(i_s \| \text{``}i_\mathrm{o}\text{''}\right) \tag{1}$$

[60] Text N° 10.

[61] Ideen I. [161]. Hua 03/1: 196–197. Lobo (2010).

This is the *terrain* where we are located. But at this stage an objection could arise. Don't the temporal changes concern all the lived experience taken as real (*reelle*)? If this were so, time would only be the form of "internal phenomena" and could not be that of objects. However, as Husserl says, time is "an objective form" and beyond, it is "the form of physical and psychophysical nature".[62] It is a form for its own sake and, moreover, a form that contributes to objectification and individuation. But what about ideal objects?

The essence of the individual is linked to this form of time which contains nothing of the modes of the "present, past, future". Time itself is therefore an "objective form" which does not pass and whose components do not pass either. Present past, future, on the other hand, represent modes of data of time, *temporal modalities*. Let us note them: c(z_s ∥ "z_o"), with z_s = Def.{data modes of z_o}.

Subjective side. Any object-temporal has a necessary relation of givenness to a egoic subject, to *any possible* egoic subject. The temporal object is, for the egoic subject, in various modes. The z_o is not only "in" time, but it occurs (*ereignet sich*) according to temporal modalities. The rest of the text examines these modes in detail. *Ereignis* is a concept that concerns the subjective side of temporality of givenness, z_s. ("*Ereignis*" *ist also ein subjektiver Gegebenheitsbegriff*). The multiple "data modes" are changing, while the ∥"z_o" remains unchanged in its rigid identity (*in seiner starren Identität*). "Time and these objects do not flow, they are, and the 'are' is rigid".[63] The temporal flow is not the flow of time, but rather the flow of 'modes of time data' and its objects". Difficult sentence, which insists on the need to separate the phenomenological flow (of data modes) from time and from temporal objects: "we must not confuse the change in objective time with the 'flow' of data modes in which everything temporal 'appears' for the subject". This seems to designate the function of temporal modalities: to constitute the "flow', the modes of appearance.

The problematic shifts, since there is also a *second objectivity* of the "lived" and a temporal constitution of its own, and therefore "a form of transcendental objectivity". This objectivity of the lived experience is the result of an activity of reflection, or at least appears in the reflection. Whatever the attitude, transcendental-phenomenological or natural, this objectivity is the "product" (in both senses of the term, both an observable event and the result of an operation) of an act of reflection. This is why, as Husserl invites us to do in *Ideas I*, we will have to *integrate this reflexive modification into our description*, whether it is done in words, in symbols or in diagrams.

[62] Ideen I, [184]. Husserl (1950a, 222).

[63] "Die Zeit und ihre Gegenstande fließen nicht, sie sind und das Sind ist starr." Husserl (2001, 182).

Intentionality as a System of Modification and Its Symbolism

In order to understand this, it is necessary to introduce a generalisation of the proposal that we have just recalled. *Any reflection*, including a pure and neutral reflection, *modifies the lived experience which is reflected on.*

Let us therefore start from this crucial example of *modification*, from the reflection itself. In the notation system that we adopt on the basis of Husserl's suggestions, the *parentheses* symbolise such a modification of neutrality. They function algebraically as an *operator* of *associativity* and they condition the syntheses of acts of the type foundation (*Fundierung*) of acts.[64] Reflection modifies the lived experience. Husserl generalises the proposition by positing that any lived experience of a new essence actually proceeds from *specific modifications*, which should be described in their specificity in order to state the laws of essence that govern them. Among these modifications, we must take into account the reflection that in its various forms intervenes in the retrospective capture of the lived experience (whether in memory, in fresh remembrance, or on the basis of an imagination, etc.). In particular, we must rely on the system of temporal modifications that contribute to constituting the lived experience as lived experience, and to begin with, to constitute it as lived experience *situated* in the flow of a consciousness, localised in a constitutive history, individuated in a hypothetical biography. Phenomenological time corresponds to the system of subjective modifications that Husserl also calls real (*reelle*) modifications. A lived experience is provided with a modification princeps and the two real and intentional constituents are themselves respectively derived from specific modifications.

As far as we are bound by our point of departure, the exploration of the varieties of modifications must take place by sinking, so to speak, into the circle of reflection and the exploration of its implications and presuppositions. This is why Husserl sums up the task of phenomenology in the following terms:

> The task of phenomenology here is to systematically elucidate all the changes in experience (*Erlebnismodifikationen*) that come under the heading of reflection, in their connection with all the changes with which they are essentially connected, and which presuppose them (*die sie voraussetzen*). This last point concerns all the changes of essence that every experience must undergo in the course of its original development, and also the various kinds of changes (*Abwandlungen*) that can be seen as being carried out in the form of 'operations' (*Operationen*) in relation to each experience.[65]

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[64] For an explication of this point see Lobo (2020). And the overall problem of syntaxes of consciousness in Lobo (2010).

[65] *Ideen I*, 148–149. "Die phänomenologische Aufgabe ist hier, die sämtlichen unter den Titel Reflexion fallenden Erlebnismodifikationen im Zusammenhang mit allen den Modifikationen, mit welchen sie in Wesensbeziehung stehen, und die sie voraussetzen, systematisch zu erforschen. Letzteres betrifft die Gesamtheit von Wesensmodifikationen, die jedes Erlebnis während seines originären Verlaufes erfahren muß, und außerdem die verschiedenen Arten von Abwandlungen, die ideell an jedem Erlebnis in der Weise von 'Operationen' vollzogen gedacht werden können." (Hua 03/1: 181–182).

Every real component of a lived experience is itself the product of a real "change". The most obvious ones that will provide the first parameters for the multiplicity of experiences are the temporal modifications. Among the real components of the experience, "temporal modification" and, among the temporal modifications, "retention" are the most obvious. But, as mentioned above, a careful distinction must be made between the retention of the reflected lived experience and the retention of the reflection itself, as lived experience.

A lived experience in general is therefore a combination of M_i and M_r. And a real lived experience itself undergoes a real (*reelle*) modification, starting with that of retention. In symbols:

$$E_r \rightarrow (E_r)_r \text{ and } (E_r)_r \neq E_r \ or \ r(E) \tag{2}$$

With r belonging to R (all actual changes, including retentions). Or in words:

> The lived experience, really lived at a certain moment, gives itself, at the moment when it comes under the gaze of reflection, as truly lived, as existing 'now'; that is not all; it also gives itself as something that has just existed (*als soeben gewesen seiend*) and, insofar as it was not looked at, it gives itself precisely as such, as having existed without being reflected upon.

Actual lived experience E:

$$R_\phi(E) \rightarrow E \text{ et}(E) \rightarrow E \tag{3}$$

Or more simply:

$$R_\phi(E) \rightarrow E \tag{3'}$$

Where ϕ denotes *phenomenological* or *pure*. But, conversely, an effective experience implies an experience that can be reflected upon and thus undergo a real temporal modification, because every effective experience is modified temporally ("it lasts"), much more, as we shall see, it submits to the system of real temporal modifications. In the context of a genetic analysis, one must beware of a typical methodological error (genetic circle or knot), which consists in "embedding" the experience in a continuous immanent temporality. Through the resources of pure reflection and the so-called phenomenological functional analysis, it is necessary to show how continuity emerges (it is the result of a synthesis, proto-synthesis), and according to the constitutive order and within the framework of correlational *a priori*, in the primitive form of the temporal continuum. This does not mean that according to the genetic order we have here an irreducible form of temporality, because, as Husserl will show on the contrary, in his later research, if we want to understand this form, we have to go deeper, and give an account according to a whole system of interlacing or complexion of the "real" temporal modifications (retention and protentions) of the constitution of this form, and of its fundamental property: its connectedness. Bernau's Manuscripts venture into an area where the modalities constituting this connectedness are at stake. Husserl explains what he calls "flow connection"[66] on the basis of the game of "filling/emptying" (*Erfüllung/Entleeren*), which has, as its necessary complement, the *zero point* of modification, which is the "lived present" (rather than the "living present"). But we will come back to this later. At this point, let us simply spell out the register of changes at play in the temporal stratum of experience—without losing sight of the fact that this real phenomenological temporality does not in any way resolve the slightest question about cosmological or physical temporality. This, whatever it maybe, can only appear in our writing in brackets and in inverted commas, for the reasons we will indicate.

$$E \rightarrow (E)_{rz} \tag{4}$$

With z for temporal (*zeitlich*). And this temporal modification can be of the M_{rz} [*zeitlich reelle Modifikation*], retention or protention type. Or a real modification with reproduction. This M_r, when it is a question of retention, can be noted *E'*.

Among the real modifications M_r, it is necessary to mention the temporal modifications (*Retention, Protention*). And a M_r differs from a M_i, in that it is *not* a function of *the ego*. *Wiedererinnerung* and *Vorerinnerung* are *intentional modifications*, but they also serve as a basis for presenting the lived experience (E) for pure reflection. Reflection on E can be oriented either on actual or on intentional modifications.

$$R_{\phi}(E) \equiv (M_r(E)) \vee (M_i(E)) \tag{5}$$

New proposition: we could assign different indices in a general way and make every lived experienced (E) a combination of $\{c_r; c_i\}$, reserving the possibility to describe M^i_{rz} i later, with the superscript indicating the intentional dominance, M_r with r in index means hyletic or "*real modification*", and the second index, the temporal type of modification, in this case. The index i designates the type of experience that is thus temporalized (perception of a sound, image, memory, idea, emotion, resolution, etc.). Under abstraction ($-$) of the intentional component c_i the phenomenological reflection is oriented toward the real component c_r and its constituents:

$$(E - c_i) = c_r \text{ and } c_r = M^i_{rz}(E) \tag{6}$$

But this notation is unnecessarily complicated and misleading and, as Husserl notes in *Ideas I*, it is useless, while dealing with the question of the consciousness of time, to bother with constitutive analyses, and conversely, when dealing with the latter, it is also possible to neglect real temporal modifications, because they are something

66 *Der fließende Zusammenhang von Retention, Protention und Urpräsentation.*

universal and common to all acts (all lived experiences). And we can note more simply

$$c_z(E) = M_z(E). \tag{6'}$$

As we shall see: a retention operates on the real components of E as well as on E as a whole, but leaving untouched the intentional components of E (noesis and noema).

Husserl explicitly returns to this point in the 1918s:

> *Ideen I*: I had then sketched out an idea of phenomenological time, of the time to which the hyletic data belong as units constituted at the same time as the noesis and their noematic correlates. I therefore believed that I could assimilate the latter, as constituted units, to the hyletic units and that I could consider them as members (*Glieder*) of one and the same phenomenological time. I had as an ulterior motive (if I describe my former point of view correctly), the original consciousness as a constituent of time, whose structures required research for themselves. And these structures, I thought, had to be the same for all the objectities of phenomenological time, the hyletic data as well as the acts. But one has to consider what kind of studies of act I had accomplished in the *Ideas*, excluding the *question of the constitution of time*. What could one properly draw from this with regard to the structures of pure consciousness? What could be obtained from it about the relations between hyletic events and noetic events, etc.?
>
> If I think about it correctly, all this was nevertheless roughly correct. And conversely, I could bring to what I have described above many additions, if not many corrections. When I sketch out a phenomenology of perception, imagination, judgement, will, etc., the *question of the temporal constitution*, *the question of how these types of experiences constitute themselves as temporal objects must be put aside*, because this constitution is something identical in essence (*Wesensgleiches*) for all acts and their noematic correlates. From this point of view, I am therefore absolutely convinced that it was right to postpone the phenomenology of the consciousness originating in time.[67]

In other words, and to put it in our symbolic terms, it is a mistake to believe that the analyses of the c_r and in particular of the temporal constituents c_z have an impact on those of the latter or vice versa, which also means from a more general point of view that the abstract nature of certain moments of lived experience does not mean that their study cannot be carried out separately. The decisive point is to bear in mind the abstract character of these areas. But the problem remains, however, to specify the nature of the relations between "hyletic and noetic events".

The groups of c_r: protention, retention, impression, etc. have the characteristic of allowing a certain squaring and of "locating" a lived experience in relation to the axes of the various lines of real temporal modifications.

From here we are able to follow sketches of the formal writing of the lived experience, E, to mark that it is itself an *operator* and not a simple element in set theoretical sense of the term. A second reason has to do with writing constraints, as we will be forced to introduce various indices in order to situate the modifications of the lived experience during the process of temporal constitution. This writing was first introduced in 1905 for the retention diagrams, but we will instead start from

[67] Husserl, (2001, 116–117). (Emphasis mine.)

the 1907 diagrams and the transcription given by Husserl, on the occasion of the revision of the 1907 diagrams and from the transcription given by Husserl, on the occasion of the revision of the binary "schema", called "apprehension-content of apprehension", which had guided phenomenology in its first steps and which has undergone profound revisions since 1909.

Analysis of the System of Continuous, i.e. Temporal Changes in 1913

But before embarking on this, let us take up again the analyses in § 81–83 of *Ideas I*. Temporal modifications are "material" "proto-syntheses". The groups of c_r: protention, retention, impression, etc. have the characteristic of allowing a certain grid to be drawn and of "locating" an experience in relation to the axes of the various lines of real temporal modifications.

At this level, there is a fundamental distinction between two types of real changes in *E*, those that are formal and those that are material. *Formal*, because we have the form of the "now" and its compositional laws in relation to the forms of the "before" and the "after" (*Vorher/Nachher*), through the regulated play of these three "horizons".

It is through the combination of R_ϕ (phenomenological reflection) and appropriate attentional modification that it is possible to orient the analytic on the real components of the modes of giving, on the "real modifications". And to begin with those which are constitutive of the actuality of *E*, and of its real component. It is in this framework that the first experiences of *linearity*, the continuous and something like an *interval*, *segment* (or "lapse of time") are constituted.

> For example, the joy that begins, ends and in the interval lasts; I can first of all hold it itself under the pure gaze; I accompany it in its temporal phases. But I can also turn my attention to its modes of givenness; notice the present mode of the 'now' and observe that in this now, and in principle in everything now, a new and always new now joins in a necessary continuity; to notice that at the same time each present now becomes an 'just now' (*Soeben*); this 'just now' becomes in turn and continuously converted into ever new 'just now' of 'just now' and so on without end.[68]

Here we enter into what should be ranged under the heading of transcendental aesthetics. An analysis of the real components, or of the modes of "combination" of the real components, or of the productivity specific to the real modifications of *E*. In contrast to Kantian transcendental aesthetics, here we explore different groups of systems or structures of modification which comprise a form which can be analysed on the transcendental level (i.e. on the table of pure reflection) and a material which is equally analysable, even if both ultimately refer to the last (original) forms: a form of flow and an original "*hylè*" (*Urdatum, Urhule*). Because of the starting

[68] Idées I. p. [164]. Hua 03/1: 199. Translation mine.

point in pure reflection, Husserl here considers the group of temporal modifications and modes, but he had already explored extensively elsewhere the groups of modes involved in the constitution of spatiality (of various levels of spatiality).

As will be explained in §§ 84 ff. of *Ideas I*, the analysis of actual changes is also part of the functional analysis. And whether the c_r is primary or secondary, we always have at work an activity of constitution, i.e. of production of a unit of meaning which can be analysed along different axes: that of the "*morphê*" and the "*hylè*", or that of the correlational *a priori* since the c_r, as we have seen, can also be distributed or divided in parallel with the division which affects the $c_{i.}$

An Example of Moving from an Analysis to a Mathematical Model

This is easy to understand insofar as the actual (e. g. retentional) changes affect the whole experience (including its c_i) and this is obvious and obviously presupposed for and by natural reflection, and even more so for pure reflection.

> In other words, the lasting experience of joy is given 'to consciousness', in a continuum of consciousness whose form is constant. An impressionistic phase there acts as a boundary phase to a continuity of retentions; these in turn are not at the same level but must be related to each other in a continuous series of intentions (*kontinilierlich-intentional*) to form a continuous '*into-one-another*' (*ein Ineinander*) of retentions of retentions. This form receives an ever-new content; therefore, to each impression, in which the now of lived experience is given, is 'added' (*fügt sich*) a new impression which corresponds to a continuously new point of duration; continuously the impression is converted into a retention, and this continuously into a modified retention, and so on.

How do we achieve this potentially infinite series of real changes? Is it by going to the limit? The resources have been indicated above; they are those of eidetic variation. But, because they are "intentional essences", this is carried out by the pure self-reflecting in the phenomenological attitude—thus under transcendental *épokhè*, under neutralisation of the regime of the original, "natural" positional activity. How can I make sure that the experience of the "performing" (*vollziehenden*) self is also subject at the level of the real components to this modification of the *etc.* (For this, I must introduce a sign designating both this necessary "addition" of a new impression, this resulting '*into-one-another*', and the continuous character of this modification. Generally speaking, everything is actually changed. For this index, like all indices corresponding to doxical modalities, we could use Greek characters, for example ε to mark that we are in the presence of an ideation under explicit neutrality).[69]

But to stick to the actual continuous modification of retention, we can notice the analogy between the operator thus brought into play and what we call in algebra. It amounts indeed to constitute a "linear operator", of the type $Au = u$, $Az = z$ and

[69] See, Lobo (2019c).

especially this generic possibility that a real modification involves and implies at the same time a real modification of real modification: $A(fAg) = (Af)\ (Ag)$. That the addition (*Fügung*) of two real modifications is a modification of modification. Let us simplify *Wr* (*real Wandlung*) by *r*.

$$E \to (E;\ \ r) \tag{7}$$

or even more directly by designating any actual change as *R*: $R(E) = E$ and $R_1(E)R_2(E) = R_2(R_1(E))$.

Or else, if we take the example of *retentions*:

$$R(E)\ R(E) = R\,(R(E)) \tag{8}$$

We have seen that we can consider the lived, *E*, as such functions, and the retention, for example, and so it is generally the case with certain other real modifications, acts as an operator, the retention, as Husserl insists, is not only a modification of the real modification, but always also of its content. For example, I retain the sound and the retention of the sound, and the retention of the retention of the sound, etc.

As I have suggested elsewhere, this is similar to an 'averaging operator' also known as the Reynolds operator, used in fluid dynamics and functional analysis and in the theory of invariants.[70] In the same way, any experience presented in the reflection is, at least, a retained experience $R_1(E)$ and the second retention $R_2(E)$ is a retention of the retention $R_1(R_2(e))$, but if we designate the second experience by (*E'*). We will have: $R_1(\mathrm{E}_1)R_2(E_2) = R_1(R_2((E_1)E_2)$, for any couple of *E, E'*. Or more simply: $R(\mathrm{E}_1)R(E_2) = R(R((E_1)E_2)$,

$$R\left(EE^{'}\right) = R(E)R\left(E^{'}\right) + R\,(E - R(E))\,\left(E^{'} - R\left(E^{'}\right)\right). \tag{9.1}$$

This can be interpreted as follows: The retention of two lives (impression of the note *C* -impression of the note *E*) (respectively, as examples of *E* and *E'*) is equal to the retention of the product of the respective retention of the note *C* and the note *E* together with the retention of the product of the difference of the do retention of *C* in relation to the impression of note *C* and the difference of the *E* retention in relation to the *E* retention. For any couple of *E*, *E'*.

$$R\left(R(E)\ E^{'}\right) = R(E)R\left(E^{'}\right) \tag{9.2}$$

[70] Reading Rota (2003, 140–151) drew my attention to this operator. A Reynolds operator is a linear operator that acts on algebraic functions. The Reynolds operator is spelled R(φ), P(φ), ρ (φ) etc. or as Rota writes it: *R* (*f*), *R* (*fg*), etc. The Reynolds operator satisfies the following conditions for ∀*f*, *g*, functions belonging to an algebra ***a*****,** (1) $R(fg) = (Rf)(Rg) + R[(f - Rf)(g - Rg)]$; (2) $R(R(f)g) = R(f)R(g)$. In functional analysis, the Reynolds operator is called an *averaging operator* if it satisfies the two conditions—which are the same as those satisfied by the retentions–, the third of which is that (3) $R(R(f)) = \mathrm{R}(f)$, for all of *f*, then *R* is an averaging operator if it is a Reynolds operator.

What we can interpret: The retention of C and E impression is equal or equivalent to the product of retention of the note C and retention of the note E.

A third condition is sometimes added:

$$R\,(R(E)) = R(E). \tag{9.3}$$

In words. The retention of the retention of *E* equals the retention of *E*.

Phenomenology and Diagrammatic of Lived Time

With the analysis of time, we are naturally on the way to a first and very primitive diagrammatisation of consciousness. The analyses of the lessons of 1905, taking as their axes the line of impressions and that of retentions, bring out two dimensions of phenomenological time, but remain fairly aporetic and incomplete, and are still in intimate debate with Brentanian psychology. This is how we are on the way to a first and very primitive diagrammatisation of consciousness. This is the case of the lessons of 1905, which takes as its axes the line of impressions and that of retentions. The 1918 diagrams introduce the lines of protentions. But as Husserl insisted at that time, and even more clearly in the lessons on passive synthesis, this diagrammatisation still remains encumbered by unclear and irrelevant presuppositions.

First of all, it is a diagrammatisation of the temporalisation of a lived experience *with* its real correlate, which supposes that a dimension of synthesis is already admitted at this stage, even if it is inferior to an active synthesis, and even to the passive syntheses which will occupy Husserl afterwards (after 1918).[71] Furthermore, we postulate the comparability and even a certain measurability of the segments of the different axes. This presupposition is particularly burdensome, since it amounts to admitting a *continuum* of a mathematical type (the "line of reals"). As Weyl will note after Husserl, the essential here is played out in the relationship between intuition and intention, in a game and a subtle conjunction of filling and emptying. Then, these diagrams are incomplete in ways that bring us into contact with the very essence or law of time. Finally, the functions and functioning of the diagrams is not the object of sufficient phenomenological reflexive attention, and this is another aspect of incompleteness.

The researches undertaken in the years 1917–1918 was to prove decisive in resolving these aporias and filling these gaps. Without engaging much deeper in the analysis of dense and difficult texts, we still need to observe how pure

[71] Husserl only belatedly began to admit such a possibility, of which the last texts of the period 1917–1918 bear many traces. In this perspective, the emergence and insistence of the motif of "passivity" and passive syntheses should be followed. We will have to renounce this, to concentrate on the evolution of the diagrams and their functioning.

phenomenological description, *symbolisation* and *diagram construction* are interact and reinforce each other. We should not forget, as Husserl never ceases to remind us, that diagrams are a way of eidetic exploration, an indispensable tool for the deepening of a necessarily momentary and local reflection. It is a question of examining, eventually of digging with dynamite "all the logical possibilities and we set out to search for eidetic possibilities and impossibilities and, finally, we go through the system of concordant eidetic necessities".[72] Under no circumstances should we substitute an indirect way, which would render the rights of eidetic intuition superfluous or challenge them; the inevitable absurdity of an interpretation that spares the consideration of the constitutive role of consciousness is clearly established by Husserl[73] in a text whose title is "*Time and Temporal Modalities. Orientation. Important ontological temporal axioms*". Husserl reminds us that the starting point of any analysis is always and by phenomenological necessity *a local observation*[74] or *a reflection,*[75] which it is advisable to put in relation with the "relativity" mentioned above and the taking into account of the conditions of "phenomenality".[76] The *terminus ad quem* of this reflection is, as always, to bring out eidetic possibilities and essence necessities. This is how we come to phenomenological axioms.

The Dialectic of Phenomenological Reflection, Symbolisation and Diagram Construction Between 1905 and 1918

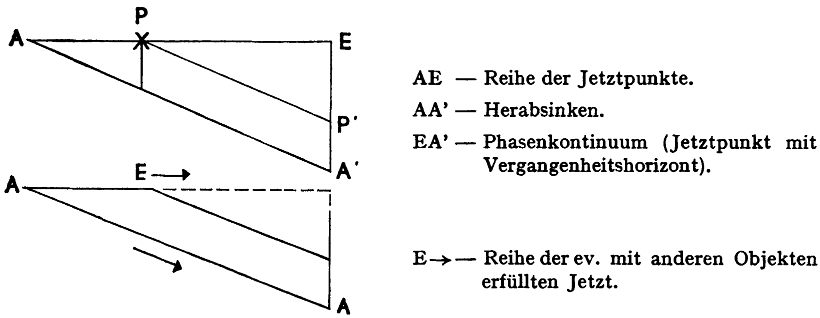
Diagrams of Retentions and Drafts of a Chronometry

As early as the lessons on the time from 1905, Husserl proposes a symbolisation that corresponds, more or less, to the one we have given, based on the *Ideas*. The *zero* noted 0 designates the *point of origin* of the temporal event or process *and* its apprehension. In this diagram, the idea is to differentiate, by abstraction, two types of flows: the flows that constitute the duration of the object (of the object-temporal temporal object), and the modes of flow specific to each point of this same duration, which are abstract moments, and therefore not strictly speaking "instants". It is possible to designate the latter as "phases" of the former, and to define them as "the continua of the flow modes of the various instants [or points in time] of the object's duration".[77]

Let us reproduce the diagrams proposed by Husserl in the lessons of 1905 in § 10, limiting ourselves to a restitution of their instructions for use.

[72] Husserl (2001, 189).
[73] Husserl (2001, 181).
[74] Husserl (2001, 28).
[75] Husserl (2001, 189).
[76] Husserl (2001, 29).
[77] Husserl (1969, 27–29).

AE—Series of points-of-now



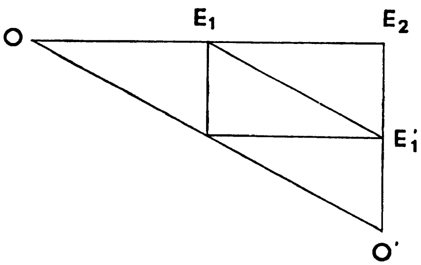
AE—Series of points-of-now [Corresponding to the line of the *reals* or the classical representation of time]

AA'—Sinking.

EA'—Continuum of phases (point-of-now with its horizon of being-passed)

E → —Continuation of the now possibly filled with other objects.

Now, if we consider these forms in their relation to the constituted "immanent units", i.e. from the point of view of the immanent content, we are led to distinguish between the original content which is "bearer of an original apprehension" and the retentively modified contents. We must therefore also distinguish between two temporal apprehensions: one which is derived ("constituted in the immanence"), the other, corresponding to the not-constituted original apprehension.

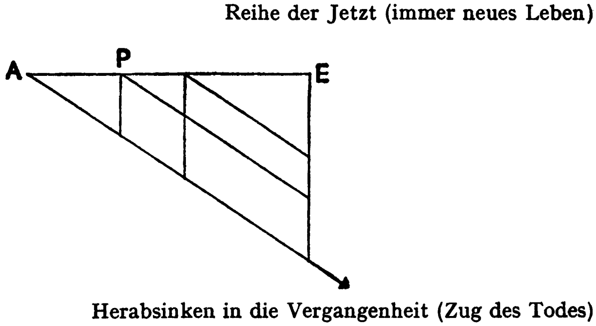


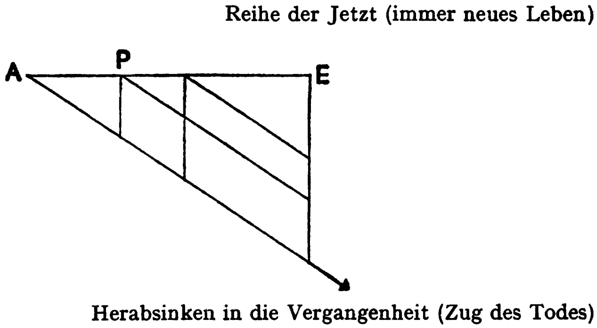
Caption: "The continuation of the now (of an ever-renewed lived experience)."

Let's stay for a few moments in this diagram. E1' designates the retention of E_1 in E_2, and O', that of O in E_2. Each E thus designates a kind of accumulation point, and each vertical segment indicates the thickness of the duration in E_2 relatively to the origin O.

The research manuscripts, known as the Seefeld manuscripts,[78] continue these investigations, and propose several diagrams, including the following two.

[78] Husserl (1969, 365).





A, P, E symbolize a series of now, in the classical linear representation of time. If the horizontal line symbolises *life*, an always renewed life, and the series of nows always arising, the oblique arrow indicates the direction of the *blurring*, operated by the retentions, for each now, which is like a progressive and continuous *dying*. The horizontal line is that of "concrete continuity" with its continuous mutations, and the previous now becomes a pure past. The oblique line shows "the series of flow modes that no longer contain any now, they grow from O", the point of origin, "to the determined interval that has the last now as its climax. Then the series of flow modes that no longer contain any now, goes down, the duration is no longer an effective duration (. . .) but a past duration that continuously sinks into the past."[79]

It is remarkable that the progression of the analyses does not necessarily result in a more complex scheme. Completing by complicating, such was the temptation or tendency of Husserl in the research immediately following the Lessons of 1905, particularly in his search for a *pure chronometry*, that is to say a theory establishing a metric (whatever) of phenomenological time considered as a fixed form of assigning temporal places and, consequently, comparable time intervals.[80] Only such a metric can account for the experience of every musician in the appreciation of durations, rhythms, pulses etc., before the construction of any metronome or clock.

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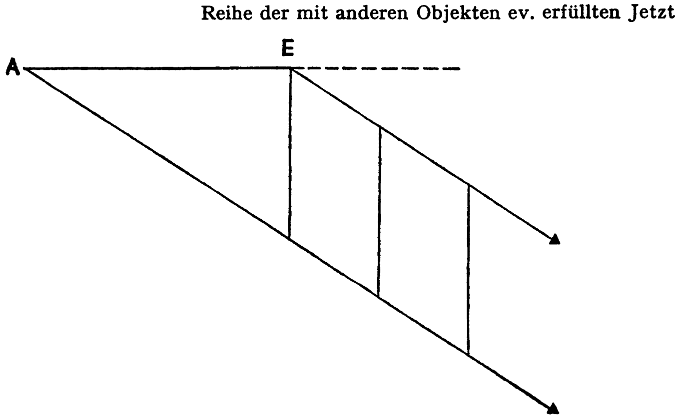
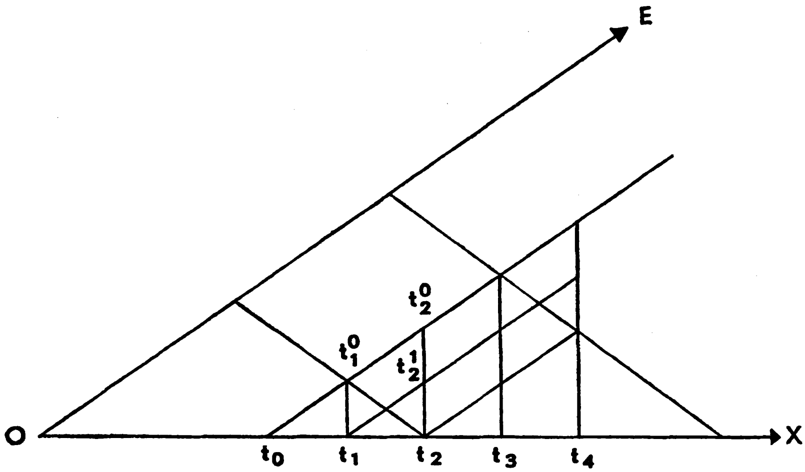
[79] Husserl (2001, 366).

[80] Husserl (1969, 235).



The same applies to the exploration of the dimensionality of phenomenological time in relation to the one-dimensionality of objective time.[81] The question that arises in the latter case, once memory and its cohorts of phenomenological flows are brought in, is whether we are not dealing with an infinity of infinities, and whether the Augustinian palace of memory[82] (which, like the soul, in some way contains all things) does not become, mutatis mutandis, a "Hilbert's hotel".[83]

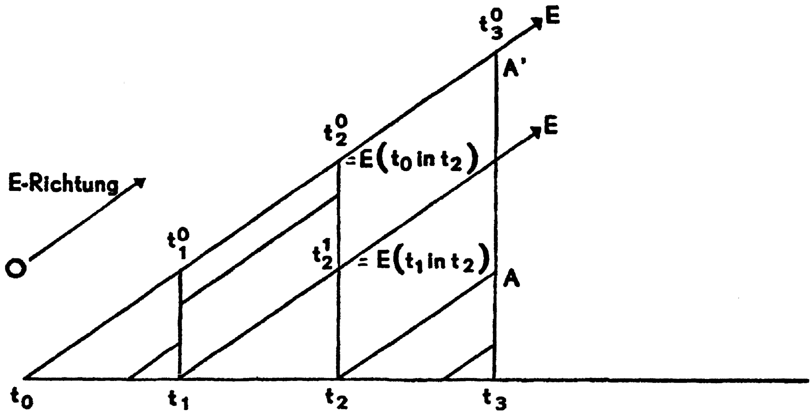


[81] Husserl (1969, 330).

[82] Augustin, *Confessions*, Book X, Chapter VIII.

[83] The question whether this analogy actually comes from Hilbert and/or first of all from Hilbert is discussed. On this subject, see an article gleaned on arXiv by Helge Kragh, "The True (?) Story of Hilbert's Infinite Hotel".

Or by introducing elements of direction combined with lengths, which would bring us closer to a *vector like theory* of temporal changes.



We are here at the source of this pure "chronometry" at which Husserl worked tirelessly and which is the counterpart of pure geometry. It provides a basic form and a system of possible forms. This phenomenological analysis should lead naturally to mathematics in so far as we are in touch with the (*potential*) infinite. More: we are at the sources of the consciousness of the *etc.,* and therefore at the phenomenological roots of complete induction.[84]

But in order to do so, we need to see how "segments" or "intervals", but also complex lines are produced,[85] by the latter, we understand a representation of an E, whose equation is $E = a + bi$ with real a, and the product bi ("imaginary") as a function of the value of b, i being $\sqrt{-1}$, where i is interpreted phenomenologically as "irreal" or "intentional", b and a as real or "hyletic", and + as the "addition" or *Fügung* introduced previously.

From Infinitesimal Analysis to Complex Analysis of Phenomenological Time

But how is it possible to understand temporal syntheses as real and continuous syntheses. As the former diagrams only deal with one type of modification (retentions), it is possible to give a simplified and more differentiated version.

One of the most important texts proposes an almost literal transcription with a beginning of formalisation or symbolisation. We can observe incidentally, how diagrammatisation and symbolisation are progressing together.

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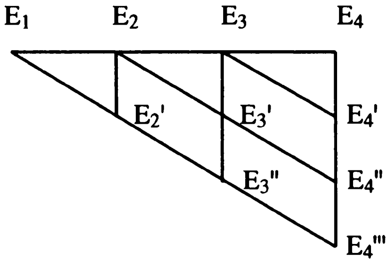
84 For example: *Rét* (*i*; [[ego]; *w* [i_s] || w[i_o] r]); (i ([ego]; *w* [is] || w[io]; r]); [...] r_n.

85 *f. g (x)* = *f (g(x), f(f'(x)* Comparison with Rota's account of "averaging operator" or "delay operators".



Let us note E_0, E_1 . . . , E_n, the orderly sequence of events or experiences *E* follow one another on the temporal line of the "real" (of hyletic flow, in the phenomenological sense of the term).

It is possible to indicate by exposing *E', E''*, etc., the degree of *sinking* in the retention of each lived experience *E*. Let us note *R* for the retention operator. We obtain the following symbolic writing: "E_0 passes to R [E_0'], this one to R[{R[E_0']}]." The experience E_0 is modified retentionally in E_0', and this retention is in turn modified retentionally in E0'. We understand immediately that this second retention of the original experience is a first retention of a second lived experience, in other words the modification of a modification of an experience is *equivalent* to the modification of a modified lived experience. Hence: "E_0" must be a momentary present content, understood as that in which the very recent past (just passed) appears. This consciousness with its content that appears in these fades should then be the present result of a change, which, for its part, would undergo a new, well-founded apprehension as the past of what has just passed. Now, E_0' changes into E_0'', and *R* into *R'*, but this *R'* would not yet be the highest degree of retention, but would be a present which would only be apprehended by a new *R*. We would therefore obtain: R{R[E_0']}' = R(R'[E_0'']) and RR' and R(E_0'') would be there.

The diagram functions as an instrument for highlighting and extracting a structure of order and dimensions that *no empirical introspection* or *self-observation* could ever reveal. New dimensions, because what is at issue is the restitution of a *temporal perspective*, analogous to the spatial perspective, where shortening becomes "tightening" (*Weiter-Zusammenrücken*). To this is added a dynamic dimension, since it can be more or less "fast". The zero point corresponds to the initial lived experience taken from the flow of continuous retentions functioning here as a *vanishing point*. We shall see that this will be important for the problematisation of the phenomenological significance of this *zero*, whose status is ambivalent, to say the least.

This analysis will open an aporetic reflection where two hypotheses are considered in turn, and which falls back on a problem already present, in another form, in Brentanian psychology: the danger of infinite regression. To understand this, it is enough to note that the retentional phases, of which we have just given some formulas for, assume that the first retention is itself *conscious and presently lived*.

It provides an opportunity for crucial methodological considerations, which reminds us that we have not left the field of phenomenological reflection, or which at least calls us not to leave it. Indeed, the problem we are considering invites us to do so. As Husserl points out, with regard to the second hypothesis, which refuses to take on the form of the momentary present, the problem it poses is in fact that of knowing "*how we can become aware of it:* for a reflection capable of noticing already presupposes a consciousness that it is already going through".

Further on, after having provisionally ruled out the danger of regression to infinity, he again faces a hypothesis which would amount to abandoning the field of phenomenological description:

> Usually, we simply have an 'accomplishment' of a modified or unmodified act, otherwise simpler or more complex, and, in relation to it, the negative modalities of accomplishment, possibly all this to be related to the components of intentionality which are themselves intentional. Such was therefore the theory, or rather the description.
>
> Will we abandon this theory, or will we change it in its essence if we say that the seizure of the original process and its phases is not facing intentional lived experiences? There would therefore be an intentional modification of the lived experiences of the original process that are not intentional and of the data originally present and the retentions *as data*. The ego can direct its gaze on the momentary attentional lived experience, and it can direct the gaze through it. But what does this mean? How does one come to understand the equating of the seizing and the being directed on an intentional consciousness and on something that is not intentional, but 'is simply is'? *On this path*, *we do not go far*. *Whoever is not shocked here has not understood the essence of intentionality*. Where do we know, this is the question we ask again, something of an original process? And from where, in it, something of the original phases?[86]

The symmetrical problem of an infinite regression on the subject of protentions is eliminated, by introducing a dynamic dimension in the processes of protentions and retentions, that is, considering "the whole process" made up of the flow of retentions and protentions, as "a flow of retentional emptying" and a flow of "protentional filling". This conjunction (*Gefüge*) will still be at the heart of the analyses on passive syntheses and announcements. We are moving from a static to a dynamic interpretation, from a rigid to a "flexible" form.[87] The following indicates the dynamics of this *Gefüge* in relation to a deeper interpretation of the relationships *intention/filling*, *empty/full*, and mentions a list of gaps in the previous diagrams and the analyses associated with them,[88] the retraction of this two-dimensionality of phenomenological time, the in-one-another (*Ineindander*) and entanglement (*Verwicklung*) and the junction or arrangement (*Gefüge*) of these dimensions.

[86] Husserl (2001, 225).

[87] Husserl (1966, 332).

[88] Husserl (2001, 8).

Gaps in the 1905 Diagrams

Let us now look at the main limitations of this diagrammatic.[89] The 1905 diagram, which symbolised (*signierte*) the consciousness of time, suffers from several limitations and gaps. It was mainly concerned with retentions as modifications of the original (impressionistic) data (of the *Urdaten* considered as hyletic). He said nothing about retentions (and other aspects developed above). Finally, it was incomplete even from the point of view of the retentions, as it did not restore their "intimate intentional structure" (*innere intentionale Aufbau*).[90]

> The diagram, which was a simple diagram of the original retentions and hyletic data, as well as of their modification, and which only indicated the consciousness of time from this point of view, was incomplete even from the point of view of the retentions, as the internal intentional structure was not adequately rendered. We have said: every constituent overall phase is a retention of a filled protention which is the limit of a horizon of an unfilled protention and, for its part, continuously mediated (of a continuum of intervals), this retention being itself a continuum of intervals, and likewise every phase, [though] in another way, as we know. (This two-dimensionality must also be found in the empty expectation, inasmuch as it is also a protention directed towards future retentions). But every retention as a retention of retention must be conscious of all this in a modified way.[91]

The revisions concern the fact that retention is exercised either globally on the temporal lived experience or event, or locally, that is, according to the objective and subjective orientation (noetic and noematic) and according to the distinction between "second objectivity" (noetic) and hyletic. According to the former, we end up with a doubling of phenomenological time into noetic and noematic time. Following the latter, we must split it into a hyletic phenomenological time and intentional phenomenological time.[92]

But multiple ambiguities then arise which lead to a revision of the schema apprehension-content of apprehension. Here we face again the objection of regression to infinity.

Let us try to understand how and why this regression occurs, and how its solution will involve a new diagrammatic work. There are many difficulties in reconstructing the real or continuous changes, as presented and set out in the *Ideas*, while they continue to be based on the 1905 diagrams.

Husserl has repeatedly positioned himself in relation to these analyses. On the two meanings of "immanent", depending on whether one speaks of the datum perceived by the constituent consciousness, e.g. a sound, or of the consciousness

[89] Husserl (2001, 7).

[90] Husserl (2001, 7–8).

[91] Husserl (2001, 7–8).

[92] Starting from text 8, which is quite clear on this point. Husserl (2001, 134) or Husserl (2001, 137–138). This also ensures the transition to the next point on modalities (Husserl 2001, 137–138). On these two points we may refer to text No. 6 and in particular to § 5 Husserl (2001, 121–124).

of sound, perception of sound, consciousness of this perception.[93] The thesis of the *Ideas*... I according to which *hylè*, *noesis* and *noema* are constituted in phenomenological time is amended. The error of the *Ideas*... I in this respect is to have separated the study of the two groups of modifications which we call real and intentional, whereas they are interrelated and interdependent. Does this compromise the 1913 analyses?[94] Answer no: they were correct in the main lines, and in the constituent analyses it was possible to ignore them, because 1) they are relatively independent and 2) the temporal constitution is something common, identical in essence and therefore incapable of distinguishing between types of consciousness. But a corrective introduced by texts 6 and 8 is necessary. This corrective[95] consists in distinguishing the temporal modalities proper from the "data modes" of the hylè. Even if they are parallel and coincident, they must be distinguished. Finally, on the status of phenomenological time which *Ideas* I have assimilated to "primary objective time". Text 12, of uncertain date, specifies[96] that the flow of modification is one with the flow of the emergence to new [that of the always new impressionistic data] (where, of course, the original emergence of new data of events can stop). So, we have an original process, with an original engendering and an original mutation of the engendered. This comes out in the essence of all the most intimate transcendental data of consciousness, including the most original intentionality in which the first "objective" time (in a certain sense) arose with the first transcendental events, phenomenological events. Among the novelties, the opposition between the two fundamental groups of modifications (*intentional* and *real*) is relativised, despite the fact that we are dealing with supposedly absolute phenomena of consciousness.

The temporal modifications are, as we have seen, operations which apparently concern the *real content* of consciousness, e.g. of sensory data, which Husserl will symbolise by a function: *Ux*. But as we know, temporalization is above all a temporalization of complete lived acts; it is therefore itself carried out according to the *noetic-noematic* correlation. It is therefore not possible to limit the study of temporal modifications, whatever they may be, to the real components alone, to the *hylè*. Within the framework of an overall analysis scheme, which guided Husserl in particular in his investigations on space, these sensory data served as a support for the constitution of objects, i.e. of "objective sense", from the moment when animation or interpretation functions, which he calls *apprehension*, entered the scene. This is the famous apprehension-sense of apprehension scheme. Now the analysis of the consciousness of time will be the occasion for a profound revision of this overall scheme. As Husserl writes: "The question arises as to whether it still makes sense to speak of apprehension. Of course, the hyletic moment is a nucleus of dependent consciousness which is clothed with functions; what counts above all

[93] Husserl (2001, 109).

[94] Husserl (2001, 119–120).

[95] Husserl (2001, 115–116).

[96] Husserl (2001, 224).

is to systematically seek, through the accomplishment of all possible reflections, the structures which are found in the pure and simple intentional object (ontological), in the noematic and really in the consciousness itself."[97]

Let us begin by correcting the error, which consists in partitioning the groups of real modifications and intentional modifications.

> Above all, we must be careful not to mix the different registers, the data of different reflections. The consciousness of now, the consciousness of the original present, the original conscious having of the object-temporal point and the conscious having that is adequate, not presumptive, so to speak complete, is assured since we are dealing with immanent objects. And we can say no less surely that the flowing consciousness of the temporal object point in the phase of which I am now conscious in the present tense actually contains the corresponding hyletic *datum*, so to speak as a nuclear content that really belongs to the consciousness of now and is the 'substrate' of the noesis that sets the temporal point with its content and in the mode of the original apprehension and position. We can then express ourselves in this way: 'the objectual content 'represents' (*repräsentiere*) itself in consciousness in this heretical moment'—just as we could also say that the formal, the temporal point and its noematic mode of the now 'represent' themselves in another sense or in another way in the real moment of the apprehension of the present. But not everything can be said about the different 'substratum' and 'representation' (*Repräsentation*); and what this means should be drawn from the living itself, from the real and intentional analysis. Otherwise one runs the risk, at first very tempting, of elaborating the consciousness constituting temporal objectivity from something like immanent objective sounds, etc., which are the basis of the consciousness of the temporal objectivity, and their apprehensions, and to ask completely false questions, such as, for example: how should we interpret in the consciousness-of-passage the self-changing of the conscious sound point in the now, of the original really there as a sound point in its fainting, if these faintings are new sensations and or reproductions, or modifications of a completely specific kind, and so on.[98]

Far from engaging us in a "material" or "hyletic" phenomenology, the study of the ultimate levels of the constitution of time, i.e. of the "flow of consciousness", rather reveals to us "*ultimate noetic formations*" which carry within them all the weight of the constitution. And as the note we follow here concludes: "*With this recognition, we also find ourselves questioning a declaration which runs through phenomenology from 1905 to 1913: that of the absolute character of consciousness and in particular of flow*. The question has every reason to be asked if one should not recognise, even in the immanent sphere released by pure reflection, *a form of relativity*. Flow is the depository of 'ultimate noetic formations or structures', whose fundamental character for consciousness and for phenomenological reflection itself must be established." For this very reason, it is necessary to "show how they carry everything in themselves and how the consciousness that carries them in itself, related to itself by objectivising, can itself make conscious..."

Far from engaging us in a "material" or "hyletic" phenomenology, the study of the ultimate levels of the constitution of time, i.e. of the "flow of consciousness", rather reveals to us "ultimate noetic formations" which carry within them all the

[97] Husserl (2001, 161).

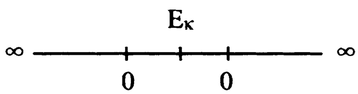
[98] Husserl (2001, 162–163).

weight of the constitution. And as the note we follow here concludes: "*Finally, noesis and noema become the central problem.*"[99] With this recognition is also called into question a statement which runs through phenomenology from 1905 to 1913, that of the absolute character of consciousness and in particular of the flow. The question has every reason to be asked if one should not recognise, even in the immanent sphere released by pure reflection, a form of relativity. Flow is the depository of "ultimate noetic formations or structures" whose fundamental character for consciousness and for phenomenological reflection itself must be established. For this very reason it is necessary to "show how they carry everything in themselves and how the consciousness that carries them in itself, related to itself by objectivising, can itself make conscious intentional objectivities through noematic data and backgrounds of meaning".[100]

An Important Gap in the Initial Retention Diagram: The Meaning of the Zero

The description of the temporal modalities and modalities from a noematic point of view is therefore unavoidable. Thus, in a text from the beginning of 1917,[101] Husserl takes up the study of the consciousness of time on a new basis to point out new gaps and ambiguities.

In this context, the zero in the axis of time acquires a new meaning and therefore becomes ambiguous. The zero is introduced in relation to the notion of real change. The zero symbolises "the original present", i.e. a consciousness of an unmodified original which is, at the same time, consciousness as unmodified.[102] But it then takes on the new meaning, that of a limit of intuitiveness. As there is a maximum of intuitiveness in relation to retentions and a minimum in relation to protentions, two zeros must be assigned to the same source experience. Hence the diagram:



The zero is introduced as a limit in the formal moment "now". The zero corresponds to the *form* or the *moment of form of the "now",* which coincides with the actuality and the accomplishment of the act, just as the *just past* (or *just*

[99] Husserl (2001, 163).

[100] « wie das Bewusstsein, das sie in sich tragt, auf sich selbst objektivierend bezogen, sich selbst durch noematische Gegebenheiten und Sinnesbestande, intentional Gegenstandlichkeiten bewusst machen kann ». (*Ibid.*)

[101] Text 8 from September 1917. Husserl (2001, 142–150).

[102] Husserl (2001, 211).

passed) or the *happening* or *becoming* are moments of form. If we remember that from the point of view of functional analysis, the intentional character of any act is determined by a modification, we are entitled to say that any present modification, and its hyletic substrate, represents a zero. In fact, the changes that we have just been talking about will be reinforced or diminished in a continuous and gradual manner. The zero indicates the *absolute starting point* of this continuous sequence of modifications. But, since temporalization (retention or protention) is a modification of this zero, insofar as the zero (the "now") is itself modified, we are led to admit *relative zeros*.[103]

The flow of consciousness is a flow of modifications and it is therefore necessary to fully assume the operative character of all modifications, including continuous real modifications (thus that of time).

> The 'modification' then refers, as it were, to an operation which always takes place in the same direction. The operation is the continuous living flow of consciousness itself and designates its particular intentional performance, in constant evolution, a continuous flow of noetic components, each of which is, according to its 'form', a constant modification of previous modifications and which exists according to its own meaning.
>
> Although this operation 'progresses in only one direction': that is, insofar as it is a flow according to the noesis and, correlatively, according to the noeme, we are dealing in each phase (we leave the marginal phases out of the picture for the moment) with an *nesting of two 'modifications' which behave in opposite ways*.[104]

We come to the reason for the *nesting* of which diagrammatic translations should be considered. But it must first be observed that the determination of a zero as the relative starting point for these two series of modifications, these two "complementary" and nested "branches", despite its apparent symmetry, must not conceal some asymmetries, which are both modal and dynamic. The temporal modality of one, that of retentions, "has the character of the 'completed' of the closed, of the determined"; the other, that of protentions, of the "not completed, of the presumptive open, of the in some way indeterminate". But especially if we take into account the dimension of intuitiveness and, consequently, the essential relations between fulfilment and intention.[105] It is then necessary to begin the analysis of both the marginal and internal phases of these series of modifications.

[103] Husserl comments on this new symbolisation in the following way: "If the continuous series of numbers starting from 0 is a symbol for the continuous modifications of the moment now or 0, each continuous number symbolizes 0 relative for the higher numbers. But this means, since the symbolism is in fact an exact expression of the noematic situation, that not only is each change in itself characterised as a change of the 'now', but also, as a relative and corresponding change, diminished in relation to a previous change of 0. Take for example 0 ... a ... b; b denotes the b^{th} change of 0 and simultaneously the b^{th} change of a. a is therefore also to be regarded as a relative 0 (a relative now) for b as its change a - $bt^{h} = c^{th}$. Each b past is not only a past now, but a *past past*, i.e. for any past that is closer to the present. The 'now' is therefore either the form of the original present tense or the form of the relative present tense, i.e. of a modification, i.e. in relation to modifications thereof". Husserl (2001, 142–143).

[104] Husserl (2001, 144).

[105] Husserl (2001, 148).

These analyses touch on a fundamental notion, that of *temporal horizon* (retentional or protentional) and with it, that of *temporal perspective*, which gives rise to an instructive development on the limits of the parallel between the phenomenology of space and the phenomenology of time, as well as to a very valuable and suggestive terminological clarification, on the notions of *aspect*, *sketch*, *interval*, *point* of time and temporal object,[106] which we must limit ourselves to indicating, and which highlight that before the spatial perspective, to which Husserl refers in passing, the roots of the "projective" character of consciousness are to be found in a more fundamental structure.

This motive insists throughout a text devoted to the phenomenon of fainting. Between the zero of fullness and the zero of cessation, the intuitive interval, with its "modal staggering"[107] is placed. Husserl comments further on: "If I attach to the objects that last the first phenomenological descriptions, it is necessary to place at the very beginning the description of the phenomenon of perspectival shortening, which occurs here in the first field of the having-been."[108] Further on, considering the passage of the field of temporal orientation and the constitution of the "temporal horizon line", Husserl characterises the latter by analogy with the horizon line of spatial perspective. Just as the latter "is the *Ultima Thulè* [sic] of distance" so that "new qualitative distances of orientation can no longer occur", so "from the now and its temporal point (. . .) the temporal intervals of the retained event increase" without the "temporal orientation" of what is retained changing after I have reached the extreme 'distance'". As Husserl observes: "The expression 'temporal orientation' is therefore ambiguous here. In a sense, it is what we better refer to as temporal perspective. (What is the horizon technically called in perspective theory?) In another sense, it is the specific orientation, the continuously changing distance (. . .) from the now." This is a new gap in the diagrams: "We have on both sides the problem of having to 'explain' these phenomena, and first of all the phenomena of temporal and spatial perspective. In our analyses by diagrams, nothing appeared about this."[109]

As for the direction of the fainting, it seems that we are heading towards another kind of zero, of a projective type: "we can only allow the apprehensions that constitute time to fade away everywhere in a single point direction; they fade away as a modification that consciously carries the modified acts within itself and that, therefore, does not require *in infinitum* new series of fainting that would be built on top of each other. Does this completely remove the difficulty? Wasn't there still the fact that each empty moment of consciousness fades away in the manner of the modification, and finally to the 0 beyond which I can no longer find anything intuitive, and so on? It seems so and therefore the answer is unsatisfactory".[110] This

[106] Husserl (2001, 151).
[107] Husserl (2001, 97).
[108] Husserl (2001, 70).
[109] Husserl (2001, 73).
[110] Husserl (2001, 81).

text is dated from the beginning of 1918 and we then begin an analysis of the status of 0 through rather aporetic developments.

Text 11 of *Bernau's Manuscripts* is particularly important in order to grasp the link between symbolisation and diagrams, the point 0 appears as an index of the originating experience given in the originating present? As Husserl writes, "the now as a form of conscious content (. . .) is continuously transformed and the 0 (the now) is the starting point of this continuous transformation; the *0 is therefore the symbol of unmodified consciousness*" (Emphasis mine). Another manuscript takes up the study of the 0 in relation to the theory of modifications and integrating from the outset the two dimensions of retention and protention. Modifications on both sides are producers of forms and the now itself is a form, the original form or fundamental form "in relation to which all other forms are modifications". We arrive at the following definition of modification: Modification here implies a proper character and a proper relation to the original form or basic-form: *just-past* implies as much as *just-past-now*, *coming* implies *coming-now*, or rather: the 'object' conscious as *past* or *coming* is characterised in this consciousness as *now-having-been* or *future-now-becoming*. In the forms as noematic sense-forms, the sense is now, but 'modified'? Modified: what is *past* is *not now*, '*no longer' now*, is a *past now*."[111]

Diagrams of the on-in-the-Other (or Nesting) and Intertwining of Retentions and Protentions

This is one of the most important revisions of the 1905 diagram. Several diagrams are proposed.[112] A first shortcoming of the 1905 diagram is the absence of representation of the lines of protention. It is a question of highlighting and formulating the laws of essence which govern "the succession of concordant momentary phases in a general style and thus also to insert particularisations in the internal structure of each momentary phase. This is the purpose of the new complete diagram of the process".[113]

However, this new diagram is still incomplete, since it does not give any expression to the phenomenon of *temporal perspective*. Nevertheless, it is with this diagram that a *symmetry* in the figuration of time has been introduced for the first time, since the protention branch and the retention branch meet at a zero point, which is also a *neutral* point, whose status we will have to clarify later.[114]

[111] Husserl (2001, 142–143).

[112] The same applies to Text 11. Beginning and middle 1917 In particular § 10, "Attempt to differentiate the graphic representation (diagram)", with the appendices, in particular Appendix VIII. Husserl (2001, 210f).

[113] Husserl (2001, 33).

[114] On this point, let us recall that neutral corresponds to a fundamental modification: neutralisation, the species of which are the imaginary modification known as modification of the quasi

Husserl then discusses what he calls the *interweaving*, *intertwining* and *interlocking* (or *nesting*) of protentions and retentions.[115] The aim is to make intelligible the way in which an awareness of the passage of time can occur. And from the outset the threat of regression to infinity looms, since such a constitutive process must be formed before the completion of the formation of a constitutive process in general, that is, before the consciousness-having of a time-object. The solution will come from the genetic path of phenomenology and the introduction of a notion that has fascinated as much as it has remained misunderstood: passive syntheses, of which temporal syntheses, continuous or possibly discrete, represent a fundamental form. The phenomenological reconstitution of the emergence of this consciousness-having from an original flow comes back to the explication of the idea of the awakening self, of a self whose life is beginning, and inversely of the mutation of this dormant life into a life of consciousness, vigilance. As with all borderline questions, phenomenological idealisation intervenes to account for a phase of constitution.

> Does this idea point to a possibility perhaps? In any case, we can assume, at the very least, that the continuous unfolding of the hyletic data—as the core data of a constitutive process—necessarily (. . .) leads to (. . .) the continual and correlatively continuous trains of trains, and that according to another fundamental legality, these retentional apprehensions are transformed into nuclei of apprehension, in the sense of our diagram. But according to a necessary law, there is not only a retention that follows the course of a 'differential', but also a protention that is directed towards what comes, [according to] a most universal law, determined according to its content (if a sound has begun to resonate, then there is also a sound to come, even though the precise manner of the intensity or quality relations remains, for the protention, indeterminate etc.). From now on, each phase of the process is a retention segment, a point of original presentation as a filled protention and an unfilled protention segment.[116]

However, it must be borne in mind that, in the middle of the process, each retention must be a retention of a previously filled protention and its empty horizon, that, in the continuous series of phases of this segment, each subsequent filling, in a retentional modification, in itself houses a point of the previous protention; that there exists, with respect to the unfilled, an overlay, and that, in the continuation, the unfilled persists, is preserved in the manner of a segment in its unfilled being, just as the filled as such persists through the segments of the retention, in spite of a temporal objectification of a higher level. Hence the conclusion, the phenomenological status of the now is becoming clearer. It is not to be confused with the irruption of pure hyletic data (impression, sensation), as suggested by a somewhat short and precipitous analysis, which constitutes a major obstacle to the understanding of the phenomenology of time. But in a form, and a form of a dynamic process

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and the various forms of épokhè or doubt that phenomenology analyses. On this point, see Lobo (2020).

[115] In a text entitled: "*Das Ineinander von Retention und Protention im ursprünglichen Zeitbewusstsein. Urprasentation und Bewusstsein der Neuheit*" (1917) Husserl (2001, 13–14).

[116] Husserl (2001, 14). (Trans. mine).

involving from the outset, at a prelogical and even preconscious level, the relations between an embryonic and partial form of intentionality and a filling. Moreover, it consists in the modification of a form through a dual dynamic process, one part of which consists in a filling and the other in an emptying. Thus, a proto-identity of temporal object is constituted, a “same”, a sound, for example, that it is likely to identify at higher reflexive levels. Husserl continues: “The now is constituted *by the form of the protentional filling*, the past by the retentional modification of this filling; in the continuity of identification, which crosses the un-filling [*emptying*], the temporal point is the same, the same as the one that has become conscious as now, become conscious as just past, etc.” (Hua 33: 14). It is therefore possible to push the problematisation further in the direction of a new hypothesis: that of a precedence of protentions over retentions, and of a kind of “originating simultaneity” corresponding not to the coincidence of two points of time on distinct time lines, but, internal and originating, between a retentional modification and the corresponding protentional modification, as well as, point by point and segment by segment, the coincidence between two processes of filling and emptying.

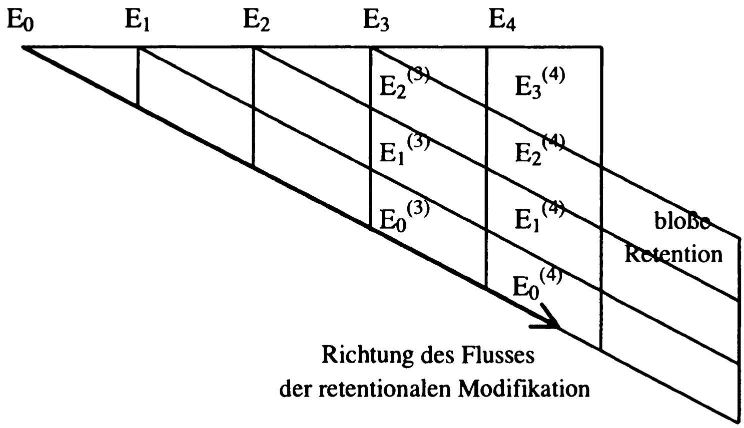
> The question closely related to this is the following: is retention in general first of all an effective retention of an object-point-temporal and of an identical point due to the fact that a protention has already produced a now and consequently something identifiable also in different modes of donation? [...] The difficulty here, however, is that in the middle of the process any retention must be retention of a previously filled protention and its empty horizon, that in the continuous series of phases of this interval, any subsequent filling in retentional modification conceals in itself a point of the protention that preceded it, that an overlapping takes place with respect to the unfilled, and that in the progression the unfilled crosses and persists in intervals in its unfilling, in the same way as the filled as such crosses through the intervals of retention, albeit with a higher temporal objectification. The now is constituted by the form of the protentional filling; what has passed [is] by retentional modification of this filling; in the continuity of the identification that crosses the recess, the temporal point is the same as the one that has been conscious as now, that has been conscious as just past, and so on. This raises the following question: *is pure and simple retention only effective retention of a point-temporal object and an identical one, only in so far as the protention has already created a now and, simultaneously therewith, something identifiable even in its different modes of givenness*?[117]

This text will lead to difficult questions that we will simply point out. One of them concerns the implication of eidetics from this primitive constitutive level. It is absolutely unavoidable for a phenomenology which wants to be eidetic, transcendental and genetic. The texts dealing with the problem of individuation

[117] Hua 33: 14. Same scruple and same answer in Hua 11:133: “But here it seems really necessary to say that all the tenses that ‘coincide’ here really coincide as an identical time that is constituted, and that all series of individuals are therefore entitled to simultaneity in that time.” Husserl (1966, 133).

deal with it explicitly, but a satisfactory solution only seems to be approached in later texts, with the lessons on passive synthesis.[118]

In order to stick to the double motif of interlocking and interlacing, let us turn to a later text[119] which presents an attempt to represent the interlacing of retentions and protentions by diagram:



Which Husserl describes and comments in these terms:

The verticals 'represent' the momentary consciousness with its original present point E_k and its retentional time interval in the manner of the layering of the past (*Vergangenheitsab-stufung*).

2) But, also the consciousness of the original present with its continuous retentional escort, as retentions of elapsed intervals, where the interval indicates at the same time the continuum of *the present and retentional original data nuclei*.

3) In addition: any partial interval, e.g. $E_1^{(4)-}$ $E_2^{(4)}$ (or in reverse order), is a retention of all previous intervals of the same flow segment, which relates to the same lower indices, i.e. up to $E_0^{1-}E_1$.

4) Even more: starting from the differential, the vertical interval increases in the continuity of the vertical rows in the primordial flow. Each interval (with each of its differentials) sinks and increases upwards, or from above, by the continuous addition of a point of origin as the origin differential. The past is enriched as long as the original presence manifests itself.[120]

From this diagram, itself enriched, we move on to the construction of a diagram of protentions, whose "dynamics" and differentials correspond to "changes of

[118] Texts 16, 17 and 18 of Hua 33. The connection between the problem of temporal individuation and that of the eidetic constitution of items identifiable as tempo-objects is made in § 28 of the SP, "Synthesen der Homogenität in der Einheit einer strömenden Gegenwart" Husserl (1966, 128–133). The joint solution is made explicit in Appendices XV-XIX, especially XVII Husserl (1966, 398–404).

[119] Appendix I to § 4 du Text 1 Husserl (2001, 15–19).

[120] Husserl (2001, 15).

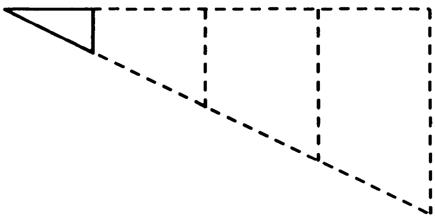
speed" (which even in a metaphorical sense remain problematic and call for elucidation).

> As for the protentions and their 'movement', it is the same as for retention. Any retentional series arises originally gushing out of its E_k, but the phrase that originally gushes out is only a point and any later point in the series has its original gushing out phase in an earlier E, so finally the whole series, e.g. E_4–$E_0^{(4)}$ in E4 to E0 on the horizontal axis. An originally gushing retentional phase is the original differential that will intentionally start in each E_k. Would we therefore also have a *protentional originating differential*, the protentions originally gushing out in each E_k? How should it be described?[121]

The first problem, however, remains that of the description, which is at one with the construction of the diagram itself. And an apparent internal contradiction arises, which enlightens us about the intentional essence and the gnoseological functioning of the diagram. The enterprise of completing the diagram comes up against an apparent impossibility here: how to complete a diagram by integrating the figuration of incompleteness into it? A first attempt and temptation is to complete the diagram of retentions, with straight lines, using dotted lines instead of solid lines.

> It should be said: E_0, as soon as it starts with a differential, an empty and already there protention which as a figure, has the form of the complete schema which is only drawn, so to speak, afterwards with the process, except that, if the process constitutes an unknown event, *its content is defined in a very incomplete, though never completely indeterminate, abstraction made of the schema (diagram).* And so it is that in all E_k, next to the schema of the retentions, there is also a schema of the protentions, which is distinguished by the fact that the protention is directed towards the future constitution, or the future event (. . .), so that the protention passes from the given and updated vertical series to the following ones.[122]

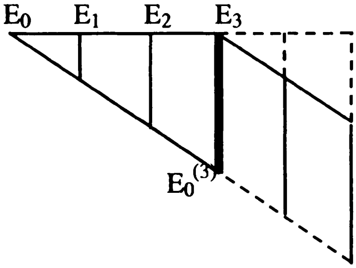
The diagram of protentions thus suggested is modulo a certain time lag with that of retentions. "So, in E_0: For E_0, we have the same diagram, but dotted."



For E_3, the current retention scheme and (purported) retentions beyond E_3 and under E_4 continue to apply. If we choose E3 as the living present, we obtain the following mixed pattern.

[121] Husserl (2001, 15). (Emphasis mine).

[122] Husserl (2001, 15). (Emphasis mine).



The diagram of the protentions is partially overlaid by the diagram of the retentions, and the attention is focused on the verticals which materialize the constituent "simultaneity" between retentions and corresponding protentions. A new diagram emerges which is meant to be complete, in which three types of lines intervene (solid, thick and dotted) which sets in motion a new phase of eidetic exploration mediated by a symbolic translation. Husserl describes it as follows:

> The dotted (blurred) lines and the *characteristic full diagram* mean that, also in the solid lines, thanks to the mediated retention, it is possible to move on to the past, which has already presented itself as filled. On both sides we have simultaneous possibilities of retro- and pre-remembrance (intuitive). The present horizon filled with the solid lines is symbolised by the broken line [*Strichelung*], and then the continuous sequence of vertical segments symbolises the medial character of the protention; the continuous order in the vertical series symbolises this medial character, which is provided in the future retentional tiering.
>
> The bold line means that the retention segment E_3-$E_0^{(3)}$ is the *living segment*, and *it emerges as the filling of the protention*, E3 as the originally flowing 'fullness' (the protention that originally deafens) and continues; there are the segment points or differential fullness of retentionally modified protentions advancing, whose horizon passes into the filling phenomenon. (Ibid.)

