Chapter 4
Cayley-Klein Spaces

In Klein's *Erlangen program* Euclidean and non-Euclidean geometries are considered as subgeometries of projective geometry. Projective models for, e.g., hyperbolic, deSitter, and elliptic space can be obtained by using a quadric to induce the corresponding metric [Kle1928]. In this section we introduce the corresponding general notion of *Cayley-Klein spaces* and their groups of *isometries*, see, e.g., [Kle1928, Bla1954, Gie1982]. We put a particular emphasis on the description of hyperplanes, hyperspheres, and their mutual relations.

4.1 Cayley-Klein Distance

A quadric within a projective space induces an invariant for pairs of points.

Definition 4.1 Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric with corresponding bilinear form $\langle \cdot, \cdot \rangle$. Then we denote by

$$K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) := \frac{\langle x, y \rangle^2}{\langle x, x \rangle \langle y, y \rangle}$$

the *Cayley-Klein distance* of any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ that are not on the quadric. We further set $K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \infty$, if $\langle x, x \rangle \langle y, y \rangle = 0$ and $\langle x, y \rangle \neq 0$. In the presence of a Cayley-Klein distance the quadric $\mathcal{Q}$ is called the *absolute quadric*.

Remark 4.1 The name Cayley-Klein distance, or Cayley-Klein metric, is usually assigned to an actual metric derived from the above quantity as, for example, the hyperbolic metric (cf. Sect. 4.4). Nevertheless, we prefer to assign it to this basic quantity associated with an arbitrary quadric.

The Cayley-Klein distance is projectively well-defined, in the sense that it depends neither on the choice of the bilinear form corresponding to the quadric $\mathcal{Q}$ nor on the

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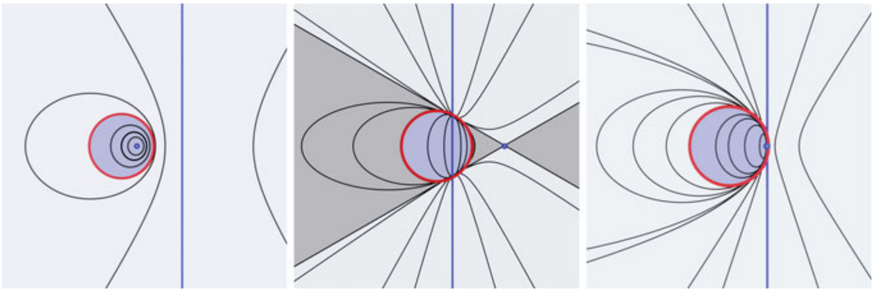

Fig. 4.1 Concentric Cayley-Klein circles in the hyperbolic/deSitter plane. *Left*: Concentric circles with hyperbolic center. *Middle*: Concentric circles with deSitter center. *Right*: Concentric horocycles with center on the absolute conic

choice of homogeneous coordinate vectors for the points $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, it is invariant under the group of projective transformations that preserve the quadric $\mathcal{Q}$, which we call the corresponding group of *isometries*.

The Cayley-Klein distance can be positive or negative depending on the relative location of the two points with respect to the quadric, cf. (3.2).

Proposition 4.1 *For two points* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{RP}^n \setminus \mathcal{Q}$ *with* $\langle x, y \rangle \neq 0$*:*

- $K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) > 0$ *if* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are on the same side of* $\mathcal{Q}$*,*
- $K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) < 0$ *if* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are on opposite sides of* $\mathcal{Q}$*.*

A *Cayley-Klein space* is usually considered to be one side of the quadric, i.e. $\mathcal{Q}^+$ or $\mathcal{Q}^-$, together with a (pseudo-)metric derived from the Cayley-Klein distance, or equivalently, together with the transformation group of isometries.

4.2 Cayley-Klein Spheres

Having a notion of "distance" allows for the definition of corresponding spheres (see Fig. 4.1).

Definition 4.2 Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric, $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$, and $\mu \in \mathbb{R} \cup \{\infty\}$. Then we call the quadric

$$S_\mu(\boldsymbol{x}) := \left\{ \boldsymbol{y} \in \mathbb{RP}^n \;\middle|\; K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \mu \right\}$$

the *Cayley-Klein hypersphere* with *center* $\boldsymbol{x}$ and *Cayley-Klein radius* μ with respect to the absolute quadric $\mathcal{Q}$.

