Chapter 5
A partial report on the controversies about the Principle of Virtual Work: from Archytas of Tarentum to Lagrange, Piola, Mindlin and Toupin.

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The efforts paid in the philological search of the most ancient sources, in which one can find the formulation of a certain theory, may seem devoid of utility. Especially if such a search is complicated by the infinite formulation variants that a theory can present, in the simplest cases because of change of notations or, in the most difficult ones, because of the presence of equivalent postulations that are chosen as starting points. To further increase the difficulties, the mathematical tools used in the development of the theory, sometimes, may appear to be very different, albeit they produce the same logical consequences.

The most famous case of different, but equivalent, mathematical tools used in a theory is represented by the synthesis operated by Cartesian geometry. The equivalence of algebraic and geometrical concepts obtained by establishing the Cartesian correspondence between geometrical and algebraic objects did allow a major advancement in human capacities to model the physical reality. It proved that one can develop equivalent theories either by using geometrical concepts or by using algebraic ones. As an instance, the concept of physical quantity can be modeled either by means of Dedekind sections (the modern name given to the geometrical definition of real numbers, which was probably due to Eudoxus of Cnidus) or by introducing Cauchy sequences of rational numbers (the modern definition based

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on algebraic concepts). A theory does not become original simply because it uses Dedekind sections instead of Cauchy sequences!

An often heard criticism against historical research regards its lack of utility in solving any kind of «practical» problem. We believe that this criticism is not well-grounded. In general, one should consider history as a source of phenomenological evidence, to be used in building theories capable to predict the behavior of social groups. In particular, one could daydream about a theory explaining the reasons for which some groups of scientists are polarized to follow some leaders in choosing a postulation scheme instead of another one when they try to found novel theories or to reorganize already existing ones.

Looking for supporters of this point of view in antiquity, one can think about Cicero. He was aware of the important role that history and its study can have in the advancement of human knowledge. One of his most famous sentences is indeed:

> «Historia vera testis temporum, lux veritatis, vita memoriae, magistra vitae, nuntia vetustatis»
> (Cicero, De Oratore, II, 9, 36)

that one can translate as follows:

> «History is true witness of the times, light of truth, life of memory, teacher of life, messenger of antiquity».

We claim that history can be our teacher also in formulating novel theories. In conclusion, albeit the study of the most ancient sources of the theory of generalized continua has surely its own scholarly interest, we are also motivated by a really practical aim: to find the most effective postulation process to be used when one wants to develop physical models.

5.1 Some «forgotten» – but not «lost» – sources in mechanical sciences.

Gabrio Piola (see [46, 39, 43, 42, 47]) considered himself a continuator of the scientific endeavors that Lagrange believed had to be performed to firmly found mechanical sciences. Therefore, Piola invested all his intellectual forces to show that the Principle of Virtual Work (or the Principle of Least Action, which can be regarded to be an important particular case of the previous one) has to be used when creating new models. Piola was aware of the request imposed by Occam's razor: a postulation scheme must use the minimum possible set of postulates, and if one has to choose between a long list of ad hoc assumptions it is much more likely to include, in the formulated theory, some logical incongruences. If the aim is to produce models for successfully predicting the observed experimental evidence and some unknown phenomena, a theory based on the fewest possible assumptions has to be preferred.

Unfortunately, Piola's point of view was not shared by the «French Geometers», as he calls the group led by Cauchy, Poisson and Navier. In this group the preferred postulation scheme was based on the balance of force, the balance of moment of forces and on some «ad hoc» (postulated) restrictions on constitutive equations. For this reason, in some groups of scholars and in some places and times, the contributions by Lagrange, Piola and all their followers were simply ignored.

The process of removal of a source, during the long period when original works were still copied manually, implied as a consequence the total loss of that source. If a librarian decided that a book was not useful, he simply did not ask the amanuenses to copy it, and after some time the book was lost. The process was sometimes made more expedite by simply erasing the «useless» book words from a parchment for reusing it to copy a more important text. This was the destiny of at least one of the most deep contributions of Archimedes to science (we refer to the famous Heiberg palimpsest).

Luckily, the commercial value of the paper on which Piola's works were written did not deserve the attention of any librarian: albeit they were ignored by too many scholars for too much long time, they did remain available in many libraries. Another fortunate circumstance is that Piola's works were originally written in Italian: a not so frequently spoken language, that can however be understood by some scholars capable to understand the sophisticated mathematical reasoning exploited in Piola's works.

As we agree with Piola's epistemological point of view and we can read Italian language, we will try in this chapter to briefly examine how the Principle of Virtual Work (called at the beginning *Principle of Virtual Velocities*) has been used to found (continuum) mechanics and how it allowed for the formulation of the theory of higher-gradient continua since its first formulation by Piola.

Piola's theories of second and higher-gradient continua were ignored, or rejected because believed to be logically inconsistent, by some scholars, followers of Poisson, who refused to accept the Lagrangian postulation of mechanics. The existence of a large group of scholars supported by public institutions did allow for the formation of «resistance» subgroups of scholars who did continue, sometimes even without knowing the existence of his works, Piola's research project. One of the leaders of Lagrangian mechanics was Paul Germain, member of the French Academy of Science in Paris (for more details about the personality of Paul Germain see [124]).

