The Filtering Approach as a Tool for Modeling and Analyzing Turbulence

Massimo Germano, A. Abbà, A. Cimarelli, Andrea Ferrero, Fernando F. Grinstein, M. Klein, Francesco Larocca, Juan A. Saenz, and Guglielmo Scovazzi

Abstract The Filtering Approach (FA) is a simple multiscale method of analysis, it extends the statistical formalism to a generic filtering operator and main ingredients are the Generalized Central Moments (GCM) homomorphic to the Statistical Central Moments (SCM). In the past this technique was intensively used to model turbulent flows in the context of the Large Eddy Simulation (LES) and at present is more and more applied to analyze turbulence and extract statistical data from under-resolved databases. In this paper we will briefly summarize the main multiscale characters of the FA, the well known identity relating GCM of the second order at different levels

M. Germano (✉)
Department Civil and Environmental Engineering, Duke University, Durham NC27708, USA
e-mail: mg234@duke.edu

A. Abbà
Politecnico di Milano, 20156 Milano, Italy
e-mail: antonella.abba@polimi.it

A. Cimarelli
Università di Modena e Reggio Emilia, 41125 Modena, Italy
e-mail: andrea.cimarelli@unimore.it

A. Ferrero · F. Larocca
Politecnico di Torino, 10100 Torino, Italy
e-mail: andrea.ferrero@polito.it

F. Larocca
e-mail: francesco.larocca@polito.it

F. F. Grinstein · J. A. Saenz
Los Alamos National Laboratory, Los Alamos, NM 87545, USA
e-mail: fgrinstein@lanl.gov

J. A. Saenz
e-mail: juan.saenz@lanl.gov

M. Klein
Universität der Bundeswehr München, Neubiberg 85577, Germany
e-mail: markus.klein@unibw.de

G. Scovazzi
Duke University, Durham, NC 27708, USA
e-mail: guglielmo.scovazzi@duke.edu

R. Örlü et al. (eds.), *Progress in Turbulence IX*, Springer Proceedings in Physics 267,
https://doi.org/10.1007/978-3-030-80716-0_9

is discussed in detail, and a new identity relating GCM of the third order at different levels is presented. Finally some recent developments are illustrated. The structure of the subfilter stresses and the decomposition of the Reynolds stresses is examined, hybrid LES modeling procedures are applied and metrics that measure the statistical homogeneity of a turbulent flow are proposed.

1 Introduction

The Navier-Stokes equations, that describe mathematically the motion of a fluid, become chaotic for a sufficiently high Reynolds number characteristic of the particular flow. As such a numerical solution, however much resolved and extended in space and time, is everything and nothing at the same time. As remarked by [1], this trajectory is apparently uncorrelated to the *true* solution of a flowfield if it is allowed to evolve over a long time, and hence is called a pseudo-orbit. Moreover there are profound and unresolved theoretical problems about the *true* solutions of the Navier-Stokes equations, their unicity and so on, see [2] for a detailed exploration of that and the related implications as regards to the numerical simulation of turbulent flows.

Only when we define a statistical operator $\langle .... \rangle_e$ we can produce averaged quantities $\langle u \rangle_e$, $\langle uv \rangle_e$, ..., and we can make comparisons. But the translation of theory in practice is not so simple. To carry out something in accordance with our theories or our ideals can be very difficult. If we open a book on turbulence, it starts with a statistical description of a turbulent flow. Usually the objectivity of turbulence relies on theories that consider fluid flow experiments that *can be repeated many times under a specified set of conditions*, as we read in Pope [3], page 34, but practically it is not so easy to have at our disposal an ensemble of realizations, so we have to recur to some surrogate in order to replace it [4]. Moreover a database produced by a numerical code is usually *coarse grained* [5], limited in *resolution* in space and time that we generally indicate with Δ, and in *extent* in time and in the homogeneous directions, that we generally indicate as T and Z respectively. If we have two or more databases of the same turbulent flow, produced by the same code or different codes for different values of resolution Δ and extent T, the basic problem is their *comparison*, in other words the extraction of their *objectivity*. Usually all that is performed face up to the *truth* represented by experimental data or Direct Numerical Simulation (DNS), ideally obtained in the limit $\Delta \to 0$ and $T \to \infty$. Long time and long space averages are the usual ingredients of an explicit averaging *statistical* operator, and the values provided by them, $u(\Delta, T)$ have to be compared with the *absolute*, the *true* quantities $u(\Delta \to 0, T \to \infty)$. All that is not so simple, and the Filtering Approach provides some help.