Remark 4.2

(i) Due to the fact that the Cayley-Klein sphere equation can be written as

$$\langle x, y \rangle^2 - \mu \langle x, x \rangle \langle y, y \rangle = 0, \tag{4.1}$$

we may include into the set $S_\mu(\boldsymbol{x})$ points $\boldsymbol{y} \in \mathcal{Q}$ on the absolute quadric.

(ii) Given the center $\boldsymbol{x}$ of a Cayley-Klein sphere one can further rewrite the Cayley-Klein sphere equation (4.1) as

$$\langle x, y \rangle^2 - \tilde{\mu} \langle y, y \rangle = 0, \tag{4.2}$$

where $\tilde{\mu} := \mu \langle x, x \rangle$. While $\tilde{\mu}$ is not projectively invariant anymore, the solution set of this equation still invariantly describes a Cayley-Klein sphere. We may now allow for centers on the absolute quadric $\boldsymbol{x} \in \mathcal{Q}$ which gives rise to *Cayley-Klein horospheres*, i.e., quadrics given by (4.2) for any fixed $\tilde{\mu} \in \mathbb{R}$. (see Fig. 4.1, right).

Proposition 4.2 *For a Cayley-Klein sphere with center $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ and Cayley-Klein radius $\mu \in \mathbb{R} \cup \{\infty\}$ one has:*

- *If $\mu < 0$ the center and the points of a Cayley-Klein sphere are on* ***opposite sides*** *of the quadric.*
- *If $\mu > 0$ the center and the points of a Cayley-Klein sphere are on the* ***same side*** *of the quadric.*
- *If $\mu = 0$ the Cayley-Klein sphere is given by the (doubly counted)* ***polar hyperplane*** *$\boldsymbol{x}^\perp$.*
- *If $\mu = 1$ the Cayley-Klein sphere is the* ***cone of contact*** *$C_\mathcal{Q}(\boldsymbol{x})$ touching $\mathcal{Q}$, which is also called the* ***null-sphere*** *with center $\boldsymbol{x}$.*
- *If $\mu = \infty$ the Cayley-Klein sphere is the* ***absolute quadric*** *$\mathcal{Q}$.*

Proof Follows from Proposition 4.1 and Lemma 3.3. □

Fixing the center and varying the radius of a Cayley-Klein sphere results in a family of concentric spheres (see Fig. 4.1).

Definition 4.3 Given an absolute quadric $\mathcal{Q} \subset \mathbb{RP}^n$ and a point $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ we call the family

$$\left(S_\mu(\boldsymbol{x})\right)_{\mu \in \mathbb{R} \cup \{\infty\}}$$

concentric Cayley-Klein spheres with center $\boldsymbol{x}$.

Proposition 4.3 *Let $\mathcal{Q} \subset \mathbb{RP}^n$ be the absolute quadric. Then the family of concentric Cayley-Klein spheres with center $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ is the pencil of quadrics $\mathcal{Q} \wedge C_\mathcal{Q}(\boldsymbol{x})$ spanned by the absolute quadric $\mathcal{Q}$ and the cone of contact $C_\mathcal{Q}(\boldsymbol{x})$, or equivalently, by $\mathcal{Q}$ and the (doubly counted) polar hyperplane $\boldsymbol{x}^\perp$ (cf. Example 3.2).*

Proof Writing the Cayley-Klein sphere equation as (4.1) we find that it is a linear equation in μ describing a pencil of quadrics. As observed in Proposition 4.2 it contains, in particular, the quadric $\mathcal{Q}$, the cone $C_\mathcal{Q}(\boldsymbol{x})$, and the hyperplane $\boldsymbol{x}^\perp$. □

This leads to a further characterization of Cayley-Klein spheres among all quadrics.

