Chapter 1
Manipulatives as Mediums for Visualisation Processes in the Teaching of Mathematics

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Introduction

The purpose of this chapter is twofold. First, it argues for a visualisation approach to teaching, using the well-known *Geoboard* as both a teaching and a visualisation tool. Second, it narrates a professional teacher development programme that formed the heart of a research project that interrogated how a visual approach to teaching can be achieved.

This chapter begins by discussing notions of visualisation in mathematics and how the use of visualisation can support a conceptual approach to teaching. I will draw from literature and discuss some of the implications that one can infer from these readings. This discussion will specifically look at the interplay between manipulatives and visualisation processes and how this can be harnessed for a visual approach to conceptual teaching. The chapter will then specifically look at how the use of manipulatives can frame a visual approach to teaching. I will draw from one particular Namibian case study research project, which used a physical manipulative (the *Geoboard* in this case) as a means to teach the properties of quadrilaterals. Research in visualisation processes in the teaching of mathematics is once again gaining traction as we continue to search for examples of best classroom practice.

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Backdrop

My observations of the many mathematics classrooms that I have visited in my long career as a mathematics teacher educator reveal that mathematics classrooms sadly often do not exude the desired inspiration and positive energy expected from an exciting learning environment. They are often drab, naked and devoid of any mathematical curiosities or interesting illustrations that relate to the beauty and intrigue of mathematics. Of course it is more difficult to adorn the walls of a mathematics classroom than a geography classroom, with, for example, interesting maps or pictures of beautiful landscapes or fascinating and obscure geomorphological features. This chapter argues that the inherent visual nature of mathematics is all too often neglected and should be revisited. Textbooks all over the world are often the only source of visual information that is at the disposal of the teacher. As Nghifimule (2016) from Namibia suggests, in many developing countries, where there is often a shortage of reading materials and a lack of access to these materials, the textbook becomes the only resource text that teachers can draw from. This implies then that the visual materials that teachers have access to are sourced mainly from textbooks. There are, however, a myriad of other opportunities to access and manufacture visualisation materials to facilitate a learning process that is genuinely conceptual and interesting. By their very nature, textbooks seem to avoid the use of visuals for concepts that are abstract in nature and not immediately visually accessible. In her analysis of the nature of visualisation objects in the algebra and geometry chapters of three Namibian mathematics textbooks, Nghifimule (2016) found that on average, 75% of these objects used pertained to geometry, a domain in mathematics that, it can be argued, is inherently more visual than algebra in any case. It is, however, this imbalance of illustrating and visually representing mathematical concepts that reinforces only a very selective use of visualisation objects in the teaching of mathematics.

Visualisation Processes in the Teaching of Mathematics

Many definitions and perceptions of visualisation abound in the literature. Often when writing and theorising about visualisation, many researchers typically define visualisation in mathematics only in the context of producing and using imagery, both physical (external) and mental (internal). For example, visualisation can be seen in the context of producing and using diagrams, graphs and figures in a mathematics environment, on the one hand, and taking a broader perspective and view of visualisation as an intricate construct that involves both product and process, on the other. For example, Arcavi (2003) quite eloquently proposes that visualisation is "the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (p. 217). Despite the emphasis on process, this definition, however, is still imagery bound. This chapter argues that visualisation processes can also apply to processes that are embodied with our actions and our ways of communicating, such as gestures and language. Either way, using visualisation processes is cognitively desirable. It stimulates learners' minds to identify patterns and trends, make connections to real-life situations and make abstract ideas more accessible and imaginable. With regard to the latter, Rudziewicz et al. (2017) aptly suggest that visualisation processes enable learners to "see the unseen". This is particularly important when engaging in mathematical concepts and ideas, such as irrational numbers, that are obscure and not as tangible as those that are imminently visible and touchable, such as geometric shapes. It is asserted that being able to "see the unseen" is a sign of deeper understanding of a particular concept (Natsheh & Karsenty, 2014). With regard to using visualisation processes, Mudaly (2010, p. 65) observes that "viewed pictures often create clearer images in our minds because of the symbols attached to what we see, accompanied by other sensory perceptions".