But there is still doubt as to the relevance and validity of such symmetry. "*It must be seen from the figure*, to what extent the protention is a reversed [turned, umgespülte] retention; it is a modification of the retention, which in any case 'presupposes' a retention in a certain way."[123]

$$E_0$$
$$E_1 V^1 (E_0)$$
$$E_2 V^1 (E_1) \quad V^2 (E_0)$$
$$E_3 V^1 (E_2) \quad V^2 (E_1)\, V^3 (E_0) \ldots$$

Or, better:

$$E_2 \;\; V^1 \{E\, V\, (E_0)$$
$$E_3 \;\; V\, [E_2 \;\; V \{E\, V\, (E_0)\}]$$

[123] Husserl (2001, 16).

This new symbolisation which puts him on the steps of theorems, or if not of phenomenological axioms.

> The original process 'symbolises' only the retentions, the protentions go from top to bottom, from series to series, thus filling up continuously. One is aware of E_0 as the limit of the retentions in their continuous mediation and in a certain way also as the limit of the conscious protention also retentively accompanying the retention as the zero limit of the series of filling.[124]

The now (E_0) acquires an ambiguous status and sense: as the limit of retentions and the limit of protentions. As a result, the items symbolised by the series E_n see their definition clarified and enriched.

> From the outset, the constituent process is indicated for the constitution of the duration segments $E_0 \ldots E_n$. But as a result the system of the original hyletic data stands out and, for the temporal phenomenological objectification, we speak of the corresponding original data, more generally, of a continuum of core data, which function as data of apprehension in the consciousness of time, of points of time of the objectified process, which are in a certain way these data in the form of points of a duration, and these are themselves constituted in variable modes of data, in a changing now, in a changing just past etc..., therefore continuously oriented relative to the now, and correlatively, to the point each time given to the consciousness in the mode of the 'now'.

Furthermore, from a point of living consciousness, the E0 becomes the index of an unconscious. Reactivating the intuitions at the heart of the analyses on the acts of collective synthesis that constitute numbers, Husserl finds at another level, some of the characteristics of the zero. Simultaneously, the Brentanian schema which guided them and hindered the development of a genetic constitutive phenomenology is again called into question.

> If the core data are symbolised by $E_0, E_1, \ldots, E_n$, then they never present themselves, except as a filling of protention, except for the starting point E0, which is in the literal sense of the word *'unconscious'* and only comes to *'consciousness'* in a mediated way, through retention; it is in this way that it comes to be grasped. But this means, however, that it is not itself as E_0 (in the original) a content of apprehension, unless the event it introduces is already awaited by a pre-remembrance = an expectation. (For this reason, just as it can be retro-directed on itself by the retro-radiation of retention, so the attentive grasper can be directed on itself and if it comes to it, then it is 'welcome').
>
> [...] The E_0 must therefore be referred to in the same way as E_0, inasmuch as we refer to the limit as consciousness. (It is in the same way that we designate the zero as a number).[125]

The E_0 is then like a *zero of consciousness*—an "*unconscious*" at the heart of presence. But other texts suggest, on *the contrary, that these are the points of original consciousness. The E's designate, in each "nodal continuity*" or "*Querkontinuität*" (in the diagram these are the lines drawn vertically) the borderline places, in which the consciousness of the now is constituted by the E_0 and its apprehension. It is preceded by a "*differential*" $E'_0 \ldots E''_0$ in which simple retention is exercised

[124] Husserl (2001, 16). Emphasis mine.

[125] Husserl (2001, 17–18).

and only stretching establishes unity, and thus motivates a protention directed towards *'continuation'*. Now each transversal series of our writing (*Schreibung*) is a filling of a protention directed towards it of the transversal continuation "immediately passed". The latter is still conscious through the corresponding piece of retention, which would be the total transverse series, except for the initial phase. But of course, in our transversal series, nothing is said about the protention that is directed onto what follows.

"If $E_k V$ ... the transverse series without protention, that is the closest line, $E_k + 1$ (V{E_kV ...}), then we must complete the writing of the first one as follows:

$$\begin{matrix} E_k & V \\ \downarrow & \downarrow \end{matrix}$$

"E_{k+1} now not only symbolises E_{k+1}, but the apprehension of it as filling, in the same way the V(E_kV ...) is presented as filling. But what is missing in this writing, then, is the fact that the perpendicular arrows that the previous E_kV had."[126]

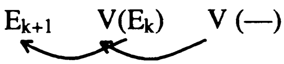
Husserl then writes:

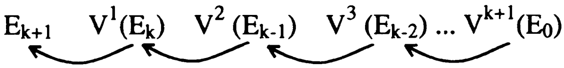
$$\{V\ (\underset{\downarrow}{E_k}\ \ \underset{\downarrow}{V}—)\}$$

But then a new question arises. Does this mean that a filling in any momentary consciousness (of the phase), for the reason that the previous momentary consciousness only comes to the surface of the consciousness in the now, only retentively?[127] Hypothesis that Husserl renders by a linear graph, with retro-directed arrows.

$$K_{k+1}\ \ V\,(E_k)\ \ V\,(—)$$

Or more clearly:





> It is surely correct that the protentions go from transverse rows to transverse rows, and that every new transverse row has a corresponding change in filling. But in the transverse rows themselves the 'filling' is formed as 'filling' and this continuously, because the retentions are continuously maintained as a phase of the transverse series. In each cross-sectional series, modifications of all the 'previous' E's occur, and a continuous fill, itself continuously modified, and with the limit point of an effective fill, with the core data (with the always

[126] Husserl (2001, 18).

[127] Husserl (2001, 18–19).

> new core data E_{k+1}) links all these data and gives the cross-sectional series, with regard to the 'E', a unity.[128]

To sum up, the given content of the living present is therefore not a pure given, but a given with a form by virtue of a system of conjugated modifications, so that it is un-realised and un-"absolutised", since every zero becomes a relative complex point.[129]

> The new *datum* as new is thus signalled to the consciousness in a special way. It is the datum in the character of the now. On the other hand, this is to be understood *cum grano salis*. For it is a simple limit of the continuum of the modes of datum and the now is only a now in the series of the 'now past'. The retentional phases of the cross-sectional series are at the same time a now reproduced at constantly different levels, a reproduced filling of intention on core data. This also applies to retentions as relative 'presences' (*Gegenwärtigkeiten*), except that the initial phase is pure impression, what is now as new happens, not as an impoverishment of an old one, as retention. It seems that the structure of the consciousness of time has in this way become clear and intelligible again.[130]

Corollary: while for Kant and certain neo-Kantians, such as Lange, the awareness of time continued to represent the ultimate link between classical transcendental aesthetics and mathematics (arithmetic), the analyses given here throw a retrospective light on the correctness of the Husserlian position at the time of the *Philosophy of Arithmetic*, in a debate that pits him against Herbart, Cantor and Frege. It is no less instructive to see with what certainty Husserl was already pointing out the logical presuppositions leading Frege to sweep aside the objections of his opponents by proposing an arbitrary nominal definition of the number ("is a cardinal number any answer that answers the question how many") without taking the trouble to observe that in addition to the positive answer, excluding negative or indeterminate answers, and consequently the modalities.[131] This passage serves both as a challenge to the Fregean foundation of arithmetic, to its restrictive definition of formal logic and, consequently, to the thesis of Frege's influence on Husserl.

Diagrams in Reflection

We have seen on several occasions that the possibility of a reflexive grasp of the elements involved here was constantly in the background of the Husserl analyses.[132]

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[128] Husserl (2001, 19).

[129] It should be noted in passing that the anchoring of classical physics in transcendental aesthetics is suspended in favour of plural and open constructive possibilities.

[130] Husserl (2001, 19–20).

[131] Husserl (1970, 129–134). But these reflections are themselves too allusive and rapid. For a more in-depth examination of some of these assertions, we would like to refer to a series of works that dig this same furrow: see Lobo (2002, 2012, 2017a, b, c, 2019b).

[132] For a more meticulous approach, the following passages should be taken into account. Husserl (2001, 28, 258–273).

In view of the reflexive nature of phenomenology, it follows that the diagram of time consciousness must simultaneously account for reflection on the original data and the original process and its constituent moments.

These reflections should be completed, taking into account the fact that these symbolised and diagrammatised phenomenological descriptions are *eidetic*, as are the ideal objects of geometry. Consequently, whatever the individuality of the data and events in question, they must have sufficient resources and content within them to support this conversion from looking at an individual element to looking at a typical and eidetic element. Of this, Husserl does not propose a diagram, for the reason that all diagrams are by definition figurations of essence relationships.

Finally, let's stick to the reflective dimension. The analysis of the diagrams of the reflexive temporalization and of the reflection on temporalization take up a variant of the above-mentioned objection addressed to the constitution of time by retentional retro-reference of the constitutive consciousness of time to itself. But it takes a radicalised form: that of an objection of regression to infinity. This is why we come to suppose that the lived, our E's would not be themselves constituted,[133] but simply given. In short, *Urdaten*, U_x, in an absolute sense.

In order to solve this problem, without any residue of naivety, Husserl thus comes to a scheme of retention integrating the transcendental reflection of the first level. The primary events are noted E_i, the retentional modifications are noted V^j.

$$\begin{array}{l|l} C_0 & E_0 \\ C_1 & E_1 \;\; V^1(E_0) \\ C_2 & E_2 \;\; V^1(E_1)\; V^2(E_0) \\ C_3 & E_3 \;\; V^1(E_2)\; V^2(E_1)\; V^3(E_0) \end{array}$$

> In the single ultimate transcendental process with its original emergence and its original disappearance, the event E and the event of the constitution of E, are conscious in different directions of reflection. The diagram shows abundantly how this is possible.[134]

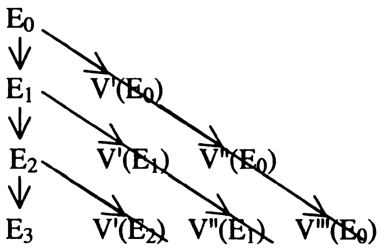
The lines take on a vectorial meaning, and the formulas that can be extracted, stating new laws of time, are similar to a vectorial calculation, whose element of combinatorial freedom is founded in the free mobility of the reflection itself. We can thus affirm "that the series of consciousness constituting series E becomes perceptibly conscious as a continuous series, as a temporal stretch (objectively given in flesh and blood) by the fact that $C_0 = E_0$ occurs first, then $C_1 = E_1V^1(E_0)$ and

[133] Hua 33: 260 (§ 2, Husserl (2001, 260). (Der Einwand des unendlichen Regresses: Sind die Erlebnisse, die die Ereignisphasen konstituieren, nicht selbst wieder konstituiert?) and before Hua 33: 256–259, the "Rekapitulation des Problemstandes: Der immanente Zeitgegenstand und der Wandel seiner Gegebenheitsweisen"). Husserl (2001, 256–259).

[134] "In dem einzigen letzten transzendentalen Prozess mit seinem Urentstehen und Urvergehen wird das Ereignis E und das Ereignis der Konstitution von E in verschiedenen Richtungen der Reflexion bewusst. Das Diagramm zeigt erschopfend, wie das moglich ist." Husserl (2001, 262).

by the vector $V_1(E^1)$ $V_2(E_0) = V^1(E_1)V^1(E0) = V^1(C_1)C_1$ is conscious as having passed, and so on. So as not to arrive at infinite regression."[135]

There are therefore two directions of transcendental reflection: one directed towards the course of the constituent flow and the other towards what follows from the constituted event.[136] We thus obtain two diagrams of reflections on the retentional processes. Diagram II.



And Diagram III.

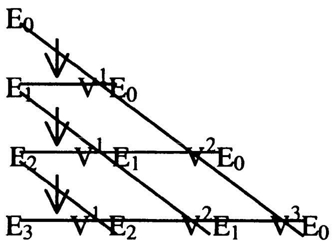
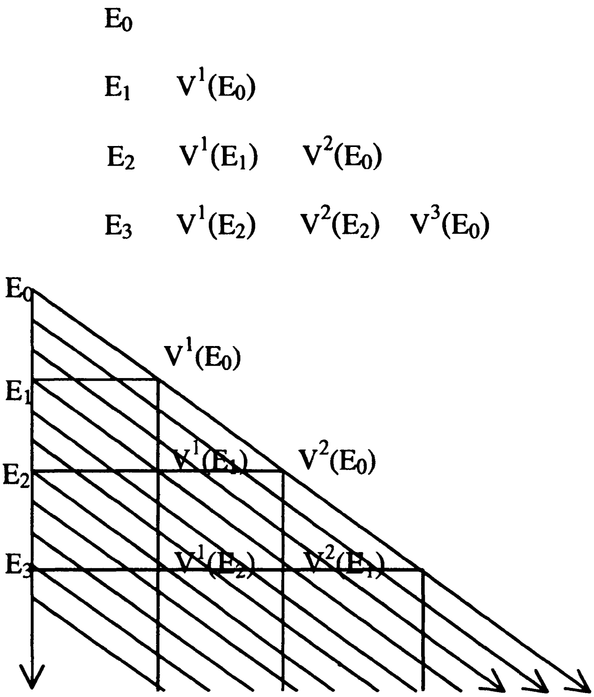


Figure III describes the following process: "As the event progresses, attention is arbitrarily directed to a phase (or part of the time series), for example a bell ringing. Similarly, on E_0. Then, all that remains is to symbolise the diagonal lines concerned, those that cross all $V_K(E_0)$. We can also take two bells (two lines) as an example."[137]

[135] Husserl (2001, 262).

[136] § 3. (Zwei Richtungen der transzendentalen Reflexion: Auf den Strom des konstituierenden Flusses und auf die Folge der konstituierten Ereignisse).

[137] III. Figur: Während das Ereignis fortschreitet, geht die auswählende Aufmerksamkeit auf eine Phase (oder ein Glied der Zeitreihe), z.B. einen Glockenschlag. Etwa auf Eo. Dann ist nur die betreffende schiefe Linie, die durch alle $V_K(E_0)$ durchgeht, zu zeichnen. Eventuell können zwei Glockenschlage zusammen herausgegriffen werden (zwei Linien) .

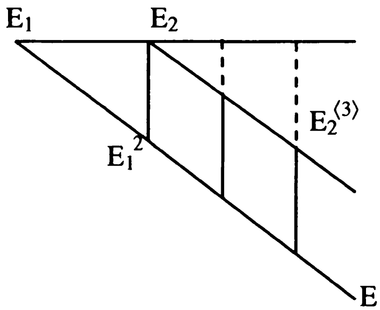


Conclusion: Perspectives for a General Theory of Possible Times

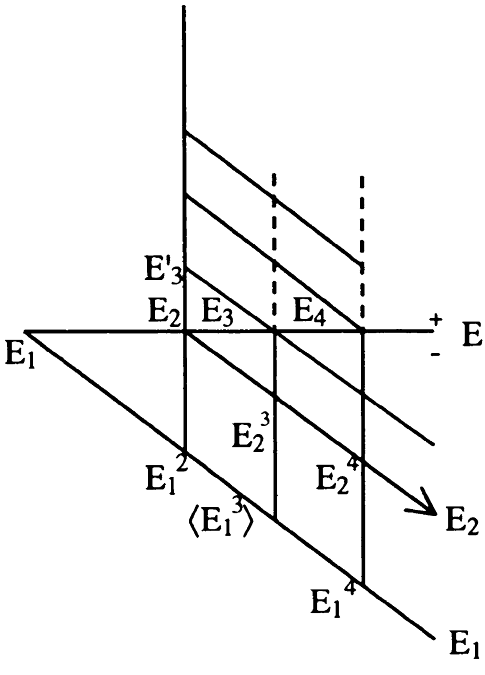
In accordance with our initial drawing and the purpose of this volume, we have limited ourselves to observing the play of the diagrams in this phase of “completion” of the previous diagrams. It is instructive to take a step back and measure the whole process, from the diagrams of 1905 to those of the end of 1918, and to resume the stages that the graphic representation of the original trial goes through.[138]

The new diagram starts from the diagram of the retentions and its symbolisation and completes it by adding a new dimension symbolised by vertical lines. The first figure takes up the figure above, and symbolises the fact that part of the event, i.e. E_1-E_2, has elapsed. The retention of this piece of duration will take place according to the series of vertical lines: E_2-$E_1{}^2 \ldots E_2{}^{(3)}$.

[138] It is the subject of text 2 of the Bernau Manuscripts Husserl (2001, 20–49).



The following figure, called "completed", therefore adds parallel lines to the verticals and diagonals.



With the following comment: "If there is to be in E_2, then, a protention [directed] to the future course of events, prescribed in its style and, in the most general way, by its type of matter, the interval $E_2E_1{}^2$ (the retention of the elapsed time) must first bear a protention which should be traced medially by the oblique band delimited by $E_2E_2{}^3$ and $E_1{}^2E_1{}^3$We correct the transcription, which is obviously faulty and omits this upper index. The following reinforces the plausibility of this correction, since what is anticipated in $E_2E_1{}^2$, is indeed $E_2{}^1E_1{}^3$."

But this nesting is a *crossover*. The anterior consciousness is a protention and the retention coming afterwards would be a retention of the anterior retention which is at the same time characterized as protention. More precisely, each retentional modification is joined to a protentional modification, which is in fact a fulfilling.[139] This diagrammatic reconstruction triggers a phase of methodological questioning: how is this phenomenalized?

> Let us now reflect on how an event becomes a phenomenological phenomenon—a phenomenon in the first, not the ultimate sense. We find in U_x, the phase of the original phenomenological process, a demarcated point *x*. How is it to be characterised? Or how is it to be characterised as an apex point of any past interval of U ... U_x, as its filling point, and how does it differ from the general replacement in which all intervals U come into play in the process according to all their points. How does this double direction of filling shed light? The result and visibly also a double sense of protention. Ux is a protention which sometimes refers to all the intervals a future in all their semicolons and sometimes is simply pro tension when each upper point, i.e. with regard to the points of the basic interval (or rather the intentional objectivity of the points of future events). and as has already been said each. of consciousness in the retention interval it is parallel with regard to all the points of the previous intervals (or these points of intentional objects) but on the other hand each, of each lower interval is only retention, and then only with regard to an intentional point of the base interval which is behind the *x* of the U_x concerned.

This phase of reflection and prospecting summons up a certain number of safeguards, in a progressive and methodical manner, such as the fact that this process is "eternal" in the sense that nothing can disturb it, that it has a strong structure of order, according to the anterior and posterior, that it is impossible to reverse, like the series of singular numbers, even if the elements here at stake are totally "individual", etc. There is therefore a fixed order of the U_x series, which is symbolised by the succession of vertical lines from left to right. There is also a fixed order of oblique lines corresponding to the processes of retention of event points, which also have a fixed direction. But these simple lines now characterise a complex process, a fundamental series and the law of continuous *fulfilling* together with an equally continuous *emptying*.

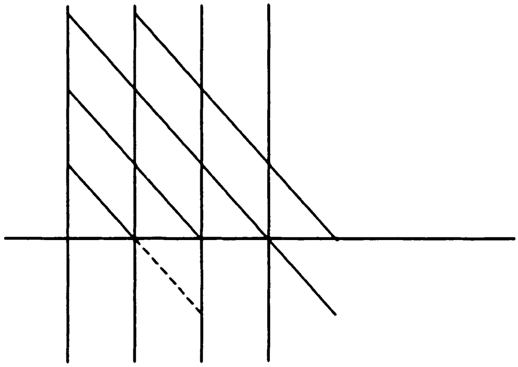
The zero-nuclei corresponding to the limits of this double process also have a double sign, positive and negative, depending on their positive or negative direction. "For they too, in spite of their null consistency, their emptiness, have their diversity as empty nuclei of the phase of consciousness U_x of each time in U_x".[140] The same diagram is now grafted on to considerations of temporal perspective and the constitution of degrees of temporal distance and proximity,[141] in which the "zero point" acquires new meanings; *zero of proximity and distance, of light and darkness*, *of filling or emptying, point of saturation, point of satiety*, *zero of tendency, neutral point*, which is "possible only as a limit point of two continuous intervals" but "which, as a limit, is the absolute proximity (a maximum of proximity) and the minimum of distance".

[139] Husserl (2001, 25–26).

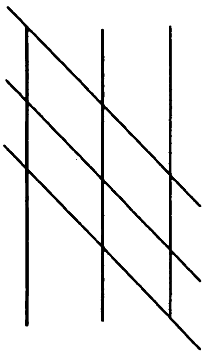
[140] Husserl (2001, 33).

[141] Husserl (2001, 39).

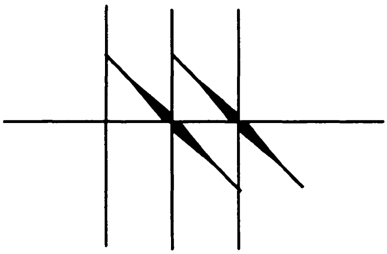
The diagram takes on a geometrical significance, a new one whose scope will prove surprising to say the least. If we symbolise the U_x trials with the figure:



Or even more simply:



It is possible to symbolise with a new diagram, "the geometric location of the maximum and minimum points (points of overlap, increasing closeness and distance)".



These points coincide astonishingly with those of the horizontal line, which in turn symbolises: the series of real phenomenological events, the original current of "intimate consciousness" (or self-consciousness), or the fundamental flow. Because the sequence of points is no longer simply a sequence of reals, but a sequence of maximum and minimum points, it is also possible, but with renewed meaning, to validate the classical diagram, which underlies the classical representations of time, but by giving it an infinitely richer content, since this flow is the original form of every being. Because it contains the actuality in the narrow sense and the ineffectiveness of the past and the potentiality of the future, it is possible to recover a classical thesis, to which we alluded above. Consciousness is in a way all things, in that it potentially and intentionally contains all things.

> The present is an omni-recovering, as if it were omniscient, as if it had consciousness of itself and all its intentional components ; its structure potentially contains within itself the omniscience of the world—as an ideal possibility, if we only take into account the fact that the horizon of darkness in which the past and the future of the flow of consciousness fade, is a contingent barrier that can be conceived as enlarged *in infinitum*, so that, as an 'idea', an omniscient divine consciousness develops which embraces itself in perfect clarity. Even the finite consciousness is omniscient, even its intentionality embraces all its past and future, but only partially clear, and for the rest, in a darkness that is a potentiality for clarity and recollection.[142]

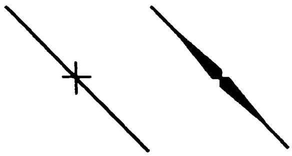
This is how we understand how an awareness of our finiteness is possible. This moment of hubris is immediately followed by a corrective, which opens up new possibilities to be explored, which touch on one of the essential characteristics of time: its continuity. Discontinuities do have to do with unpredictability, but this is not directly dependent on indeterministic options. They are in fact phenomenological breaks in time that disrupt the "law of time", which consists of a combination of hollowing out and filling in. We come to a "complementary question" "inconsiderately neglected, but knowingly and knowingly in order to simplify things": the continuous nature of the future. This does not invalidate previous analyses, but it does force us to explicitly mark this additional abstraction. What can a discontinuity consist of? Answer: "in a break (*pausen*) in a temporal event, a pause which is constructed as a term from individual temporal objects".[143]

The possibility of such a break or judgement poses a formidable problem, that of the intuitiveness of such a protention: "Is protention, foresight, intuitive in nature, and how, if nothing comes from what has been foreseen (if something of the sort actually exists)?" The line of the original phenomenological events or data, called the line of maximum points, thus becomes the line of nodal points, in which the double process of retentional recessing and protentional filling can also be interrupted, continuous time is suspended, and where, as Hamlet laments, time is out of joint, disjointed, out of its hinges.[144]

[142] Husserl (2001, 45).

[143] Husserl (2001, 49).

[144] This possibility poetically and politically enunciated by Hamlet who refers to a possibility that Derrida (1993, 39 ff.) tried to formalise under the concept of *messianic horizon* understood as a complexification of the structure of the horizon.



Husserl suggests that this possibility might find a rationale in taking into account a group of modifications that we have hitherto dismissed, and which involve taking into account the experiences that emanate from the self, the attentional modifications. It would also be necessary to explore, in the same perspective, the intervention of passive and active modalities (of taking a position or an attitude, which imply an active participation of the self), both doxical and affective. But this problem leads to a formidable question which will only be studied in the course of the 1920s, and which is only insisted upon in Bernau's Manuscripts: that of passive syntheses, of which temporal modalities are only one category. In relation to this, it would also be necessary to articulate temporal modalities and doxic and affective modalities (passive and active).

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Part VI
Diagrams, Gestures and Subjectivity

A Topological Analysis of Space-Time-Consciousness: Self, Self-Self, Self-Other

Hye Young Kim

Abstract This paper attempts to explore a possibility to visualize the structure of time-consciousness in a knot shape. By applying Louis Kauffman's knot-logic, the consistency of subjective consciousness, the plurality of now's, and the necessary relationship between subjective and intersubjective consciousness will be represented in topological space.

Keywords Diagrams · Knots · Space · Time · Self · Consciousness

For a long time, I believed that time must flow like this, without much doubt (Fig. 1):

But, does it? What if time's arrow doesn't flow straight forward towards the future? I couldn't be sure if time doesn't flow like this, for example, in the shape of a knot (Fig. 2):

What is time? What do we understand by 'time'? Is time different than my time-consciousness? In other words, is there objective and subjective time? If my time-consciousness refers to subjective time, what is objective time? Can I understand time beyond my perception of time?[1]

Let's say now is 19:13, June 8, 2018 at Gare de l'Est in Paris, France. It takes 30 min to go to the Notre Dame from here by bus. Is this objective time? I remember

[1] In this sense, the question of time is basically in the same structure as the question of being. What is being? Can I understand 'being' beyond the realm of my understanding of my own being?

This paper attempts to explore a possibility to visualize the structure of time-consciousness in a knot shape. By applying Louis Kauffman's knot-logic, the consistency of subjective consciousness, the plurality of now's, and the necessary relationship between subjective and intersubjective consciousness will be represented in topological space.

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Fig. 1 A straight arrow as the flow of time

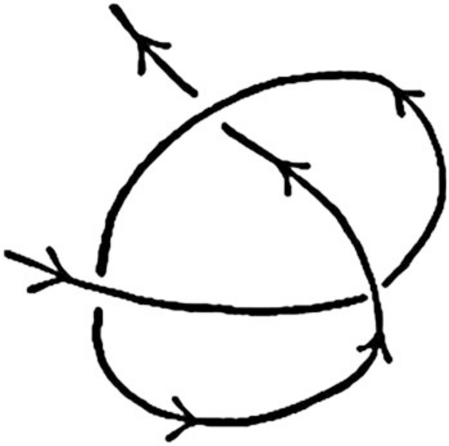


Fig. 2 A knotted arrow as the flow of time

yesterday when I was walking home in the rain, and I think about what to have for dinner in about 10 min. Is this subjective time? Here I take my time-consciousness as time and time as my time-consciousness.

Then, why time? Or why time-consciousness? If consciousness refers to the perception of the subjective experiences by an autonomous individual, consciousness is, or is based on, time-consciousness. And our time-consciousness is spatial.

This is an attempt to prove that the structure of time is not a straight arrow that flows forwards. In doing so, I will show how an identical subjective consciousness is possible based on time-consciousness and reveal the necessary relationship between subjective and intersubjective consciousness. The problem of 'now' is in the center of the discussion.

In my discussion, I deal with two knot models: one is trefoil and the other is Borromean rings. A trefoil knot is the simplest example of a nontrivial knot, which means that it is not possible to untie this knot in three dimensions without cutting it. It is one seamless line, i.e. one ring (Fig. 3).

Borromean rings consist of three topological circles which are inter-linked with each other (Fig. 4).

Fig. 3 Trefoil knot

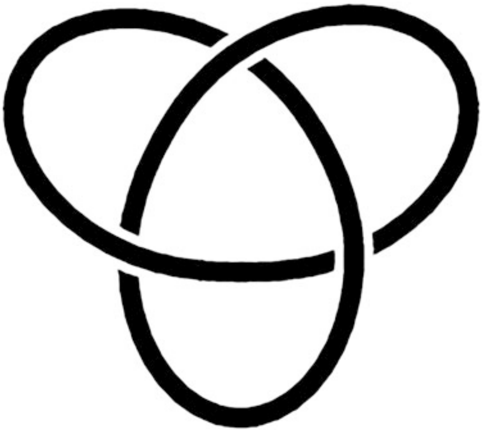
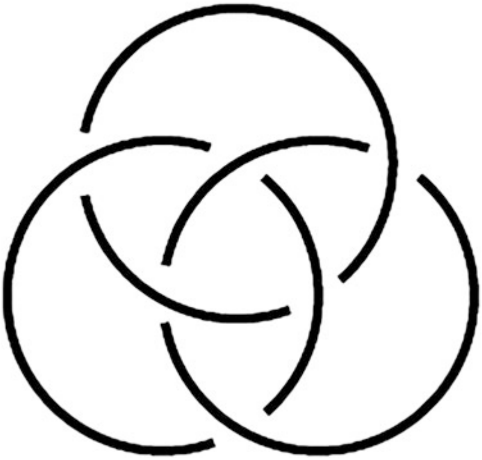


Fig. 4 Borromean rings



Knot Logic: Linking as Mutuality

Let's name the rings A, B, and C. They are woven in the way that ring A surrounds ring C, ring C surrounds ring B, and ring B surrounds ring A.[2] Or, more precisely, "ring A is attached to ring C (by ring B), and ring C is attached to ring B

[2] Colin McLarty (Truman P. Handy Professor and Chair of Philosophy, Case Western Reserve University) pointed out that the expression "surround" could cause confusion, because the point of Borromean rings is that no two are linked to each other, but they are only connected by way of the third. The same point was raised by Roberto Casati (Director of Institut Jean Nicod, Ecole Normale Superieure, Paris, tenured senior researcher). I am agreed with this criticism and add an extra explanation that was suggested by McLarty.

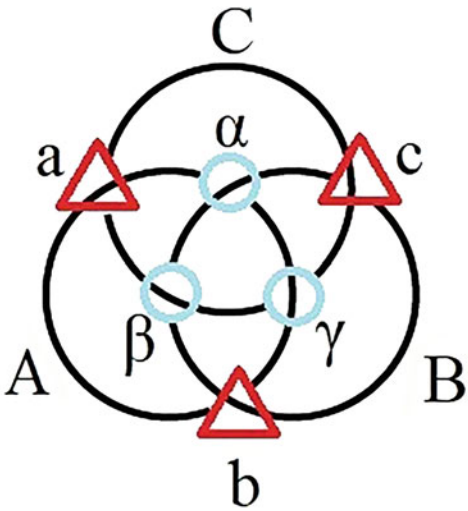


Fig. 5 Borromean rings with 6 crossings

(by ring A)."[3] There are six crossings in this model where the three individual rings are connected and separated (Fig. 5).

Each crossing, i.e. the link between the rings, manifests the mutual relationships between the rings. Here I apply Louis Kauffman's Knot Logic, which is a variant of set theory that allows mutual relationship (Kauffman 1995).

Kauffman presents knot set theory as a diagrammatic alternative to Venn diagrams that models a non-standard set theory and explains its relationship. Venn diagram shows the possibility of a logical comprehension of the connective tissue in topology and geometry (Fig. 6).

Each marker (where A intersects B and B intersects C) is placed in two regions and indicate that at least one of these regions is not empty. Therefore, some A are B and some B are C (Kauffman 2015, 33).

Kaufmann articulates its logical connection in relation to theory of knots and links in three dimensional space. Let's say there are undefined objects 'a,' 'b' and a notion of membership is denoted 'a b,' which means 'a *belongs* to.' It will be possible for 'a' to belong to itself or 'a belongs to b' while 'b belongs to a' (Kauffman 1995, 32).

[3] Colin McLarty.

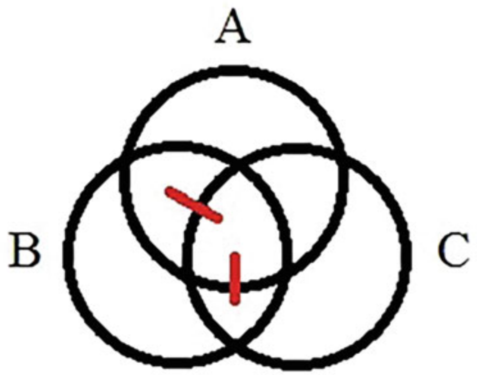


Fig. 6 Venn diagram

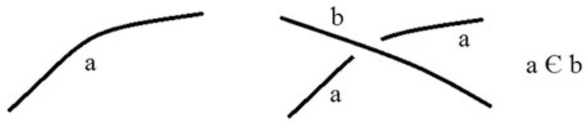


Fig. 7 Membership indicated by crossing-over convention

Objects will be indicated by non-self intersecting arcs in the plane and a given object may correspond to a multiplicity of arcs, which will be labelled with the label corresponding to the object. And membership is indicated by the diagram as below (Kauffman 1995, 32–33) (Fig. 7):

Here we see that ‘a belongs to b.’ The arc ‘b’ is unbroken, while ‘a’ labels two arcs that meet on opposite sides of ‘b,’ which represents that ‘a passes under b’ according to the convention of illustrating one arc passing behind another by putting a break in the arc that passes behind. This **pictorial convention** is important for the logic and the relationship with three dimensional space (Kauffman 1995, 33).

Kaufmann shows this with the example of von Neumann construction of sets of arbitrary finite cardinality that starts with an empty set Φ={ } and building a sequence of sets Xn with X_0={ } X_1={{ }} X_2={{ },{{ }}}. By using overcrossing convention for membership, one can draw a diagram of this construction (Kauffman 1995, 33) (Fig. 8):

The three rings of Borromean rings mutually belong together and the crossings between the rings show their belongingness to each other. I marked the three crossings on the outer edge as a, b, c, and the other three inside as α, β, and γ. At the crossing ‘a,’ where the ring C is placed under the ring A, C belongs to A. At each crossing, there is a membership relationship. See Fig. 5.

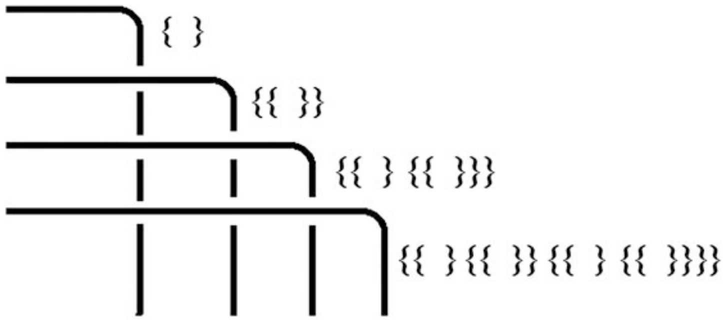


Fig. 8 Diagram of von Neumann construction

Fig. 9 Mutual membership between different rings

The mutuality of the three rings are as below at each crossing:

Crossing	Belongingness	
a	C A	B = {A, C}
b	A B	A = {B, C}
c	B C	C = {A, B}
α	A B	A = {B, C}
β	B C	B = {A, C}
γ	C A	C = {A, B}

The mutual membership relationship between different rings is as below, for example, in the Borromean rings (Fig. 9):

These sets are the sets that are members of each other. But there are sets that are members of themselves as well (Kauffman 1995, 34) (Fig. 10):

These diagrams indicate sets that may have a multiplicity of identical members (Kauffman 1995, 34) (Fig. 11):

Instead of regarding the multiplicity of identical members as all equivalent to one another to condense them ({ . . . a, a . . . } = { . . . a . . . }), Kauffman suggests another solution, in which identical members cancel in pair. Thus { . . . a, a . . . } = { . . . . . . }, which is {a, a} = { }. And it looks like below in diagram (Kauffman 1995, 34–35).

Fig. 10 A set can be a member of itself

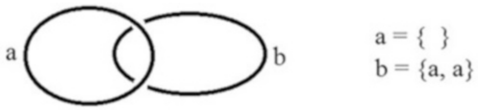


Fig. 11 Sets may have multiplicity of identical members

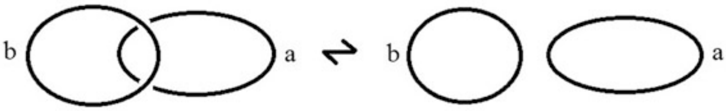


Fig. 12 Rule of cancellation

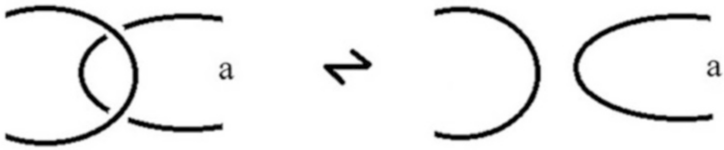


Fig. 13 Second Reidemeister move

But note that "this move is only allowed when no other lines come in between the places where b passes over a" (Fig. 12).[4]

The rule of cancellation of identical members is fundamental to knot set theory, as in the second Reidemeister move (Fig. 13):

These are the three Reidemeister moves (Fig. 14):

A knot consists in a single closed curve and a link may have many closed curves and a tangle has arcs with free ends. Reidemeister proves that any knot or any link in three dimensional space can be represented by a diagram containing only crossings of the type indicated as in the Fig. 7.

And the first Reidemeister move allows the creation or cancellation of self-membership in the corresponding knot set. But if the loop is a physical loop in a rope, "the cancellation of the loop in the first move must be paid for by a corresponding twist in the rope" (Kauffman 1995, 38) (Fig. 15).

[4] Colin McLarty (personal communication).

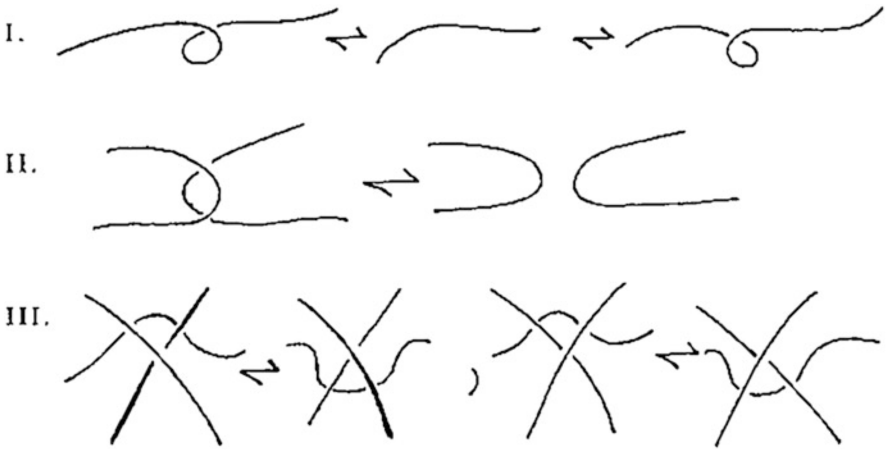


Fig. 14 Three Reidemeister moves (Kauffman 1995, 36)

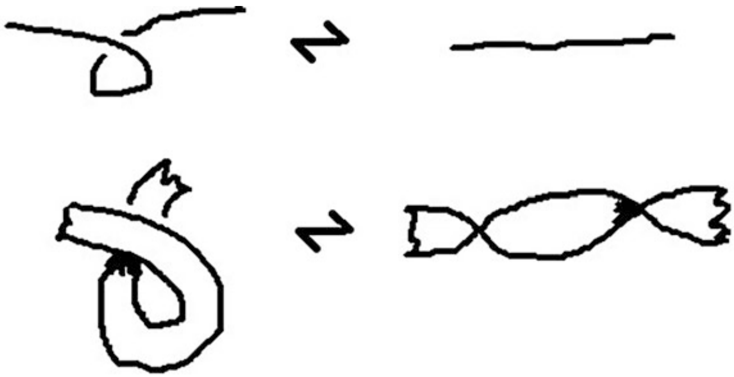


Fig. 15 Cancellation in a rope paid for by a twist

According to Kauffman, one can regard it as an *exchange* rather than an elimination or creation of the loop. If this is applied to the diagram that represents a twisted band, the self-membership is not lost as we move to the topology (Kauffman 1995, 39). "*Any knot set has a representative that is a member of itself and the states of self-membership and non-self membership are equivalent*" (Kauffman 1995, 40). Therefore, a radical knot set is a member of itself and only if it is not a member of itself (Kauffman 1995, 40). And this knot set gives a way to conceptualize non-standard sets without recourse to infinite regress through a twist in the boundary. The self-membering set is represented by a curl, where the observer on the curl itself goes from being container to being a member. This shows that membership becomes *topological relationship* (Kauffman 1995, 41). The same logic works in a Möbius [sic] band as well: A = −A (Fig. 16).

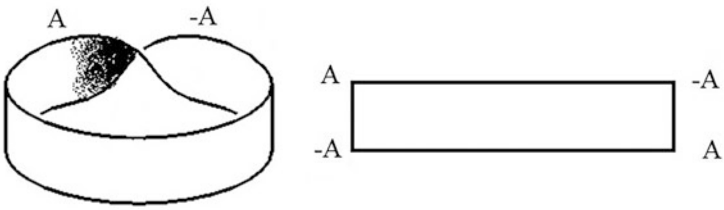


Fig. 16 Möbius band

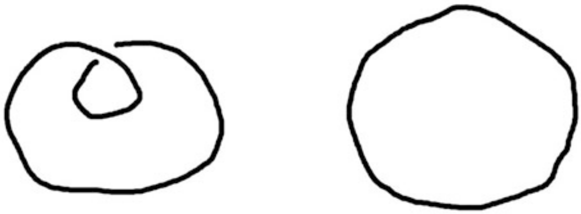


Fig. 17 Resolving the paradox

Kauffman shows that this self-membership solves Russell's paradox. Let R be the set that contains all the sets that do not include themselves. If R contains itself, R does not include itself, but if R does not include itself, R should be a member of itself. But we can solve this paradox in the domain by having every set as a member of itself and **not** a member of itself (Kauffman 1995, 40). One could regard it not as 'solving' of the paradoxical contradiction but as rather resolving the contradiction (Fig. 17).[5]

Belongingness: Not-I, Knot-I

What is interesting about the two-dimensionalized model of the Borromean rings is that inside the Borromean rings, a trefoil knot is nestled, i.e. in the inner structure of the mutual relationships of the rings is a trefoil knot with the outer edges open. "The rings are a mutuality of three distinct entities, while the trefoil diagram is a self-mutuality of its *three internal arcs*. Note that the structure of the Borromean rings requires all *three rings* to be linked. The knot in space projects out the three arcs when it is drawn in the plane" (Fig. 18).[6]

[5] Based on comments from Atocha Aliseda (Professor of Philosophy at the Institute for Philosophical Research at UNAM, Mexico) for the author's lecture "Knotted Space-Time-Consciousness: Intersubjective Subjectivity" at Instituto de Matemáticas, UNAM, Mexico (Oct 9, 2018).

[6] Louis Kauffman (personal communication with Louis Kauffman).

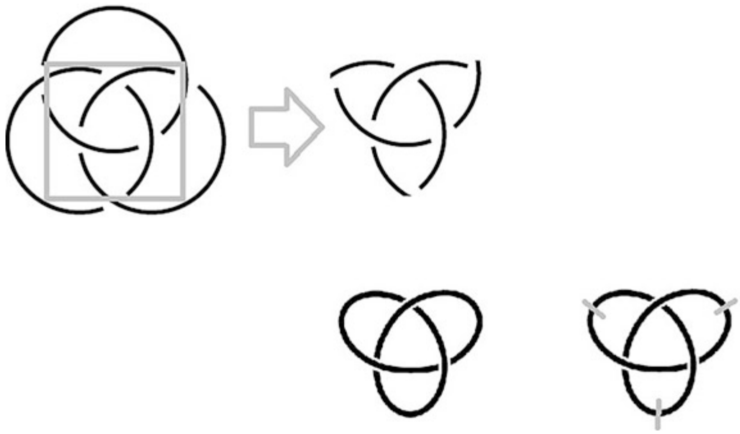


Fig. 18 A trefoil knot in Borromean rings

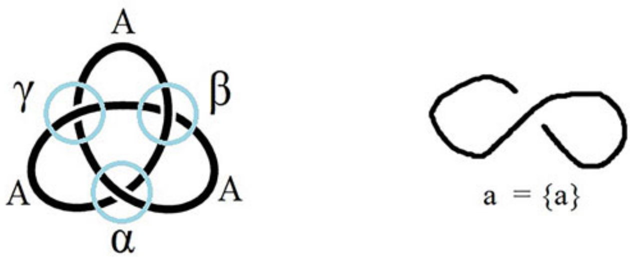


Fig. 19 Trefoil knot representing a stable self-mutuality in three loops

In a trefoil knot, there are three crossings as in the inner structure of the Borromean rings: α, β, γ.

This shows the self-mutuality, because a trefoil knot consists of one seamless ring. Therefore, a trefoil knot represents a stable self-mutuality in three loops about itself: a = {a} (Fig. 19).

I apply this logic of self-mutuality, i.e. self-membership to explain the consistency of a subjective consciousness throughout the change of time.

How can a subjective 'I' at each different temporal locus a, b, and c be identical? This means that different 'I's at each point of now *is* the same 'I' with continuity. 'I' at 'a: now (−1),' 'I' at 'b: now (0),' and 'I' at 'c: now (1)' mutually belong together (Fig. 20).

'I' at each moment has to be able to observe 'I' at another moment which, as the observed, is not the same 'I' from the observant's point of view; instead these different 'I's conform to an identical 'I' that understands this connectivity and the continuity between different moments.

Fig. 20 I-consciousness on different moments of now

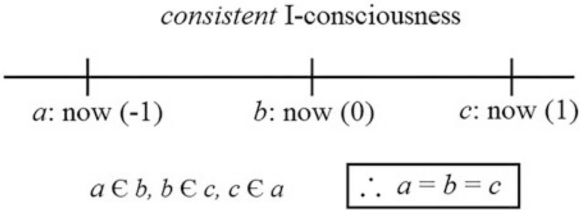
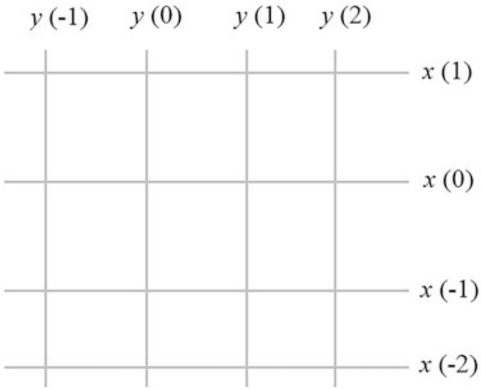


Fig. 21 Plane with axes of time(y) and space(x)



To see how this works more precisely, I will place the knot on a plane with axes of time and space. Let's say that vertical lines indicate temporal (y), horizontal lines spatial locations (x) (Fig. 21).

By having a plane of both vertical and horizontal axes of time and space, we can understand the space-consciousness which is necessarily attached to my time-consciousness. Time-consciousness is always spatial. If time is perceived through change which is caused by movements, it always presumes the space that is either the (back)ground of the movement to 'take place' or the act of movement itself creates (results in) space. A linear time-consciousness does not manifest the dimension of space on its single-linear structure. But when we move, either actually moving (our body) from here and there, or reconstructing our memory from then or pre-grasping our future from now, we always and necessarily locate ourselves spatially. When my time-consciousness at each now is constructed through the modification[7] of my memory and anticipation, it is impossible for me to recall my experience or expect the experience in the future without its spatial location because these experiences 'take place.'

[7] Edmund Husserl uses this term as well to explicate his theory of inner time-consciousness.

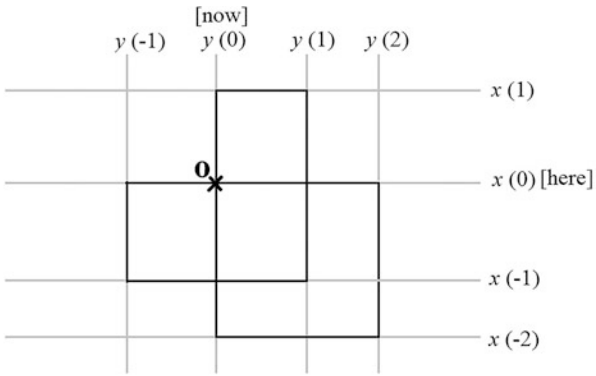


Fig. 22 A trefoil knot on a plane with axes of time(y) and space(x)

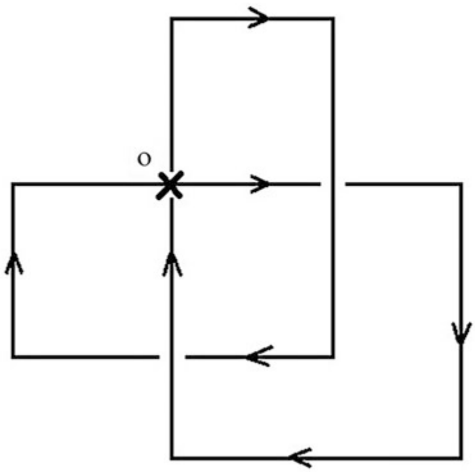


Fig. 23 My movement starting from the point 'o' where I am now here: x(0):y(0)

Now, place a trefoil knot on this plane (Fig. 22):

The point 'o' is placed on the point where x and y meet. At 'o,' I am now here: x(0): y(0) (Fig. 23).

I move—time flows clockwise. Starting from 'o' the flow passes countless moments on the line including 'p, q, r, s, t, u, v' (Fig. 24).

If we understand 'now' as a point of 'now,' which is 'o,' from there one sees oneself in different temporal and spatial locations—in other words, I 'locate' myself (Fig. 25).

The most interesting act of seeing happens between the points of 'o' and 's' at the crossing, because they are overlapped—they share the same temporal and spatial location (Fig. 26).

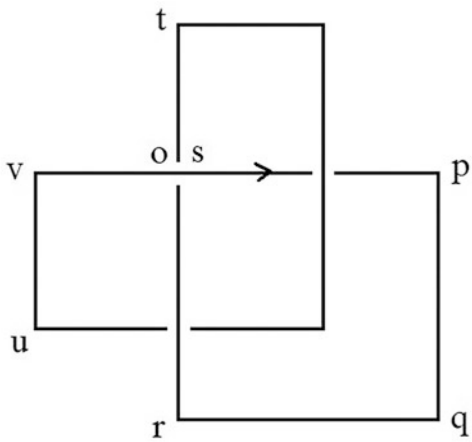


Fig. 24 The flows passes countless moments including o, p, q, r, s, t, u, v

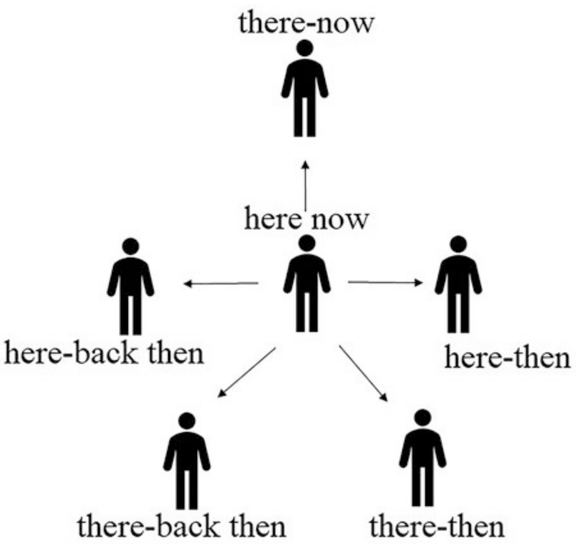


Fig. 25 Each point representing different spatial and temporal locations of myself

The act of seeing—I (o) see my (present) self (s)—is the basis of the formation of 'self-ness.' In other words, I have to be able to see that I am there as I, for me to be able to perceive myself as I. The I that is observed is I but at the same time not the I, the observer. The *subjective* understanding of (my)self is based on the distinction of I and not-I and its identity. We can solve the paradox of I = −I with the knot logic of self-mutuality: not-I as *knot*-I (Fig. 27).

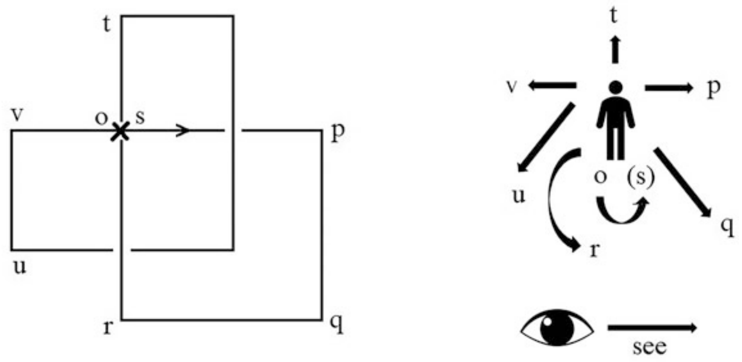


Fig. 26 Observation of myself at different spatial and temporal locations

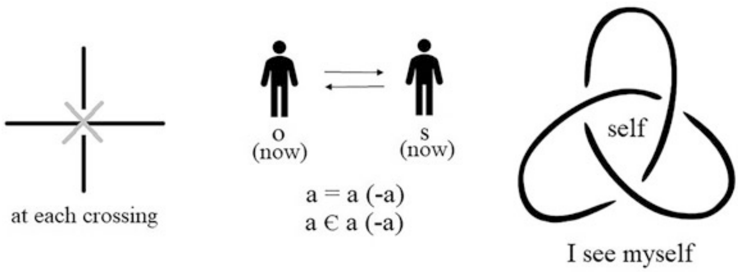


Fig. 27 At 'o (s)' I see myself

This is what happens when one sees oneself at different points from where they are standing now, here—spatial/temporal locations. We can call this process 'modification.' This is, however, not only the modification process of time-consciousness but *the formation process of consciousness*.

Vorstellung[8]	I imagine myself being here/there then.
Beobachtung	I see myself being here now.
Erinnerung	I remember myself being here/there back then.

A trefoil knot as a whole is each moment of now, in which there are different now's (the past and the future), i.e. different temporal and spatial location (self-location) in a continuous structure (Fig. 28).

[8] This is my own use of "Vorstellung," "Beobachtung," "Erinnerung." The reason that I chose "Vorstellung" instead of "imagination" is to avoid the strongly connected connotation of the word related to "image" or "imago." And the "vor-" in "Vorstellung" could be interpreted as "pre-" or "fore-," expressing the "fore-grasp" of the future.

Fig. 28 A trefoil knot as a 'whole' now

This connection between now's at each now indicates the modification process and this modification creates the self-ness (subjective) of the consciousness.

Conclusion 1. *one trefoil knot represents the modification of each now (point of time). The generation of consciousness (self-ness) is based on one's awareness of temporal and spatial location which happens through the modification of multiple points of now's. My consciousness is, therefore, my spatial and temporal consciousness.*

In this context, however, Roberto Casati[9] raised a question of significant importance: "if you travel through a knotted tunnel, you cannot discover there is a knot. Shouldn't the same apply to time?"

I try to answer this question in six points as follows:

1. I thought of a knotted tunnel as a "band" or a rope in three dimensions, and you walk on top of this band which has a "twist" when it is spread out. A link where the line(path) is separated and linked on a knot is paid for by a twist when it is in three dimensions. And as you walk on this band, you have two paradoxical identities as a container and a member, the observer and the observed.
2. Here the crucial point of the knotted band(rope) lies in the fact that when you are on it, what you do is "to observe" at each moment (of time—you would call this "now"), and this observation necessarily comes with memory and anticipation, through which the relationship of the observer with the observed can be aroused at all.
3. Therefore, the discovery of the "knot" is possible through the process of "self-othering," in other words, when I can observe myself on a different spot of the band, where I used to be or will be. In this sense, the problem of time becomes the problem of awareness of time, or perception of time. It was my stance to start with the notion that the problem of time is as the problem of being, in the sense that it is either actually not there, or does not matter before it is perceived or understood by the subject. The "othered" self plays the role of the so-called "third eye" to see the knot (and the twist).

[9] Institut Jean Nicod.

4. If the tunnel of time is knotted and we travel inside the tunnel, I assume, the twisted or knotted time flow wouldn't be discovered, but if the tunnel is knotted and see-through, actually, it would work even better because then you would be able to actually see out and see the overlapped tunnels that pass either under or above the tunnel on which you are traveling in the moment of now. And this is also the question of whether we "see" only to the front. Isn't it a different problem, if we admit that we look back or look up or look on the side?
5. If each moment of now is plural, and each individual travels alone in a separate (knotted) tunnel of time, how do we get to see the other tunnels of time where other individuals travel in? How can we explain or describe the "shared" time (or time experience) by being there together in the same space?
6. Also, in a way, I take this question as the question of "spatiality of time" related to the "realization of dimension," as knots are conceivable only in three dimensional space, because that is when (where) the twist on a knot is created and recognizable.

Self-Mutuality as Mutuality: Mutuality as Self-Mutuality

The mutual relationship in the Borromean rings between three different rings could possibly represent the mutual relationship between different 'now's of different subjects: my 'now,' your 'now,' hers, his, etc (Fig. 29).

I have already shown that the center of the Borromean rings forms a trefoil knot. This trefoil knot in the Borromean rings has individual sides formed from two different rings that surround it. Therefore, in the self-mutual structure of the identical A, we can find the mutual relationships between three individual rings of the Borromean structure.

My (subjective) 'now' is composed of different 'now's, not only mine, but also others' as well. At the same time, the self-mutual (subjective) now is placed in the heart of the (intersubjective) 'mutual' now. The Borromean rings with a trefoil

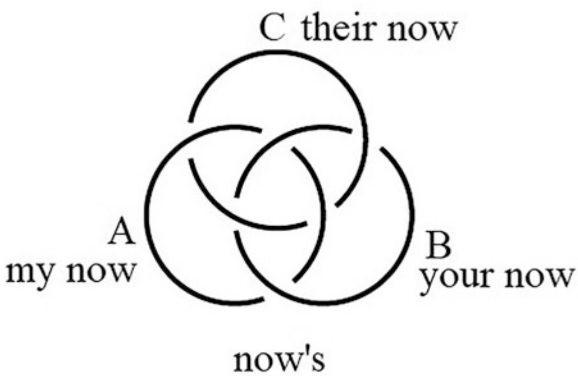


Fig. 29 Borromean rings with different subjects

Crossing	Belongingness	
α	A A / A B	A = {A} = {A, B}
β	A A / B C	A = {A} = {B, C}
γ	A A / C A	A = {A} = {C, A}

structure within itself manifests the fundamental and necessary relation between mutuality and self-mutuality.

Conclusion 2. *The logic of mutuality and self-mutuality in the Borromean rings and the trefoil knot models visualizes the fundamental inter-relation of the subjective (self) and intersubjective consciousness and their structure, through which we understand how we are aware of myself and others at the same time. Consciousness 'happens' through the interaction of the perception of self and the perception of not-self (including other objects and observed self). This double perception occurs simultaneously. Consciousness = Self (-consciousness).*

Plurality of Now's

Traditionally (in philosophy), we have understood the 'flow of time' as the line of points of now, a seamless succession of the now moments. Each now is believed to have its own plane (Fig. 30).

On the plane of each now, spatial locations are marked. If time is to be understood as the fourth dimension that explains spatial changes, e.g. my being here now and then, my being there now and then, then the flow of time is explained by connecting the points of here and there at each plane of now, and my memory acts as the mediator of connections, i.e. the connecting lines between different points on this and that plane of now's (Fig. 31).

These multiple lines between different planes of now's can be crossed or paralleled, but these lines do not describe the 'backward' or 'repeatable' movements

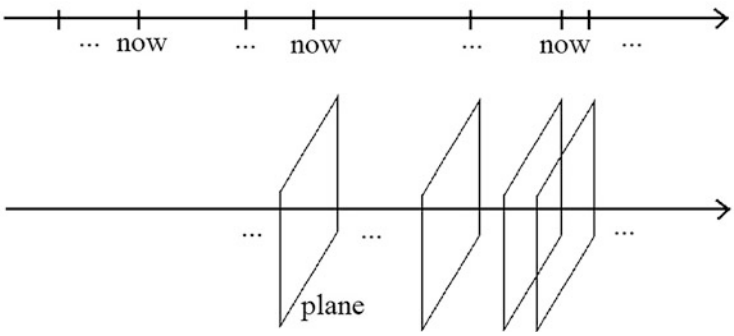


Fig. 30 Each now with its own plane

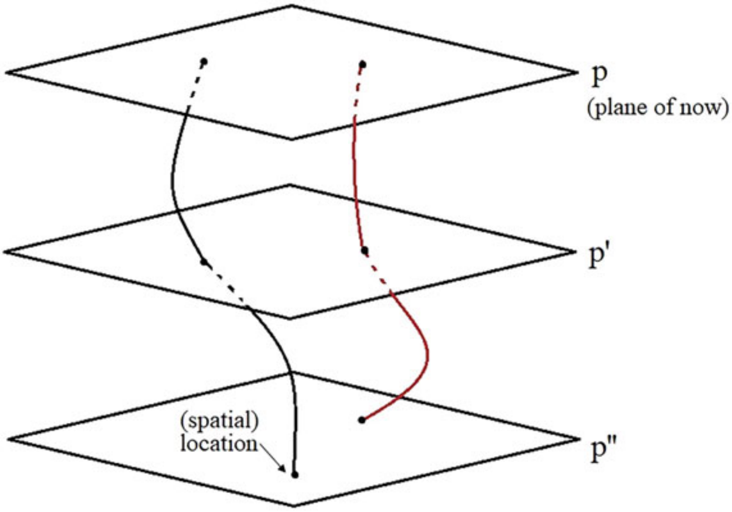


Fig. 31 Connecting parallel planes of now

of temporal comprehension (modification of consciousness) between the points which modify one's construction of time, understanding it as a flow. At each point of now, there can be multiple 'I's as the subject of time-consciousness. I not only remember being just now, or before a certain amount of time, but I conceive my being in the very next moment through my imagination and expectation.

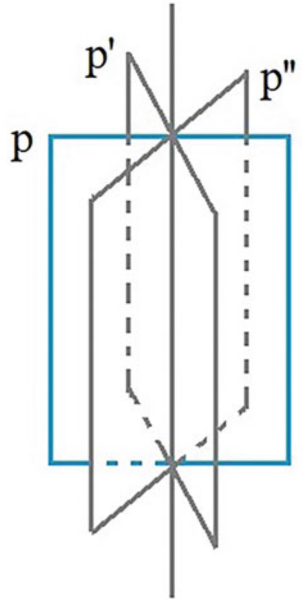
On each plane of now, there are countless points that represent 'now'-ness of my time-consciousness. Firstly, different 'now's of different subjects, and secondly different 'I'-ness of the same subject on the same plane of now. In this context, I raise two questions:

a) Are the 'I' on the P and the 'I' on the P′ the same 'I's?
b) Can 'I' be here and there at the same time? If so, how?

If the second hypothesis is true, there can be multiple 'I's of the identical subject on each plane. On the plane of time and space above, the point 'o' and 's' represent the identical here and now but at the same time different here and now, because there is space for the act of observation between the *different* points of 'o' and 's.' I will come back to this 'space' between 'o' and 's' more specifically.

One might be able to regard each plane as each subject's plane of now. However, neither the multiplicity of now that are scattered here and there on the identical plane of each now, nor their inter-connection between different planes, is explained in the model with paralleled planes of now. The problem of paralleled planes of now is that on each separate plane, my understanding of 'now' in relation to the past and the future, i.e. my continuous time perception of each moment of now is not fully clarified.

Fig. 32 Planes of now on a shared axis



In short, here we ought to be able to explain

a) how the planes belong together
b) how multiple different points of now on a plane belong together.

But, what if the planes of 'now' at each crossing would rather look more like this, if there is a plane at all (Fig. 32):

Let's 'sew' a seamless string—continuous flow of the time-consciousness of an identical subject into these planes, then it would look like below (Fig. 33):

And there can be many of them, because there are a great number of individuals (Fig. 34).

Then if we see it from above—it would look like these tangled strings (Fig. 35):

And there are knots! and therefore we see crossings and links (Fig. 36).

The links of these countless knots and their mutual relationships could be explained by Borromean space (connective space) (Fig. 37):

Conclusion 3. *The multiplicity of now's and their inter-relations can be proven in the knots and their connective space. Time flows in the shape of knots.*

The connection between now's at each now indicates the modification process, and this modification creates the self-ness (subjective) of consciousness. The generation of consciousness (self-ness) is based on one's awareness of temporal/spatial location which happens through the modification of multiple points of time, which were, are, and will be all now's. When we understand the temporal modification at

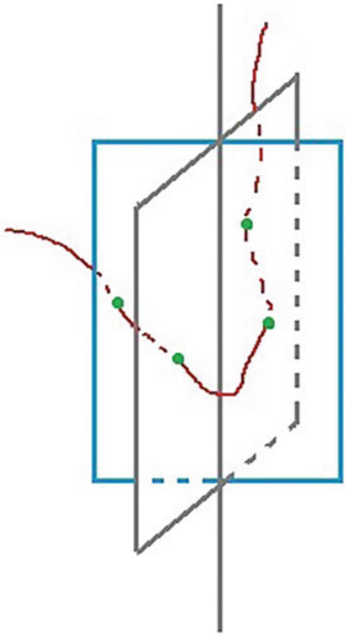


Fig. 33 A seamless string of consciousness going through the planes

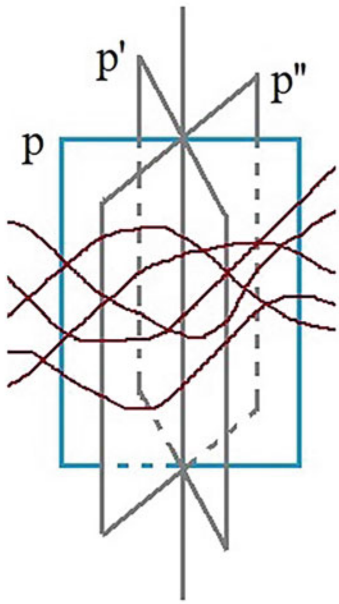


Fig. 34 Multiple strings representing multiple individuals (subjects of consciousness) going through the planes

each now between the past and the future in a knot form, we can explain the identity of the same 'I's on different planes of now. I belong to myself, yet I don't belong to

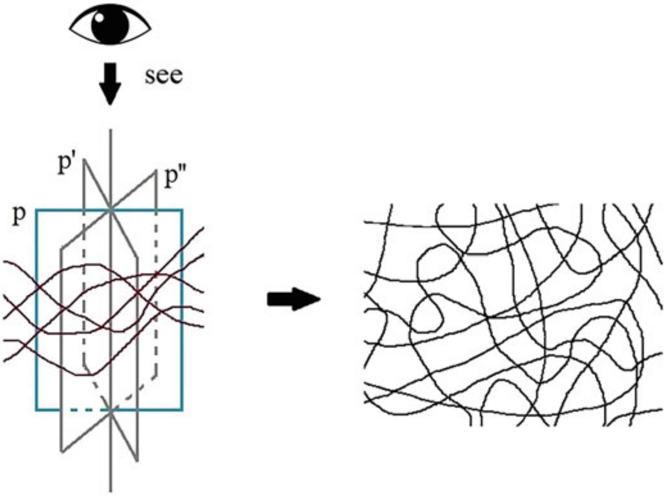


Fig. 35 Viewing the multiple strings from above

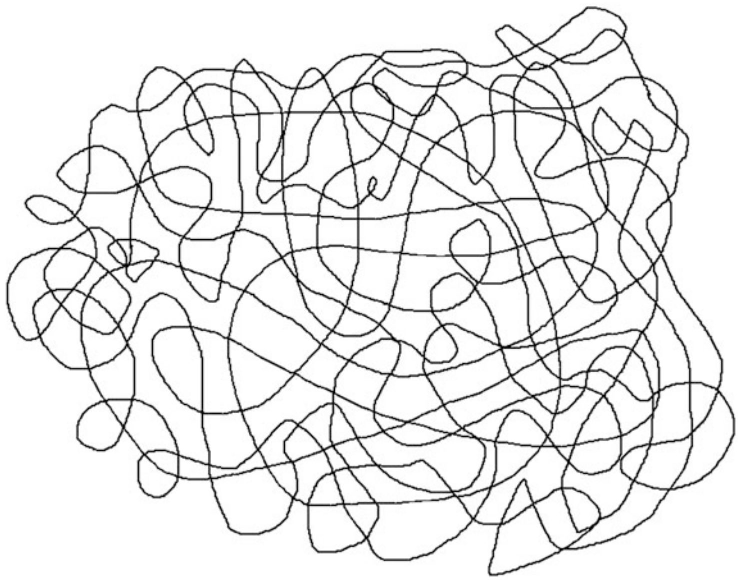


Fig. 36 Tangled strings viewed from above with crossing and links

myself. This act of self-observation generates ‘space’ in itself. For example, in the diagram of the trefoil model of time-consciousness, the ‘space’ between the point ‘o’ and the point ‘s’ which are the same present moment is not explicable with the model of paralleled planes of ‘now.’ This space between I and not-I is explainable through the links and their diagrams of the knots, because “links and their diagrams

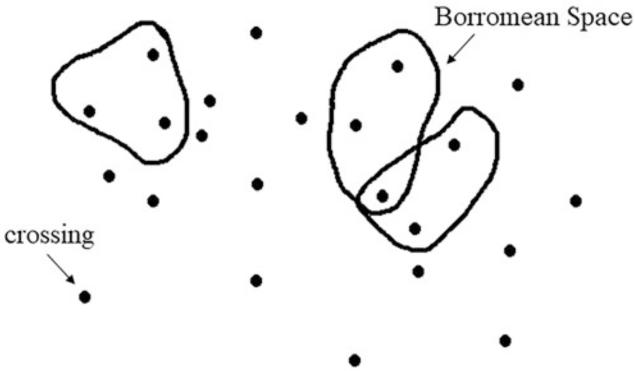


Fig. 37 Borromean space (connective space)

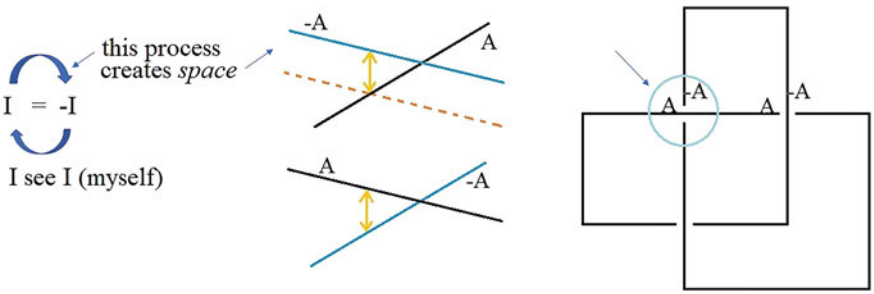


Fig. 38 The simultaneous perception of self and not-self

encode three dimensional manifolds" (Kauffman 1995, 111). The process of self-perception is always three-fold, which creates three-dimensional space.[10]

Conclusion 4. *Time-consciousness is spatially constituted and perceived.*

Seeing myself, i.e. locating myself (objectively) necessarily requires a third person view-point, even if it is after all only 'I' who plays the third person role by observing myself in the 'not-I' position. Self-observation is the process of *self-othering*. As the necessary relation of the mutuality and self-mutuality logic in the Borromean rings and the trefoil knot, self-consciousness, which is subjective, is always, at the same time, intersubjective, for the intention of my self-perception is based on and points at my othered self. In this sense, the perception of my self-consciousness takes place simultaneously with my perception of not-self and vice versa (Fig. 38).