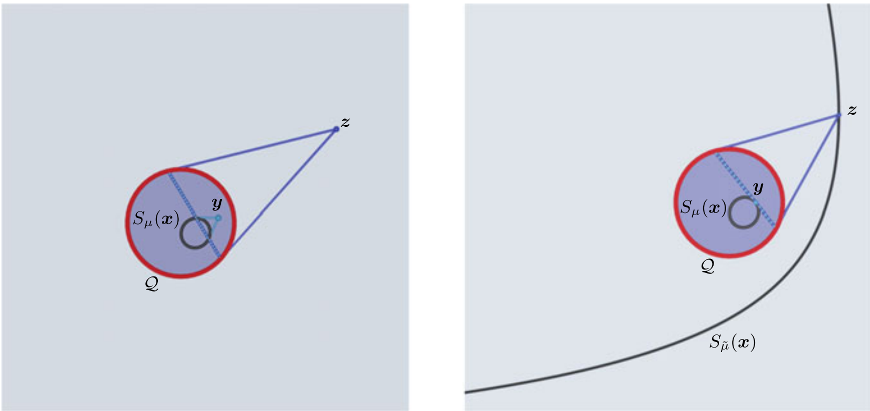

Fig. 4.2 *Left*: Polarity with respect to a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ and the absolute quadric $\mathcal{Q}$. *Right*: A Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ and its (concentric) polar Cayley-Klein sphere $S_{\tilde{\mu}}(\boldsymbol{x})$

Corollary 4.1 *Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a non-degenerate absolute quadric. Then a second quadric is a Cayley-Klein sphere if and only if it is tangent to $\mathcal{Q}$ in the (possibly imaginary) intersection with a hyperplane (not tangent to $\mathcal{Q}$).*

Proof Follows from Proposition 4.3 and Example 3.2. Indeed, by Proposition 4.3, concentric Cayley-Klein spheres are exactly the pencils of the form $\mathcal{Q} \wedge C_{\mathcal{Q}}(\boldsymbol{x})$. On the other hand, Example 3.2 states that a pencil of the form $\mathcal{Q} \wedge C_{\mathcal{Q}}(\boldsymbol{x})$ consists exactly of the quadrics tangent in the intersection with the polar hyperplane of $\boldsymbol{x}$. □

Remark 4.3 A pencil of *concentric Cayley-Klein horospheres* with center $\boldsymbol{x} \in \mathcal{Q}$ is spanned by the absolute quadric $\mathcal{Q}$ and the (doubly counted) tangent hyperplane $\boldsymbol{x}^\perp$, which yields third order contact between each horosphere and the absolute quadric.

4.3 Polarity of Cayley-Klein Spheres

To describe spheres in terms of their tangent planes we turn our attention towards polarity in Cayley-Klein spheres. A quadratic form of a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ with center $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ and Cayley-Klein radius $\mu \in \mathbb{R}$ is given by (4.1):

$$\Delta(y) := \langle x, y\rangle^2 - \mu \langle x, x\rangle \langle y, y\rangle .$$

The corresponding symmetric bilinear form is obtained by the polarization identity:

$$b(y, \tilde{y}) = \frac{1}{2}(\Delta(y + \tilde{y}) - \Delta(y) - \Delta(\tilde{y})) = \langle x, y\rangle \langle x, \tilde{y}\rangle - \mu \langle x, x\rangle \langle y, \tilde{y}\rangle ,$$

for $y, \tilde{y} \in \mathbb{R}^{n+1}$.

Lemma 4.1 *Let $\boldsymbol{Y}$ be the polar hyperplane of a point $\boldsymbol{y} \in \mathbb{RP}^n$ with respect to $S_\mu(\boldsymbol{x})$. Then the pole $\boldsymbol{z}$ of $\boldsymbol{Y}$ with respect to the absolute quadric $\mathcal{Q}$ is given by (see Fig. 4.2, left)*

$$z = \langle x, y \rangle\, x - \mu \,\langle x, x \rangle\, y. \tag{4.3}$$

Proof The polar hyperplane of $\boldsymbol{y}$ with respect to $S_\mu(\boldsymbol{x})$ is given by

$$\boldsymbol{Y} = \left\{ \tilde{\boldsymbol{y}} \in \mathbb{RP}^n \;\middle|\; b(y, \tilde{y}) = \langle z, \tilde{y} \rangle = 0 \right\}.$$

□

For every point on a Cayley-Klein sphere the tangent hyperplane in that point is given by polarity in the Cayley-Klein sphere. Now the tangent hyperplanes of a Cayley-Klein sphere, in turn, may equivalently be described by their poles with respect to the absolute quadric $\mathcal{Q}$.

