Chapter 18
Mathematics Through Play: Reflection on Teacher Narratives

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Introduction

Issues in mathematics teaching and learning, and debates related to mathematics as a field, have been broad and diverse. Globally, these issues and debates have placed at their centre the need to enable learners to access mathematics. It is being recognised that one cannot succeed in mathematics, and indeed in any field of endeavour, if one does not have access to that field. Therefore, while success appears to be the underlying driver behind the issues and debates in mathematics education and research, it is the notion of access that is clearly key. The centrality of the notion of access became a theme, for the first time since its inception, of the International Group for the Psychology of Mathematics Education (PME) at its 43rd Annual Meeting in 2019. As stated in the introduction to the conference:

> The theme of the conference is *Improving access to the power of mathematics*. Since this is only the second time the conference will be hosted on the African continent, we would like to give the conference a strong African focus – focusing on access, which is very relevant in South Africa as well as in the rest of Africa. However, we would also like to focus on the power of mathematics, thereby giving the conference a strong mathematics flavour. It is hoped that the deliberations at this meeting will affect the course of future mathematics education on all levels worldwide and that this conference will greatly contribute toward closer collaboration between African countries in future (PME43 Conference programme, Graven et al., 2019, p. 1)

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Deliberations at the meeting were diverse, collaborative and all broadly connected to access. They focused on the following themes, in the keynote addresses: (1) mathematics and mathematical literacy, (2) transition zones in mathematics education research for the development of language as resource, (3) representational coherence in instruction as a means of enhancing students' access to mathematics and (4) institutional norms: the assumed, the actual and the possible. The key message from the keynote addresses is that there are foundational structures and norms that define what a classroom looks like. There is therefore a need to 'tear down these norms' (Liljedahl, 2019, p. 1-1) in order to create new possibilities in student behaviours in mathematics classrooms. Key to these norms not only involves the way mathematics is taught but, more critically, how mathematics is conceptualised and what contexts are made available in order to promote new and different interactions among students and teachers. A way of getting to 'tear down these norms' involves revisiting our understanding of the nature of mathematics and the contexts in which it needs to be introduced and experienced by learners and teachers.

In attempts to enable learners to gain access to mathematics, formal and more traditional approaches are still more prevalent. As our topical review of the 2019 PME conference proceedings shows, very few approaches accommodate the more aesthetic dimensions of mathematics that have been widely reported to be effective in earlier years of mathematics teaching. The aesthetic dimension of mathematics, especially play, has become an implicit and often long-forgotten ingredient of high school mathematics instruction. During the 2019 PME conference, very few presentations directly addressed the aesthetic dimensions of mathematics (see, e.g. Antonini, 2019; Bruns & Gasteiger, 2019; Edwards, 2019; Krausel & Salle, 2019; Nakawa et al., 2019). In this chapter, we discuss the performative nature of mathematics, its play and playfulness-related nature. We locate our discussion within a specific play-related activity of mathematics which involved a cohort of high school mathematics teachers. We frame our discussion by using teachers' conceptions of mathematics, after participating in the activity that involved play.

In the next sections, we make some observations about mathematics and the concept of play. Our observations are intended to respond to the question: To what extent should play need to form part of school mathematical activity?

What Is Play?

Our review of the concept of play has indicated that an understanding of play needs to be considered from several dimensions: cultural, creative, imaginational and developmental. The grounding research work of Bert Van Oers (1998) has been widely acknowledged in the literature related to the interconnection between mathematics and play and is therefore reviewed more closely in this chapter.

Play and Culture

Van Oers and Duijkers (2013) have described play as ‘a specific format of cultural activities’ and that ‘all cultural activities can be formatted in different modes, organizing the actions of the participants in characteristic ways’ (p. 515). They also argue that ‘learning and play cannot be separated’ since cultural activities embed the function of learning. According to Van Oers and Duijkers (2013), ‘changes in the playing actions or in the content of the activity of play are demonstrations of learning’ (p. 515). Play starts from what is culturally given and what individuals are familiar with. In other words, and according to Van Oers and Duijkers (2013, p. 190):

> … culturally given activities (practices) are made playful in the child’s role play. This is, by the way, consistent with the cultural-historical analysis of the relation between culture and play by the famous Dutch cultural-historian Huizinga (1938/1951): it is not play that is the origin of culture, but cultural practices, playfully accomplished, should be seen as the origins of cultural developments.