Conclusion 5. *The phenomenon of the necessary inter-relation of the consciousness of self and the consciousness of the other is visualized, i.e. zur Erscheinung kommen, in knots*

[10] Johann Gottlieb Fichte introduced the equation of "I = −I."

Fig. 39 Impression of listening to music

Fig. 40 Experience of listening to music at each moment

and their links as seen above, because "knots and links form a calculus that is inherently self-referential and mutual" (Kauffman 1995, 112).

Music

And now why music? Because music is the most intuitive but at the same most analyzable way of our understanding of time. Once I wrote that we listen to music like this (Fig. 39) (Kim 2017):

But, in fact, this is how we 'listen' at each moment (Fig. 40):

We hear the flow, but we listen to each moment *as a whole*. Each moment of music not only comprises the whole music but itself *is* the whole music. The note A then, the note B now, and the note C afterwards become *one* in the moment, creating space where they are interwoven as in the Borromean rings.

Another important aspect of music lies in the relationship between the performer and the listener of the music. This always includes the self-mutual relationship because the performer is always a listener too. The performer and the listener mutually belong together at each moment of music. This is a temporal unity, but one which is always spatial because of the distance between the performer and the listener.

For a performer, either as solo or with other performers, performing music is the process of creating and experiencing *each moment as a whole*, where the whole universe falls into the present moment. This experience is not explained through the theory of successive notes and the flow of the melody, but it is 'creating' each moment as a whole, where the (spatial) movement of my body turns into one note, and each note constructs the whole (temporal) structure of music; my consciousness as a performer and as a listener are separated (difference), but become one (identity) again.

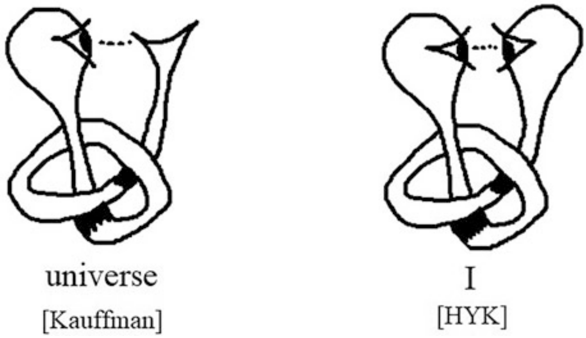


Fig. 41 Knotted Wheeler's Universe (Kauffman) and knotted self (Kim)

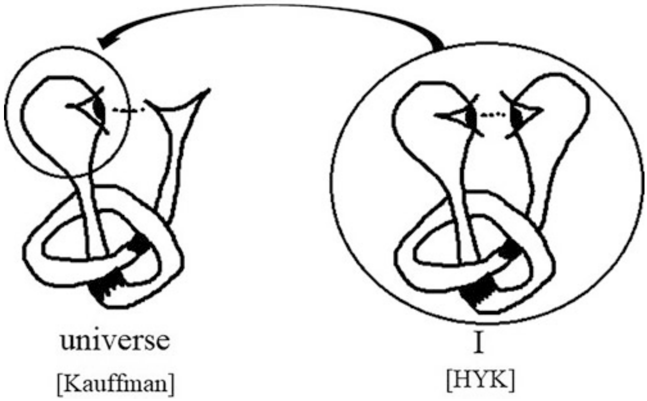


Fig. 42 Knotted self (consciousness) in the eye of the observer of the knotted universe

Kauffman's Universe and HYK's Self

Kauffman has his knotted version of Wheeler's Universe, and here you go, my version of knotted 'Self' (consciousness) (Fig. 41).

And this knotted 'consciousness' is placed in the eye that observes the very beginning (arche)—big bang (Fig. 42).

The riddle of *identity–difference* and *subjectivity–intersubjectivity* seems to have one variant of its answers in knots. *Why knot.*

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Gestes, diagrammes et subjectivité

Philippe Roy

Abstract En prenant appui sur les recherches philosophiques au sujet des gestes et des diagrammes que j'ai menées de pair avec celles dans le physico-mathématique, dans les champs politique ou artistique (et psychanalytique), je propose d'établir comment la subjectivité peut se penser au travers de nouvelles articulations des dualités "Individu/Société" ou "Unité individuelle/Multiplicité du devenir" et autrement que par la catégorie de substance. Dans ce cadre, la substance devient alors centre(s) d'indifférence (concept d'origine schellingienne). En conséquence, je montre en quoi un centre d'indifférence est en rapport avec les gestes et les diagrammes. Ce qui me conduit à faire appel au concept de virtuel-Temps pour soutenir la dimension événementielle, mémorielle et affective de la subjectivité ainsi pensée. Dès lors, au lieu de "Qui suis-je ?" sont posées des questions plus spécifiquement gestuelles-et-diagrammatiques : "Comment je me porte ?" et "Qu'est-ce qui m'emporte ?".

Keywords Geste · Diagramme · Centre d'indifférence · Virtuel · Événement · Affect

L'objectif est de proposer quelques repères de ce que pourrait être une pensée diagrammmatique, gestuelle de la subjectivité. Traditionnellement, l'idée de sujet est prise, comprise dans les fourches de certaines dualités. En effet, elle est comprise d'un côté entre l'unité individuelle et la multiplicité sociale (identité sociale) et d'un autre côté elle est comprise entre l'unité individuelle et la multiplicité du devenir propre à cet individu. Ces deux côtés étant liés.

Le problème de la subjectivité est donc d'arriver à penser à la fois la permanence d'une individualité et la multiplicité variable qui en forme le contenu (propre au social et au devenir individuel). Par exemple, c'est cette synthèse du un et du multiplie que tente Aristote avec la substance individuelle et ses différents types de forme. Les existants que sont les gestes et les diagrammes permettent une nouvelle

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réponse à ce problème, se substituant à l'image de la pensée reposant sur l'idée de substance. En quoi ceci va dans le sens du thème de ce colloque. Sauf qu'à l'idée que la forme devient substance je serai plus enclin à affirmer que la substance est devenue centre d'indifférence (concept d'origine schellingienne). Dès lors, il me faudra donc montrer aussi en quoi un centre d'indifférence est en rapport avec les gestes et les diagrammes. Ceci m'amènera à les articuler avec les concepts de virtuel et d'événements qui forment à eux deux une synthèse de la permanence et du devenir.

Un exemple de diagramme social (et ses gestes)

Je propose tout d'abord de me tourner vers un diagramme social, politique pour saisir comment est pensable gestuellement et diagrammatiquement le rapport entre individu et société. En effet, un individu est ce qui peut venir *prendre place* dans un diagramme social, si on m'accorde qu'un diagramme se réclame de la topologie. Il est donc compris dans une multiplicité formée par des places, plus exactement par des placements (rapports entre places).

Je ne proposerai pas quelque chose de très original en évoquant ici Michel Foucault, même si on n'attribue pas habituellement la gestualité à sa pensée politique. Ce n'est pas le cas du diagramme. C'est maintenant bien connu, il mentionne ce concept dans *Surveiller et punir* auquel Deleuze, dans son livre *Foucault*, a donné une plus grande extension que cette seule occurrence. Dans *Surveiller et punir,* Foucault désigne le *Panopticon* comme étant le diagramme du *pouvoir disciplinaire*. Je rappelle rapidement que le *Panopticon* est un modèle de prison inventé par Bentham au XVIII^{e} siècle. La prison est construite circulairement, les cellules occupent la circonférence et la tour des surveillants est au centre du cercle. Une ouverture grillagée est agencée dans chaque cellule, tournée du côté intérieur du cercle vers la tour centrale. La lumière venant de l'extérieur de la prison pènétre chaque cellule par sa fenêtre, éclairant donc l'intérieur de la cellule. Les surveillants peuvent voir les prisonniers par de petites ouvertures percées dans leur tour, ces derniers pouvant ainsi être constamment surveillés. Ce pourquoi les surveillés en viennent à se surveiller eux-mêmes, par soupçon d'y être. Le *Panopticon* est le diagramme des sociétés disciplinaires, qui est en partie la nôtre, puisqu'il est encore en jeu dans tous les lieux où s'exerce ce type de pouvoir : prison, école, usine, mais aussi dans le rapport que chacun a avec soi. Michel Foucault peut donc écrire « le Panopticon ne doit pas être compris comme un édifice onirique : c'est le diagramme d'un mécanisme de pouvoir ramené à sa forme idéale ».[1] Surveillant et surveillé sont à la fois objets du diagramme et sujets de celui-ci en tant qu'ils en permettent l'activation. Ils sont sujets-objets. Foucault, en employant les termes de « forme idéale », tient bien à distinguer le diagramme de toutes ses effectuations concrètes telle la prison panoptique elle-même ou tels types d'école, de prison etc. qui sont plutôt des *dispositifs* effectuant le diagramme.

[1] Michel Foucault, *Surveiller et punir,* Paris, Gallimard, 1975, p. 239.

Le diagramme *Panopticon* est donc bien un certain enveloppement spatial, une topologie marquée par des places et des intensités variables qui seraient ici les rapports disciplinaires (normes de conduite variant temporellement en fonction de l'ordre des tâches) intégrés par tous les sujets-objets qui prennent place dans le diagramme. Mais si diagramme il y a, y a-t-il un geste associé à ce diagramme disciplinaire ? Oui, Foucault écrit que « chaque fois qu'on aura affaire à une multiplicité d'individus auxquels il faudra imposer une tâche ou une conduite, le schéma panoptique pourra être utilisé ».[2] « Imposer une tâche ou une conduite à une multiplicité d'individus » est bien le geste du pouvoir disciplinaire (les places faisant le partage entre les normes de ceux qui imposent des tâches, il y a des manières d'imposer et de ceux à qui sont imposées des tâches). Ce geste est le dehors intérieur à la multiplicité comme à chaque individu. Il varie en s'effectuant en chaque point de son diagramme, de même que ce diagramme varie lui-même en fonction des résistances ou dysfonctionnements qu'il rencontre. Il y a une plasticité diagrammatique comme il y a une plasticité gestuelle. Cette plasticité diagrammatique est une des différences qu'il y aurait avec une structure (qui est plutôt caractérisée par des jeux de déplacement d'éléments de la structure). Si bien que l'individu change avec la société, tout en se disant d'une permanence en lui qui est celle du *site* d'effectuation des gestes. Ce que Gilles Châtelet, à la suite de Schelling, appelle des centres d'indifférence. Les centres d'indifférences sont les centres du virtuel. Je vais donc chercher à m'expliquer sur ce point, jusqu'à la fin de ce texte.

Prendre sur soi, le 0-milieu

D'un point de vue individuel, on peut justement accéder à ces centres par certains gestes. Ainsi le geste du prendre sur soi[3] s'exprime par un diagramme. Exemple : je prends sur moi pour ne pas rire (je me retiens de rire). Ce geste implique trois coordonnées : une intensité (le désir de rire), une intensité opposée (force que j'oppose à ce désir) et une entité qui tient ensemble cette opposition, qui est proprement ici le sujet, c'est-à-dire aussi le site d'effectuation du geste de prendre sur soi. C'est en ce site qu'ont lieu à la fois le rapport aux différences (opposition des intensités) et leur équilibre (identité). Il est au *milieu* des opposés. Le moi n'existe pas en s'opposant au non-moi (Fichte) mais en tant qu'il est identité de la différence et de l'identité (Schelling).[4] Cette identité est centre d'indifférence en tant qu'elle

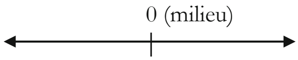
[2] *Ibid.*, p. 240.

[3] Cf Gilles Châtelet, *Les enjeux du mobile*, Paris, Editions du Seuil, 1993, pp. 124–129 "Une question de Kant : quels diagrammes pour le négatif ?".

[4] Sur ces questions on se reportera aux *Leçons de métaphysique allemande*, Tome 1, Paris, Éditions Grasset & Fasquelle, 1990 de Jacques Rivelaygue, pp. 185–192 « L'idée du sujet chez Fichte et Schelling » et sur la critique importante d'Hölderlin du moi pensé par Fiche dont hériteront Schelling et Hegel voir pp. 208–219 « Commentaire de "Être et jugement" ».

ne met l'accent ni sur la différence, ni sur l'identité, mais sur le milieu (le centre) qui les affirme tous deux. Ce site n'est pas un étant qui m'apparaît puisque je suis en lui pour effectuer le geste, ce pourquoi je le désigne par le 0. Pensons ici au 0 des nombres relatifs, il est bien impliqué par mon geste qui est *composé* de celui d'avancer de trois pas et de reculer du même nombre de pas, pourtant il ne m'apparaît pas, je fais seulement l'expérience en pensée des intensités de marcher et de reculer. On ne s'en sortira pas en disant qu'il apparaît en tant que j'ai fait du surplace car je pourrais être encore à la même place sans n'avoir rien fait. Le 0 n'est pas un résultat, il est le site présent dès le début de l'effectuation du geste, site d'un projet et non terme du geste, site du faire et non du fait. Convenons alors que ce geste s'exprime par le diagramme suivant:

Prendre sur soi



Le geste de prendre sur soi met donc au jour l'existence de notre ancrage subjectif. Or, si j'ai pu dire tout à l'heure que le 0 n'apparaissait pas (il n'est pas phénoménal), il faut dire aussi qu'il n'est pas soumis au devenir. En effet, seules passent les intensités, le 0 n'est pas emporté par le passage du temps, raison pour laquelle il est disponible pour tout nouveau geste de prendre sur soi. Il possède donc cette permanence qu'on attribue au sujet. Si bien que l'on peut même identifier ce geste dans l'expérience de pensée dite du cogito cartésien. En effet, et sans vouloir proposer une énième exégèse du cogito, je voudrais porter simplement mon attention sur le fait que la condition pour que Descartes se mette en quête de raisons pour ne pas suivre les voies qui pourraient sembler nous donner accès à une certitude absolue (par les sens, le raisonnement, les vérités mathématiques et même par l'existence de toute réalité qui pourrait n'être qu'illusion), eh bien cette condition est justement de *prendre sur lui* de ne pas suivre ces voies, ces directions de pensée. Cette résistance, cette résolution ne pas suivre ce que tout le monde suivrait est ce qui donne à la démarche cartésienne ce côté extravagant qu'est cette obstination à traquer le moindre petit doute. Avec le geste de prendre sur soi Descartes s'ancre et nous ancre dans un centre et par là nous incite à résister à suivre certaines directions de la pensée. Le "je pense, je suis" suppose lui encore ce centre en tant qu'il est la condition, le point immobile, pour que se déploie les diverses modalités de la pensée (douter, concevoir, imaginer, sentir, affirmer, nier etc.). Rappelons que Descartes affirme dans *Les méditations métaphysiques* qu'il recherche, comme Archimède, un point fixe. Toutefois il ne tiendra pas complètement son engagement en identifiant ce point fixe à une substance qui pense. On aurait pu imaginer un autre Descartes mettant au jour le point 0 du prendre sur soi se demandant s'il peut prendre sur soi de prendre sur soi comme il se demande s'il peut douter qu'il doute ? Mais prendre sur soi de prendre sur soi n'est-ce pas encore prendre sur soi ? Cet accès par un geste à un centre comme condition de la pensée (et aussi de la liberté) est aussi identifiable chez d'autres philosophes que Descartes.

Ainsi, pour Platon, l'âme se met en état de penser par un certain *geste* qui est propre à cet état, qu'il décrit ainsi dans le *Phédon* : « se concentrer elle-même en elle-même ».[5] Dans *Donner la mort*, Jacques Derrida qualifie ce geste comme étant un geste de remembrement : « ce mouvement de rassemblement sur soi de l'âme, cette fuite du corps vers le dedans d'elle-même où elle se replie pour se rappeler à elle-même, pour être auprès d'elle-même, pour se garder dans ce geste de remembrement ».[6] L'âme forme le lieu où elle se rassemble. Pas de rassemblement sans lieu où se rassembler, sans lieu d'incurvation. Or ce lieu d'incurvation tend à être un point puisque l'âme se *concentre* en ce lieu. Ce point d'incurvation attire l'âme à soi, l'âme n'est plus qu'elle-même, pure, indissoluble. Le geste de concentration pointe un lieu qui met l'âme dans l'attitude du « penser », qui la prédispose à la motricité dialectique ou dianoétique et l'affecte du désir d'apprendre (elle s'élance[7]). Autre exemple d'évocation d'un centre pour la pensée : dans *La Raison dans l'histoire* Hegel dit que l'Esprit « est précisément ce qui trouve en soi son centre. Il tend également vers le centre, mais il est lui-même ce centre. Il ne trouve pas son unité en dehors de lui. Il la trouve durablement en lui-même ».[8] Toujours selon Hegel, cette centration oppose l'Esprit et la matière (la nature) : « l'être matériel consiste précisément dans le fait de poser son centre *hors de soi*. Ce n'est pas ce centre, mais ce fait de tendre vers lui qui est immanent à la matière ».[9]

Toutefois, contrairement à Descartes, Platon et Hegel, je soutiens qu'il faut dire du centre, pour moi celui du prendre sur soi, qu'il concerne aussi le corps en tant qu'il est impliqué par tout geste effectué corporellement, chaque geste ayant un centre pour s'effectuer, un prendre sur soi implicite, son centre de gravité. Le centre du geste est ce que le centre de gravité est pour un mobile, bien plus, il est le centre de gravité du geste, ce par quoi le geste *se porte*. Si on appelle "gesticulation" le fait de ne pas arriver à effectuer un geste alors il faut dire que lors d'une gesticulation le centre gestuel fait défaut, en quoi elle est inhabitable et n'est donc pas un geste. C'est un peu comme dans un accident de voiture, le mouvement devient incontrôlé, nous perdons le centre du geste de conduire et si la voiture fait des tonneaux c'est le centre de tous nos gestes que nous perdons. Notre corps s'agite en tous sens, ses mouvements nous échappent, l'accident nous laisse sans gestes. De ce centre invisible, au cœur du geste, il faut donc affirmer qu'il est le point d'où s'origine l'effectuation d'un geste. Il est ce qui ne bouge pas dans ce qui bouge, il est ce à partir de quoi est dirigée l'effectuation d'un geste (dans les deux sens du verbe « diriger »). Il est ce qui ne passe pas dans ce qui passe, comme un point de suspension

[5] Platon, *Phédon,* Paris, Garnier Flammarion, 1991, p. 242.

[6] Jacques Derrida, *Donner la mort,* Paris, Éditions Galilée, 1999, p. 30.

[7] *Phédon, op.cit.,* 79-d, p. 242.

[8] Hegel, *La Raison dans l'histoire,* Paris, Hatier, 2012, p. 15.

[9] Hegel, *Encyclopédie des sciences philosophiques, II. Philosophie de la nature*, Paris, Vrin, 2004, p. 205. C'est Hegel qui souligne.

dans l'exclamation du geste. Il est un point virtuel qui préside à la succession d'états actuels qui composeront le mouvement du geste.[10]

Le geste du prendre sur soi se dit donc de la pensée et des corps. Il n'est pas sans faire écho, non pas à la glande pinéale, mais à sa position *centrée* dans le cerveau qui justifie son choix pour Descartes. Le geste du prendre sur soi, étant à la fois dans le corps et l'âme, aurait aussi quelque chose de la substance spinoziste qui s'exprime, pour l'homme, sous deux attributs. Ce geste se dit de la nature corporelle et de la pensée, via le diagramme. Le geste en pensée et le geste effectué corporellement s'expriment par un diagramme *réel*. Il existe un niveau diagrammatique (c'est ce que les neurologues découvrent de plus en plus en mettant en évidence des simulateurs présidant à l'effectuation d'un geste et même lors de la seule perception d'un geste). Puisque le diagramme se dit des corps et de l'esprit on saisit pourquoi la réalité diagrammatique et gestuelle (en laquelle nous sommes sujet-objet) peut prétendre à la fondation d'une philosophie de la nature. C'est la grande intuition qu'a eue Gilles Châtelet, allant dans le sens de Schelling pour qui l'identité absolue de l'identité et de la différence se réclame de l'Esprit et de la Nature (identité d'indifférence sujet-objet et donc esprit-nature) d'où la fondation d'une *Naturphilosophie*. J'ai pu montrer dans mon livre *L'immeuble du mobile*[11] que le 0 du prendre sur soi est le nouvel ancrage de la physique galiléenne. Mais laissons ceci de côté et revenons à la question de la subjectivité. Est-ce que ce 0 du prendre sur soi est le premier et dernier mot de la subjectivité ?

Rapport à soi, la verticale

Pour amorcer une réponse à cette question je reviens aux gestes corporels. Il est clair que l'*ensemble* des gestes que j'effectue, qu'ils se fassent en même temps ou successivement, doivent avoir un centre d'équilibre. Bien plus, parler de mes gestes est insuffisant puisqu'ils s'effectuent souvent avec ceux des autres. Mes placements gestuels sont conditionnés par les placements des autres, tout en les conditionnant aussi. Le 0 du prendre sur soi de chaque geste est comme traversé par un centre d'indifférence qui les unit. Ce centre, ce site d'indifférence unifiant c'est en un premier lieu la *verticale*. Je vais le justifier. Remarquons tout d'abord, qu'être en équilibre suppose d'être en rapport avec sa verticale. La verticale gravitaire est l'horizon sous et par lequel nos gestes se coordonnent avec équilibre pour s'effectuer, elle est ce par rapport à quoi chacun des gestes successifs ou simultanés

[10] On remarquera que Hegel quand il aborde l'organisme animal dans *L'encyclopédie* ne traite à aucun moment de la question du centre de gravité propre à un corps organique (alors qu'il en a parlé pour la matière inerte).

[11] Philippe Roy, *L'immeuble du mobile. Une philosophie de la nature avec Châtelet et Deleuze*, Paris, Presses Universitaires de France, collection "MétaphysiqueS", 2017.

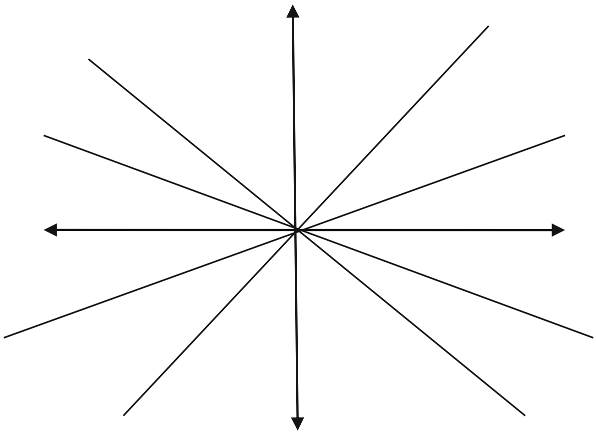
se positionnent. Si bien que la verticalité unifie, met en rapport ce qui n'est pas simultané, identité de l'identité et de la différence.

La fonction subjectivatrice de la verticale est bien marquée par Jacques Lacan quand il commente le célèbre stade du miroir. : « C'est la stabilité de la station verticale, le prestige de la stature, le caractère impressionnant des statues qui nous donnent le style d'identification dans lequel le moi trouve son point de départ et qui y laissent leur empreinte pour l'éternité ».[12] C'est aussi la verticale que l'on peut mettre au jour dans le diagramme de la subjectivité collective souveraine (du peuple) telle que Rousseau la présente dans le *Contrat social*. La subjectivité est celle du peuple mais est aussi en chacun de nous. On aurait ici une constitution de subjectivité souveraine et légitime au sein d'un diagramme politique. Je m'y attarde un peu. Soit l'énoncé du pacte social par Jean-Jacques Rousseau : « *Chacun de nous met en commun sa personne et toute sa puissance sous la suprême direction de la volonté générale; et nous recevons en corps chaque membre comme partie indivisible du tout.* A l'instant, au lieu de la personne particulière de chaque contractant, cet acte d'association produit un corps moral et collectif composé d'autant de membres que l'assemblée a de voix, lequel reçoit de ce même acte son unité, son *moi* commun, sa vie et sa volonté ».[13] Rousseau se réfère ici implicitement à l'idée d'un centre de gravité en tant que les volontés particulières de chaque individu ne doivent pas faire pencher le pacte de leur côté, chaque individu doit *prendre sur lui* (sans cela il n'y aurait même pas de pacte possible). La volonté générale passe donc par le centre de gravité qui se forme chez chaque individu de ce geste-événement collectif que tout le monde effectue et par le centre du geste qu'est la verticale souveraine, neutre, à laquelle chaque individu *se* soumet (et non pas est soumis puisque les individus l'effectuent). Suprême direction de la verticale gravitaire de la souveraineté en tant que centre d'effectuation du geste. Rousseau a donc ici une conception gestuelle de la politique. Il décrit la manière d'être du geste, son diagramme, que devra avoir tout acte de souveraineté populaire (qui prendra la forme pour Rousseau d'une déclaration, d'une convention).

Multiplions encore les approches. On peut montrer le relais du 0 par la verticale en suivant la dialectique géométrique de Gilles Châtelet.

[12] Jacques Lacan, « Some Reflections on the Ego », *International Journal of Psychoanalysis*, 1953, n°34, p. 12.

[13] Jean-Jacques Rousseau *Du contrat social*, Paris, Éditions G-F Flammarion, 1966, pp. 51–52. C'est Rousseau qui souligne.



La verticale est la direction moyenne, la direction d'indifférence à l'inclinaison et depuis laquelle je peux apprécier les différences d'inclinaison, identité de l'identité et la différence (que nous pourrions appeler tout aussi bien, et même mieux : identité de l'indifférence et de la différence). Elle est l'ancrage pour effectuer le geste de déploiement des obliques. Je dois m'imaginer, me projeter en elle pour l'effectuer. Elle est l'immobilité, la neutralité, à partir de laquelle se donnent les différences d'inclinaison. La verticale n'est pas emportée avec ce qu'elle déploie, sans cela le déploiement serait impossible. Elle a donc un statut comparable au point 0. En se plaçant en son site, notre geste tient ensemble des différences. Les obliques existent en étant dans un certain rapport à la verticale. C'est donc ce geste de rapport à soi effectué en se plaçant en la verticale qui vient à la suite du geste de prendre sur soi, celui-ci étant inclus dans le diagramme de la verticale (chaque direction et sa direction opposée sont tenues ensemble en 0).

En physique, la verticale est celle de la lumière de la relativité restreinte après le 0 galiléen.[14] Plus haut, je disais que la verticale gravitaire met en rapport les 0 d'indifférence de différents gestes. Eh bien, là aussi la verticale de la lumière met en rapport différents référentiels galiléens. La lumière (et sa célérité) est ce qui assure une invariance, une *permanence* en faisant communiquer ce qui varie. Bien plus, ce rapport produit des effets. La figure rigide d'un trait oblique est perçue plus courte à cause de ce rapport à la lumière. C'est la lumière et sa vitesse invariante absolue (dans le vide) qui est directrice. Je voudrais alors montrer rapidement, mais j'espère suffisamment, que la verticale peut se substituer à la substance spinoziste marquée par l'idée d'éternité (et non à proprement parler de permanence en tant que l'éternité est hors du temps chronologique). Parlons tout d'abord du rapport entre la substance et la lumière. Deleuze fait explicitement le rapprochement de la substance et de la lumière (le Lumineux substantiel) dans un article intitulé «

[14] Je me permets encore de renvoyer à mon livre, *L'immeuble du mobile. Une philosophie de la nature avec Châtelet et Deleuze, op.cit.,* Chap. 1 Sect. 4.

Spinoza et les trois éthiques » publié dans *Critique et clinique*. Les essences sont « *pures figures de lumière* produites par le Lumineux substantiel ».[15] Ce concept de *figure de lumière* vient tout droit de Bergson (*Durée et simultanéité*). Comme je l'ai affirmé plus haut, la relativité donne la priorité à la propriété lumineuse des figures sur celle de leur rigidité. Deleuze propose donc une lecture bergsonienne de Spinoza et aussi, tout aussi anachronique, pré-einsteinienne. Il est bien question ici des essences et non des individus en tant que composés de rapports entre parties extensives (qui sont plutôt à envisager sous la forme d'une mécanique des chocs). Deux choses donc : la production de la substance est celle de la lumière et les essences seraient purement lumineuses, en tant qu'elles sont à considérer sur le plan de la substance, en Dieu : « ce sont en elles-mêmes des "contemplations", c'est-à-dire qu'elles contemplent autant qu'elles sont contemplées, dans une unité de Dieu, du sujet ou de l'objet ».[16] Deleuze souligne bien que la vitesse absolue (étant celle de la Lumière) est associée aux figures de lumière. Cette vitesse absolue est aussi celle de la pensée du troisième genre (Deleuze donnant comme exemple la démonstration de la proposition 30 de la partie V de l'*Ethique* dont je reparlerai plus bas[17]). Et cette pensée a l'expression correspondante suivante dans l'attribut spatial : « Les essences [. . .] sont des vitesses absolues qui [. . .] emplissent [l'espace] en une fois, d'un seul coup ».[18] J'avoue que je ne comprends plus vraiment bien ce que veut dire Deleuze ici. De quel espace peut-il être question ? Ce ne peut être celui de l'étendue, on ne comprend en effet pas du tout ce que signifierait qu'une essence se meut à vitesse absolue dans l'espace extensif. Mais Deleuze ne dit toutefois pas qu'elles *ont* une vitesse mais qu'elles *sont* des vitesses absolues, ne faut-il pas comprendre alors qu'elles se manifestent par une vitesse absolue ? N'est-ce pas dire que l'essence, sans se déployer, pourrait saisir d'un coup ce qui se passe en elle dans l'étendue, comme si l'espace se contractait en elle ? (c'est ce que montre la théorie de la relativité, l'espace est comme compactifié pour un observateur qui se trouve sur la ligne de lumière). Deleuze irait dans ce sens quand il écrit plus loin : « la vitesse absolue, c'est la façon dont une essence survole *dans l'éternité* ses affects et ses affections (vitesse de puissance) ».[19] Ne doit-on pas ajouter que

[15] Gilles Deleuze, *Critique et clinique*, Paris, Editions de Minuit, 1993, p. 183. Le rapprochement de la lumière et de la substance peut sembler cavalier dans la mesure où le mot "lumière" n'apparaît que 4 fois dans l'*Ethique* plutôt sous forme de métaphore. Il a pourtant été souvent proposé, par exemple dans cet article de Pierre Sauvanet qui mentionne d'autres philosophes l'ayant fait avant lui: (https://journals.openedition.org/philosophique/271#bodyftn15). Deleuze se réfère à Yvonne Toros et plus particulièrement à son travail *Espace et transformation : Spinoza* (Paris-1) où elle confronte Spinoza et Vermeer, esquissant une théorie projective de la couleur en fonction du *Traité de l'arc-en-ciel* de Spinoza.

[16] *Critique et clinique*, *op.cit.*, p. 184.

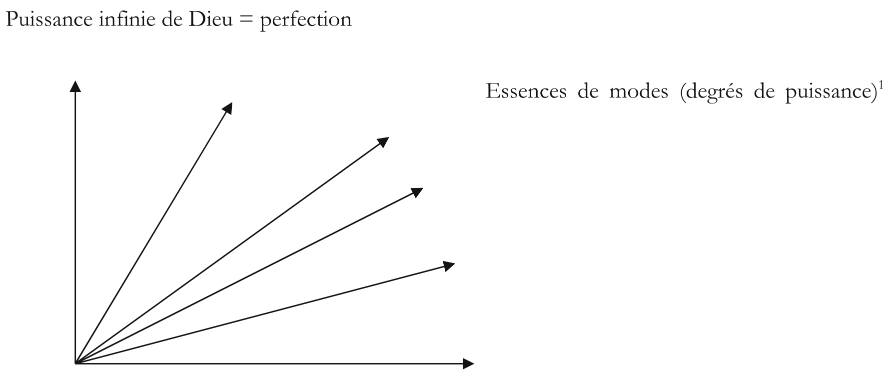
[17] « Dans la mesure où il se connaît lui-même et où il connaît son Corps sous l'espèce de l'éternité, notre Esprit a nécessairement la connaissance de Dieu et il sait qu'il est en Dieu et qu'il se conçoit par Dieu.» *Ethique*, Paris-Tel-Aviv, Éditions de l'éclat, 2007, p. 312.

[18] *Critique et clinique*, *op.cit.*, p. 184.

[19] *Ibid*. Je souligne.

la vitesse absolue que ces essences sont est en fait celle de Dieu par laquelle elles passent lors d'une intuition ? (raison pour laquelle Deleuze écrirait entre parenthèses "vitesse de puissance"). Il me semble qu'il faudrait compléter aussi ces arguments en affirmant que la lumière ne doit pas être vue uniquement comme une certaine matérialité mais aussi comme une certaine spiritualité. La puissance lumineuse est aussi puissance de penser. Et puisque l'intuition est extratemporelle, se disant de l'éternité, on est donc amené à dire de ce Lumineux substantiel qu'il n'est pas affecté par le temps qui passe, de même que les physiciens nous apprennent qu'un rayon lumineux, dont l'énergie n'est pas absorbée par un corps, va à l'infini. Mais de quel espace propre à la substance lumière-pensée peut-il être question ici ? Ce ne peut être celui de l'étendue. Ne serait-ce pas plutôt un espace diagrammatique propre aux essences elles-mêmes ?, un diagramme où coexisteraient la lumière-substance, les essences et leurs pensées. Je vais montrer qu'il est possible de tracer ce diagramme en identifiant à présent substance et verticale (après l'identification de la substance et de la lumière). Bien plus, je m'engage ensuite à faire ressortir l'aspect gestuel de la substance spinoziste.

Commentant le troisième genre de connaissance chez Spinoza, Gilles Deleuze caractérise *l'essence* de chaque mode (de chaque *individu*) comme étant irréductible à une autre et qu'elle doit être comprise comme un degré de puissance de la puissance absolue de Dieu. De plus si « chaque essence convient avec toutes les autres. C'est que toutes les essences sont comprises dans la production de chacune », enfin « non seulement chaque essence exprime toutes les autres dans le principe de sa production, mais elle exprime Dieu comme ce principe lui-même qui contient toutes les essences et dont chacune dépend en particulier ».[20] Cet ensemble de caractérisations est donc encore exprimable par le diagramme de la verticale:[21]



[20] Gilles Deleuze, *Spinoza et le problème de l'expression*, Paris, Éditions de Minuit, 1968, p. 282 et p. 283.

[21] On prendra garde ici de ne pas interpréter les flèches de chaque oblique comme désignant un mouvement dans chacune de leur direction. Il faut n'avoir en vue que l'obliquité en tant que telle.

Chaque puissance de mode en tant qu'elle est ordonnée affirme l'ordre lui-même donc chacun de ses degrés et par conséquent l'horizon qui les ordonne (Dieu=substance=nature). C'est en effet par l'horizon que tout s'ordonne, il induit une connexité entre les puissances. Chacun des degrés peut être dit plus ou moins parfait ou puissant en tant qu'il est un certain degré de participation à la perfection, à la puissance infinie de Dieu comme les obliques de ce diagramme sont plus ou moins inclinées (grandeur intensive) puisque participant plus ou moins à l'inclinaison parfaite, infinie, qu'est la verticale. Chaque essence prend place sur ce diagramme. Cependant le renvoi aux autres essences et à Dieu n'existe *pour une essence* que si elle fait l'expérience de ce diagramme : c'est le troisième genre de connaissance.

C'est ce qu'exprime la proposition 30 de la partie V de l'*Ethique* : « Dans la mesure où il se connaît lui-même et où il connaît son Corps sous l'espèce de l'éternité, notre Esprit a nécessairement la connaissance de Dieu et il sait qu'il est en Dieu et qu'il se conçoit par Dieu ».[22] Se concevant par Dieu, comme degré de puissance de la verticale, notre Esprit peut donc concevoir les autres Esprits et Corps comme d'autres degrés de puissance renvoyant à Dieu et donc à lui-même. Toutefois, la condition d'accès à ce troisième genre est que l'individu soit en grande partie *actif* c'est-à-dire cause adéquate de ce qui lui arrive, ce qui signifie pour moi « habiter un geste ». C'est déjà l'enjeu du second genre de connaissance. « Habiter nos gestes » est alors étendu ici à tous nos gestes et *leur enchaînement* (d'où l'ancrage implicite en la verticale). Étant actif (= vertueux pour Spinoza), nous désirons aussi que les autres y soient, nos gestes veulent résonner chez les autres et inversement chaque geste d'un autre résonne en nous. « Le bien que tout homme recherchant la vertu poursuit pour lui-même, il le désirera aussi pour les autres, et cela d'autant plus qu'il aura une plus grande connaissance de Dieu ».[23] Un homme actif cherche à favoriser les gestes des autres. Comme l'écrit Pascal Sévérac : « être actif signifie, pour un mode, non pas rendre passif un autre mode—mais bien plutôt le rendre nécessairement actif : *agir, c'est faire agir*—au sens où être actif, c'est nécessairement, pour soi-même, à la fois rendre actif un autre et être rendu actif par un autre, lui-même actif [. . .] nous pouvons être la cause adéquate d'un effet en dehors de nous, c'est-à-dire en un autre, si et seulement si cet autre est lui-même également cause adéquate de cet effet ».[24] Un individu actif suit nécessairement de bons enchaînements de gestes, or les gestes qu'il favorise chez les autres seront à la fois compatibles avec son enchaînement et propres à leurs enchaînements. Il y a une fructification des enchaînements et ceci grâce aux images et aux paroles qui communiquent inter-individuellement les gestes à travers la *lumière* et le son (si bien que nous les simulons inconsciemment). Si bien qu'à l'extrême tous les individus appartiennent à un individu les totalisant dans leurs différences. Ou encore, c'est

[22] *Ethique*, *op.cit.*, p. 312.

[23] *Ibid.*, partie IV, proposition 37, p. 251.

[24] Pascal Sévérac. *Le devenir actif chez Spinoza*, Paris, Éditions Champion, 2005, p. 79. C'est Pascal Sévérac qui souligne.

comme si tous dansaient une danse commune tout en dansant chacun pour leur compte une danse différente, favorisant la danse des autres.

Comme je l'ai laissé entendre, Spinoza va associer à la substance (ou à la verticale pour moi) un geste, le geste de Dieu. C'est un des grands mérites d'Alfonso Cariolato dans son livre intitulé justement *Le geste de Dieu* d'avoir localisé ce terme de « geste de Dieu » dans le texte même de l'*Ethique* de Spinoza. À la fin du scolie de la proposition 49 de la deuxième partie, il est effectivement fait usage du mot latin *nutu*. Robert Misrahi traduit la fin du scolie ainsi : « Il reste à montrer combien la connaissance de cette doctrine est utile à la vie, ce que nous dégagerons aisément de ce qui précède. Elle est utile en effet : 1° en ce qu'elle nous apprend que nous agissons par le seul commandement de Dieu, que nous sommes des participants de la nature divine, et cela d'autant plus que nous accomplissons des actes plus parfaits et comprenons Dieu de plus en plus ».[25] Or la locution « nous agissons par le seul commandement de Dieu » est rendue autrement par Charles Appuhn « nous agissons par le seul *geste* de Dieu ».[26] C'est un des rares traducteurs à avoir identifié ce thème du geste dans cette phrase (suivi par Guéroult dans le premier volume de son commentaire de l'*Éthique*). Pourtant le substantif *nutus* était en latin médiéval, et donc dans le latin des traités, un autre terme pour *gestus.* L'expression geste *de* Dieu doit faire l'objet de précautions car, à mal la lire, Dieu semble être différent de son geste, or Dieu *est* son geste, il n'est pas un hyper-sujet qui voudrait ce geste. Le geste (de) Dieu est pure production des modes et de leur manière d'être, de leurs gestes. Le geste (de) Dieu est la cause de soi intérieure à tous les gestes. Dieu est *causa sui* car il est geste. Il est cause de soi comme un événement qui n'en finirait plus d'arriver et qui pour Spinoza n'a même jamais commencé ou a toujours déjà commencé (ce pourquoi il ne se dit pas du temps chronologique mais de l'éternité).

En mettant l'accent sur le geste, les gestes, il est alors entendu que se manifeste aussi la dimension du désir, de la persévérance dans l'être. Nos gestes, la gestualité que nous sommes nous poussent. Par ailleurs, et pour être fidèle à Spinoza, il est nécessaire d'affirmer aussi que cette gestualité substantielle ne doit pas simplement s'exprimer dans les corps mais aussi dans la pensée. Les idées impliquent des gestes. C'est une conséquence qu'énonce aussi Alfonso Cariolato : « L'agir de "nous", les hommes, qui devient le geste même de Dieu, ouvre donc au centre de l'œuvre de Spinoza l'exigence d'une pensée elle aussi d'une certaine façon "gestuelle". Dit en d'autres termes : non pas une pensée *sur* l'existence et peut-être même plus *de* l'existence mais à chaque fois un geste de pensée qui se met en relation avec l'existence comme à ce qui immanquablement la provoque et lui échappe

[25] *Ethique*, *op.cit.,* p. 152.

[26] Alfonso Cariolato indique d'autres traductions parmi les traductions françaises : « "nous n'agissons que par la volonté de Dieu" (E. Saisset), "nous agissons par la seule volonté de Dieu" (R. Caillois), "nous agissons par le seul commandement de Dieu" (B. Pautrat), "nous agissons sur la seule initiative de Dieu" (P. Macherey) » *Le geste de Dieu*, *Chatou, Les éditions de la transparence, 2011,* p. 113.

».[27] Ce rapport à soi gestuel met donc l'accent sur la dimension éthique de notre subjectivité.[28] J'ai laissé entendre plus haut que ce rapport à soi n'était pas sans impliquer la composition avec les gestes des autres, je me dois d'ajouter qu'il ne faut pas perdre de vue aussi que nous pouvons être acteurs de gestes (corps et pensée) d'une société (nous l'avons vu d'emblée avec Foucault). Spinoza propose aussi de penser les sociétés politiques comme des individus.[29] Le rapport à soi éthique est aussi politique (c'est bien un certain rapport à soi politique que Rousseau a en vue dans *Du contrat social*).

Après le centre d'indifférence du 0 du geste de prendre sur soi nous sommes donc passés au centre d'indifférence de la verticale du geste éthique et politique du rapport à soi qui fait communiquer et met en rapport les gestes. Avec la verticale se fait jour une connexité gestuelle. Les gestes sont mis en rapport dans une forme de coexistence qui ne se réduit pas à la simultanéité. Ce 0 et cette verticale sont les sites en laquelle s'ancre le sujet constitué par des gestes. Ces sites ou centres d'indifférence fondent la permanence subjective au sein de la variété des gestes effectués.

Je vais proposer à présent de faire un pas supplémentaire et passer à un nouveau type de site d'indifférence, à un nouveau substitut de la permanence substantielle en pensant la subjectivité à travers ses autours, en accentuant donc sa topologisation. La dimension du désir que nous avons vu apparaître avec la persévérance dans l'être gestuelle va y être plus prépondérante.

Tourner autour de soi, le trou

Une des voies de passage à ce nouveau palier serait de suivre encore le physico-mathématique. On sait qu'à la communication lumineuse propre à la relativité restreinte fera suite la communication lumineuse géodésique de la Relativité Générale au sein d'un espace topologique riemannien. Or c'est cette voie que David Rabouin a fait le choix d'emprunter pour proposer un néo-spinozisme (post euclidien, riemannien) dans son livre *Vivre ici. Spinoza, éthique locale*. Dans mes termes, il est passé du centre d'indifférence qu'était la verticale à un type de centre d'indifférence impliqué par la topologie. Bien plus, la surface topologique, surface

[27] *Ibid.,* pp. 57–58. C'est Alfonso Cariolato qui souligne. J'ai défendu cette idée d'une conception gestuelle de la pensée chez Spinoza en plusieurs endroits. Par exemple dans *Trouer la membrane. Penser et vivre la politique par des gestes*, Paris, L'Harmattan, 2012, pp. 121–147, « Individu et communauté chez Spinoza » ou encore dans ma thèse « Gestes et diagrammes politiques » soutenue à l'université de Paris 8 en 2014.

[28] Même l'éthique idéelle platonicienne ne fait pas l'économie du passage du prendre sur soi de la concentration de l'âme à la verticale qui la tourne vers la hauteur divine des Idées (sous le soleil souverain du Bien).

[29] C'est un point que je développe aussi, à la suite d'Alexandre Matheron dans *Trouer la membrane. Penser et vivre la politique par des gestes, op.cit.*

d'indifférence, va impliquer une multiplicité de centres d'indifférence. Voyons ceci de plus près.

Dans le cadre d'un diagrammatisme du genre de la Relativité Générale la communication lumineuse, géodésique, va rendre possible ou pas des connexions en fonction du diagramme. David Rabouin écrit « Dans la Relativité Générale [la] partition de l'univers en régions avec lesquelles je peux ou non être connecté [. . .] dépend du point où l'on se trouve et de la courbure de l'univers qui s'y manifeste ».[30] Or c'est ce type de diagrammaticité qu'il faut, selon David Rabouin, rapporter aux affects des individus, plus précisément à ce qu'il appelle « leurs espaces affectifs ». À suivre ces diagrammes topologiques, il devient alors possible de penser des chemins formant des proximités, des voisinages entre des affects qui sont pourtant distants chronologiquement. Le diagramme propre aux espaces affectifs permet de penser *quels sont* les états affectifs qui se connectent au sein d'une vie individuelle. Dans son livre *Vivre ici* David Rabouin écrit : « Imaginons qu'étant petit enfant, disons vers l'âge de six ou sept ans, vous vous soyez retrouvé en classe dans une situation particulièrement humiliante. Par exemple, il vous fallait prendre la parole à l'estrade et vous aviez fait pipi dans votre culotte [. . .] L'important, et le surprenant quand on y songe, est que cette expérience enfantine puisse nous *toucher* "ici et maintenant" (consciemment ou inconsciemment peu importe)—par exemple sous la forme d'une appréhension à prendre la parole devant un auditoire. [. . .] L'enfant que j'étais est là, tout près de moi, avec sa honte et sa peur, et des sentiments similaires me "reprennent" dans des situations que je sais pourtant être totalement différentes ».[31] Deux moments chronologiquement très éloignés peuvent être topologiquement proches sur le diagramme d'un espace affectif (ils sont dans un même voisinage, ce qui ne veut pas dire identiques : « ils sont suffisamment "proches" pour que je puisse les considérer comme se trouvant dans un même voisinage »[32]). Une connexion diagrammatique relie ces deux lieux, ces deux *points de vue affectifs*, comme les appelle David Rabouin. Ce diagramme connectif peut alors être selon lui assimilé à une variété riemannienne, les lieux étant les points de vue affectifs. Et comme un mobile suit la ligne sur le diagramme de la Relativité Générale, un homme suit la ligne de sa vie affective sur le diagramme de l'espace affectif.

Seulement, à s'en tenir à la seule connexion des différents points de vue affectifs, rien ne nous permet de comprendre pourquoi ceux-ci forment la ligne d'*une* vie qui s'expérimente comme telle (unifiée). La ligne n'est pas seulement une connexion d'affects mais une *intégration* des affects dans un affect d'ensemble : un affect d'affects, la joie d'être joyeux ou la tristesse d'être triste (qui, quand elle l'emporte, crée un état mélancolique). « Lorsque nous avons l'impression, par exemple, que *tout* va bien (ou, à l'inverse, que *tout* va mal), nous ne considérons que le résultat,

[30] David Rabouin, *Vivre ici. Spinoza, éthique locale*, Paris, PUF, 2010, p. 137.

[31] *Vivre ici. Spinoza, éthique locale*, *op.cit.*, pp. 129–130.

[32] *Ibid.*, p. 133.

d'une évaluation d'ensemble de nos affects.[33] » David Rabouin dénomme cet affect d'ensemble, un *affect de second ordre*. Cet affect appelle une réaction, un geste. « Le simple fait que je sois triste d'être triste induit un désir [. . .] l'effort pour maintenir sa puissance ou "persévérer dans son être".»[34]

David Rabouin insiste beaucoup dans son projet sur l'indépendance de cet espace riemannien affectif au regard du monde perçu et sa spatialité *extrinsèque*. « Il n'y a pas des affects attachés à un espace-temps préalablement donné qui permettrait de les repérer dans un "monde" toujours déjà là.[35] » Il y a un espace propre à notre vie affective, avec ses lieux dont les distinctions relèvent du passage à un *autre* affect (points de vue affectifs). C'est donc l'affect qui fait la différence et non des situations spatiales-et-chronologiques pré-données. Cet espace affectif, s'il double chaque moment, chaque acte, est donc aussi celui sur lequel se connectent les affects d'autres moments, dégagés des actes (actions + perceptions), des énoncés qui y étaient associés, dans une temporalité qui n'est donc pas chronologique comme celle du temps des actes (donc de la perception) : « si vous acceptez que votre mémoire vous fait constamment faire de curieux voyage ("dans le temps", comme on dit), que l'enfant que vous étiez peut être présent là, tout près de vous, comme peuvent être présents vos proches morts, vos rêves lointains ou vos promesses, alors vous avez saisi tout ce qu'il y a à saisir dans le fait qu'un espace affectif n'est pas un espace-temps comme ceux auxquels vous donne accès le "poids mort" de la perception ».[36] Ce n'est pas le poids mort de la réalité massique, propre à la perception, mais plutôt la masse nulle de l'*imagination*, comme celle des images du cinéma ou des images-souvenirs ou même des diagrammes en tant que tels (sans la part du geste effectué), qui est plus proche de la nature des gestes affectifs. « L'idée géniale du romancier, dit en substance Proust, aura été de comprendre que comme les affects se déroulent de toutes façons au niveau de l'imaginaire, et donc dans un régime irréductible de confusion, on pouvait jouer avec eux bien mieux en se servant d'images qu'en se plaçant au niveau du prétendu réel, où ils sont toujours lestés du "poids mort" de la réalité perceptive ».[37] Par-delà la réalité perceptive et les idées représentatives qui y sont associées, il y a donc les points de vue affectifs qui sont même la condition grâce à laquelle nous pouvons nous rapporter à des points de vue affectifs qui ne sont plus les nôtres. « Comment se fait-il que nous reconnaissions ces affects comme "proches" des nôtres alors qu'ils sont pourtant, au niveau de la représentation, si évidemment distants : qui irait croire, si les affects sont liés aux objets et aux valeurs, que l'amitié d'Achille et de Patrocle, la colère du même Achille, la fidélité de Pénélope, la ruse d'Ulysse, etc., toutes choses prises si intimement dans les systèmes de valeurs que je reconnais comme étrangers, puissent

[33] *Vivre ici. Spinoza, éthique locale*, *op.cit.*, p. 171. C'est David Rabouin qui souligne.

[34] *Ibid.*, p. 173.

[35] *Ibid.*, p. 149.

[36] *Ibid.*, p. 188.

[37] *Ibid.*, p. 131. Cette idée de « poids mort » est de Proust.

me sembler une illustration de “l’amitié”, de la “colère”, de la “fidélité”, de la “ruse”, etc.[38] »

Il me semble important de remarquer ici, pour saisir le nouveau geste en jeu, que chaque point de vue affectif prend son sens avec la pensée de son voisinage, de son autour (que serait la colère d’Achille sans Agamemnon, Hector, leurs gestes etc.). La teneur existentielle d’un point de vue existe par ce qui tourne autour de lui, par ce qui le voisine (rappelons qu’il ne faut pas se laisser aller à entendre par « voisinage » une proximité spatiale ou chronologique). David Rabouin est bien inspiré quand il rappelle cette fameuse phrase de Proust au début de la *Recherche* : « Un homme qui dort, tient en cercle *autour* de lui le fil des heures, l’ordre des années et des mondes ». Il commente : « Lorsque ces cercles se mettent en mouvement, dans la chambre de Marcel, au moment de s’endormir, ce n’est pas seulement le petit enfant attendant le baiser de sa mère qui est là. Il y a aussi Golo et Geneviève de Brabant, ou encore une église, un quatuor, la rivalité de François Ier et de Charles Quint. Eux aussi sont là, près de lui, et partagent avec lui un univers d’affects où les “distances” ne sont pas celles que nous livrent la prétendue “réalité” extérieure ».[39] Les points de vue affectifs, leur couleur existentielle dépendent de leur autour, de leur tourner autour. Quels sont alors les nouveaux centres d’indifférence de ce geste de tourner autour de soi ? Ils sont les trous autour desquels tournent les voisinages, les boucles des heures, des années, des mondes. En effet, le trou entraîne la giration. Comme l’écrit Gilles Châtelet, se référant à l’analyse complexe, le trou « est ce autour de quoi quelque chose doit tourner ».[40] Évidemment ces trous ne viennent pas trouer un espace pré-donné comme le feraient des trous dans un mur, ces trous viennent avec leurs boucles, avec leur surface (ici un espace affectif), ce sont des trous diagrammatiques. Puisque j’ai parlé de Lacan plus haut et qu’il est question d’espace affectif, il serait intéressant de se pencher sur l’espace pulsionnel ayant une précédence ontologique sur la verticale phallique. En effet la pulsion, que Lacan nomme objet « a », comporte la présence d’un creux; une boucle tourne autour d’un vide, l’ensemble des béances des objets « a » formant selon les termes de Lacan « une communauté topologique ».[41]

Tournons-nous aussi vers l’espace topologique des monades, le *spatium*, de Leibniz. On peut voir sur ce diagramme, réduit à deux points de vue monadiques, que Deleuze place les points de vue dans des trous autour desquels se trouve une inflexion.

[38] *Ibid.,* p. 132.

[39] *Ibid.*, p. 130.

[40] Gilles Châtelet, *L’enchantement du virtuel*, Paris, Éditions Rue d’Ulm, 2010, p. 107.

[41] Jacques Lacan, *Les Quatre Concepts fondamentaux de la psychanalyse*, Paris, Seuil, 1973, pp. 199–206.

Deleuze commente : « Partant d'une branche de l'inflexion, nous déterminons un point qui n'est plus celui qui parcourt l'inflexion, ni le point d'inflexion lui-même, mais celui où se rencontrent les perpendiculaires aux tangentes dans un état de la variation. Ce n'est pas exactement un point, mais un lieu, une position, un site, un "foyer linéaire", ligne issue de lignes. On l'appelle *point de vue* pour autant qu'il représente la variation ou inflexion. Tel est le fondement du perspectivisme. Celui-ci ne signifie pas une dépendance à l'égard d'un sujet défini au préalable : au contraire, sera sujet ce qui vient au point de vue, ou plutôt ce qui demeure au point de vue ».[42] En précisant que la subjectivité vient prendre place là où un point de vue *demeure*, en ce trou sur le diagramme, Deleuze met bien en évidence une autre figure de la permanence à travers ce type de centre d'indifférence.

Ces trous-points de vue coexistent donc dans une certaine texture diagrammatique impliquant et mêlant des diagrammes individuels, collectifs, politiques (comme le laisse entendre Proust avec le cercle du fil des heures, de l'ordre des années et des mondes). L'exemple donné par David Rabouin de la connexion de l'enfant parlant en classe avec l'adulte intervenant devant un auditoire, n'est pas sans être pris dans un diagramme politique dans et par lequel on doit un jour être élève et un autre orateur, là aussi il y a un chemin qui connecte les affects propres à ces deux points de vue sociaux. Le diagramme individuel et le diagramme politique sont ceux d'une certaine coexistence affective et coexistent aussi ensemble.

Donnons un exemple de diagramme affectif politique. Récemment dans son livre *Capitalisme, désir et servitude* Frédéric Lordon a établi implicitement un diagramme affectif pour penser le capitalisme. Je dis "implicitement" parce qu'il n'emploie pas le mot "diagramme". Ce diagramme est celui de connexions de désirs (donc d'affects). Ainsi, le désir du salarié est enrôlé au service du désir du patronat. « Faire entrer des puissances d'agir tierces dans la poursuite de son désir industriel à soi, voilà l'essence du rapport salarial ».[43] La mobilisation de ces puissances d'agir « n'est possible qu'en faisant de la médiation de l'argent le point de passage obligé, le point de passage exclusif du désir basal de la reproduction

[42] Gilles Deleuze, *Le pli*, Paris, Éditions de Minuit, 1988, p. 27. C'est Deleuze qui souligne.

[43] Frédéric Lordon, *Capitalisme, désir et servitude. Marx et Spinoza*, Paris, Éditions La Fabrique, 2010, p. 19.

matérielle ».[44] « Tout est fait pour prendre les agents "par les affects joyeux" de la consommation en justifiant toutes les transformations contemporaines—de l'allongement de la durée du travail ("qui permet aux magasins d'ouvrir le dimanche") jusqu'aux déréglementations concurrentielles ("qui font baisser les prix")—par adresse au seul consommateur en eux ».[45] Le diagramme est cependant composé de beaucoup plus de places. « La grande entreprise est un feuilletage hiérarchique structurant la servitude passionnelle de la multitude salariale selon un gradient de dépendance. Chacun veut, et ce qu'il veut est conditionné par l'aval de son supérieur, lui-même s'efforçant en vue de son propre vouloir auquel il subordonne son subordonné, chaîne montante de dépendance à laquelle correspond une chaîne descendante d'instrumentalisation ».[46] Là encore les actions de chacun sont dépendantes d'un diagramme affectif, feuilleté. Le diagramme est même plus étendu car les désirs extérieurs font pression : ceux des actionnaires mais aussi ceux, opposés, des concurrents (concurrence qui est aussi effective entre les sous-traitants). Ces pressions extérieures se transmettant alors à chaque étage du feuilletage.

Le but ici n'est pas de proposer le diagramme complet de Frédéric Lordon mais de se donner une idée de la dimension politique que peut revêtir un diagramme affectif, c'est un véritable complexe affectif, déterminant pour nos conduites. Les déroulements chronologiques de celles-ci sont pilotés par les connexions diagrammatiques : le point de vue affectif de l'actionnaire ou de l'entrepreneur doit trouver des modes de connexion avec les affects du salarié, le plus efficace étant de motiver le salarié, de le rapprocher de ses affects plus que de commander par la crainte. Les actes suivront, il se tuera au travail joyeusement.[47] L'affect de consommation de ce même salarié n'étant pas sans voisiner aussi avec le point de vue affectif de l'actionnaire partageant avec lui les joies de l'argent. Le geste affectif du capitalisme est donc celui de ce diagramme, dont la manière d'être change, inventant toujours de nouvelles manières pour mobiliser le désir des salariés-consommateurs, pour connecter les désirs.[48]

Pensons aussi au petit enfant chez qui il a été montré que les premières relations avec son entourage sont de l'ordre d'un diagramme familial affectif dans lequel les

[44] *Ibid.*, p. 25.

[45] *Ibid.*, p. 50.

[46] *Ibid.*, p. 41.

[47] C'est dans ce sens que va le régime de désir néolibéral qui « se donne pour tâche spécifique de produire à grande échelle des désirs qui n'existaient pas jusqu'alors [...] désirs du travail heureux ou, pour emprunter directement à son propre lexique, désirs de "l'épanouissement" et de la "réalisation de soi" dans et par le travail ». *Ibid.*, p. 76. Pour cela le mimétisme des désirs des images valorisées de l'entrepreneur, du winner, sont capitales.

[48] Frédéric Lordon rappelle que son approche a, sous le nom d'*obséquium*, été déjà celle de Spinoza. « Spinoza nomme *obsequium* le complexe d'affects qui fait se mouvoir les corps assujettis vers les objets de la norme, c'est-à-dire qui fait faire aux sujets – où "sujet" est à comprendre au sens de *subditus et* non de *subjectum,* sujet *du* souverain et non sujet souverain – les gestes conformes aux réquisits de la persévérance de son empire » *Ibid.,* p. 87.

intensités toniques sont indiscernables des tonalités affectives (mimiques, spasme des rires, postures des corps). L'enfant est dans une forme d'*accordage* avec les tonicités-tonalités de ceux qui s'occupent de lui (ici ce n'est pas seulement les gestes affectifs du visage). Il vient prendre place dans ce diagramme affectif qu'il active aussi, puisqu'être dans un diagramme c'est aussi l'activer, le faire varier. Il y a covariation affective des acteurs.

Allons alors à l'essentiel, l'accent est donc à présent mis sur la consistance ou étoffe subjective qui provient de nos voisinages, de ce qui tourne autour des trous qui nous composent. Charge à nous de modifier, d'agencer nos voisinages pour mieux exister affectivement (augmenter notre puissance en s'efforçant d'organiser de bonnes rencontres dirait Spinoza). Ce déclare encore ici une dimension politico-éthique de la subjectivité, qui passe aussi par des voisinages imaginés, fictifs, artistiques ou même de pensée (je vais y venir plus loin). Par ailleurs, il faut en déduire que cette coexistence ne se dit pas du temps chronologique mais plutôt d'une forme temporelle ne passant pas avec le temps qui passe. De même que le diagramme de la Relativité Générale ne passe pas avec les temps chronologiques des mobiles puisqu'il les préside et les dirige. Ce temps permanent de la coexistence qui n'est pas engagé dans la successivité chronologique étant donc en suspens, en retenue, est celui du *virtuel*.

Le virtuel-Temps événementiel

Le diagramme virtuel préside aux actualisations, il rend possible le passage des présents parce qu'il ne passe pas lui-même. On peut alors affirmer maintenant que c'est le virtuel qui fonde la permanence et la retenue propre aux centres d'indifférence, aux ancrages subjectifs que j'ai passés en revue depuis le début (le 0, la verticale, le trou). Le virtuel se dit donc du temps en tant que temps qui ne passe pas, temps de la coexistence. C'est ici que se situe la subjectivation propre à ce que Deleuze a appelé « un cristal » qui rend possible de penser les virtualités en faisant l'épreuve de les devenir.[49] On crée les conditions pour faire l'expérience du « tourner autour » d'une virtualité, circuit d'un cristal grâce auquel nous devenons la virtualité, nous remontons le temps que son actualisation descend. La colère d'Achille est saisissable car, lisant ou repensant à certains passages de l'*Illiade*, nous devenons Achille en colère. Nous jouons son expérience en pensée car nous l'actualisons tout en remontant vers ce qui s'actualise : la colère en tant que virtualité non chronologique, virtualité constituée elle-même par les autours d'autres circuits

[49] Cf Gilles Deleuze, *L'image-temps*, Paris, Editions de Minuit, 1985, pp. 92–128, chapitre « Les cristaux de temps » ou dans un cadre pas essentiellement cinématographique mais aussi physico-mathématique, Gilles Deleuze et Clairet Parnet, *Dialogues,* Paris, Éditions Flammarion, 1993, pp. 117–185 « L'actuel et le virtuel ». Je consacre aussi un chapitre au cristal dans *L'immeuble du mobile. Une philosophie de la nature avec Châtelet et Deleuze, op.cit.*, pp. 143–159, chapitre 6 « Le virtuel et le temps ».

(voisinages d'Agamenon, d'Hector etc.).[50] Penser c'est alors devenir. « Ce qu'on voit dans le cristal est le temps en personne, le jaillissement du temps. La subjectivité n'est jamais la nôtre, c'est le temps, c'est-à-dire l'âme ou l'esprit, le virtuel. L'actuel est toujours objectif, mais le virtuel est le subjectif : c'était d'abord l'affect, ce que nous éprouvons dans le temps; puis le temps lui-même, pure virtualité qui se dédouble en affectant et affecté, "l'affection de soi par soi" comme définition du temps.[51] » Le virtuel, cet en-soi du temps, est une figure nouvelle de l'Absolu.[52] Cet Absolu nous subjective si nous devenons ses affections de soi par soi (tels le prendre sur soi, le rapport à soi, le tourner autour de soi comme liste non exhaustive de modalités de l'affection de soi par soi).[53]

Mais cette figure est aussi celle de la liberté. Le virtuel se réclame de l'événementialité.[54] Selon l'approche leibnizienne, le virtuel est l'ensemble des événements du monde qui s'actualisent dans les âmes et se réalisent dans les corps. Je cite Deleuze au sujet de Leibniz dans *Le Pli* : « Nous ne pouvons parler de l'événement que déjà engagé dans l'âme qui l'exprime et dans le corps qui l'effectue, mais nous ne pourrions pas du tout en parler sans cette part qui s'en *soustrait* ».[55] Chaque événement a une part de lui qui ne passe pas, qui ne s'effectue pas dans les âmes ou les corps, ce qui rend raison de cette propriété de retenue propre aux centres d'indifférence. Rappelons-nous aussi ce que je disais du geste (de) Dieu

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[50] On pourrait défendre que le prendre sur soi possède déjà une caractérisation cristalline, en tant que point d'arrêt d'une effectuation (je n'effectue pas le rire, je me retiens de rire). Ce point d'arrêt me met alors en rapport avec la virtualité freinée. N'avons-nous pas là une origine possible de la conscience ? N'y a-t-il pas conscience réflexive parce que je ne suis pas engagé dans l'action ? « Il est couramment admis qu'il y a une sorte d'enregistrement mental (en général hors de toute conscience) de l'existence d'un projet de mouvement avant l'action. L'existence de ce projet atteint très facilement la conscience quand son exécution est inhibée » Daniel N. Stern, *Le monde interpersonnel du nourrisson*, Paris, PUF, 1997, p. 105. On entrevoit aussi pourquoi est envisageable ici une origine morale de la conscience (puisqu'elle impliquerait le geste de prendre sur soi moral).