Duval (2014) argues that making use of visualisation processes is a cognitive process that is underscored by reasoning, whereby learners think through what they see and hear (including words) in order to make sense of the mathematical idea under consideration. This chapter thus argues that it is a process integral to any form of mathematising. It therefore goes without saying that visualisation facilitates a process in learners to develop a deeper and richer mathematical understanding than they otherwise might have developed from other learning processes. As Figueiras and Arcavi (2014) so appropriately assert that in the context of teaching, if applied meaningfully, visualisation can "visually evoke real-life experiences" that form the basis of contextualising mathematics in the learner's own environment. Visualisation can thus be a process of harmonising what learners see in their environment to what they think or imagine in their minds (Nemirovsky & Noble, 1997).

Notions of the interplay between visualisation and mathematics are of course nothing new in the context of theorising and researching strategies of teaching and learning. Cobb et al. (1992) referred to the connection between mathematics in learners' minds (i.e. their imaginations) and the mathematics in their environment as a form of dualism. Zazkis et al. (1996) also expounded on the concept of visualisation as a link between what learners think and what they see by stating that:

> [a]n act of visualization [that is, the process of visualization] may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, chalkboard, or computer screen, of objects or events which the individual identifies with object(s) or process(es) in her or his mind. (p. 441)

From a teachers perspective, the pedagogy of visualisation has interesting and significant implications. The plethora of different curriculum statements all over the world refer to the use of visualisation processes in different ways. The National Curriculum for Basic Education of Namibia, for example, aligns visual skills with communication and graphicacy:

> A high level of communication skills, more than just functional literacy, numeracy and graphicacy, is essential in a knowledge-based society..... Visual communication plays an increasingly important role in a knowledge-based society, and learners need to develop good visual communication skills in understanding, investigating, interpreting, critically analysing, evaluating, and using a wide range of visual media and other sources of aural and visual messages. (Namibia. MoE, 2010).

Similarly, the National Curriculum Statement Grades R – 12 in South Africa aims to produce learners that are able to:

...communicate effectively using visual, symbolic and/or language skills in various modes. (RSA. DoBE, 2011).

Incorporating a visual pedagogy thus seems appropriate and indeed desirable if the classroom is to be a space where learners are indeed offered an opportunity to learn and hone the skills that the curriculum statements above are calling for. It is thus incumbent on the mathematics teacher to create learning and teaching environments where such skills can be developed.

One way of embedding a visual approach to teaching is through the use of manipulatives.

Manipulatives

Manipulatives are objects, physical or virtual, that the user can consciously or unconsciously handle and transform to meet certain objectives. It is in the manipulation of these objects that learning and teaching outcomes can be achieved. For manipulatives to foster meaningful learning or teaching, they should be objects that have a visual appeal that inspire curiosity and an interest to explore and experiment. Manipulatives do not only describe and illustrate a mathematical idea, but they can also develop mathematical concepts and can be used as mathematising devices.

Manipulatives have been used as learning and teaching aids in many classroom and home environments over the centuries. Instinctively, young babies soon after birth clutch and manipulate objects as they maneuvre them close to their mouths in order to suckle them – the first tentative learning steps in perceiving their environment, on the one hand, and satisfying their sucking instinct, on the other. Teaching manipulatives or teaching aids are featured in all corners of the world as devices that enable the teacher to conceptually illustrate or model mathematical ideas and concepts. We should all be familiar with the Cuisenaire rods for configuring numerical proportions in the early-grade mathematics classroom, for example.

But the mystery remains as to why then these manipulatives or teaching aids remain a relatively rare sight in my classroom observations.

It goes without saying that manipulatives are inherently visual, whether these manipulatives are tactile, as in physical manipulatives, or digital as in virtual manipulatives. In both cases the visuality of the manipulative is key for the conceptual development of a mathematical idea.