The Filtering Approach (FA) is a simple multiscale approach to the analysis of turbulent flows. As usual its origins are deeply rooted in the past, see [6] for some historical notes on that, and a recent important motivation was to understand what the computer was producing in the first numerical simulations of turbulent flows. The Leonard [7] idea of representing the Large Eddy Simulation of a generic turbulent quantity $a(\mathbf{x}, t)$, computed with a code characterized by a grid length h, with a filtered representation $\langle a \rangle_f$ due to a filtering operator $\mathcal{F}$ characterized by a filtering length $\Delta \approx h$

$$\mathcal{F}[a] \equiv \langle a \rangle_f = \int F(\mathbf{x}, \mathbf{y}; \Delta)\, a(\mathbf{y}, t)\, d\mathbf{y} \quad ; \quad \int F(\mathbf{x}, \mathbf{y}; \Delta)\, d\mathbf{y} = 1 \qquad (1)$$

was in our opinion a major step in the turbulence studies. On its wake an operational formulation was proposed based on a hierarchy of filtering operators and on the Generalized Central Moments, (GCM) [8], that extend the Statistical Central Moments (SCM) and the statistical formalism to a generic filtering operator. Fundamentals to the multiscale approach are some simple identities that relate the GCM at different resolution levels, in particular with the statistical one.

In this paper some recents developments are presented. The structure of the subfilter stress [12] and a new decomposition of the Reynolds stress [14] are examined, a new dynamic modeling procedure is applied to the LES of Shock Driven Turbulent Mixing [20] and indices that measure the statistical homogeneity of a turbulent flow are proposed [23]. Some conclusions are finally provided.

2 Statistical and Generalized Central Moments

Main ingredients of the statistical approach are the Statistical Central Moments associated to the statistical operator $\mathcal{E}$, defined as

$$\begin{aligned} \tau_e(a, b) &\equiv \langle ab \rangle_e - \langle a \rangle_e \langle b \rangle_e \\ \tau_e(a, b, c) &\equiv \langle abc \rangle_e - \langle a \rangle_e \tau_e(b, c) - \langle b \rangle_e \tau_e(c, a) - \langle c \rangle_e \tau_e(a, b) - \langle a \rangle_e \langle b \rangle_e \langle c \rangle_e \\ \tau_e(a, b, c, d) &\equiv \cdots \end{aligned} \qquad (2)$$

and it is easy to see that due to the operational rules of the statistical averaging operator, rules that in turbulence are often referred as the Reynolds rules of the mean, $\langle \langle a \rangle_e \rangle_e = \langle a \rangle_e$ and $\langle a \langle b \rangle_e \rangle_e = \langle a \rangle_e \langle b \rangle_e$, we have an equivalent formulation of the SCM given by

$$\tau_e(a, b) = \langle a' b' \rangle_e \quad ; \quad \tau_e(a, b, c) = \langle a' b' c' \rangle_e \quad ; \quad \cdots \qquad (3)$$

where the brackets indexed e stand for the statistical average $\langle a \rangle_e = \mathcal{E}[a]$ and the apex stands for the statistical fluctuations $a' = a - \langle a \rangle_e$.

In the general case of a linear and constant preserving filtering operator $\mathcal{F}$ the Reynolds rules of the mean are usually not satisfied. We can however associate formally to $\mathcal{F}$ the *Generalized Central Moments*, GCM, algebraically homomorphic to the SCM

$$
\begin{aligned}
\tau_f(a,b) &\equiv \langle ab\rangle_f - \langle a\rangle_f\langle b\rangle_f \\
\tau_f(a,b,c) &\equiv \langle abc\rangle_f - \langle a\rangle_f\langle bc\rangle_f - \langle b\rangle_f\langle ca\rangle_f - \langle c\rangle_f\langle ab\rangle_f + 2\langle a\rangle_f\langle b\rangle_f\langle c\rangle_f \\
&\equiv \langle abc\rangle_f - \langle a\rangle_f\tau_f(b,c) - \langle b\rangle_f\tau_f(c,a) - \langle c\rangle_f\tau_f(a,b) - \langle a\rangle_f\langle b\rangle_f\langle c\rangle_f \\
\tau_f(a,b,c,d) &\equiv \cdots
\end{aligned}
\tag{4}
$$

and that is the starting point of the Filtering Approach. The GCM have the same algebraic properties as the SCM, and in a sense they extend the statistical formalism to a generic filtering operator $\mathcal{F}$. We refer for more detail on that to [6, 8] and we also remark a recent paper where the formal relationships between filtering and averaging are defined using generalized central moments in the more complex case of variable density flows [9].

