Nonlinear Modal Analysis Through the Generalization of the Eigenvalue Problem: Applications for Dissipative Dynamics

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1 Introduction

The concept of nonlinear modal analysis (NMA) has become a very popular analytical as well as computational tool in the study of nonlinear structural systems. There have been several formulations of a computational procedure for this [1, 2], including time-domain [3] and frequency-domain methods [4]. Quasi-static approaches have also gained popularity in recent years, and this chapter considers such a formulation that is based on earlier work in [5].

Recent efforts by the authors [5] have indicated that a generalization of the constrained minimization of Rayleigh quotients provides an interesting nonlinear modal analysis approach (termed Rayleigh quotient-based NMA (RQNMA)) closely related to some of the other methods above. This chapter considers improvements to this formulation in three areas, namely: (1) modal analysis under dynamical operation; (2) forced response synthesis; and (3) investigation of modal coupling. A 1-dimensional (1D) finite element model of a bar with a frictional end will be used for numerical demonstration.

The rest of this chapter is organized as follows: Sect. 2 provides an overview of the methodologies including RQNMA and the chosen example; Sect. 3 presents results for nonlinear modal analysis, forced response synthesis, and an investigation of a case with modal coupling; and finally, Sect. 4 draws conclusions from the study.

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2 Methodology

The current section provides an overview of the computational procedures involved in RQNMA and a description of the numerical benchmark that will be used for demonstrations in the rest of this chapter.

2.1 *Rayleigh Quotient-Based Nonlinear Modal Analysis*

Consider a (discrete, multiple degree-of-freedom) nonlinear dynamic system governed by the equations of motion,

$$\begin{aligned} \mathbf{M}\ddot{\bar{u}} + \mathbf{C}\dot{\bar{u}} + \mathbf{K}\bar{u} + \bar{f}_{nl}(\bar{u}, \dot{\bar{u}}, \dots) &= \bar{f}_{ex}(t) \\ \mathbf{M}, \mathbf{C}, \mathbf{K} &\in \mathbb{R}^{n\times n} \\ t \in \mathbb{R}_+;\ \bar{u}, \bar{f}_{ex} : \mathbb{R}_+ \to \mathbb{R}^n;\ \bar{f}_{nl} : \mathbb{R}^n \times \dots &\to \mathbb{R}^n. \end{aligned} \tag{1}$$

Here, n denotes the number of degrees of freedom (DoFs) in the system, and the solution, excitation, and nonlinearity are functions denoted above. The mass matrix ($\mathbf{M}$) is taken to be symmetric positive definite; the stiffness matrix ($\mathbf{K}$), symmetric positive semi-definite, and the damping matrix ($\mathbf{C}$), symmetric.

The definition for the nonlinear modes of this system used in [5] was the extremizing eigenpair of the corresponding eigenvector-dependent nonlinear eigenproblem (NEPv). Mathematically, this can be expressed as an algebraic system representing the first-order optimality conditions of the corresponding Rayleigh quotient extremization problem (constrained potential energy minimization),[1]

$$\begin{aligned} \mathbf{K}\bar{u} + \bar{f}_{nl} - \lambda\mathbf{M}\bar{u} &= 0 \\ \bar{u}^T\mathbf{K}\bar{u} - \bar{u}^T\bar{f}_{nl} - \lambda q^2 &= 0. \end{aligned} \tag{2}$$

In the above algebraic system of equations, parameterized by the modal amplitude q, the multiplier λ was interpreted as the square of the natural frequency of the nonlinear mode under consideration. This interpretation presents certain difficulties in the applicability of the procedure for problems with even nonlinearities (nonlinear forces unsymmetric about the equilibrium). Furthermore, the relationship to dynamic operation is not very clear, introducing difficulties in the incorporation of nonlinear terms that are fundamentally dynamic such as rate-dependent friction models, etc. Although some preliminary efforts were undertaken in [5, 6], these aspects need to be investigated further.

[1] Equation (2) is a restatement of the optimality conditions in [5], with the second equation modified to a form more suitable for numerical implementations.