In his research Dzambara (2012) engaged with 75 mathematics teachers from 25 secondary schools in Windhoek, the capital city of Namibia. The purpose of his research study was to audit the availability and use of mathematics teaching aids in secondary schools in that city. He analysed the distribution and use of 12 different types of teaching aids ranging from different chalkboard instruments, charts and posters to physical objects such as geometric models. In broad brushstrokes Dzambara (2012) found that whilst some teaching aids such as charts and posters, chalkboard 30 and 60 degree set squares, chalkboard rulers, protractors and compasses, mathematical sets for learners and improvised teaching aids were used regularly (i.e. on a daily basis), other teaching aids were only used moderately (i.e. used as frequently as possible) to never used. These included physical manipulatives such as geometric models and computers. Devices such as graph boards, interactive white boards and, surprisingly, the *Geoboard* were used by only a handful of teachers. On average, across all the teaching aids, Dzambara (2012) found that 11% of teaching aids were used on a daily basis, 40% were used as frequently as possible, and 49% were never used. More accurately, the chalkboard 30 and 60 degree set squares were used daily by 25 teachers, as frequently as possible by 39 teachers and never by 11 teachers. Charts and posters were used daily by 12 teachers, as frequently as possible by 51 teachers and never used by 12 teachers. Physical objects other than geometric models were used daily by 5 teachers, as frequently as possible by 27 teachers and never used by 37 teachers. Geometric models/shapes were used on a daily basis by 2 teachers, as frequently as possible by 29 teachers and never used by 44 teachers. It is interesting to note that 47% of the teaching aids were school purchases, 35% accounted for personal purchases and only 11% of the teaching aids were bought directly by the Ministry of Education. The rest were donations or gifts. From Dzambara's (2012) work, it is apparent that quite overwhelmingly, teaching aids are not used in in Namibia's most urban area. This despite the fact that 96% of his participating teachers agree that the use of teaching aids in mathematics classes promotes learners' participation and interest in mathematics. This is corroborated by the work of Suydam and Higgins (1977) on the use of manipulatives, suggesting that "lessons using manipulative materials have a higher probability of producing greater mathematical achievement than do non-manipulative lessons" (p.83). This however presumably assumes that the use of the relevant manipulative was appropriate, well planned and aligned with the learning and teaching objectives. An unmediated manipulative operating in a pedagogical vacuum has only limited value. It would be interesting to extend Dzambara's work to the rural areas of Namibia where access to resources and infrastructure is compromised and limited and where the poverty gap is most pronounced. Kraft (2014) quotes the Namibia Statistic Agency that "people in rural areas are twice as likely to be poor compared to those in urban areas with about 37.4 percent of people living in rural areas being poor compared to 14.6 percent in urban areas". This would indicate that the purchase and acquisition of teaching aids may not be seen as a priority and thus be neglected.

The Interplay between Manipulatives and Visualisation

Due to the tactile nature of manipulatives, the link between using them and visualisation seems self-evident. Manipulatives can provide the entry point to visualising a particular mathematical concept, particularly if the concept is at first obscure and a little nebulous. Take the well-known problem of calculating the shortest distance for a spider to crawl from the top corner of a room to the diagonally opposite corner at the bottom of the room. The use of a physical model of a rectangular prism may enable the problem solver to visualise the problem quite easily as opposed to sketching this three-dimensional scenario on a plane piece of paper first. The strength of manipulatives lies in their potential to model real-life situations and thus provide for a visual representation of a particular problem-solving situation. Essentially there are two different types of manipulatives. These are physical and virtual manipulatives.

Physical Manipulatives

Physical manipulatives, also sometimes referred to as concrete objects or physical models, are objects or devices that are made of concrete materials such as wood, plastics, paper, clay and polystyrene. Drawings on paper would also fall under this category. Heddens (1986) makes the point that physical manipulatives from the learners' real world are strategically and typically used to visualise and represent mathematical ideas in a way that can clarify these ideas more simply and effectively than without them. It is thus unclear why more teachers do not make use of physical manipulatives in their teaching. Sets of physical manipulatives in many colours, shapes and sizes are available on the market. These however come at a financial cost, which often deters schools and teachers from purchasing them. As will be discussed below, substantial costs can be saved when physical manipulatives are self-made using materials that are readily available in the immediate surroundings of the school and the learners.

In their narrative about proficient teaching, Kilpatrick et al. (2001) posit that the use of physical manipulatives should be a key feature of teaching as they provide a means to link informal knowledge and intuition to mathematical abstraction; they can be used as mathematical representations to clarify ideas and support reasoning and build understanding; and they enhance and enrich conceptual understanding and inspire mathematical talk in the classroom.

The physical manipulative that is featured in this chapter is the *Geoboard*.