2.1 Multiscale Identities

Let us now derive some important identities that are the basic ingredients of the Filtering Approach and that characterizes its multiscale nature. Let us consider the GCM associated to the product $\mathcal{P} = \mathcal{G}\mathcal{F}$ of two filtering operators. It is easy to derive the following important relations that connect the GCM at different scales of resolution

$$
\tau_p(a,b) = \langle \tau_f(a,b)\rangle_g + \tau_g(a_f, b_f) \tag{5}
$$

$$
\begin{aligned}
\tau_p(a,b,c) = {} & \langle \tau_f(a,b,c)\rangle_g + \tau_g(a_f, \tau_f(b,c)) + \tau_g(b_f, \tau_f(c,a)) \\
& + \tau_g(c_f, \tau_f(a,b)) + \tau_g(a_f, b_f, c_f)
\end{aligned}
\tag{6}
$$

where we use the simplified notation $a_f \equiv \langle a\rangle_f$, $b_f \equiv \langle b\rangle_f$, $c_f \equiv \langle c\rangle_f$. We remark that if we also assume that $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$

$$
\langle\langle\cdots\rangle_g\,\rangle_f = \langle\langle\cdots\rangle_f\,\rangle_g \tag{7}
$$

we can additionally write

$$
\begin{aligned}
\tau_{fg}(a,b) &= \langle \tau_g(a,b)\rangle_f + \tau_f(a_g, b_g) \\
&= \langle \tau_f(a,b)\rangle_g + \tau_g(a_f, b_f) = \tau_{gf}(a,b)
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\tau_{fg}(a,b,c) = {} & \langle \tau_g(a,b,c)\rangle_f + \tau_f(a_g, \tau_g(b,c)) + \tau_f(b_g, \tau_g(c,a)) \\
& + \tau_f(c_g, \tau_g(a,b)) + \tau_f(a_g, b_g, c_g) \\
= {} & \langle \tau_f(a,b,c)\rangle_g + \tau_g(a_f, \tau_f(b,c)) + \tau_g(b_f, \tau_f(c,a)) \\
& + \tau_g(c_f, \tau_f(a,b)) + \tau_g(a_f, b_f, c_f) = \tau_{gf}(a,b,c)
\end{aligned}
\tag{9}
$$

Moreover when $\mathcal{GF} = \mathcal{G}$ we have more simply

$$\tau_g(a, b) = \langle \tau_f(a, b) \rangle_g + \tau_g(a_f, b_f) \tag{10}$$

$$\begin{aligned} \tau_g(a, b, c) = \langle \tau_f(a, b, c) \rangle_g + \tau_g(a_f, b_f, c_f) + \tau_g(a_f, \tau_f(b, c)) \\ + \tau_g(b_f, \tau_f(c, a)) + \tau_g(c_f, \tau_f(a, b)) \end{aligned} \tag{11}$$

and we remark again that these identities represent very important multiscale properties of the GCM. The first one is well known, it has played an important role in the development of the dynamic modeling approach [10] and we remark here that the flexibility of this approach was recently discussed and demonstrated also in the context of modeling the commutation errors in LES [11]. The second one, the relation (6) connecting GCM of the third order, is new, and it is very important in the case of compressible flows, where $a = \varrho$ represents the density and $b = u_i$, $c = u_j$ are the velocity components.

3 Recent Contributions

3.1 Structure of the Subfilter Stress

The second order GCM associated to a velocity field $\tau_f(u_i, u_j)$ are better known under the name of *subfilter stresses*. Their intimate structure has been studied [12] based on the following considerations. Due to the additive properties of the GCM we can write

$$\tau_f(u_i, u_j) = \left[\tau_f(\tilde{u}_i, u_j) + \tau_f(u_i, \tilde{u}_j) + \tau_f(w_i, u_j) + \tau_f(u_i, w_j)\right] / 2 \tag{12}$$

where

$$u_i = \tilde{u}_i + w_i \tag{13}$$

is a generic additive decomposition of the turbulent velocity field u_i. All that is exact, and a first approximation is