This chapter explores an alternative interpretation of the multipliers λ, assessing its ramifications using a numerical example. Upon solving Eq. (2) for a range of modal amplitudes $q \in [-Q, Q]$, one obtains parameteric relationships $\bar{u}(q)$ and $\lambda(q)$, denoting the deflection shape and multiplier (Rayleigh quotient), respectively. Note that hysteretic nonlinearities, at this stage, are only evaluated *quasi-statically* along the *modal backbone*.[2] These are interpreted to describe the terms in the equations of motion of the system at the modal level as

$$\ddot{q}(t) + \lambda(q(t))q(t) = 0. \tag{3}$$

This system, by construction, is conservative and does not possess any dissipative characteristics, eventhough the original system may be dissipative. The interpretation here is that this represents the conservative *part* of the complete system. Since $\lambda(q)$ is known from the above, this equation may be solved using any NMA technique applicable to conservative systems. For example, using a single (cosine-) harmonic expansion, an estimate of the modal natural frequency may be made. Assuming $q(t) = Q\cos(\omega_n(Q)t)$ (with natural frequency ($\omega_n(Q)$ depending on the harmonic amplitude of the solution), one obtains

$$\omega_n(Q) = \sqrt{\frac{\mathcal{F}^{(c)}\{\lambda(q(t))q(t)\}}{Q}}, \tag{4}$$

where $\mathcal{F}^{(c)}(.)$ denotes the discrete Fourier cosine transform. $\omega_n(Q)$ represents the *effective* natural frequency of an oscillation of the system that extremizes the nonlinear Rayleigh quotient *at each instant of oscillation*.

The effective mode shape of the nonlinear mode is taken to be the gradient of the solution $\bar{u}(q(t))$ with respect to q averaged over a cycle. This can be computed as the zeroth harmonic (denoted by $\mathcal{F}^{(0)}(.)$) of the gradient,

$$\bar{\phi}(Q) = \mathcal{F}^{(0)}\left\{\frac{\partial \bar{u}}{\partial q}(Q\cos(\omega_n t))\right\}. \tag{5}$$

Here, $\omega_n(Q)$ and $\bar{\phi}(Q)$ denote the undamped natural frequency and the mode shape.

The dissipative characteristics of a mode can be estimated by obtaining an effective coefficient describing the non-conservative forces from a time-domain evaluation of the nonlinearities. Following the procedure above, assuming $q(t) = Q\cos(\omega_n t)$ allows for the definition of $\dot{q}(t)$, which can then be used to evaluate the nonlinear force in the modal domain ($f^{(m)}(t)$ below) as

$$\dot{\bar{u}}(q(t)) = \frac{\partial \bar{u}}{\partial t}(Q\cos(\omega_n t))(-Q\omega_n \sin(\omega_n t)),$$

[2]Details on hysteretic evaluation are left out for brevity. The interested reader is directed to [5].

$$f^{(m)}(t) = \bar{\phi}^T(Q)\left(\mathbf{K}\bar{u}(q(t)) + \mathbf{C}\dot{\bar{u}}(q(t)) + \bar{f}_{nl}(\bar{u}, \dot{\bar{u}}, \dots)\right). \tag{6}$$

Introducing a damping term in the modal level of the form

$$\ddot{q}(t) + c(Q)\dot{q}(t) + \underbrace{\omega_n^2(Q)}_{\lambda(Q)} q(t) = 0 \tag{7}$$

allows the estimation of the coefficient $c(Q)$ through the sine harmonics of $f^{(m)}$ as

$$c(Q) = \frac{\mathcal{F}^{(s)}\{f^{(m)}(t)\}}{-Q\omega_n(Q)}. \tag{8}$$

This can also be used to estimate a "modal damping factor" given by $\zeta(Q) = c(Q)/(2\omega_n(Q))$, which is a quantity that is often used in system identification and modal testing practice.