Virtual Manipulatives

These manipulatives, also referred to as digital manipulatives, are virtual and non-tactile in nature as they occur on digital devices such as tablets, computers and smartphones. The extent of their manipulation potential and capacity depends on the complexity and comprehensiveness of the software that drives the manipulative. In computer-aided design (CAD) programmes, for example, it is possible to move a three-dimensional representation of, say, a cuboid, and view it from any perspective or angle one wishes. This capacity can of course be very powerful in enabling the user or consumer to visualise obscured sides and corners of shapes and thus enable him/her to see the unseen. The plethora of manipulation games available for children's use on their tablets can, however, at times be quite overwhelming. The quality and efficacy of these games vary greatly from game to game and require a critical eye to determine this quality. +The same applies to mathematical software. A useful indicator for quality is the extent of the versatility of the software and the degree to which the user is able to take control of the manipulation capacity to suit their needs.

Case Study: The Geoboard

Introduction

This interpretivist case study took place in the North Eastern Region of Kunene in Namibia. It was framed by an intervention programme that involved three selected Grade 7 teachers from three different schools.

The aim of the study was to investigate and analyse the use of *Geoboards* as visualisation tools to teach the properties of quadrilaterals to Grade 7 learners. The research questions that framed the study were as follows:

- What are the affordances of the utilisation of *Geoboards* as visualisation tools in the teaching of the properties of quadrilaterals in Grade 7 classes?
- What are selected teachers' experiences of using *Geoboards* as visualisation tools in teaching the properties of quadrilaterals, as a result of participating in an intervention programme?
- How do the participating teachers make use of the Van Hiele phases in their teaching of quadrilaterals using the *Geoboard*? (Matengu, 2018).

Due to limited space, the third research question will not be reported on in this chapter.

The case study was constructed within the context of an intervention programme that involved working with three purposefully selected teachers over a period of one term. The intervention programme involved regular meetings where *Geoboards* were designed and manufactured. Ideas about how these *Geobaords* could be put to good use were discussed and workshopped. Each teacher then planned a short teaching and learning programme which consisted of each teacher teaching three lessons which were video-recorded. These video-recordings were then analysed in conjunction with stimulus-recall and focus group interviews in order to answer the research questions above.

The Geoboard

The *Geoboard* is a well-known manipulative that unfortunately has not found much traction in Namibia. Matengu (2018) cited Miranda and Adler (2010) who observed that in general, "Namibia is one of the many countries in which the use of manipulatives in mathematics classrooms is not common practice".

The *Geoboard* usually consists of a flat square piece of wood into which small nails are driven in a pattern of repeated squares. The nails then serve as little posts around which elastic bands can be placed to form various plane shapes. This manipulative is easy to manufacture and serves the exploration of shapes very well as it is quite versatile and transportable. It is particularly useful to explore area and perimeter of shapes such as quadrilaterals.

The Intervention Programme

The intervention programme consisted of the researcher and the three teachers meeting on a regular basis. In the first instance, they all made a *Geoboard* themselves. A local carpenter was solicited to assist with the sourcing and cutting of the square base boards to size. A square lattice of points was then constructed on a piece of paper and placed on the square board. The points served as markers where the nails were then hammered in. See Fig. 1.1.

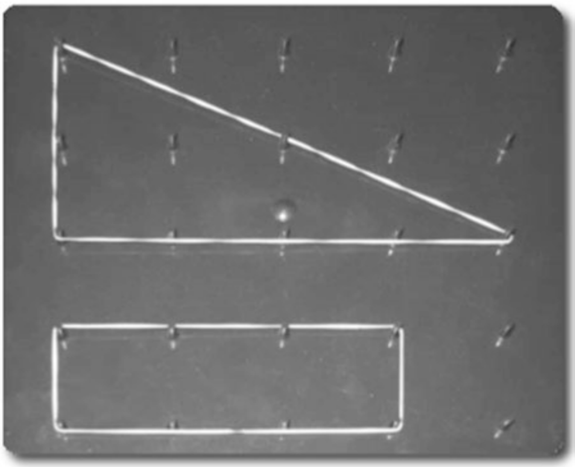

Fig. 1.1 A typical *Geoboard* consisting of a 4 × 4 lattice of squares

The nails now serve as posts around which elastic bands can be placed to form plane geometry shapes, such as a right-angled triangle and rectangle as can be seen in Fig. 1.1.