His works in continuum mechanics [88, 90, 58] and his leadership inspired a group of scholars to pursue the Lagrangian scientific program (see the works [1, 86, 5, 12, 159, 15, 50, 49, 135, 2, 157, 87, 98, 99, 100, 69, 123, 122, 125, 80]). In fact Paul Germain was not the only scholar who believed that Lagrange program had to be applied in Continuum Mechanics. A very partial list can be attempted: Lev Davidovič Landau [116] and Leonid Ivanovich Sedov [156, 154, 155] in URSS, Heinz Parkus [133] in Austria, Ekkehart Kröner [113, 114] in Germany, Rivlin [143, 144, 145], Green [102], Pipkin [84, 107, 108, 109, 110, 137, 138, 139], Mindlin [126, 127, 128, 129], Steigmann [54, 162, 166, 163] in USA. Paraphrasing Piola: all listed scholars «could manage to accept the use of the powerful abstract concepts given to us by the genius of Lagrange». A special mention needs to be given here to Richard Toupin (for more details see [43, 37]). He could fully master the mathematical techniques needed to handle variational principles, since he had studied the famous textbook by Landau. He managed to include within the book [170] a chapter about variational principles, albeit he had to accept Truesdell's very negative remarks. This move was very wise as, thanks to this chapter, many scholars could learn about and appreciate them, notwithstanding the negative remarks by Truesdell. Probably, this circumstance became clear to Truesdell and Noll: in fact, in the subsequent textbook [169] the authors simply erase any trace of this very important part of continuum mechanics. Another eminent personality in the valorization of variational principles in Continuum Mechanics is Ernst Hellinger: his fundamental entry in the *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (1913, Bd. IV-4, Hft. 5) has been essential in preserving the scientific heritage of Lagrange and Piola (see [73, 75, 76, 74, 71]). Like Piola was (see the preface of his work written in 1848 as translated in [46]), one must be extremely surprised in discovering that the Principles of Virtual Work and Least Action still need to be supported and that there are ongoing controversies about its validity and role in continuum mechanics. The social phenomenon that led to their rejection in so many scientific groups, notwithstanding that they were adopted as the most fundamental postulate by undisputed scientific authorities like D'Alembert, Lagrange, Hamilton, Landau, Feynman [85], and Sedov, should be investigated. A remark seems necessary in this regard: all the scholars advocating the importance of variational principles in continuum mechanics have a background in Theoretical Physics. Therefore, it seems that in continuum mechanics there are rather distinct cultural subgroups that are competing for affirming their point of view.

As the memory of the works of Piola, Hellinger, Kröner, Sedov, Mindlin, Green, and Pipkin seems to have been somehow removed by some groups of scholars, we believe that it is needed to reaffirm what follows: if one wants to model microscopically strongly inhomogeneous systems, he has (see e.g. [24, 18, 135, 20]) to consider, in the Principle of Virtual Work, internal work functionals (see Germain [89] and Salençon [148]) depending linearly also on second and higher gradients of virtual displacements. Albeit one could remark that this is a statement already found in the work by Piola (published between 1822 and 1856), it is sure that it has been systematically ignored or overlooked. In our opinion, the theory of higher-gradient continua gives a further example of the «erasure, loss or removal and rediscovery» of human (scientific) knowledge. This kind of processes have been observed for many other scientific theories (see e.g. [147]).

Unfortunately, and this is a big loss for science, we do not have enough sources to prove that, already after its first formulation, the Principle of Virtual Work caused a sharp debate. This debate most likely had some bitter moments, as the complete loss of Archytas works may indicate. It is fortunate that an unknown amanuensis did save for posterity the Pseudo-Aristotelian on *The Mechanical Problems*. Attributing it to Aristotle, who could not be in any way his author (in many of his works, Aristotle shows that he is not versed in mathematics), this amanuensis showed to have understood deeply both the importance of the text that he was saving and the nature of human psychology and social behavior. It is sure, however, that the debate about the importance of the Principle of Virtual Work, and its impact on science, has been taking place without interruption until our days.

5.2 An Italian secondary source: Vailati. While underestimating Hellenistic mechanics, he recognizes in it two different ways for studying Statics problems.

In his paper [174] on the history of the Principle of Virtual Work, Vailati starts his introduction with the words

> «The works in which we had preserved the memory of the ideas of Greeks on Mechanics and of the degree of elaboration that, because of their contribution, the theories concerning the equilibrium did reach, can be divided neatly in two categories, that correspond to two, radically different, directions in the manner of considering and solving the questions of Statics».[1]

Therefore, it seems that Vailati believes to be able to distinguish two competing methodologies (or postulations) in the Hellenistic texts of mechanics. On this point we agree completely with him: probably, already in Hellenistic Mechanics both the Principle of Virtual Work together with the balances of force and moment of force were systematically used in the study of problems of Mechanics. Vailati's introduction then continues by stating that:

> «Greek writers did not manage to reach even the enunciation of the most elementary among the principles of Dynamics: the law of Inertia».[2]