The natural frequency ($\omega_n(Q)$), mode shape ($\bar{\phi}(Q)$), and damping coefficient ($c(Q)$) estimated above represent quantities that may be readily employed for systems operating close to the nonlinear resonance corresponding to the considered nonlinear mode. The forced response of the system can be *synthesized* in such regimes using the so-called single-mode theory [7], which will briefly be presented here for completeness. For a complex excitation of the form $\bar{f}_{ex}e^{i\Omega t}$ (with Ω and $\bar{f}_{ex}$ being the excitation frequency and some complex amplitude), the solution is assumed to be of the form, $u(t) = Qe^{i\theta}e^{i\Omega t}$, with θ representing the phase of the response (assumed constant in time). Substituting this into the equations of motion and conducting an inner product with $\bar{\phi}(Q)$ will yield the complex algebraic equation in terms of unknown amplitude Q and phase θ

$$(\lambda(Q) - \Omega^2) + i(c(Q)\Omega) = \frac{\bar{\phi}^H(Q)\bar{F}_{ex}}{Q}e^{-i\theta}. \tag{9}$$

This can be solved analytically to yield,

$$\begin{aligned} \Omega^2 &= (\lambda(Q) - \frac{c^2(Q)}{2}) \pm \sqrt{\frac{c^4(Q)}{4} - \lambda(Q)c^2(Q) + \frac{|\bar{\phi}^H(Q)\bar{F}_{ex}|^2}{Q^2}} \\ \theta &= Arg\left(\frac{\bar{\phi}^T(Q)\bar{F}_{ex}}{Q}\right) - \tan^{-1}\frac{c(Q)\Omega}{\lambda(Q) - \Omega^2}, \end{aligned} \tag{10}$$

provided Ω is real. Note that $()^H$ indicates the Hermitian transpose in the above.

In the described approach, the requirement of solving a potentially large nonlinear system is only for solving the RQNMA equations (Eq. (2)), and all of the other steps only involve *post-processing* the RQNMA results through interpolation and single-time function evaluations, which are typically several orders of magnitude faster than nonlinear equation solving. Further, the use of analytical gradients of the solutions of Eq. (2) with respect to q enables one to employ Hermite interpolants, improving accuracy of the post-processing steps.

In order to consider two modes concurrently, the following formulation is proposed (functional dependence on (Q_1, Q_2) is dropped for brevity of notation):

$$\begin{bmatrix} 1 & m_{12} \\ m_{12} & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \bar{\phi}_1^T \bar{f}_{ex} \\ \bar{\phi}_2^T \bar{f}_{ex} \end{Bmatrix}. \tag{11}$$

Since nonlinear modes may not always be expected to be exactly orthogonal to each other, an off-diagonal mass matrix term m_{12} is included in the above (diagonal terms are 1 from definition). The terms $k_{ij}(Q_1, Q_2)$ and $c_{ij}(Q_1, Q_2)$ represent the nonlinear damping and stiffness terms and are assumed to be functions of the harmonic amplitudes of both the modes under consideration. The mode shapes $\bar{\phi}_1(Q_1)$ and $\bar{\phi}_2(Q_2)$ are, however, assumed to only be dependent on the amplitudes of their respective modes.

Given a pair of harmonic amplitudes (Q_1, Q_2), the solution ansatz $u(t) = \sum_{i=1}^{2} Q_i \bar{\phi}_i(Q_i) \cos(\omega_{n,i}(Q_i)t)$ will be used to evaluate the nonlinear forces in the system. Since $\omega_{n,1}/\omega_{n,2}$ need not be a rational fraction, one cannot use a periodic assumption here. Although multi-component Fourier techniques may be employed, they are avoided here since the evaluation of frictional nonlinearities is not very straightforward. Therefore, the nonlinear forces are generated over an *arbitrarily long* period of time (covering several cycles of each mode), and the parameters $c_{ij}(Q_1, Q_2), k_{ij}(Q_1, Q_2)$ are estimated using linear regression (iterations not necessary). Note that $m_{12}(Q_1, Q_2) = \bar{\phi}_1^T(Q_1)\mathbf{M}\bar{\phi}_2(Q_2)$ by definition.