The group then discussed and planned a total of nine lessons, i.e. three lessons per teacher, where the first lesson constituted a pilot lesson. Each lesson focused on the properties of different quadrilaterals.

All the nine lessons were video-recorded at times that suited the researcher and the teachers. Stimulus-recall interviews were then conducted to collaboratively analyse the lessons in order to critically reflect upon them. After this process of interviews, a focus group interview enabled the participating teachers to collectively deliberate about and reflect on their experiences.

Analysis of the Lessons

In order to enable the researcher and the teachers to make sense of the lessons in terms of how the *Geoboard* was utilised as a visualisation tool, the analytical tool in Fig. 1.2 was used as a lens through which to observe each lesson.

The work of Presmeg (1986) and Duval (1995) were the main sources that inspired and informed this framework.

Concrete pictorial imagery (PI) refers to the formation of concrete images or representations. Teahen (2015) argues that concrete visual representations are important objects for learners to visualise mathematical operations. The *Geoboard* is an ideal device to form concrete visual representations of geometric shapes. The characteristics of these shapes can then be deliberated on.

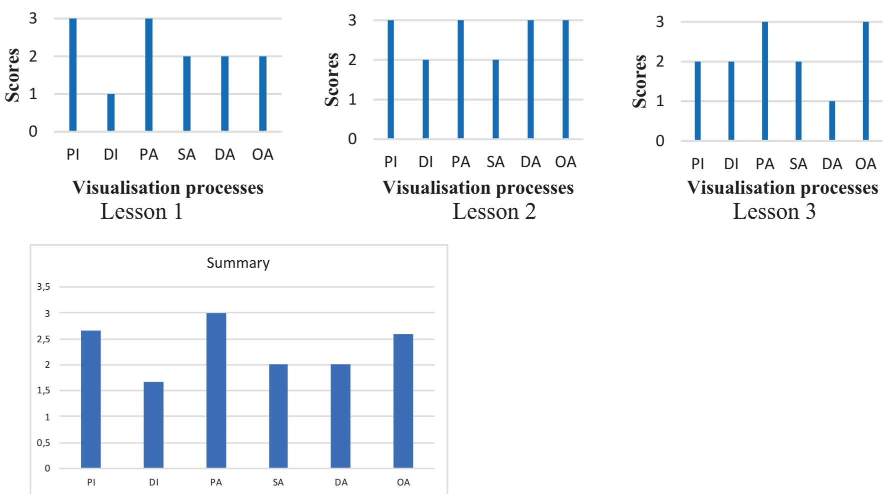

Fig. 1.2 Bar graphs showing evidence of the visual imagery and apprehensions used in Mr. Jones' three lessons

Dynamic imagery (DI) refers to the dynamic nature of a visual representation. Typically in classroom settings, static images such as those in textbooks or posters on the wall are used. Dynamic images however are those that change and move and thus can capture the imagination in exciting and novel ways. Once again the *Geoboard* is an ideal instrument where the user can change the look of a shape in an instant by repositioning and moving the elastic bands to different positions, thus creating a new shape or altering an existing shape, thereby manipulating its original characteristics.

Perceptual apprehension (PA) is what an image evokes. Matengu (2018) argues that it is often the very first perception that an individual experiences when confronted with an image, such as a geometric shape on the *Geoboard.* For any image to have an impact on a learner, it must evoke perceptual apprehension (Duval, 1995). As Duval (1995) continues to suggest, "nothing is more convincing than what is seen" (p.12). It is however important that a learner needs to move beyond only perceptual apprehension. Samson and Schäfer (2011) suggest that one way of moving beyond perceptual apprehension is by seeing (or showing) an image in multiple ways. The *Geoboard* is well suited to this as a learner can construct a triangle, for example, in multiple ways, using different configurations of the pins and stretching the elastic bands in different ways.

Sequential apprehension (SA) is understanding the mathematical implications of the constructed shape or figure. Once again, the *Geoboard* lends itself well to exploring the mathematical implications of a constructed shape. In Fig. 1.1, taking a 1×1 square as one-unit square, for example, enables a learner to explore the area of the triangle by counting the number of squares and then conjecturing the relationship of the sides of the triangle with respect to this area.