[51] Gilles Deleuze, *L'image-temps, op.cit.*, pp. 110–111.

[52] Deleuze (ainsi que Félix Guattari) l'affirme implicitement dans *Qu'est-ce que la philosophie ?*, Paris, Éditions de Minuit, 1991. Il écrit que le virtuel « c'est l'immanence pure » p. 148 et que « le plan d'immanence [. . .] constitue le sol absolu de la philosophie » p. 44. Cet Absolu avait été déjà mis en avant par Deleuze commentant la durée (et au fond le virtuel) bergsonien. Gilles Deleuze, *Le bergsonisme*, Paris, PUF, 1966, p. 27.

[53] On pourrait proposer à ce stade l'hypothèse suivante : le 0, la verticale, le trou sont des centres directeurs propres à d'autres êtres naturels que l'homme (y compris la physique, nous l'avons vu) mais celui qui effectue en plus les gestes *du virtuel*, d'affection de soi par soi, qui correspondent à ces centres, aurait la possibilité d'ouvrir des mondes. Comme si en devenant les gestes du virtuel on le démultipliait. Pour parler comme Leibniz, le virtuel ne serait plus *le* monde impliqué par l'expression de notre monade et les autres mais un pluri-virtuel, un virtuel qui s'enfante. C'est d'ailleurs cet enfantement du virtuel que la physique quantique manifeste. Cela demanderait évidemment d'être plus élaboré et ne reste pour l'heure qu'à l'état embryonnaire.

[54] Là aussi on pourra se reporter à Deleuze et Guattari, *Qu'est-ce que la philosophie ?*, *op.cit,* pp. 147–151. Toutefois ils ne vont pas jusqu'à associer gestes et virtuel événementiel.

[55] Gilles Deleuze, *Le pli, op.cit.,* p. 142. Je souligne.

de Spinoza, de la substance comme événement qui a de toujours commencé, cause de soi. Le geste (de) Dieu étant au cœur de tous les gestes des modes c'est donc bien l'événementialité substantielle qui passe par tous les centres d'indifférence gestuels. La substance étant toujours dans une forme de retrait non séparé des modes, une causalité immanente dira Spinoza. Cependant, contrairement à Spinoza, pour lequel il n'y a pas de virtuel ni d'événements à proprement parler (puisque son approche est celle de la nécessité, du déterminisme), je soutiens, sans pouvoir m'y étendre ici, qu'il y a des gestes qui font événement, que l'événementialité est même inhérente à la gestualité, étant entendu que les gestes ont tous une part virtuelle (ce pourquoi ils s'actualisent et arrivent en s'enchaînant et parfois par surprise, et ce pourquoi aussi il y a des gestes en pensée et une mémoire des gestes). Nous ne sommes plus sous l'horizon *du seul* grand geste (de) Dieu.

Il me faudrait continuer et montrer aussi que les diagrammes des « tourner autour » ne sont pas le fin mot de l'histoire, et remonter alors encore vers plus d'événementialité. Ainsi on pourrait emboîter le geste d'un autre régime diagrammatique, impliqué en physique par l'électrodynamique des champs qui, avec les *diagrammes* de Feynman, montre qu'une particule virtuelle peut mettre en connexion, en *croisement,*[56] deux moments d'une particule. La particule est dite virtuelle parce qu'elle est événementielle (événement qu'est la mise en connexion) et qu'elle suit un chemin dans le virtuel. Mais cette connexion peut être aussi celle entre deux points de vue affectif. On pourrait là encore suivre Proust et ses fameuses mises en résonance affective entre deux moments d'une vie. Ou alors pensons à ce que les psychanalystes appellent l'après-coup. J'en donne seulement une indication. Jean Laplanche dans son livre *Vie et mort en psychanalyse* cite Freud qui évoque un traumatisme lié à ce qu'il appelle un attentat sexuel (ici l'attouchement commis par un épicier sur une petite fille qu'il désigne du nom d'Emma), voilà ce qu'écrit Freud : « un souvenir est refoulé qui ne s'est transformé qu'*après coup* en traumatisme ». Jean Laplanche commente en faisant justement usage d'une comparaison avec la physique quantique : « C'est là l'essentiel du raisonnement : nous cherchons à pister le traumatisme, or le souvenir traumatisant ne l'a été que secondairement : nous n'arrivons pas à repérer historiquement l'événement traumatisant. On pourrait illustrer ce fait par l'image d'une “relation d'indétermination” de Heisenberg : si on veut localiser le trauma, on ne peut plus apprécier son impact traumatisant, et vice versa ».[57] Nous retrouverions bien en tout cas l'exploration d'un temps non chronologique, des retentissements du virtuel et donc aussi un nouveau palier diagrammatique de la pensée de la subjectivité.

Rétrospectivement, on conviendra que le problème de la subjectivité, pensé diagrammatiquement et gestuellement, n'est plus « Qui suis-je ? » mais plutôt

[56] La place manque ici, mais il aurait fallu montrer que le nouveau centre d'indifférence est le cœur du croisement associé au geste de se repasser soi-même dessus dessous (d'où l'intérêt de la théorie mathématique des nœuds). Entre le tourner autour de soi et ce dernier geste, s'intercalerait le geste de « se mettre en vrille ». Là encore, je ne peux m'en remettre qu'à renvoyer le lecteur intéressé à mon livre *L'immeuble du mobile. Une philosophie de la nature avec Châtelet et Deleuze*.

[57] Jean Laplanche, *Vie et mort en psychanalyse*, Paris, Flammarion, 1970, p. 68.

« Comment je me porte ? », « Qu'est-ce qui m'emporte ? »[58] ou « Qu'est-ce qui m'arrive ? ». J'ai voulu montrer que penser ainsi, amène à défendre que la permanence du sujet n'est pas celle de la substance mais du virtuel, et qu'est sujet celui qui se porte par les gestes d'affection de soi par soi tout en étant subjectivité par les diagrammes, au fond toujours affectifs, des gestes qu'il habite et qui l'habitent (en quoi nous sommes bien des *manières* d'être comme le laisse entendre Spinoza pour les modes) et par ceux qu'il fait arriver (conception événementielle de la subjectivité, dans la filiation de Bergson). Ce n'est qu'une perspective, je n'irais pas jusqu'à dire que c'est la seule perspective qui traverse et fonde la subjectivité (pensons à celles qui prennent racine dans les représentations sociales, le langage, le logos, la conscience). Disons qu'elle permet au moins d'éclairer certains aspects qu'on ne pourrait pas apercevoir sans elle.

[58] On rappellera qu'étymologiquement « geste » vient du latin *gero* qui signifie « porter ».

Some Prolegomena for a Contemporary "Critique of Imagination"

Filipe Varela

Abstract From some great protagonists of the philosophical tradition to some contemporary tentative approaches in cognitive sciences and neurosciences, I present some grounds to further understand what imagination is, what it does, how it possibly works and where it may be located in our nervous systems. Such an approach follows the imperative to correlate the precious insights coming from Philosophy, with those of the sciences that nowadays study our perceptual and cognitive apparatus from a mainly physiological and bio-chemical perspective. I believe that such correlation, and even cooperation, is vital to a comprehensive understanding of this elusive thing, imagination.

Keywords Imagination · Leibniz · Hume · Kant · Husserl · Neuroscience

Six Guiding Theses

My aim is to address some prolegomena, or introductory remarks, for a "Critique of Imagination", by identifying the characteristic features which render imagination autonomic as an object of study. Ideally, I should begin by stating what I believe to be both its elements and its logic. But there are some remarks that I would like to make beforehand and this paper will only deal with them.

I am convinced that imagination is our most important cognitive resource. This statement amounts to a thesis—let's call it α—and, moreover, not an obvious one. So, it has to be grounded. Also, such thesis comes together with another one, β, supporting it, and which is perhaps even more challenging: that imagination is not a "faculty", i.e., it is not one of the several mind processing units, corresponding to a more or less defined area of our brains, or, in post-Fodorian terms, a "module". Instead, let's say γ, and using a metaphor, that imagination is the energetic field produced by each of the "elements" of our representational activities which attracts

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and binds them together both in actual and virtual ways. And therefore, if we take our brain as a representational system which has evolved in view of an optimal interaction with its environment and where all mental activity has a seat, then, δ, imagination is the very activity of articulating representations and pieces of representations in our representational organ, in our brains.

We see how the study of the elements and the logic of imagination could fit the picture. It would identify and describe the elements of representations in our brains, and then would identify and describe the relationships established between those representations. This obviously resembles the general approach to a computational modeling of our representative structures, as pursued by neurosciences, cognitive sciences or artificial intelligence theories. Those approaches already exist and are being theoretically developed and empirically tested by several researchers as I write.

The interesting thing is that, if we stick to δ—and, as I will try to show, this definition is not new in the philosophical thought—then, imagination should immediately stand out has the general "logic" of our representing ability, in both its actual and virtual instances. And if I was so bold as to assimilate brain and mind, we could even play with the hypothesis of imagination being the very "logic of the mind", to use an idea of Gottfried Leibniz regarding, precisely, imagination. Such role of imagination, even if not going that far, looks something big, something deserving one's attention. So, if one is to make use of Gaston Bachelard's advice that the philosopher's apprentice should, instead of following the metaphysician into his chamber, feel more tempted to follow a scientist that enters his laboratory, then I should learn what neurosciences and cognitive sciences have to say about imagination.

Here the plot thickens. There are a few main subjects shared by neurosciences and cognitive sciences: representations, memory, neural networks, belief production, creativity, problem solving, and so on. I was surprised to realize that imagination is not among them. In my view, and in a bit oversimplified terms, the main reason for that is ε: these disciplines suffer from too broad conceptions, misconceptions, and even lack of conceptions, about imagination. First, imagination is usually equated with higher functions of our cognitive behavior, a place where the current state-of-the-art of these disciplines does not allow them to enter yet. Also, we make very general indistinctions and confusions about imagination; conversely, we still don't have a proper systematic theory about imagination, and mainly about what imagination truly is and does; and finally, it seems we haven't stressed enough and accounted for, in our contemporary epistemological theories, the importance that some of the greatest philosophers awarded to imagination. Another main reason, quite obvious, is that the conception of imagination, as I presented it, seems way too vague and general to be epistemologically useful and relevant in those disciplines. What has been stressed, and still is to a large extent, is imagination's obscurity, its stubborn capacity to bypass our theoretical efforts. My claim is ζ: the ground for the notion of imagination can be precisely located in our nervous system, at the very heart of our representative activity, through what neurosciences already know about our brain.

So, in order to articulate connections between a general, common sense notion of imagination, the philosophers' thoughts on it, and the empirical discoveries and theoretical modelings of our cognitive processes, I decided to essay an answer for the most elementary question in the scope of this research project: what is imagination? And I will address it through the theses α and δ: that imagination is our most important cognitive resource, and that it is the very activity of articulating representations and pieces of representations.

Imagination: Common Sense Notions

The Cambridge Dictionary, for example, offers three meanings for the entry imagination. The first, as a verb (*to imagine*) "the ability to form pictures in the mind". This one clearly refers to imagination as a representational resource. That's what would happen if I would invite you to picture a cat, even though there's no cat around. But it also refers to the desired outcome of this paper: that you, who are reading, "picture" a mental unified representation of the ideas, conceptions, assumptions, etc., that I'm bringing to your attention in a more or less coherent way. Of course that such "picture" would not so much be a visual one, but of another kind. The second meaning, as a noun (*an imagination*), accounts for "something that you think exists or is true, although in fact is not real or true". Bluntly put, it seems to say: what deceives you. However, if broken by its disjunctives, it has two senses: "something that you think that exists although in fact is not real" and "something that you think is true, although in fact is not true". An example of the first sense may be: "I believe that mathematical entities exist, although I cannot locate them empirically". An example of the second sense may be: "I believe that a monster inhabits Loch Ness' waters". So, what it tells us is that "imaginations" are representations that only exist in our minds, and can either be trustworthy or deceiving. The third meaning, also as a verb (*to imagine*), is the most widely acknowledged and praised about imagination: "the ability to think of new ideas". This one is self-explanatory, and usually equated with creativity. In a word, and using Liane Gabora's idea about creativity, imagination is also the ability to take out of your brain something that was never put there before.

Considering these meanings and considering that although referring to different things they are all called "imagination", then there must be a connection, or resemblance, between them, because we tend to group things perceived as related in congruent intelligibility devices, like concepts or categories.

These interesting features and others falling under the general notion of *imagination* are also implicit in some of the words with which different languages refer to it, and some of them were particularly striking to philosophers. In Portuguese, Spanish, English or French, the words *imagination* and its idiomatic variants derive etymologically from the Latin *imaginatio*, which, as a verb, refers to the ability of producing images, to picture oneself, to represent even in the sensory absence of what is being represented; and as a noun, to the outcome of the acts of imagination, either real or unreal. These notions were developed, for example, by Jean-Paul Sartre, in the realm of a consciousness that deals with unreal objects. The English term *fancy* accounts for the production of something unreal and, moreover, extravagant, far-fetched. Hume meditated about it. The Greek term *εἰκασία* referred to the ability to know through images. Plato famously criticized it, at least in the *Republic*, because true knowledge can only come from apprehending ideas themselves, not images of those ideas. *Φαντασία* is another Greek word still present in western languages, this time emphasizing the unreal nature of some of the mind's productions, *φάντασματα*, a use that Husserl, ex., deepened. John Llewellyn made the insightful remark that the German term Kant used the most for imagination, *Einbildungskraft*, means, analytically, the power of bringing a manifold into a unity, *in-eins-Bildung-kraft*, which uncovers synthetical features of imagination that Kant appraised, and which are perhaps the most important features of imagination.

Imagination: Philosophic Insights from Leibniz, Hume, Kant and Husserl

We're starting to see imagination has an ability to represent, both real and unreal objects, some of them quite useful, others quite strange and even unreliable, but also has an ability to know through images, or representations, that can't be found in our empirical reality. Moreover, we see it as the ability to synthesize and associate different representations in order to take something new out of them. That this doesn't amount to a sheer psychological, viz. *subjective*, working of human mind was extensively argued by Husserl, namely through the argument that most of these imagined productions, or idealities, specifically mathematical entities, became reified in language, then in culture, and were intersubjectively agreed on by a community. Through their undeniable evidence, they became objective products of culture.

Put in these terms, imagination seems very relevant to epistemology, and even more particularly to philosophy of mathematics. It is not accidental that some of the most philosophically stimulant discussions about imagination sprung from philosophers trying to understand the nature of mathematical objects and mathematical (and also logical) reasoning.

If it was possible to state imagination as a broad argument for philosophical reflection, I think it would have this form:

1. We perceive and know reality through representations, because, as Aristotle puts it in *On the Soul*, it is *not the stone* that is in the thought, but *an image* of it.
2. Those representations, whatever they might be, are therefore necessary for perception, thought and knowledge to be possible, because they are the interface between phenomena and the knowing subject, and because it is through them that such subject rises to more complex representations.

3. Those representations, or pictures, or images, are a product of a human ability, which is usually referred to as imagination.

∴ Therefore, imagination is a fundamental requirement for the production of knowledge.

I believe that it was a reasoning of this kind that led philosophers to find a particular interest in imagination. However, like Plato or Malebranche, some philosophers focused primarily on the dangerous or deceptive features of our ability to produce representations. Others, like Fichte or Schopenhauer, not only focused on the epistemic virtues of imagination, but eventually considered them the ultimate ground of true knowledge, and for quite different reasons. In the middle, somewhat elevated, of this spectrum, fall those philosophers who weighted imagination as a free, unconstrained source of representations, providing for instance, artists with their most important tool, but also as a constrained, rule-dependent source of representations, which provided for instance scientists with the ability to go beyond the raw data they observed, and enabled them to articulate what they knew in order to achieve what they didn't know. Einstein's now famous declaration "imagination is more important than knowledge" means precisely that you may be very knowledgeable and educated, but if you don't exercise your imagination, you won't get anything new out of all that knowledge. Moreover, the philosophers sharing this view were led to believe that a deep structural resemblance united, for instance, the imagination of artists and scientists, in spite of the flagrant differences between their practices. However, these philosophers were also the ones who met more difficulties and hesitations regarding this subject, and eventually faced more dead-ends and aporias. Suspending the different metaphysical agendas, those would be the cases of Aristotle, Descartes, Leibniz, Hume, Kant, Bachelard, Bergson, Husserl, Wittgenstein, Piaget, Sartre, Bataille, in a sense also Peirce, and in due justice, of Heidegger as a Kantian interpreter. I cannot obviously give a synopsis and certainly not give a reasonable account of the many hypothesis, conclusions and aporias that these thinkers worked out, in sharply different philosophical and methodological frameworks. Moreover, much of the discussion comes from parallel themes, like the methods of *aphairesis*, or abstraction, the problem of the *imaginary*, imagination's view as a faculty and its rank in the hierarchy of faculties, the idea of *Gedankenexperiment*, aesthetic experience, among others. So, I will briefly evoke some ideas of four of them, which, I believe, deserve particular attention. They are Leibniz, Hume, Kant and Husserl.

The power of bringing manifold to a unity, a version of the old problem of particulars' relation with universals and of *genus* and *species*, calls for a capacity to discriminate similitude, analogy or affinity, performed in the interface between mind and phenomenon. This feature was attributed to imagination by Kant, and before him Hume, and after them Husserl. Leibniz, preceded them all and, as usual, paved the way for the most far-reaching insights, especially considering, in the words of Jane Kneller, the «psychological misapprehensions of phobic proportions» against imagination in that period.

Leibniz' view of imagination, following the reconstruction made by Dennis Sepper, accounts for the necessity of the connection between each element in his monadologic universe; he needed to infer a logic of relations between monads that could be expanded to the foundations of another of his groundbreaking projects, the alphabet of thought, the *characteristica universalis*, and its combinatorial possibilities. The understanding of such a structure of relations was, to Leibniz, a task of imagination. Imagination is conceived as a fundamentally associative set of rules, and therefore as a fundamentally logical faculty, not a mere random association of representations. And considering that this would provide human knowledge with its highest achievement, imagination was the most important epistemological faculty of all. This is perhaps the first time that imagination was placed at the height of human's capacities.

David Hume kept the bar high. In Jonathan Cottrell's reading, Hume seems to, at some point, have considered Reason as a part of the broader power which is imagination, as a sub-faculty of it responsible for "probable or demonstrative reasoning", being the whole of *Imagination*, namely memory, will, whims, prejudices, much ampler: the whole of reason. So, in a sense, the whole of cognition is seen as a general, polyform operation of imagination, thanks to what he considered to be his most glorious discovery, the principle of association of ideas and its three laws, performed by imagination, transversal to all perceptual and mental activity. Such conception of imagination was manifestly too comprehensive, and that is why Hume is considered an *eliminativist* of mental faculties, and of imagination in particular: all faculties are but different manifestations of imagination.

Kant, in his turn, inherited some of Leibniz and Hume's conceptions, quite hesitantly and at a very late stage of the first *Critique*'s meditations. Imagination came forward in his system of human reason as a dynamic representational power, but also as an associative one, a synthetical one, a kind of energy fueling the cognitive interests of the other faculties, especially understanding, reason and judgement, open to different forms of *play* in order to produce special intelligibility devices, namely *schemes*, *examples* and *symbols*. Yet, more importantly, Kant considered imagination a transcendental power, i.e., the performer of an activity without which the acts of knowing could not take place, because it was the common root to the two stems of knowledge, sensibility, our capacity to be affected by phenomena, and understanding, our capacity to think them. In that interface, imagination performed an astonishing set of synthesis required by basically every other faculty, like apperception, sensibility, understanding, faculty of judgment, reason, faculties of taste, of volition, and so on. The uncovering of so many transcendental links led Kant to believe that imagination was a mystery well-kept in the rock-bottom pit of our soul. Basically, it seemed to be everywhere, but nowhere clearly visible. In *CPR*'s *Appendix to the Transcendental Dialectic*, in general evoking a deeply Leibnizian flavor, Kant allows Hume's hypothesis and states that all our cognitive forces work under such an agreement that it's like they converged in their apparent diversity to a single unitary power—the fundamental comparative forces of the spirit—, and, Kant admits, it wouldn't surprise him that such power was in fact imagination in disguise. He embraces Hume's principle of imagination, association, and further refined it through analogy and affinity. However, Kant never considered imagination the most important faculty, not even one of the higher faculties. Never wrote a "Critique of Imagination". Heidegger famously accused him of cowardice whenever facing imagination, because Kant refused to face and acknowledge the obvious, that is, that imagination is the root of the two fundamental stems of knowledge, sensibility and understanding, and therefore the true transcendental source of knowledge.

When Husserl took imagination, or, in his preferred term, *Phantasie*, as a subject of meditation, he surely kept an intense and fertile discussion with Aristotle, but also with these three previous philosophers. Like Leibniz, Husserl wrote *Formal and Transcendental Logic* over the background of a relational ontology where the goal was to apprehend a system of contact between the mind and phenomena where expression and meaning were one. Like Hume, he felt that imagination was the source of every possible cognitive connection. Like Kant, he saw imagination as a far-reaching synthetical energy. However, Husserl called our attention to imagination's role in our *virtual* knowledge, in the horizon of open, potential possibilities that always come attached to our experiences and the *essences* or *eidos* we extract from it. Not only does imagination forges these unreal objects, essences, but it also performs an eidetic variation of essences that enhance its intelligibility and its given, actual, content of sense. So, considering this particular eidetic variation, and considering that imagination for Husserl belongs to the realm of pure conscience and its intrinsic intentionality, it is conceived as the ability to *a priori* represent, even if unconsciously, a horizon of potential virtuality of the given phenomena that always comes together with the actual given. That is, Husserl's emphasis is on the exploratory, eminently "hallucinatory" character of imagination, its vital role in engendering new knowledge, that may well be "unreal", yet apodictic. That was particularly visible, for Husserl, in mathematical practice and in mathematical objects, where imagination must play a decisive role.

Imagination: Hints from Neuroscience

From these four briefly outlined conceptions, we get some probable vertebrae of imagination's spine. They were also at the core of my suggestion, presented elsewhere, that imagination is our ability to perceive a certain phenomenon as a horse; but also to perceive a horse in a zebra; to perceive from a horse a unicorn, a Pegasus or a centaur; and finally to perceive a certain stable configuration of stars as reminding a centaur, and agree to call it that, even though none of those who sees the constellation has ever seen a centaur in her empirical reality. I like this image because it feeds my romantic idea that reality naturally demands us to imagine, like saying "I'm too vast and rich for your mind, so make me fit in". But her terms seem too demanding and unfair, like, "I'll give you an alphabet with a thousand letters, each with a different meaning, and I want you to, in, say, a period of 70 years, write the best description of what you perceive in and around you". Also adding:

"I also dare you to say why I choose those letters and not others". Mankind has, quite successfully, embraced this challenge. Not only has been trying to write and to describe what comes to her, but even creates associations with those letters that, at the same time, economize its use, but also enlarge what reality offers and what we can directly perceive in it. I will try to make this romantic *flânerie* more tangible.

As I claimed earlier, imagination is not a main theme of neurosciences. However, some established empirical findings and theoretical models of our nervous systems do, I believe, provide us with the elements for an interpretation of what imagination, physiologically, might be.

It begins with neurons and the way they relate to each other so as to represent. Neurons exist in our brains in an estimated number of 100 billion, and consist of a body, or *soma*, an axon, and the dendritic ramifications which link it with other neurons. Dendrites are bridges between neurons, established through electrical and biochemical signals. These bridges are connected whenever a neuron, in an excitatory state, polarize or depolarize, i.e., calls for the excitation of other neurons to collaborate in the coding of a particular range of microfeatures. It is largely accepted that our brains decompose stimuli in parts, being each part assigned to a particularly specialized set of neurons. These work together within their groups in higher degree and with other groups simultaneously. That is, synaptic exchanges take place. Synapses are the expression of the team-work of these players, for our representational capacities do not rely on the activity of isolated neurons, but on the collaboration between many neurons.

This collaboration is established and kept through the dendritic outgrows of neurons, whenever they are presented with stimuli. Such connections between neurons, and networks of neurons, are believed to be excited or inhibited depending on the resemblance of the features of the stimulus that each neuron is charged with. These workings and connections, and their precise nature and triggering mechanisms, are one of the difficult and exciting areas of research in neuroscience, dealing with several types of chemical neurotransmitters, ion channels and other fine-grained subjects that do not fit here. The bottom line is: neurons interact apparently due to the resemblance of the features of the stimuli that they code.

Each neuron seems to be sensitive to different, particular ranges of the microfeatures of stimuli (*coarse coding*) which means that when we're presented with a stimulus, a network of neurons is activated to process several different microfeatures of it, and the pattern of the connected activation of those neurons is what properly puts forward a representation. The connections build during this representation, a synaptic exchange that ties together a constellation of cells in a representation, establishes a path between neurons and groups of neurons which becomes a memory. Or, reversely, a memory is the representation provided by the concerted activation of groups of neurons. So, different memories may share the same neurons, because neurons have a property called *neural reentrance*. They are reused over and over again in different representations that nevertheless share features, which is an ingenious way of a finite organ to deal with a manifold of possibly infinite stimuli and representations, and to be able to work even if parts of it are damaged. Yet, since each neuron is tuned to respond maximally to specific ranges of microfeatures, neurons are grouped in assemblies which share a maximal sensitivity to certain features. This specialization of certain assemblies is what grounds the division of our cortex in broad, more or less specialized regions. This also grounds the "faculty psychology" and its evolution to module theory, a retake on faculties. In a word, the coding process of neurons is distributed, but not randomly: it is distributed, but it is *content-addressable*, i.e., specific microfeatures are dealt with by more or less specialized neurons to those microfeatures. Since these specialized neurons are preferably triggered by certain microfeatures, different stimuli may trigger the same neurons. That is, different stimuli may lead to the activation of overlapping neurons, and that means that those stimuli, although different, share common microfeatures.

Consequently, we see that groups of neurons in certain regions of our brain are excited or inhibited thanks to the affinity of the microfeatures they code, and the affinity between distinct stimuli is perhaps a result of an overlapping activation of the neurons that code those features. When two stimuli trigger largely overlapping assemblies of neurons, then it is highly probable that those stimuli are related or similar. If there wasn't this hardwired ability of neurons and neuron assemblies to overlap, there was no way of establishing similitude or resemblances between stimuli. That is to say that our brains are naturally interested in finding resemblances between stimuli, and that is empirically detected by seeing the activation of roughly the same neuron assemblies when presented with similar or resembling stimuli.

Now, when a certain stimulus is coded in a memory, that is, in a pattern of neural activation, we also know that such pattern, or coding, doesn't become written in the stone. Neuron patterns, or memories, are extremely flexible, which means that when a memory is retrieved, it is not so much reproduced, but reconstructed. This neural feature seems to be at the heart of our ability to introduce new data in our stored memories, not only to enrich them, but also to relate them with newly formed memories. To put it simply, our memories have an "actual" character—this specific neural coding by these specific neuron assemblies—and a "virtual" or possible character—other relationships that this pattern may establish with other neurons. That virtual character is what opens the field for new relationships between neurons.

This ability of our neurons to establish new connections and new patterns from stored ones is also something deeply intrinsic to our brain behavior. J.-P. Changeux calls it "spontaneity": our brains spontaneously play with stimuli coded in memories, and that is particularly visible, throughout human life, in dreams. There's an inner motivation of our brains to play with what it codes, precisely in the realm of possible connections between neurons that weren't invoked in the actual stimulus interpreted. Liane Gabora dwell on these particular neurons, that are called to "simulate" other connections between stimuli and called them *neurds*. This top-down influence means that our brains naturally enjoy the possibility of rearranging its data in connections different from those that were experienced in a stimulus.

We start to see in this behavior something that matches my most general notion of imagination.

If, considering our current knowledge of our brain, neurons are our basic physiological representational "devices", then representations are conjoined efforts of neurons. And since we represent different stimuli with the same neurons, the difference between representations is actually a function of different collaborations between neurons, or of different pattern activations. That is, the specificity of representations lies in the relationships that neurons and neuron assemblies establish between themselves. The specific relationships that largely overlapping regions of neurons establish with one another is what makes the specificity of each representation. Moreover, these representations are opened to other possible interactions between the neuron assemblies that coded them, which means that a particular representation is like a *schematic* access, in Kant's fundamental term, to other possible real or unreal representations, to take Husserl's notion.

It is my belief that neurons are what I'll call *imaginators*, the elements or agents of representations; and that the specific activity of relating and activating themselves in representations, in both actual and possible ways, is indeed what I believe imagination to be. Imagination is the realm of both actual and possible relationships that neurons establish between themselves when coding stimuli in order to build a representation. Moreover, it is also the spontaneous play of neurons and neuron assemblies between themselves.

That this actual and virtual character is intrinsic to our brain representing behavior is not only visible in its spontaneous activity—and there are many examples, like dreaming, mind-wandering, hallucinating, perceptive errors, completing incomplete data, among others—but also in the reconstructive character of memories when recalled, and in the very possibilities which come up with every representation, that is, the possibility of this representation to be related with other representations with which it shares common features.

Imagination is the interval of possible variations of activations. I didn't even dare to calculate a figure for the sum of those different activations in a base of 100 billion neurons. These variations occur in all parts of the brain, and possibly not only in the brain; they're not specific of a region. So, it is not a faculty, nor a higher module. It is everywhere. Basically, imagination is another universe inside our brain. But, much like the universe, it has patterns, constants, regularities. So, I believe, in the spirit of John Sallis, that imagination is the binding force under the big picture of our inner cephalic universe.

This conception, in my view not only naturalizes imagination. It places it at the ground of our neuronal processes, taking neurons as its elements and the set of their relationships as its logic. Suspending the many critics, objections, warnings that can be made to this sketchy, tentative, proposal, I think it is a good starting point for an investigation about imagination. Especially because it agrees with the insightful conceptions of those previous great minds. Accords with Leibniz, as the ground of a structural, all-governing logic; with Hume, as the idea that imagination is the whole of Reason: it undergrounds all our cognition, in what it has of structurally unifying; with Kant, as the idea that imagination is an *a priori* source of knowledge, hardwired in the brain: without these connections the mind cannot represent; and finally, with Husserl, as the horizon of all variations that a representation can possibly, or virtually, undergo.

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Le langage diagrammatique au-delà de la différence phénoménologique

Fabien Ferri

Abstract En 1992, dans *Le contrôle dans les systèmes à base de connaissances*, Bruno Bachimont fonde tout son programme de recherche dans le cadre d'une pratique de l'intelligence artificielle qu'il nomme « artéfacture », sur un principe qu'il nomme « différence phénoménologique ». Ce principe renvoie selon lui à deux types de connaissances, irréductibles les unes aux autres, bien qu'on puisse les articuler dans le cadre d'une ingénierie : d'une part les connaissances scientifiques, exprimées dans des langages formels, où « dire, c'est calculer » (et mesurer pouvons-nous ajouter) ; d'autre part les connaissances phénoménologiques, exprimées dans les langues naturelles, où « dire, c'est signifier ». Cette contribution interroge ce principe, pour savoir sous quelles conditions il peut être dépassé, non pour récuser le programme de recherche qui en a découlé, mais pour le poursuivre, à travers un nouveau programme de recherche que nous nommons, à la suite de W. J. T Mitchell et Frederik Stjernfelt, la « diagrammatologie ».

Keywords Connaissance phénoménologique · Connaissance scientifique · Diagrammes · Diagrammatologie · Différence phénoménologique

Introduction

Le fil conducteur de cet article est le problème de la connaissance, tel qu'il a été reposé de façon philosophique, épistémologique et technologique au début des années 1990 par Bruno Bachimont, dans le cadre d'une pratique technologique de l'intelligence artificielle : celle de la construction de systèmes à base de connaissances devant permettre l'instrumentation de la résolution de problèmes

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dont on ne connaît pas de solution algorithmique.[1] Ces systèmes font intervenir des connaissances de nature phénoménologique, *a priori* irréductibles aux connaissances scientifiques, car ces dernières sont de nature calculatoire et métrologique, c'est-à-dire mathématico-expérimentale. Cette différence entre la connaissance phénoménologique et la connaissance scientifique, Bruno Bachimont l'a nommée « différence phénoménologique[2] ».

La différence phénoménologique renvoie selon lui à deux types de connaissances, irréductibles les unes aux autres, bien qu'on puisse les articuler dans le cadre d'une ingénierie : d'une part les connaissances scientifiques, exprimées dans des langages formels, où « dire, c'est calculer » (et mesurer peut-on ajouter) ; d'autre part les connaissances phénoménologiques, exprimées dans les langues naturelles, où « dire, c'est signifier ».

Cet article programmatique discute le bien-fondé de cette différence, pour savoir sous quelles conditions elle peut être dépassée, non pour récuser le programme de recherche qui en a découlé,[3] mais pour le prolonger en le radicalisant, à travers un nouveau programme philosophique, épistémologique et technologique de recherche que nous nommons, à la suite de W. J. T. Mitchell et Frederik Stjernfelt, « diagrammatologie[4] ». La diagrammatologie désigne selon nous une phénoménologie de la connaissance opératoire. Elle doit être en mesure de nous donner des contenus intuitifs nouveaux, en l'occurrence des contenus opératoires exprimés graphiquement. Cette phénoménologie de la connaissance, que Charles Sanders Peirce appelait *phanéroscopie*,[5] s'exerce dans un espace bien particulier, celui de la surface d'inscription, qui peut tout aussi bien être une feuille de papier qu'un écran d'ordinateur.[6]

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[1] B. Bachimont, *Le contrôle dans les systèmes à base de connaissances. Contribution à l'épistémologie de l'intelligence artificielle* (1992), 2e éd., Paris, Hermes Science, 1994, p. 3.

[2] *Ibid.*

[3] Programme de recherche qui a été nommé par B. Bachimont « Artéfacture ». Voir : B. Bachimont, « L'artéfacture entre herméneutique de l'objectivité et herméneutique de l'intersubjectivité: un projet pour l'intelligence artificielle », *in* J.-M. Salanskis, F. Rastier et R. Sheps (éd.), *Herméneutique: textes, sciences*, Paris, Presses universitaires de France, 1997, p. 301–330.

[4] Une des premières occurrences du terme *diagrammatology* semble remonter à l'un des pères du tournant iconique, W. J. T. Mitchell. Voir : W. J. T. Mitchell, « Diagrammatology », *Critical Inquiry*, Vol. 7, No. 3, The University of Chicago Press, 1981, p. 622–633. Le terme a été repris par Frederik Stjernfelt dans : *Diagrammatology. An Investigation on the Borderlines of Phenomenology, Ontology and Semiotics,* Springer, Netherlands, 2007.

[5] Pour une étude comparative synthétique des phénoménologies de la connaissance de C. S. Peirce et E. Husserl, voir : B. Bachimont, « L'artéfacture entre herméneutique de l'objectivité et herméneutique de l'intersubjectivité: un projet pour l'intelligence artificielle », article cité.

[6] Sur les graphes existentiels de Peirce comme outils d'aide à la compréhension sémiotique et logique des écrans interactifs actuels, voir: J. Vogel, *Sémiotique de l'information chez Charles S. Peirce*, thèse de doctorat en sémiologie, Montréal, Université du Québec à Montréal, 2014. URL: http://www.archipel.uqam.ca/6457

La première forme canonique de notre rapport au monde : la signifiance

Suivant la « différence phénoménologique », on peut se rapporter au monde de deux manières différentes par la médiation d'un langage. Or se rapporter au monde par la médiation d'un langage, c'est dire le monde. La première question sous-jacente à la différence phénoménologique est donc : comment peut-on dire le monde ? Selon le partage opéré par cette différence première, il y a deux formes canoniques au moyen desquelles le monde peut se dire : la signifiance et le calcul. La signifiance désigne de façon générique notre manière de nous rapporter au monde en usant de la langue naturelle : dans cette perspective, dire le monde, c'est le signifier. Le régime de la signifiance peut se décliner de nombreuses façons, au sens où il peut se référer à des systèmes de signes entièrement hétérogènes (langage, mythe, religion, science, etc.), ce qu'Ernst Cassirer a appelé des « formes symboliques[7] ». Le régime de la signifiance ne se réduit donc pas à l'usage du langage articulé.

Les deux catégories de systèmes de signes que nous retenons pour les besoins de notre argumentation sont les suivantes : la catégorie des systèmes de signes iconiques et celle des systèmes de signes linguistiques. Dans la première catégorie, l'unité élémentaire, c'est l'image ; dans la seconde, c'est le mot. Le caractère commun à ces deux types d'unités élémentaires, c'est qu'ils se rapportent au monde suivant la modalité de la signifiance. Mais dire le monde de façon non arbitraire, cela suppose que le monde se dise, c'est-à-dire s'exprime et se manifeste. Cette expression du monde induisant la modalité signifiante comme expression véridique de notre rapport au monde suppose donc un troisième terme faisant la jointure entre le monde qui s'exprime et le système de signes qui l'exprime : ce troisième terme, la tradition philosophique l'appelle *phénomène*.

Il y a en effet phénomène lorsque quelque chose du monde apparaît. Or une chose du monde qui apparaît, apparaît nécessairement à quelqu'un. Ce quelqu'un, c'est le *sujet phénoménologique*. Le sujet phénoménologique, en tant qu'il se rapporte au monde de façon signifiante par la médiation de phénomènes, passe par un médium, qui est le support des signes qu'il manipule et destine dans des actes de communication : parole vivante ou support matériel externe d'inscription, tel une tablette d'argile ou un parchemin. Lorsque l'arrangement d'un jeu de signes conduit à la reconnaissance d'une chose du monde phénoménologique par une pluralité de sujets, il y a production d'une connaissance phénoménologique. Cette connaissance permet d'accéder à un objet phénoménologique de sens commun, linguistiquement catégorisé (lorsque les signes sont des mots) ou sémiotiquement catégorisé (lorsque les signes sont par exemple des pictogrammes). Les systèmes

[7] E. Cassirer, *La philosophie des formes symboliques* (1923–1929), Paris, Minuit, 1972. 3 vol. E. Cassirer, *La philosophie des formes symboliques. 1, Le langage*, Paris, Minuit, 1972. E. Cassirer, *La philosophie des formes symboliques. 2, La pensée mythique*, Paris, Minuit, 1972. E. Cassirer, *La philosophie des formes symboliques. 3, La phénoménologie de la connaissance*, Paris, Minuit, 1972.

de signes linguistiques et iconiques nous permettent ainsi d'exprimer des connaissances générales phénoménologiques à travers une activité de catégorisation des objets de notre expérience phénoménologique qui prend la forme d'énoncés linguistiques signifiants ou d'enregistrements graphiques schématiques non linguistiques. Les deux objets culturels qui symbolisent par leur existence matérielle ces deux systèmes de signes en suivant une exigence de totalisation étant le dictionnaire et l'encyclopédie. En effet la vocation du dictionnaire, c'est d'enregistrer l'ensemble des ressources linguistiques de la langue, tandis que celle de l'encyclopédie est d'enregistrer l'ensemble des connaissances, en articulant système de représentations linguistiques et système de représentations non linguistiques, à travers des planches et des schémas par exemple, ou des maquettes numériques, depuis le développement de l'informatique et des interfaces numériques.

Autrement dit, lorsque le monde est exprimé à travers le prisme d'un système de représentations linguistiques ou d'un système de représentations non linguistiques, ou d'un système de représentations qui articule les premières aux secondes, nous avons affaire à la forme canonique de la signifiance. Cette forme canonique est productrice de connaissances phénoménologiques. En thématisant cette première forme de communication signifiante, on ouvre la possibilité d'une phénoménologie de la connaissance phénoménologique qu'on peut dès lors nommer *phénoménographie.*[8] La phénoménographie a ainsi pour objet la description des relations entre les expressions linguistiques et les expressions schématiques. C'est donc une phénoménologie du second ordre dont l'espace de présentation est la surface d'inscription, et l'espace combinatoire, l'espace des signes. Dans cet espace combinatoire, les signes deviennent ce qu'il faut nommer des *phénoménogrammes*. Dès lors on peut distinguer *phénoménogrammes linguistiques* et *phénoménogrammes schématiques* : les premiers enregistrent le vocabulaire technique permettant de décrire les phénoménogrammes schématiques ; les seconds permettent d'enregistrer les ressources cognitives visuelles du monde phénoménologique, ses saillances, prégnances et régularités perceptives, grâce à des pictogrammes plus ou moins standardisés.

Lorsque Reviel Netz a fait l'histoire cognitive de la mise en forme de la déduction dans les mathématiques grecques,[9] il a montré que c'est l'articulation de ces deux types de ressources cognitives qui a été au principe de la construction des diagrammes grâce auxquels ont été opérées les premières productions théorématiques, fondées sur le transfert de nécessité. Ce transfert, on peut le reformuler dans les termes de la philosophie de Gilbert Simondon et dire qu'il s'agit d'une

[8] Sur la justification de l'usage du terme « phénoménographie », voir: F. Ferri, « Matérialiser le schème et dynamiser le schéma : penser et agir par le diagramme », in F. Ferri, M. Crevoisier & C. Widmaier (dir.), *Philosophique 2020*, Hors-série, Besançon, Presses Universitaires de Franche-Comté, coll. « Annales Littéraires », n° 1009, 2020, p. 42, note 33.

[9] R. Netz, *The Shaping of Deduction in Greek Mathematics. A Study in Cognitive History*, Cambridge, Cambridge University Press, 1999.

transduction contrôlée sur des symboles,[10] car la transduction est une opération fondée sur la saisie de relations transitives, donc de relations d'équivalence.[11] Par transduction contrôlée, nous entendons l'actualisation analogique par un lecteur d'une opération capturée dans un système de signes iconiques qui en exhibe le schème de fonctionnement irréductible. L'opération transductive correspond à l'actualisation d'une essence opératoire saisissable par un sujet interprétant dans un acte de réinvention. Par exemple, c'est par transduction contrôlée que Claude Shannon a traduit concrètement les fonctions logiques de l'algèbre booléenne en portes logiques dans les circuits électroniques.

Le problème qui se pose est le suivant : qu'est-ce qui fait qu'un assemblage de signes linguistiques ou non linguistiques est l'expression d'une connaissance phénoménologique ? Comment la transcription phénoménographique d'un donné phénoménologique sous une forme idéogrammatique ou pictogrammatique conduit-elle à l'obtention d'un résultat scientifique pouvant être calculé théoriquement et mesuré expérimentalement ?

L'essence de ce problème est celui de la représentation, celui du lien qui unit un signe à ce qu'il signifie. C'est aussi celui de la manipulation de la suite de traces grâce auxquelles on passe d'un signifiant originel à son signifié en acte, tel qu'il se donne dans la plénitude d'une expérience événementielle indubitable et même cruciale.[12] Notre hypothèse de travail est que le dépassement de la différence phénoménologique doit passer par une théorie de la transduction contrôlée, c'est-à-dire par une théorie de la connaissance analogique validée par des critères permettant de distinguer l'usage heuristique de l'analogie de son usage rhétorique.[13] Cette théorie se fonde sur un seul postulat ontologique, consistant à donner une valeur d'être à la relation.[14] En donnant une valeur ontologique à la relation de façon explicite, on se donne comme postulats la réflexivité et l'analogicité du réel. Si la relation « a valeur d'être » comme l'écrit Simondon, alors la relation entre deux relations a elle-même valeur d'être, donc l'analogie a valeur d'être. Pour que l'intuition d'une telle relation, entendue comme saisie intelligible dans un acte de compréhension irréductible à un acte d'interprétation d'un énoncé linguistique

[10] G. Simondon, « L'amplification dans les processus d'information », *in* Jean-Yves Chateau (éd.), *Communication et information*, Chatou, La Transparence, 2010, p. 175, note 2.

[11] Voir : G. Simondon, « Allagmatique », in *L'individuation à la lumière des notions de forme et d'information*, Grenoble, Jérôme Millon, 2013, p. 529–536. Dès à présent abrégé en *ILFI* suivi des numéros de pages.

[12] Sur cette question, voir : S. Neuwirth & G. Wallet, « Enquête sur les modes d'existence des êtres mathématiques », *Philosophia Scientiae*, 2019/3, 23-3, p. 83–108.

[13] F. Ferri, « Technique, science, philosophie : les conditions d'exercice de l'analogie valide », *Philosophique 2017*, n° 20, Besançon, Presses Universitaires de Franche-Comté, coll. « Annales Littéraires », n° 969, 2017, p. 79–94.

[14] Ce postulat métaphysique est nommé « réalisme des relations » et prend sa source dans l'ontologie et l'épistémologie bachelardiennes. Sur cette question, voir : J.-H. Barthélémy, *Simondon ou l'encyclopédisme génétique*, Paris, Presses universitaires de France, 2008, Chap. 1 ; ou encore: J.-H. Barthélémy & V. Bontems, « Relativité et réalité. Nottale, Simondon et le réalisme des relations », *Revue de synthèse*, vol. 122, n° 1, janv.-mars 2001, p. 27–54.

signifiant, ait elle-même valeur d'être, il faut donc qu'elle soit elle-même une relation. L'intuition analogique est donc la relation qui rend possible la saisie de la relation entre deux relations.

Le « jugement par perception intuitive » dont parlait Goethe[15] désigne selon nous une telle saisie analogique réelle. Cette saisie correspond à l'analogie réelle entre le devenir du sujet connaissant et le devenir des objets connus qu'il met en relation dans le processus de recherche et d'apprentissage qu'il met en œuvre. Elle correspond à la saisie simultanée : (1) de l'*analogie opératoire* entre deux relations objectives ; (2) de l'opération analogique faite par le sujet connaissant au cours de la morphogenèse adaptative de son processus d'apprentissage s'opérant à même l'expérience qui l'informe et le transforme. Le jugement par perception intuitive dont parlait Goethe serait donc une analogie entre au moins trois opérations : deux opérations objectives et une opération subjective, dont un système d'inscriptions diagrammatique permet d'exhiber l'« équivalence transopératoire[16] ». Autrement dit, un système d'inscriptions présentant des rapports iconiques et dont la manipulation conduit à la mise en fonctionnement d'une machinerie sémiotique correspond selon nous à l'enregistrement de la connaissance opératoire saisie par intuition analogique. L'appropriation de l'inscription correspondant, en tant qu'acte d'interprétation opératoire, à la mise en œuvre du faire sens de cette inscription, à travers une action efficace située en contexte, autrement dit au moyen d'une opération pratique succédant à l'interprétation de cette inscription.

En couplant ces postulats ontologiques simondoniens aux principes de la sémiotique de Peirce, notre hypothèse est que le schématisme morphogénétique à l'œuvre dans le cadre d'une ontogenèse relationnelle (morphogenèse) équivaut au schématisme diagrammatique à l'œuvre dans le cas d'une genèse théorématique (accroissement de la connaissance apodictique). Si une telle hypothèse venait à être vérifiée, elle confirmerait et expliciterait l'intuition de Goethe selon laquelle il existe un « jugement par perception intuitive » (*Anschauende Urteilskraft*) saisissant la signification de totalités organiques au-delà du périmètre du calculable, exhibant ainsi des gestes qui ne possèdent pas de formalisation algorithmique (passage à la limite, récurrence transfinie, etc.).

L'enjeu théorique du programme de la diagrammatologie est donc celui du statut scientifique de la connaissance, dont la question directrice est celle de savoir dans quelle mesure science et connaissance peuvent s'unir dans un langage et une écriture qui leur sont communs. Son enjeu pratique est celui de l'apprentissage de savoir-faire au sens fort, à travers une activité de lecture et d'écriture diagrammatiques.

Revenons à la différence phénoménologique. Cette différence part du constat que les catégories de la langue et les phénomènes de la conscience[17] ne sont pas

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[15] J. W. von Goethe, « Anschauende Urteilskraft » (1820), dans *Die Schriften zur Naturwissenschaft*, Abt. 1, Bd. 9: Morphologische Hefte 1817–1824, Leopoldina Ausgabe, Böhlau, Weimar, 1954.

[16] G. Simondon, « Allagmatique », in *ILFI*, op. cit. p. 531.

[17] B. Bachimont, *Le contrôle dans les systèmes à base de connaissances*, *op. cit.*, p. 233.

réductibles à des mesures et à des lois mathématiques qui en fourniraient des modèles d'observation et d'explication sous une forme équationnelle ou algorithmique. Elle sépare donc deux types d'objectivité : l'« objectivité phénoménologique » et l'« objectivité scientifique[18] ». L'objectivité scientifique s'exprime dans les connaissances scientifiques : ces connaissances s'écrivent dans les langages formels de la science, qui permettent de légaliser et de quantifier les régularités du réel au moyen de la notation symbolique ; les phénomènes étant capturer expérimentalement par des opérations de mesure. C'est donc la métrologie comme technique et science de la mesure qui fait le lien entre le monde de la grandeur physique où se produisent les phénomènes naturels (étudiés par les sciences expérimentales) et les modèles théoriques élaborés pour expliquer ces phénomènes. L'objectivité phénoménologique s'exprime quant à elle dans les connaissances véhiculées à travers les systèmes sémiotiques, qui ne sont pas réductibles aux seules expressions linguistiques.

D'une certaine manière, on peut dire que l'histoire de la différence phénoménologique est l'histoire de l'épistémologie entre science et ontologie. C'est l'histoire du fondement de la science entre cumulativité de la connaissance positive et réflexivité de la connaissance philosophique. Autrement dit, cette histoire commence avec Aristote : elle passe par ces jalons incontournables que sont Descartes, Leibniz, Kant, Husserl. Poser la question du dépassement de la différence phénoménologique suppose d'avoir bien identifié au préalable les termes de la différence. Cette différence, c'est celle entre le catégorial et le calculatoire. Or le paradigme de la logique catégoriale nous est donné par la logique aristotélicienne, qui est la science de l'objectivité phénoménologique. Que fait-elle essentiellement ? Elle systématise les connaissances en catégories.[19] En revanche, le paradigme de la logique calculatoire nous est donné par la logique mathématique, dont on peut situer l'acte de naissance dans l'œuvre de George Boole et plus généralement dans le réseau des algébristes britanniques du 19e siècle, sur lequel on dispose maintenant de nombreuses connaissances historiques.[20]

De la signifiance au calcul : brève histoire de la différence phénoménologique

Pendant presque deux mille ans, on a tenu le système aristotélicien des formes valides de déduction pour complet en pensant qu'il n'était pas possible de l'améliorer. Le renouveau des études logiques à l'époque moderne est dû à Boole grâce au développement de l'algèbre de la logique. Comprendre comment on passe

[18] *Ibid.*, p. 5.

[19] B. Bachimont, *ibid.,* p. 249.

[20] Pour des études approfondies sur le réseau des algébristes britanniques du 19e siècle, voir les travaux de M.-J. Durand-Richard.

de la logique d'Aristote à la logique contemporaine, c'est comprendre comment on passe d'une logique empirique fondée sur la perception et l'usage du langage naturel (*i.e.* d'une logique qu'on peut qualifier de « catégoriale » dans la mesure où elle ordonne les contenus de la perception en catégories de la langue qui enregistrent et organisent ces contenus à travers la technique de l'écriture alphabétique) à une logique formelle qui n'est plus simplement une idéalisation de la syntaxe[21] ordonnant ces contenus de nature phénoménologique (à travers des règles logico-grammaticales), mais un langage algébrique réglé par des lois de combinaison nous obligeant à faire abstraction du sens des signes manipulés pour ne prêter attention qu'à leur forme symbolique et aux lois qui contraignent leur manipulation.

L'algébrisation de la logique est ce qui initie le développement de la logique dite symbolique, dont l'essence est d'être un calcul opérant sur des symboles généraux, à la différence de la logique catégoriale, dont les règles syntaxiques ne valent que pour des idiomes particuliers.[22] La différence entre logique catégoriale et logique algébrique est celle qui nous permettra de distinguer une logique régie par des règles variables (logique grammaticale) d'une logique régie par des lois uniformes (logique calculatoire). Tout l'enjeu de l'écriture diagrammatique est de savoir s'il est possible de réconcilier ces deux approches de la logique (logique ancienne vs logique contemporaine, logique catégoriale vs logique calculatoire, etc.) dans une nouvelle approche qu'on pourrait alors qualifier de « compositionnelle ».

La logique catégoriale fondée sur le donné phénoménologique peut-elle épouser la logique mathématique fondée sur la formalité, le calcul et la mesure ? Si oui, comment le peut-elle et sous quelles conditions ? Ce problème est celui de la coïncidence non arbitraire entre une objectivité perçue et une objectivité réelle, entre une objectivité phénoménologique et une objectivité physico-mathématique. Dans l'histoire de la différence, le moment cartésien est celui où l'opération du doute vient remettre en question le donné phénoménologique, objectivé et systématisé dans le canon de la logique aristotélicienne et explicité dans les contenus linguistiques des langues écrites.[23] La généralisation de ce doute, dans sa pratique hyperbolique, conduisant Descartes à l'appliquer non seulement aux contenus transmis par la tradition,[24] mais aussi à toute donnée immédiate de la conscience.[25]

Dans la mesure où la notion de symbole physique permet d'introduire une dimension spatiale dans le donné phénoménologique, une naturalisation du donné devient envisageable, si le donné correspond au résultat d'un calcul symbolique sur des représentations et si réciproquement ce calcul correspond à une physique matérielle des symboles, donc à une mécanique symbolique. C'est l'hypothèse

[21] Par idéalisation de la syntaxe, nous entendons la formalisation des règles logico-grammaticales de la langue telle qu'elle est prise comme objet d'étude logique.

[22] Idiome : instrument de communication linguistique utilisé par une communauté.

[23] La doxographie scolastique ayant été garante de la vérité préscientifique transmise à travers les siècles antérieurs à la révolution galiléo-cartésienne.

[24] Descartes, *Discours de la méthode*, 1637.

[25] Descartes, *Méditations métaphysiques*, 1641.

leibnizienne de la mathématique universelle, solidaire de la métaphysique monadologique et du système de l'harmonie préétablie. Dans cet horizon métaphysique, l'univers est une immense machine computationnelle dont Dieu est l'ingénieur et la logique événementielle est un calcul des prédicats monadologiques. *Mathesis universalis* veut alors dire : les mathématiques équivalent à l'ontologie.[26] Dire l'être suppose de maîtriser un langage universel qui est celui des mathématiques (*lingua characteristica universalis*). Autrement dit, c'est Leibniz qui permet d'arriver à la seconde forme canonique au moyen de laquelle ce n'est pas seulement le *monde*, mais l'*univers* qui se dit : cette forme canonique est le calcul. La caractéristique universelle de Leibniz est le système de notation rêvé permettant l'entre-expression de la pensée des choses et des choses pensées par la médiation des caractères expressifs, dans la proportion de leurs rapports internes essentiels.

Refusant l'harmonie préétablie de Leibniz, la philosophie kantienne est une recherche de la médiation entre la pensée et le réel suivant la méthode transcendantale.[27] Le problème devient le suivant : comment la vérité peut-elle se voir sans être immédiatement évidente et sans pour autant être médiatisée par la déduction logique ? Il faut passer par un type de déduction conduisant à l'intuition d'une vérité au terme d'une synthèse objectivante. Pour Kant, ce type d'inférence est celui que mettent en œuvre les mathématiciens dans leur pratique, car les mathématiques résolvent selon lui le problème d'opérer une synthèse universelle dans la contingence empirique, dans la mesure où elles portent sur les opérations de construction productrices des connaissances synthétiques *a priori*, c'est-à-dire universelles, nécessaires et donatrices d'objets unifiés pouvant être rencontrés dans l'expérience réelle. Autrement dit, sortir de la métaphysique, c'est distinguer le contenu linguistique d'une notion de son contenu scientifique, son contenu analytique (logique) de son contenu synthétisé (mathématique). Cela signifie qu'à partir de Kant, la logique définit le *pensable*, alors que la mathématique définit le *pensé*, c'est-à-dire le contenu réel de la science comme objet de synthèse répétable. Peut-on concilier l'inconciliable, c'est-à-dire les termes de la différence phénoménologique, sans commettre de paralogisme ?

C'est là qu'il faut dissiper une confusion et ne pas assimiler le calcul symbolique à une mécanisation de la pensée. Car le calcul est une opération symbolique effectuée par l'esprit qui interprète et non une procédure physique effectuée par la nature. En effet le calcul est mis en œuvre par une puissance interprétative qui manipule des signes porteurs de sens alors que le mécanisme opère sur des signes matériels dépourvus de significations.[28] Par conséquent, ce n'est pas la pensée qui est une opération de calcul, mais le calcul qui est une opération de la pensée. Tout l'enjeu étant de savoir quel supplément enrôle avec elle l'opération de pensée supra-calculatoire. C'est la question que posait Martin Heidegger au semestre d'hiver

[26] G. W. Leibniz, *Mathesis universalis. Écrits sur la mathématique universelle*, éd. D. Rabouin, Paris, Vrin, 2018.

[27] E. Kant, *Critique de la raison pure*, 1781.

[28] E. Husserl, *Articles sur la logique (1890–1913)*, Paris, Presses universitaires de France, 1995.

1951–1952,[29] peu de temps après la publication de l'article séminal d'Alan Turing, qui signait l'acte de naissance de l'intelligence artificielle.[30]

Pour que cette opération de la pensée soit congruente à une opération de la nature, il faut reposer le problème du schématisme en des termes nouveaux, pour être en mesure de mettre au jour « l'art caché des profondeurs de l'âme humaine[31] » dont parlait Kant dans la première *Critique*, et faire d'une opération de la pensée une opération de la nature dont la schématisation n'est pourtant pas « légalisable » par un algorithme. Or telle est la fonction des représentations diagrammatiques selon nous : schématiser de manière non algorithmique une finitude non calculable.

Pour échapper au paralogisme, il faut réussir à montrer comment les catégories de la pensée possèdent un contenu phénoménologique, correspondant à une expérience vécue par la conscience, c'est-à-dire un contenu intentionnel. Or un vécu est intentionnel lorsqu'il porte sur quelque chose, c'est-à-dire lorsque la conscience accède à un objet. C'est la tentative de la phénoménologie husserlienne, pour laquelle l'objet n'existe pas en dehors de l'acte qui le vise. Le but de la phénoménologie husserlienne a été de constituer l'objectivité réelle de la science à partir d'une phénoménologie de la conscience en fondant la science à partir de l'intentionnalité. Alors que Kant a réduit la Nature aux phénomènes en fondant dans les structures transcendantales du sujet la connaissance scientifique, Husserl a réduit le Monde à un phénomène visé suivant les modalités intentionnelles d'une subjectivité incarnée. Dans sa distinction entre essence phénoménologique obtenue par idéation et idées scientifiques obtenues par idéalisation, Husserl a thématisé la différence phénoménologique.[32] Les premières sont anexactes car déterminées linguistiquement par une opération de catégorisation ; les secondes sont exactes car légalisées mathématiquement. La différence phénoménologique est donc chez Husserl la différence entre le monde de la vie décrit par nos langues naturelles et le monde de la science décrit par le langage mathématique. En fondant la réduction phénoménologique comme méthode, Husserl a élaboré une objectivité de la conscience différente de l'objectivité kantienne en montrant que les phénomènes de la conscience ont une essence et non une nature. Mais la phénoménologie husserlienne a échoué, dans la mesure où elle n'a pas été capable de penser l'invention, c'est-à-dire la genèse des structures eidétiques, autrement dit l'historicité de la raison.[33] Malgré cet échec, elle a toutefois donné une stratégie

[29] E. Hörl, « La destinée cybernétique de l'occident. McCulloch, Heidegger et la fin de la philosophie », *Appareil*, [En ligne], 1 | 2008. URL: http://journals.openedition.org/appareil/132; DOI: 10.4000/appareil.132

[30] A. M. Turing, « Computing Machinery and Intelligence », *Mind*, Volume LIX, Issue 236, 1 October 1950, p. 433–460.

[31] Kant, *Critique de la raison pure*, Paris, GF Flammarion, 2001, p. 226.

[32] Husserl, *Idées directrices pour une phénoménologie pure et une philosophie phénoménologique*, Paris, Gallimard, 2018.

[33] R. Boirel, *Théorie générale de l'invention*, Paris, Presses universitaires de France, 1961, p. 60–61.

cognitive féconde pour aborder la différence phénoménologique ; Kant ayant fourni quant à lui une stratégie épistémologique.

En mettant en œuvre d'une part une stratégie transcendantale d'inspiration kantienne visant à produire une philosophie de l'informatique ; d'autre part une stratégie cognitive d'inspiration husserlienne visant à produire une phénoménologie de la connaissance, Bruno Bachimont a réussi à démontrer[34] (1) que l'informatique est une science de la nature qui a le calcul comme objet matériel (thèse 1) ; (2) que toute connaissance ne se constitue que par la médiation d'une inscription matérielle dont elle est l'interprétation (thèse 2). Parce que le calcul est un objet matériel, il représente un nouveau support d'inscription des connaissances dont l'opérationnalisation computationnelle dans les dispositifs informatiques autorise, au sens phénoménologique du terme, la constitution de nouvelles connaissances, ouvrant ainsi le projet de ce que Bruno Bachimont a nommé une « herméneutique matérielle ». Ce projet a trouvé son fondement théorique dans une théorie matérielle de la connaissance—la théorie du support—et sa réalisation concrète dans l'accomplissement d'une critique de la raison computationnelle.[35]

Dépasser la différence phénoménologique ne signifie pas la nier, car l'histoire de la différence montre qu'une telle négation fondée sur un optimisme épistémologique conduit à un paralogisme. Dépasser la différence ne pourra se faire qu'en trouvant un point de contact entre monde de la vie et monde de la science, langue naturelle et langue formelle artificielle scientifique, permettant de réarticuler les termes de la différence, plutôt que de les assimiler. Un tel point de contact existe : c'est la notion de symbole ou de forme symbolique, telle que nous en héritons de Cassirer et sur laquelle travaille actuellement Jean Lassègue.[36]

Dans la mesure où la science est aussi un langage, elle doit avoir un fonctionnement herméneutique. La communauté herméneutique des langages formels et des langues naturelles, c'est en effet qu'ils élaborent des réponses à des problèmes. Dire, c'est n'est donc pas seulement signifier et calculer, c'est aussi échanger comme on peut le faire dans une conversation : dire, c'est alors argumenter pour convaincre l'autre, au-delà de la différence phénoménologique, dans l'élément herméneutique de l'échange intersubjectif.

[34] B. Bachimont, *Herméneutique matérielle et Artéfacture : des machines qui pensent aux machines qui donnent à penser. Critique du formalisme en intelligence artificielle*, Thèse de doctorat en Épistémologie, Palaiseau, École Polytechnique, 1996.

[35] Un abrégé de la théorie du support se trouve dans le mémoire suivant : B. Bachimont, *Arts et sciences du numérique. Ingénierie des connaissances et critique de la raison computationnelle* [En ligne], mémoire d'HDR, Université de technologie de Compiègne, 2004, p. 61–117.

[36] J. Lassègue, *Cassirer, du transcendantal au sémiotique*, Paris, Vrin, 2016.

Conclusion

Nous pensons que la matérialité symbolique des diagrammes constitue un milieu apte à favoriser le développement d'une nouvelle activité herméneutique, mixte d'herméneutique formelle de l'objectivité[37] et d'herméneutique informelle de l'intersubjectivité. Nous nommons cette activité herméneutique sémiotique effectuée à même les diagrammes *herméneutiques opératoire*. Nous parlons d'herméneutique opératoire au sens où le faire est l'interprétation du signe (herméneutique) et où le signe est l'expression d'une opération (opératoire). L'enjeu de la phénoménographie telle que nous l'entendons, c'est donc la mise en forme graphique des éléments notationnels pour rendre possible un accès rapide et efficace aux connaissances à véhiculer ; l'enjeu de l'herméneutique opératoire étant l'appropriation analogique des schèmes opératoires dont sont porteurs les diagrammes.

Le principe du dépassement de la différence phénoménologique est fourni par Peirce : en effet, si tout interprétant sémiotique est de nature iconique[38] dans la mesure où sa structure formelle internalise la relation de correspondance entre un signe et son objet, alors le développement d'une écriture diagrammatique peut permettre d'articuler sur un même plan sémiotique le catégorial et le calculatoire avec comme critère de convergence le caractère iconique des langages formels et des langues naturelles, c'est-à-dire leur caractère analogique. Une première écriture diagrammatique nous a été donnée par les graphes existentiels de Peirce,[39] et plus récemment par les graphes conceptuels de Sowa.[40] Posé dans les termes de la pensée diagrammatique, le problème du schématisme tel que nous en héritons depuis Kant est selon nous un problème de design de l'information[41] visant à véhiculer un contenu de connaissance opératoire qui n'est pas réductible à une opération de calcul et dont l'écriture diagrammatique est le mode d'expression privilégié.

[37] Cf. J.-M. Salanskis, *L'Herméneutique formelle* (1991), Paris, Klincksieck, 2013.

[38] Peirce classifie les signes en trois grandes catégories : indice, icône, symbole. Au sein de la catégorie de l'icône, il distingue l'image et le diagramme. Le diagramme, sous-catégorie de l'icône, est une figure abstraite puisqu'il consiste à représenter la forme d'une chose, c'est-à-dire sa structure, au moyen d'une structure notationnelle médiate isomorphe à cette forme, qui entretient donc avec cette forme première une relation d'analogie structurale. C'est pourquoi Peirce définit le diagramme comme une « icône de relations intelligibles ». Cf. Peirce, *Collected Papers*, 4.531.

[39] D. D. Roberts, *The Existential Graphs of Charles S. Peirce*, The Hague, Mouton, 1973.

[40] J. F. Sowa, *Conceptual Structures*, Boston, Addison-Wesley, 1984.

[41] A.-L. Renon, *Design & Sciences*, Saint-Denis, Presses Universitaires de Vincennes, 2020.

Part VII
Diagrams: from Mathematics to Aesthetics

Ars diagrammaticae

De la mathématique à l'esthétique & retour

Charles Alunni

Abstract *Ars diagrammaticae* interroge la prégnance de l'art du diagramme et ses implications philosophiques à travers l'analyse d'exemples tirés essentiellement de la physique mathématique moderne et contemporaine. Nous interrogeons plus spécifiquement les notions de « technogramme », de court-circuit de la main et de la pensée, de « preuve », de métaphore, de « géométrie » (au sens de Pierre Cartier), de « proto-diagramme », proposant l'idée d'une philosophie symplectique qui ne peut se fonder que de la désintrication des concepts d' « image », de « figure » et de « diagramme ».

Keywords Image · Figure · Diagramme · Technogramme · Métaphore · Preuve · Philosophie symplectique

> **Pour un mathématicien, comprendre une démonstration ce n'est pas *refaire* une à une les étapes ou les lignes qui la constituent, mais *trouver un geste qui comprime*, qui permette de *saisir d'un seul coup* l'ensemble de la démonstration.**

Alain CONNES, *Séminaire « Pensée des sciences »*, Ens, 2003.

Je prendrai cette déclaration d'Alain Connes comme une sorte de *fil conducteur* compacté de la notion de diagramme.