On this point we do not agree with Vailati. This statement does not seem well-founded. We will not delve here in this subject, and we refer to [147] for a discussion of the big amount of secondary sources of Hellenistic Science, where the concept of inertia is formulated. Refraining from any polemics, we simply comment here that, in available secondary sources of Hellenistic texts (once translated into Latin or a more modern language), one, obviously, cannot find the exact words (force in particular) which Vailati expected to find for describing the concept. This circumstance has been the cause of many misunderstandings in the history of mechanics. Unfortunately, it has to be remarked that Vailati's opinion has been followed by the majority of scholars. For instance, in the foreword of [56] by Louis de Broglie, one finds stated that:

> «The history of mechanics is one of the most important branches of the history of science. From earliest times man has sought to develop tools that would enable him to add to his power of action or to defend himself against the dangers threatening him. Thus he was unconsciously led to consider the problems of mechanics. So we see the first scholars of ancient times thinking about these problems and arriving more or less successfully at a solution. The motion of the stars which, from the Chaldean shepherds to the great Greek and Hellenistic astronomers, was one of the first preoccupations of human thought, led to the discovery of the true laws of dynamics. As is well known, although the principles of statics

[1] «Gli scritti nei quali ci è stata conservata memoria delle idee dei Greci sulla Meccanica e del grado di elaborazione che raggiunsero per opera loro le teorie relative all'equilibrio, si possono, [...] dividere nettamente in due categorie, corrispondenti a due indirizzi radicalmente diversi nel modo di considerare e di risolvere le questioni di Statica».

[2] «..[..] gli scrittori greci [...] non seppero assorgere neppure all'enunciazione del più elementare dei principi della Dinamica: la legge d'inerzia.»

> had been correctly presented by the old scholars those of dynamics, obscured by the false conceptions of the Aristotelian school, did not begin to see light until the end of the Middle Ages and the beginning of the modern era».

We believe that the perfection obtained by Hellenistic Astronomy could not be attained without a clear understanding of a version of the law of inertia. (see e.g. [146, 134, 9, 10, 19, 117]). Instead Vailati, still in his introduction, adds that:

> «The questions of Statics seem to have been the only ones of which the Greek writers tried to pursue a general treatment, that is scientific in the modern sense of the word;[i)] as, for what that concerns the study of the laws of motions, they seem to having been contented of [ii)] gross descriptions and classifications of phenomena»[3]
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> [i)] The observations on the composition of movements, which one finds in the works by Aristotles, and the more elaborated theories as elaborated by the astronomers on the same subject (see about this subject the classical monograph by Schiaparelli: *On the homocentric spheres by Eudoxus, Callippus and Aristotle*) belong rather more to Geometry than truly to Mechanics. To the same class belong also the researches of Archimedes «On Spirals», that are also based on kinematical assumptions.[4]
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> [ii)] And maybe it is only because of the fault of unskillful compilers and commentators that aforementioned theories and researches did take, later, the aspect and the pretension of scientific theories. And it is interesting, in this regard, to consider as close the considerations developed by Aristotle, in his III book Περὶ οὐρανοῦ, on the distinction between heavy and light bodies, with the following sentence with which he concludes his answer to the 33-rd of the Mechanical Problems, in which one demands: Why does anything get carried its own course when the propulsion does not follow along and keep pushing? [as it is very short, we report here the whole answer as translated by Winter] Perhaps it is clear that the first has done such as to push another, and that another, but it stops when what is propelling the carried object is no longer able to push, and when the weight of the object being carried slopes more than the forward force of the pushing.[5]

Aimed at helping the reader to frame the previously reported sentence by Vailati, we recall here that the Aristotelian text: On the Heavens (Περὶ οὐρανοῦ; De Caelo or De Caelo et Mundo), is the Aristotelian effort to describe the universe. It was

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[3] «[...] le questioni di Statica [...] sembrano esser state le sole delle quali gli scrittori greci di Meccanica abbiano intrapresa una trattazione generale e scientifica nel senso moderno della parola; [i)] poiché, per ciò che riguarda lo studio delle leggi del moto, essi sembrano essersi accontentati di[ii)] grossolane descrizioni e classificazioni dei fenomeni».

[4] «Le osservazioni sulla composizione dei movimenti, che si trovano nelle opere d'ARISTOTELE, e le teorie più elaborate degli astronomi su questo stesso soggetto (cfr. in proposito la classica monografia dello SCHIAPARELLI: *Sulle sfere omocentriche d'Eudosso, Calippo, ed Aristotele*) appartengono piuttosto alla Geometria che non alla Meccanica propriamente detta. Alla stessa classe appartengono pure le ricerche di ARCHIMEDE «Sulle spirali», basate anch'esse su considerazioni cinematiche.»

[5] «E forse solo per colpa dei compilatori e commentatori imperiti che queste assunsero più tardi l'aspetto e la pretensione di teorie scientifiche. E interessante a questo riguardo riavvicinare le considerazioni svolte da ARISTOTELE, nel III libro Περὶ οὐρανοῦ, sulla distinzione fra i corpi pesanti e i leggeri, colla seguente frase con cui egli chiude la sua risposta alla 33a delle Questioni meccaniche, nella quale si domanda: Perché i corpi scagliati non continuano a muoversi indefinitamente? A Greek sentence follows whose translation by Winters is given in the text».

written in 350 BC and includes an astronomical catalog together with Aristotelian ideas (based on those by Eudoxus) on how the terrestrial world was constituted.