2.2 Benchmark Model Description

The procedures described in Sect. 2.1 will be demonstrated using a nine-noded finite element model of a linear bar with a frictional end as shown in Fig. 1. The governing partial differential equations are

$$\rho A \frac{\partial^2 u}{\partial t^2} + \left(\alpha \frac{\partial u}{\partial t} - \beta \frac{\partial^3 u}{\partial x^2 \partial t}\right) - EA \frac{\partial^2 u}{\partial x^2} = f_{ex}(t)\delta(x - \ell) \quad x \in (0, \ell), \tag{12}$$

where $\delta(x)$ denotes the dirac delta distribution indicating excitation at the end. This is discretized using eight linear finite elements. The last node is connected to the ground using an elastic dry friction element [8] parameterized by the stiffness k_t and slip force μN. Two cases are considered:

1. $k_t = 2.5$ MN m^{-1}; $\mu N = 0.75$ MN.
2. $k_t = 6$ MN m^{-1}; $\mu N = 0.75$ MN.

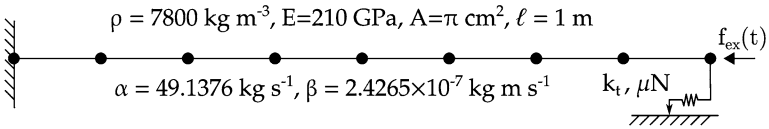

Fig. 1 Bar model

The parameters of case 2 are set in such a way as to highlight mode-coupling effects, whereas such effects are intentionally avoided in case 1.

3 Results

The current section presents the results of the application of the procedures described in Sect. 2.1 to the 1D bar model described above. The nonlinear modal analysis results are compared against frequency-domain references based on the periodic motion concept [4] (EPMC). All frequency-domain computations are conducted using the first five harmonics with harmonic balance.

3.1 Nonlinear Modal Analysis

Figure 2 presents the nonlinear modal characteristics for mode 1 of the system for both the cases. Case 1 (Fig. 2a) is a scenario where a single mode responds in an *isolated manner* around resonance, while case 2 (Fig. 2b) is a scenario where this is not so. A "tongue-like" projection typical of modal coupling in periodic motion backbones can be seen in the EPMC backbone in the boxed region in Fig. 2b. This feature is not captured by the RQNM formulation due to its fundamentally single modal definition.

Figure 3 depicts the frequency characteristics of the first mode and the 1/3rd frequency characteristics of the second mode, indicating a 1:3 internal resonance.

3.2 Synthesis of Frequency Responses

Figure 4 plots the frequency responses for the two cases (simulated using single harmonic forcing with different amplitudes solved using HBM), along with the nonlinear modal backbones from EPMC and RQNM. It can be seen that both the backbones follow the frequency response peaks closely.

The frequency response synthesis formulation in Eq. (10) can be traced back to classical multiple-scale approaches [9] and is presently employed to determine the