Sur mon titre :

Ars **:** latin signifiant « façon d'être » et « façon d'agir ». Apparenté au grec ἄρθρον (« articulation »). La racine *armus* a donné le latin *ritus* « compte », le grec *ari-thmos*, « nombre ». Ce sera aussi le sens de « talent ». Le mot a servi d'équivalent au grec *technè*. En français, à la fin du X$^{\text{ème}}$ siècle, le mot prend le sens de « moyen, méthode, connaissance », puis celui de « ruse, artifice »

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L. Boi, C. Lobo (eds.), *When Form Becomes Substance*,
https://doi.org/10.1007/978-3-030-83125-7_18

Le phénomène de compactification

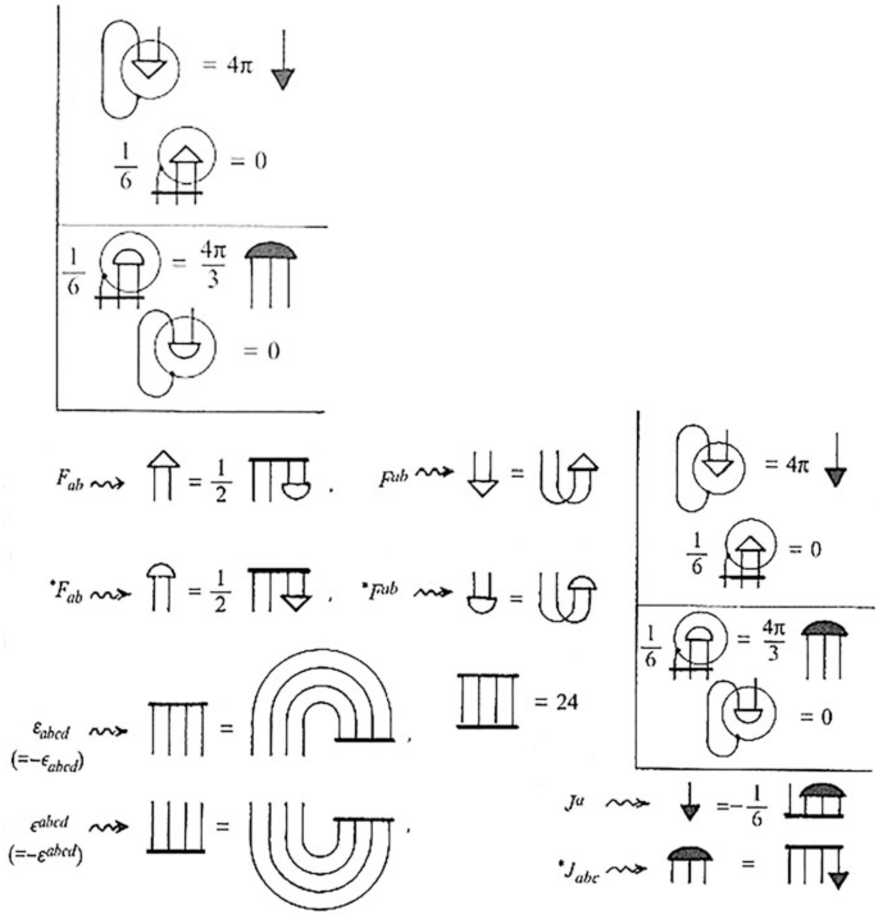
Les équations de Maxwell pour l'électromagnétisme — qui rendent compte, entre autres, de la lumière visible — sont passées de 255 signes-symboles chez Einstein en 1905 — dans sa formulation déjà « compactée » — à 6 aujourd'hui !

TABLE 7.1. Maxwell's Equations in the Course of History
The constants c, μ_0, and ε_0 are set to 1, and modern notation is used for the components.

The Homogeneous Equation	The Inhomogeneous Equation
Earliest Form	Earliest Form
$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$	$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho$
$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\dot{B}_x$	$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = j_x + \dot{E}_x$
$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\dot{B}_y$	$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = j_y + \dot{E}_y$
$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\dot{B}_z$	$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = j_z + \dot{E}_z$
At the End of Last Century	At the End of Last Century
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{E} = \rho$
$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$	$\nabla \times \mathbf{B} = \mathbf{j} + \dot{\mathbf{E}}$
At the Beginning of This Century	At the Beginning of This Century
${}^*F^{\beta\alpha}{}_{.\alpha} = 0$	$F^{\beta\alpha}{}_{.\alpha} = j^\beta$
Mid-Twentieth Century	Mid-Twentieth Century
$dF = 0$	$\delta F = J$

Compactification des équations de Maxwell

Par ailleurs, il existe une représentation *encore plus épurée*, comme « l'écriture diagrammatique » de Roger Penrose, élaborée dans les années 1970, puis reprise plus tard, vers 2003, par Carlo Rovelli et les théoriciens de la *Loops Quantum Theory*.



Diagrammes Maxwell-Penrose

Les variantes successives, correspondant à des niveaux d’abstraction et de généralisation toujours plus élevés, se traduisent par une condensation toujours plus poussée de l’écriture : *moins nombreux sont les signes figurant dans la formule, plus ils signifient*. Aussi ne faut-il pas croire que plus une formule mathématique ou physique est longue et complexe, plus elle est ardue et profonde – c’est même l’inverse qui est vrai.

En vérité, les signes mathématiques sont des « technogrammes » *qui condensent sous une forme scripturale simple, un réseau complexe de concepts et de procédures.*[1]

[1] Voir sur ce point, Charles Alunni, « De l’écriture de la mutation à la mutation de l’écriture : de Galileo Galilei et Leonardo da Vinci au “technogramme” », in *Les Mutations de*

Hermann Weyl nota que Minkowski comparait *le principe du minimum*, formulé et employé à l'origine par Lord Kelvin, à ce qu'il appelait « le véritable principe de Dirichlet, celui de résoudre un problème par un minimum de calcul aveugle et un maximum de visions d'idées ».[2]

Cela engage évidemment la question de la « vision théorique », que ce soit celle des enchaînements de plus en plus « épurés » des formes symboliques dans le droit-fil de ce qui est au cœur même de l'instauration galiléenne. C'est ici la reconduction de ce que je qualifie de perpétuation du dispositif de *manu*facture et de manu*tension* des formalismes et de leurs apparats de traces. Cette approche philosophique s'accorde sur la « nature intransitive » du langage : le « signifié » se présente toujours avec une certaine « physionomie » corporelle (le « corps du signifié ») qui « transite » ensuite dans ses emplois variés. Mais il en va ici de *la main* et de ses tensions — *pour la pensée* : « La pensée est essentiellement l'activité d'opérer avec des signes. Cette activité est exercée par la main quand nous pensons en écrivant [. . .] *La pensée est quelque chose comme une activité de la main* ».[3]

Si l'on veut approfondir ce que John Archibald Wheeler appelle « *Philosophy of Pictures* », « *Pictorial Representation* », « *Pictorial Technics* » en topologie différentielle (c'est-à-dire dans sa définition d'objets géométriques dans un espace-temps non-métrique ou non géodésique en Relativité Généralisée), si l'on prend philosophiquement au sérieux son feuilletage de la Géométrie Différentielle selon les trois axes (les trois « perspectives ») du « purely *pictorial* », de l'« *abstract* differential » et des « components *mani*pulations »,[4] si l'on veut saisir tous les enjeux de la « *notation diagrammatique* » de Roger Penrose et des « formules de nœuds » de Louis H. Kauffman, il convient d'opérer la connexion avec cet aphorisme wittgensteinien tiré de ses *Vermischte Bemerkungen* : « I think with my pen » : « Je pense en fait avec la plume. Car ma tête, bien souvent, ne sait rien de ce que ma main écrit ».[5]

Il faut en saisir la « lettre » (et l'« esprit ») en se demandant *ce qui passe dans la pensée* et *pour la pensée.* Se demander si sa capacité d'élaboration diagrammatique et « imagée » d'entités physico-mathématiques (ce que Cartier appelle « géométrie ») telles que les « tenseurs » aurait quelque

l'écriture, Paris, Publications de la Sorbonne, « Logique Langage Sciences Philosophie », éd. François Nicolas, 2013.

[2] Hermann Weyl, « Über den Symbolismus der Mathematik und mathematischen Physik », *Studium Generale*, 1953, 6, p. 219-228, in Hermann Weyl, *Le Continu et autres écrits*, Paris, Vrin, « Mathesis », 1994 [éd. Jean Largeault], p. 255.

[3] Ludwig Wittgenstein, *Le Cahier bleu et le Cahier brun*, Paris, Gallimard, 2004.

[4] Cf. John Archibald Wheeler, in Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, San Francisco, W. H. Freeman, 1973.

[5] Ludwig Wittgenstein, *Vermischte Bemerkungen* [*Remarques mêlées*, TER, 1990], Oxford, Basil Blackwell, 1977, p. 29.

chose « à voir » avec le fait que l'acte mécanique de l'inscription comme telle, saisi comme geste pragmatique d'écriture sur une feuille ou sur un tableau noir, est lui-même le résultat d'une application de « forces tensorielles » sur le corps du stylo ou sur le corps de la craie.

On pourrait dire qu'en un sens le « tenseur » décrit à la fois (mathématiquement) le système des tensions aux points de contact, et qu'il l'« indexe » et qu'il le « code » dans la pensée, par le mouvement et la dynamique même de son inscription. Notre question pourrait se formuler ainsi : « qu'est-ce qui, de cette "algèbre tensorielle au bout des doigts", se "schématise", se diagrammatise dans la pensée et pour la pensée de la *manu*facture et de la manu*tension* ? » (« [...] comment pourrais-je jouer du piano en lisant les notes si ces dernières n'avaient pas déjà quelque rapport avec une certaine espèce de mouvements des mains ? » questionnait encore Wittgenstein).

Gilles Châtelet est très précis sur ce point : « J'ai donc ainsi en quelque sorte *propulsé une main par la pensée* et on serait tenté de dire que la pince ou le compas donnent *un point de vue à la main* ».[6]

C'est la problématique ouverte du *court-circuit de la main et de l'esprit* dans la reprise et l'élaboration contemporaines des matérialités symboliques, des stratégies manipulatoires de leurs traces algébriques, et de leurs représentations géométriques.

Concluons ici avec Federico Zuccaro :

> Et si ce *disegno* ne commandait ni ne maîtrisait notre intellect, en particulier notre intellect pratique ; si ce dernier ne commandait, c'est-à-dire ne dirigeait notre volonté, et si celle-ci ensuite ne commandait à nos forces (*virtù*) et à nos puissances inférieures, ainsi qu'aux parties du corps, *en particulier aux mains*, nous ne pourrions découvrir l'ordre et la façon d'opérer droitement en nous-mêmes.[7]

Du diagramme comme preuve par l'image

> Mes lectures – rares [...] surtout le passionnant *Électromagnétisme* du dernier grand théoricien (mort) Maxwell. Je dis *passionnant*. Un livre tout fait d'une métaphore originelle, initiale, puis uniquement les formules et *les diagrammes* – un *ornement extraordinaire*.

Paul Valéry, novembre 1893, in *André Gide—Paul Valéry. Correspondance 1890–1942*, Paris, Gallimard, 1955, Édition Robert Mallet, p. 190.

> Un diagramme peut immobiliser un geste, le mettre au repos, bien avant qu'il ne se blottisse dans un signe, et c'est pourquoi les géomètres ou les cosmologistes contemporains aiment les diagrammes et leur pouvoir d'évocation péremptoire. Ils

[6] Gilles Châtelet, *Les Enjeux du mobile*, *Mathématique, physique, philosophie*, « Des Travaux », Seuil, Paris, 1993, p. 221.

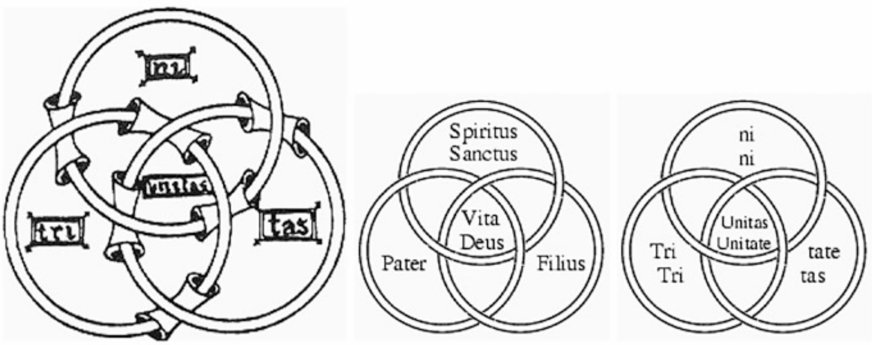
[7] Federico Zuccaro, *L'Idea dei pittori* (1607).

saisissant les gestes au vol ; pour ceux qui savent être attentifs, *ce sont les sourires de l'être*.

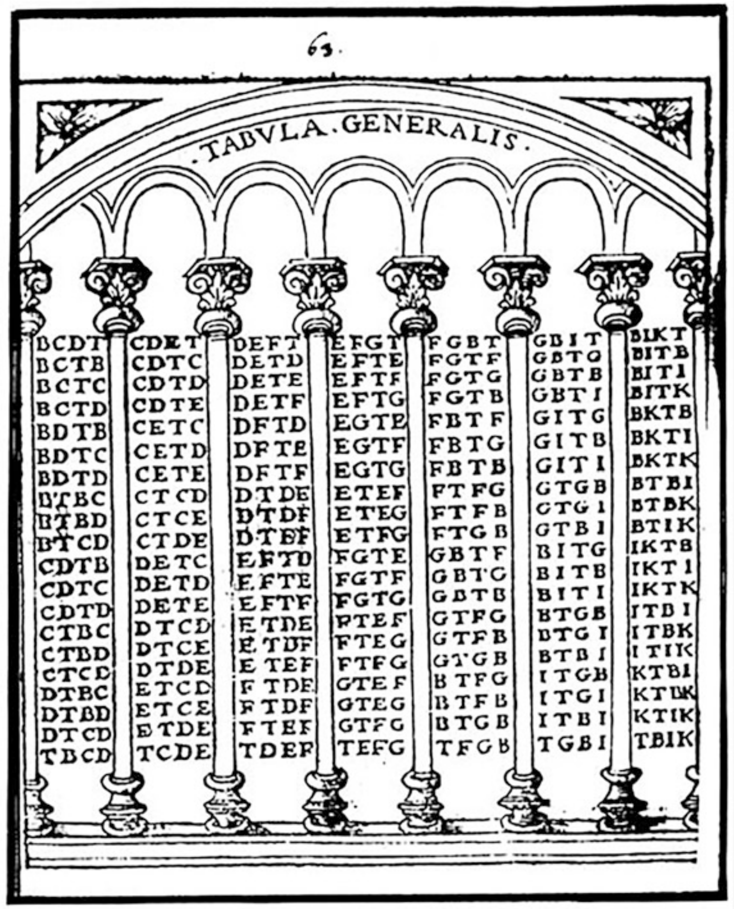
Gilles Châtelet, Les enjeux du mobile. Mathématique, physique, philosophie, « Des Travaux », Seuil, Paris, 1993.

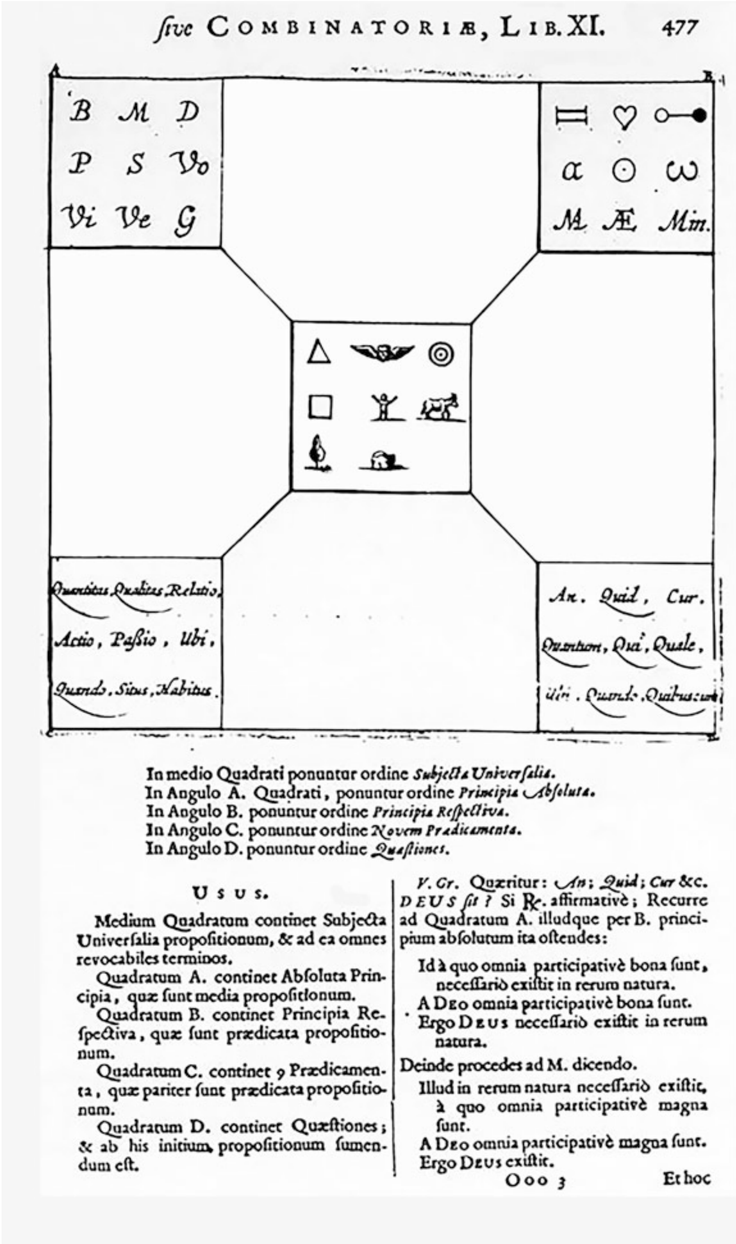
Me limitant ici au « diagramme scientifique », je n'aborderai pas en profondeur la question des protodiagrammes (du partage platonicien entre *skiagraphia* et *skenographia*, des usages métaphoriques de l'image ou du *dessin skiagraphiques* dans les arts mimétiques à la « scénographie » engageant la perspective optique – à la mise en place des *tabulae* lulliennes, des tableaux *pré-diagrammatiques* de Pierre de la Ramée ou des diagrammes de Nicolas Oresme —, en passant par le statut médiéval du « diagramme » chez Joachim de Flore qui, *au-delà de l'opposition du texte et de l'image*, apparaît comme un véritable programme politico-religieux et un nouveau support « médiatique » des *preuves onto-théologiques de la Trinité*. Simplement quelques exemples visuels :

1. Un exemple de nœud boroméen trinitaire. Hugues de Saint-Victor (vers 1160).



2. Raymond Lulle (1235–1315)





sive COMBINATORIÆ, LIB. XI. 477

In medio Quadrati ponuntur ordine *Subjecta Universalia.*
In Angulo A. Quadrati, ponuntur ordine *Principia Absoluta.*
In Angulo B. ponuntur ordine *Principia Respectiva.*
In Angulo C. ponuntur ordine *Novem Prædicamenta.*
In Angulo D. ponuntur ordine *Quæstiones.*

USUS.

Medium Quadratum continet Subjecta Universalia propositionum, & ad ea omnes revocabiles terminos.

Quadratum A. continet Absoluta Principia, quæ sunt media propositionum.

Quadratum B. continet Principia Respectiva, quæ sunt prædicata propositionum.

Quadratum C. continet 9 Prædicamenta, quæ pariter sunt prædicata propositionum.

Quadratum D. continet Quæstiones; & ab his initium propositionum sumendum est.

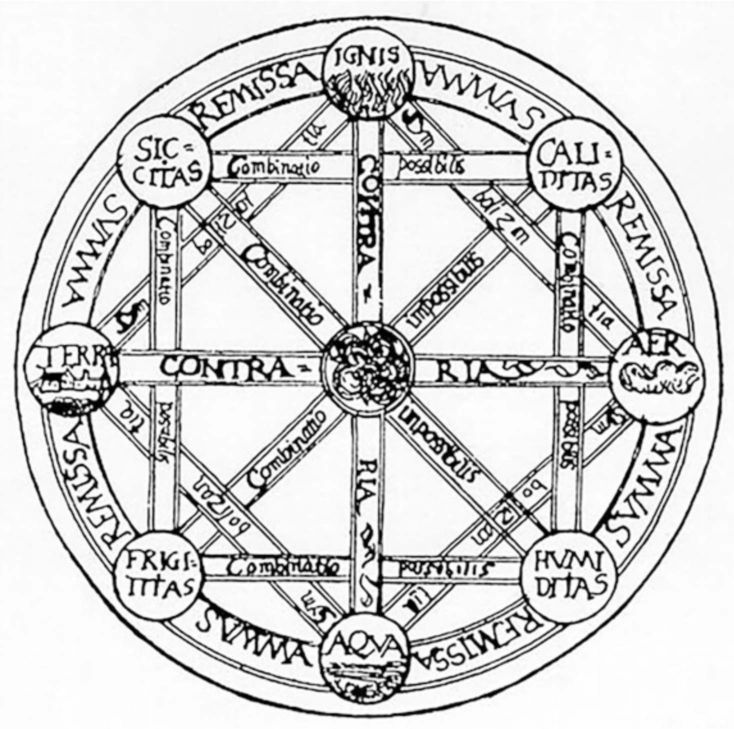
V. Gr. Quæritur: *An; Quid; Cur* &c. *DEUS sit?* Si Rc. affirmativè; Recurre ad Quadratum A. illudque per B. principium absolutum ita ostendes:

Id à quo omnia participativè bona sunt, necessariò existit in rerum natura.
A DEO omnia participativè bona sunt.
Ergo DEUS necessariò existit in rerum natura.

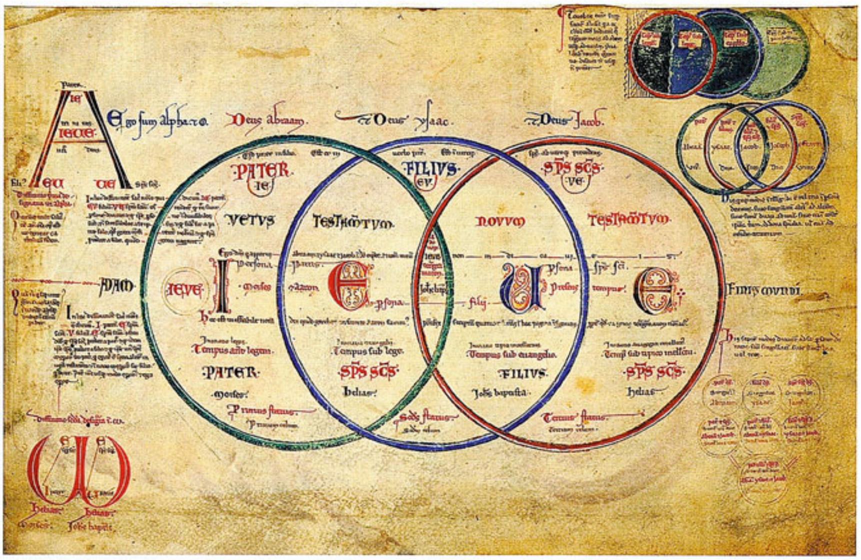
Deinde procedes ad M. dicendo.

Illud in rerum natura necessariò existit, à quo omnia participativè magna sunt.
A DEO omnia participativè magna sunt.
Ergo DEUS existit.

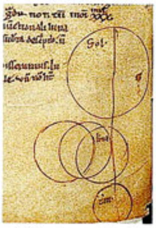
Ooo 3 Et hoc



3. Joachim de Flore (1130 ?–1202)

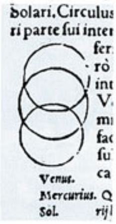


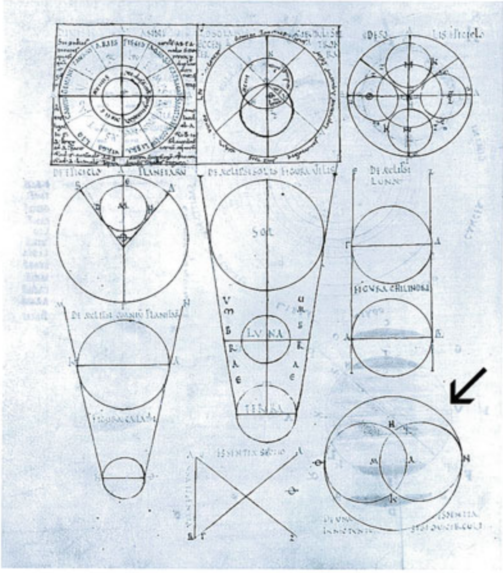
AP 30
Drei-Kreise-Diagramm, Joachim von Fiore, Liber Figurarum; Oxford, CCC, MS 255A fol. 7r.



AP 28a
Drei-Kreise-Diagramm, zu Wilhelm von Conches, Philosophia mundi II 23; nach Johannes Herold, D. Honorii Augustodunensis presbyteri libri septem (Basileae 1544) pag. 175.

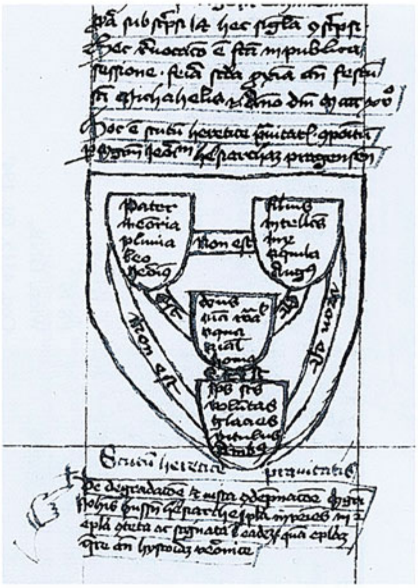
AP 28b
Drei-Kreise-Diagramm, zu Wilhelm von Conches, Philosophia mundi II 30; Biblioteca Apostolica Vaticana, Vat. Reg. Lat. 72 fol. 96ʳ.



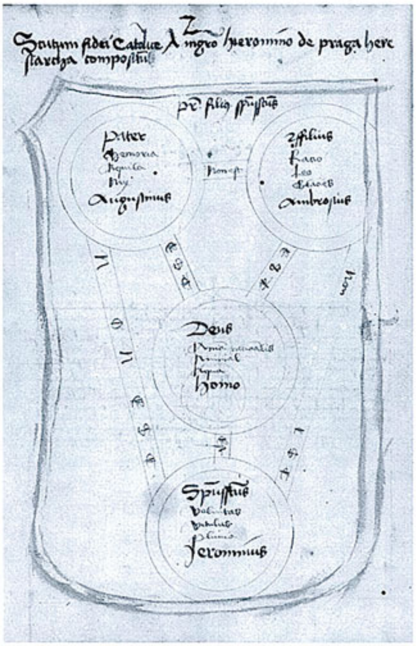


AP 29
Drei-Kreise-Diagramm; nach dem Kommentar des Calcidius zu Timaeus, übernommen von Abbo von Fleury; Berlin, Staatsbibl., Phillipps 1833 fol. 36ʳ.

4. Guillaume de Conche (1080 ?–1150)

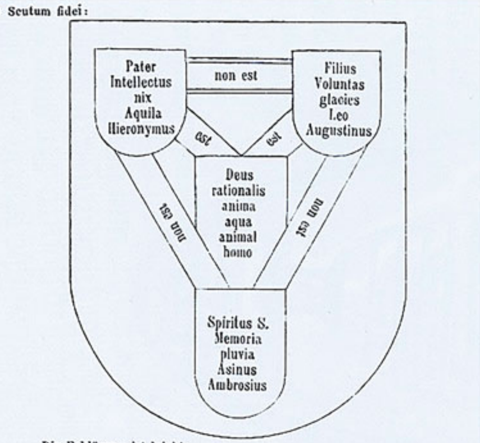


FS 20
Scutum heretice pravitatis; Biblioteca Apostolica Vaticana, Ottobon. Lat. 2087, fol. 241va.



FS 21
Hamburg, Staats- und Univ.-Bibl., Cod. hist. 31e, fol. 7v.

1) Scutum fidei:

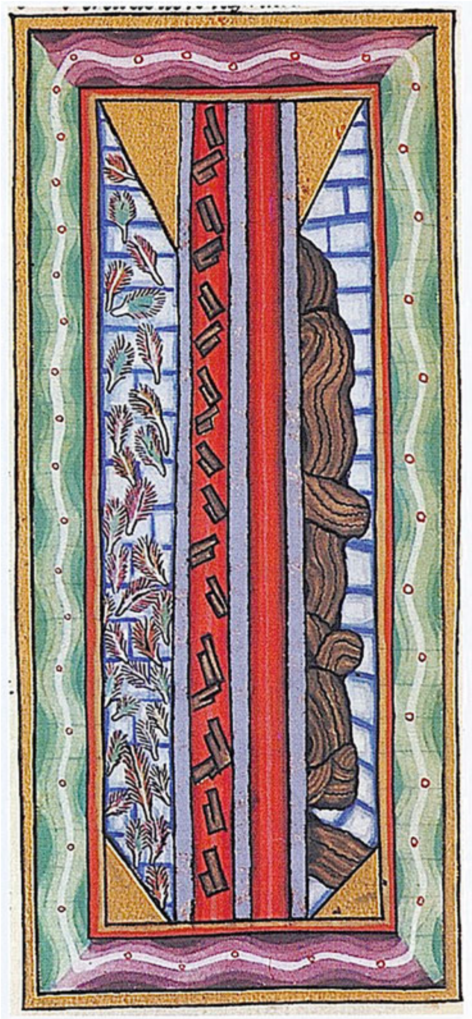


Die Erklärung ist leicht:
Pater non est Filius, non est Spiritus S. — est Deus.
Intellectus non est Voluntas, non est memoria — est rationalis anima.
Nix non est glacies, non est pluvia, est aqua und so fort.

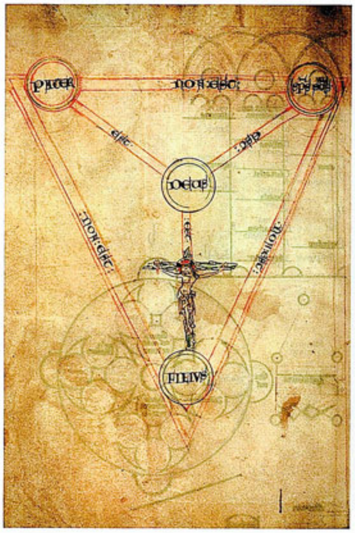
FS 22
HÖFLER, Geschichtschreiber 3, S. 19f.: »Cod. Universit. Monac. n. 186f. 34«.

5. Hildegarde de Bingen

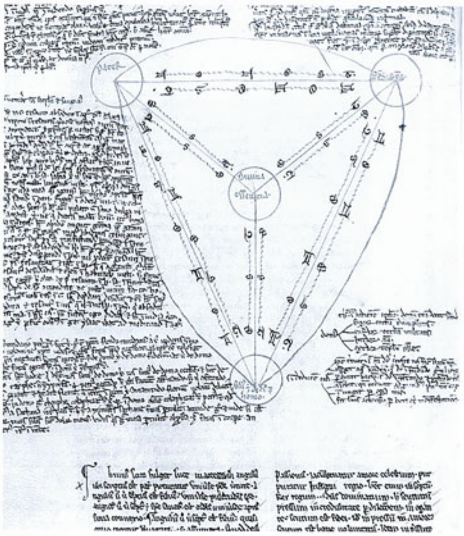
1098–1179. Forme symbolique hybride de laTrinité



6. Pierre de Poitiers (1130–1215)



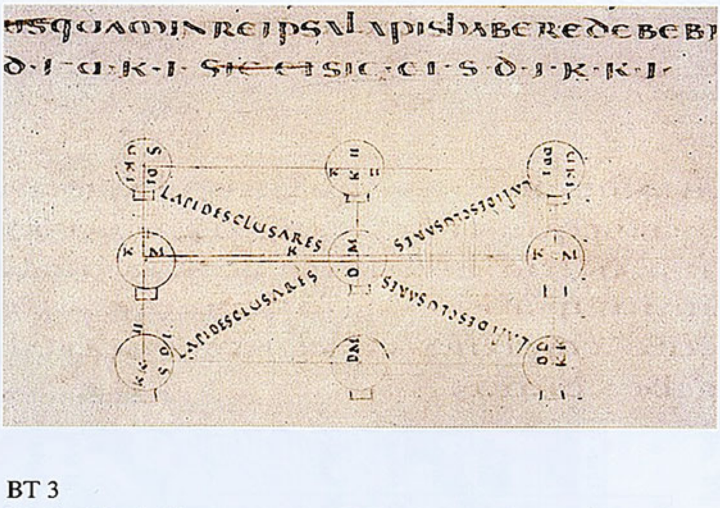
FS 2
Petrus von Poitiers, Compendium historiae in genealogia Christi, London, BL, Cotton Faustina B VII, fol. 42v.



FS 3
Robertus Grosseteste, Dicta, Durham Cathedral, A. III. 12, fol. 14v (vor 1231).

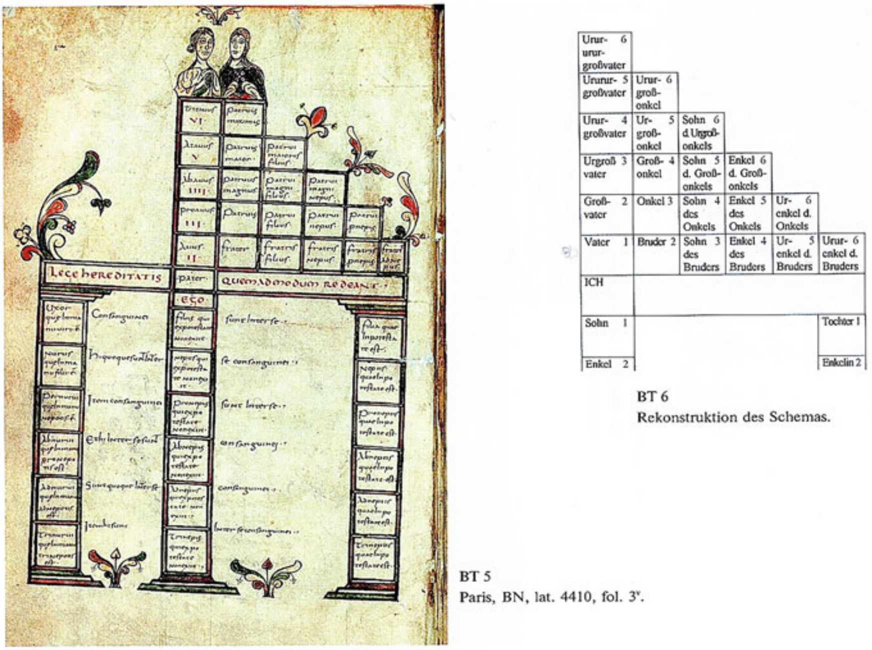
264 Abbildungen

7. On a également les exemples d'une émergence de tels « proto-diagrammes » dans la *pensée juridique médiévale* : du *Corpus Agrimensorum Romanorum* **(Clic)** aux *Arbores consanguinitatis et affinitatis*.



BT 3
Wolfenbüttel, Herzog-August-Bibliothek, Codex Arcerianus, Cod. Guelf. 36.23 Aug. 2, fol 58r.

Corpus Agrimensorum Romanorum. Daté du VIème siècle



Arbores consanguinitatis. IXème siècle tardif reproduisant un schème du IIème siècle romain

8. Que la métaphore et le diagramme s'avèrent structurellement solidaires a aussi été signalé par Gilles CHÂTELET :

> Si les stratagèmes allusifs peuvent prétendre définir un nouveau type de systématicité, c'est parce qu'ils donnent accès à un espace d'entrelacement de la singularité du diagramme et de la métaphore [. . .]. Cet entrelacement est un dispositif où chaque composante s'adosse aux autres : sans le diagramme, la métaphore ne serait qu'une fulguration sans lendemain parce qu'incapable d'opérer ; sans la métaphore, le diagramme ne serait qu'une icône gelée, incapable de sauter par-dessus ses traits gras qui retiennent les images d'un savoir déjà acquis ; sans la subversion du fonctionnel par le singulier, rien ne viendrait contrarier la force de l'habitude.[8]

Cet entrelacement, qui investit non seulement la métaphore et son diagramme, mais la pensée mathématique effective dans ce qu'on appelle aujourd'hui une « *géométrie de l'entrelacement* », nous conduit à parler de *philosophie symplectique* (des « algèbres de Clifford » aux « nœuds » de

[8] Gilles Châtelet, Séminaire Ens, 1997. Publié en anglais sous le titre, « *Interlacing the singularity, the diagram and the metaphor* », in [Duffy ed.] *Virtual Mathematics*, Manchester, Clinamen Press, 2006, p. 31–45.

Kauffman).[9] J'indique en passant qu'on doit ce mot à Hermann Weyl, qui titre le chapitre VI de son *Classical Groups* de 1938. Quelle que soit son étymologie, l'adjectif « symplectique » signifie fondamentalement « tressés ensemble » ou « tissés » ; et c'est cet *effet de tresse* « par self-induction » qui nous intéresse ici et avant tout.

Je voudrais préciser ici ce qui différencie quelque peu mon approche de celle de Gilles Châtelet. Je formulerai cet écart en disant qu'il s'agit selon moi de *faire remonter plus radicalement l'acte diagrammatique du tracé d'un dessin* (explicité en sa pure figuration géométrique) *à la lettre prise dans l'économie discursive de la formule, toute formule constituant déjà un diagramme complexe* qui n'a de sens qu'au centre pointé d'un contexte théorique et d'uneformeconceptuelle.

[9] Il suffit de rappeler que le grec συμπλεκτικος signifie proprement « qui entrelace », en particulier dans le contexte des sciences naturelles (« qui est entrelacé avec un autre corps ou une autre partie ») ; ce qui a donné le latin *complexus* (enlacement) et *complex* (uni, joint), eux-mêmes dérivés de *complector* (qui signifie, en son sens figuré, « SAISIR par l'intelligence, par la pensée, par la mémoire ou par l'imagination » et « embrasser [comprendre] dans un exposé » : *una comprehensione omnia complecti* = « comprendre tout dans une formule unique »). L'utilisation de ce mot en mathématiques est due à Hermann Weyl qui, dans un effort pour éviter une confusion sémantique, a rebaptisé l'obscur (pour l'époque) « groupe du complexe linéaire », « groupe symplectique » ; Cf. Hermann Weyl, *The Classical Groups. Their Invariants and Representations* [1938], Princeton University Press, Princeton, 1946[2], Chap. VI, « The symplectic group », p. 165. Quelle que soit son étymologie, l'adjectif « symplectique » signifie fondamentalement « tressés ensemble » ou « tissés » ; et c'est cet effet de tresse « par self-induction » qui nous intéresse ici et avant tout.

Quatre exemples physico-mathématiques :

§ 4] SIMULTANEOUS IDENTIFICATIONS 55

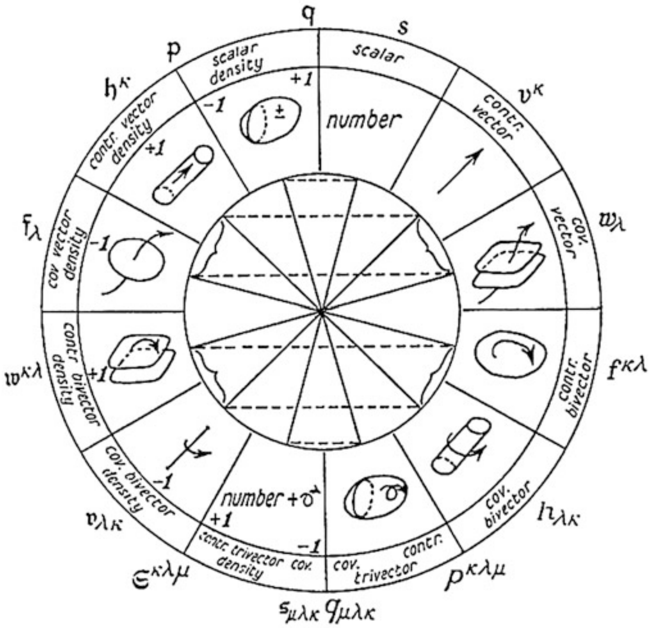
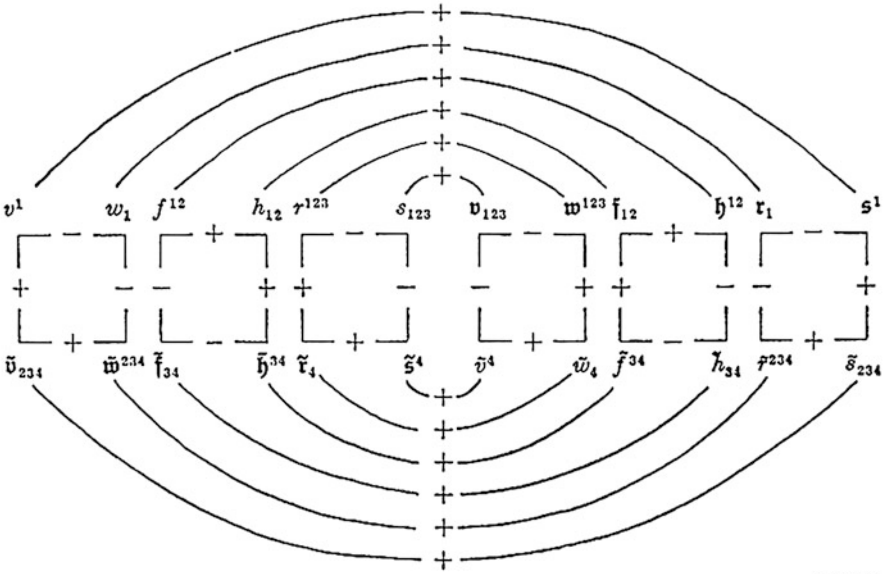


FIG. 13.

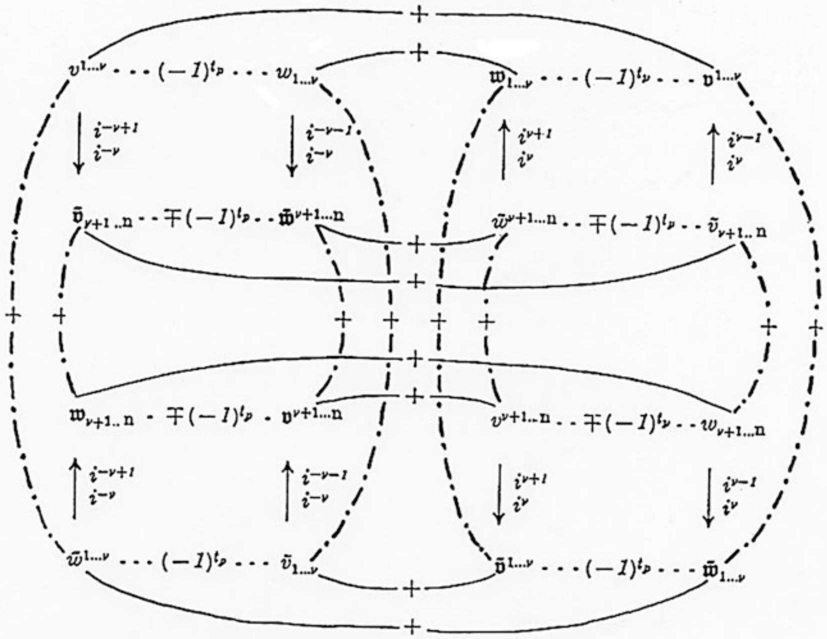
and the following table:



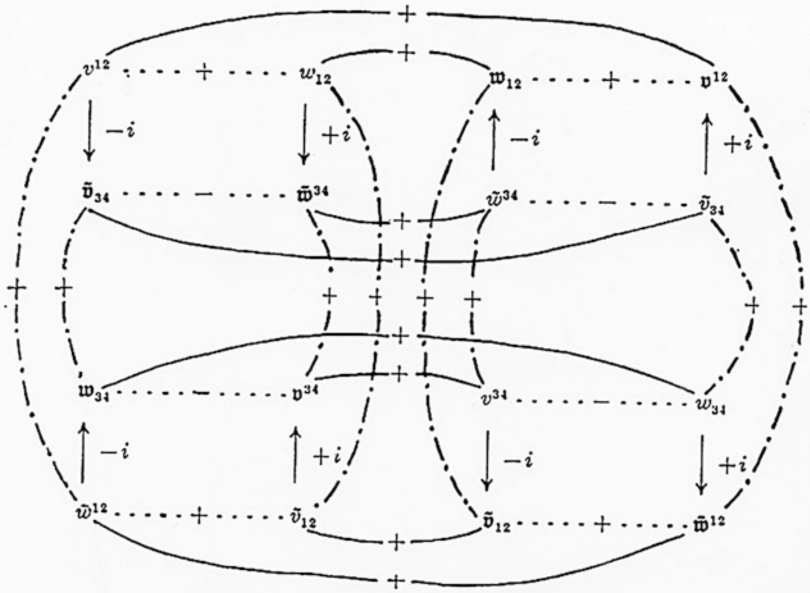
(4.24)

J. A. SCHOUTEN. ***Tensor Analysis for Physicists*****, 1951**

50 IDENTIFICATIONS OF QUANTITIES IN E_n [Chap. III



for odd/even signature. Accordingly we have for $n = 4$ and signature $---+$



(4.15)

J. A. SCHOUTEN. Idem

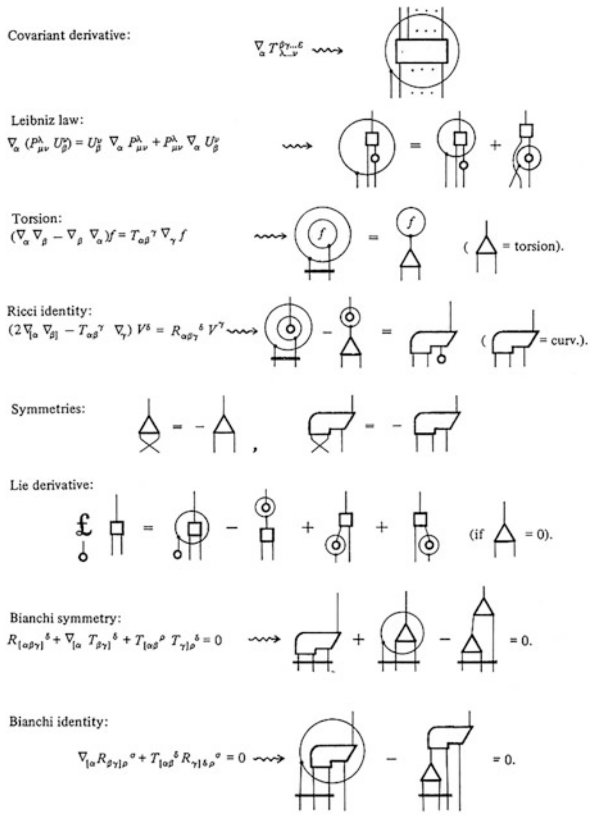


Fig. A-8. The 'loop' notation for (covariant) derivative; curvature, torsion.

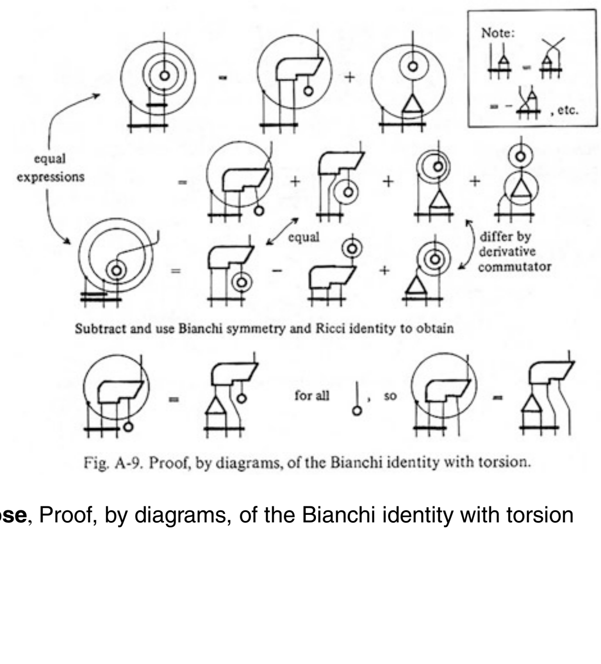


Fig. A-9. Proof, by diagrams, of the Bianchi identity with torsion.

Roger Penrose, Proof, by diagrams, of the Bianchi identity with torsion

Je prends enfin l'exemple de la « formule » du Lagrangien – aujourd'hui "fonctionnel" au CERN de Genève

Modèle Standard

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \frac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu$$

$$+\bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu$$

$$-\frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^-$$

$$-\frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right)$$

$$+\frac{2M^4}{g^2}\alpha_h - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu)$$

$$+Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$-A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu))$$

$$-\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu)$$

$$+g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$-2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right)$$

$$-\frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\right)$$

$$-gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H$$

$$-\frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right)$$

$$+\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right)$$

$$+\frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+)$$

$$+igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+)$$

$$+igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right)$$

$$-\frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right)$$

$$-\frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+)$$

$$+\frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+)$$

$$-g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - g^2 s_w^2 A_\mu A_\mu\phi^+\phi^-$$

$$-\bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j$$

$$+igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right)$$

$$+\frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda)$$

$$+(\bar{u}^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - 1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)\}$$

$$+\frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}\frac{m_e^\lambda}{M}\left(-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)\right)$$

$$-\frac{g}{2}\frac{m_e^\lambda}{M}\left(H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^+\left(-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j)\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j)\right)$$

$$-\frac{g}{2}\frac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \frac{ig}{2}\frac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)$$

$$+\bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \frac{M^2}{c_w^2})X^0$$

$$+\bar{Y}\partial^2 Y + igc_w W^+_\mu(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W^+_\mu(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+ Y)$$

$$+igc_w W^-_\mu(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W^-_\mu(\partial_\mu\bar{X}^- Y - \partial_\mu\bar{Y}X^+)$$

$$+igc_w Z^0_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-)$$

$$-\frac{1}{2}gM\left(\bar{X}^+ X^+ H + \bar{X}^- X^- H + \frac{1}{c_w^2}\bar{X}^0 X^0 H\right)$$

$$+\frac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-\right)$$

$$+\frac{1}{2c_w}igM\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right)$$

$$+igMs_w\left(\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-\right)$$

$$+\frac{1}{2}igM\left(\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0\right).$$

(le « lagrangien étant ce qui donne les lois du mouvement [position-vitesse] ou la dynamique du système) dans le *modèle dit standard* de la physique actuelle.

Je le compare à sa compactification dans le contexte de la *géométrie non-commutative* développée par Alain Connes.

Modèle Standard en couplage minimal

$$\mathcal{L}_E + \mathcal{L}_G + \mathcal{L}_{GH} + \mathcal{L}_H + \mathcal{L}_{Gf} + \mathcal{L}_{Hf}$$

Action Spectrale (ac+ac)

$$S = \int d^4x \sqrt{g}\ (1/2\kappa_0^2\, R - \mu_0^2 (H^* H)$$

$$+ a_0\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} + \ b_0\, R^2 + c_0\, {}^*R^*R + d_0\, R;_\mu{}^\mu$$

$$+\ e_0 + 1/4\, G^i_{\mu\nu}\, G^{\mu\nu i} + 1/4\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha}$$

$$+\ 1/4\, B_{\mu\nu}\, B^{\mu\nu} + |D_\mu H|^2 - \xi_0\, R|H|^2 + \lambda_0 (H^* H)^2)$$

Je situerai donc *en amont de tout acte diagrammatique, la lettre ou gramme*. C'est ce qui peut être rendu lisible par une remontée généalogique jusqu'à la métaphore inouïe et fondatrice du *Livre de la nature* et à ses divers prolongements.

Dans cette perspective, le diagramme apparaît dès lors comme manifestation de la *structure articulée et dialectique du gramme*.

δια-γραφω = dessiner ; décrire || enregistrer ; attribuer || effacer.
δια-γραμμα (το) = 1° figure dessinée || 2° registre || 3° décret.
Statut du **δια-** : prép. signifiant *à travers*. Ici, *ce qui perce* (dans) le gramme et *traverse* l'écriture (et la « formule »).

Cette pulsation « vitale » et « initiale » du gramme, l'acte inducteur de sa mobilité potentielle (ce que Châtelet pointe comme « enjeux du mobile ») se recueille dans le sens originel du *dia-* qui signifie (à la lettre) « en divisant », puis, « en traversant », en passant « à travers ».

Cette « division » opère comme « différence de potentiel » engendrant la tension et l'« autonomie de son centre d'indifférence » : ce qui donne sa réserve à l'acte, dans le mouvement, sans jamais l'épuiser.

Ce qui recolle également à l'emprunt (1584) au latin *diagramma* (déverbal de *diagraphein*), attesté à l'époque impériale au sens d'« échelle des tons » musicaux, c'est l'idée d'un *scaling* à la fois de ses niveaux opératoires et de ses investissements catégoriaux.

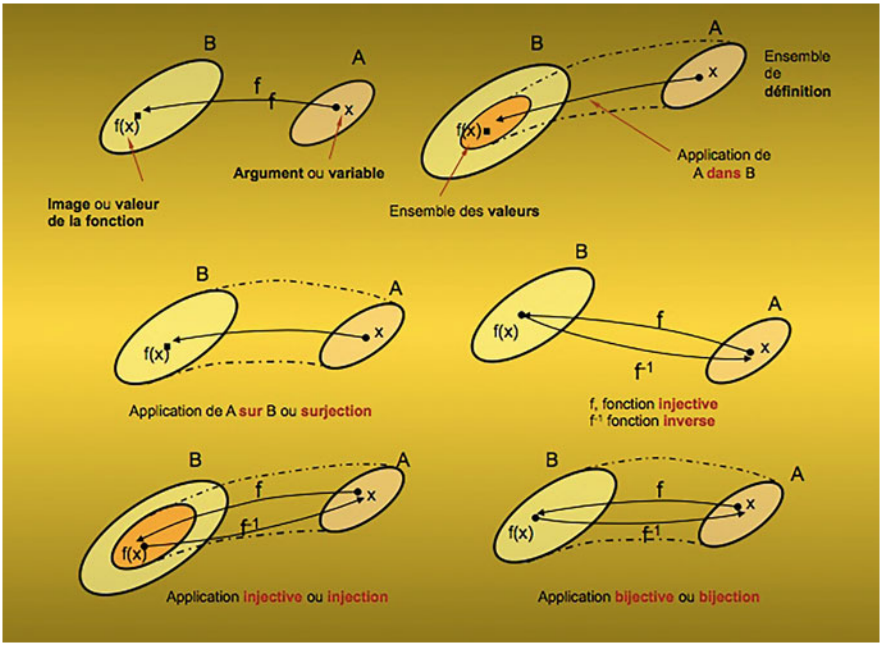
Dernier exemple, très bref, que je tire de la Théorie des catégories.

> **The fundamental idea of representing a function by an arrow first appeared in topology [. . .]. The arrow $f : X \to Y$ rapidly displaced the occasional notation $f(X) \subset Y$ for a function. It expressed well a central interest of topology. Thus a notation (*the arrow*) led to a concept (category).**

Sanders Mac LANE, *Categories for the Working Mathematician*, Springer, Berlin, 1971, p. 29.

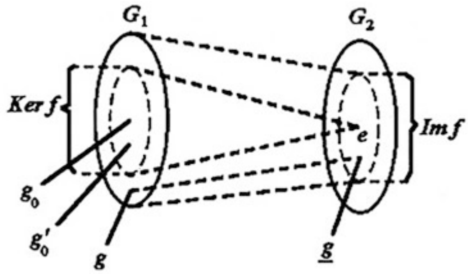
*Q*UELQUES PRÉCISIONS MAINTENANT CONCERNANT LA DÉSINTRICATION DES CONCEPTS D'*Image,* DE *Figure &* DE *Diagramme.*

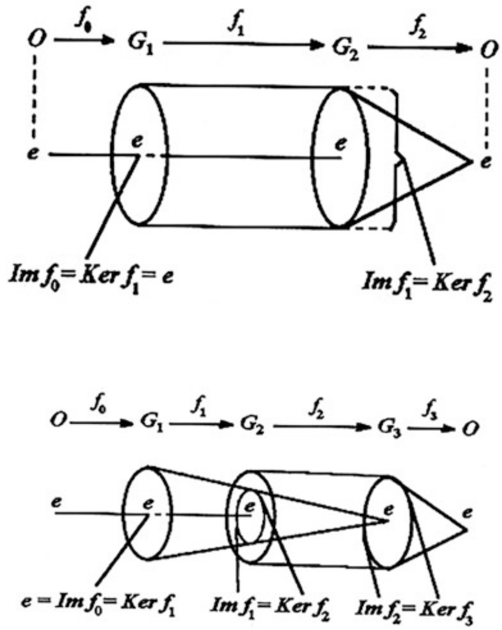
Image : j'ai abondamment utilisé la notion d'« image », mais en un sens qu'il me faut maintenant préciser. C'est plus ici l'« image » au sens mathématique (et ensembliste) du terme – comme *forme épurée et « abstraite »*, extension des diagrammes de Venn – qu'il faut avoir à l'esprit, que l'image « illustrative ».



Ainsi, l'IMAGE est *dans* le diagramme, et le diagramme *enveloppe* l'image comme l'un de ses éléments fonctionnels. Le diagramme inscrit dès lors l'image comme *son noyau* — on peut prendre ici encore le « noyau » en son sens proprement mathématique de « noyau d'homorphisme ».

$$
\begin{array}{c}
f: \\
G_1 \longrightarrow Im\, f \in G_2 \\
g_0 \longrightarrow e \in Im\, f \\
g'_0 \longrightarrow e \in Im\, f \\
g \longrightarrow \underline{g} \in Im\, f \\
g^{-1} \longrightarrow \underline{g}^{-1} \in Im\, f \\
g\, g \circ g^{-1} \longrightarrow \underline{g}\, e\, \underline{g}^{-1} = e \in Im\, f \\
\therefore \quad g\, g \circ g^{-1} \in Ker\, f.
\end{array}
$$





Séquences homomorphiques & structure des groupes d'homotopie. *Théorème des séquences exactes*

Wang RONG, Chen YUE, *Differential Geometry and Topology in Mathematical Physics*, World Scientific, Singapore, 1998.

Jean-Toussaint Desanti déclarait dans *Philosophie : un rêve de flambeur. Variations philosophiques 2*, Paris, Grasset, 1999, p. 138 :

> Le rapport entre “noyau de fruit” et “noyau d’homomorphisme” n’est pas de simple et arbitraire homonymie. [. . .] “le noyau enferme et maintient un arbre virtuel conforme à son espèce”. C’est cette fonction de “signal” d’invariance, au sein d’un domaine en devenir, qui me paraît autoriser les extensions de signification aux champs les plus étrangers au départ à nos expériences naturelles usuelles. Ainsi l’usage du mot a été admis pour désigner, au sein de l’atome, la région où se trouve concentrée sa charge, au sein de la cellule la région où se trouve concentré et préservé son matériel génétique : toujours conformément à une exigence d’invariance, chaque fois spécifique, et repérable selon des procédures appropriées. Dans la langue des algébristes, un noyau prend le sens de : invariant.

Quant à l’image, on pourrait dire qu’elle *développe* le diagramme.

On retrouve alors la suture image / diagramme, la double structure invaginante enveloppement / développement, et la *question de la preuve* (ou preuve par l’image), sujets que je ne traiterai pas ici, mais dont on peut avoir une idée avec la Thèse de Saunders Mac Lane sur la question de l’algorithme logico-mathématique et de sa preuve :

> Une preuve n’est pas seulement une série de pas individuels, mais un groupe de pas, rassemblés en vue d’un plan et d’une intention définis [. . .]. Nous affirmons ainsi que toute preuve mathématique possède une idée directrice (prédominante) <*leitende Idee*> qui détermine chaque pas individuel et qui peut être donnée comme un plan de la preuve <*Beweisplan*>.
>
> [. . .] De nombreux styles fondamentalement différents peuvent être utilisés pour donner n’importe quelle preuve – le style précis, symbolique et détaillé utilisé dans les *Principia* et dans bien d’autres parties des Mathématiques, qui requiert un exposé rigoureux des pas d’épreuve au prix des idées sous-jacentes –, et le style intuitif, conceptuel qui déploie toujours les idées et les méthodes centrales d’une preuve, de manière à comprendre les manipulations individuelles à la lumière de ces idées. Ce style est pratiqué en particulier dans les ouvrages et dans les cours d’Hermann Weyl.

Saunders Mac Lane, Abgekürzte Beweise in Logikkalkul , 1934, PhD Thesis, Georg August- Universität zu Göttingen. Réédité in I. Kaplansky [éd.], Saunders Mac Lane Selected Papers, New York, Springer Verlag, 1979, p. 1-62.

Je rappelle ici que cette Thèse, soutenue devant Hermann Weyl, s’intégrait à un projet de *théorie des structures* pour les Mathématiques qui devait se fonder sur les principes d’idées directrices permettant d’établir une preuve intuitive plus proche de la logique formelle.

FIGURE : Il y a plus dans le diagramme que dans la figure. Alors que la figure est condamnée à sa triste fonction illustrative, le diagramme s’efforce d’articuler *une « compénétration de l’image et du calcul ».* On pourrait dire que le diagramme pousse le calcul par ses « stratagèmes allusifs » et que, contemporainement, le calcul suit et poursuit l'image diagrammatique qui est *orientation* vers la preuve.

Si le calcul cale et calque l'image, inversement (et dans un même geste solidaire), « la notation contamine le calcul ».

Le diagramme se distingue de la figure par la machinerie qui le fait fonctionner. C'est une puissance machinique qui lui est propre. Il ne s'agit pas d'un simple codage ou d'une information rassemblée en un lieu singulier qui illustrerait ou résumerait simplement un modèle, mais c'est un déploiement de gestes virtuels. Le diagramme est une structure covariante, c'est-à-dire indépendante des référentiels et valable dans tous les mondes possibles. En ce sens, le diagramme véhicule ses propres règles syntaxiques. C'est un « "indicateur" d'exigences de relations », selon la belle formule de Jean-Toussaint Desanti.

C'est ici le lien essentiel du diagramme à la relation, au virtuel, non à une identification objectale. Ainsi, pour Charles Sanders Peirce :

> Le diagramme représente une Forme définie de Relation. La relation est ordinairement une relation qui existe, comme dans une carte, ou bien une relation qu'on a l'intention de faire exister <is intended to exist>, comme dans un plan.

Charles Sanders Peirce, Prolegomena for an Apology to Pragmatism [1906], in *The New Elements of Mathematics*, C. Eisele [ed.], La Haye, Mouton Publishers, 1976.

Le diagramme combine à la fois une interprétation des signes figurés dans un schéma, et la façon dont ces signes fonctionnent entre eux par connexion structurale des actes intentionnels spécifiques qui les ont engendrés. Cette « indication » diagrammatique des enchaînements d'actes (comme enchaînements d'actes potentiels) se caractérise toujours par une très grande plasticité qui fait qu'elle ne dévoile jamais totalement, procédant toujours par indices et indexations dont la dialectique opératoire (et manipulatoire) reste extrêmement subtile. L'« indication », loin de figer une réalité, enveloppe de l'indétermination qui n'est pas un manque de détermination, mais une promesse de virtualités à éveiller.

On voit *ce qui sépare le diagramme de la figure* : si la figure est apte à « illustrer » de façon statique, le diagramme s'impose comme une « figure-calcul » purement dynamique, du simple fait qu'il mobilise ensemble image et calcul. Non seulement le diagramme épouse une réalité déjà là, déjà donnée, mais il est en outre capable de s'adapter, par l'anticipation que lui donne toute sa puissance de feu virtuelle, une réalité encore à venir qui sera celle d'un savoir ou d'un problème inédit. Aptes à se réactiver eux-mêmes, les diagrammes se constituent ainsi en de véritables « multiplicateurs de virtualités ».

Il existe une puissance opératoire tout à fait particulière aux diagrammes. Ils ne se contentent pas de visualiser des algorithmes ou de coder et de compactifier "l'information" pour la restituer sous forme de modèles ou de "paradigmes". Le diagramme est bien ce grouillement de gestes virtuels :

pointer, boucler, prolonger, strier le continuum. Une simple accolade, un bout de flèche et le diagramme saute par-dessus les figures et contraint à créer de nouveaux individus. Le diagramme ignore superbement toutes les vielles oppositions “abstrait-concret”, “local-global”, “réel-possible”. Il garde en réserve toute la plénitude et tous les secrets des fonds et des horizons que sa magie tient toujours pourtant en éveil.

C’est la raison pour laquelle Alain Connes peut expliquer ce que nous avons vu en exergue **:** « Pour un *working mathematician*, comprendre une démonstration ne consiste pas à refaire une à une les étapes ou les lignes qui la constituent, mais à trouver un geste qui comprime, qui permette de saisir d’un seul coup l’ensemble de la démonstration ». Or, le diagramme est précisément en mesure de compacter un ou plusieurs gestes, tout en exhibant dans le visible des opérations jusque-là restées muettes.

Libérant l’implicite opératoire, le diagramme suggère des connexions nouvelles, à la fois structurales et ontologiques. C’est là tout l’enjeu immense d’une dignité ontologique propre au figural et à ses différentes lignées. Différant de la figure, il ne s’identifie pas non plus à un graphe. Alain Connes (dans *Triangle de pensées*, Paris, Éditions Odile Jacob, p. 132), considère ainsi que la théorie des graphes ne constitue pas à proprement parler une théorie, mais simplement « un savoir, une série de faits ». La théorie des graphes apparaît plus comme une théorie « horizontale » que « verticale » au sens où elle se disperse autour de thèmes apparemment sans lien, au lieu d’échafauder une pyramide de résultats fortement dépendants — ce qui est le cas du diagramme physico-mathématique.

Le diagramme rend ainsi familier, concret et sensible à l’œil de l’esprit ce qui pourtant relève purement de l’ordre du contre-intuitif. Ces dispositifs « producteurs d’ambiguïté » sont ainsi en mesure de condenser et d’amplifier l’intuition par leur profonde richesse allusive. C’est pourquoi Kepler a pu affirmer par une géniale prévision, que le nombre n’était autre que « l’œil de l’esprit par lequel, seul, la réalité est rendue visible » !

Le diagramme apparaîtra dès lors comme une véritable opération d’*épure esthétique* liant les gestes de scription, d’inscription et de compactification ; et ce statut suggestif et imaginant (sinon invaginant) du diagramme scientifique viendra alors rencontrer à nouveau la pratique artistique de l’acte pictural comme tel. Je conclurai par une citation d’ordre *strictement esthétique* :

> Très souvent les marques involontaires sont beaucoup plus profondément suggestives que les autres, et c’est à ce moment-là que vous sentez que toute espèce de chose peut arriver. — Vous le sentez au moment même où vous faites ces marques ? — Non, les marques sont faites et on considère la chose comme on ferait d’une sorte de diagramme. Et l’on voit, *à l’intérieur* de ce diagramme, *les possibilités* de faits de toutes sortes *s’implanter*.

Francis BACON, cité par Gilles Deleuze dans *Francis Bacon. Logique de la sensation*, note 87, ch. 12 « Le diagramme », p.94. Il reprendra le thème baconien au ch. 17, « L’œil et la main », p. 145.

Grid Diagram: Deleuze's Aesthetics Applied to Maggs's Photographs

Jakub Zdebik

Abstract Gilles Deleuze's diagram is an abstract concept without a corresponding visual manifestation. Yet, it is also a concept interrelated with his aesthetic philosophy. The grid, a nonfigurative structure, serves to articulate the aesthetics of the diagram. Applied to grids manifest in Arnaud Maggs's art, the diagram is revealed to be, in turn, an art historical methodology, a capture of information and an instance of pure functionality. With the theories of Buci-Glucksmann, Joselit, Krauss, Elkins and Genosko, the structural visual apparatus beneath the philosophical diagram devised by Deleuze and Guattari, with the support of Foucauldian and Kantian notions, is made to take shape.