Play is birthed from activities that are cultural entities. Play is therefore ‘essentially a cultural phenomenon dependent on cultural decisions that allow … playfulness in an activity’ (Van Oers & Duijkers, 2013, p. 190). Decisions made in play are cultural. They are not self-evident but depend on cultural values and images and relevant norms of society. According to Van Oers and Duijkers (2013, p. 191), play is organised by the following rules (p. 191): social rules (indicating how to interact with each other, with what resources and under what conditions), technical rules, conceptual rules and strategic rules. From this cultural perspective, Van Oers (2008) has indicated that ‘play is most certainly the basis for all human culture’ and that ‘institutions such as the law, constitutions, art, crafts, science, sport, and trade are basically rooted in human playful activities’ (Huizinga, 1938, in Van Oers, 2008, p. 192). Van Oers, quoting Huizinga (1938), emphasises that human civilization ‘emerges and develops in play, as play’. Also, ‘play has a culture-creating function’. This is because:

> In all play activities, “new or modified meanings are attributed to familiar objects and actions, which subsequently become the basis for new objects and actions”. (p. 370)

Play, Creativity and Imagination

According to Van Oers and Duijkers (2013), play is also a creative act. This is because individuals ‘construct novelty within the constraints and provisions of the situation’ (p. 516). Play, conceived from an activity theory perspective, is ‘a specific mode of activity defined by a format that includes three basic parameters (rules, degrees of freedom and involvement)’ (Van Oers & Duijkers 2013, p. 185). Oers and Duijkers conceive play as ‘a form of human activity’ whose basic characteristics can be recognised as human behaviour that is repetitive, flexible, affect, non-literal, preferential and pleasurable. These characteristics are essential in creative acts.

A key characteristic of play is that it involves imagination. In play, 'people can free themselves … from conventional meanings, and … are allowed to explore alternatives' (Van Oers, 2008, p. 370). According to Vygotsky (1978), imagination emerges in the context of the activity of play involving the creation of an imaginary situation. The activities of human consciousness and meaning-making capacity are inextricably connected to the development of imagination. According to Van Oers (2005), imagination is defined as the constructive process that 'creates new configurations out of known elements of thought' (p. 5). It is noted that because 'products of imagination can oppose ordinary reality', they are highly useful since they form a strong 'basis for critical thinking that confronts everyday reality with new meanings' (p. 5). Imagination therefore has a creative and transformative power. According to Van Oers (2005):

> When people think of imagination as the act of making images, they are basically talking about a process of constructing mental forms that represent some object, and that presumably could be dealt with as if they were that object. Imagination is not creation out of the blue, but it is based on reconstructions with well-known objects within a familiar activity context: by combining old, conventional objects or ideas into a new construct the [individual] creates new things … Imagination indeed produces new objects (new means, actions, or subthemes). (p. 6)

There is therefore a close relationship between imagination and play: both of these are concerned with the processes of abstraction and divergent thinking intended at producing new objects.

A critical component of imagination involves substitution. In 'object-substitution', a person 'takes an object and treats it as if it is something else' (Van Oers, 2005, p. 6). As an illustration of object substitution, Van Oers (2005) gives the example of a child who takes a stick as a horse. In this case, the child 'does not see a real horse in the stick, but it performs some of the actions that relate to a horse'. Quoting Bodrova and Leong (1996), Van Oers (2005) indicates that:

> for the process of imagination, object-substitution in an activity context is an essential element … The object to be substituted should not necessarily be replaced by another physical object; the substitutes can be symbols, drawings, pictures, mental images, narratives, schemes, models, and so on. (p. 7)

Van Oers (2005) indicates that the 'symbolic character of these substitutions supports the formation of imaginations in children's play activities and—by the same token—provides the psychological means for surpassing the limitations and determinants of the actual setting'. In his emphasis of the substitution act as being 'fundamental for every act of imagination', Van Oers notes that 'by substituting an object for something else, the original object actually gets a new image that can be dealt with instead of the original object' (p. 7). The key aspect of this process of substituting is that it results in the generation of new and unexpected meanings related to common objects. Substitution creates spaces for an individual to go beyond the information given. As Kohl (1994, in Van Oers, 2005, p. 16) argues, 'it is this ability to imagine the world as other than it is, that leads to hope and the belief that even the most oppressive and difficult of conditions are not absolute'.