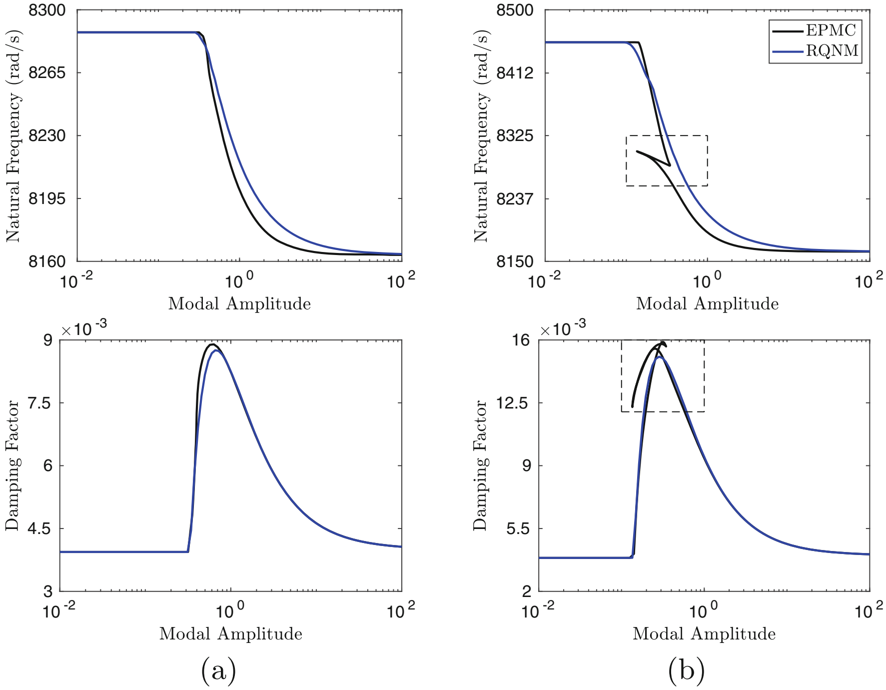

Fig. 2 The nonlinear frequency–amplitude and damping–amplitude relationship for (**a**) case 1 and (**b**) case 2. Boxed region(s) in (**b**) indicates relics of modal-coupling effects

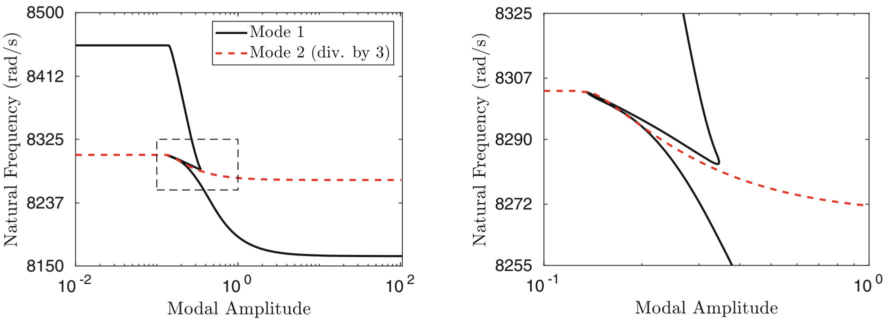

Fig. 3 The EPMC frequency–amplitude plot of mode 1 along with the mode 3 backbone (with frequency divided by 3) with an enlarged version

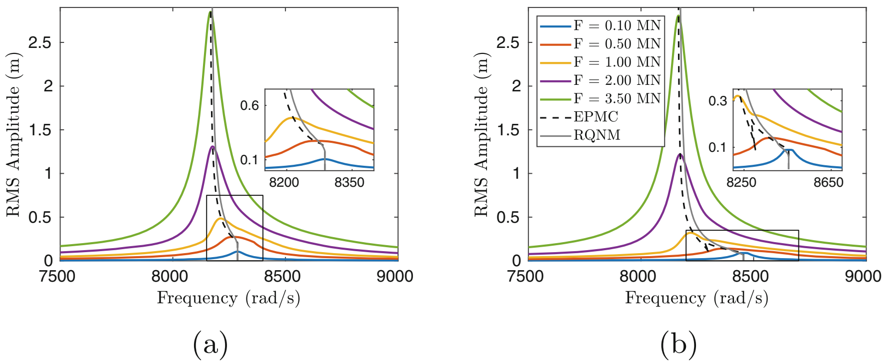

Fig. 4 The frequency response of the system in terms of the RMS displacement amplitude of the forcing node along with the modal backbones for (**a**) case 1, and (**b**) case 2