$$\tau_f(u_i, u_j) \approx \frac{\tau_f(\tilde{u}_i, u_j) + \tau_f(\tilde{u}_i, u_j)}{2} \tag{14}$$

that we can formally write in terms of a tensorial eddy viscosity ν_{ki}

$$\tau_f(u_i, u_j) \approx -\nu_{kj} \partial_k \tilde{u}_i - \nu_{ki} \partial_k \tilde{u}_j \tag{15}$$

given explicitly by

$$\nu_{ki} = -\frac{1}{4}\int\int F(\mathbf{x},\mathbf{y})F(\mathbf{x},\mathbf{z})\,(y_h - z_h)\,\big(u_j(\mathbf{y}) - u_j(\mathbf{z})\big)\,d\mathbf{y}d\mathbf{z} \tag{16}$$

This last relation is due to the fact that we can explicitly write [13]

$$\tau_f(\tilde{u}_i, u_j) = \frac{1}{2}\int\int F(\mathbf{x},\mathbf{y})F(\mathbf{x},\mathbf{z})\,(\tilde{u}_i(\mathbf{y}) - \tilde{u}_i(\mathbf{z}))\,\big(u_j(\mathbf{y}) - u_j(\mathbf{z})\big)\,d\mathbf{y}d\mathbf{z} \tag{17}$$

and we can approximate at the first order

$$\tilde{u}_i(\mathbf{y}) - \tilde{u}_i(\mathbf{z}) \sim (y_h - z_h)\,\frac{\partial \tilde{u}_i(\mathbf{x})}{\partial x_h} \tag{18}$$

3.2 Decomposition of the Reynolds Stress

The second order SCM associated to a velocity field $\tau_e(u_i, u_j)$ and to a scalar $\tau_e(u_i, \varphi)$ are better known under the name of *Reynolds stresses* and *Reynolds fluxes*. A fundamental problem in the analysis of turbulence is to extract the Reynolds stresses and the Reynolds fluxes from a database obtained by a numerical simulation or an experimental exploration, and in [14, 15] the previously derived operational relations between SCM and GCM are applied to this particular problem. Let us indicate the Reynolds averaging operator as $\mathcal{E}$, and the filtered average as $\mathcal{F}$. Let us first assume that $\mathcal{E}\mathcal{F} = \mathcal{E}$. In terms of SCM and GCM we define the following quantities

$$\begin{aligned} R_{ij} &\equiv \tau_e(u_i, u_j) \\ \tau_{ij} &\equiv \tau_f(u_i, u_j) \\ T_{ij} &\equiv \tau_e(\langle u_i\rangle_f, \langle u_j\rangle_f) \end{aligned} \tag{19}$$

where u_i are the components of the velocity field at a given time and location. We have

$$R_{ij} = \langle \tau_{ij}\rangle_e + T_{ij} \tag{20}$$

Let us now only assume that $\mathcal{E}\mathcal{F} = \mathcal{F}\mathcal{E}$, a much weaker, very general and respected assumption. We define the following additive GCM

$$\vartheta_{ij} \equiv \tau_f(\langle u_i\rangle_e, \langle u_j\rangle_e) \tag{21}$$

and by applying the identity (5) we have

$$\langle R_{ij}\rangle_f + \vartheta_{ij} = \langle \tau_{ij}\rangle_e + T_{ij} \tag{22}$$

This new identity has been recently applied to the study of the decomposition of the Reynolds stress from filtered data [14] when filtering is not applied in homogeneous statistical directions or in time for statistically stationary turbulent fields.

We can extend all that to compressible flows. As in [16, 17] we will not introduce Favre averages. Let us first of all assume that $\mathcal{EF} = \mathcal{E}$, and let us define the following GCM

$$\begin{aligned}
&R_{\varrho i} \equiv \tau_e(\varrho, u_i) \;\; ; \quad R_{ij} \equiv \tau_e(u_i, u_j) \\
&R_{\varrho ij} \equiv \tau_e(\varrho, u_i, u_j) \\
\\
&\tau_{\varrho i} \equiv \tau_f(\varrho, u_i) \;\; ; \quad \tau_{ij} \equiv \tau_f(u_i, u_j) \\
&\tau_{\varrho ij} \equiv \tau_f(\varrho, u_i, u_j) \\
\\
&T_{\varrho i} \equiv \tau_e(\langle\varrho\rangle_f, \langle u_i\rangle_f) \;\; ; \quad T_{ij} \equiv \tau_e(\langle u_i\rangle_f, \langle u_j\rangle_f) \\
&T_{\varrho ij} \equiv \tau_e(\langle\varrho\rangle_f, \langle u_i\rangle_f, \langle u_j\rangle_f)
\end{aligned} \tag{23}$$