Play and Development

The significance of play can also be considered from a developmental perspective. Play is a major source of development. It is noted that 'a child's greatest achievements are possible in play' and that these achievements 'become her basic level of real action and morality' (Vygotsky, 1978, in Van Oers, 2008, p. 371). Meaning is also central in play. Meaning (contextualised in an imaginary situation) regulates actions of the individual. 'These actions produce new objects that may be a starting point for new explorations of meanings and the production of new tools' (Van Oers, 2008, p. 371). According to Van Oers, contradictions arise between the meanings in play and the real actions. However, these contradictions are essential because 'contradictions between the meanings in play (emerging from the imaginary situation) and the real actions are the main basis for children's development' (see Vygotsky, 1978, p. 101).

Connections Between Mathematics and Play

Researchers such as Kuschner (2012) have highlighted that integrating play into the school curriculum is not an easy task. This is because, according to Kuschner, 'play is natural to childhood but school is not' (p. 242). From this perspective, one needs to conceive play as a context that can be recontextualised for school practice. As Van Oers (2013) has noted, there is a value in conceiving play 'as a context for young children's learning' (p. 186). This recontextualisation (through play) can take two forms: horizontal and vertical. Horizontal recontextualisation arises 'when a new situation is recognized as an opportunity for an alternative realization of a well-known activity' (Van Oers, 1998, p. 138). An example of horizontal recontextualisation involves a discovery that one can apply the mathematical notion of area to irregular geometrical forms. On the other hand, vertical recontextualisation is a process of progressive continuous contextualising in which:

> New problems in an activity may arise and become new pivots of action patterns. These action patterns often lead to the invention of new goals, new means for action, and new strategies. These new action patterns develop into new activities and new contexts for acting that, although emerging from a well-known activity, are not directly a new, alternative realization of that activity. (pp. 138–139)

As an example, Van Oers (1998) cites a study that involved a shoe-store play activity in which actions of measuring emerged. During the shoe-store play activity:

> Measuring became a separate activity … including forms of measuring and conversations about measuring that the children never could have heard in a real shoe-store. Measuring as a new activity gradually emerged out of the shoe-store play activity, leading to a new, even more 'abstract' activity and context of acting. (p. 138)

Researchers such as Dowling (1996) have described the practice of using contexts in mathematics as 'recontextualised practice'. Dowling, however, notes that this 'recontextualized practice relating to one activity cannot retain its structure under the gaze of another activity' (p. 410). The esoteric domain of mathematics usually proves difficult to be accessed by many learners because of its highly technical and complex nature and its processes. Textbook authors and teachers then design non-mathematical settings through which the non-arbitrary content can be realised. The arbitrary settings can be transformed and recontextualised to conform to the particular non-arbitrary mathematical content. The aim of this recontextualisation is to enable learners to access the subject matter knowledge of mathematics.

It needs to be noted that aspects of play in mathematics have been widely recognised and utilised in areas of mathematics involving younger learners as they interact with mathematics in primary and elementary mathematics classrooms. Very little activity and discussion of mathematics linked to play has been at the centre of interactions at higher levels of mathematics teaching and learning.

In the next section, we present an activity, a Crossing the River Activity, as an example of play activities that are possible at higher levels of mathematics learning. The participants in the Crossing the River Activity were in-service mathematics teachers engaged in an Education Honours programme at a South African tertiary institution. Thirty-five (35) participants were involved in the activity. The teachers were from schools from different provinces in South Africa and had differing levels of experience in teaching secondary school mathematics.

An Activity of Mathematics Involving Play: Crossing the River Task Activity

The following activity was given to the teachers:

> Two adults and two children come to a river on their way to a family wedding. The children find a small canoe/boat on the river-bank. They discover that the boat will hold one or two children, but only one adult and no children. How must they cross the river? Mr. Shinghaih, a businessman, the owner of the boat, charges R50.00 per trip for making use of the canoe/boat. How much will the group of 8 adults and two children pay?