It must be remarked that Winter (see [175]), whose interpretation and analysis we support completely, does interpret the answer given in the pseudo-Aristotelian work in the completely opposite way: Winter recognizes in the answer to the 33-rd mechanical problem a description of the principle of inertia.
Let us now quote the relevant part of Vailati's work on which we agree completely. In fact, Vailati describes in a very involute way the differences of the two approaches to Statics found in Hellenistic textbooks. In his work we read:

> «In the first of previously mentioned directions is characterized by the tendency to proceed to the determination of the equilibrium conditions by directly examining, for each mechanism, the relationships that subsist among the compatible motions of its parts and tracing the analogies that, from this point of view, one can find in the various devices to which the human intelligence recurs to win with small efforts the great resistances. It is represented, first of all, by the short work on Mechanical Problems (Μηχανικά Προβλήματα) and secondly by another work, which is not less important for the history of mechanics, which has been transmitted to us only by means of a Latin compilation, having as title, *De ponderibus*, due to Jordanus Nemorarius, a mathematician working during the XIII century.»[6]

The reader will remark how quickly and roughly the formulation of the Principle of Virtual Work is formulated by Vailati. One has to master it fully to recognize that Vailati is talking really about it. One may suspect that Vailati is attributing to his Hellenistic sources the confusion which most likely is, instead, in his own formulation. Correctly, Vailati attributes to Hellenistic science the *De ponderibus*. In fact, he continues by stating that:

> «The Greek origin of the *De ponderibus*, albeit cannot be considered as completely ascertained, is, nevertheless, admitted by authoritative scholars as really likely [...].)»[7]

The description of the second line along which Statics was developed by Hellenistic Science is then described by Vailati:

> «The peculiarities of the second direction are, instead, from one side, the intention to place as exclusive foundation of Statics the consideration of the centers of gravity, and, on the other side, the preoccupation to build this science following the model of the Euclidean Geometry, by presenting it under the form of a series of theorems one linked to the others that can be obtained by deduction of a certain number of fundamental propositions having the same character of immediate evidence as the axioms of Geometry».[8]

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[6] «Il primo dei suddetti due indirizzi è caratterizzato dalla tendenza a procedere alla determinazione delle condizioni di equilibrio esaminando direttamente, per ciascun meccanismo, le relazioni che sussistono tra i moti compatibili delle sue parti e rintracciando le analogie che presentano, da questo punto di vista, i vari ordigni a cui l'industria umana ricorre per vincere con piccoli sforzi grandi resistenze. Esso è rappresentato anzitutto dall'operetta sulle Questioni meccaniche (Μηχανικά Προβλήματα). attribuita ad Aristotele [...] e in secondo luogo da un altro scritto, non meno importante per la storia della Meccanica, che ci è giunto solo attraverso a una compilazione latina, portante il titolo *De ponderibus*, dovuta a Giordano Nemorario, matematico del XIII secolo.»

[7] «L'origine greca del De ponderibus, sebbene non possa considerarsi come completamente accertata, è nondimeno ammessa da critici autorevoli come assai probabile [...].»

[8] «Caratteri del secondo indirizzo sono invece da una parte il proposito di porre ad esclusivo fondamento della Statica la considerazione dei centri di gravità, e dall'altra la preoccupazione

Then Vailati attempts attributing to Archimedes the second direction which he is referring to:

> «This second direction that can be attributed to Archimedes, who probably was his first initiator, is represented not only by his works *On the Equilibrium of Planes* and *On Floating Bodies* but also by the fragments concerning the Statics that are found gathered in the eight book of the Συναγωγή by Pappus».[9]

Finally and unfortunately, Vailati concludes with a statement that seems completely devoid of any philological and logical support:

> «The finding, due to the Orientalist Carra de Vaux, of the previously cited oeuvre by Heron, whose Arabic translation was laying forgotten among the manuscripts of the library of Leiden (to which it had arrived by means of the erudite traveler and Dutch mathematician Golius [1596-1667]) [...], albeit supplying to us a document, being as more precious as it is unique, of a treatment of Statics in which the two methods of which I have spoken cooperate one with the other and are applied simultaneously sometimes also to the solution of one and the same question, does not, however, gives any new information for what that concerns the historical relationships of the two aforementioned directions, which seem to have been developed with perfect independence one from the other, albeit it is difficult to believe that they have not had ever any reciprocal influence.»[10]

It is therefore clear that, even though being blurred by the standard prejudices about Hellenistic science, also Vailati finds in the available sources the traces of the very ancient interaction and counterposition between the Postulation based on the Principle of Virtual Work and the Postulation based on The Balance of Forces and Moments of Forces.

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di costruire questa scienza sul modello della geometria di Euclide esponendola sotto la forma d'una serie di teoremi concatenati ed ottenibili per deduzione da un certo numero di proposizioni fondamentali aventi lo stesso carattere di evidenza immediata che presentano gli assiomi della geometria.»

[9] «Questo secondo indirizzo che fa capo ad Archimede, il quale secondo ogni probabilità ne fu il primo iniziatore, è rappresentato oltreché dalle sue opere *Sull'equilibrio delle figure piane* e *Sui Galleggianti*, anche dai frammenti riguardanti la Statica che si trovano raccolti nel libro ottavo delle Συναγωγή di Pappo.»