where ϱ is the density and u_i are the components of the velocity field at a given time and location. We have

$$R_{\varrho i} = \langle\tau_{\varrho i}\rangle_e + T_{\varrho i} \quad ; \quad R_{ij} = \langle\tau_{ij}\rangle_e + T_{ij} \tag{24}$$

$$R_{\varrho ij} = \langle\tau_{\varrho ij}\rangle_e + T_{\varrho ij} + \tau_e(\langle\varrho\rangle_f, \tau_{ij}) + \tau_e(\langle u_i\rangle_f, \tau_{\varrho j}) + \tau_e(\langle u_j\rangle_f, \tau_{\varrho i}) \tag{25}$$

We remark finally that this decomposition of the Reynolds stresses for compressible flows is very simple if compared to the usual ones expressed in terms of the statistical or the Favre fluctuations.

3.3 Dynamic Coarse Grained Modeling

Coarse Grained Simulation [5] combines classical large eddy simulations based on explicit sub-grid scale models and implicit LES (ILES) relying on subgrid models provided by physics-capturing numerics. We can apply this method to the Flow Simulation Methodology [18, 19] in a hybrid ILES/RANS approach. All that is motivated by the study of a challenging problem, the turbulent mixing driven by a shock wave [20, 21].

The role here of the Filtering Approach is to suggest dynamic blending procedures based on the following main points. We have remarked that the multiscale identity that connects the turbulent subgrid stress at different levels

$$\tau_{fg}(a, b) = \langle\tau_f(a, b)\rangle_g + \tau_g(a_f, b_f) \tag{26}$$

has played an important role in the development of the dynamic modeling approach [10]. If the test filter $\mathcal{G}$ is the statistical operator $\mathcal{E}$ we could imagine dynamic models [22] constrained by the Reynolds stress. Following Speziale [18] we assume that

$$\tau_{ij}^{LES} \sim f(\Delta/L)\,\tau_{ij}^{RANS} \tag{27}$$

where f is the so called *contribution function*. Empirically we could write

$$f(\Delta/L) = [1 - \exp(-\beta\Delta^2)]^m \tag{28}$$

where Δ is a grid length and m and β *ad hoc* coefficients, but a dynamic procedure could remove such arbitrariness. We can write

$$\tau_{ij}^{RANS} = \tau_{ij}^{RES} + \langle\tau_{ij}^{LES}\rangle \tag{29}$$

and to exploit this multiscale identity in order to determine dynamically the contribution function f.

3.4 Statistical Homogeneity Indices

Many important benchmark turbulent flows are provided with one or more homogeneous direction. Sustained homogeneous turbulence is provided with homogeneity in time and three space directions, homogeneous decaying turbulence and decaying Taylor Green vortex flow are homogeneous in three directions, plane channel turbulent flow is homogeneous in time and two space directions, turbulent flow past a cylinder is homogeneous in time and the spanwise direction. In all these cases the objective in these chaotic simulations is to extract from the obtained database some statistical quantities, typically the mean values and the Reynolds stresses for the velocity components.

In the case of the turbulent flow past a cylinder we have two homogeneities, the time t and the spanwise spatial direction z, and we can average along one or both of them. Given the turbulent velocity field $u_i(x, y, z, t)$ we will introduce the following space and time *statistical* filtering operators $\mathcal{T}$ and $\mathcal{Z}$

$$\begin{aligned}
\mathcal{Z}\{u_i\} &\equiv \langle u_i\rangle_z \equiv \frac{1}{Z}\int_0^Z u_i(x, y, z', t)dz' \\
\mathcal{T}\{u_i\} &\equiv \langle u_i\rangle_t \equiv \frac{1}{T}\int_0^T u_i(x, y, z, t')dt'
\end{aligned} \tag{30}$$

and we define $\mathcal{E}$ as the product of the two, $\mathcal{E} \equiv \mathcal{T}\mathcal{Z}$, where Z and T are the extents of the computational domain in the spanwise direction z and in time t.