As can be seen, at face value, this activity seems to have no obvious link to traditional school mathematics. As such, it has the potential to enable participation of all. Coming to a river on the way to a wedding should be fun. Children finding a small boat should also be fun and demonstrates curiosity that is inherent in children! The constraints on the use of the boat make crossing the river appeal to problem-solving abilities of children! However, having to pay for unnecessary trips should not be much fun; it can be expensive, especially if the family is from a poor background! In any case, they would need to reserve as much money for upcoming transactions at the wedding!

Central to this activity is imagination. This can be seen in the following excerpts arising from the dialogue among the teachers while engaged in the activity.

1 T1: Who is going to be a mother and father?

2 T5: You are going to be …

3 T3: Anyone can be a parent here … [laughter …]

4 T2: A girl cannot be a father or a boy a mom.

5 T7: For us to follow and understand eeesh! One lady here will be a mother, … I volunteer to be a father in this activity.

6 T10: Then I will be the mom in the activity

7 T9: Lets be serious and think about the problem mathematically. … not wasting time.

In mathematics anyone can represent anything like x and y variables. In drama a boy can act as girl but does not mean it's impossible. …

8 T4: We need to keep record of our steps so that we can check if there is any mistakes.

9 T9: Since we start with a family of four people, dad, mom, daughter and son, … Thandile and James[not real names] are the children. [The class claps hands agreeing.]

10 T8: But I think, … we need to read the instructions and questions carefully. Why are they giving us the mass of each member in this house. …?

11 T5: I am lost. The problem is talking about money, weight of people, a family of four people, someone is a businessman, eeish! Yaaah ne.

12 T6: Aaah!!, I see now, the boat as well can only carry so much. … Becoz, if too much it will definitely sink and people will die. …

13 T9: That's why we can't be wasting time, we all are in it and must think and work as one. Let's allow the boy to go first and …

14 T2: No no no! it will be expensive, … and who will bring back the boat?

15 T9: Oooh! I see why T8 was talking about weights of each person.

16 T1: Lets make dad and mom go first. Aah no, but we can try first.

17 T3: From the conditions, a dad and mom cannot go, you see dad weighs 70 kilogram and mom is 78 kilogram. If we add 70 and 78 we get what? … it's 148 kg. This will be too much for the boat. The owner needs R15 for one-way trip.

….

The group members agree to now physically attempt to demonstrate their strategies. They agreed to draw a line on the floor with a white chalk to represent a river and used a loose desktop plank as a boat, where two members (as agreed in their earlier conversations) will hold the boat side by side as they move past the line marking. All of these actions involved imagination, improvisation and creativity. This kind of dialogue is rare in typical mathematics lessons:

26 T7: Recalls, we are starting with children only right? [the members echoed yaaah]

27 T8: Boy and girl can now go, … and stop there for now …, that's one trip you see. Now let's choose who should bring back the boat ne, that's second trip.

28 T2: I think it does not matter who brings back the boat between the children. … but if a child and parent go together, yes it does. So a boy can bring back the boat.

29 T9: Are you recording that in a table?

30 T4 & T5: Yes yes yes.

31 T8: Lets see the table … No this is not a table. Make it like table of values in graphs of functions in algebra.

32 T9: We can do that afterwards when putting together like presenting our findings.

33 T8: Ooh!! I see.
34 T3: Let us continue the activity …
35 T1: I will now go with mom.
36 T6: That's a third trip [the members agree …]
37 T7: Its becoming tricky now, as you can see Thandile and Mom cannot go back.

A Formal Solution to the Activity

In describing a formal solution to the activity, one could denote the movements as follows: Since there are two children and two parents, the letter C could indicate child, P as parent, F as going forward and B as going backward. The first trip would consist of both children going forward to the other side of the river (i.e. 2CF). In the second trip, one child would return (go back) with the boat to the original side (i.e. 1CB), and then one parent would get into the boat to get to the other side (i.e. 1PF). Proceeding in this way, we have a representation of the trips from one side of the river to the other side: 2CF, 1CB, 1PF, 1CB, 2CF, 1CB, 1PF, 1CB and 2CF. This gives a total of nine trips when we have two children and two parents/adults. [Therefore the family would pay 9 × R50 = R450 for the use of the boat.] When an uncle joins the family, how many trips would be required? Upon observation, one finds that we need an extra four trips. This means there would be 13 trips when there are three adults and two children. Proceeding in this way, and when we increase the number of adults, we come up with a pattern that can be summarised in Table 18.1.