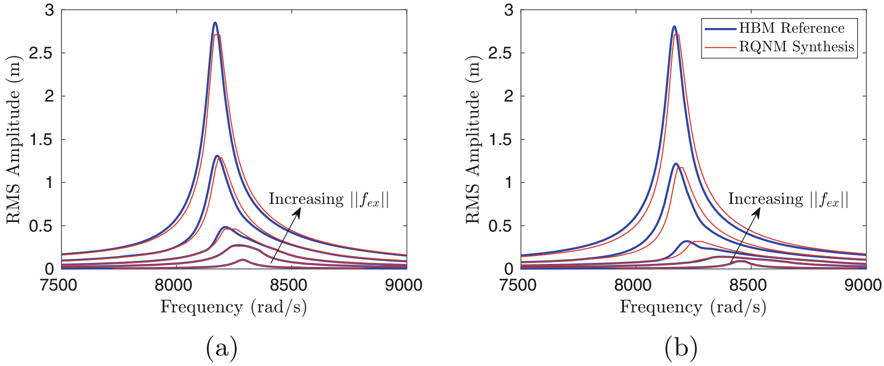

Fig. 5 Synthesis of forced responses: (**a**) case 1 and (**b**) case 2. See Fig. 4 for forcing amplitudes

accuracy of the developed reduced equations at the modal level. The accuracy of the synthesized responses can be taken to be indicative of the utility of the underlying nonlinear modal characteristics.

Figure 5 depicts the frequency responses (amplitudes) synthesized using Eq. (10) for the two cases under consideration. Although the synthesized responses follow the references closely, small discrepancies may be seen, especially for case 2. This is thought to be due to the internal resonance noted above.

3.3 Nonlinear Modal Modeling in the Presence of Modal Coupling

The possibility of using a multi-modal nonlinear expansion is explored here. Using harmonic amplitudes Q_1 and Q_2 for modes 1 and 2, respectively, the solution ansatz that will be used is

$$\bar{u}(t) = \bar{\phi}_1(Q_1)Q_1\cos(\omega_1(Q)t) + \bar{\phi}_2(Q_2)Q_2\cos(\omega_2(Q)t). \tag{13}$$

Note that both modes are assumed to have only a cosine harmonic for simplicity. The next task will be to fit the parameters c_{ij}, k_{ij} to the internal forces of the system. As already mentioned, $\bar{u}(t)$ cannot be expected to be periodic for arbitrary Q_1, Q_2. Therefore, a time series is generated with $t \in (0, T)$ with an arbitrarily large T. Using this, the internal forces are evaluated and transformed to the *modal* domain to yield modal forces as follows:

$$\begin{aligned} f^{(m,1)}(t) &= \bar{\phi}_1^T(Q_1)\left(\mathbf{K}\bar{u}(t) + \mathbf{C}\dot{\bar{u}}(t) + \bar{f}_{nl}(t, \bar{u}, \dots)\right); \\ f^{(m,2)}(t) &= \bar{\phi}_2^T(Q_2)\left(\mathbf{K}\bar{u}(t) + \mathbf{C}\dot{\bar{u}}(t) + \bar{f}_{nl}(t, \bar{u}, \dots)\right). \end{aligned} \tag{14}$$

This is fit to parameters c_{ij}, k_{ij} by solving the linear regression problem

$$\begin{bmatrix} q_1(t) & q_2(t) & 0 & \dot{q}_1(t) & \dot{q}_2(t) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & q_1(t) & q_2(t) & 0 & \dot{q}_1(t) & \dot{q}_2(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} k_{11} \\ k_{12} \\ k_{22} \\ c_{11} \\ c_{12} \\ c_{22} \end{Bmatrix} = \begin{bmatrix} f^{(m,1)}(t) \\ \vdots \\ f^{(m,2)}(t) \\ \vdots \end{bmatrix}. \tag{15}$$

Figure 6 depicts an example of the performance of such a fit (2D maps parameterized by Q_1, Q_2). Note that this approach (along with the cosine assumption in Eq. (13)) is justified only for time-invariant nonlinearities. Some observations that may be made from the parameter estimates are:

- The parameters have non-trivial relationships to the modal amplitudes.
- The non-diagonal modal stiffness terms $k_{1,2}$ (which are zero in linear modal analysis) seem to play an appreciable role in at least some regimes.
- The $c_{2,2}$ damping term even takes negative values in the presence of small mode 2 and intermediate mode 1 amplitudes.