Keywords Deleuze · Diagram · Schema · Faciality · Grid · Maggs · Photography

When we think of diagrams, we do not necessarily think of art. When one hears the term "diagram," images of architectural plans, maps, or even charts or graphs, come to mind. These could be considered as visual examples of diagrams. In philosophical terms, the theory of diagrams allows us to understand and examine what all of these examples have in common, namely that the diagram is a concept that aims at capturing zones of potentiality between the virtual and the actual, the abstract and the concrete, and information and its visualization. Of course, this space between the conceptual and the real is the subject of visual works by artists who use the device of the artistic diagram. One artist that straddles the divide between art and diagram, visualization and information, as well as how art captures the objective subjectively, is Arnaud Maggs whose photographic oeuvre is an expression of the diagrammatic in grid forms.

The relationship between the grid and the diagram, if it is to be understood as a graphic display of information, has been established as essential. In fact, the grid

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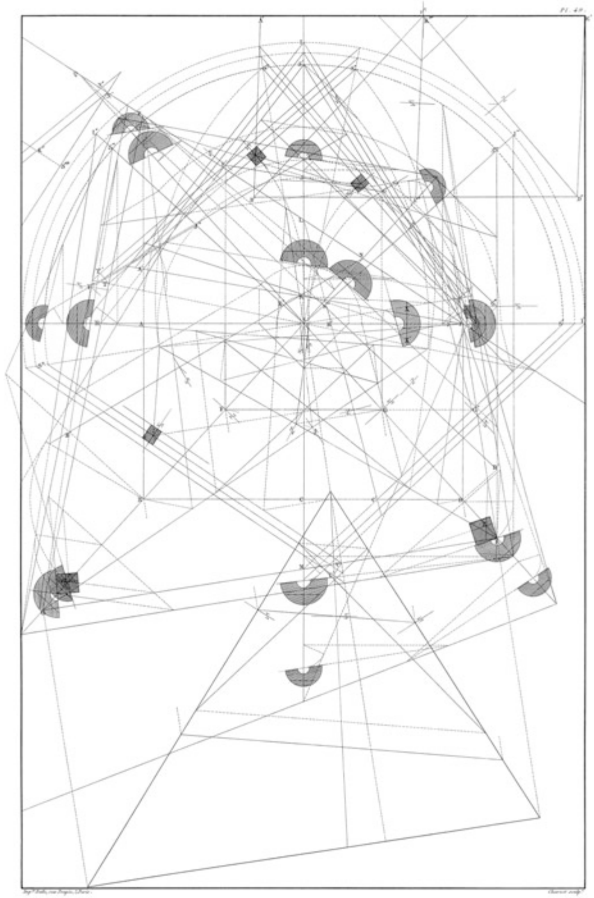

Arnaud Maggs, The Dada Portraits: Marcel Duchamp, detail, 2010 © The Estate of Arnaud Maggs / courtesy Stephen Bulger Gallery

is an integral part of the very beginnings of diagrammatic visualization. Howard Wainer explains how the roots of the grid in the data-based graphic visualization stretch all the way to 1400 BCE in geographical terms: "... a primitive coordinate system of intersecting horizontal and vertical lines that enable a precise placement of data points was used by surveyors of the Nile flood basin."[1] Then in 140 BCE Hipparchus mapped the skies with a coordinate system using longitudinal and latitudinal axes. Wainer tells us of Roman surveyors who used a "coordinate grid to lay out their towns on a plane."[2] He also reminds us of the chessboard originating

[1] Wainer, Howard. Graphic Discovery: A Trout in the Milk and Other Visual Adventures, (Princeton: Princeton UP, 2006), 10–11.

in seventh century India, the coordinate system of musical notations of the ninth century, and how the late seventeenth century sees the "earliest examples of printed graph paper . . ."[3] in the form of a standardized grid.[4]

This relationship between graphic display and the grid is present in Maggs's photographic œuvre. The grid pattern informs the arrangement of repetitive photographs from his earliest exhibition, *64 Portraits* (1976–1978), where anonymous models are captured in elegant mugshot-like poses and the photographs are arranged in an orderly fashion.[5] It is also the arrangement of his single-subject portraits such as Joseph Beuys or André Kertész: *100 Profile Views* (1980), in which photographs of the German artist staring forward are arranged in a grid with five rows of seventeen views and one with fifteen views giving the impression that the series is not finished.[6] *144 Views* (1980) capture the French photographer Kertész from a rotating perspective, as if the model was spinning on his chair at a slow pace, and display stilted dynamism in an arrangement that resembles a contact sheet. Apart from organizing the repetitive visual display of the photographs, the grid is also a characteristic part of the objects that Maggs photographs: agendas—Eugène Atget's lined pages of a client book is the subject of *Répertoire* (1997); receipts—*Les factures des Lupé* (1999–2001)—showing gridded and unfolded receipts of a nineteenth century family's ordered financial life; and ledgers—*Contamination* (2007)—a ledger notebook is photographed for the coral pink mold spreading across the predetermined horizontal and vertical lines like a slowly unfolding Rorschach flip book. The grid also appears through repetitive accumulations such as in *The Complete Prestige 12" Jazz Catalogue* (1991) (frames filled with numbers); *Hotel* (1991) (photos of hotel signs that say 'hotel' vertically); and *Travail des enfants dans l'industrie* (1994) (nineteenth century tags with notes and smudges). The repetition here is akin to what Craig Owens called an allegorical "projection of the structure as a sequence" which has for effect a counter-dynamic "static, ritualistic, repetitive" result.[7] Information conveyed through the pattern of the grid is an armature for information—it sorts data. In his survey of graphic visualization, Edward Tufte explains how a display with multiple repetitions of a variation on an image focuses the viewer on self-reflexivity both in the meaning of the image: "multiple images reveal repetition and change, pattern and surprise—the defining elements in the idea of *information*" as well as the medium's space: "Multiples enhance the dimensionality of the flatlands of paper and computer screen, giving

[2] Ibid.

[3] Ibid.

[4] Ibid.

[5] Langford, Martha. "Turning Colours," *Nomenclature* (Oshawa: the Robert McLaughlin Gallery; Winnipeg: Gallery One One One; Hamilton: McMaster Museum of Art, 2006), 16: "Structure (the grid), systematization (the neutral treatment), and copiousness (the number of numbers) formed a bridge between The Complete Prestige 12" Jazz Catalogue and the earlier cycle of work that had brought Maggs to international attention."

[6] Sutnik, Miai-Mari, "Portraits by Arnaud Maggs," (Toronto: The Power Plant, 1999), 111–112.

[7] Owens, Craig. The Allegorical Impulse: Toward a Theory of Postmodernism. October, 12 (1980), 72.

depth to vision by arraying planes and slices of information."[8] This repetitive display can be found in the art of other conceptual artists such as Sol LeWitt, whose works, especially in the 1960s and 1970s, often display grid patterns in two and three dimensions, or Hanne Darboven, whose works of documentation and repetitive bureaucratic archivist aesthetic often took the shape of walls filled with framed documents in a tight grid. Both these artists work with the transmission and display of information: "It is thus the epitome of counter-narrative, for it arrests narrative in place, substituting a principle of syntagmatic disjunction for one of diegetic combination. In this way allegory superinduces a vertical or paradigmatic reading of correspondences upon a horizontal or syntagmatic chain of events."[9] Creating, in effect, a diagram of relations based on an abstract grid. Owens's reading of the grid as visual instance of allegorical fragments is revelatory because it suggests that it is through art that the relationship of the grid to the philosophical diagram can become apparent since the diagram can be understood as a visual apparatus.

Maggs's conceptual art, seemingly austere, cold, calculated, holds up a naked diagram—an arrangement of abstract traits of a function that can be reconfigured from one system to another. Maia-Mari Sutnik acknowledges that vertical and horizontal interconnections in Maggs's works function through distance and presence: "Maggs's series of horizontal and vertical rows—the organizing grid—is a compelling composite in which each subject is distant, yet insistently present."[10] This paradox between distance and presence resembles Rosalind Krauss's formulation of the grid's centrifugal and centripetal properties.[11] Furthermore, Krauss touches the core trait of the grid, the temporal aspect—explained by Owens—and shows how it inhabits the structure through its genealogy.[12]

The time aspect points to a future, a potentiality. If Krauss located the grid's origins in the past—the nineteenth century saw artists using optics manuals in order to further their knowledge of colour and subconsciously absorbing grids indicative of the authority of scientific texts. In many ways, however, the diagram is antithetical to the grid yet the grid nears the diagram. Hubert Damisch, for example, sees the structure of the grid as yielding a potential that is associated with the diagram: "The grid provides a pattern which is quasi universal, while at the same time allowing for different uses."[13] By invoking his interest in graphic archaeology and how the grid is omnipresent across civilizations, he explains that the grid transcends its seemingly constraining parameters: "A grid, a chessboard, is not a structure, but the

[8] Tufte, Edward R. Visual Explanations: Images and Quantities, Evidence and Narrative. (Cheshire: Graphics Press, 1997), 105.

[9] Ibid.

[10] Sutnik, Portraits, 110.

[11] Krauss, Rosalind, "Grids," *October.* 9 (1979), 63.

[12] Krauss, Grids, 52.

[13] Damisch, Hubert. "Genealogy of the Grid." *The Archive of development*. Eds. Henk Slager, Annette W. Balkema. (Amsterdam: Rodopi, 1998), 51.

possibility of it."[14] Damisch explains how the grid, from its earliest manifestations is at the crossroads of historical becomings, a diagrammatic approximation open to potentials: "I am in search of a genealogy of the grid. A genealogy that starts with the procedures of divination: the way the ancient peoples and among them the Greeks played with pebbles and cards, and how they distributed them in a given space turned into a more or less regular scheme that led to the use of grid patterns."[15] Damisch sees divination as the art of potentiality: "Divination meant dealing with the future in terms of form and structure."[16]

The grid as a historical vehicle—for Krauss the scientific grids absorbed by art and formalized by the avant-garde; for Damisch, the more esoteric essence of the grid pointing towards its future functionality—can be aligned with Sanford Kwinter's determination of the diagram's role in a historical potential: "diagram gives us the power to program historical becoming . . ."[17] Kwinter, in order to visualize the shuttling motion of the diagram between the imaginary and the real, proposes to conceptualize the diagram as hammer as well as song.[18] The diagram, from an aesthetic point of view, has two sides: one rigid and one poetic. It is no wonder that Kwinter pinpoints the origin of diagrammatic aesthetics in Goethe's theories:

> From Goethe then, we were supposed to have learned that diagrams do not themselves produce form (at least in no classical sense of this word) but rather that diagrams emit formative and organizational influence, shape-giving pressures that cannot help but be "embodied" in all subsequent states of the given region of concrete reality upon which they act.[19]

Understood another way, the diagram is self-generating and self-reflexive, not formal but bending towards formalization. One should not forget the diagrams in the shape of grids in Goethe's theory of colour. Gary Genosko's explanation of Guattari's diagram reinforces Kwinter's originary concept of the Goethean diagram and its formalizing and organizational impulse:

> It is via diagrams that the passage from modeling to meta-modeling take place; this passage is none other than that of expression plane to content plane. The diagram's productivity entails that meta-modeling is productive of new references; it functions; forces things together; doesn't need meaning, just the manufacture of it.[20]

The passage-function of the diagram, as Kwinter is going to explain through Deleuze and Foucault, is akin to another concept—that of Kant's schema:

[14] Ibid.

[15] Ibid.

[16] Ibid.

[17] "Diagrams must be conceived as songs as well as hammers," Sanford Kwinter, "The Genealogy of Models: The Hammer and the Song," *ANY: Architecture New York* 23(1998), 62.

[18] Kwinter, Genealogy, 62.

[19] Kwinter, Genealogy, 58.

[20] Genosko, Gary. *Félix Guattari: A Critical Introduction*. (London: Pluto Press, 2009), 11.

> The role that the concept of diagram is now playing in our attempts to theorize material reality in the late 20th century is not so different from the way the concept of the "schema" was used by Kant to theorize Newtonian reality in the late 18th century. Both seek to serve as synthetic explanatory devices (though they are no less real for that) that open up a space through which a perceptible reality may be related to the formal system that organizes it, whether this latter is a priori or a posteriori as in the Kantian/Humian version.[21]

The Kantian schema converts objects of intuition into concepts, it bridges the incommensurability between empirical objects and pure understanding. The schema is a mediating representation that shuttles between the sensible and the intellectual. This passage between empiricism and understanding is done by the schema by providing an image to the concept, an image, that is a bit sketchy, an outline, a monogram. Its rendering is streamlined but it carries more possibility of information.[22]

The Kantian schema is very interesting here since it is something that will produce the functionality of passage (later, connectivity) that I want to explore through Deleuze's diagrams in Maggs's abstract structures present in two of his images.

The first one is from his series *Dada Portraits* (2010), specifically his portrait of Duchamp. The second is photograph of a page of the manual of the geologist Abraham Gottlob Werner titled *Werner's Nomenclature of Colour* (2005). The source of the first work, *Portrait Dada Marcel Duchamp,* is a carpentry manual that was published in mid-nineteenth century, a copy of which Maggs purchased while travelling to Paris a few years before his death in 2012. The manual, *Charpente Générale: Théorie et pratique* was written by B. Cabanié and published in 1864.[23] The intertwining of sober and precise lines, the abstraction of these technical illustration and the complexity of the visual information were off-putting to Maggs at first, but their promising function became apparent: they resembled all to portraits of dada artists, all he needed to do was to apply a name to the face: "By merely naming the abstractions of the drawings' exploded lines, he configures a series of portraits. Profiles render Dada men such as Marcel Duchamp, Max Ernst and Man Ray; frontal views depict Dada women such as Sophie Täuber, Hannah Höch and Mary Wigman."[24] This naming is a paradigmatic association upon a syntagmatic activity: "... he *reads* these found objects differently—but not for traces of their stories. Now he makes one up instead."[25] I will analyze the paradoxes of these non-representational diagrams by focusing on a text by David Joselit where he analyses diagrams of artists like Man Ray and Francis Picabia through the theories of Brian Rotman dealing with mathematical diagram aesthetics.

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[21] Kwinter, Genealogy, 57.

[22] Zdebik, Jakub. The Manifold Dimensions of Janice Kerbel's Architectural Diagrams, in Isabelle Wallace and Nora Wendl (eds). *Architectural Perspectives in Art*, (London: Ashgate, 2012): 296.

[23] I would like to thank Katyuska Doleatto from the Estate of Arnaud Maggs and Spring Hurlbut for providing this source.

[24] Monk, Philip. "Elegiac Pantomime: Arnaud Maggs After Nadar," *Canadian Art* (2014), 88.

[25] Ibid.

The other work is a photo of an organizational grid that comes from the nomenclature book by Werner first published in 1814 and then comprehensively titled *Werner's Nomenclature of Colours: with additions, arranged so as to render it highly useful to the arts and sciences, particularly zoology, botany, chemistry, mineralogy, and morbid anatomy: annexed to which are examples selected from well-known objects in the animal, vegetable, and mineral kingdoms so as to indicate its uses*. The indexicality of the photo captures this scientific work that projects its intended objectivity trough the grid. In order to analyze this work, I will look at the writings of Hubert Damisch on the structure and form of grids and Hannah Higgins on the relationship between art and functional grids. Furthermore, the taxonomies evaporate in subjective approximation and quasi-poetics in Werner's grid, I will also explore the idea of the optical subconscious that Rosalind Krauss discovers in the grid, as she traces a historical trajectory from nineteenth century optics manuals to avant-garde abstractions at the beginning of the twentieth century.

What is the difference between a carpentry diagram presented as a portrait of a famous Dadaist and a photograph of a grid explaining the colour equivalencies in the animal, vegetable and mineral domains? One is an unsettled grid pattern that rests into a portrait while the other is a grid liberating a *poesis* through its mediation as a photograph. One is an abstract diagram of connectivity (paradigmatic, to use Owens's terms indicating a mutually exclusive relationship between terms, in this case the technical drawing, the name and the portrait designation), while the other is an abstract machine, a diagram of potentiality (syntagmatic, again in Owens's terminology involving a sequential relationship that is made manifest by the technical grid's release as photograph). And they both illustrate the potentially paradoxical diagrammatic impulse of the grid.

According to Philip Monk, the regimented patterns of the photographic portraiture displays of Maggs's early works is reconfigured in a new format in his *Dada Portraits*: "... their graphic structure" is "haunting the surrounding 'conventional' portraits ..."[26] Monk suggests that their "gird-like solidity dissolves into ghostly demarcations."[27] The grid patterns transcend one photographic system to reconfigure themselves in a new system, or as Monk puts it: "All that is solid melts into air, to be recomposed in a new understanding of the ephemerality of Maggs's enterprise."[28] It sounds suspiciously like the diagram working syntagmatically by connecting between heterogeneous works through the grid, focusing not on the subject nor the form but the meta-modeling that takes place between the works, and invoking the notion of passage: the atomizing into traits of the essential function of a work and reconfiguring it in another milieu.

The elements that haunt Maggs's early works are present in Deleuze and Guattari's definition of the diagram. Deleuze and Guattari's diagrams are particular types of images: assembled heterogenous elements, they are not representational

[26] Ibid.

[27] Ibid.

[28] Ibid.

of a particular form and essentially, they are instructions that shuttle between the idea of an object and an actual object. It is fruitful to look to Christine Buci-Glucksmann, who sees this diagrammatic function in the grids of paintings like those of Vermeer,[29] and who distinguishes between two versions of the diagram: the Foucauldian and the Baconian.[30]

The function of the diagram is traced by Deleuze to the use of the term by Foucault in *Discipline and Punish* to describe how Jeremy Bentham's Panopticon, a prison system which involves a central tower with a single guard who can observe floors of stacked cells that are opened to light from both sides such as boxes ticked with dots. This prison does not function through a system of brick, mortar and steel but is distilled into its primary function—visibility. The grid aspect of the visual apparatus might not be immediately obvious in the prison whose dominant shape is circular. But in military camps, the grid pattern cannot be denied. It is the use of the term diagram in relation to army camps that also brings it clearly to a relation with the grid that shape the camps illustrating Foucault's theories.[31]

Traditionally, the diagram is not an image that is made to be contemplated for itself; rather it is a series of practical instructions that guide us towards the potential realization of an object. Instead of representing something in an illusory fashion, the diagram represents something that does not yet exist. It is at the crossroads between an idea of an object—its intellectual representation—and the produced actual object. At this transitional stage, the plan remarks reflexively on the nature of representation itself.

The second instance of the diagram, the Baconian diagram pinpointed by Buci-Glucksmann, the passage of pure function is still present but here, it is visible in the form of traits, even if they are designating something abstract and hard to visualize.

A transitional site between abstraction and representation such as the one found in art is described in *A Thousand Plateaus*. Posited within the pages of that book is the peculiar position of the abstract machine between pure abstraction and concrete articulation.[32] The abstract machine can be read as a work of art that imbibes from a heterogeneous source and is on the way to become a recognizable object: "Abstract machines consist of *unformed matters and nonformal functions*. Every abstract machine is a consolidated aggregate of matters-functions (*phylum* and *diagram*)."[33] This device is not meant to be understood as concretely as, say, a particular painting. But a painting, like one by Bacon, can be an illustration of the principle at work of an object-image standing between the abstract and the concrete if we are fully aware that by using a painting as an example we are firmly standing within the

[29] Buci-Glucksmann, Christine. "On the Diagram in Art," *ANY: Architecture New York*, 23(1998), 35.

[30] Buci-Glucksmann, On the Diagram, 34.

[31] Foucault, Michel. Surveiller et punir: Naissance de la prison, (Paris, Gallimard, 2004), 202.

[32] Deleuze, Gilles, and Félix Guattari. *A Thousand Plateaus: Capitalism and Schizophrenia II*. Translated by Brian Massumi. (Minneapolis: University of Minnesota Press, 2005), 510–14.

[33] Deleuze and Guattari, A Thousand Plateaus, 511.

formalized side of the abstract machine (the unformed heterogeneous side, or the virtual element of the abstract machine, is non-representational, therefore not possible to illustrate in expressive terms nor accessible to concepts). It is important to note that Bacon' subject matter consists of images only dovetailing figurative art. The nature of the image in these paintings consists of passages between modes of representation—abstract machines rendered visible.

In painting, Deleuze sees the diagram in the zone of indiscernibility: he is able to give us an example of this in Bacon's 1946 *Painting*. He explains how Bacon set out to paint a bird but ended up with an umbrella shading a Mussolini-figure flanked by a spread-eagle meat arrangement. The intended bird had taken flight and come to rest in another figure: "The diagram-accident has scrambled the intentional figurative form, the bird: it imposes nonformal colour-patches and traits that function only as traits of birdness, of animality."[34] Those traits are still present in the umbrella, the figure, and the wing-like opening of the meat. The peculiarity in this painting is that we can witness the diagram: it "can be found, not at the level of the umbrella, but in the scrambled zone, below and to the left, and it communicates with the whole through the black shore."[35] We are here close to Monk's description of the grid's function throughout Maggs's oeuvre. We read the painting through its traits, a map with a black shore from which spreads the diagram.

This function of passage will turn into the notion of connectivity as our examination of grids as diagrams continues and it will be our guiding thread in the analysis of Maggs's works.

Grid as Diagram in *Portraits*

A sweeping poetic interrogation of the diagram by Deleuze in the context of art shows the expansive terrain the concept covers:

> What can be said, first of all, of that invisible force of coupling that sweeps over two bodies with an extraordinary energy, but which they render visible by extracting from it a kind of polygon or diagram? And beyond that, what is the mysterious force that can only be captured or detected by triptychs? It is at the same time a force (characteristic of light) that unites the whole, but also a force that separates the Figures and panels, a luminous separation that should not be confused with the preceding isolation. Can life, can time, be rendered sensible, rendered visible?[36]

First of all, Deleuze here compares a diagram to a polygon, giving the abstract mechanism a two-dimensional rendering. The time aspect is important too, bringing us back to what Damisch, Kwinter ("It is in time, I would argue, where the

[34] Deleuze, Gilles. *Francis Bacon: The Logic of Sensation*. Translated by Daniel F. Smith. London: Continuum, 2003, 126.

[35] Deleuze, Bacon, 126.

[36] Deleuze, Bacon, 63.

diagram operates"[37]), and Genosko see as the potential and dynamic expansion of the diagram, or, more specifically, in Guattari's *la grille*, a timetable spreadsheet operating like an abstract machine. However, the overall description is one of the aesthetic forces that are present when there is a multiplicity of works (diptych, triptych) which can easily translate into Maggs's multiple arrangement in grid pattern of his works. As Sutnik states, the grid reveals a multiple viewpoint of the subject of the work: "His system of assembling photographic grids is to reveal something of the human subject that one single portrait view could not fully accomplish"[38] More forcefully, Sutnik suggests that the grid itself is the apparatus that bears on seeing: "The modular grid structure, as much as the camera, becomes a tool for expanded seeing."[39] We can think of the viewfinder framing grid that flattens the space beyond the lens. Can it render the sensible visible?

Deleuze goes on to give a primary function to the diagram its meta-modelling; it is approximate and connective: "The diagram is thus the operative set of asignifying and nonrepresentative lines and zones, line-strokes and color-patches. And the operation of the diagram, its function, says Bacon, is to be 'suggestive.'"[40] Very simply put, this diagrammatic suggestiveness is the engine fueling Maggs's *Dada Portraits*. The grid here as a tool has an unspecified function. The function is the defining parameters of the diagram as Deleuze explains. Let us remind ourselves that the *Dada Portraits* are carpenter's diagrams that have been given a person's name. The connection between a name and a gridded diagram reveals the function of the diagram through a work of art, a portrait. This "portrait" needs to be addressed as are the diagrammatic elements it contains.

Faciality of *Portraits*

But what we are dealing with is a photo of a grid. In *A Thousand Plateaus*, Deleuze and Guattari don't have a very positive view of photography. Nevertheless, the photo and the tracing are necessary to capture the pure potentiality of the diagram: "The difference between them is not simply quantitative: short-term memory is of the rhizome or diagram type, and long-term memory is arborescent and centralized (imprint, engram, tracing, or photograph)"[41] The function of the photo, here put on the same plane as imprints, engrams and tracings is to capture and fix. These are contrary to the map, that functions rhizomatically. But it is a necessary part of the "diagram type" so that it can communicate information. In another part of the book, Deleuze and Guattari explain how the tracing's functions is contrary to the map

[37] Kwinter, Genealogy, 60.

[38] Sutnik Portraits, 109.

[39] Sutnik, Portraits, 109.

[40] Deleuze, Bacon, 101.

[41] Deleuze and Guattari, A Thousand Plateaus, 16.

because it is a reproduction: "It is instead like a photograph or X ray that begins by selecting or isolating, by artificial means such as colorations or other restrictive procedures, what it intends to reproduce."[42] This notion of reproduction is important because even if the diagram is unreproducible, in order for it to be communicable, it has to be somewhat part mapping/part tracing.

Are Maggs's portraits reproductions? Rather appropriated ready-mades. Maggs's portrait of Duchamp's is a readymade in the way that the function of the grid is already present in the *Portrait* according to Deleuze and Guattari's concept of faciality: "[The faciality machine] carries out the prior gridding that makes it possible for the signifying elements to become discernible, and for the subjective choices to be implemented. The faciality machine is not an annex to the signifier and the subject; rather, it is subjacent (*connexe*) to them and is their condition of possibility."[43] Faciality is doubled: it is connective and subjacent — a gridded undercoat of potentiality.

Tom Conley explains that faciality is the crux of subjectivation and significance where subjectivation deals with a living being negotiating with milieu and significance. Here subjectivation and significance are apparent in Maggs's portrait when the connection between gridding and name are possible because of "signs that disseminate infinite meaning" and that are "not under the control of language rules."[44] In fact, the elements Conley isolates in his definition of faciality resemble elements of Deleuze's Baconian diagram: "Subjectivation and signifiance are correlated, respectively, with the 'black hole' or unknown area of the face in which the subject invests his or her affective energies (that can range from fear to passion) and with the 'white wall', a surface on which signs are projected and from which they rebound or are reflected."[45] The white wall is the canvass from which clichés must be erased and the black hole is the diagrammatic zone of indiscernibility. What is important here, in Conley's definition, is that there is a zone of subjective mediation in the interpretation process, that which connects the carpenter's plan with an artist's name.

The face structure is a carpentry drawing, in fact, Deleuze and Guattari stress the relationship of possibility through the redundancy that is the binary obverse of the cartographic principle of rhizomatic understanding. These notions and their complexities are repeated in the concept of faciality: "Facial biunivocalities and binarities double the others; facial redundancies are in redundancy with signifying and subjective redundancies."[46] In fact, Deleuze and Guattari provide a diagrammatic understanding of the facial function by explaining how it operates as an abstract machine: "It is precisely because the face depends on an abstract machine

[42] Deleuze and Guattari, A Thousand Plateaus, 13.

[43] Deleuze and Guattari, A Thousand Plateaus, 180.

[44] Conley, Tom. "Faciality," *The Deleuze Dictionary.* Ed. Adrian Parr (Edinburgh: Edingburgh UP, 2010), 101.

[45] Conley, Faciality, 101.

[46] Deleuze and Guattari, A Thousand Plateaus, 180.

that it does not assume a preexistent subject or signifier; but it is subjacent to them and provides the substance necessary to them."[47] The diagrammatic armature is closer to Maggs's portraiture strategy, it is not the subjects that pick faces but rather: "it is faces that choose their subjects. What interprets the black blotch/white hole figure, or the white page/black hole, is not a signifier, as in the Rorschach test; it is that figure which programs the signifiers."[48] A variation of making the sensible visible of the diagram, faciality is interconnecting visibilities that hide invisibilities: "Faciality, the Deleuzian concept of the signifying process, is the assemblage of the visible that conceals the invisible."[49] The viewer's intermediary role is crucial in the reading of the face: "The viewer disassembles the invisible signified and then reassembles it according to the viewer's signification structure."[50] At the crux of combinatory activity, the viewer is the zone of indiscernibility: "The viewer accumulates this combination of assemblage and disassemblage until they stop viewing." But in this case, the viewer also has a role in the overall recognition of significance of the reassembled object: "Consequently, faciality is the viewer's impression that results from the mutual re-assemblage of invisible signification systems."[51] We can see how through faciality, it is visibility as a function that is able to transition between milieus. So in this paper on grids, the concept of faciality brings us back full circle to Foucault and visibility.

Grid and Virtuality

In the diagram, the virtual zone stands between the disassemblage and reassemblage of traits of a pure function of a system. As Charles Sanders Peirce explained, the diagram has an important facet: the crucial aspect of the diagram, is its function of connectivity.[52] And this connectivity was already explained in the subject/face relationship.

The diagram's superiority over the icon stems from the hybrid and multidimensional terrain it covers. It also includes spectators and their experiences. On a purely graphic level, the diagram's capacity to adapt comes from the way the lines and the curves reach equilibrium between the large quantity of information and the economy of design it transmits.

[47] Deleuze and Guattari, A Thousand Plateaus, 180.

[48] Deleuze and Guattari, A Thousand Plateaus, 180.

[49] Kim, Younggeun and Joonsung Yoon. "A Grin without a Cat: The Faciality of Electronic Art" *Leonardo*. 44.3 (2011): 288.

[50] Kim, Grin, 288.

[51] Kim, Grin, 288.

[52] Zdebik, Jakub. Future Archaeology: The Speculative indexicality of Adrian Göllner's Conceptual Artefacts. *Revue d'Art Canadienne/Canadian Art Review*. (2017) 42.1, 48–63.

Genosko explains how models, or coordinates of thought, are put into motion by diagrams. He tells us that the diagram is meta-modelled—it is not representational (here it has some of the function of Kant's schema: "... mappable yet a bit sketchy ..."[53]). The diagram cannot be represented, it is virtual: "Diagrammaticity is like a slice of chaos released by a meta-model and which opens up a new universe through which it coils itself along, like a virtual worm."[54] So how do we see the diagram in the domain of art? In the domain of art, Buci-Glucksmann explains how the virtuality of the diagram shapes the space of the work: "In essence, the new pictorial abstraction reinscribes the powers of the technological virtual into painting by creating heterogeneous spaces, multiple connections, disconnections, and undecidable zone."[55] The answer is connectivity. Genosko circles back to Foucault's panoptic grid which he plugs into the *grille:* "Deleuze emphasized that, understood abstractly, panopticism in Foucault is a diagram, an abstract machine, a spatio-temporal map of *relations between forces* immanent to the fields of their application. Any timetable is a diagram's concrete assemblage of time and task in a given institutional matrix. And at first glance the grid fits into this schema."[56] Buci-Glucksmann explains the grid's relationship to the diagram: "To be certain, the diagram as '*relationship of forces*' implies an abstract machine that grids the social and engenders an 'intersocial in the making.'"[57] As Genosko explains, the abstract machine ("contact between form-matter unmediated by representation; that is, it is not anchored by an anterior referent, and it's irreducible to a finished substance"), just as Buci-Glucksmann sees it, accounts for the dynamism in a grid: "It is important not to freeze the grid in a synchronic slice, stripping it of all dynamism and diachrony in order to fit it into the timetables described by Foucault that contract into smaller and smaller units, and thus greater and greater control over bodies stripped of the ability to innovate, indeed, even to move freely."[58] This verticality seen in the time-grid is precisely part of the dada diagram explained by Joselit from the perspective of a cubist gridding and then a machine-like drawing superimposed with writing.

Dada Diagrams/*Dada Portraits*

Dada diagrams originated with cubist facets, according to Joselit. With Cubism, we are shifting from sequence to simultaneity and, as Marshall McLuhan reminds us,

[53] Genosko, Guattari, 11.

[54] Genosko, Guattari, 11–12.

[55] Buci-Glucksmann, On Diagrams, 36.

[56] Genosko, Guattari, 57 (Italic's mine).

[57] Buci-Glucksmann, On Diagrams, 34 (Italic's mine).

[58] Genosko, Guattari, 58.

once this happens, we are aware of structures and configurations.[59] In dada art, the grid, a vestige of the cubist facet represents time in a static medium and shows cubist simultaneity replaced with diagrammatic relationality: “In other words,” Joselit writes, “the diagram reconnects the disconnected fragments of cubist representation. This reconstituting act does not function as a reestablishment of coherence but rather it is a free play of polymorphous connections that, until this day, is the main motif of modern and postmodern art.”[60] In a similar way, Maggs makes the link between heterogeneous elements: portraiture and carpentry, or even a technical diagram and an individual portrait, or an explosion in a shingles factory and a face.[61]

But the diagram is an antithesis of the rigid grid: “The diagram is unstable, formless, and fluctuating, subject to ‘micro-movements,’ variations, and points of resistances.”[62] Yet, at the level of the dada art, the relationship between diagram and grid resides precisely in the liminal state between fluctuation and rigidity. Taking as an example of dada drawings in which ambiguous (not to say abstract) contraptions (not to say machines) are made up of grids and writing which convey an image as an assemblage of diachronic and synchronic elements: time and space multiply the polyphony of sensations made visible, as Deleuze mentioned in his book on Bacon, underlying relationality: “it emphasizes pure relationality between things rather than directly assaulting their objectivity. Diagrammatic visuality produces an interstitial space . . . ”[63] This interstitial space, which can be observed in Maggs’s grid patterns, resides somewhere between carpentry and art history.

Between carpentry and art history, a fissure is created along the two disciplines.[64] According to Joselit, this division can be summarized as that between science and imagination: the diagram stands at the crux of this duality. For scientists, the diagram is not formalized enough and for artists, the diagram is associated with an unwavering belief in the possibility of a universal objective truth.[65] But a fissure is also present in *Werner’s Nomenclature* since here the distinctly poetic tones of the description belie the austerity of the scientific grid which organizes them. This division is what brings us back to visual diagrams in the scientific domain and how art history can interpret them.

[59] Marshall McLuhan. “The Medium is the Message.” Understanding Media: The Extensions of Man. (New York: Signet Books, 1964), 28.

[60] David Joselit, “Dada’s Diagram”, The Dada Seminar, ed. Leah Dickerman and Matthew S. Witkovsky, (Washington, 2005), 232.

[61] The “explosion in a shingles factory” was used by a critic of Duchamp to characterize *Nu descendant un escalier* (1912).

[62] Buci-Glucksmann, On Diagrams, 34.

[63] Joselit, Dada’s Diagram, 234.

[64] Joselit, Dada’s Diagram, 236.

[65] Joselit, Dada’s Diagram, 236.

Routes of Reference: Technical Diagrams as Art

This relationship between visualization, art and information is the kernel of James Elkins's "Art History and Images That Are Not Art" and his negotiation between the expected objectivity of scientific images and the way that art history offers a methodology to decipher the diagrammatic aspect of informational images.

Elkins connects art and information via a two-way thoroughfare. First, he looks at the established relationship between art and science. Then, he looks at complex, opaque, and arcane scientific images to read them through art historical methodologies. One of the two directions is made in relation to visual art history. Here, artists are inspired by science for the aesthetic value that scientific imagery brings to art while not being wholly cognizant of the complexities of science, a gesture he deems Romantic (Poe, Redon, Kandinsky, Picasso, Ernst, Duchamp, etc.). In all fairness, science is also accused of unsophistication in relation to art, as when Mandelbrot states that his fractal models resemble minimalist paintings.[66] In another instance, Elkins links non-art images inherently to the history of art, such as medical images from the Renaissance or computer graphics that seek to perfect the mimicry of nature. Finally, he mentions that one of the relations between scientific images and art is that of illumination and inspiration (maps, panoramas, botanical illustrations, etc.).[67]

In X-ray crystallography, for example, "shining X-ray through [a] crystal" and capturing the refracted light results in another black and white "picture" of questionable aesthetic value. A "picture" (in quotations since it looks more like a clumsy black and white imprint of a doily) of a crystal contains several superimposed annotations; the central pattern is indexical of the X-ray refracting through the crystal, resulting in a concentric pattern of semi-circular dustings of dots. The initial pattern is then reconfigured and superimposed in straight lines emanating from the center, following crisscrossing axes; this is finally annotated on a grid. Here, Elkins reads into this repetitive coding a flattened perspective of Neoclassical painting: "from Neoclassical modes of picture making to spare convention of descriptive geometry, and from the austerities of crystal symmetries to the techniques of scientific engraving"[68] constitute the range of influences that can be read into this image. Images that carry information can be read both subjectively and objectively, bringing the diagram square in the middle of the science vs art duality.

This leads Elkins to break down ways of looking at informational images through an art historical methodology—two zones of knowledge that are hard to reconcile. If a Renaissance sketch and a Renaissance painting by two different artists can nevertheless resemble each other, it is more difficult for a non-initiated observer

[66] Elkins, James. 'Art History and Images That Are Not Art', *The Art Bulletin*. 77:4, 1995, 556.
[67] Ibid., 557.
[68] Ibid., 560.

to see the resemblance between a DNA autoradiograph ("recording the seepage of chemicals through a viscous jelly"[69])—the familiar image from court-room television dramas consisting of short lines of various grey value distributed along columns—and a sketch tracing the steps to determine the inconsistencies within the image in need of edits. These 'routes of references' between initial visual data annotation and its visual clarifying organization are harder to find for the uninitiated in scientific images: such as a sonar chart in which fish are illustrated with umbrella-like patterns of various sizes or even patterns inscribed in 5000-year-old clay tablets. These routes of references are what we are witnessing in Maggs's works: we have to be privy to Dadaist aesthetics to be in on the joke of the *Dada Portraits* which are not directly dependent on the carpentry structure. As for the grid of the nomenclature of colours, the scientific document, to interpret the object as it is intended, we must appreciate its aesthetic dimension underscored by the photographic medium that highlights this very quality.

In both cases, scientific and aesthetic, Elkins suggests, "art history could help elaborate the working concepts such as picture, decoration, landscape, and naturalism and the history of science . . . could elucidate how the routes of reference are combined." A complicated relationship between art and science is seen through more examples of issues of interpretation based on knowledge, manifested through diagrammatic annotations of atomic particles which then turn out to be taken for anexact of illustrations of phenomena (Feynman diagrams) or through Sewell Wright's choice of a visual convention in topographic mapping to illustrate the dynamics of genetic evolution which is then repurposed for an altogether different field of inquiry.[70] Here we can fit Maggs's repurposing of Cabanié's carpentry manual and Werner's geological reference book. Elkins provides an example of a picture become theory, like Erlich diagrams of how cells defend themselves against toxins, the patterns from which were then used in later studies of immunology,[71] and finally he considers the notion of unrepresentability as when a single virus can be illustrated in wildly varied ways, each capturing an accurate facet of the object: "In more radical terms, what is unrepresentable can *never* be adequately put in an image because it is nonpictorial, unimaginable, forbidden, or transcendental."[72] I will stop here at Elkin's notion of unrepresentability and refocus our efforts on images that are based on information in the realm of art.

Maggs's *Duchamp Portrait* seems to be an illustration of pure connectivity without a recognizable object upon which function should converge, instead it is suspended in a holding pattern in a zone of incongruence between the line and the title. It is as if the functional traits of Bacon's paintings that atomize themselves and take off from one system to fly over to another heterogeneous system would never have landed. Here we have the connections that allow us to maybe imagine

[69] Ibid., 562.

[70] Ibid., 565–566.

[71] Ibid., 567.

[72] Ibid., 568.

the lines of a face, the way Maggs did himself when he was leafing through the source document for the first time, according to the title, the name and the textual aspect that is added.

The *Dada Portraits*, a diagram that plays with the rigid elements of the carpenter's gridded designs, reflects the diagrammatic aspects of Deleuze and Guattari's notion of faciality, "the visible concealing the invisible," it makes us reflect on what the virtual "looks like," by playing of the old dada invention of abstract, technical designs and seemingly mismatched words, which all leads to the routes of reference that connects art to scientific illustration.

Nomenclature as Poetic Potential

Maggs's *Werner's Nomenclature* is quite different: here the photograph is obviously that of a page from a book, the grid is discernible and functional, the title of the work does nothing to be playful, instead explicitly naming the subject of the work. And yet, the interconnectivity between the various milieus where the colours can be found creates a centrifugal effect of casting the grid far beyond itself.

Hannah Higgins, in her book on grids, makes an interesting connection between functional, pragmatic grids and their use in art. The grid pattern of the ledger and that of the gridded screen allowing for the construction of perspective in painting, are both devices that mitigate information: "Indeed, both the ledger and the screens of perspective paintings are grids that create spatial balances among many kinds of information."[73] In fact, she connects the vanishing point of perspective to empty space in the ledger:[74] "Like perspective painting, ledger entries balance entries on either side of a space that is disarticulated—filled by an object in the painting or left blank in the ledger."[75] In a passage reminiscent of the defining attribute of the diagram in Foucault and Deleuze, the analogy between the ledger and the painting remains at the level of pure function in both organizational visual systems: "In both, diverse pieces of information are made to appear homogenous in the geometrized space of the field."[76] Krauss also explained the presence of the grid screen in the picture plane of the Renaissance. But it is the direct representation of the grid, not one that is purely functional and that will be painted over, that interests her in the twentieth century, as she explains in her 1979 essay "Grids." This avant-garde grid comes from the optical-subconscious absorption of functional grid-patterned visualization of scientific information on the subject of optics by painters in the late nineteenth century and emerges as the grid-as-subject of avant-garde abstraction.

[73] Hannah B. Higgins, The Grid Book, (Cambridge, MIT Press, 2009), 136.

[74] "Alberti's geometric formula for calculating perspective in terms of a vanishing point is structurally analogous to the empty space between ledger entries Paciolo required," Ibid.

[75] Ibid.

[76] Ibid.

In this case, we could also make a correlation between Higgins rapprochement between the ledger and the perspectival grid and the equally functional grid of the scientific treatise (like for example that of Goethe) and the works by Mondrian, that are perhaps the most well-known grid-patterned paintings. Damisch also sees Mondrian's picture plane as a diagram-function (transposing functions from one system to another, in this case the painting, memory and reality). Damisch writes that Mondrian's paintings have a very precise function, "which is to impress upon a visual memory a spatial organizational schema that would function as a *grid*, that would in turn be transferred onto the world to inform anew."[77] According to Damisch, this turning away from nature is part of the artist's ordering impulse that controls nature in order to define an exact world. This turning away from nature is also how Krauss saw the grid in art.[78] But in another essay in which Krauss writes about the grid, she puts it on the subjective side of the objective/subjective duality spectrum. She explains that the grid, in its various manifestation in painting, whether it is the pointillist "all-over network" of grids which resemble "light hitting the retina," or the window lattice as an entry for light, "the grid occupies a subjective rather than an objective pole, and it is in this sense that it becomes the perfect vehicle for the mapping of opiticality, or what came to be labeled 'vision as such.'"[79] Krauss, in this 1995 essay on Piero della Francesca's *The History of the True Cross* (1447–1466), explains how the grid erases time and sequence: "The grid's structure as pure simultaneity, erasing everything sequential even as it pressed out the last vestiges of shadow or tactility, made it in this sense the treasured emblem of the visual."[80] We are back to our early definition of the grid as arresting sequence (Owens) and presence/absence (Sutnik), or, in this case, the absence of presence. Finally, Krauss settles on the grid's quality of structure which for her is an interconnection of both the subjective and the objective.[81] Interconnectivity is important to dada diagrams, as explained by Joselit. The connective function is something both the grid and the diagram share, even if circuitously. Taking issue with Krauss's notion of structural definition of the grid (the 1979 essay and not the 1995 one), Damisch follows her up to a point especially in the way that the grid decenters the focus on the plane: "The grid does more and goes further than just breaking with the axial and centered model of the root-tree. The regime of the grid is of another order, of another level than that of the sign."[82] Damisch pushes the grid onto the level of the diagram by explaining semiotically how the

[77] Damisch, Hubert, *Fenêtre jaune cadmium ou les dessous de la peinture*, (Paris: Seuil, 1984), 55–56 (Translation's mine).

[78] "It is what art looks like when it turns its back on nature." Krauss, Grids, 50.

[79] Krauss, Rosalind. "The Grid, the True Cross, the Abstract Structure," *Studies in the History of Art*, 48 (1995), 308.

[80] Krauss, True Cross, 308.

[81] Krauss, True Cross, 308. "For the grid to operate as a structure, it must put into play both these aspects—subjective and objective—in an interconnection . . . "

[82] Damisch, Hubert. "Remarks on Abstraction," *October*, 127 (2009), 153.

grid operates. The perspectival screen that Krauss touched upon, but ultimately turned away from, in her essay in order to focus on the outward subject of the grid, is an "armature of the organization" of represented reality and therefore still holds up the relationship between the real and the artificial, contrariwise to the twentieth century avant-garde abstract grid opposing the relationship between the painting and the world it represents.[83] It is precisely this perspectival grid that Damisch wants to focus on since it offers a diagrammatic space to the picture, an abstract function that is the armature of a potential image: "In the manner of the form called "chessboard," the grid works as a "support," an "under-board"—underpinning *comme en sous-jeu*—of the picture."[84] In that state, Damisch explains that the grid is: "Less structure than form (form as Wittgenstein stated it, which is the condition of possibility for structure). Less icon than hypo-icon, in the sense Peirce understands."[85] Which chimes with the statement that a grid is a potential/virtual structure. Pierce's hypoicon is composed of two different elements: the image and the diagram. Here, the diagram's function is that of connectivity which was issued at the beginning of this discussion on the perspectival grid with its connection to the ledger and their informational functions, it is also present in the interconnectivity of the objective/subjective grid structure and, finally, dissolved in the pure functionality of the diagram.

Maybe this tension can be neutralized by the coexistence of either diagrammatic poles in its formalization as a grid, or matrix open to a function.

This relation between science and the diagram is observed by Krauss in the grid as a devise that is equally scientific and spiritual. In her example, Krauss demonstrates how grids from optical treatises like the ones by Chevreul, Rood, Helmhotlz or even Goethe are absorbed subconsciously by art.[86] These grids, at first purely functional, obtain a symbolic function in the representation of grids as windows in symbolist art. The functional purity is finally liberated in these abstract representations at the beginning of the twentieth century.

Krauss makes the link between optical science of the nineteenth century, the works on mythologies of Claude Lévy-Strauss captured in a grid and the grid of symbolist style that finally hatch in the form of abstract grids in twentieth century painting. Krauss describes her methodology as an etiology, or the study of the causes of a malady. But it is rather the transversal plan of her analysis, that shatters the fourth wall and puts on the same footing different ways of seeing the grids across ages, cultures, disciplines that render the grid, trait upon trait, in fact a function that crosses milieus (in the deleuzoguattarian sense) and lands upon the work of art in all of its spiritual and geometric dimensions—making it, in effect, a diagram.

[83] Krauss, Grids, 52.

[84] Damisch, Remarks, 153.

[85] Damisch, Remarks, 153.

[86] Of course, Maggs also made a series of works based on Chevreul's chromatic diagrams of colours.

The time aspect is part of the photograph's of the grid: "A book, or a scientific demonstration, that we should normally apprehend diachronically is represented synchronically, resulting in a first impression of complexity and mutability in which we are invited to play a part."[87] Take for example Maggs' work dealing with the colour green. Apple green can be found "Under the Side of Wings of a Green Broom Moth." Or that Emerald green is located on the "Beauty Spot on Wing of Teal Drake." And, as Martha Langford explains, it is the very thing that the photograph of organizational grids allows us to witness in Maggs work:

> *Werner's Nomencalture* presents the colours of the natural world as an orderly linguistic system: [the colours] remain within their squares, only pointing at something outside the system of mechanical reproduction that has produced the handbook.[88]

We have come back to optical manuals, the sources of the artistic grids described by Krauss, but here they are appreciated for their esthetic side, like a work of art, framed and mounted. The photo bears witness to the sensuality of the diagram. As Langford observes: "At some time, in some place increasingly remote from our experience, these pages were turned, and earnestly consulted for their correlations."[89] And now, it is these connections that pass between the scientific origins into art to demonstrate the structure at the basis of the visual diagram.

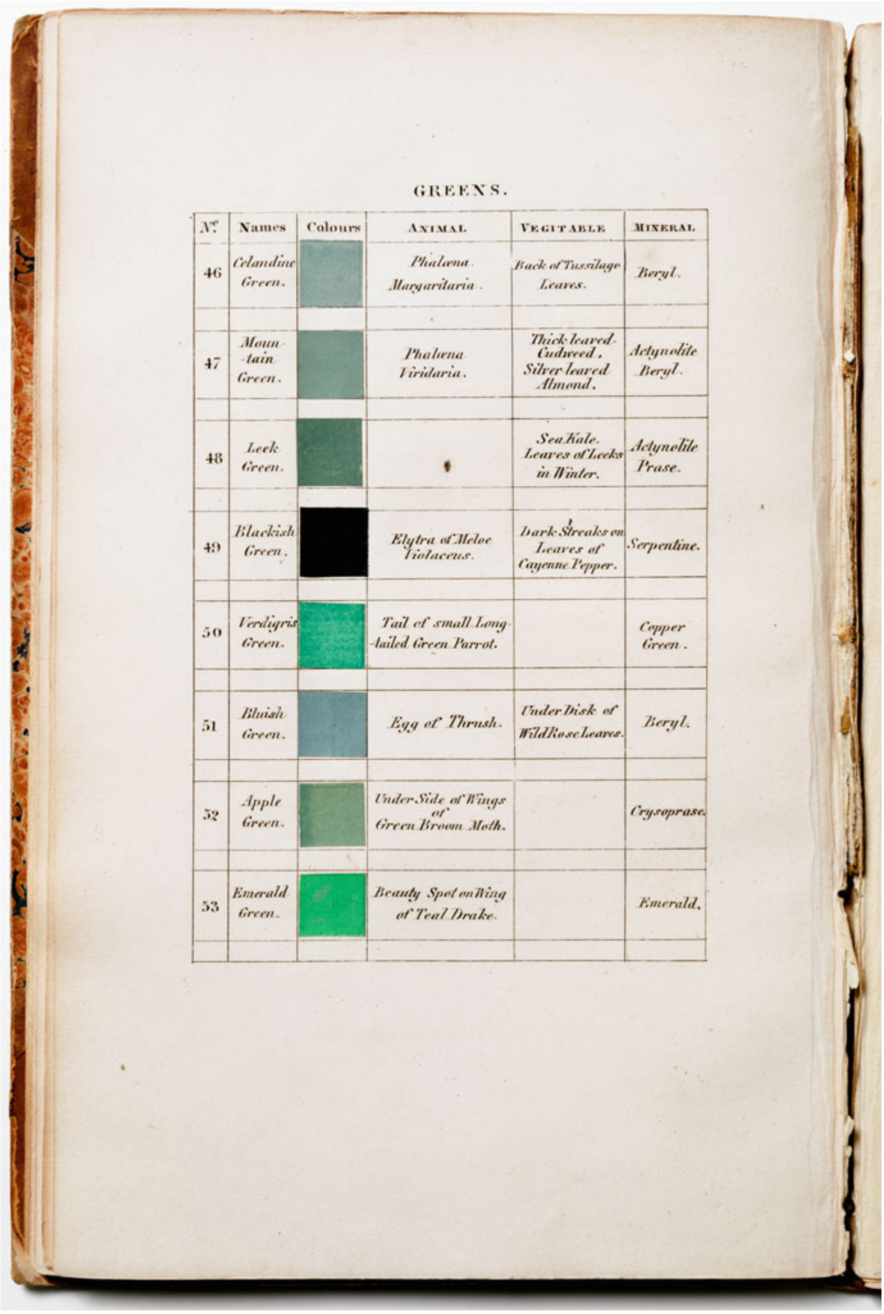
Conclusion

Maggs's photography reveals how the grid is the structural visual apparatus under the philosophical diagram. If the diagram, with its harnessing of abstract functions and their transposition from system to system, with its zone of indiscernibility and overall immaterial connective activity remains a hard to pin down philosophical concept, the grid, and the way it operates in the domain of art specifically, provides a strategy to start imagining it. The grid, in art but especially in Maggs, is both rigid, defined and orderly and yet, full of poetic potential, abstract connectivity and centrifugal spatial possibility. The grid is sequential and repetitive but communicates temporality and simultaneity. The grid, finally, is much more than its immediate visual imprint much like the diagram is far more than its graphic manifestation.

[87] Langford, Turning Colours, 16.

[88] Langford, Turning Colours, 24.

[89] Langford, Turning Colours, 24.



GREENS.

Nº	Names	Colours	Animal	Vegitable	Mineral
46	Celandine Green.		Phalæna Margaritaria.	Back of Tussilage Leaves.	Beryl.
47	Mountain Green.		Phalæna Viridaria.	Thick leaved Cudweed. Silver leaved Almond.	Actynolite Beryl.
48	Leek Green.			Sea Kale. Leaves of Leeks in Winter.	Actynolite Prase.
49	Blackish Green.		Elytra of Meloe Violaceus.	Dark Streaks on Leaves of Cayenne Pepper.	Serpentine.
50	Verdigris Green.		Tail of small Long-tailed Green Parrot.		Copper Green.
51	Bluish Green.		Egg of Thrush.	Under Disk of Wild Rose Leaves.	Beryl.
52	Apple Green.		Under Side of Wings of Green Broom Moth.		Crysoprase.
53	Emerald Green.		Beauty Spot on Wing of Teal Drake.		Emerald.

Arnaud Maggs, *Werner's Nomenclature of Colours*, 2005 with permission from © The Estate of Arnaud Maggs / courtesy Stephen Bulger Gallery

Les jeux de l'unilatère

A la limite du discursif, le réel se dessine-t-il?

Amélie de Beauffort

Abstract J'ai choisi d'adopter le parti pris du dessin pour exposer la nécessité d'un regard plasticien pour la pensée, pour la création, mais aussi pour habiter le monde. Ce parti se prend à partir d'une pratique graphique qui s'appuie sur une morphologie plastique de la manipulation des nœuds et des gestes qui font et défont le dessin. Fernand Deligny apprit auprès d'enfants mutiques, que l'être humain n'est pas uniquement un être discursif, que quelque chose échappe aux mots mais parvient parfois à émerger à même la trace d'un dessin. J'expliciterai le développement de quelques gestes plastiques et conclurai par un parallèle avec les fils arachnéens et topologiques de Deligny. L'enjeu sera de se laisser prendre par les jeux de l'unilatère et de tenter d'y saisir quelque chose de cela que Lacan nomme le réel.

Keywords Dessin · Topologie · Art · Fernand Deligny · Unilatère · Nœud · Intersensorialité

J'ai choisi d'adopter le parti pris du dessin (à moins qu'il ne m'ait choisie) pour exposer la nécessité d'un regard plasticien pour la pensée, pour la création, mais aussi pour habiter le monde. Ce parti se prend (s'éprend) à partir d'une conception singulière du dessin qui prend appui sur des considérations topologiques. Je travaille à partir d'une morphologie plastique de la manipulation des nœuds et des gestes qui font et défont le dessin et où la feuille de papier devient un acteur du processus. La vie du papier oscillera entre potentialité et épuisement. Il n'y a pas d'opposition entre les jeux de la nature, les formes naturelles (le monde objectif) et les artefacts (le monde subjectif).

L'être humain n'est pas uniquement un être discursif. Quelque chose de la fulgurance de la pensée et de la création creuse le langage, échappe aux mots mais parvient parfois à sourdre à même la trace d'un dessin.

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Par le présent exposé, je tenterai d'expliciter le développement de quelques gestes plastiques et conclurais par un parallèle avec les fils arachnéens de Fernand Deligny qui accompagna des enfants mutiques dans les Cévennes. L'enjeu sera de se laisser prendre par les jeux de l'unilatère et d'y être pris, de tenter d'y saisir quelque chose de cela que Lacan nomme le réel.

Je vous présenterai ma pratique à travers quelques exemples. Mais je tiens à préciser d'emblée qu'une démarche artistique s'éclaire dans l'après-coup. Les intentions se formulent de plus en plus précisément à mesure que la recherche plastique se développe. Au départ, il y a ce qui tombe sous la main, s'organise, s'assemble et se transforme. Ça commence comme un bricolage

A mes yeux, le dessin a plus à voir avec un début qu'avec une fin. Une vision d'un état naissant, comme une ouverture.

Il reste toutefois souvent défini comme une forme de projection mentale. Il n'en demeure pas moins qu'il y a un clivage entre l'image et le langage. Le langage plastique ne peut se réduire au discursif. Il importe de maintenir cet écart, de l'activer.

Quelles seront les conditions de possibilités pour que quelque chose agisse dans la pratique du dessin qui la dépasse et la déplace ?

Le dessin s'élabore inséparablement de son support, dans le sens d'une mise en question de celui-ci. L'espace du dessin se constitue dans le faire depuis les premiers choix. Surtout, le papier est extrêmement actif. Lui aussi dessine : par son indiscipline, il indiscipline le dessin. Cela n'est pas sans effet sur le dessinateur, qui loin d'être le maître du jeu, établit une relation avec le papier. Le dessinateur dessine avec, plutôt que sur la feuille de papier. Elle est un partenaire.

La feuille est sensible (une évidence pour le papier photo). Je veux simplement dire que le papier n'est ni un simple écran ni une surface neutre, il a une épaisseur, il se froisse, se frotte, se déchire, il boit, se détend, gondole . . . Le papier s'anime.

Deleuze donnait une perspective vitaliste au non organique à laquelle je suis sensible:

> Il y a un lien profond entre les signes, l'événement, la vie, le vitalisme. C'est la puissance de la vie non organique, celle qu'il peut y avoir dans une ligne de dessin, d'écriture ou de musique. Ce sont les organismes qui meurent, pas la vie. Il n'y a pas d'œuvre qui n'indique une issue à la vie, qui ne trace un chemin entre les pavés.[1]

La feuille de papier est déterminée tant par sa fragilité que par sa dynamique. Je la considère comme une membrane, une surface entre, un milieu ; et un milieu qui n'est pas neutre, dans le sens du *Umwelt* de Jacob von Uexküll qu'il définit comme relatif à l'organisme qui y vit.

Une membrane, ce peut-être la feuille de papier. Mais cette citation de Beckett témoigne qu'il y va du sujet.

> [. . .], c'est peut-être ça que je sens, qu'il y a un dehors et un dedans et moi au milieu, c'est peut-être ça que je suis, la chose qui divise le monde en deux, d'une part le dehors, de l'autre le dedans, ça peut être mince comme une lame, je ne suis ni d'un côté ni de l'autre,

[1] Gilles Deleuze, Pourparlers, Éditions de Minuit, Paris, 2003, p. 196.

> je suis au milieu, je suis la cloison, j'ai deux faces et pas d'épaisseur, c'est peut-être ça que je sens, je me sens qui vibre, je suis le tympan, d'un côté c'est le crâne, de l'autre le monde, je ne suis ni de l'un ni de l'autre. Samuel Beckett[2]

En poursuivant ces observations, dessiner suppose alors comme pour l'araignée chère à Von Uxekull et à Fernand Deligny, d'ourdir sa surface d'inscription. Ourdir vient du latin : *ordiri* « faire une trame ; ourdir sa toile (de l'araignée) » signifie « commencer, entreprendre (quelque chose) ».[3] Le terme appartient au monde du textile. Tresser et nouer sont parfois utilisés dans un sens proche. D'ailleurs il est simple de passer du nœud à la tresse (un coup de ciseaux suffit).

C'est amusant de penser alors à la spéculation de Semper qui annexe l'origine de l'art à une origine « textile ». L'homme tressait, nouait des roseaux, les assemblait pour se protéger et diviser l'espace. Il remarque que, dès cette origine, le geste manuel est d'emblée un geste conceptuel qui fait advenir de l'espace. Pensée à situer en écho à celle d'Heidegger, pour qui l'espace ne préexisterait pas aux choses.

Prendre au sérieux cette articulation entre geste manuel et geste conceptuel comme origine de l'art entraine une conséquence : il en découle une esthétique du faire où gestes et concepts ne vont pas l'un sans l'autre, ils interagiront avec les matériaux et les transformeront.

Par ailleurs en fait de transformation, mon travail a la particularité de modifier l'espace de la feuille en un espace noué et unilatère. Un simple demi-tour constitue une aubaine pour le dessinateur qui découvre que les deux faces de la feuille se trouvent alors mises en continuité. Je parle ici du ruban de Möbius d'où découlent mes objets d'étude topologique et artistique. J'y reviendrai.

Le dessin privilégie la proximité de la main et du papier, donc le toucher et par conséquence l'aveuglement (je fais ici référence aux développements de Derrida autour des dessins d'aveugles, où il montre bien comment la main du dessinateur cache le tracé à la vue de l'artiste au moment où il trace). Ces privilèges du toucher et de l'aveuglement éclipsent le primat accordé habituellement à la vision. Elle s'en trouve modifiée et elle s'oriente vers un mode de perception haptique qui déstabilise nos repères, un regard qui se rapproche et d'où nait une nouvelle aptitude : un tact du regard. Ce nouveau paradigme de la vision travaille à rebrousse-poil de nos intuitions, de nos représentations spatio-temporelles euclidiennes habituelles. Il appelle une conception topologique de l'espace. On y croise des surfaces unilatères dont la fameuse bande de Möbius. Fondamentalement paradoxal, le topologique maintient une tension irrésolue entre local et global, dedans / dehors, continu / discontinu. Il problématise et qualifie les voisinages, frontières et passages. Ces termes ont un écho très fort à des enjeux anthropologiques et sociétaux actuels. « Un être frontière qui n'a pas de frontière »,[4] un être dont la mise en jeu permet

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[2] Samuel Beckett, L'innommable, Paris, Minuit, 1953, p. 158.

[3] Du lat. pop. *ordire,* class. *ordiri* « faire une trame; ourdir sa toile (de l'araignée) » d'où au fig. « commencer, entreprendre (quelque chose) ». http://www.cnrtl.fr/etymologie/ourdir

[4] G. Simmel, La tragédie de la culture et autres essais, Poche, 1993. Cette citation conviendrait tout aussi bien à certaines surfaces topologiques.

au-delà de la métaphore, l'irruption du réel. Pour Lacan, le réel est ce qui échappe aux registres symbolique et imaginaire, je dirai plus simplement que c'est ce qu'on n'avait pas imaginé, mais convoqué dans l'acte de création. Le réel excède nos représentations.

La topologie qui m'intéresse n'a rien à voir avec l'algèbre. Elle se pratique à partir de gestes simples qui ne demandent que du papier, des ciseaux, de la colle et des pigments.

Dans un premier temps, j'avais commencé à nouer des bandes de papier avec l'idée de les représenter.

Qu'y avait–il de si captivant, de si intrigant, voire de si inquiétant pour paraphraser Gilles Châtelet dans ces nouages ? Pour Châtelet, le nouage incarne un espace de pensée irréductible à la lettre, un espace « rebelle » au registre symbolique.

> Un nœud n'est pas une trajectoire: nous ne le maitrisons pas en nous mettant « à la place » d'un point mobile qui décrirait la courbe. Le nœud pousse à l'extrême la tension entre le mobile, le latéral et l'épaisseur. [. . .] Le secret du nœud emporte bien au-delà de l'espace dans lequel il est censé baigner: il va jusqu'à briser les rapports classiques de la lettre et de l'image. [. . .], la géométrie moderne et la découverte des relations insoupçonnées entre théorie des nœuds et statistiques visent à capturer toute la puissance allusive du nœud, à s'installer au cœur même du croisement. Un croisement n'est pas un point, mais éclate comme un événement géométrique: quelque chose s'est passé et suggère un espacement. C'est peut-être ce type d'expérience que recherchaient les moines irlandais, en se risquant dès le VIIème S. dans leurs labyrinthes. On a pu montrer que les lettres animées, les enluminures des manuscrits n'étaient pas adjointes au texte comme une "décoration" ou comme un "supplément", mais prétendaient solliciter la patience du lecteur, toujours pressé de recevoir l'information contenue dans un texte. Nous savons qu'il y a une espèce de pudeur géométrique de l'entrelacs qui impose un effort pour saisir le relief qu'il déploie. Nullement commises au "remplissage", les torsades des enluminures entendent bien rappeler que le spéculatif est irréductible au littéral et ne s'épuise pas en alignant des unités signifiantes. Les recherches contemporaines n'en sont pas éloignées. Elles redécouvrent la dimension comme ce qui se rebelle à tout processus verbal et ne se laisse saisir que par des notations nouvelles. Celles-ci ne sont pas des « ornements » mais possèdent le privilège de discipliner les allusions et le pressentiment des formes.[5]

Ces enluminures sont du côté de la lettre mais comme pour mieux en montrer l'opacité. Il s'agit bien d'affirmer une importance de la forme, de l'opacité et la complexité qu'elle peut engendrer par rapport au sens littéral. Une forme qui si elle se laisse appréhender par des notations, ne s'y réduit jamais. Un réel surgit aussi de cette écriture. Cette résistance retient l'attention. Si le nœud résiste à l'écriture mathématique, il se laisse néanmoins manipuler et dessiner.

Et voilà ce qui fit pivot pour mon travail : La bande nouée prit le lieu et la place du dessin et c'est ainsi que s'ouvrent les jeux de l'unilatère.

Entre les mains, une bande de papier unilatère nouée sur elle-même, a la dimension d'un objet. Mais on ne peut nier sa dimension spatiale puisque l'espace y

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[5] Gilles Châtelet, *op. Cit.*

Fig. 1 Demi-torsions de surfaces, matériaux divers, 2012–2016, crédit Hannane Housni

est modifié à part entière. Un espace s'y dessine, un espace paradoxal où jouer/errer et souvent se perdre (Fig. 1).

Dès le début de ces explorations, j'ai compris 2 choses:

1. Il fallait mobiliser des séquences de gestes signifiantes à la fois par rapport aux enjeux topologiques et aussi par rapport au « dessiner ». Ces gestes se déclinent comme des verbes, par exemple : nouer, couper, flécher, percer, plier, coller, border/déborder, déposer /percoler de la couleur. Il y a du processus, presque un programme, mais qui ne sait pas l'essentiel de ce qui sera produit. On attend ce qu'on n'attendait pas; mais que d'heures de travail préparatoire passées pour ménager une place à l'événement, cet événement qui ne pouvait pas être formalisé en amont, et qui advient en même temps qu'il émerge comme énonciation d'une pensée plastique par son inscription même. Je pense que c'est un des points qui montre quelque chose de ma position d'artiste : ce premier degré, cette attente de quelque chose qui émerge plastiquement.

Châtelet évoque un dispositif d'extraction de gestes qui est également un stratagème allusif,[6] c'est ce qu'il appelle diagramme. Ne serait-ce pas entre permanence, variabilité (mobilité) que se situe la dimension diagrammatique ? Chaque dessin mobilisera un certain parcours et une temporalité singulière de gestes, en jouant sur la variabilité de certains composants et l'invariabilités d'autres.

Le programme de l'œuvre a beau être projeté, il est mené quasi à l'aveugle quant à ce qui pourra en résulter. Les gestes, pour la plupart, bien que considérés avec la plus grande attention sont posés de façon indirecte, en retardant le regard sur l'image et le privant de son pouvoir dominant. Ce qui se fomente, ne l'est pas

[6] Gilles Châtelet, L'enchantement du virtuel, L'enchantement du virtuel: Mathématique, physique, philosophie, Edition de Charles Alunni et Catherine Paoletti, 2010, p. 64.

pour donner une nouvelle assurance à l'image, ou dans un but esthétique, mais pour que ce qui échappe alors soit porteur d'une question.

Une énonciation qui se donnera à même la trace de son inscription, telle que les gestes ne se livreront que plastiquement à la visibilité et à la spatialité. Ce voir dans l'après coup, qui débouche parfois sur du savoir, n'est pas produit par une pensée qui passerait d'abord par les mots, mais bien produit par une main qui « pense » en traçant.

2. Pour que des séries de dessins élaborées à partir de différentes séquences de gestes se différencient mais surtout qu'elles donnent une relative lisibilité de leurs écarts, il fallait certes des variables mais également de l'invariant.