It can be seen in Table 18.1 that since there is a common difference of four trips as we move from one column to the next, we obtain the following as a general pattern (formula) for describing the number of trips (T) when we change the number of adults but keep the number of children to two. We obtain the formula $T = 4n + 1$ (where n is the number of adults (parents)).

As observed from the dialogue, as an illustration of modelling, participants pretended to be adults and children. They had to cross a river using a small boat that could hold either one or two children or one adult only. After working out how to do it, they discovered that the number of crossings required was four times the number of adults, plus one, giving $Tn = 4n + 1$ as a generalisation that represents the mathematical essence of the activity.

Table 18.1 Total number of trips (T) when no. of adults increases but no. of children is constant

Number of parents (n)	1	2	3	4	5	6	7	8	9
Number of children[a]	2	2	2	2	2	2	2	2	2
Number of trips (T)	5	9	13	17	21	25	29	33	37

[a]We keep the number of children constant. This is a critical principle of mathematical thinking and problem-solving

Teachers' Comments Linked to the Crossing the River Activity

We now present voices from what the mathematics teachers said after taking part in the performance of the activity. After the performance, the teachers were asked to respond to a questionnaire that had five close-ended and open-ended questions. The instructions given to the teachers were as follows:

> You are required to reflect on the Crossing-the-River activity and write detailed comments based on what you observed and experienced in today's maths class. Write down your reflections using the following leads:
>
> (i) According to my observations from the Crossing-the-River activity, mathematics is …
>
> (ii) According to my observations from the Crossing-the-River activity, mathematics is about …
>
> (iii) I learned the following mathematics ideas from my participation in the Crossing-the-River activity, …
>
> (iv) I liked the Crossing-the-River activity because …
>
> (v) My other comments on the Crossing-the-River activity are as follows …

Comments Linked to the Nature of Mathematics

Key comments from the teachers involved perceptions of mathematics that considered mathematics as a subject that 'does not come from another planet' and that mathematics can be role-played or demonstrated through everyday life situations. Comments from teacher T1 below illustrate this observation:

> I have learnt that mathematics problems are interrelated with our daily lives so, in order to solve problems, we must demonstrate the problems or give the problems live; we can do this by either <u>role playing</u> or demonstrations. And also I learnt that when it's easier not to rush to formulas right away but to solve the problems manually where applicable, because by doing this you will understand how the formula came about or how it was derived … Mathematics is not abstract or from another planet but this activity showed me that things that we do in our daily lives have mathematics. Again it showed me that most of <u>mathematics problems can be demonstrated or role-played</u> in order to make someone understand better.

This activity made T1 to see mathematics in a different way from before. He noted:

> My mind has shifted from thinking that mathematics is about equations and formulas; rather it showed me that mathematics is interrelated with our daily lives … I think as teachers we also should stop teaching mathematics as an abstract subject but try to make demonstrations and role plays that learners would understand. And one thing for sure, learners enjoy a subject when they realise that they interrelate with it. I think this will make learners enjoy the subject compared to before. We should make examples that they see in their daily lives. This kind of activities they make you think, they make you participate and I think if you took part in something you will never forget it in your life. I now know that <u>when I see a mathematics problem I should give it life by demonstrations or role-plays</u>.

The above comments seem to indicate that, for mathematics to be enjoyable and readily relevant to learners, it needs to be rooted and find its application in modes of engagement that incorporate the aesthetic and artistic forms of thinking and acting that are embedded and promoted through play.

Teacher T2's comments highlighted the communicative, conversational and observational aspects of mathematics. She noted:

> Mathematics is a subject that **communicates** with individuals based on practical observations. It is the subject that requires full attention of the student and gives more of the attention to what is being observed. Mathematics in this activity is an eye-opening and mind-opening subject that enables one to come up with their own ideas and extend further in visualising from observation so that one may not forget the next time they solve problems. Mathematics is about paying attention and giving full focus on scenarios, and also making notes and in every observation. It is about understanding **what is needed and what is not needed** or allowed in the process of solving problems. It is also about understanding patterns.