Proposition 4.4 *Let $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ and $\mu \in \mathbb{R} \setminus \{0, 1\}$. Then the poles (with respect to the absolute quadric $\mathcal{Q}$) of the tangent hyperplanes of the Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ are the points of a concentric Cayley-Klein sphere $S_{\tilde{\mu}}(\boldsymbol{x})$ with*

$$\mu + \tilde{\mu} = 1,$$

and vice versa.

Proof Let $\boldsymbol{y} \in S_\mu(\boldsymbol{x})$ be a point on the Cayley-Klein sphere. Then the tangent plane to $S_\mu(\boldsymbol{x})$ at the point $\boldsymbol{y}$ is the polar plane of $\boldsymbol{y}$ with respect to $S_\mu(\boldsymbol{x})$. According to Lemma 4.1 the pole $\boldsymbol{z}$ of that tangent hyperplane with respect to $\mathcal{Q}$ is given by (4.3). Computing the Cayley-Klein distance of this point to the center $\boldsymbol{x}$ we obtain

$$K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{z}) = \frac{\langle x, z \rangle^2}{\langle x, x \rangle \langle z, z \rangle} = \frac{\langle x, y \rangle^2 (1-\mu)^2}{\langle x, y \rangle^2 (1 - 2\mu) + \mu^2 \langle x, x \rangle \langle y, y \rangle} = 1 - \mu,$$

where we used $\langle x, y \rangle^2 = \mu \langle x, x \rangle \langle y, y \rangle$. □

Definition 4.4 For a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ we call the Cayley-Klein sphere $S_{1-\mu}(\boldsymbol{x})$, consisting of all poles (with respect to the absolute quadric $\mathcal{Q}$) of tangent hyperplanes of $S_\mu(\boldsymbol{x})$, its *polar Cayley-Klein sphere* (see Fig. 4.2, right).

Remark 4.4 The two degenerate Cayley-Klein spheres $\boldsymbol{x}^\perp$ and $C_{\mathcal{Q}}(\boldsymbol{x})$ corresponding to the values $\mu = 0$ and $\mu = 1$ respectively, may be treated as being mutually polar. Then polarity defines a projective involution on a pencil of concentric Cayley-Klein spheres with fixed points at $\mu = \frac{1}{2}$ and $\mu = \infty$.

4.4 Hyperbolic Geometry

Let $\langle\cdot,\cdot\rangle$ be the standard non-degenerate bilinear form of signature $(n, 1)$, i.e.

$$\langle x, y\rangle := x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

for $x, y \in \mathbb{R}^{n+1}$, and denote by $\mathcal{S} \subset \mathbb{R}\mathrm{P}^n$ the corresponding quadric. We identify the "inside" of $\mathcal{S}$, cf. (3.2), with the n-dimensional *hyperbolic space*

$$\mathcal{H} := \mathcal{S}^-.$$

For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ one has $K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) \geq 1$, and the quantity $d(\boldsymbol{x}, \boldsymbol{y})$ given by

$$K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 d(\boldsymbol{x}, \boldsymbol{y})$$

defines a metric on $\mathcal{H}$ of constant negative sectional curvature. The corresponding group of isometries is given by $\mathrm{PO}(n, 1)$ and called the group of *hyperbolic motions*. The absolute quadric $\mathcal{S}$ consists of the points at (metric) infinity, i.e., at infinite distance from any given point in $\mathcal{S}^-$. We call the union

$$\overline{\mathcal{H}} := \mathcal{H} \cup \mathcal{S}$$

the *compactified hyperbolic space*.

In this *projective model* of hyperbolic geometry *geodesics* are given by intersections of projective lines in $\mathbb{R}\mathrm{P}^n$ with $\mathcal{H}$, while, more generally, *hyperbolic subspaces* (totally geodesic submanifolds) are given by intersections of projective subspaces in $\mathbb{R}\mathrm{P}^n$ with $\mathcal{H}$. Thus, by polarity, every point $\boldsymbol{m} \in \mathrm{dS}$ in the "outside" of hyperbolic space,

$$\mathrm{dS} := \mathcal{S}^+,$$

which is called *deSitter space*, corresponds to a hyperbolic hyperplane $\boldsymbol{m}^\perp \cap \mathcal{H}$.