[10] «Il ritrovamento, dovuto all'orientalista Carra de Vaux, dell'opera di Erone dianzi citata, la cui traduzione araba giaceva dimenticata tra i manoscritti della biblioteca di Leida (alla quale era pervenuta per mezzo dell'erudito viaggiatore e matematico olandese Golius [1596-1667]) [..] pure fornendoci un documento, tanto piu prezioso in quanto è unico, d'una trattazione della Statica nella quale i due metodi di cui ho parlato cooperano l'uno accanto all'altro e sono promiscuamemte applicati talvolta anche alla soluzione d'una stessa questione, non ha tuttavia recato alcun nuovo dato per ciò che riguarda le relazioni storiche dei due suddetti indirizzi, i quali sembrano essersi svolti con perfetta indipendenza l'uno dall'altro, sebbene sia difficile credere che essi non abbiano mai avuta alcuna influenza reciproca.»

5.3 The Principle of Virtual Work as formulated by Archytas of Tarentum in the Mechanical Problems.

We believe, however, that the presumed date of publication attributed to *The Mechanical Problems* in the Corpus of Aristotle by Winter (who argues that the author is Archytas of Tarentum, 428–347 BC) and the fact that this work solves all the studied mechanical problems by using the Principle of Virtual Work indicate that this Principle was formulated before the concepts of force and moment of forces were introduced in any form.

The Mechanical Problems (in Greek Μηχανικά) is nearly unanimously considered the most ancient oeuvre in mechanical sciences to be produced by Western Culture of which we have any information. Although it was attributed to Aristotle (and transmitted to us in the Corpus Aristotelicum) – albeit it must be acknowledged that there are some disputes about its attribution to Archytas of Tarentum – it is nowadays nearly universally accepted that its author wrote it at the end of the fourth century before Christ, and that already in antiquity it had been falsely attributed to Aristotle. On this false attribution many speculations can be made, and we leave them to the judgment of the reader, as absolutely no evidence is available.

Its introduction begins with a short and traditional appeal to the "marvel" that must guide any kind of research (this is a *topos* in scientific Hellenistic works) and it is argued that, to understand and describe phenomena, it is necessary linking them to their causes. After that, one finds the range of applicability and investigations of the "mechane", and the principles on which this theory is based. Following again another well-established topos, used since the pre-Hellenistic times and also nowadays in physics textbooks, the presentation of arguments is carried out via the traditional scheme constituted by questions and detailed answers, based always on the same fundamental principle. Here is the translation – by Winter [175] – of the first part of the introduction of *The Mechanical Problems:*

> «One marvels at things that happen according to nature, to the extent the cause is unknown, and at things happening contrary to nature, done through art for the advantage of humanity. Nature, so far as our benefit is concerned, often works just the opposite to it. For nature always has the same bent, simple, while use gets complex. So whenever it is necessary to do something counter to nature, it presents perplexity on account of the difficulty, and art [techne] is required. We call that part of art solving such perplexity a mechane.».

Unfortunately, the translation from an ancient language never manages to be really faithful. However, we believe that Winter's translation manages to transmit the original spirit of the text. *The Mechanical Problems*, as translated into English by Winter (written in Italic), with our comments are the following:

- 1. *So first the circumstances about the yoke are confusing, through what cause are the larger yokes more accurate than the smaller?*
- 2. *Why does a balance beam return when you remove the weight if the string is set from the top, and not return, but stay put when supported from below?*
- 3. *Why is it that small forces can move big weights with a lever?*

These first three questions immediately show the style of Greek investigations. Contrarily to what is read very often in some books of History of science, Greek spirit did not delve only into theoretical questions that are not connected with experimental evidence or applications. The principle of lever is explained and formulated having in mind probably its most ancient application: the balance.

- 4. *Why do the men at the middle of the boat move the boat most? Is it because the oar is a lever?*
- 5. *Why does a steering oar, small as it is, and at the end of the boat have such force that with one little handle and the force of one man, and that gentle, it moves the great bulk of ships?*
- 6. *Why when the yardarm is higher does the boat sail faster, with the same sail and the same wind?*
- 7. *Why when out of the wind they wish to run across, the wind not being at their back, do they tighten [send, furl] the sail toward the steers-man and, having made it a foot wide, let it out toward the prow?*

After having explained the principle of the lever, other applications are proposed to the science of sailing, that was so important for all Greek people. Again, one can ask where the myth of Greeks not being interested in applications comes from.

- 8. *Why are round things easier to move than things of other shapes?*
- 9. *Why, with larger circles, whether wheels, pulleys, or rollers, do we move more easily and quickly the things which are lifted or pulled?*
- 10. *Why is an empty balance beam easier to move than a weighted one?*
- 11. *Why do burdens go easier on rollers than on wagons, despite wagons having large diameter wheels and rollers small?*

Here is a peculiarity of the mechanical problems studied in this textbook: the author wants to underline the power of the use of the mathematical concept of circle in solving mechanical problems.