Je me suis appuyée sur un invariant qui ne s'épuise pas depuis le début de ce travail : il s'agit à chaque fois de nœuds de 8 (Figs. 2 and 3).

Fig. 2 *Persée, (8 × 8).8, par exemple*, calque polyester,encre, punaises,dim. Variable, 2011 crédit Paul Louis

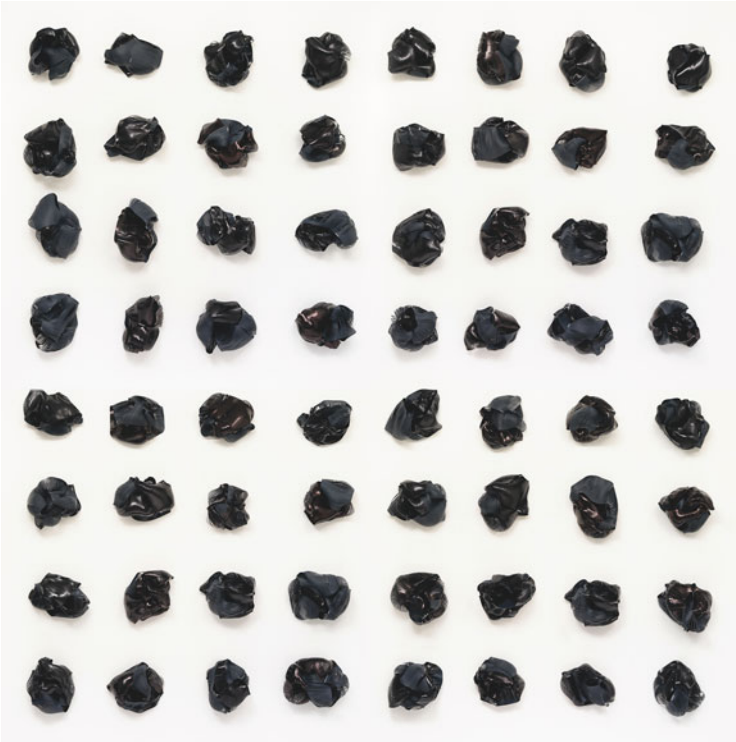


Fig. 3 *Persée, (8 × 8).8, par exemple*, calque polyester,encre, punaises,dim. Variable, 2011 crédit Paul Louis

L’invariant, Persée (8 × 8). 8, par exemple

Feinte ou leurre, paradoxe : localement, on y voit deux versants, un mat et un brillant, les 2 faces sont en continuités et la bande striée de fentes est unilatère. Elle est meuble, comme une terre peut l’être (on peut en transformer la forme) mais aussi en tant qu’elle se laisse mouvoir et tourner sur elle-même. L’objet est animé d’une résistance qui lui est propre. Déposée, elle s’abandonne comme une méduse échouée. La surface recourbée, fendue, ne prend forme que lorsqu’elle est épinglée au mur. On peut y piquer comme dans le creux d’une boutonnière, des punaises à tête large, noires et brillantes. Et si par mégarde, ce nœud, qui ne manque pas de ressort, se déboutonne ; alors il n’y a plus qu’à rejouer !

La surface se courbe en saillie et en concavité. Les fentes permettent une certaine élasticité. La forme que prend le nœud se négocie entre tension, résistance et gravité. Petite parenthèse : cette pièce : chevelure, ou bouclier, miroir noir de lumière, a été le premier pas vers un rapprochement que cette image résume par le côte à côte d’un corail, appelé gorgone (cnidaire, une espèce animale assez simple appelées aussi ortie (urticantes) ou éventails de mer) (Fig. 4).

Fig. 4 Gorgone, cnidaire, photomontage, collection de l'artiste, 2016 (Paul luois et Hannae Housni)

Le mythe raconte que le corail nait mythologiquement du sang de Méduse versé sur une sorte d'algue. Cet être souple dans le règne marin se transforme, se solidifie dans le règne terrien, un de ces êtres qui débordent les catégorisations exclusives. Un espace poétique s'est ouvert entre biologie, récit mythologique et art. Je ferme ici la parenthèse car je n'ai pas le temps de développer cet aspect aujourd'hui.

Je reviens au développement de la série *Persée, (8 × 8).8, par exemple*. Je décidais de resserrer la question autour de cet invariant. La répétition allait-elle épuiser le huit ?

Où s'arrêter ? Plutôt quand ? Le plus simple était de s'en tenir à un nombre de décision minimum. Le huit étant déjà donné, je m'y suis tenue. Qu'allait produire un mur de 8 par 8 nouages de huit « identiques » ? Installer, réinstaller un mur de nœud de 8: une vie des formes se tient dans la réserve. Celle donnée lors d'un accrochage, n'est que prêtée, puisqu'elle est en définitive non répétable à l'identique.

A Plat

De cette série en dérive une autre, à partir de ce dessin en volume et de sa projection en 2 dimensions. J'ai alors posé ce type d'enroulement sur du papier photosensible pour en réaliser un photogramme (un photogramme permet de prendre une empreinte de l'ombre négative de l'objet sans appareil) (Fig. 5).

Il se trouve que le processus du photogramme, si rapide, produisait à mes yeux beaucoup de choses banales, répétitives. De la foule images qui s'amoncelaient sur ma table de travail, ne surgissait pas à chaque fois cette corrélation évoquée plus haut entre geste et concept (Fig. 6).

Le principe de ce grand désordre est d'avoir chaque chose à sa place c'est-à-dire sous les yeux, à portée de main.

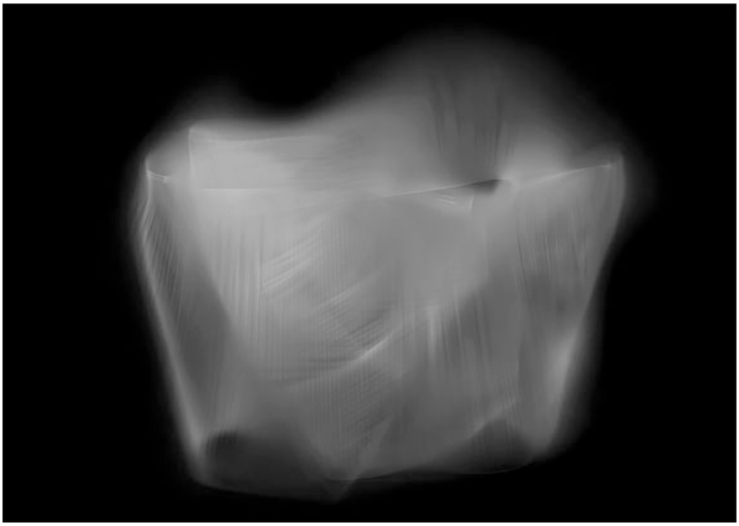


Fig. 5 *Gélatine*, photogramme, 30 × 40 cm, 2013 crédit A de Beauffort

Fig. 6 Vue d'atelier crédit A de Beauffort

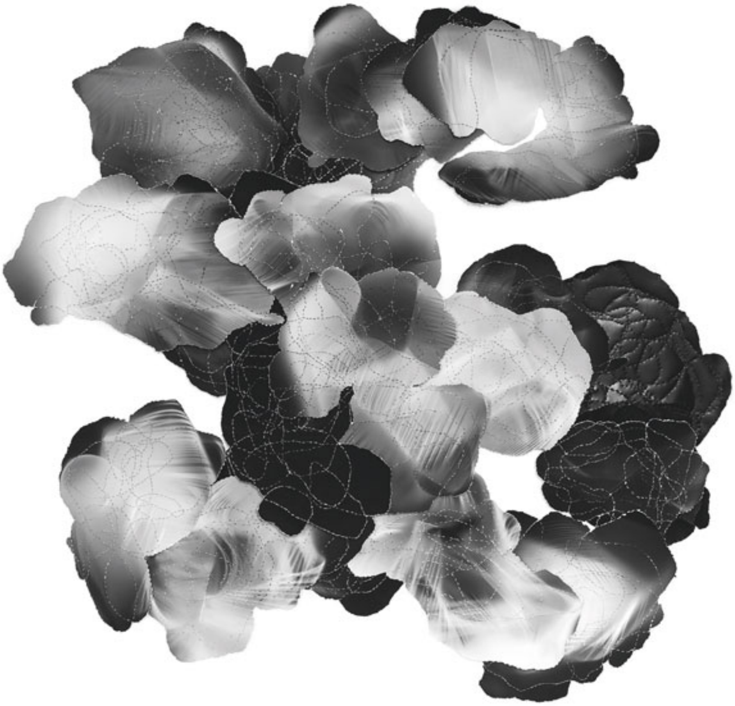


Fig. 7 Cd'8, photogramme, punaises, 170 × 90 cm, 2018 crédit A de Beauffort

De nouveaux dessins vont dériver de gestes de collage, dont une série appelée C d'8 (Figs. 7 and 8)

Il s'agissait d'assembler de façon quelque peu opportuniste des fragments prélevés dans ce matériau de restes de photogramme. J'y cherchai des continuités ou au contraire des ruptures avec pour conséquence que lorsque le regard s'approche et qu'il pense saisir une forme, celle-ci fugace, lui échappe toujours un peu. Il y a un jeu de texture dans ces fragments : ils sont troués, poinçonnés. Comme une sorte de réitération du premier mouvement qui fut le dépôt d'un nœud sur le papier sensible pour faire le photogramme. Ici une boucle en ficelle nouée (nœud de 8) tombe sur l'avers ou le revers de la surface. Son tracé est relevé, et poinçonné parfois du dos vers l'avant (à l'aveugle) parfois de l'avant vers l'arrière. Et le nœud tel un dé est relancé, en boucle.

Ce procédé, où il s'agit de percer une surface pour reporter un dessin, fait penser à celui du poncif : une technique qui permet de reporter un dessin piqué de trous sur un autre support au moyen d'une poudre. Cette technique permettait par exemple de reporter en pointillés les contours du dessin comme repères préparatoires à la fresque.

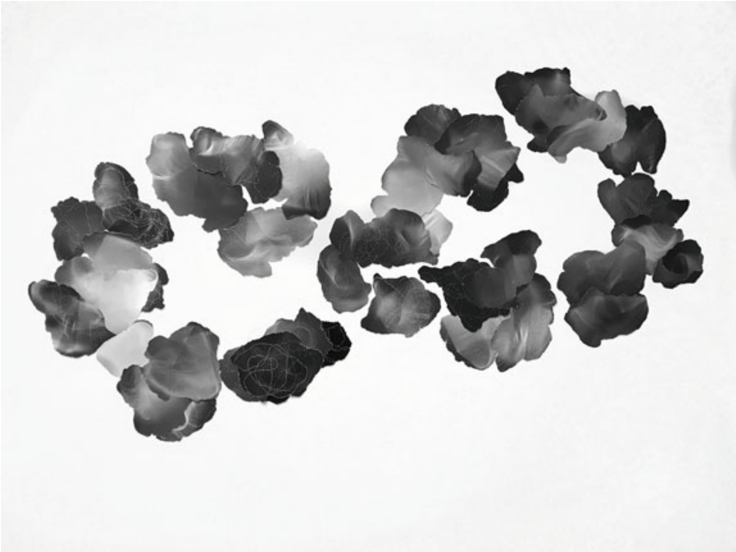


Fig. 8 Cd'8, détail. crédit A de Beauffort

Poudre aux yeux J'ai systématisé quelques gestes dans une série de dessin appelée *de la Poudre aux yeux :* Poinçonner (trouer-traverser) retourner, recommencer (en boucle ou presque). Le fusain pulvérisé, pris entre les couches, laisse son dépôt par frottements directs, par empreintes ou encore par la dispersion poudreuse engendrée au rythme des traversées du support (à chaque coup de marteau sur le poinçon). Les différentes couches, dont l'ordre (dessus / dessous) et sens (recto / verso) peuvent échanger leurs positions, sont tantôt ici utilisées comme poncif, tantôt comme martyr[7] ou encore comme support de transfert : le poncif retourné, détourné, devient lui-même le support du dépôt résiduel (Figs. 9 and 10).

Piqure, trait ou flèche, les poinçons se réitèrent et font du dessin un art de la passoire : le médium fuit de toute part et échappe à sa détermination de support et réinvente sa fonction de surface (et non d'écran) de projection, elle devient projetante et contaminante. Chaque coup de poinçon fait palpiter les couches de papier, fait et défait tout autant formes et matières. Chaque coup porté retire ici, dissémine là, et fait traverser la poussière de fusain entre les différentes strates (Figs. 11 and 12).

Ces tracés de 8 peuvent être répertoriés comme des motifs (Fig. 13).

[7] *Sa définition le renvoie un matériau destiné à être usé ou abîmé. Par exemple en ébénisterie ce sera un plateau ou une cale fait d'un matériau assez tendre destiné à supporter les dépassements d'usinage, notamment les perçages ou les fraisages ; il est placé au contact (et généralement en dessous) de la pièce à usiner.*

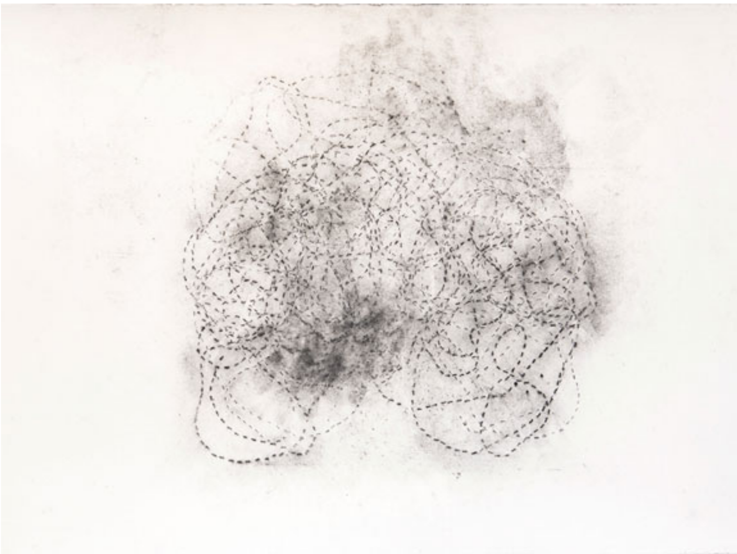


Fig. 9 *De la poude au yeux*, papier,fusian, poinçons, détail crédit Hannane Housni

Fig. 10 De la poudre aux yeux #15.1, papier poinçonné, pigment, 2015 (et détail) crédit Hannane Housni

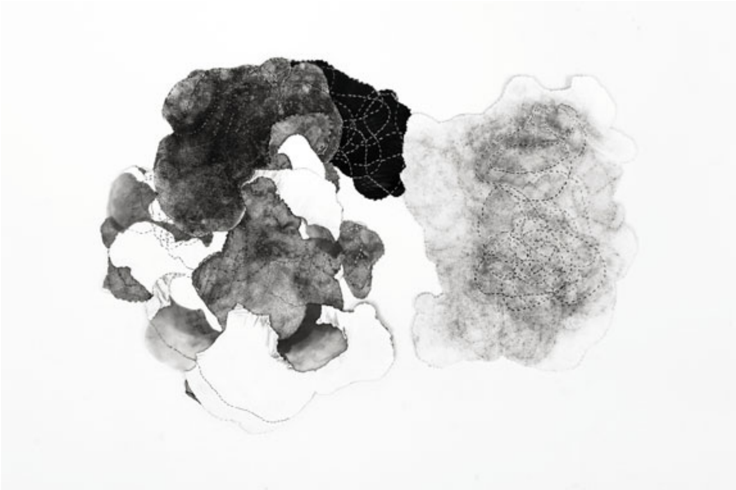


Fig. 11 De la poudre aux yeux #15.1, papier poinçonné, pigment, 2015

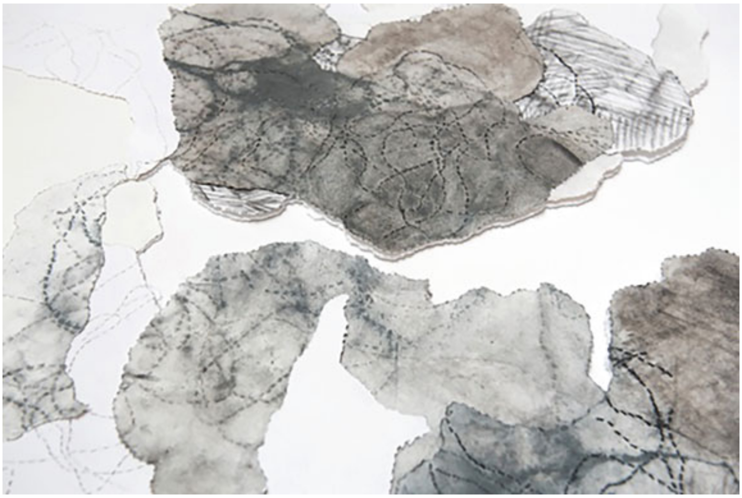


Fig. 12 *De la poudre aux yeux* #15.1, papier poinçonné, pigment, 2015

Du calque et de la ficelle de ces dessin pour évoquer les images ci-dessous qui ne se présentent pas comme des œuvres d’art (mais qui sont à mes yeux des œuvres de l’art- œuvres sans artistes).

Ce sont les lignes d’erre de Fernand Deligny

Fernand Deligny, proposait une prise en charge totalement novatrice à une époque où celle l’autisme infantile est encore mal assurée. La vie s’organisait auprès d’adultes qui n’étaient ni particulièrement diplômés, ni liés au milieu médical ou éducatif. Ce pouvait être des ouvriers, des paysans, des étudiants . . . Deligny les

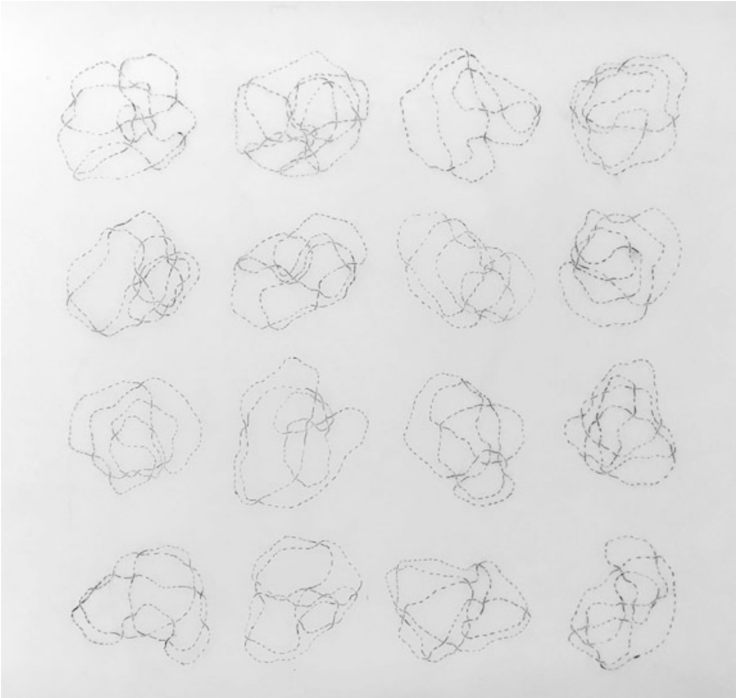


Fig. 13 16 noeud de 8, calque polyester, pierre noire, 2017, crédit A.de Beauffort

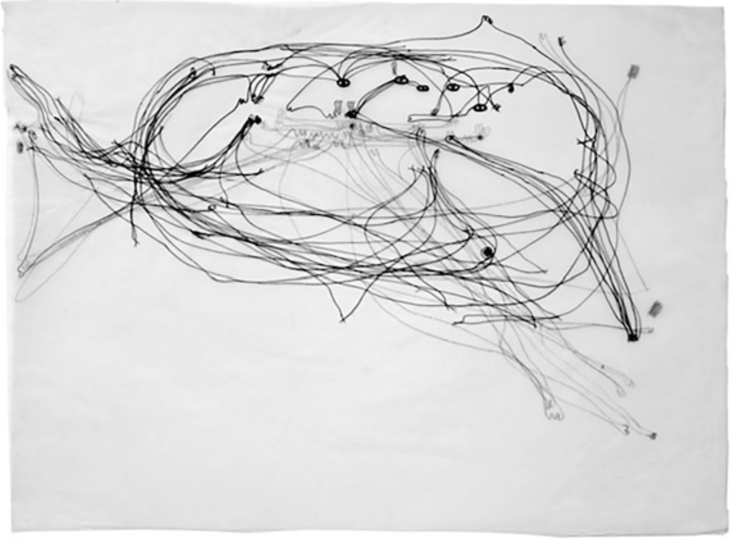
appelait des présences proches. Ils vivaient collectivement le quotidien dans des aires de séjour aménagées, situées à quelques km les unes des autres au milieu de la nature dans les Cévennes.

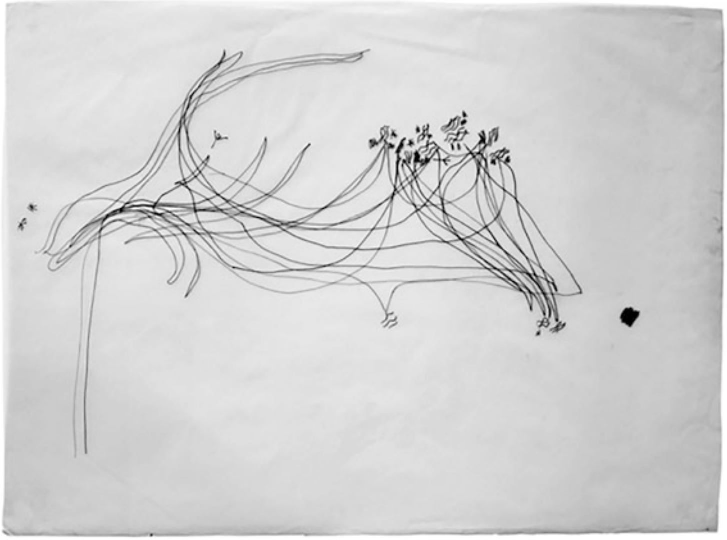
Deligny dans son questionnement sur la nature humaine, (qu'il préfère nommer l'humain de nature), repère à partir de comportements autistiques répétitifs, des rituels de dessins de boucles. Il évoque des mouvements du corps dans l'espace, des tours autour de soi, ou des « tracer » sur une feuille de papier. Il y voit une forme d'inné inscrit dans le génome de l'espèce. Cette persistance se montre du côté de ce qu'il nomme l'agir. Le faire serait chargé d'intention alors que l'agir non, le geste n'aurait alors de valeur que pour lui-même.

Deligny situe le geste de nouer comme à la limite entre faire et agir.

> Lorsque « ces enfants-là » se mettent à nouer,[8] la première boucle se passe bien mais tout se gâche au moment de la seconde. C'est comme si deux mains n'y suffisent pas, il suffirait d'une troisième, « la troisième étant celle de l'autre que chacun est et qui vient se (sup)poser au moment opportun au cours du nouer ». Pour faire un nœud les deux mains de l'agir n'y suffisent pas, n'y arrivent pas alors qu'elles sont pourvues d'une dextérité surprenante.

[8] Deligny évoque ici des lacets de chaussure.





> Nouer, cela se dit aussi pour ce qui concerne les relations . . . .[9]

Dans les Cévennes, les présences proches relevaient et dessinaient au quotidien leurs propres déplacements, ensuite ils relevaient sur des calques les *lignes d'erre* des enfants.

> Pour rien, pour voir, pour n'avoir pas à en parler, des enfants – là, pour éluder nom et prénom, déjouer les artifices du IL dès que l'autre est parlé. » Ces cartes ne servent ni à comprendre ni à interpréter des stéréotypies; mais à « voir » ce qu'on ne voit pas à l'œil nu, les coïncidences ou chevêtres (lignes d'erre qui se recoupent en un point précis, signalant qu'un repère ou du commun se sont instaurés), les améliorations à apporter à l'aménagement de l'espace, le rôle des objets d'usage dans les initiatives des enfants, leur degré de participation à telle tâche coutumière au fil des jours, l'effet sur eux du geste pour rien d'un adulte . . .[10]

Outre l'importance des travaux de Deligny dans le champ des sciences humaines, ces diagrammes, ces *lignes d'erre* recèlent des qualités plastiques où la trace se substitue (sans équivalence) aux mots qui manquaient à ces enfants mutiques. Ces dessins produits par Deligny et son équipe offrent une forme d'écriture des traces de vie et des déplacements des corps alors que ceux-ci ne disposaient pas de la parole. Ces dessins n'ont d'autres fin que de rendre visible comment les déplacements enregistrés font image, ils dessinent un territoire, un milieu. Deligny se présente comme « poète et éthologue ». Le monde autistique a quelque chose de particulier puisqu'il ne se dit pas, mais il peut faire image. (À entendre en écho avec Châtelet et le nouage rebelle. Fernand Deligny dont on connaît la défiance à l'égard du langage était également attentif à ce qui lui opposait de la résistance.)

Les lignes d'erre sont l'expérience fragmentaire d'un nouage sensible au monde et d'une forme d'intelligence collective, ou de la recherche d'un commun d'espèce. Ils se présentent comme des cartographies mais mobilisent plus que cela : une attention au déplacement à la façon de partager et d'habiter l'‘espace, dans les lieux de vie communs, mais aussi l'affirmation que l'être humain ne se réduit pas au langage discursif.

Certains comportements autistiques répétitifs dessinent des boucles, Deligny les désignent comme des gestes innés, une permanence, un ressassement enregistré dans le génome de l'espèce.

> Si l'autisme est une pensée en image, observe Deligny, c'est parce qu'il différencie un flot de sensations, le flot de sensations qui est [. . .] dans un espace, un territoire, vécu comme le développement du corps. L'expérience de l'autiste, son attrait pour les jeux de lumière sur l'eau, pour les moindres accords entre la forme creuse qui recueille et le frisson du fluide, est une forme extrême de l'animisme [. . .]. Dans cet échange non verbal entre les formes substantielles et actives, la métaphore est inscrite dans les interactions sensorielles: le creux

[9] Deligny, Œuvres, Arachnéen, 2017, p. 1314.

[10] *Cartes et lignes d'erre, traces du réseau de Fernand Deligny, 1969–1979*, Sandra Alvarez de Toledo, Arachnéen, 2013.

> de la main est équivalent à un trou dans la terre, la porte ferme un conduit, la pierre est une peau plus ou moins lisse . . . ».[11]

Le non-sens de l'image autiste est ce qui caractérise un mode d'apparition du réel. En contrepartie, l'autiste ne supporte aucun événement qui vienne perturber la permanence des choses, aucun trou qui vienne percer le réel. Aucune altération, aucun manque n'est supportable car il sera perçu à même le corps, à même le réel hors de toute dimension symbolisable.

Chevrier, dans « L'image, "mot nébuleuse" », texte publié dans *Fernand Deligny œuvres*, texte publié aux éd L'Arachnéen parle d'une conception saisissante de l'image incarnée comme inscription *a-symbolique* du réel en opposition à l'image au sens de représentation.

Que nous reste–t-il, ou que reste-t-il de nous lorsque le langage se retire ?

Deligny nous intéresse car il forge une pensée extrême de l'image, une pensée de l'image-trace, une image hors représentation (ou délivrée de l'emprise du regard). Une extrémité qui n'est pas la nôtre (nous qui ne sommes pas mutiques, nous ne sommes pas si proche de l'individu (qui pour Deligny n'est pas un sujet puisqu'il n'est pas divisé par le langage) mais qui dévoile à l'état « pure » une continuité primitive, infra-verbale, une topologie entre monde sensible, image et corps. Un point où la nature de la pensée et la nature sont en continuité.

Le langage échoue à dire le réel, si ce n'est, sous forme d'éclats, par cet échec et par ses failles même. Par ses trous.

Trouer, c'est défaire l'unité : celle du support, ou celle, syntaxique du langage. Malmener le langage est aussi une forme de spatialisation tel que le diagramme peut l'accomplir.

Le diagramme cultive une certaine poésie en cela qu'il disloque la linéarité et la logique syntaxique du langage. Le langage est alors considéré en dehors de la fonction de communication et cela permet dès lors de desserrer l'étreinte du savoir établi, de se dégager et de la mémoire et de la raison, de se tenir au bord du vide pour que surgisse l'inconnu.

Trouer le langage est une opération poétique (rappelons-nous que Deligny se dit autant poète qu'éthologue.).

Beckett dans une *Lettre* en 1937 écrivait:

> Etant donné que nous ne pouvons éliminer le langage d'un seul coup, nous devons au moins ne rien négliger de ce qui peut contribuer à son discrédit. Y forer des trous, l'un après l'autre, jusqu'au moment où ce qui est tapis derrière, que ce soit quelque chose ou rien du tout, se mette à suinter.[12]

Qu'est-ce qui peut bien suinter ? Si ce n'est le réel.

Certains rapprochements ne sont pas innocents. Je citerai encore le texte de Chevrier qui relève que « Fernand Deligny opposait l'imprégnation du discours par

[11] J.J.F. Chevrier, « L'image, "mot nébuleuse", dans Œuvres, Arachnéen, 2017, p. 1781–1782.

[12] Samuel Beckett cité par Gilles Deleuze dans l'Epuisé, Les Editions de Minuit, p. 70.

le monde sensible à l'imprégnation de l'individu par le discours ».[13] Il me semble intéressant de penser ces 2 mouvements, cette double fertilisation du discours par le monde et de l'individu par le discours en face à face. Ou dos à dos et néanmoins en continuité pour faire un clin d'œil à la structure moëbienne.

Trouer et nouer sont ici et là, si je peux me permettre, liés ; ce n'est pas un hasard : former une boucle dessine un trou.

Pour conclure, disons que présenter des objet/espaces topologiques plutôt que d'en représenter des indices, c'est appeler ce qui est tapi dans l'angle mort du sujet, et attendre que le réel se mette à suinter. Et peut-être cette exsudation fera-t-elle trembler « l'image » même de l'objet topologique, ce qui serait un des effets inattendus de l'haptique.

[13] Samuel Beckett in « German letters », dans Bruno Clément, *L'œuvre sans qualité. Rhétorique de Samuel Beckett*, Paris, Seuil, 1994, p. 238.

Le diagramme à l'œuvre

Farah Khelil

Abstract Cette contribution traite de l'émergence d'un raisonnement par le diagramme dans la pratique artistique de Farah Khelil et en particulier dans un travail intitulée *Point d'étape*. Elle en éclaire la singularité dans la relation de l'art à la philosophie et son rapport au réel, et s'appuie pour cela sur la lettre fictive de Paul Cézanne envoyée au mathématicien Felix Klein, où il révèle le diagramme à l'œuvre dans sa peinture, mais aussi le dialogue avec les mathématiques et en particulier la topologie , et dont l'enjeu est de questionner la peinture et son rapport à l'espace, et par la même la pratique contemporaine d'agencement et de traitement de données qui prennent des formes hybrides et polymorphes.

Keywords Diagramme · Point d'étape · Processus · Peinture · Topologie · Agencement

« De tous les arts, la peinture est sans doute le seul qui intègre nécessairement, « hystériquement », sa propre catastrophe, et se constitue dès lors comme une fuite en avant »,[1] affirme Gilles Deleuze. Dans sa propre histoire, la peinture a toujours été confrontée à autre chose qu'elle-même : l'imprimerie, la photographie analogique ou au numérique qui l'ont toujours mise au défi de se redéfinir jusqu'à la menacer à disparaître dans sa condition même d'être-au-monde. À travers cette histoire de la peinture, on peut lire celle de la modernité. Des questions, d'ordre ontologique, relatives à la nature de la peinture se posent toujours : pourquoi de la peinture aujourd'hui encore ?[2] C'est dans ce sens que dans les années 80, la philosophie questionne l'esthétique contemporaine en usant du concept de

[1] Gilles Deleuze, *Francis Bacon, Logique de la sensation*, Paris, Du Seuil, L'ordre philosophique, 2002, p. 96.
[2] *Ibid.*

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diagramme. Comment « penser par le diagramme »[3] permet-il au geste pictural d'aller au-delà des frontières techniques ? et comment le geste de la peinture permet-il de penser au-delà d'une fonction spécifique, une certaine forme de *laisser être* et d'indétermination dans un monde contemporain désormais fait de mélange technique ?

La présente contribution propose de réaliser une coupe dans ce mélange et de s'imprégner de cette atmosphère technique à travers la peinture comme outil d'observation. Il s'agit d'une lecture du diagramme à l'œuvre dans la peinture de Paul Cézanne, d'une part, en s'appuyant en particulier sur sa lettre adressée au mathématicien Felix Klein.[4] Dans ce courrier, le peintre explicite sa méthode de morcellement et révèle le dialogue qu'il entretien avec la topologie et en particulier la géométrie projective, dont l'enjeu est de questionner la peinture dans son rapport au lieu et à la nature. Dans un deuxième temps, il sera question de mettre en rapport ce diagramme toujours à l'œuvre mais dans un système différent, celui de la monstration et de la visualisation des données dans l'installation en *display*. C'est-à-dire, la disposition et l'agencement de formes et d'objets hétérogènes sur un plan, où il en éclaire la singularité dans sa relation en général au traitement et à la synthèse des données ainsi qu'à une certaine « ontologie plate ».[5] Ce mode d'être de l'œuvre d'art, serait-il un moyen d'exposer le diagramme ou de le projeter en dehors d'elle-même ?

Dans une lettre adressée à Félix Klein, Paul Cézanne présente sa méthode de morcellement en peinture, où il est question que tout dans la nature se modèle selon la sphère, le cône et le cylindre. Cette méthode permet d'après lui de peindre sur ces figures simples, traiter la nature par elles, le tout mis en perspective. Il écrit : « si on ne veut pas se perdre [dans l'océan des variétés], il faut une synthèse, une synthèse qui ne simplifie pas, mais qui généralise. Il faut organiser nos sensations ».[6] Synthétiser signifie associer et combiner par une synthèse, recomposer les éléments d'un tout. En chimie, il s'agit d'une action de composer un corps avec des éléments. Et par extension, c'est une généralisation, dans le sens de donner un caractère général à une méthode ou un procédé. On parle aussi d'image de synthèse.

[3] Noëlle Batt (dir.), *Penser par le diagramme. De Gilles Deleuze à Gilles Châtelet*, TLE N° 22, PUV, 2004.

[4] Maurice Matieu, *La banalité du massacre, suivi de la lettre de Paul Cézanne à Felix Klein et autres textes*, Actes sud, 2001. Cette lettre est signée Paul Cézanne (P.c.c Jean Borreil-Maurice Matieu et datée à Aix-en-Provence, le 17 avril 1906. D'après les sources : Paul Cézanne, *Correspondance*. Emilie Bernard, *Souvenir sur Paul Cézanne*. Maurice Denis, *Journal et Théories*. P. M. Doran, *Conversations avec Cézanne*. Félix Klein, Bd. 43 der *Mathematische Annalen*, 1893.

[5] « Contrairement aux « ontologies plates » proposées jusqu'ici, on ne se contentera pourtant pas d'un plan d'identités individuées et non hiérarchisées, en ayant recours aux concepts d'« interaction » ou d' « émergence » pour expliquer l'apparition de totalités et de structures organisationnelles; on associera à notre ontologie formelle de l'égal une ontologie objective de l'inégal », Tristan Garcia, *Forme et objet. Un traité des choses,* PUF, 2011, p. 13.

[6] Maurice Matieu, *La banalité du massacre, suivi de la lettre de Paul Cézanne à Felix Klein et autres textes*, Actes sud, 2001, p. 21–22.

Alors de quelle synthèse s'agit-il ? D'après Deleuze, la synthèse chez Cézanne est une « *Analytique* des éléments ».[7] D'après le philosophe, le langage analogique serait un langage de relations, qui comporte les mouvements expressifs, les signes paralinguistiques, les souffles et les cris. Et c'est dans ce sens que Cézanne fait un usage analogique de la géométrie, et non pas un usage digital comme d'autres peintres comme Pollock par exemple. Cet usage analogique de synthèse, que le peintre nomme « motif » est le diagramme. Tandis que chez d'autres peintres, cette synthèse est basée sur le code et serait donc digitale. Les préoccupations de Cézanne sont la profondeur, la sphéricité et la localité. Il écrit : « Ce qui compte, ce ne sont pas ces abstractions que le cercle, le triangle et le parallélogramme, ce qui compte, ce sont les volumes. Au fond, toute la peinture est là, céder à l'air ou lui résister. Lui céder, c'est nier les localités; lui résister c'est donner aux localités leur force, leur variété ».[8] Pour Cézanne, l'air est la profondeur caractéristique de la nature, il est le chaos, l'irreprésentable, le dehors qu'il faut non pas modeler mais *moduler*[9] ou ordonner en suivant des principes et une méthode afin de trouver la *juste distance*. Il ajoute : « c'est la logique qui compose le tableau »,[10] une logique de la sensation, dirait Deleuze. Cette logique permet de synthétiser la masse de données qu'offre la nature, afin de non pas la reproduire mais en dégager l'atmosphère. « L'atmosphère est fondamentalement un fait ontologique qui concerne le statut et le mode d'être des choses et non la manière dont elles sont perçues. Si tout acte de connaissance est, en lui-même, un fait atmosphérique puisqu'il est un acte de mélange d'un sujet avec un objet, l'extension du domaine atmosphérique va bien au-delà de tout acte de connaissance ».[11]

D'après Cézanne, il faut de l'air entre les objets pour bien peindre. Dans un système analogique, il associe l'air tantôt au plan, tantôt à une somme suffisante de Bleutés entre les couleurs chaudes pour faire sentir l'air et permettre une profondeur dans le tableau. Il explique : « je fais mes plans avec mes tons, comprenez-vous ? Pour la peinture, les volumes seuls importent. Ils signifient que les plans perspectifs disparaissent et que les valeurs d'atmosphère s'atténuent ».[12] Afin de ne pas se perdre dans cette unité atmosphérique, Cézanne élabore une synthèse avec les volumes, comme une « pictosynthèse » qui transformerait le dedans en dehors et inversement. Comme si, à travers cette méthode de mélange entre localités et volumes d'air, le milieu pictural se faisait sujet et le sujet milieu, à la fois mélangés, reconnaissables et unis par une même atmosphère. « Le mélange n'est pas simplement la composition des éléments, mais ce rapport d'échange topologique. C'est lui qui définit l'état de fluidité. Un fluide n'est pas un espace ou un corps

[7] Gilles Deleuze, *Francis Bacon, Logique de la sensation*, p. 107.

[8] Ibid. p. 21.

[9] Ibid. p. 24.

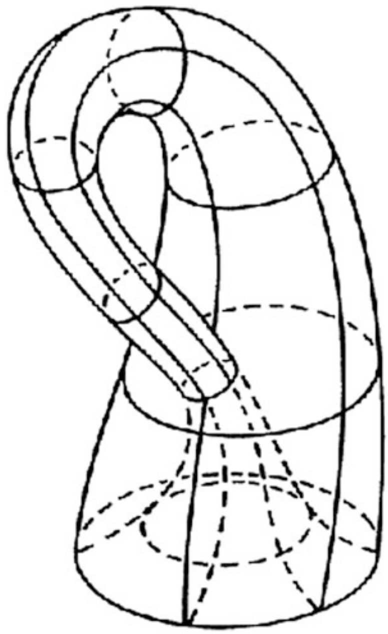
[10] Ibid. p. 25.

[11] Emanuele Coccia, *La vie des plante*s, éd. Rivages, 2017, p. 87.

[12] Maurice Matieu, *La banalité du massacre, suivi de la lettre de Paul Cézanne à Felix Klein et autres texte*s, p. 21.

défini par l'absence de résistance. Il n'a rien à voir avec les états d'agrégation de la matière : les solides aussi peuvent être des fluides, sans devoir passer de l'état gazeux ou liquide. Fluide est la structure de la circulation universelle, le lieu dans lequel tout vient au contact de tout, et arrive à se mélanger sans perdre sa forme et sa propre substance »,[13] affirme Emanuele Coccia dans sa métaphysique du mélange. Cézanne a installé par cette méthode une sorte de climat dans sa peinture où il se passe une inversion topologique entre les éléments et défini ainsi un état de fluidité. Cet état est diagrammatique, il est éminemment instable et fluant, ne cessant de brasser matières et fonctions de façon à constituer des mutations.

La bouteille de Klein



Dans la bouteille de Klein, il y a un chemin d'inversion qui permet au dehors d'être à l'intérieur même du dedans et à la fois d'en sortir, par emboîtement de réflexions. En ce sens, Cézanne installe une topologie de la peinture, il la rend pure lieu topologique et, par le geste de peindre, il fait diagramme. Cela peut s'interpréter comme une ambiguïté fondamentale inhérente au diagramme, qui admet au moins deux lectures contradictoires, à la fois, celle qui unit et celle qui sépare. Il ne fonctionne jamais pour représenter un monde préexistant, il produit un nouveau type de réalité, un nouveau modèle de vérité. « On peut toujours modifier la surface, dit Cézanne, mais on ne peut pas toucher à la profondeur sans toucher à la vérité

[13] Emanuele Coccia, *La vie des plantes*, p. 42.

».[14] Dans ce sens, Cézanne essaie de synthétiser les éléments de la nature à travers les volumes afin de créer une nouvelle atmosphère et une apparence de tous ses changements afin que le « tableau se module tout seul ». Le peintre a trouvé dans sa méthode un rapprochement avec les recherches topologiques de Felix Klein. La formalisation mathématique est nécessaire parce qu'elle offre la possibilité de concevoir ce que l'on ne voit pas. Ce que Cézanne a tenté de faire avec sa méthode de la modulation chromatique est que le chromatisme soit raisonné, qu'il y ait une logique de la sensation, où la méthode prime sur le sujet. Car ce qui rapproche la méthode du peintre de la géométrie projective du mathématicien, c'est la vision hétérogène. C'est lorsque peindre un corps humain ou une pomme s'appréhende de la même façon et avec la même intensité. « Au fond, Monsieur Klein, nos méthodes se ressemblent. Ce que vous nommez le groupe de [transformations spatiales] est, si j'ai bien compris, un invariant indépendant de l'objet. La transformation spatiale laisse inchangées les qualités géométriques de l'objet. Cela doit signifier que celles-ci sont indépendantes de leur situation dans l'espace, de leur orientation ou de leur retournement, c'est-à-dire de leur rapport en miroir, si je traduis dans mon langage. Mais alors si l'objet est le même ici et dans le miroir, l'objet comme spécifié disparaît, il n'a plus aucune utilité. Il devient ce que j'appelle un motif ».[15] Le motif chez Cézanne est le diagramme : c'est l'entrecroisement de la géométrie comme charpente et de la couleur comme sensation colorante. Seule la géométrie est abstraite, et la sensation n'est qu'un point de vue éphémère, elle est confuse, elle manque de durée et de clarté. C'est ce que Cézanne reprochait aux impressionnistes, leur sensation confuse. Le motif est la méthode diagrammatique qui permet de rendre la géométrie concrète ou sentie et de donner à la sensation la durée et la clarté nécessaire afin que quelque chose de nouveau en sorte. Ce motif diagrammatique est analogique, modulaire, car il permet de mettre en connexion immédiate des éléments hétérogènes. Il introduit entre les éléments une possibilité de connexion proprement illimitée dans un plan fini et dont tous les moments sont actuels et sensibles. D'après Deleuze, la peinture est l'art analogique par excellence. Elle est même la forme sous laquelle l'analogie devient langage, trouve un langage propre : en passant par un diagramme. Car « si la représentation est indépendante de l'intuitif, des figures hétérogènes peuvent représenter la même chose. Si l'espace est défini par une métrique, l'espace est défini par une méthodologie et cette méthodologie c'est le morcellement. Il n'y a plus de surface dans sa totalité, mais morceau par morceau. Du coup, c'est l'infini lui-même qui va être atteint. Si en effet, dans la géométrie projective, la section du cône fait image de l'espace dans sa totalité, alors l'infini de la géométrie élémentaire, le plan, devient une section de cône que moi j'appellerais imaginaire. Alors aussi, il n'y a plus d'homogénéité de l'infini mathématique »,[16] explique l'artiste.

Cézanne termine sa lettre avec un doute : « je suis vieux et je suis inquiet. Longtemps, j'ai pensé qu'il serait bien de former des élèves à ma méthode.

[14] Ibid. p. 22.

[15] Ibid. p. 28–29.

[16] Ibid. p. 29–30.

Maintenant, l'affreuse pensée me vient que, si le morcellement auquel j'ai été conduit sur le plan du tableau était absolument nécessaire, j'ai peut-être introduit par là le morcellement de la peinture elle-même, j'ai peut-être précipité la peinture sur une juxtaposition. Finalement, cher Monsieur Klein, en voyant l'abstraction chez vous, c'est la catastrophe que je vois chez moi. Je vais détruire la peinture ».[17] Cela nous renvoie ici à Louis Marin qui écrit : « M. Poussin ne pouvait rien souffrir du Caravage et disait qu'il était venu au monde pour *détruire la peinture* ».[18] C'est à partir de la notion de « catastrophe », opérée chez chacun des peintres Turner, Cézanne, Van Gogh, Klee, Kandinsky et Bacon, que Deleuze aboutit dans son raisonnement à penser l'acte de peindre comme affrontement à une catastrophe : « Le diagramme est bien un chaos, une catastrophe, mais aussi un germe d'ordre et de rythme. C'est un violent chaos par rapport aux données figuratives, mais c'est un germe de rythme par rapport au nouvel ordre de la peinture ».[19] Par ailleurs, ce terme dans la théorie des catastrophes fondée par René Thom, désigne le lieu où une fonction change brusquement de forme. La construction d'une méthode en peinture, ou dans d'autres formes d'expression, permet de saisir au mieux ces changements brusques de forme, ces singularités, et de capter la profondeur atmosphérique en instaurant un système de lignes et de plans afin de composer dans le chaos. Cette catastrophe dans l'acte de peindre est inséparable d'une naissance, dans le cas de Cézanne, celle de la couleur.

« La hiérarchie qui ordonnait l'échelle des différences entre une chose et ses qualités, par exemple, a été réduite à néant. Les diverses qualités d'une chose sont devenues des choses au même titre que la chose. Auparavant, le rouge était subordonné à la table dont il constituait la couleur; et puis le rouge est devenu un quelque chose au même titre que la table »,[20] écris Tristan Garcia. C'est dans ce sens que nous pouvons lire l'histoire de la peinture, lorsque la couleur rouge cessa d'être peinte pour représenter une table rouge mais plutôt pour ce qu'elle est, comme dans les peintures de Mark Rothko ou d'Yves Klein. Aussi, avec Marcel Duchamp, les choses sortent du cadre d'un tableau pour être exposées telles quelles. Mais avec Cézanne, et à travers sa méthode, avait déjà été à l'œuvre cette méthode que Deleuze nomme *diagramme*. La notion de diagramme, permet d'installer dans le travail artistique un dispositif critique, beaucoup plus qu'esthétique. Ce régime de pensée permet d'étudier le processus de variation des choses, d'expliciter par le geste un phénomène dans son devenir, comme l'intuition. Il permet aussi de saisir intuitivement le réel et de le tracer sans se préoccuper des dialectiques entre contenu, expression, analogique, numérique, ou sans se préoccuper encore du style, pour se pencher enfin sur la mise en forme de la machine de pensée pure à travers une

[17] Ibid. p. 31–32.

[18] Louis Marin, *Détruire la peinture*, Flammarion, Champs arts, p. 11.

[19] Gilles Deleuze, *Francis Bacon, Logique de la sensation*, *op. cit.*, p. 95.

[20] Tristan Garcia, *Forme et objet. Un traité des choses*, *op. cit.*, p. 41. Il ajoute : « Une qualité, un mot, un concept, un jugement, un avis—plus rien n'est rien, plus rien n'est tout, plus rien n'est moins-que-chose, plus rien n'est plus-que-chose, et tout est simplement quelque chose », p. 42.

œuvre protéiforme que conduit le geste. Il représente pour l’artiste qui y s’intéresse un instrument précieux à la fois de neutralité et de mesure.

Farah Khelil, *Point d’étape*, 2017, photographie encadrée, documents, carte postale, verre, bois, billes de jeu. © Farah Khelil et ADAGP

Aujourd’hui l’artiste fait face à une masse très importante de données à travers documents préexistants, images fixes ou mobiles, textes ou sons, qu’il utilise avec ou dans son œuvre. Lorsqu’ils ne sont pas représentés ou utilisés comme document informatif, ces médiums sont exposés, composés dans le lieu de monstration et font œuvre. Lorsqu’on ouvre le système de représentation vers un système de présence des choses, nues et en relation avec le lieu d’exposition, une forme s’impose, celle des *diplays*. Le *display*, contrairement au modèle moderne de l’œuvre d’art accompagnée de ses documents informatifs, permet la naissance d’une atmosphère mixte; un mélange d’œuvre-exposition et d’exposition-œuvre. Cet état hybride, porte en lui les questions ontologiques des œuvres et des médiations, ainsi que des gestes et des actions qui les font naître. Le *display* permet de questionner l’exposition à travers les gestes de mise en relation et de formalisation dans l’espace. Ce n’est pas un format ou une norme, mais un dispositif de pensée. Il nécessite une action et une performance qui accompagnent l’intuition, le hasard et la nécessité du lieu.

C’est en questionnant une pratique personnelle de la peinture au-delà de l’illustration et de la représentation, que *Point d’étape* (2016) a pris forme comme une réflexion de cette question : « pourquoi faire peinture ?»; ou encore pour y échapper. Il est question ici de technique de visualisation de données, d'une méthode au-delà d'une tradition, vers une stratégie de fuite. Cette technique n'est pas digitale, mais analogique et c'est dans ce sens qu'elle émane d'une pensée-peinture. *Point d'étape* est un dispositif agencé qui trace des expériences de pensée et tente d'organiser et de synthétiser, sous forme de séquences, des sensations souvent indisciplinées. Il s'agit, d'après les termes de Gilles Châtelet, de mettre en scène « une désorientation pour orienter et imposer un projet, qui se donnera ensuite pour le plus évident ».[21] *Point d'étape* fonctionne de cette façon, un peu comme une carte mentale sur laquelle se projettent des gestes de lectures, des matériaux qu'on utilise lors des recherches en atelier ou en bibliothèque, ainsi qu'une sélection d'objets collectés, posés ou déposés en relation. *Point d'étape*, nous renvoie en anglais au terme *waypoint*, « point de cheminement » ou « point de passage », ce qui renvoie dans le jargon des navigateurs à une balise qui attire l'attention sur un emplacement avant de changer de cap. Aussi il y a une racine commune avec le sens d'*itération*, qui signifie en mathématique l'action de répéter un processus. Le terme *itération* est issu du verbe latin *iterare* qui signifie « cheminer », de *iter* « chemin ». Agencées et éclatées dans l'espace, les choses se composent tantôt par séparation, tantôt par relations entre différents objets hétérogènes. C'est cette hétérogénéité des formats et médiums qui composent *Point d'étape* et appellent l'œil du spectateur à se mouvoir dans la pensée, dans un système de mise en abîme, de superposition et ainsi à prendre les différents chemins de pensée parcourus. *Point d'étape* ne restaure pas une forme d'expression d'un réel en particulier, ou une forme de contenu précis. Il s'agit avec cet agencement visuel, de créer une image de pensée insaisissable et perplexe, où tous les formats et médiums (peinture, dessin, vidéo, documents, objets, etc.) ou (solide, fluide, vivant ou artificiel) peuvent coexister. Chaque composition représente une séquence, dans laquelle chaque élément est en rapport avec d'autres éléments et créent ainsi une Figure. Comme une image de pensée, *Point d'étape* est persistante, visuelle, immédiate et dans le même temps elle possède une durée, un après-coup qui invite à y faire retour et à la percevoir sur le mode d'un processus. Il peut s'agir de documents d'archive, de livres, d'une citation, mais souvent il s'agit de rebuts, comme des billes de jeux de Solitaire, des billes antimites, des faux-cils (cosmétiques) ou encore des chutes d'anciennes cartes postales découpées, des plantes, des graines de Pensées ou encore des dessins à l'aquarelle, une vidéo ou un arbre artificiel pour maquette d'architecture. À partir d'anciens travaux et recherches, il s'agit de composer un nouveau type de réalité afin de donner à voir le devenir d'un raisonnement, et peut-être d'inventer un nouveau cycle d'images en rapport avec le réel, ou de projeter une nouvelle perspective visuelle des formes.

Dans *Point d'étape*, généralement les objets ont plusieurs fonctions; des photographies et documents, encadrés ou pas, jouent le rôle de plateaux qui accueillent des objets divers, des volumes en verre ou en bois tiennent des livres ouverts comme

[21] Gilles Châtelet, *Les enjeux du mobile*, *Mathématique, physique, philosophie*, Paris, Seuil, 1993, p. 35.

des marques-pages, ou encore une feuille d'arbre est glissée dans le plissement d'une page comme une épingle à nourrice. Mettre en relation ces objets permet de créer du sens : ils évoquent, à travers leurs rapports, des sensations. Cette méthode s'articule autour de confrontations, de tissage, de transversalité et de gestes de montages d'éléments hétérogènes aussi bien dans leurs matérialités que dans leurs spatialités ou leurs temporalités. Les hétérogénéités sont principalement des images empruntées, d'autres produites (dessin, aquarelle, photographies, notations, vidéo), utilisées à la fois pour ce qu'elles véhiculent et comme matériaux; ainsi que de nombreuses lectures utilisées comme sources de mots ou de concepts. Il y a là une dimension paradoxale à l'épreuve de la lecture, qui vient imposer la lenteur, l'attention, voire de l'immersion dans l'image, et dans son système de suggestion allusive. Le lecteur est amené à opérer par stations successives, à plonger en profondeur dans la matérialité, la conceptualisation et le rébus que chaque image contient, s'activant grâce à une dynamique du regard, de la pensée et de l'imagination. *Point d'étape* est un format qui a la capacité de s'inscrire dans une typologie des images de pensée à travers leurs formes tantôt abstraites, tantôt figuratives.

Les séquences de *Point d'étape* sont placées à un niveau de regard assez bas, à plat, au niveau des mains, voire à même le sol. Pour voir l'ensemble, le spectateur doit baisser les yeux et la tête, voire s'accroupir pour se pencher sur des détails et découvrir des jeux de trompe-l'œil. En développant une esthétique « plate », où tous les éléments sont indifférenciés et disposés au même niveau, *Point d'étape* permet de faire coexister, de mettre en relation des éléments hybrides et hétérogènes, et ceci sans formatage; dans le sens d'un support en vue d'accueillir des données, une forme de stockage et un contenant permettant l'usage de la matière et des données d'une recherche. Dans ce milieu, tous les formats peuvent exister ensemble et faire une multiplicité spatio-temporelle. Dans son livre intitulé *Simondon*, Jean-Hugues Barthélémy rappelle que « Les prétendues « choses », facilement considérées par le bon sens comme ce qui existe avant d'entrer en rapport, ne sont pourtant faites que de relations ».[22] Ce sont ces relations qui sont mises en évidence dans les agencements de *Point d'étape*, qui portent en eux un potentiel imaginaire. Dans ce dispositif polymorphe de la pensée, il y a une dialectique de montage. Lorsqu'on photographie ce champ en vue plongeante, on aplatit tous les objets qui deviennent—ou redeviennent—image. Les billes réfléchissantes et miroitantes, récupérées des plateaux de jeu de Solitaire, sont à la fois disposées sur les documents et représentées en photographie. Il se crée alors, comme l'écrit Deleuze, une « zone d'indiscernabilité » ou d'indéterminabilité objective entre deux formes, dont l'une n'était déjà plus et l'autre pas encore. Se substituent alors à la forme initiale des rapports originaux dans un système d'itération, un processus persistant, un motif redondant ou encore une mise en abyme.

D'après Cézanne, les plans et les volumes permettent ce traitement, et jouent un rôle d'instrument de neutralité par rapport au sujet et au contenu. Ses objets

[22] Jean-Hugues Barthélémy, *Simondon*, PUF, 2014, p. 48–49.

sont un prétexte. Il dit : « Ce qui m'intéresse, c'est la synthèse rationnelle, la généralisation rationnelle qui permet le style. L'objet représenté disparaît devant la méthode, je n'ai plus de sujet. Qu'est-ce que j'ai ? Un motif qui est un prétexte. Je suis une sorte de tapisserie persan ».[23] Ces agencements sont conçus suivant une dimension géométrale, de gestes assez disciplinés, qui quadrillent les éléments en suivant des lignes et des niveaux de stratifications, invisibles mais qui sautent au regard, et c'est à partir de cette construction spatiale que les objets finissent par se chevaucher et par se superposer, en suivant un jeu de transparence et d'opacité. Bien que discipliné et ordonné, ce geste de disposer et d'agencer les différents éléments reste principalement guidé par une intuition, tâtonnant, doutant, répétant jusqu'à se convaincre d'un résultat, même inachevé. Par ces gestes, il y a un besoin d'ouvrir les images et les documents qui habitent la mémoire mentale ou physique. Il s'agit d'éclater cette masse de données afin de composer autrement avec des sources d'ailleurs, du dehors. Par ces gestes, il y a un besoin d'échapper au système classique de la représentation; celui qui fige une identité et un point de vue unique aux choses.

D'après Tristan Garcia, « pour qu'il y ait des régimes objectifs de la connaissance, de la pensée, de l'action, de la mémoire, de la volonté, de l'intention, de la perception, de la proprioception, du désir ou de tout autre rapport actif à des objets, il faut que ces objets soient aussi d'emblée des choses seules : leur multiplicité est impossible sans la solitude exclusive de chacun. Si chaque chose n'était pas exclusivement seule, il n'y aurait multiplicité de rien ou il n'y aurait qu'une multiplicité compacte de tout. Pour qu'existent des multiplicités données ou construites d'objets (objets matériels, objets historiques, objets de langage, objets idéels), il faut que ces objets soient assez distincts pour être ensemble : cette condition de distinction pour une appartenance à un ensemble, c'est justement la solitude au monde. Et si chaque chose est dans le monde, c'est parce qu'aucune n'est en soi ».[24] Ce qu'on peut retenir de la pensée diagrammatique, ce n'est pas tant l'esthétique visuelle des graphes ou des schémas, que la mécanique du processus de création, la synthèse du temps de l'œuvre, le geste de pensée, l'intensité dynamique des forces et le dynamisme qui relève de cette pensée critique qui met en commun toutes les techniques. Les gestes de relations induits par l'expérience de pensée se cristallisent dans les différentes techniques, allant de la peinture au *display*, à travers des agencements expérimentaux et polymorphe de quelques choses au pluriel qui témoignent de l'état de pensée dans lequel se trouve l'artiste à un moment, face au réel, dans une solitude peuplée.[25]

[23] Ibid. p. 26.

[24] Tristan Garcia, *Forme et objet. Un traité des choses*, op. cit., p. 68.

[25] « C'est aussi pour une autre raison que "penseur privé" n'est pas une bonne expression : car, s'il est vrai que cette contre-pensée témoigne d'une solitude absolue, c'est une solitude extrêmement peuplée, comme le désert lui-même, une solitude qui noue déjà son fil avec un peuple à venir, qui invoque et attend ce peuple, n'existe que par lui, même s'il manque encore . . . », Gilles Deleuze et Felix Guattari, *Mille plateaux*, Paris, Éditions de Minuit, 1980, p. 467.

Farah Khelil est artiste. Elle est auteure d'une thèse de doctorat en Art et Science de l'art qu'elle a soutenue en 2014, intitulée « *L'artiste en traducteur. La pensée du diagramme comme expérience de création* » à L'École des Arts de la Sorbonne, Paris 1 Panthéon-Sorbonne. Elle participe à de nombreuses expositions personnelles et collective à l'échelle internationale.

Part VIII
Poetics and Politics of Diagrams

Diagrammes du possible : de l'espace des phases au sujet politique

Tatiana Roque

Abstract Nous vivons un temps de renfermement des possibles. C'est d'un tel renfermement du possible qu'il s'agit lorsqu'on remarque l'allure inexorable avec laquelle les politiques néolibérales se présentent–de plus en plus aidés par des gouvernements autoritaires. There is no alternative, slogan de Thatcher, revient mais maintenant comme une imposition. Le possible, ou sa fin, fait débat. C'est justement dans ce face-à-face avec ce qu'on ne sait pas où ce que cela va donner que la notion de diagramme devient utile, dans cette transition vers de nouvelles possibilités de dire. Parce que faire diagramme est –en fait–moins que dire. L'expression est diagrammatique lorsqu'elle ne se constitue pas encore en programme. Tout de même des opérations peuvent avoir lieu à partir des tendances, des pistes, des suggestions ; des opérations qui se prennent directement à des ébauches, à ce qui est à peine esquissé. La fonction du diagramme est de suggérer, comme remarque Deleuze: une tâche qui introduit des possibilités de fait. Le diagramme doit être opératoire et contrôlé en même temps. Produisant une catastrophe qui ne submerge pas la violence du fait.

Keywords Diagramme · Politique · Sujet · Espace des phases

Nous vivons un temps de renfermement des possibles. Noyés dans l'excès de réel—un réel qui nous semble à chaque fois plus surréel—la question se pose sur comment s'inventer un possible. Non pas un possible qui soit le contraire du réel, qui soit trop loin du réel de notre expérience. Mais développer des instruments pour repérer des fourmillements dans ce réel, des saillances qui ressortent de ce réel qu'on croit pourtant imperméable à des nouveaux possibles. C'est d'un tel renfermement du possible qu'il s'agit lorsqu'on remarque l'allure inexorable avec laquelle les politiques néolibérales se présentent—de plus en plus aidés par des gouvernements

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autoritaires. *There is no alternative*, slogan de Thatcher, revient mais maintenant comme une imposition.

Le possible, ou sa fin, fait débat aussi souvent dans la pensée qui se veut anti-système, comme dans les écrits sur la non-mort du capitalisme[1] ou des livres comme celui de Mark Fisher *Capitalist Realism: Is there no alternative?* Il est aussi question de se trouver une place pour le possible dans les discussions sur des nouveaux réalismes ou l'accélerationnisme, dont le manifeste revendique la libération de l'horizon: « L'effondrement de l'idée d'avenir est symptomatique du statut historique régressif de notre époque, bien davantage que d'une maturité sceptique, comme les cyniques essaient de nous le faire croire de tous les bords du champ politique ».[2]

C'est donc un choix politique d'insister sur l'urgence de parler du possible. « Soyons réalistes, exigeons l'impossible », était l'emblème d'il y a 50 ans. Ensuite, il n'y a pas très longtemps, dans des mouvements altermondialistes, c'était « un autre monde possible » qu'il fallait créer. Maintenant, tout cela semble fini. Un nouveau réalisme s'impose, non sans une certaine souffrance, à tous ceux qui, pris par la mission de changer le monde, ont investi le possible sous le mode du rêve ou de l'utopie. Tandis que quelque chose nous dit que notre temps est plutôt celui de la dystopie.

Au cours de nos investissements politiques, nous ressentons une division qui capture notre rapport au réel ; quelque chose entre le « on ne peut rien faire » et le « on sait déjà ce qu'il y a à faire ». Personne ne sait (au moins parmi ceux qui s'opposent à la droite). Ceux qui disent savoir ce qu'il faut faire en disent toujours trop. En deçà des bavardages, il nous reste l'engagement dans une possibilité de dire et de faire sans que les termes soient donné d'avance, sans garantie, sans point d'appui, sans référence. Ni dans le domaine du langage ni dans celui des institutions.

C'est justement dans ce face-à-face avec ce qu'on ne sait pas ce que cela va donner que la notion de diagramme devient utile, dans cette transition vers de nouvelles possibilités de dire. Parce que *faire diagramme* est—en fait—moins que dire. L'expression est diagrammatique lorsqu'elle ne se constitue pas encore en programme. Tout de même des opérations peuvent avoir lieu à partir de tendances, de pistes, de suggestions ; des opérations qui se prennent directement à des ébauches, à ce qui est à peine esquissé. La fonction du diagramme est de suggérer, comme remarque Deleuze en citant Francis Bacon: une tâche qui introduit des possibilités de fait. Le diagramme doit être opératoire et contrôlé en même temps. Produisant une catastrophe qui ne submerge pas la violence du fait. C'est ainsi que, dans le cas de Bacon, une figuration sort du tableau du réel pour constituer une nouvelle précision.

C'est à peu près cette puissance opératoire du diagramme que je veux récupérer dans la politique. Aussi parce qu'elle peut donner lieu à un nouveau pragmatisme,

[1] Colin Crouch, *The Strange Non-death of Neo-liberalism*, Polity, 2011.

[2] Alex Williams, Nick Srnicek, "Manifeste accélérationniste", in Multitudes 2014/2 (n° 56), Traduction Yves Citton, p. 23–35.

qui ne revient pas seulement à « faire avec », en partant des conditions déjà données, mais qui s'approche d'un pari—un pari à partir des conditions à peine esquissées. Je dirais que l'opération diagrammatique qui m'intéresse est de l'ordre d'un pari. Un pari sur un possible qui est encore en train de se faire. Un possible qui ne peut nous affecter que sous forme des restes de croyance dans le monde—dans ce monde. On ne sait pas quoi faire pour le changer, mais cela ne nous arrête pourtant pas. J'ai envie de dire qu'on cherche des brèches, mais il s'agit là encore d'une idée négative, la brèche étant ce moindre espace entre des blocs gênants du réel. On cherche plutôt des positivités, des points protubérants—exubérants—qui sortent du réel, et qui ne s'y adaptent pas très bien. Voici un principe de définition du possible: un morceau de réel qui veut s'en échapper. Et on en fait un pari. Un morceau du réel qui pointe vers quelque chose d'inconnu encore, et qui peut même nous faire peur, mais qui est déjà là. Quels instruments permettent de plonger là-dedans ? C'est la question à laquelle le diagramme aide à répondre.

Il n'existe pas, d'un côté, des blocs de réel qu'il nous revient d'interpréter et, de l'autre côté, le possible comme utopie, vécue de façon mélancolique. La réalité est là, et elle presse. S'allier avec le possible est se trouver une place à partir de laquelle risquer un pari. Un point d'appui. Une prise dans le réel pour faire le saut. Sans se soucier de savoir si cela va marcher ou pas. Il faut le faire. De toute façon, c'est tout ce qu'on peut faire. (Un peu comme du surf.) Tracer des positivités pour se permettre de ne pas se noyer dans ce réel. Et cela en même temps qu'on trace dans le possible des diagrammes des dires et des faires.

C'est en agissant sur des conditions atmosphériques que le diagramme acquiert un rôle pilote dans la constitution d'un réel à venir. L'énoncé suppose déjà un agencement, souligne Guattari: je dis une chose et cela me rapproche de quelqu'un, crée un monde possible, sans intention. Il s'agit, dit-il, d'une condition atmosphérique. Une diagrammatique présente des degrés d'intensité, de résistance, de conductibilité, d'échauffement, de vitesse, bref, de mobilité (comme cela sera développé par Gilles Châtelet).

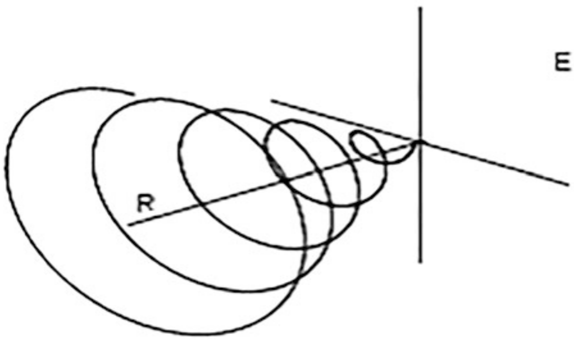
Il faut que des morceaux de réalité soient suffisamment sortis de leur place pour qu'ils gagnent de nouveaux traits d'expression, disent Deleuze et Guattari dans *Mille Plateaux*. La diagrammatisation fait ce passage. Lorsqu'une matière sort de son inertie, en acquérant une certaine mobilité, elle donne lieu au signe. Quelque chose se présente et instaure, d'un seul coup, des formes d'expression et des formes de contenu qui lui sont spécifiques. La notion de diagramme est proposée dans ce livre justement pour penser un régime de signes qui échappe, en même temps, du schéma de la représentation et des catégories du langage. Comment engendre-t-on une réalité sans passer par un mécanisme de représentation ?