Comments Linked to 'Performing' Mathematics

Teacher T3 now viewed mathematics as a subject that needs to be performed. One needs to perform mathematics, rather than stick to the textbook:

> I learned that you can teach maths without using any tool such as a textbook because maths is a practical-based learning area and not a theoretical one. It is very interesting and more enjoyable. I have also learned that as a mathematics teacher you can use your learners as teaching aids in class in order to perform practical examples so that your learners can understand your lesson far more better and enjoy it. I liked the crossing river activity because it has taught me a lesson that as a mathematics teacher you need to make the subject to be more interesting by ensuring that you don't always stick to the textbook whenever you are teaching the kids, but perform some of the activities practically so that the learners can see the nature and the importance of mathematics.

Comments Linked to Culture and Everyday Experiences

Teachers T3 and T4 made comments that highlighted cultural aspects involved in a mathematical activity:

> T3: Mathematics is part of nature. In this activity there was a family that needed to cross the river so that they can attend a wedding. They were using a boat to get to the other side of the river … The number of adults kept on increasing until we got a linear number pattern.
>
> Mathematics is formulated from what is really happening in our life. From this activity I think people who invented mathematics observed what is happening around us, then came up with possible solutions for the problems that human beings may come across. I am saying this because in this activity what the family went through in order for them to reach their destination might happen to a real family who live in rural places where there are rivers.
>
> T4: Mathematics is doable. It is difficult yet doable. From the river crossing activity I realised that mathematics is what we do every day, the way [we] think and the decisions we

have to make. Mathematics is about dealing with the logic of numbers, shape, quantity and arrangement. Maths is all around us, in everything we do. It is the building block of everything in our daily lives, including art, as well as the arrangement of objects.

Comments Linked to Pedagogy

T5: It was interesting, enjoyable and a little bit challenging. At first it was challenging to figure out the solution on my own, but after the first trip I was able to see what was going on. The way it was enjoyable, I couldn't stop to add more adult relatives in order to see how many trip I would have. It also made me see that every problem that one may come across have solutions … learning is more interesting when you learn looking/using what is happening around you … one figure[s] out solutions more wisely when he put himself in the situation.

T6: You can teach maths without using any tool such as a textbook because maths is a practical-based learning area and not a theoretical one. It is very interesting and more enjoyable. I have also learned that as a mathematics teacher you can use your learners as teaching aids in class in order to perform practical examples so that your learners can understand your lesson far more better and enjoy it. I liked the crossing river activity because it has taught me a lesson that as a mathematics teacher you need to make the subject to be more interesting by ensuring that you don't always stick to the textbook whenever you are teaching the kids, but perform some of the activities practically so that the learners can see the nature and the importance of mathematics.

Comments Linked to 'Creative and Critical Thinking'

T7: The activity involves creativity and imagination to discover the solutions. When crossing the river the family had to consider money they were supposed to pay. Therefore creativity was also needed.

T8: I liked the cross river activity because help me acquire problem-solving skills, to be a critical thinker or think out of the situation. It also help to understand maths in real life situations, reminds me why I am doing mathematics as how to teach it such as creating scenarios in class that can help learners to understand mathematics better.

It is important to think out of the box as to understand certain principles in order to solve a problem. The Crossing the River activity was reminding us the qualities that are nurtured by mathematics which are power of reasoning, creativity, abstract or spatial thinking, critical thinking, problem-solving ability and effective communication skills.

Comments Linked to Change of Mindsets

T9: Most learners will realise that mathematics is not a monster that most of them think it is. Through this activity my reflection in mathematics is that, mathematics is practical and relevant to all our daily activities, we just need to realise it and embrace this great subject called mathematics. We eat maths, walk maths and talk maths.