- 12. *Why are spears or pellets carried farther from the sling than from the hand?*
- 13. *Why, around the same capstan, are longer spikes moved more easily? And likewise, thinner winches, by the same force?*
- 14. *Why is wood the same length broken over the knee more easily if you break it while holding it having set it equidistant from the ends rather than being close alongside the knee? And if you set it on the ground and step into it, you break it farther from the hand rather than near?*
- 15. *Why are pebbles at the seashore rounded?*
- 16. *Why is it that the longer a board is, the weaker it gets? and, lifted, bends more, even if the short one – say, two cubits – is thin, and the long one – say, 100 cubits – is thick?*
- 17. *Why are big heavy bodies split by little wedges?*
- 18. *Why, if someone makes two pulleys working together on two blocks, and puts a rope around them in a circle, one block hanging, the other getting lifted up/let down, and hauls on the end of the rope, does he draw up great weights, even if the lifting force is small?*

- 19. *Why, if you put a large ax on wood, and a large burden on that, doesn't it pull apart the wood, no matter how considerable the burden is? But if you raise the ax and hit with it, you split the wood even if you have less weight than you put on the ax in the first place?*
- 20. *Why is it that phalanxes balance big heavy meats hanging from a stub?*
- 21. *Why do doctors pull out teeth more easily even adding weight – that of the tooth-puller – than with the bare hand?*
- 22. *How do they crack nuts easily, without even hitting, in the tools which they make for cracking them?*
- 23. *Why, when both terminal points of a parallelogram carry two vectors, don't they go an equal straight line, but instead one goes many times the other?*
- 24. *It is confusing why the larger circle describes a line equal to a smaller circle's when they have been put on the same center.*
- 25. *Why do they make beds the way they do, sides two-to-one – one side six feet and little more, the other three? And why don't they web them on the diagonal?*
- 26. *Why, given that the weight is the same in each case, is it more difficult to carry long boards at the end on one's shoulder than by the middle?*
- 27. *Why, given two burdens of equal weight, is the one too long harder to carry on the shoulder – even if one carries it at the middle – than if it were shorter?*
- 28. *Why at water wells do they make shadoofs*[11] *as they do?*
- 29. *Why, when two men carry an equal weight on a board or some such, do they not labor the same unless the weight be at the middle, but it is more work for the one of the carriers who is closer?*
- 30. *Why, standing up, do we all first make an acute angle with calf and thigh, and with thigh and torso, and if we don't we cannot arise?*
- 31. *Why is it easier to move something moving than something at rest? As for instance, they pull wagons faster moving than starting. Because it is most difficult to move a weight which is moving the opposite way?*
- 32. *Why do objects thrown stop?*
- 33. *Why does anything get carried its own course when the propulsion does not follow along and keep pushing?*

Questions 32 and 33 and their answers clearly show that the concept of inertia was not ignored by Greek scientists. A harsh debate on this point has been started since Middle Ages, which is still not settled.

- 34. *Why, when thrown, do neither smaller nor larger objects go further, but always must have some symmetry to the one throwing?*
- 35. *Why in eddying water does everything end up getting carried into the middle?*

The reader will note that not all the questions are dealing with the use of machines employed in engineering applications and are aimed to get a "mechanical gain".

[11] Note of the Authors: The shadoof, or sweep, is an early crane-like tool with a lever mechanism, used in irrigation since around 3000 BCE by the Mesopotamians, 2000 BCE by the ancient Egyptians, and later by the Minoans, Chinese (c. 1600 BCE), and others. The sweep is used to lift water from a water source onto land or into another waterway or basin. The mechanism comprises a long counterbalanced pole on a pivot, with a bucket attached to the end of it.

Many questions are aimed at simply understanding how and why some natural phenomena occur, how some object of common use can be employed and why they are useful as they are. Other questions are aimed at explaining real world situations, the functioning of certain instruments or the evolution of the configurations assumed by certain bodies. There are several questions involving sailing techniques, a very important subject for the trading tradition of Greeks.

In conclusion, we can dare to conjecture that *The Mechanical Problems* seems to be a text including exercises to be solved, like some modern Solved Exercises textbooks that are accompanying textbooks where the theory is fully exposed. If this conjecture is well-grounded, it is a pity that the original theoretical textbook was not transmitted to us. The Authors believe also that it would be interesting to elaborate the answers to all the questions with modern methods.

5.4 D'Alembert: the rediscovery of the Principle of Virtual Velocities (or Virtual Work)

While Lagrange, in his *Mécanique analytique* (first edited in 1788) [115] acknowledges Johann Bernoulli as the one who formulated the Principle of Virtual Work already in 1717 in a letter to Varignon, we believe that, most likely, one can track the first modern formulation of the Postulation of Mechanics in which such a principle is the basic postulate in the *Traité de Dynamique* (1768) by Jean-Baptiste Le Rond d'Alembert. As the reader may have already agreed, it is not possible to establish how novel D'Alembert treatise is when compared with Hellenistic sources, which have been lost, and whose existence we can simply conjecture by reading secondary sources. It is in fact rather unlikely, also on the basis of Vailati's considerations, that *The Mechanical Problems* did not produce a large subsequent literature. We can rather safely state that the most relevant Hellenistic sources have been lost (see also [56]): the few fragments and references can be found in very corrupted sources whose interpretation has been very often debated.