Consider two hyperbolic hyperplanes with poles $\boldsymbol{k}, \boldsymbol{m} \in \mathrm{dS}$.

- If $K_{\mathcal{S}}(\boldsymbol{k}, \boldsymbol{m}) < 1$, the two hyperplanes intersect in $\mathcal{H}$, and their hyperbolic intersection angle $\alpha(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp)$, or equivalently its conjugate angle $\pi - \alpha$ is given by

$$K_{\mathcal{S}}(\boldsymbol{k}, \boldsymbol{m}) = \cos^2 \alpha(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp).$$

- If $K_{\mathcal{S}}(\boldsymbol{k}, \boldsymbol{m}) > 1$, the two hyperplanes do not intersect in $\mathcal{H}$, and their hyperbolic distance $d(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp)$ is given by

$$K_{\mathcal{S}}(\boldsymbol{k}, \boldsymbol{m}) = \cosh^2 d(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp).$$

  The corresponding projective hyperplanes intersect in $(\boldsymbol{k} \wedge \boldsymbol{m})^\perp \subset \mathrm{dS}$.

- If $K_{\mathcal{S}}(\boldsymbol{k}, \boldsymbol{m}) = 1$, the two hyperplanes are parallel, i.e., they do not intersect in $\mathcal{H}$ but do intersect in $\overline{\mathcal{H}}$.

Finally, the hyperbolic distance $d(\boldsymbol{x}, \boldsymbol{m}^{\perp})$ of a point $\boldsymbol{x} \in \mathcal{H}$ and a hyperbolic hyperplane with pole $\boldsymbol{m} \in \mathrm{dS}$ is given by

$$K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{m}) = -\sinh^2 d(\boldsymbol{x}, \boldsymbol{m}^{\perp}).$$

It is occasionally useful to employ a certain normalization of the homogeneous coordinate vectors:

$$\begin{aligned} \mathbb{H}^n &:= \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1} \;\middle|\; \langle x, x \rangle = -1,\ x_{n+1} \geq 0 \right\}, \\ \mathrm{d}\mathbb{S}^n &:= \left\{ m = (m_1, \ldots, m_{n+1}) \in \mathbb{R}^{n,1} \;\middle|\; \langle m, m \rangle = 1 \right\}. \end{aligned}$$

Then $\mathrm{P}(\mathbb{H}^n) = \mathcal{H}$ is an embedding and $\mathrm{P}(\mathrm{d}\mathbb{S}^n) = \mathrm{dS}$ is a double cover. For $x, y \in \mathbb{H}^n$ and $k, m \in \mathrm{d}\mathbb{S}^n$ above distance formulas become

$$\begin{aligned} \langle x, y \rangle &= -\cosh d(\boldsymbol{x}, \boldsymbol{y}), \\ |\langle k, m \rangle| &= \cos \alpha(\boldsymbol{k}^{\perp}, \boldsymbol{m}^{\perp}), \quad \text{if } |\langle k, m \rangle| \leq 1, \\ |\langle k, m \rangle| &= \cosh d(\boldsymbol{k}^{\perp}, \boldsymbol{m}^{\perp}), \quad \text{if } |\langle k, m \rangle| \geq 1, \\ |\langle x, m \rangle| &= \sinh d(\boldsymbol{x}, \boldsymbol{m}^{\perp}). \end{aligned}$$

Remark 4.5 The double cover $\mathrm{P}(\mathrm{d}\mathbb{S}^n) = \mathrm{dS}$ of deSitter space can be used to encode the orientation of the corresponding polar hyperplanes, e.g., by endowing the hyperbolic hyperplane corresponding to $m \in \mathrm{d}\mathbb{S}^n$ with a normal vector in the direction of the hyperbolic halfspace on which the bilinear form with points $x \in \mathbb{H}^n$ is positive: $\langle x, m \rangle > 0$. Using the double cover to encode orientation one may omit the absolute value in $\langle x, m \rangle = \cos d$ to obtain an oriented hyperbolic distance d between a point and an hyperbolic hyperplane. Similarly, one may omit the absolute value in $\langle k, m \rangle = \cos \alpha$ which allows to distinguish the intersection angle α and its conjugate angle $\pi - \alpha$.