Discursive apprehension (DA) is about articulating the mathematical ideas inherent in an image or representation. Duval (1995) makes the interesting observation that it is through speech and engagement, i.e. through a discursive process, that the mathematical properties of a shape start to make sense to a learner. Although this discursive process can be silent and personal, it goes beyond just a personal apprehension.

Operative apprehension (OA) involves the actual operation on a figure. According to Duval (1995) this can involve mental or physical manipulation depending on the nature of the figure or representation. Operative apprehension can also involve transforming the figure.

In the analysis of the three lessons per teacher, the researcher together with the teachers used the above framework to reflect on their respective lessons. This proved to be a very fruitful exercise to not only answer the research questions but also as a means to critically reflect on their practice. Each interview took the form of a stimulus-recall interview where the conversation between the researcher and the teacher was guided by the video recording of the specific lesson they were both watching. The interview questions were loosely framed by and aligned with the criteria in the analytical framework (see Table 1.1 below). The video recordings could be paused at any time to probe either further or deeper.

Table 1.1 Analytical framework

Type of visual imagery and apprehension criteria	Visualisation process indicators	Coding			
		0	1	2	3
Concrete pictorial imagery PI	There is evidence of the use of *geoboards* that encourages learners to form mental pictures of the properties of quadrilaterals.				
Dynamic imagery DI	There is evidence that the teachers encouraged the manipulation of static *geoboard* figures to dynamic processes by changing the position(s) of rubber bands and number of pegs to transform shapes.				
Perceptual apprehension PA	There is evidence that the teacher used the *geoboards* to assist learners to simply recognise basic shapes. These are not necessarily relevant to the constructions of quadrilaterals.				
Sequential apprehension SA	There is evidence that the teachers facilitated the independent construction of shapes using the *geoboards*. Learners are encouraged to construct and describe shapes on their own.				
Discursive apprehension DA	The teachers encouraged learners to verbally describe the properties of the constructed shapes.				
Operative apprehension OA	The teacher sets problems for the learners to solve using the *geoboard.*				

Adopted from Presmeg (1986) and Duval (1995).

Table 1.2 Coding descriptors

Coding	Categories	Descriptions (visualisation process)
0	No evidence	There is no evidence of visualisation processes in the use of *geoboards*
1	Weak evidence	There is little evidence of visualisation processes in the use of *geoboards* (1–2 incidences)
2	Medium evidence	There is sufficient evidence of visualisation processes on the use of *geoboards* (2–3 incidences)
3	Strong evidence	There is abundant evidence of visualisation processes, more than three incidences on the use of *geoboards*

Although the interviews and their analysis were mostly qualitative in nature, the researcher and the respective teachers also quantitatively captured the extent to which the visual imagery and apprehension criteria were evident. This was done by using the coding framework with coding descriptors in Table 1.2 above.

Findings and Discussion

Mr. Jones' Lessons

In all of Mr. Jones' three lessons, there was strong evidence that his use of the *Geoboard* evoked perceptual apprehension – see Fig. 1.2. He made extensive use of the *Geoboard* to illustrate properties of various quadrilaterals. He often demonstrated the construction of a quadrilateral in a step-by-step fashion using sequential apprehension to extend his original construction. On one occasion he constructed two perpendicular lines and asked the learners to complete the shape. Mr. Jones used the *Geobaord* dynamically relatively seldomly. On one occasion he asked the learners to transform a square with five pins on each side to one with seven pins on each side. On another, he asked the learners to change the size of a square without dismantling the original shape. Mr. Jones often encouraged learners to discuss the properties of the quadrilateral he constructed on the *Geoboard.* He mostly encouraged discursive apprehensions by asking strategic questions. Learners were given tasks that required them to construct lines of symmetry, diagonals and missing line segments of shapes, as can be seen in Fig. 1.3. There was thus medium to strong evidence of using operative apprehensions in Mr. Jones's lessons.