Let us first of all examine the differences between space and time averages. Consistently with the filtering approach based on the GCM let us define the following turbulent stresses associated to the filters $\mathcal{T}$ and $\mathcal{Z}$

$$\begin{aligned}
\tau_z(u_i, u_j) &\equiv \langle u_i u_j \rangle_z - \langle u_i \rangle_z \langle u_j \rangle_z \\
\tau_t(u_i, u_j) &\equiv \langle u_i u_j \rangle_t - \langle u_i \rangle_t \langle u_j \rangle_t \\
\tau_e(u_i, u_j) &\equiv \langle u_i u_j \rangle_{zt} - \langle u_i \rangle_{zt} \langle u_j \rangle_{zt} \\
\tau_z(\langle u_i \rangle_t, \langle u_j \rangle_t) &\equiv \langle \langle u_i \rangle_t \langle u_j \rangle_t \rangle_z - \langle u_i \rangle_{zt} \langle u_j \rangle_{zt} \\
\tau_t(\langle u_i \rangle_z, \langle u_j \rangle_z) &\equiv \langle \langle u_i \rangle_z \langle u_j \rangle_z \rangle_t - \langle u_i \rangle_{tz} \langle u_j \rangle_{tz}
\end{aligned} \tag{31}$$

We note that due to $\mathcal{E} = \mathcal{Z}\mathcal{T} = \mathcal{T}\mathcal{Z}$ we have the two identities

$$\begin{aligned}
\tau_e(u_i, u_j) &\equiv \tau_z(\langle u_i \rangle_t, \langle u_j \rangle_t) + \langle \tau_t(u_i, u_j) \rangle_z \\
&\equiv \tau_t(\langle u_i \rangle_z, \langle u_j \rangle_z) + \langle \tau_z(u_i, u_j) \rangle_t
\end{aligned} \tag{32}$$

and we can define two measures of turbulence resolution, the first related to the time average and the second to the spanwise average, given by

$$M_t(x, y) = \frac{\langle \tau_t(u_i, u_i) \rangle_z}{R_{ii}} \quad ; \quad M_z(x, y) = \frac{\langle \tau_z(u_i, u_i) \rangle_t}{R_{ii}} \tag{33}$$

where $R_{ij} \equiv \tau_e(u_i, u_j)$. We remark that by definition $M = 0$ corresponds to a perfect DNS and $M = 1$ to a perfect RANS. As such $M_t(x, y)$ and $M_z(x, y)$ measure and can be defined as *indices of the statistical homogeneity* in time $\mathcal{T}$ and in the spanwise direction $\mathcal{Z}$. They are related to the total Reynolds RANS stress produced by the joint average $\mathcal{E} = \mathcal{Z}\mathcal{T} = \mathcal{T}\mathcal{Z}$.

We remark finally that in the general case

$$\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2 = \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3 = \cdots \tag{34}$$

we can read all that as a multiscale analysis of the variance, where $\mathcal{E}$ is the statistical operator and $\mathcal{E}_i$ are different nested partitions of the probability. In this case the relations

$$\langle u \rangle_e = \langle\langle u \rangle_2 \rangle_1 = \langle\langle\langle u \rangle_3 \rangle_2 \rangle_1 = \cdots \tag{35}$$

$$\begin{aligned}
\tau_e(u, v) &= \tau_1(\langle u \rangle_2, \langle v \rangle_2) + \langle \tau_2(u, v) \rangle_e \\
&= \tau_1(\langle u \rangle_2, \langle v \rangle_2) + \langle \tau_2(\langle u \rangle_3, \langle v \rangle_3) \rangle_1 + \langle\langle \tau_3(u, v) \rangle_2 \rangle_1 \\
&= \cdots
\end{aligned} \tag{36}$$

are better known in applied statistics respectively as the Law of Total Expectation and the Law of Total Variance, the *Adam's and Eve's laws* [24–26]. The total statistical mean $\langle u \rangle_e$ is the average of the partial means, and the total statistical covariance $\tau_e(u, v)$ is the average of the partial covariances plus the statistical covariances of the partial mean values. Our multiscale identities are the generalization of the Adam's and Eve's laws to a hierarchy of generic filtering operators, and we really think that this simple multiscale approach could be usefully applied not only to modelling turbulent flows but also to the analysis of turbulent databases.