Deleuze voit dans la proposition d'une nouvelle strate (ni de contenu ni d'expression) le trait originel de l'œuvre linguistique de Hjelmslev. « On projette la forme d'expression et la forme de contenu sur le sens ». Deleuze souligne que ce que Hjelmslev appelle « sens » est une matière non linguistiquement formée. En effet, dès qu'elle devient linguistiquement formée, cette matière devient substance de contenu ou substance d'expression. Le sens est donc une matière non linguistiquement formée qui pourtant peut être parfaitement formée du point de vue sémiotique. Et pour qu'on puisse parler de quelque chose sémiotiquement formée qui ne soit pas linguistiquement formée il faut une sémiotique a-signifiante. Une sémiotique qui fait fonctionner la signification mais qui n'est pas encore de l'ordre de la signification, donc ni expression ni contenu, ni signifiant ni signifié. « Ordonner, conseiller, promettre, donner la parole, faire l'éloge, prendre au sérieux ou à la légère, ricaner sont des actes diagrammatiques, des sémiotiques a-signifiantes qui font fonctionner la signification ». (*Révolution moléculaire*, p. 443).

L'espace des phases: exemple des diagrammes qui ne sont pas représentation

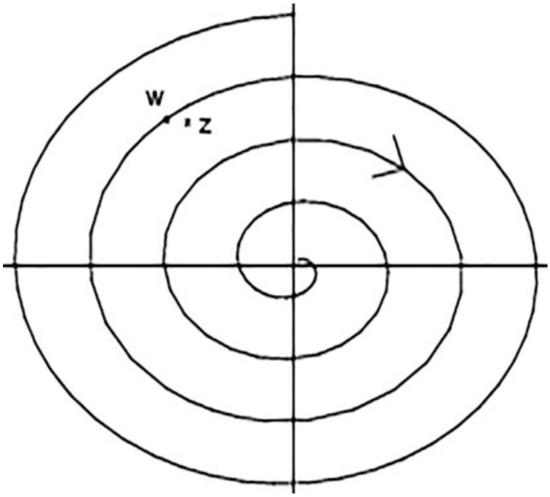
L'exemple aide à montrer que les diagrammes ne sont pas de l'ordre de la représentation. Cela s'obtient par la description de l'espace des phases en contraposition aux graphiques de type cartésien.[3]

Si l'on représente une trajectoire dans un graphique cartésien, le temps sera une coordonnée: la variable indépendante. Supposons un graphique de trois dimensions, dans la figure, dont deux de ces dimensions définissent la position bidimensionnelle d'une particule (sur le plan E) et le troisième axe (R) représente le temps.



Au-delà de cette représentation cartésienne du mouvement, il est possible de tracer un champ de vecteurs, défini, dans ce cas, dans un espace à deux dimensions. Lorsque les valeurs sur l'axe R augmentent, la courbe de la figure tourne avec des rayons de plus en plus grands. Ce mouvement peut être projeté sur le plan E. Si les valeurs sur R déterminent des instants successifs et si en parcourant R en direction positive on réalise une évolution dans le temps, la courbe trace une dynamique dans le plan E décrite par le mouvement ci-dessous:

[3] Livre Deleuze Philosophe des Multiplicités.

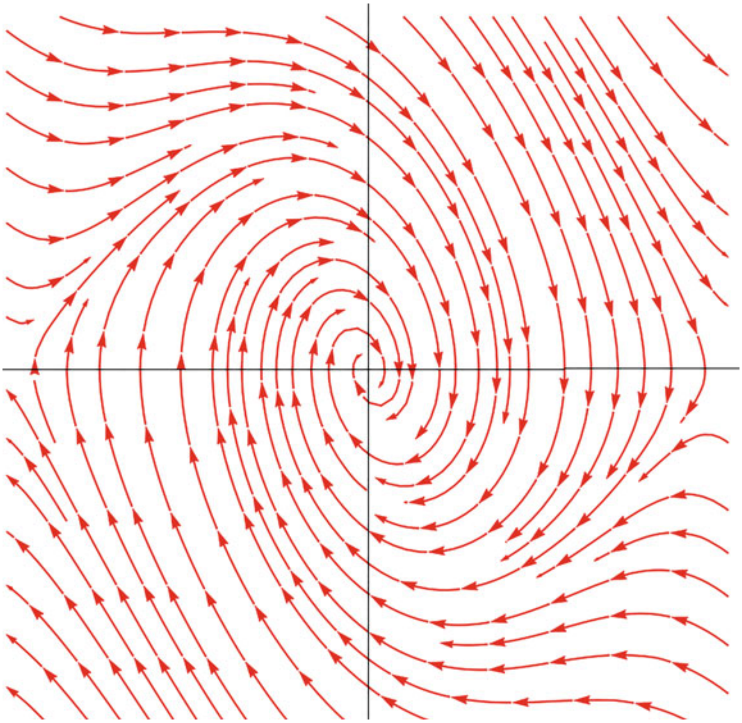


Le temps dans cet espace n'est pas explicite, c'est-à-dire, qu'il n'est pas une coordonnée indépendante. Dans l'espace des phases, le temps n'apparait pas comme une variable indépendante. Il devient intrinsèque à la courbe: il est la flèche. Dans un même graphique, on peut alors tracer l'ensemble de toutes les dynamiques possibles à partir des conditions initiales distinctes.

Dans le premier graphique, cartésien, on ne saurait pas représenter la variation des conditions initiales sur un même dessin. Cela parce qu'il est impossible de savoir, pour une courbe, où sera la trajectoire à l'instant correspondant à une autre courbe. On sait pourtant comment la trajectoire se comportera qualitativement dans le futur. Pour cette raison, on peut négliger le temps comme variable explicite.

Cette opération est à l'origine de l'analyse qualitative des solutions des équations différentielles, proposée par Henri Poincaré.[4] Cette approche implique une nouvelle manière de concevoir les solutions des équations différentielles. Ce qu'on perd en précision, en vue de ne pas savoir exactement où sera la trajectoire à un instant donné, on le gagne par la possibilité de connaître l'allure de l'ensemble des solutions définies par des conditions initiales distinctes. Ainsi, on n'aura plus un graphique pour chaque solution mais l'on envisage toutes les solutions à la fois dans l'espace des phases (comme dans la figure suivante).

[4] Sur les courbes définies par les équations différentielles, *Comptes rendus hebdomadaires des séances de l'Académie des sciences de Pari*s, 93, 1881, p. 951–952, et 98, 1884, p. 287–289. Et aussi, *Journal de mathématiques pures et appliquées*, 2, 1886, p. 151–217.



Où la trajectoire sera exactement à un instant donné t ? On ne sait pas. Mais ce « ne pas savoir » ne nous arête pourtant pas, ne gâche pas la possibilité de dire quelque chose d'important sur les comportements des trajectoires. Dans cette nouvelle manière de concevoir une *solution,* il y a un déplacement du « on ne peut rien dire » (où exactement chaque solution va être à un instant donne) à un « voilà ce qu'on peut dire » (ces sont toutes les solutions possibles). Avant, il s'agissait de représenter les solutions effectives. Maintenant, ces sont les solutions possibles qu'on envisage. On ne les représente pourtant pas, faisons attention, parce qu'on ne peut pas connaître tous les données de réalité pour dessiner les solutions effectives. C'est le « ne pas connaître » en soi-même qui est joué pour faire ressortir un nouveau mode de connaître, un mode qui est celui du possible. Pour cette raison, l'espace de phases n'est pas une représentation. C'est un diagramme ou, plus précisément, c'est le produit par un diagramme, par une diagrammatisation qui rend dicible et connaissable toute sorte des choses dont il n'était pas question auparavant (quand on ne disposait que de la représentation cartésienne et analytique des solutions).

Le diagramme a à voir justement avec cette façon de retracer la ligne entre le dicible et le non dicible. Ce n'est pas question de dire de nouvelles choses sur une réalité déjà donnée, mais de produire d'autres dicibles, du connaissable, en se plongeant sur un espace de possibilités. Ce déplacement se lie aussi à des facteurs contingents, historiques. Cette nouvelle manière de concevoir une solution pour les équations différentielles vient des problèmes issus de la mécanique céleste, dans lesquels les questions importantes ont trait à des comportements asymptotiques, comme c'est le cas du problème de la stabilité du système solaire ou du problème des trois corps.

L'espace des phases, habité par des trajectoires, peut recevoir différentes interprétations selon des problèmes physiques particuliers. C'est une entité mathématico-physique. Dans l'espace des phases, chaque courbe est donnée avec ses perturbations. Avec un autour. C'est son atmosphère. Chaque trajectoire est tracée avec d'autres solutions possibles à partir des conditions initiales que l'on obtient par la perturbation de celle qui a définit la trajectoire de départ. Puisque la condition initiale de départ ne peut être jamais connue avec une précision illimitée, chaque solution s'exhibe déjà avec les déviations qu'elle peut souffrir si sa condition initiale changeait d'un tout petit peu. Chaque solution vient entourée d'une foule de possibles.

Le diagrammatique fait que l'être mathématique est donné en même temps que son champ des possibles. Il n'est pas figé, il n'est pas une abstraction définie de manière axiomatique. La condition initiale, qui est un point, est enceinte de toutes les perturbations possibles, des différents types d'ébranlement qu'elle peut souffrir. L'impossibilité de droit de connaître cette condition avec une précision infinie, qui ne pourrait être qu'approchée par la physique-mathématique, devient inscription, elle passe dans le graphique même, avec ses flèches, et se convertit dans une multitude des trajectoires possibles. Pour cette raison, les mathématiciens disent qu'une telle manière de concevoir les solutions des équations différentielles est *qualitative*, par contraste avec la tendance analytique précédente qui voulait que les solutions soient explicites. Les solutions sont acceptables, du point de vue des systèmes dynamiques, si elles se constituent comme des possibilités de fait—des diagrammes donc.

Faute de programmes on trace des diagrammes

Je reviens au problème du possible en politique. La gauche est en crise, une crise due en bonne partie aux changements du sujet politique sur lequel ses théories se sont appuyées. Les nouveaux mouvements n'ont pas réussi non plus à créer des institutions nouvelles, qui soient à la hauteur des soulèvements post-crise de 2008. Du point de vue des programmes, ces mouvements, intensifiés à partir de 2011, n'ont pas eu de grand succès. La reprise du pouvoir en Égypte, après l'occupation de la place Tahir ; Erdogan en Turquie, après les mouvements du parc Ghezi ; Trump après *Occupy Wall Street* et la lutte des 99%. Il en est de même de la déposition de Dilma, l'arrestation de Lula et maintenant l'élection de l'extrême-droite au Brésil, tout cela après les journées de protestation qui ont secoué le Brésil en juin 2013.

Le titre du nouveau livre de Naomi Klein reconnaît à lui seul qu'il faut faire un pas en avant: il ne suffit pas de dire non (*no is not enough*).[5] La faillite des

[5] Naomi Klein, No is not enough, Allen Lane, 2017.

institutions va de pair avec le vide des nouveaux programmes de gauche. Il est vrai que les mouvements récents ne se sont pas organisés autour de programmes clairs ; on a eu même du mal à comprendre exactement s'ils proposaient un agenda. Cela a été aussi le cas de Juin 2013 au Brésil et de *Nuit Debout* en France. Un des slogans récurrents pendant l'occupation des places, « nous n'avons rien à revendiquer », indique la force de destitution incarnée dans ces mouvements, qui vise plutôt à signaler aux gouvernants qu'ils ne sont pas considérés comme interlocuteurs légitimes. Le cas des Gillets Jaunes, plus récent, semble avancer quelques points précis de revendication, même si on ne sait pas encore en prévoir le déroulement.

Les journées de Juin 2013 au Brésil ont témoigné d'une forte dispute au niveau de l'énonciation. Un des plus forts slogans à rassembler les manifestants de toutes tendances fut: « não vai ter Copa » (la Coupe du Monde n'aura pas lieu). Dans le pays du football, tout d'un coup, les gens se mettent à hurler contre la Coupe du Monde. Est-ce que les brésiliens n'aiment plus le football, est-ce qu'ils sont en train de critiquer le marché du foot devenu milieu d'affaires ? Tout cela et rien de cela. Le cri était l'expression intense d'une mutation subjective. Il n'aidait pas trop à comprendre le phénomène de se demander « qu'est-ce que ça veut dire ? ». Parce que la mutation subjective en question a lieu lorsque la division entre un plan de contenu et un plan d'expression n'est pas encore tracé. « Qu'est-ce que ça veut dire ? » est alors une tentative de faire rentrer l'expression dans le régime de la représentation.[6]

Tout de suite après les premiers soulèvements de 2013 au Brésil, une guerre d'énonciations a mobilisé l'opinion publique, les intellectuels et les médias, ce qui a fini par opérer des reconversions dans le domaine du signifiant. La mutation subjective devrait à tout prix se rabattre sur des formes déjà connues de l'action et de l'organisation politique. Peu à peu la mutation subjective se renfermait alors complètement sur des polarisations, renvoyées aux appareils binaires de la politique institutionnelle. En faisant passer la diagrammatique au deuxième plan, il ne restait qu'un système des significations déjà disponible, qui s'efforçait pour maintenir les clivages politiques, amenant la gauche à désigner le mouvement comme étant de droite. À partir de là, des démarcations surdéterminées par de telles significations prenaient de plus en plus le devant de la scène, en désamorçant la force subjective de l'événement.

Les mouvements de 2013 au Brésil ont pratiqué des manières de dire et de faire, des modes d'être ensembles et des modalités d'être contre qui ont été sans arrêt menacés par des positions polarisées, des blocages de l'expression et des désaffections du désir. C'est tout le système des significations dominantes qui hantait l'événement et les agencements sémiotiques qu'il tentait de mettre en place: « On refuse de considérer que les agencements sémiotiques de toute nature doivent nécessairement s'organiser en phrases compatibles avec le système des significations dominantes », rappelait Guattari.[7] Ce système a fini par réussir, en

[6] Philosopher au Brésil aujourd'hui, Rue Descartes 76, 2012/4.

[7] Félix Guattari, *Lignes de fuite: Pour un autre monde de possibles*, Éditions de l'Aube, 2011.

faisant rentrer l'indignation dans une moule reconnaissable: il fallait alors choisir de quel « côté » se placer. Il y a eu ainsi une retombée de la dimension diagrammatique de l'événement sur des programmes—des vieux programmes, qui n'étaient pas à la hauteur de l'événement.

La déposition de Dilma, le jugement de Lula et l'élection de Bolsonaro ont quelque chose à voir avec ça. Les cris des rues en 2013 indiquaient des chemins pour la correction des défauts des politiques publiques d'alors, dommage que personne au gouvernement n'ait entendu. Premièrement, par-dessous le slogan contre la Coupe du Monde, on critiquait le choix du modèle économique, un modèle dépendant de grands événements et de ce qu'ils apportent des grands travaux, en favorisant donc les entrepreneurs du secteur de la construction civile. Justement des entrepreneurs qui, en prison, ont fini par dénoncer Lula comme ayant été favorisé personnellement par leurs entreprises. Le procès contre l'ancien président du Brésil est très partiel et il est évident qu'il découle en grande partie des intérêts des secteurs politiques de reprendre le pouvoir. La deuxième indication des rues en 2013, et qui n'a pas été entendu par le gouvernement de l'époque, avait trait au fonctionnement de la démocratie. Partout dans le monde une crise s'annonce des mécanismes traditionnels de la représentation politique ; une crise qui dans plusieurs pays—dont le Brésil—est traité par l'extrême-droite en termes populistes. L'indignation généralisée contre la corruption qui a commencé à monter au Brésil à partir de 2013, et qui ne visait pas spécialement le PT, était le symptôme de l'insatisfaction contre l'absence de formes de contrôle et de participation démocratique qui puisse rendre plus transparent les jeux du pouvoir. Un problème qui n'est pas facile à résoudre et constitue le cœur même de la crise démocratique qui s'étend à plusieurs pays à l'heure actuelle. En France, en ce moment, il semble que les revendications référendaires des Gilets Jaunes vont dans ce sens—celui d'un approfondissement de la démocratie. La réponse à de telles questions n'est pas donnée d'avance et on ne peut donc pas imputer la responsabilité au Parti des Travailleurs de n'avoir pas su répondre à chaud aux désirs des rues en 2013. Ayant été pourtant un des gouvernements de gauche existant dans le monde en ce moment, on attendait au moins plus d'écoute et d'ouverture à ce genre d'insatisfaction. Mais le contraire a eu lieu. Les discours des intellectuels et des politiciens du PT ont accusé chaque fois plus fortement le mouvement comme étant de droite (ce qui n'était pas vrai en 2013). À partir de 2014, une vraie droite a commencé à s'organiser et a pris le relais de l'indignation par la suite, ayant été à la tête des manifestations de 2015 qui demandaient la déposition de Dilma et la prison de Lula. Ils ont gagné. Et ils ont élu Bolsonaro en 2018.

La crise du sujet politique

Les mouvements politiques de cette époque post-crise de 2008 n'ont pas réussi à créer des programmes, des organisations ou même des langages pour répondre aux sentiments de malaise avec le cours actuel de la politique et de l'économie. Autant la bataille politique à l'intérieur des mouvements, autant le rapport entre les différents niveaux de subjectivation qu'elle engageait appellent l'invention des discours politiques nouveau, des nouvelles tactiques et stratégies. Il faut encore créer un cadre institutionnel qui puisse supporter, organiser, défendre et donner consistance aux changements qui affectent pourtant déjà la subjectivité. La question de l'événement politique se montre à chaque fois plus lié au problème de l'institution et à la prolifération de nouveaux champs de possibles.[8]

L'*Anti-Œdipe* nomme trois types d'investissements subjectifs qui peuvent correspondre à trois attitudes politiques qui se manifestent lors des ruptures politiques. Deleuze et Guattari en parlent au sujet de mai 68, mais ces réflexions sont toujours utiles. À partir de la coupure subjective provoquée par un événement politique, trois investissements peuvent se composer:

- Des investissements préconscients qui portent sur les « intérêts » (la position des syndicats ou des partis) ;
- Des investissements subjectifs qui voient la possibilité et les conditions d'un nouveau *socius*, mais qui maintiennent la lutte sur le seul niveau molaire (la vision politique « programmatique » qui veut un nouveau programme pour un nouveau *socius*) ;
- Des investissements révolutionnaires inconscients qui opèrent en passant de l'autre côté du *socius*, sur son versant moléculaire, c'est-à-dire, une « rupture de causalité qui force à réécrire l'histoire à même le réel et produit ce moment étrangement polyvoque où tout est possible » (ce que Guattari appelle une politique « diagrammatique »).

Dès 1968, Guattari réfléchissait à des manières de conjuguer différents types de mouvements sociaux. Les traditionnels (syndicats et partis), les à peine nés (minorités, écolos, fous, pour l'école) et les subjectivités en mutation (qui ne se constituaient pas encore en tant qu'un mouvement organisé). Ces différents types d'investissements peuvent coexister mais peuvent aussi s'affronter, puisqu'ils concernent des subjectivités qui « interprètent » et « sentent » de façon différente le possible nouvellement crée, en voulant le réaliser selon des logiques hétérogènes. Comment inventer des modes d'organisation ou, au moins, comment créer un plan institutionnel d'un nouveau type devant une telle multiplicité ? Cette hétérogénéité

[8] Je raconte plus dans un article écrit avec Maurizio Lazzarato pour Rue Descartes, ce que je dis dans les paragraphes suivants vient de ce travail conjoint. L'événement est aussi le livre de Lazzarato expérimentations politiques. « Ruptures subjectives et investissements politiques: juin 2013 au Brésil et questions de continuité », *Rue Descartes*, 92, 2017/2. Maurizio Lazzarato, *Expérimentations politiques*, Éditions Amsterdam, 2009.

est un trait à la fois incontournable et insoluble—au moins dans les termes usuels de la politique—de la question du sujet politique à l'heure actuelle. Pour cette raison on parle d'une crise du sujet politique.

La notion de classe rendait les choses bien plus faciles. Le sujet politique était le travailleur, point. Mais tout le monde sait aujourd'hui que la notion de classe ne marche plus sans qu'on revienne à la définition même de travail. Une conséquence immédiate est qu'on ne sait pas encore quel nouveau diagramme peut remplacer l'antagonisme de classe, qui donnait une ligne facile à saisir en recoupant le capital d'un côté et le travail de l'autre. Tracer la ligne pour définir l'antagonisme a toujours été un des enjeux plus importants dans la définition du sujet politique: bourgeois/prolétaire a déjà été une réponse satisfaisante. À l'heure actuelle, on parle des 99% contre le 1% (qui est une façon de dire les pauvres contre les riches) ou on soutient un populisme de gauche comme manière de dépasser la conception usuelle de l'antagonisme en politique. Ces tentatives, dont on ne connaît pas encore l'efficace politique, témoignent de la pertinence du problème.

Si l'on prend comme point de départ une cartographie des expérimentations politiques qui font défi à l'heure actuelle, il est impossible de contourner la question des minorités. Il faut alors rendre compte des questions de race, de sexe et de genre. Une question qui s'impose pourtant à notre moment politique est de savoir quelles puissances politiques sont en jeu dans des mouvements dits minoritaires et comment les rassembler ? Comment faire pour que ces mouvements ne renforcent pas la fragmentation typique de notre époque et ne retombent dans des politiques exclusivement identitaires ? Deleuze et Guattari ont inventé la notion de devenir-minoritaire pour tenter de résoudre ce problème. Ils soulignaient qu'au niveau des organisations politiques les énoncés de pouvoir n'appartiennent à personne: c'est un fait de majorité. La bataille sur des énoncés est alors tout aussi politique que linguistique. La linguistique maintient comme préalable intangible une langue adulte, normale, masculine, hétérosexuelle, blanche et capitalistique ; une langue fondée sur des systèmes d'universaux. Cette abstraction ne fait que masquer le caractère historiquement contingent des pouvoirs. Les mouvements minoritaires sont donc inséparables de l'enjeux de créer une langue mineure. Et si on fait attention à l'histoire ces sont les mouvements de femmes, de noirs, des indigènes, des immigrés qui ont fait face à ce défi.

Mais comment dépasser le stage dans lequel une langue mineure est utilisée comme dialecte ? Car, ils le reconnaissent, « ce n'est certes pas en utilisant une langue mineure comme dialecte, en faisant du régionalisme ou du ghetto, qu'on devient révolutionnaire ». C'est en utilisant beaucoup d'éléments des minorités, mais en les connectant, en les conjuguant. C'est ainsi qu'on peut inventer un devenir autonome, imprévu. On ne peut pas échapper au cynisme du capitalisme en rentrant dans le ghetto. Mais on ne mobilise pas non plus des nouvelles forces subjectives en renonçant à la singularité de chaque groupe social. Deleuze et Guattari parlent alors d'« une figure universelle de la conscience minoritaire ».[9] Cette figure est la

[9] Guillaume Sibertin-Blanc, Deleuze et les minorités: quelle « politique » ?, Cités 40, 2009/4, p. 39–57.

variation continue, une amplitude qui ne cesse de déborder l'étalon majoritaire et le seuil représentatif. Mais le mot « universel » me gêne encore dans cette formule. Le choix semble prendre un raccourci par la voie des catégories considérées impossibles à dépasser. La variation continue, les lignes et les traces qui constituent le devenir minoritaire de tout le monde (par opposition au fait majoritaire) sont de l'ordre du diagramme. Mais il faut en dire plus.

Dans la pratique politique à l'heure actuelle, on a trop besoin des nouveaux paramètres pour évaluer, de façon immanente, l'effectivité des luttes et des organisations. Ces paramètres doivent inclure les modes d'existence rendus possible, les problèmes posés, ainsi que leur puissance de connexion. S'il s'agit d'une lutte entreprise par un groupe minoritaire, il serait intéressant d'évaluer son aptitude pour s'articuler aux problèmes d'autres minorités. C'est ainsi qu'une fonction diagrammatique opère des rapports transversaux, en s'opposant à ce qui constitue des communautés renfermées sur elles-mêmes, qui produit des ségrégations à l'intérieur des mouvements.

Les difficultés de notre moment politique sont d'autant plus profondes que cette multiplicité des sujets doit inclure aussi les personnages en exode du monde du travail. La dissolution de la figure du travailleur, conséquence des nouvelles formes d'accumulation capitaliste, pose des graves problèmes aux modes usuels de concevoir le sujet politique. À l'époque de Marx, pourtant, il y avait aussi une multiplicité de personnages qui pouvait rendre compliquée la tâche de définir un sujet politique unique. C'est même surprenant, du point de vue historique, que Marx ait regardé autour de lui autant des modes différents de travailler et ait sorti de là un sujet politique: le travailleur industriel. Il a fallu mettre de côté le *lumpen-prolétariat*, les artisans, ceux qui offraient des services, les domestiques ou même les travailleurs des manufactures.[10] Le travailleur industriel était encore numériquement minoritaire, au milieu du XIXe siècle, puisque l'industrie n'était pas la forme hégémonique de la production. Il avait pourtant une caractéristique essentielle favorisant l'unification des sujets politiques: seulement dans le cas de l'industrie le travail devenait interchangeable et homogénéisé, en rendant les divers acteurs dépourvus d'identités particulières. L'enjeux était alors plus politique que socioéconomique. Marx voulait renforcer le sujet politique du travailleur afin de rendre disponible une catégorie apte à organiser les acteurs. Et le plan a marché pendant longtemps, puisque la théorie a aidé l'organisation des travailleurs. Même si la puissance politique réelle de la classe ouvrière n'allait se montrer que bien après ses premiers écrits sur le sujet (soit dans le *Manifeste communiste* ou dans *Le Capital*).

Autour de la moitié du XIXe siècle, quand Marx a écrit sur la puissance politique de la classe ouvrière, la figure du travailleur qu'il décrivait était un pari: une tendance.[11] Est-ce qu'on pourrait dire que le travailleur était un diagramme ?

[10] Comme le montre, par exemple, plusieurs travaux historiques, comme ceux de Marcel van der Linden.

[11] Antonio Negri, *Marx au dela de Marx*, Paris, L'Harmattan, 1979.

Le fait est que c'est Lénine qui va mettre au point le dispositif permettant que le travailleur devienne concrètement le protagoniste d'une dualité de classe: la coupure opérée par le parti révolutionnaire (présent dès *Que faire ?)*. Deleuze et Guattari soulignent que la dualité de classe était inventée par Lénine et c'est elle qui va fonctionner comme plateforme de reconversion des subjectivités, rendant possible que les travailleurs agissent en tant que sujets politiques antagonistes de la bourgeoisie.[12] C'est seulement ainsi, par l'intermédiaire du dispositif qu'était le parti, que le travailleur en tant que diagramme pouvait se déployer en tant que classe ouvrière (porteuse d'un programme politique).

Si l'on disloque la méthode—et non pas la réponse—de Marx au présent, il y a lieu de demander sur la possibilité de tracer des nouveaux diagrammes pour saisir les sujets politiques de nos jours. Le travailleur à lui-seul ne fournit pas une réponse satisfaisante et il semble difficile qu'une figure unificatrice puisse le faire. Une bonne partie du drame de la gauche repose sur cette difficulté. Les promesses du monde du travail sont désespérantes: fin des travaux intermédiaires (tâches de routine remplacées par des robots), prolifération des sous-travaux, précarisation, délocalisation, intermittence, ubérisation etc. Tout cela doit être prise en compte. Et additionné aux mouvements des femmes, des noirs, des immigrés, ainsi qu'aux luttes écologistes. Il faut rendre compte aussi de ceux qui s'auto-intitulent « entrepreneurs » (sous différentes modalités d'entreprise) et qui ne font pas toujours partie des strates les plus riches. Devant une telle multiplicité, il semble impossible de revendiquer une seule figure, comme c'était le cas du travailleur. Avec en plus l'embarras du manque croissant d'identification d'un individu avec son travail, qui ne fournit guère la collectivisation nécessaire à l'établissement des liens politiques. Rien de cela n'existe et n'existera plus, ainsi qu'un espace-temps spécifique (comme c'était le cas pour l'usine) qui rend le travail, une activité bien délimitée (et démarquée) de la vie.

La question du sujet politique est d'autant plus complexe à l'heure actuelle que l'affaire d'en tracer un diagramme doit se prévenir des traits fondamentaux du pari marxiste: universalité (possibilité de parler d'un travailleur en général) ; délimitation dans un espace-temps (l'usine en tant qu'espace de collectivisation) ; projection sur une figure des relations déterminantes de l'économie (la relation capital-travail identifiable à partir du seul rapport d'exploitation du travailleur). Rien de cela n'est possible maintenant. Les sujets potentiellement porteurs d'une nouvelle pratique politique sont multiples et singuliers. Ils demandent davantage que ces singularités soient prises en compte. Ses histoires forcément multiples—que Chimamanda Adichie nous prévient de ne pas éclipser sous une histoire unique. Comment prendre tout cela en compte pour inventer une nouvelle théorie du sujet politique ? Peut-être qu'elle peut s'annoncer en termes de diagrammes. Puisque le pari marxiste était unitaire, on n'avait pas besoin des catégories pour saisir des modes d'expression non linguistiquement formés. Nos sujets politiques pourtant s'expriment déjà mais dans des termes pas encore disponibles. Le diagramme sert

[12] Gilles Deleuze, Félix Guattari, *L'Anti-Œdipe*, Paris, Minuit, 1972.

justement à ébaucher des saillances du possible que le langage et le système des significations disponibles ne rendent pas énonçable. Il permet donc de faire un pari, qui ne se laisserait mesurer que par le brassage de mobilisation politique dont il est capable (en détournant un peu les mots de Gilles Chatelet sur le virtuel mathématique).

Bref, on pourrait résumer ainsi le défi politique contemporain: s'emparer du possible, en portant des instruments de diagrammatisation, pour tenter un pari. Dire en faisant, faire même sans savoir à dire. Se noyer et ressortir pour reprendre de l'air. Le diagramme aide l'expression au long d'un tel parcours incertain ; il rend possible un mode d'expression plus convenable à l'expérimentation radicale convoquée par la pratique politique dans l'actualité. Nous ne pouvons pas dire comment cela va se passer, à quoi cela va aboutir, mais nous pouvons définir les conditions d'un processus. Le diagramme sert alors d'instrument à une nouvelle pragmatique—à une nouvelle conception même du pragmatisme qui fait question en politique.

Comment retrouver l'enchantement de faire de la politique si nous sommes bourrés par le réalisme, le pragmatisme et la lucidité que les temps actuels requièrent ? Il parait décevant de dire qu'il faut être pragmatique et faire seulement ce qui est possible. En même temps, l'impossible, l'imaginaire associé aux projets trop utopiques semblent irresponsables devant les urgences du présent. Redevenir pragmatique est alors s'inventer un possible, ici et maintenant.

La dimension diagrammatique de l'écriture littéraire : un formalisme dynamique inscrit dans la sensorialité du langage

Noëlle Batt

Abstract After briefly recalling the evolution of the concept of diagram in the philosophical work of Gilles Deleuze, we shall first present the hypothesis that literary and more particularly poetic writing has a diagrammatic dimension. The hypothesis is sustained by a number of resonances between, on the one hand, the features and morpho-dynamic processes which are prominent in the definition of diagram proposed by different disciplines (mathematics, physics, architecture, fine arts, . . .), and, on the other hand, those which characterize the specific reorganisation of signs and infra-semantic elements of the language which constitute the material of the literary text. This being done, we shall confirm the value of the hypothesis by confronting it to the conclusions drawn from the analysis of two poems : «A une passante» by Charles Baudelaire et «Exultation is . . . » by Emily Dickinson. We'll note then, that the diagrammatisation of meaning in a literary text is often the opportunity for the poet or writer to convey a daring or even subversive statement.

Keywords Diagramme · écriture · Langue · Logique · Sens

> Essayez de vous figurer ce que suppose le moindre de nos actes. Songez à tout ce qui doit se passer dans l'homme qui émet une petite phrase intelligible, et mesurez tout ce qu'il faut pour qu'un poème de Keats ou de Baudelaire vienne se former sur une page vide, devant le poète.
>
> Songez aussi qu'entre tous les arts, le nôtre est peut-être celui qui coordonne le plus de parties ou de facteurs indépendants : le son, le sens, le réel et l'imaginaire, la logique, la syntaxe et la double invention du fond et de la forme . . . , et tout ceci au moyen de ce moyen

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> essentiellement pratique, perpétuellement altéré, souillé, faisant tous les métiers, le *langage commun* [. . .][1]

Il s'agit donc ici d'aborder la question de la diagrammatisation dans un domaine rarement mentionné quand il est question de ce thème : l'écriture et la lecture littéraires, écriture créative et lecture *insistante* comme l'a dénommée Jean Bollack, insistante parce qu'elle insiste aux lieux problématiques du texte, ceux qui résistent le mieux à l'élucidation.

DE DIAGRAMME EN DIAGRAMME

De nombreux articles de ce recueil abordent la question des diagrammes. J'ai moi-même beaucoup écrit sur le sujet depuis 2004[2] où j'ai tenté de saisir les mutations du concept dans l'évolution de l'œuvre philosophique de Gilles Deleuze, un concept d'abord exposé en 1975 dans un article de *Critique*[3] après avoir été prélevé chez Foucault dans *Surveiller et punir*, puis théorisé en 1981 dans l'ouvrage *Francis Bacon. Logique de la sensation*[4] à partir des propos tenus par le peintre Francis Bacon à David Sylvester[5] concernant la phase préparatoire de ses tableaux, —ce moment où il trace au hasard sur la toile des marques qui constituent « a sort of graph » (que Michel Leiris traduit : « une sorte de diagramme »), et d'où doit « sortir » le tableau—, concept encore une fois remis sur le métier en 1986, en conservant les acquis du travail sur Bacon mais en revenant à Foucault, dans le livre intitulé *Foucault*.[6] A chaque étape de la pensée deleuzienne, le concept de diagramme s'accouple ou « travaille » avec d'autres concepts : dans la 1ère phase Foucault, avec la machine abstraite, l'immanence, les mutations ; dans la phase Bacon, avec le devenir, la zone d'indiscernabilité, la modulation ; dans la 2ème phase Foucault, avec la dimension, l'informel, les multiplicités. J'illustrerai chaque phase avec une citation :

— Foucault 1

> *Un diagramme ne fonctionne jamais pour représenter un monde objectivé* ; au contraire il organise un nouveau type de réalité. Le diagramme n'est pas une science, il est toujours

[1] Paul Valéry : « Poésie et pensée abstraite », conférence à l'Université d'Oxford, *The Zaharoff lecture for 1939*, Oxford, Clarendon Press, 1939, repris dans *Variété V*, 1944, *Œuvres*, Bibliothèque de la Pléiade, tome 1, Paris, Gallimard, p. 1339.

[2] Noëlle Batt, « *Penser par le diagramme. De Gilles Deleuze à Gilles Châtelet* », *TLE* 22, Presses Universitaires de Vincennes, 2004.

[3] Gilles Deleuze, « Ecrivain non : un nouveau cartographe », *Critique* n°343, Paris, Editions de Minuit, 1975.

[4] Gilles Deleuze, *Francis Bacon. Logique de la sensation*, Paris, Editions de Minuit, 1981.

[5] David Sylvester, *Interviews with Francis Bacon*, London, Thames & Hudson, 1975; tr. fr. Michel Leiris, David Sylvester, *Entretiens avec Francis Bacon*, Paris, Flammarion, 1976.

[6] Gilles Deleuze, *Foucault*, Paris, Editions de Minuit, 1986.

> affaire de politique. Il n'est pas un sujet de l'histoire ni ne surplombe l'histoire. Il fait de l'histoire en défaisant les réalités et les significations précédentes, constituant autant de points d'émergence ou de créationnisme, de conjonctions inattendues, de continuums improbables. On ne renonce à rien quand on abandonne les raisons. Une nouvelle pensée, positive et positiviste, le *diagrammatisme*, la *cartographie*.[7]

— Bacon

La citation qui suit concerne la fabrication du tableau de 1946 intitulé « Peinture » et dont Bacon dit qu'il voulait « faire un oiseau en train de se poser dans un champ » mais que des traits tracés composant le diagramme est sorti quelque chose de tout-à-fait différent : l'homme au parapluie devant l'animal de boucherie écartelé :

> C'est la série ou l'ensemble figural qui constitue l'analogie proprement esthétique : les bras de la viande qui se lèvent comme des analogues d'ailes, les tranches de parapluie qui tombent ou se ferment, la bouche de l'homme comme un bec dentelé. A l'oiseau, s'est substitué non pas une autre forme, mais des rapports tout différents qui engendrent l'ensemble d'une figure comme l'analogue esthétique de l'oiseau. Le diagramme-accident a brouillé la forme figurative intentionnelle, l'oiseau : il impose des taches et des traits informels qui fonctionnent comme des traits d'oisellité, d'animalité. Et ce sont ces traits non figuratifs dont, comme d'une flaque, sort l'ensemble d'arrivée [...] Le diagramme a donc agi en imposant une zone d'indiscernabilité ou d'indéterminabilité objective entre deux formes, dont l'une n'était déjà plus et l'autre pas encore. Il détruit la figuration de l'une et neutralise celle de l'autre. Et entre les deux, il impose la Figure sous ses rapports originaux. Il y a bien changement de forme, mais le changement de forme est déformation, c'est-à-dire création de rapports originaux substitués à la forme : la viande qui ruisselle, le parapluie qui happe, la bouche qui se dentelle. [...] D'où le programme de Bacon : produire la ressemblance avec des moyens non ressemblants. »[8]

— Foucault 2

Voici comment a été remaniée pour le livre *Foucault* (1986) la citation de l'article de *Critique* (1975) donnée précédemment :

> C'est que le diagramme est éminemment instable et fluant, ne cessant de brasser matières et fonctions de façon à constituer des mutations. Finalement, tout diagramme est intersocial, et en devenir. Il ne fonctionne jamais pour représenter un monde préexistant, il produit un nouveau type de réalité, un nouveau modèle de vérité. Il n'est pas sujet de l'histoire ni ne surplombe l'histoire. Il fait l'histoire en défaisant les réalités et les significations précédentes, constituant autant de points d'émergence ou de créativité, de conjonctions inattendues, de continuums improbables. Il double l'histoire avec un devenir. (p. 43)
>
> Il y a une histoire des agencements, comme il y a un devenir et des mutations de diagrammes. » (p. 49)
>
> [...] l'histoire des formes, archive est doublée d'un devenir des forces, diagramme. » (p. 51)

J'avais aussi inclus dans ce premier essai une étude sur Gilles Châtelet, philosophe des mathématiques, auteur des *Enjeux du mobile* qui rejoint Deleuze dans sa

[7] *Critique*, op. cit., p. 1223.

[8] *Francis Bacon. Logique de la sensation*, op. cit., p. 100–101.

conception d'un diagramme non représentatif, non illustratif, véritable opérateur de pensée, en insistant sur une dimension très importante, celle du corps qui parle à travers le geste :

> Le diagramme ne se démode jamais : c'est un projet qui vise à ne s'appuyer que sur ce qu'il esquisse ; cette exigence d'autonomie en fait le complice naturel des expériences de pensée (. . .) épreuve[s] par l[es]quelle[s] le physicien-philosophe prend sur lui de se désorienter, de connaître la perplexité inhérente à toute situation, où le discernement ne va nullement de soi. Il s'agit pour lui d'orchestrer une subversion des habitudes associées à des clichés sensibles [. . .] et de se transporter par la pensée dans les enceintes hors causalités, à l'abri des forces, pour se laisser flotter entre mathématiques et physique [. . .] de mettre en scène la désorientation pour orienter et imposer un projet physico-physique qui se donnera *ensuite* pour le plus évident.[9]

Le diagramme deleuzien a eu une postérité importante chez les créateurs, —les architectes (Peter Eisenman, UN studio, Rem Koolhaas), les artistes, principalement dans le domaine des arts plastiques (Smithson, Basbaum, Perjovschi) et de la musique (Pascale Critton et Pascal Dusapin)— pour ne donner que quelques exemples.

Ayant beaucoup travaillé préalablement dans ma spécialité, l'analyse des textes littéraires et la théorie littéraire, j'ai vite vu les articulations possibles entre la théorisation du diagramme et celle de l'écriture littéraire et j'ai commencé à écrire sur le sujet en réintroduisant aussi dans le jeu la contribution première du logicien Charles Sanders Peirce qui, dans le cadre de son système de signes triadique (signe, symbole, icône) valant aussi bien pour le langage mathématique que pour le langage naturel, présente le diagramme comme l'une des deux sous-catégories de l'icône, laquelle opère avant tout par la similitude entre le signifiant et le signifié. Alors que, dans l'image, le signifiant représente « les simples qualités des signifiés », dans le diagramme, la ressemblance entre le signifié et le signifiant « ne concerne que les relations entre leurs parties ». Le diagramme est donc défini comme « un *representamen* qui est de manière prédominante *une icône de relation* et que des conventions aident à jouer ce rôle ». J'ai développé la position de Peirce dans un article en anglais intitulé « Diagrammatic Thinking in Literature and Mathematics »[10] et dans un autre en français intitulé : « La pensée du texte littéraire : une pensée diagrammatique. Iconicité et abstraction »,[11] dans lequel je traite aussi du commentaire éclairé apporté sur la question par Christiane Chauviré, spécialiste de Peirce et de Wittgenstein. Je la cite :

> C'est au diagramme, modélisation, figure stylisée d'un état de choses imaginé par le mathématicien, qu'il revient d'expliquer la pensée mathématique dans son aspect fécond, aussi bien d'ailleurs que la création artistique. Nous avons là un cas de mentalisme élégant. La pensée mathématique consiste à tracer sous l'œil de l'esprit des figures qu'elle déforme

[9] Gilles Châtelet, *Les Enjeux du Mobile*, Paris Editions du Seuil, 1993, p. 35.

[10] « Diagrammatic Thinking in Literature and Mathematics », *European Journal of English Studies*, vol. 11, n°3, december 2007.

[11] « La pensée du texte littéraire : une pensée diagrammatique. Iconicité et abstraction ». *Visible* 9, Presses Universitaires de Limoges, 2012.

> et reforme, non certes au hasard, mais conformément à des règles et dans le but d'obtenir une configuration finale, un diagramme où se lira la conclusion nécessaire du raisonnement [. . .]. Le diagramme n'est pas, comme la figure du géomètre, simple illustration, support concret d'un raisonnement se déroulant ailleurs, dans la pensée pure et abstraite : la formation et déformation de diagrammes est constitutive de la pensée mathématique qui s'y épuise tout entière, la pensée en général n'étant aux yeux de Peirce, comme plus tard à ceux de Wittgenstein, rien d'autre qu'une manipulation de signes (qui peuvent être des signes mentaux ou externes). [. . .] La construction de diagrammes successifs, chacun apportant une modification [. . .] au précédent permet de lire dans la configuration finale, des propriétés inattendues ou des relations mathématiques insoupçonnées jusqu'alors.[12]

Enfin, j'ai eu l'occasion de revenir sur la question du diagramme en mars 2017 lors d'une intervention dans le séminaire de Franck Jedrzejewski au collège de philosophie. J'y ai tenté une mise au point sur les usages du concept de diagramme dans diverses disciplines —sémiotique, mathématique et physique, architecture, littérature — depuis son développement par Gilles Deleuze. Et j'ai dressé, à cette occasion, une liste des traits qui me semblaient faire consensus pour ces disciplines:

* Le diagramme ne représente pas le passé ou l'existant. Il n'est pas illustratif. Il esquisse, anticipe, indique quelque chose qui n'est pas encore visible (qui donc appartient au futur, est en devenir, virtuel), mais que l'on va pouvoir commencer à entrevoir en scrutant le diagramme. Le diagramme permet de penser l'impensé.

** Le diagramme est un lieu de transition qui assure le passage entre des effectuations différentes d'une même réalité, qui fait communiquer des séries divergentes. Le diagramme n'est un lieu que pour les mutations.

*** Il translate les éléments ; redistribue les rapports. Pour reprendre une formule de Deleuze, : « il n'y a pas enchaînement par continuité mais ré-enchaînement par-dessus les discontinuités. »

**** Il permet de penser le concept dans son processus d'engendrement. La dynamicité du diagramme mathématique apparaît comme l'un de ses traits majeurs.

***** Il requiert la présence du corps et de la main, mais l'expérience des architectes montre qu'avec l'usage de l'ordinateur la main peut perdre sa prédominance.

****** On reconnaît au diagramme une synopticité, une synchronicité. Il rapproche, comprime, contracte des univers différents dans un seul espace ou un seul plan (compactification).

******* Il peut être vu comme l'analogue dans le tracé du processus mental, de la *cogitatio*.

[12] Christiane Chauviré, « Perception visuelle et mathématiques chez Peirce et Wittgenstein » dans J. Bouveresse et J.J. Rosat (sous la dir. de), *Philosophies de la perception. Phénoménologie, grammaire et sciences cognitives*, Paris, Odile Jacob, 2003, p. 202–203.

LA DIMENSION DIAGRAMMATIQUE DE L'ECRITURE LITTERAIRE

C'est donc à convaincre que l'écriture littéraire possède une dimension diagrammatique impliquant certains des traits ci-dessus que je vais m'employer maintenant, et je vais le faire en trois étapes :

– tout d'abord, compte tenu de la vocation interdisciplinaire de ce recueil, rappeler un certain nombre de faits concernant l'écriture littéraire qui ne sont pas toujours connus des non-spécialistes ;
– ensuite, désigner les traits, dispositifs et processus morpho-dynamiques qui signent la dimension diagrammatique de l'écriture littéraire ;
– enfin, exposer brièvement les conclusions de deux analyses de poèmes pour pointer concrètement la mise en œuvre de cette dimension diagrammatique dans un texte particulier.

SPÉCIFICITÉ DE L'ÉCRITURE LITTÉRAIRE

Deux grands thèmes sont souvent repris par les écrivains ou les philosophes à propos de la littérature :

– Le langage littéraire est « une autre langue dans la langue ».
– Le langage littéraire est une « idée sensible » (Kant), une « forme-sens » (Meschonnic).

Iouri Lotman, sémioticien russe défendra pour sa part l'idée que la littérature est un « système de modélisation secondaire » qui s'appuie sur un matériau complexe, le langage, lui-même système de modélisation primaire. En passant du système linguistique au système littéraire, le signe linguistique change de nature (il devient iconique) et change de limites (il ne coïncide plus avec le mot, et peut être défini comme une configuration de mots ou de composants de mots appartenant à des niveaux linguistiques hétérogènes). Mais le système littéraire n'annule pas le système linguistique. Les deux systèmes sont perçus de façon concomitante et c'est leur différence qui est signifiante.

Une autre langue dans la langue

La citation la plus célèbre est celle de Proust :

> Les beaux livres sont écrits dans une sorte de langue étrangère.[13]

[13] Marcel Proust, *Contre Sainte Beuve,* Paris, Gallimard, 1954, Editions de poche, p. 296.

Elle sera reprise par Gilles Deleuze :

> Ce que fait la littérature dans la langue apparaît mieux : comme dit Proust, elle y trace précisément une sorte de langue étrangère, qui n'est pas une autre langue, ni un patois retrouvé, mais un devenir-autre de la langue, une minoration de cette langue majeure, un délire qui l'emporte, une ligne de sorcière qui s'échappe du système dominant. Kafka fait dire au champion de nage : je parle la même langue que vous et pourtant je ne comprends pas un mot de ce que vous dites. Création syntaxique, style, tel est ce devenir de la langue : il n'y a pas de créations de mots, il n'y a pas de néologismes qui vaillent en dehors des effets de syntaxe dans lesquels ils se développent. Si bien que la littérature présente déjà deux aspects, dans la mesure où elle opère une décomposition et une destruction de la langue maternelle, mais aussi l'invention d'une nouvelle langue dans la langue, par création de syntaxe.[14]

On peut aussi citer Valéry :

> [...] étranges discours qui semblent fait par un *autre* personnage que celui qui les dit et s'adresser à un *autre* que celui qui les écoute. En somme, c'est *un langage dans un langage*.[15]

ou encore Mallarmé :

> Le vers, qui de plusieurs vocables refait un mot total, neuf, étranger à la langue et comme incantatoire, achève cet isolement de la parole : niant, d'un trait souverain, le hasard demeuré aux termes malgré l'artifice de leur retrempe alternée en le sens et la sonorité, et vous cause cette surprise de n'avoir ouï jamais tel fragment ordinaire d'élocution, en même temps que la réminiscence de l'objet baigne dans une neuve atmosphère.[16]

Iconicité et abstraction

Comment se crée cette autre langue dans la langue ? Deleuze nous dit : « Création syntaxique, style, tel est ce devenir de la langue ». C'est juste mais c'est insuffisant. La différence entre les deux langues vient essentiellement du fait que la langue littéraire investit dans le travail de création du sens *tous* les niveaux infra-sémantiques (syntaxique, mais aussi phonétique, morphologique, rythmique ...) et que ce sont les éléments de tous ces niveaux qui sont organisés par l'écrivain, mis en résonnance les uns avec les autres et conjointement sémantisés dans une nouvelle configuration verbale.

La langue dessine, trace, et c'est au nom de l'iconicité qu'elle produit que Kant peut parler à propos des idées littéraires, d' « idées sensibles », que Meschonnic peut dire du poème qu'il est une « forme-sens », que Valéry peut nous proposer la très belle image du pendule pour indiquer l'indissolubilité du son et du sens dans le texte littéraire :

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[14] Gilles Deleuze, *Critique et Clinique*, Paris, Editions de Minuit, 1993, p. 15–16.

[15] Paul Valéry, op. cit. p. 1324.

[16] Stéphane Mallarmé, *Variations sur un sujet*, « Crise de Vers », 1886/92/96, dans *Œuvres complète*s, Bibliothèque de la Pléiade, 1945, p. 368.

> Pensez à un pendule qui oscille entre deux points symétriques. Supposez que l'une de ces positions extrêmes représente la forme, les caractères sensibles du langage, le son, le rythme, les accents, le timbre, le mouvement —en un mot, la *Voix* en action. Associez, d'autre part, à l'autre point, au point conjugué du premier, toutes les valeurs significatives, les images, les idées ; les excitations du sentiment et de la mémoire, les impulsions virtuelles et les formations de compréhension— en un mot tout ce qui constitue le *fond*, le sens d'un discours. Observez alors les effets de la poésie en vous-mêmes. Vous trouverez qu'à chaque vers, la signification qui se produit en vous, loin de détruire la forme musicale qui vous a été communiquée, redemande cette forme. Le pendule vivant qui est descendu du *son* vers le *sens* tend à remonter vers son point de départ sensible, comme si le sens même qui se propose à votre esprit ne trouvait d'autre issue, d'autre expression, d'autre réponse que cette musique même qui lui a donné naissance.[17]

Mais tout cela n'a rien de très extraordinaire pour un artiste ; cela signifie simplement que le poète ou le romancier investit *totalement* le matériau à sa disposition.

Or, une langue, ce n'est pas seulement du sens, c'est aussi du son, une matière rugueuse ou caressante, des formes rondes ou aiguës, un rythme régulier ou heurté, des points et des contrepoints, des moments d'intensité et des moments de calme plat. Et la spécificité de la littérature, c'est de créer du sens avec tout ce qui *a priori* n'en est pas, du sens qui redouble, excède, ou contredit les significations simples véhiculées par la communication linguistique ordinaire, quotidienne, pragmatique. Cette création implique une coopération de l'esprit et de la main, de l'œil et de la main, de l'oreille et de la main, et même de la bouche et de la main. La littérature se fomente dans la manipulation des mots, des phrases, et plus encore des unités plus petites qui les composent, morphèmes, phonèmes et même traits distinctifs (par exemple le trait de la tension *versus* celui du relâchement qui assure la distinction entre /sheep/ et /ship/). Et si les phonèmes ont une dimension acoustique, ils ont aussi une dimension articulatoire. Les sons du langage poétique ne sont pas seulement dans l'oreille du poète et du lecteur, ils sont dans sa bouche et dans sa gorge.[18] C'est avec toutes les parties de son appareil phonatoire que le lecteur du poème 1755 de Emily Dickinson[19] expérimente la différence radicale entre *revery* d'un côté et *clover*, *bee* et *prairie* de l'autre. S'il connaît un peu de phonétique et a

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[17] Paul Valéry, op. cit., p. 1331–1332.

[18] Le poète Francis Ponge a attiré l'attention sur le ressenti du son dans la bouche : « [. . .] donné à jouir à ce sens qui se place dans l'arrière-gorge : à égale distance de la bouche (de la langue) et des oreilles. Et qui est le sens de la formulation, du verbe. [. . .] Ce sens qui jouit plus encore quand on lit que quand on écoute (mais aussi quand on écoute), quand on récite (ou déclame), quand on-pense-et-qu'on-l'écrit. Le-regard-de-telle-sorte-qu'on-le-parle. », « My Creative Method », *Méthodes*, Paris, Gallimard, 1961, Edition de référence, 1988, p. 20.

[19] To make a prairie it takes a clover and one bee –

One clover and a bee,
And revery.
The revery alone will do
If bees are few.

déjà entendu parler de la différence entre les consonnes fricatives et les plosives, il comprendra tout de suite ce qui cause la différence. Dans le cas contraire, ce n'est pas grave ; cela ne l'aura pas empêché d'éprouver et de ressentir pourquoi, si les abeilles venaient à manquer, la *revery* à elle seule, pourrait fabriquer la prairie . . .

Mais un problème demeure dans notre état des lieux, qui tient au fait de parler de langue. Car la littérature n'est une langue que par comparaison, une comparaison qui occulte une de ses spécificités bien soulignée par Benveniste dans ses *Dernières Leçons*[20] comme dans son livre sur Baudelaire,[21] et qui est d'être *une écriture*. Or langue et écriture ne sont pas synonymes, pas plus que ne le sont discours et texte. Et qui dit écriture, dit composition, construction de rapports, création de *patterns*, de motifs, de figures qui reposent sur des relations de similitude, d'opposition, d'inversion, de gradation, *etc.* . . .

Diagrammatisation de l'écriture ; « sensorialisation » d'une nouvelle logique

Il s'agit donc chaque fois, dans un texte artistique comme d'ailleurs dans un tableau ou un morceau de musique, de communiquer quelque chose d'inouï. Et pour rendre cet inouï audible, il faut le rendre convaincant. Or, si la philosophie convainc par la rigueur de l'argumentation, la poésie convainc par la séduction du verbe choisi, disposé, composé.[22]

Pour faire accepter un nouveau paradigme ou une autre logique, le poète a recours à un jeu d'écriture singulier, qu'il invente pour la circonstance, un jeu qui se joue avec et contre les règles de la langue pragmatique, mais aussi avec et contre celles de la langue poétique de l'époque gouvernée par les usages de la rhétorique et de la prosodie.

Ecrire est donc d'une certaine manière, « sensorialiser » un nouveau cheminement logique, l'inscrire dans la matérialité multisensorielle d'une langue en lui donnant forme, orientation, rythme, énergie, en lui donnant de l' « effet », ce qui n'annule pas l'abstraction des relations logiques qui restent parfaitement perceptibles au lecteur du moment. Mais l'abstraction se dessine, elle acquiert une visibilité au fur et à mesure de l'engendrement du poème. *Camina caminando*. Le chemin se fait en marchant, disait le poète Antonio Machado. Le poème se fait en s'écrivant, et ce faisant, il inscrit la dynamique d'engendrement du concept, de

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[20] Emile Benveniste, *Dernières leçons. Collège de France 1968 et 1969*, Paris, EHESS, Gallimard, Seuil, 2012.

[21] Emile Benveniste, *Baudelaire*, (Présentation et transcription de Chloé Laplantine), Limoges, Lambert-Lucas, 2011.

[22] On renverra sur cette question à une réflexion d'Alain Badiou parue en deux versions : « Que pense le poème ? », dans *L'Art est-il une connaissance ?* Paris, Editions Le Monde, 1993, et « Philosophie et poésie : au point de l'innommable », *Po&sie* n° 64, 1993.

l'idée, de la relation logique dans une physicalité qui rend perceptible au lecteur les forces déployées, les chemins frayés, les trajets empruntés, les dispositifs créés par l'écrivain ou le poète.

L'art déforme et forme. Il convertit. Il s'agit donc de suivre le trajet des mutations en cours, lisibles dans les dispositifs textuels ou les processus morpho-dynamiques suivants :

– l'association de l'iconicité et de l'abstraction qui rend audible et visible l'idée ou la relation nouvelle en l'inscrivant dans la sensorialité du langage,
– la création d'une continuité sémantique non pas en « enchaînant par continuité, mais en ré-enchaînant par-dessus les discontinuités ». C'est le principe d'une écriture et d'une lecture littéraires qui ne sont plus linéaires, mais *tabulaires* comme l'ont dit en leur temps les membres du groupe μ,[23]
– la subversion de la grammaire qui fait de la langue déformée, mutante, un véritable opérateur de pensée. Si l'on en croit Valéry : « On dira, rigoureusement, que [. . .] la phrase accomplit un travail sur l'esprit du sujet »,[24]
– l'inscription dans le déroulement du texte, de sa dynamique d'engendrement,
– l'extension de la performativité potentielle du langage (Austin, « Quand dire c'est faire . . . ») au texte entier.

Avec Charles Baudelaire et Emily Dickinson

Je vais donc maintenant commenter brièvement la performativité textuelle à l'œuvre dans deux poèmes, l'un fort connu : « A une Passante » de Baudelaire, et l'autre beaucoup moins, d'une poétesse américaine du XIXè siècle : le poème 76 « Exultation is the going . . . » de Emily Dickinson, afin d'expliciter ce que j'appelle la dimension diagrammatique du texte littéraire et plus particulièrement poétique. L'espace imparti ne me permet pas de faire une analyse exhaustive des poèmes et le lecteur me pardonnera de le frustrer de nombreux agencements qu'il aurait découverts avec plaisir. Ne seront soulignés ici que les traits qui participent de la diagrammatisation majeure du poème.

1. **Baudelaire, « A une passante ».**[25]

La rue assourdissante autour de moi hurlait.
Longue, mince, en grand deuil, douleur majestueuse,
Une femme passa, d'une main fastueuse
Balançant, soulevant le feston et l'ourlet ;

[23] Groupe μ, *Rhétorique de la poésie. Lecture linéaire, lecture tabulaire*, Editions Complexe, Bruxelles, 1977.

[24] Paul Valéry, *Ebauches sur Mallarmé*, 1900.

[25] Poème de 1860, d'abord publié dans la revue *L'artiste*, puis dans *Les Fleurs du mal* ($2^{\text{ème}}$ édition).

> Agile et noble avec sa jambe de statue.
> Moi, je buvais, crispé comme un extravagant,
> Dans son œil, ciel livide où germe l'ouragan,
> La douceur qui fascine et le plaisir qui tue.
>
> Un éclair . . . puis la nuit ! - Fugitive beauté
> Dont le regard m'a fait soudainement renaître,
> Ne te verrai-je plus que dans l'éternité ?
>
> Ailleurs, bien loin d'ici ! trop tard ! *jamais* peut-être !
> Car j'ignore où tu fuis, tu ne sais où je vais,
> Ô toi que j'eusse aimée, ô toi qui le savais !

Coup de foudre condamné par les circonstances : l'anonymat, le grand deuil, la classe sociale. Les faits ne permettent rien. Le bouclage de la situation est annoncé par le premier vers borné par deux séquences sonores en miroir « la rue »/ « hurlait », comme deux parois qui se renvoient le bruit urbain agressant un sujet masculin en position d'objet : « moi » situé dans la deuxième partie du vers, tandis que le deuxième sujet, féminin, parasite à l'état d'anagramme la première partie du vers, comme on le verra plus tard.

Les deux premiers quatrains sont organisés avec une rigueur implacable qui fait place (1) aux figures répertoriées (l'hypallage « main fastueuse » ; la métaphore figée du coup de foudre : « son œil, ciel livide où germe l'ouragan ») ; (2) à la rime sémantique : « –tueuse » avec deux occurrences (« majestueuse » et « fastueuse »), « –tue » (« statue ») dans le premier quatrain, confirmée par une autre rime du deuxième quatrain (le verbe « tue »), (3) au rythme très travaillé. Le verbe d'action (« passa ») est au passé simple. Le personnage féminin, décrit avec une progressivité qui est du grand art,[26] est nommé dans le troisième vers seulement, et à la troisième personne. La bienséance est respectée. Seule, transparaît l'émotion du sujet masculin victime du coup de foudre (au deuxième quatrain dans le balbutiement « moi »-« je », dans l'expression oxymoronique « crispé comme un extravagant » (à la fois immobilisé et dérivant [extra-vagare], dans la connotation dramatique des termes qui décrivent le coup de foudre (« ciel livide où germe l'ouragan ») et ses conséquences (« fascine » et « tue »).

Les deux tercets vont nous entraîner dans une autre aventure. Une femme passa. Au passage, elle déclenche une passion. Mais elle est passée. *It is a matter of fact* et l'amoureux pétrifié et devenu fou, doit vivre avec cela. Après avoir été le théâtre d'un véritable débordement d'émotion où les mots s'entrechoquent dans des phrases sans verbes et sur-ponctuées, (il faut aussi noter que l'espace et le temps y progressent de manière de plus en plus désespérante : « ailleurs, bien loin d'ici »/ « trop tard ! *jamais* peut-être ! »), les deux tercets vont devenir le lieu où se construit

[26] Quiconque a vu les vidéos de Bill Viola montrant au fond de l'horizon un point qui petit à petit prend de l'épaisseur, devient une ligne, puis une silhouette, puis le mouvement d'une robe longue, puis une femme, se dira qu'elles illustrent parfaitement l'arrivée de la « passante » telle que Baudelaire la décrit dans le poème.

une fantasmagorie. Une relation imaginaire de celui qui a été frappé d'amour avec celle qui est passée à tout jamais, va se développer dans le cocon protecteur de la langue et de l'énonciation. L'intimité frappée d'interdit dans le réel va pouvoir trouver une expression dans la langue via les pronoms personnels de deuxième personne. L'adresse directe et le tutoiement apparaissent, d'abord le pronom objet « te », puis le pronom sujet « tu » répété deux fois dans l'avant-dernier vers, le pronom objet de nouveau : « toi » deux fois aussi dans le dernier vers. Et le chiasme, la figure la plus sensuelle de la rhétorique permet l'enlacement symbolique avec l'aimée avant que sa justification n'en soit fournie : si les circonstances l'avaient permis, il l'eût aimée mais surtout : elle le savait.

Il faut tout de suite évoquer comme participant de la même logique, le phénomène anagrammatique et paragrammatique qui conduit à inscrire la dénomination de la femme aimée, la passante, dédicataire du poème à même son titre : « A une passante », dans l'ensemble du poème. Le premier hémistiche du premier vers en donne un modèle exemplaire : « L**a** r**u**e **ass**ourdi**ssante »** (a-u-ass-ssante), modèle qui sera ensuite reproduit tout au long du poème en séquences variables dans lesquelles les phonèmes seront plus ou moins dispersés —les phonèmes vocaliques : [a], [u], [an], et les phonèmes consonantiques [s], [t] comme dans « majestueuse » ou « fastueuse », « feston », « statue », etc.

Ce phénomène assure, à la passante, par le biais des sonorités composant sa dénomination, une mutation de l'absence charnelle à la présence textuelle. La réalité scripturale du poème est une dénégation active de ce que son discours met en scène : l'écriture figurale réalise (*performs*) ce que le réel interdit.

2. **Emily Dickinson : poème 76, « Exultation is** . . . ».[27]

Exultation is the going
Of an inland soul to sea,
Past the houses – past the headlands –
Into deep Eternity –

Bred as we, among the mountains,
Can the Sailor understand
The divine intoxication
Of the first league out from land ?

Ici aussi, la langue poétique va constituer une scène parallèle où s'inscrira, dans le cas présent, une pensée subversive du dogme religieux. En effet le poème nous parle de la vie terrestre et de la vie éternelle, et pour tenter d'expliquer ce que peut être le passage de l'une à l'autre, Dickinson utilise la métaphore spatiale de la terre et de la mer en distribuant les traits de temporalité sur des mots voisins (par exemple, dans le troisième vers, la préposition polysémique « past » dans « Past the houses

[27] Emily Dickinson, *Poems* (Mabel Loomis Todd and T.W. Higginson eds.), 1890; *The Complete Poems*, (Thomas Johnson, ed.), Boston, Little, Brown & Co, 1955. Le poème 76 est daté approximativement de 1859.

— past the headlands »). Entrer dans la vie éternelle, pour une âme, est comparé au cheminement du terrien qui va prendre la mer. L'émotion ressentie à l'occasion de ce voyage est exprimée dans chacune des deux strophes par un mot abstrait d'origine latine : *Exultation* dans la première strophe et *Intoxication* dans la seconde. Ce qui se joue ici est une subversion d'un aspect institutionnel de la religion, qui concerne la différence entre l'expérience d'un chrétien intégré dans une communauté et celle d'un chrétien plus mystique qui vit son rapport à Dieu directement, de manière non médiée. On note ainsi dans la deuxième strophe, une opposition entre le Sailor qui représente le prêtre de la communauté, le passeur d'âmes, et le je individuel. Et la question posée est de savoir si le professionnel, qu'il soit de la mer ou de la religion, peut encore ressentir l'intensité de l'émotion du passage de la terre à la mer, de la vie terrestre à la vie éternelle. Poser la question est audacieux, mais y répondre serait dangereux. C'est donc le poème qui va répondre à la place de l'énonciateur individuel représenté. Et il va y répondre par une opération textuelle de mise en correspondance croisée d'une opposition syntaxique et d'une opposition morphologique qui, je crois mérite bien le nom de diagramme puisque c'est le dessin parfait de cette icône de relation qui agit sur le lecteur pour le persuader que le moment stratégique de ce passage de la vie terrestre à la vie éternelle n'est pas comme il aurait pu le penser, le moment où l'on accède au Paradis, mais celui où l'on expérimente le passage d'un lieu à un autre, d'un temps à un autre. Lorsqu'on n'est plus en A et pas encore en B. Lorsque l'on se vit dans l'entre-deux. L'entre-deux, moment de jouissance et de vertige.

Regardons et écoutons.

L'*entrée* est exprimée *syntaxiquement* dans la *première* strophe : « going to sea » ; « [going] into deep eternity ».

La *sortie* est exprimée *syntaxiquement* dans la *seconde* strophe : « out from land ».

Mais il est aussi question d'entrée et de sortie dans les préfixes des mots anglais d'origine latine qui traduisent l'émotion de l'âme.

La *sortie* est exprimée *morphologiquement* via le préfixe *ex-* dans le mot *Exultation* qui figure dans la *première* strophe.

L'entrée est exprimée *morphologiquement* via le préfixe *in-* dans le mot *Intoxication* (*in-toxicare*) dans la *seconde* strophe. De fait, c'est encore mieux que cela car la première syllabe de *to*xicare venant après le préfixe latin *in* forme ...... la préposition anglaise *into* !! mais cela c'est le petit supplément de chance que réservent parfois les langues aux poètes ...).

Les mouvements d'entrée et de sortie se trouvent donc exprimés deux fois dans chaque strophe de façon croisée : une fois au moyen de la syntaxe, et une autre fois au moyen de la morphologie.

Les deux mouvements se neutralisent et c'est bien le suspens de l'entre-deux qui est célébré ici.

C'est donc un dispositif morphosyntaxique assez sophistiqué (un croisement entre syntaxe et morphologie impliquant les deux strophes du poème), qui signifie et donne à voir un changement de paradigme général : une valorisation de l'entre-deux plutôt que du territoire bien défini, ainsi qu'un changement de paradigme chrétien : une dévalorisation de l'entrée au Paradis au profit du grand saut dans l'inconnu.