We now turn our attention to the Cayley-Klein spheres of hyperbolic/deSitter geometry. First, consider a pencil of concentric Cayley-Klein spheres $S_{\mu}(\boldsymbol{x})$ with center inside hyperbolic space $\boldsymbol{x} \in \mathcal{H}$, $x \in \mathbb{H}^n$. Depending on the value of $\mu \in \mathbb{R} \cup \{\infty\}$ we obtain the following types of hyperbolic/deSitter spheres (see Fig. 4.1, left):

- $\mu < 0$: A *deSitter sphere* with hyperbolic center.
- $0 < \mu < 1$: $S_{\mu}(\boldsymbol{x})$ is empty.
- $1 < \mu < \infty$: A *hyperbolic sphere* with center $\boldsymbol{x} \in \mathcal{H}$ and hyperbolic radius $r = \operatorname{arcosh} \sqrt{\mu} > 0$:

$$S_{\mu}(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathcal{H} \;\middle|\; K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 r \right\} = \mathrm{P}\left(\left\{ y \in \mathbb{H}^n \;\middle|\; \langle x, y \rangle = -\cosh r \right\} \right).$$

Second, consider a pencil of concentric Cayley-Klein spheres $S_\mu(\boldsymbol{m})$ with center outside hyperbolic space $\boldsymbol{m} \in \mathrm{dS}$, $m \in \mathrm{d}\mathbb{S}^n$ (see Fig. 4.1, middle):

- $\mu < 0$: A *hypersurface of constant hyperbolic distance* $r = \operatorname{arsinh} \sqrt{\mu} > 0$ to the hyperbolic plane $\boldsymbol{m}^\perp \cap \mathcal{H}$:

$$S_\mu(\boldsymbol{m}) = \left\{ \boldsymbol{y} \in \mathcal{H} \,\middle|\, K_\mathcal{S}(\boldsymbol{m}, \boldsymbol{y}) = -\sinh^2 r \right\} = \mathrm{P}\left(\left\{ y \in \mathbb{H}^n \,\middle|\, |\langle m, y \rangle| = \sinh r \right\} \right).$$

- $0 < \mu < 1$: A *deSitter sphere* tangent to $\mathcal{S}$. All its tangent hyperplanes are hyperbolic hyperplanes.
- $1 < \mu < \infty$: A *deSitter sphere* tangent to $\mathcal{S}$ with no hyperbolic tangent hyperplanes.

Third, a pencil of concentric Cayley-Klein horospheres with center on the absolute quadric $\boldsymbol{x} \in \mathcal{S}$, $x \in \mathbb{L}^{n,1}$ consists of *hyperbolic horospheres* and *deSitter horospheres* (see Fig. 4.1, right).

4.5 Elliptic Geometry

For $x, y \in \mathbb{R}^{n+1}$ we denote by

$$x \cdot y := x_1 y_1 + \ldots x_n y_n + x_{n+1} y_{n+1}$$

the standard (positive definite) scalar product on $\mathbb{R}^{n+1}$, i.e. the standard non-degenerate bilinear form of signature $(n+1, 0)$. The corresponding quadric $\mathcal{O} \subset \mathbb{RP}^n$ is empty (or "purely imaginary", cf. Example 3.1 (i)), as well as the set $\mathcal{O}^- = \varnothing$, while

$$\mathcal{E} := \mathcal{O}^+ = \mathbb{RP}^n$$

is the whole projective space, which we identify with the n-dimensional *elliptic space*. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$ one always has $0 \leq K_\mathcal{O}(\boldsymbol{x}, \boldsymbol{y}) \leq 1$ and the quantity $d(\boldsymbol{x}, \boldsymbol{y})$ given by

$$K_\mathcal{O}(\boldsymbol{x}, \boldsymbol{y}) = \cos^2 d(\boldsymbol{x}, \boldsymbol{y})$$

defines a metric on $\mathcal{E}$ of constant positive sectional curvature. The corresponding group of isometries is given by $\mathrm{PO}(n+1)$ and called the group of *elliptic motions*.

In this *projective model* of elliptic geometry *geodesics* are given by projective lines, while, more generally, *elliptic subspaces* are given by projective subspaces. By polarity, there is a one-to-one correspondence of points $\boldsymbol{x} \in \mathcal{E}$ in elliptic space and elliptic hyperplanes $\boldsymbol{x}^\perp$.