> T10: Mathematics is a subject derived from a common activity. It is concepts that can be derived from any activity that forms a pattern, from that activity of Crossing the River, a pattern was formed while it was just a common thing that can involve one capability of thinking. Taking down all results and observations made it easy for us to come up with a sequence or pattern for any given number of kids or adults. So I viewed mathematics different from that activity, it makes me to note everything that I do every day, something that is a routine to think a mathematical concept for it and come up with something or pattern for it. It also opened my mind business-wise, that when I see someone doing something over and over again, what can I do to actually improve the situation. So the Crossing over River activity was a mind-blowing for me.
>
> The activity also made me to actually be conscious about the results in any activity you are doing, because in that activity you were supposed to always have a child moving with the boat. I mean the restrictions were in such a way that you think beyond the current trip in order to sustain them.

The voices of these teachers highlight the importance of 'bringing learners to mathematics'. We claim that the underperformance that often characterises the problematic classroom experiences of many learners arises from pedagogies that strive to 'bring mathematics to learners'. Such pedagogies pay little recognition to the need to bring learners to mathematics. To bring learners to mathematics means that we need to know not only what mathematics is and is about, but also who the people are. We argue that play is the stage that levels the playing field for learners and teachers of mathematics. Play is the stage that unveils the inner stage that bonds mathematics and learners. Through play, learners become engaged and sufficiently glued to that stage.

Reflections on Teacher's Comments

At the centre of this Crossing the River play activity was imagination. The activity involved imagining a family of four going to attend a wedding, possibly on a Saturday morning, after a rainy night. They possibly needed to cross a flooded river that connects their home to a neighbouring village. The activity also involved imagining having to pay for using a boat to cross the river. This imagination, though a product of the mind, has a strong appeal to the everyday world and contexts in which the teachers who participated in this study are located. Being adults themselves, the teachers are more likely to have had the experience of attending an actual wedding (possibly of their own!). They are likely to appreciate the significance of this story activity as a context for everyday living. However, there is an immediate gap that is inherent in this story context. Although the story immediately appeals to the every day, it does not readily connect itself to any aspect of formal mathematics as we may know it. It is this gap that is critical in this chapter.

In our opinion, a gap like this one is critically and pedagogically necessary for mathematics teaching and learning. In the context of this chapter, we propose that play fills the gap between the everyday and the mathematical worlds. Play plugs in the gap! We argue here, in agreement with Popkewitz (1988), that there is a gap between school mathematics and mathematics. He observes that ‘School Mathematics is shaped and fashioned by social and historical conditions that have little to do with the meaning of mathematics as a discipline of knowledge’ (p. 221). School mathematics is a recontextualised practice. It is a recontextualisation of mathematics. There is a gap, a space, between school mathematics and mathematics, between the everyday world and the world of mathematics (the esoteric domain of mathematics). We are arguing here that play or playfulness in the mathematics classroom makes visible the aesthetics of the gap or spaces between school mathematics and esoteric mathematics. In other words, play creates aesthetic spaces between the everyday world and the mathematical world.

In this chapter, we argue that there are gaps that are always present in any mathematical experience or activity in a classroom or curriculum. As long as mathematics teaching is intended to move students from one state of knowing or experience to a different or possibly ‘better’ state of functioning in the mathematical world, there will always be gaps to contend with. There will be gaps involving several aspects such as gaps between routine and non-routine procedures in mathematical problem-solving, gaps between the so-called ‘easy’ and ‘difficult’ content, gaps between the concrete and abstract, gaps between method and content, gaps between known and unknown and gaps between one mathematics content area and another content area. Considered in this way, we argue that mathematics and mathematics education generally are practices that are ‘pregnant’ with gaps and that play gives birth and creates aesthetic spaces from these gaps. The aesthetic spaces so created, and as can be seen from the data in this chapter, have a strong potential to ‘fill’ the gap between the everyday and the mathematical world that we are intending to get students closer to as they attend mathematics lessons. Teaching mathematics should therefore be more concerned with recognising and bringing into lessons the aesthetic spaces and experiences that are relevant to the lesson topic. Learning mathematics should in turn be about productively responding to the aesthetic spaces or experiences that are made available. Where relevant aesthetic experiences may be unavailable or non-existent, the teacher and his/her students need to recognise that they have a collective responsibility to search for or create the aesthetic spaces that may be critical for the mathematical experiences that are intended to be gained.