Nobody can reasonably doubt, however, that the masterpiece by D'Alembert managed to produce a renewed impetus to mechanics and mechanical sciences. The fame of the *Traité de Dynamique* has been so widespread that the Principle of Virtual Velocity (later and more often called also the Principle of Virtual Work) therein enunciated, is often named after his author: the D'Alembert Principle. It has to be explicitly remarked here that also mechanical systems where a part of energy is dissipated can be described by means of models based on the Principle of Virtual Work: this was probably the reason for which already D'Alembert and Lagrange did prefer to use it, instead of the Principle of Least Action, as their most fundamental postulate.

We believe that it is suitable to quote here an excerpt from the *Traité de Dynamique* (see also [42]). This is an excerpt that reflects the spirit of D'Alembert in considering science. We recognize some stylistic affinities with the words found in Greek scientific texts, notwithstanding how deformed and corrupted they may have reached us. We believe indeed that D'Alembert is providing a "Manifesto" for mechanicians and that, at the same time, he guides mechanicians towards the correct methodologies, techniques and perspective to be used in (Continuum) Mechanics:

> «The certainty of mathematics is an advantage which these sciences owe to the simplicity of their object. [...] the most abstract notions, those which the layman regards as the most inaccessible, are often those which carry with them the greatest light»

Here D'Alembert warns about the need of using abstraction and abstract objects in formulating mathematical models. He then continues stating that:

> «[...] in order to treat following the best possible method [...] any Science whatsoever it is necessary [...] to imagine, in the most abstract and simple way possible, the particular object of this Science, [it is necessary] to suppose and admit in this subject anything else, than the properties which this same Science treats and supposes. »

Subsequently, D'Alembert introduces a "Principle", and this unique Principle is the founding principle of Mechanics, as a deductive theory. This Principle is precisely presented in his treatise together with its important consequences and applications. In his *Mécanique analytique*, Lagrange continues and develops the oeuvre by D'Alembert and continues to call such a principle the Principle of Virtual Velocities, which somehow evokes its ancient Greek origin. Its name was later changed into *Principle of Virtual Work*. Probably, this was not a good choice, it being possibly suggested by some echoes of the debate involving the supporters of the Postulation based on the Laws of Balance.

D'Alembert seems to have suffered because of the criticism formulated by the inductivists. The detractors of falsificationist–deductive physical theories base their arguments always on the same refrain: the deductive theory is too much abstract, "far from experience" and "devoid of physical content". The detractors of "falsificationism"[12] claim that a sound physical theory is based on "solid" experimental grounds and that these solid experimental grounds are encoded within the theory starting from the formulation of its basic principles. Instead, D'Alembert, a follower of the Archimedean vision of science, considers mathematics as a fundamental tool for formulating logically well-posed theories to be, subsequently, verified experimentally.

D'Alembert then continues his *Traité de Dynamique* by stating that:

> «From this standing two advantages result: the principles receive all clarity to which they are susceptible: [and these principles] are finally reduced to the smallest number possible [...] as the object of a Science is necessarily determined, the principles will be more fecund if they will be less numerous [...]. »

[12] Falsificationism is the scientific philosophy claiming that a theory is scientific only if it is based on a set of hypotheses implying a cascade of falsifiable deductions. If a statement cannot be refuted (i.e. it is not falsifiable) then, following the point of view formulated by Karl Raimund Popper – probably the 20th century's most influential epistemologist –, it is not a scientific claim. Popper indeed rejects the classical inductivist description of the scientific method and replaces it with the concept of "empirical falsification". He claims that, while a scientific theory cannot be certainly proven, it should be in principle falsifiable, that is proven to be false by some experiments.

In writing these words D'Alembert refers to an epistemological (meta-)principle dating back to Hellenistic science that is often referred to as the Occam's razor[13]. It has to be remarked here that the supporters of postulations of mechanics based on balance laws are ready to multiply the number of principles on which mechanics should be based.

In the words that we will read next, D'Alembert particularizes his reasoning and refers more specifically to Mechanics, claiming its special need, among all exact sciences, for a clear and solid foundation:

> «Mechanics, above all, seems to be (the Science) which has been more neglected from this point of view: also the great majority of its principles either obscure by them-selves, or enunciated and demonstrated in an obscure way, have given place to several spiny problems [...] I proposed to my-self to move back the limits of Mechanics and to make its approach easier, [I proposed to my-self] not only to deduce the principles of Mechanics from the most clear notions, but also to apply them to new uses, to make it clear at the same time both the inutility of the many and various principles which have been used up to now in Mechanics and the advantage which can be drawn by the combination of others (principles) in order to have the progress of this Science in one word (I want to make clear which is the advantage) of extending the principles by reducing them.»

Choosing some more technical excerpts from the treatise by D'Alembert may need a difficult work of interpretation, as the nomenclature used by D'Alembert is changed nowadays. He applies the Principle of Virtual Velocities to a wide range of cases and shows how powerful it is in inventing new models and providing new explanations for observed phenomena. It will be easier to comment the text written by Lagrange. He indeed spent all his active life to write and rewrite his *Méchanique Analytique*, and his style of presentation is so elegant and precise that one can easily read it even nowadays. Lagrange repeatedly credits D'Alembert for having clarified the importance of the Principle of Virtual Velocities.