Two hyperplanes in elliptic space always intersect. If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$ are the poles of two elliptic hyperplanes, then their intersection angle $\alpha(\boldsymbol{x}^\perp, \boldsymbol{y}^\perp)$, or equivalently its conjugate angle $\pi - \alpha$ is given by

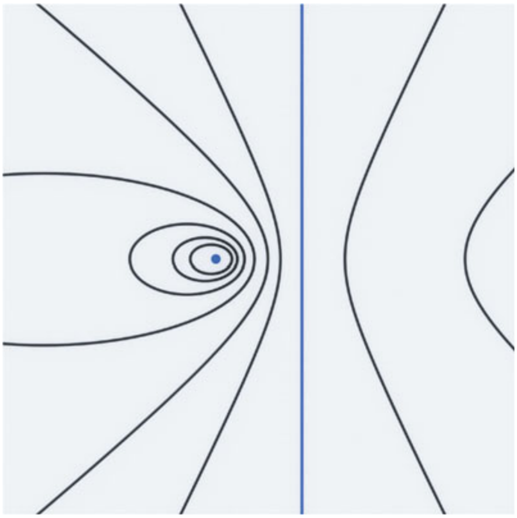

Fig. 4.3 Concentric Cayley-Klein circles in the elliptic plane

$$K_{\mathcal{O}}(\boldsymbol{x}, \boldsymbol{y}) = \cos^2 \alpha(\boldsymbol{x}^\perp, \boldsymbol{y}^\perp).$$

The distance $d(\boldsymbol{x}, \boldsymbol{y}^\perp)$ of a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n$ and an elliptic hyperplane with pole $\boldsymbol{y} \in \mathbb{R}\mathrm{P}^n$ is given by

$$K_{\mathcal{O}}(\boldsymbol{x}, \boldsymbol{y}) = \sin^2 d(\boldsymbol{x}, \boldsymbol{y}^\perp).$$

One may normalize the homogeneous coordinate vectors of points in elliptic space to lie on a sphere:

$$\mathbb{S}^n := \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x \cdot x = 1 \right\}.$$

Then $\mathrm{P}(\mathbb{S}^n) = \mathcal{E}$ is a double cover, where antipodal points of the sphere are identified. In this normalization elliptic planes correspond to great spheres of $\mathbb{S}^n$, and it turns out that elliptic geometry is a double cover of *spherical geometry*. For $x, y \in \mathbb{S}^n$ above distance formulas become

$$|x \cdot y| = \cos d(\boldsymbol{x}, \boldsymbol{y}),$$
$$|x \cdot y| = \cos \alpha(\boldsymbol{x}^\perp, \boldsymbol{y}^\perp),$$
$$|x \cdot y| = \sin d(\boldsymbol{x}, \boldsymbol{y}^\perp),$$

Remark 4.6 The pole $\boldsymbol{x} \in \mathcal{E}$ of an elliptic hyperplane $\boldsymbol{x}^\perp$ has two lifts to the sphere, $x, -x \in \mathbb{S}^n$, which may be used to encode the orientation of the hyperplane (cf. Remark 4.5). This allows for omitting the absolute values in above distance formulas, while taking distances to be signed and distinguishing between intersection angles and their conjugate angles.

A Cayley-Klein sphere in elliptic space $S_\mu(\boldsymbol{x})$ with center $\boldsymbol{x} \in \mathcal{E}$, $x \in \mathbb{S}^n$, is not empty if and only if $0 \leq \mu \leq 1$ (see Fig. 4.3). In this case it corresponds to an *elliptic sphere* with center $\boldsymbol{x} \in \mathcal{E}$ and elliptic radius $0 \leq r = \arccos \sqrt{\mu} \leq \frac{\pi}{2}$:

$$S_\mu(\boldsymbol{x}) = \left\{\boldsymbol{y} \in \mathcal{E} \;\middle|\; K_{\mathcal{O}}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \cos^2 r\right\} = \mathrm{P}\left(\left\{y \in \mathbb{S}^n \;\middle|\; |x \cdot y| = \cos r\right\}\right).$$