Conclusion: Mathematics and the Aesthetic

The Crossing the River Activity reflected on here is an attempt to align school mathematics activity with aesthetic approaches currently being proposed in mathematics education reforms. A key aspect of the activity analysed in this chapter involves foundational elements of play, creativity and imagination that have been advocated particularly in the seminal work of von Oers. The Crossing the River Activity is an illustration of the creative and transformative power of play when linked to key mathematics concepts and learning:

> Imagination is not creation out of the blue, but it is based on reconstructions with well-known objects within a familiar activity context: by combining old, conventional objects or ideas into a new construct the [individual] creates new things … Imagination indeed produces new objects (new means, actions, or subthemes). (Van Oers, 2005, p. 6)

We have illustrated in this chapter the imaginative and playful aspects of mathematics that lead to learning, recontextualising and development of new ideas. The recontextualisation aspects involved in play not only lead to new ideas and meanings but also give rise a kind of mathematics described by Gadanidis et al. (2016, p. 225) as:

> Mathematics worthy of attention, worthy of conversation, worthy of children's incredible minds, which thirst for knowledge and for opportunities to explore, question, flex their imagination, discover, discuss and share their learning.

There is a notable thirst for aesthetic mathematics experiences that need quenching, arising from a recognition that school mathematics 'commonly lacks an aesthetic quality' (p. 226). Learners have 'never experienced the aesthetic quality of quenching a thirst for mathematics'. Quoting Papert (1978), Gadanidis et al. (2016) argue that 'common views of mathematics exaggerate its logical face and devalue all connection with everything else in human experience', thus missing 'mathematical pleasure and beauty' (p. 226). In agreement with Brown et al. (1989), Gadanidis et al. also argue that 'many of the activities students undertake are simply not the activities of practitioners and would not make sense or be endorsed by the cultures to which they are attributed' (p. 34). In this connection, Root-Bernstein (1996) has noted that 'students rarely, if ever, are given any notion whatever of the aesthetic dimension or multiplicity of imagining possibilities of the sciences' (p. 62). Our reflections of the Crossing the River Activity illustrates the aesthetic experiences that result from involving learners in play. In their research, Gadanidis et al. (2016) used a number of empirical examples to suggest that, in order to enhance the aesthetic quality of (school) mathematics, educators need to learn from the arts and from artists. In doing so, they first recognise the need to distinguish between mathematics and school mathematics and express a need to be 'cautious about suggesting that mathematics or the work of mathematicians might be devoid of "art"' (p. 227). According to Root-Bernstein (1996), the sciences (mathematics) and the arts are 'very similar, certainly complimentary, and sometimes even overlapping ways of understanding the world' (p. 49). Gadanidis et al. (2016) have argued in agreement with Root-Bernstein and have identified that the problem may not be with mathematics, but with school mathematics:

> The problem with school mathematics is not that it lacks the arts, but rather that it lacks the aesthetic that is common to mathematics, the arts, and other disciplines: the aesthetic that makes the experience of these disciplines human. (p. 227)

In fact, Popkewitz (1988) has argued that there are limitations with school mathematics. He observes that 'school Mathematics is shaped and fashioned by social and historical conditions that have little to do with the meaning of mathematics as a discipline of knowledge' (p. 221). Gadanidis et al. (2016) have proposed a model for designing aesthetic mathematics experiences that draw on 'important connections among elements of narrative (what makes for a good story) and mathematics (what makes for a good math experience)' (p. 228). The activities mentioned in Gadanidis et al. (2016) present opportunities for students to experience mathematics worthy of 'attention' and 'conversation' and present opportunities to 'explore, question, flex their imagination, discover, discuss and share their learning' (p. 238).

We propose that the Crossing the River Activity that we have reflected upon in this chapter represents an attempt towards capturing the story-based nature (Gadanidis & Hoogland, 2003) of human cognition that needs to form part of mainstream school mathematics education. The comments from the teachers indicate the aesthetic spaces that the activity attempted to create in order to enable them to access the mathematical concepts intended to be developed through the activity. The positive experiences that this activity generated among teachers suggest that activities such as these need not only form an integral part of the mainstream higher level of school mathematics but also part of mathematics teacher professional development. This chapter demonstrates that play, and mathematics through play, creates an embodied stage for simultaneously showcasing the nature of mathematics and enabling access to mathematics.

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