For frequency response synthesis, analytical approaches like in Eq. (10) are not trivial for multi-mode expansions. Therefore, a HBM implementation is developed that interpolates (using linear interpolants) the quantities $k_{ij}(Q_1, Q_2)$, $c_{ij}(Q_1, Q_2)$, $m_{12}(Q_1, Q_2)$, and $\bar{\phi}_i^T(Q_i)\bar{f}_{ex}$ to evaluate the internal forces. Since these quantities are estimated from the above regression over a uniform grid of (Q_1, Q_2), the interpolation can be carried out in a very fast manner.[3]

[3]NLvib [10], an open-source MATLAB nonlinear vibration/continuation package, was used for the numerical continuation in this case.

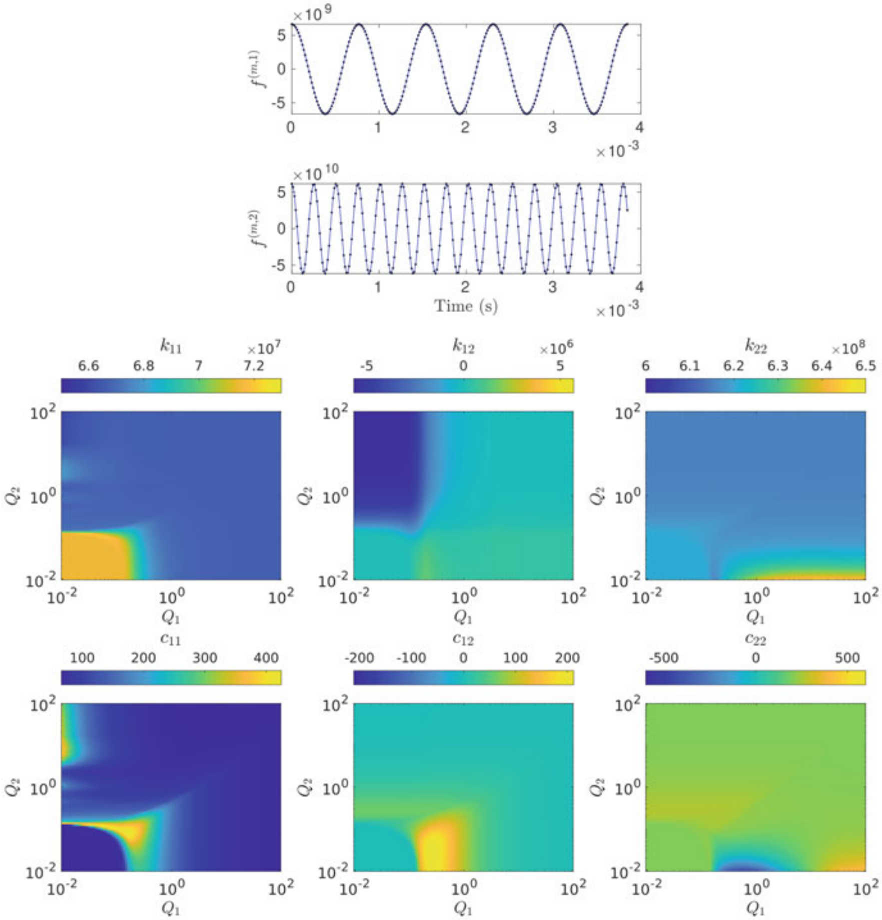

Fig. 6 (**a**) An example of the fitting procedure (case 1, $(Q_1, Q_2) = (100, 100)$). The dots depict the reference forces, and the continuous lines depict the fitted forces. (**b**) Parameter fits as functions of Q_1, Q_2 for case 2

Figure 7a, b plots the frequency response of the original system along with the synthesized frequency responses from the single-mode and two-mode expansions, and Fig. 7c, d shows the contributions of the individual modes in the response for the two cases. The two-mode expansion performs much better than the single-mode formulation in predicting the frequency responses, especially toward the higher amplitude regimes. However, the modal interaction features in case 2 are still not captured by this formulation. The reason is possibly due to the fact that such features may not be explained using just a single harmonic formulation. It will therefore be meaningful to explore multi-harmonic implementations of the modal analysis procedure outlined in Sect. 2.1, starting with obtaining multi-