Ms. Ruth's Lessons

Ms. Ruth's lessons showed strong evidence of using the *Geoboard* in the context of both perceptual and operative apprehension. She however did not make use of the *Geoboard* dynamically – see Fig. 1.4. On one occasion she turned the *Geoboard* on its side, however, to demonstrate a different orientation of a particular shape. In her second lesson, she used the shapes on the *Geoboard* to engage the learners in articulating their observations about the similarities and differences of the shapes, thus

Fig. 1.3 Learners engaging in operative apprehension when doing activities on the *Geoboard*

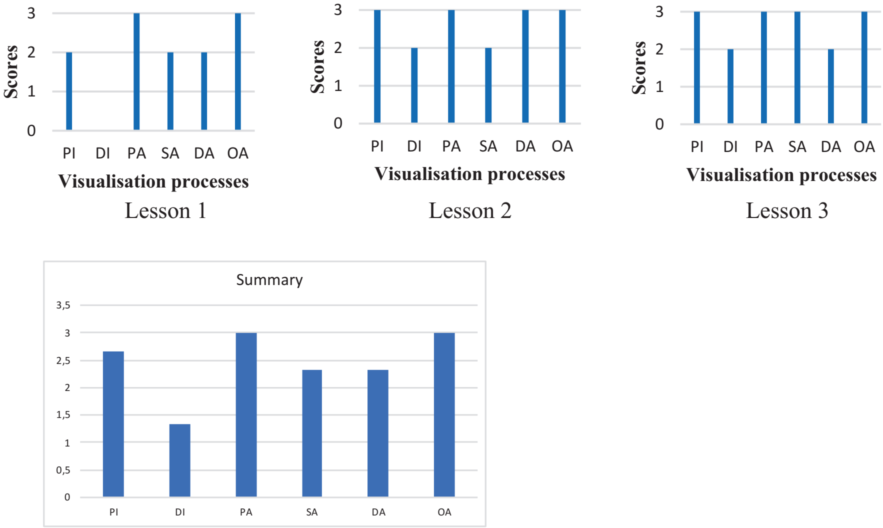

Fig. 1.4 Bar graphs showing evidence of the visual imagery and apprehensions used in Ms. Ruth's three lessons

Fig. 1.5 The use of dot papers to supplement the *Geoboard*

eliciting from them the essential properties of these shapes. It was interesting to note that Ms. Ruth also made use of dot papers (dots arranged in a similar square lattice as the nails on the *Geoboard*) to supplement the *Geoboard* activities she gave to the learners – see Fig. 1.5.

Ms. Smith's Lessons

Ms. Smith was very effective in exploiting the pictorial imaging capacity of the *Geoboard* – see Fig. 1.6. The images that she produced were very striking and evoked strong perceptual apprehensions in the learners – see Fig. 1.7.

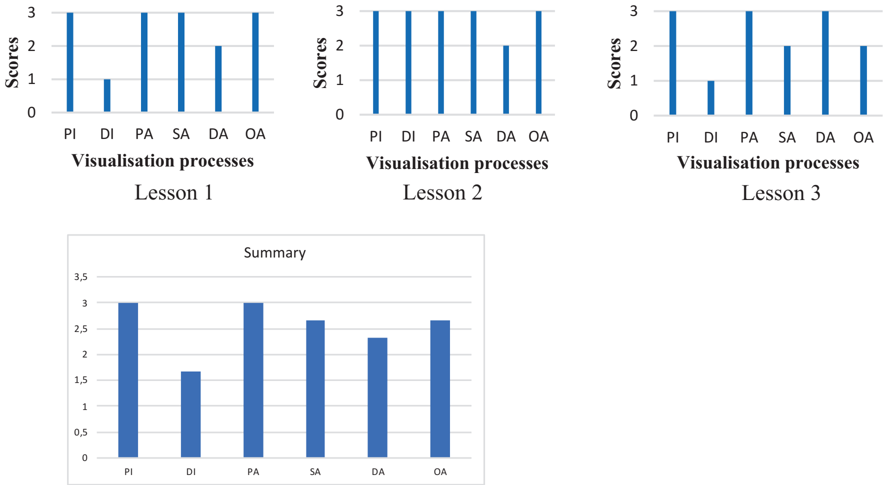

Fig. 1.6 Bar graphs showing evidence of the visual imagery and apprehensions used in Ms. Smith's three lessons