4 Conclusions

In the paper the main peculiarities of the Filtering Approach based on the Generalized Central Moments are summarized and some recent applications both in the analysis and modelling of turbulent flows are presented. All that in the spirit of illustrating a *unified theory linking the direct approach to the statistical one by a continuous interval of intermediate steps* [8].

References

1. J. Lee, Chaos and direct numerical simulation in turbulence. Theoret. Comput. Fluid Dyn. **7**, 363–395 (1995)
2. J.L. Guermond, J.T. Oden, S. Prudhomme, Mathematical perspectives on large eddy simulation models for turbulent flows. J. Math. Fluid. Mech. **6**, 194–248 (2004)
3. S.B. Pope, Turbulent flows. Cambridge University Press, pag. 603–604 (2000)
4. N. Aubry, On the hidden beauty of the proper orthogonal decomposition. Theoret. Comput. Fluid Dyn. **2**, 339–352 (1991)
5. F.F. Grinstein, *Coarse Grained Simulation and Turbulent Mixing*, Cambridge (2016)
6. M. Germano, Fundamentals of large eddy simulation, Advanced Turbulent Flows Computations, R Peyret and E Krause eds., CISM Courses and Lectures 395:81–130, Springer (2000)
7. A. Leonard, Energy cascade in large-eddy simulations of turbulent fluid flows. Adv. Geophys. **18A**, 237–248 (1975)
8. M. Germano, Turbulence: the filtering approach. J. Fluid Mech. **238**, 325–336 (1992)
9. J.A. Saenz, D. Aslangil, D. Livescu, Filtering, averaging, and scale dependency in homogeneous variable density turbulence. Phys. Fluids **33**, 025115 (2021)
10. M. Germano, U. Piomelli, P. Moin, W.H. Cabot, A dynamic subgrid-scale eddy viscosity model. Phys. Fluids **A3**, 1760–1765 (1991)
11. M. Klein, M. Germano, Analysis and modelling of the commutation error. Fluids **6**(1), 15 (2021)
12. A. Cimarelli, A. Abbà, M. Germano, General formalism for a reduced description and modelling of momentum and energy transfer in turbulence. J. Fluid. Mech. **866**, 865–896 (2019)
13. M. Germano, A direct relation between the filtered subgrid stress and the second order structure function. Phys. Fluids **19**, 038102 (2007)
14. M. Klein, M. Germano, Decomposition of the Reynolds stress from filtered data. Phys. Rev. Fluids **3**, 114606 (2018)
15. M. Klein, C. Kasten, M. Germano, Decomposition of turbulent fluxes from filtered data and application to turbulent premixed combustion modelling. Flow Turbul. Combust. **103**, 503–517 (2019)
16. M. Germano, On the hybrid RANS-LES of compressible flows, Progress in Hybrid RANS-LES Modelling, S Girimaji et al. eds., Springer, 253–263 (2014)
17. M. Germano, A. Abbà, R. Arina, L. Bonaventura, On the extension of the eddy viscosity model to compressible flows. Phys. Fluids **26**, 041702 (2014)

18. C.G. Speziale, Turbulence modeling for time-dependent RANS and VLES: a review. AIAA J. **36**(2), 173–184 (1998)
19. M. Germano, Comment on “Turbulence modeling for time-dependent RANS and VLES: a review”. AIAA J. **36**(9), 1766 (1998)
20. F.F. Grinstein, J.A. Saenz, R.M. Rauenzahn, M. Germano, D.M. Israel, Dynamic bridging modeling for coarse grained simulations of shock driven turbulent mixing. Comput. Fluids **199**, 104430 (2020)
21. F.F. Grinstein, J.A. Saenz, M. Germano, Coarse grained simulations of shock-driven turbulent material mixing. Phys. Fluids **33**, 035131 (2021)
22. M. Germano, A statistical formulation of the dynamic model. Phys. Fluids **8**, 565–570 (1996)
23. A. Ferrero, F. Larocca, G. Scovazzi, M. Germano, *A numerical study of the spanwise turbulence past a cylinder flow*, 17th ETC, 3–6 Sept. 2019, Torino, ITALY (2019)
24. R.A. Fisher, Statistical methods for research workers, Oliver and Boyd, Edinburgh, Scotland (1925)
25. J.A. Rice, Mathematical statistics and data analysis, Third Edition, Thomson (2007)
26. J.K. Blitzstein, J. Hwang, Introduction to probability, CRC Press LLC (2014)