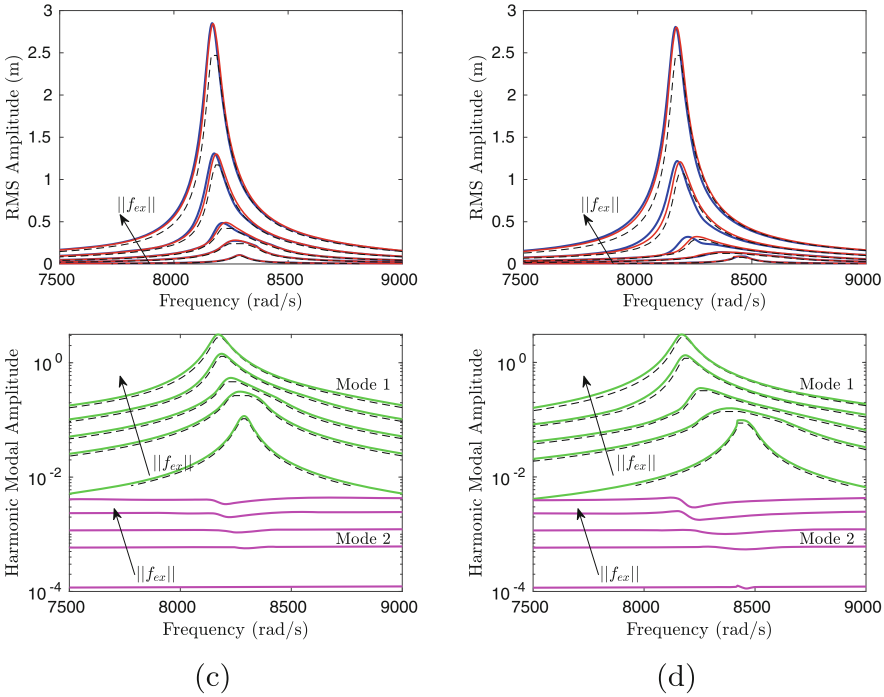

Fig. 7 The reference frequency response of the system (blue solid lines), single-mode synthesis (black dashed lines), and two-mode synthesis (red solid lines) for (**a**) case 1 and (**b**) case 2. Also plotted are the harmonic modal amplitudes for mode 1 (green) and mode 2 (magenta) along with the single-mode expansion (black dashed lines) for the two cases in (**c**), (**d**), respectively

harmonic mode shapes and/or natural frequency and damping estimates from a multi-harmonic approximation of Eq. (3) (see [7, 11], for instance).

4 Conclusions

An improved formulation of the Rayleigh quotient-based nonlinear modal analysis has been presented and its applicability demonstrated using a numerical benchmark.

Using a numerical multi-degree-of-freedom (MDoF) benchmark, response synthesis using single-mode and two-mode expansions has been compared, showing that the two-mode expansion offers superior overall accuracy. Both techniques, however, fail to detect nonlinear modal interaction/internal resonance phenomena. The results indicate that a multi-harmonic multi-modal expansion could yield better results. Time- domain (transient response) synthesis will require the extension of modal models of the form in Eq. (11), where the nonlinearities are dependent on the harmonic amplitude and not on the instantaneous amplitude. Results and techniques from classical multiple time-scale approaches [9] are promising for such extensions.

The major advantage with the proposed approach comes from the fact that the computational requirements for large problems are minimized. The only nonlinear algebraic solution operations with sizes as large as the system are necessary in the initial NEPv stage. Further, evaluations of the nonlinear functions (which could be expensive in some cases) are only necessary for the modal model regression, which does not involve any further large nonlinear solutions. Therefore, the construction as well as utilization of the modal models is both relatively very cheap in comparison to evaluation of the full-order model.

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