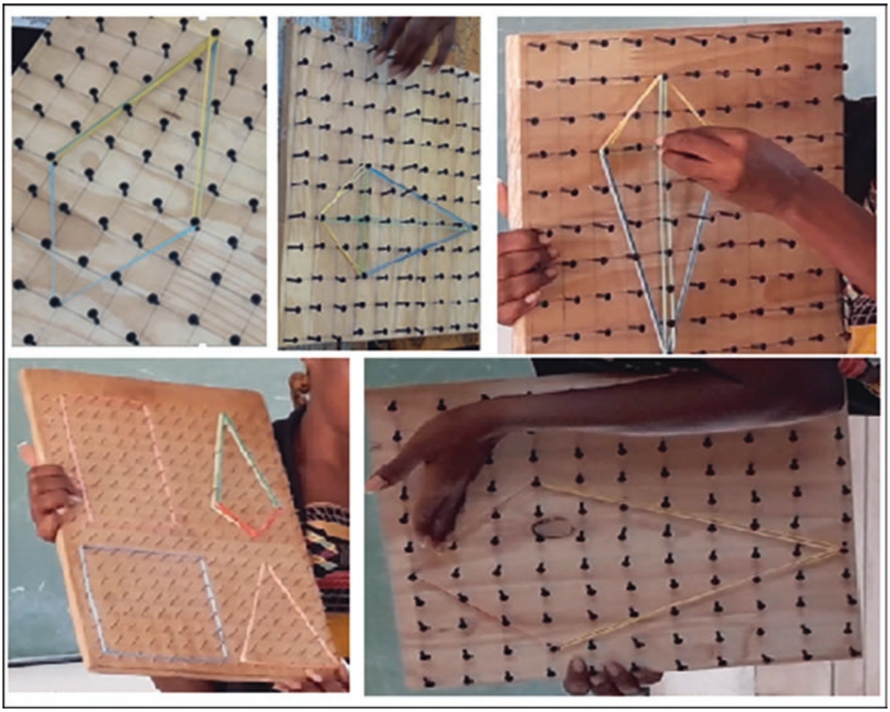

Fig. 1.7 Shapes with strong visual appeal on Ms. Smith's *Geoboard*

Interviews

In the focus group interview, the teachers were generally very positive about their experiences of teaching with a *Geoboard.* They found it a powerful manipulative to visually represent quadrilaterals and other geometric shapes. Matengu (2018) wrote that they alluded to the dynamism that a *Geoboard* affords, particularly in "forming and transforming shapes" (p.110). Ms. Smith particularly commented on the flexibility of the *Geoboard*. She often turned and re-orientated her board.

The teachers also alluded to the practicality of the *Geobaord.* Mr. Jones, for example, emphasised that a *Geoboard* does not need cleaning – it is just a matter of replacing and changing the elastic bands to create a new shape. The teachers also acknowledged that the construction and manipulation of quadrilaterals on the *Geoboard* was much easier than on a chalkboard.

All the teachers observed that the *Geoboard* brought "excitement and increased participation during teaching" (Matengu, 2018, p. 110). Interestingly they also suggested that the *Geoboard* helped cater for diverse learners. They said that the manipulative was user-friendly and thus encouraged interaction amongst learners.

There were also limitations to using the *Geoboard* that were identified by the teachers. Ms. Ruth, for example, found that her *Geoboard* should have been bigger in size for all her learners to be able to observe her constructions. In relation to this, it is important that all the learners have access to a *Geoboard.* As much as the teachers used the *Geoboard* as a demonstration tool, it should be used as an exploration tool by the learners. The teachers also noted that not all geometric shapes are constructible on a *Geoboard.* It is thus important that teachers plan carefully and strategically. It was noted by Mr. Jones, for example, that when exploring lines of symmetry, the sides of a selected quadrilateral should consist of an odd number of pins in order to easily find a mid-point.

Conclusion

The above case study illustrated how the use of a simple, easy-to-make manipulative enriched a series of lessons that focused on the properties of quadrilaterals. The manipulative enabled the participating teachers quickly and effectively to illustrate and facilitate visual explorations into the properties of quadrilaterals, thus adopting a visual approach to teaching mathematics. The above case is of particular significance in the context of a rural school environment where reliable and quality access to digital technologies, such as the Internet and computer software, is still very limited. It is thus incumbent on mathematics teachers in these environments to look for alternative means to either source or make manipulatives that can be used as tools for a visual pedagogy. This case study showed that the *Geoboard* is a highly appropriate and effective manipulative that enables teachers and learners to make mathematics just a little more visible.

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