We conclude by recalling that, at beginning of D'Alembert's *Traité de Dynamique*, one can find the following statement that is also a scientific program:

> «I have proscribed completely the forces relative to the bodies in motion, entities obscure and metaphysical, which are capable only to throw darkness on a Science which is clear by itself. »

However, D'Alembert is not an extremist. As proven by the following statement that we find in his work:

[13] Occam's razor, Ockham's razor, Ocham's razor (Latin: novacula Occami) or law of parsimony (Latin: lex parsimoniae) is a principle on which the meta-theory guiding the formulation of theories is based: "logical entities and fundamental principles should not be multiplied without necessity". This meta-theoretic principle is linked to the name of the English Franciscan William of Ockham (c. 1287–1347). However, this principle can be found in many authors, and probably dates back – at least – to Hellenistic culture. Equivalent statements are often found like, for instance, "the simplest explanation is most likely the right one", which is often used as common sense statement in discussions. This meta-theoretic razor advocates that when choosing among different competing set of hypotheses giving the same consequences, or predictions, the set including the fewest assumptions has to be chosen. Of course, this does not mean that one has to choose a smaller set of hypotheses when these are producing wrong predictions! Unfortunately, this last misunderstanding is still diffused in modern times.

> «I must warn [the reader] that in order to avoid circumlocutions, I have used often the obscure term force, and some other terms which are used commonly when treating the motion of bodies; but I never wanted to attach to this term any other idea different from those which are resulting from the Principles which I have established both in the Preface and in the first part of this treatise.»

We approve this use of the term "force": it is a purely mathematical object introduced as a linguistic shortcut for: "linear continuous functional defined on the set of admissible virtual velocities". This last is the precise definition of "force" in terms of kinematical quantities, previously defined or introduced as primitive concepts.

5.5 The formulation of Continuum Mechanics by Lagrange

The treatise *Méchanique Analytique* by Lagrange has been his lifelong endeavor. Lagrange continued to write it until the last moments of his life. One reads at the beginning:

> «We will use, in general, the word 'force' or 'power' [puissance] for denoting the cause, whatever it will be, which is impressing or tends to impress motion to the bodies to which it is assumed to be applied.»

The nomenclature chosen by Lagrange has to be deciphered: he uses the word "force" as a synonym of the word "power". This choice may cause (and in fact did cause) a lot of misunderstandings. Some scholars, who wanted to discuss the work of Lagrange in their historical accounts of the development of mechanics, while reading Lagrange textbook without having read its first pages, were confused by Lagrange nomenclature and, therefore, concluded that the ideas of Lagrange were not clear about some fundamental concepts. Some of them arrived to conclude that Lagrange was not capable to distinguish between force and power: this can happen when somebody reads a book of mathematics jumping the pages where definitions and notations are introduced.

Instead, Lagrange tried (unfortunately without success!) to introduce a nomenclature paralleling the nomenclature used by Galileo. In fact, Lagrange (following Galileo) chooses the word 'moment' for meaning what, in the modern nomenclature, is called 'power'. In subsequent initial pages of the *Méchanique Analytique* one reads, in fact,

> «Galileo uses the word 'moment' of a weight or a power applied to a machine the effort, the action, the energy, the 'impetus' of this power for moving this machine [...] and he proves that the moment is always proportional to the power times the virtual velocity, depending on the way in which the power acts.»

This sentence clarifies the use of words chosen by Lagrange. Lagrange adds some comments to his choice of nomenclature:

> «Nowadays one uses more commonly the word 'moment' for the product of a power times the distance along its direction to a point or a line, that is the lever arm by which it acts [...], but it seems to me that the notion of moment given by Galileo and Wallis is much more

natural and general, and I do not see why it was abandoned for replacing it by another which expresses only the value of the moment in certain cases».

The readers who have studied generalized continuum theories, and in particular second gradient continua, will recognize how clearly Lagrange understands the concepts that are at the basis of mechanics: with his choice he wants to open the door to future generalizations of his models. Then Lagrange formulates Archytas-D'Alembert general Principle for Mechanics:

«The Principle of virtual velocities can be formulated in a very general way, as follows: If a system whatsoever constituted by bodies or points each of which is pulled by powers whatsoever is in equilibrium and if one impresses to this system a small motion whatsoever, in virtue of which every point will cover an infinitesimally small distance which will express its virtual velocity, then it will be equal to zero the sum of the powers each multiplied times the distance covered by the points where it is applied along the line of application of this same power, when considering as positive the small distances covered in the same direction as the power and as negative the distances covered in the opposite direction.»

Albeit a modern formulation of this principle usually includes the use of concepts from functional analysis, tensor algebra and mathematical analysis, one must agree that: i) Lagrange's formulation seems so general that it actually includes all versions that have been formulated up to now, ii) it uses the minimum possible mathematical concepts, i.e. only concepts from Euclidean geometry, that are sufficient to express in a rigorous way the principle in its full generality.

5.6 The controversy between Poisson and Piola about the deduction of the equation of the equilibrium of fluids: Piola's contact interactions in continua

Lagrange himself and then Gabrio Piola applied the Principle of Virtual Velocities to deduce the equations of the motion for compressible first gradient fluids, without viscosity. Unfortunately, as already remarked by Vailati, Archimedes had formulated the concepts relative to the equilibrium of fluids in terms of a postulation approach based on the necessary condition for equilibrium that can be deduced from the principle of minimum energy: i.e. the balance of force.

We do not have the sources describing how the Principle of Virtual Work had been studied in Hellenistic science and how (and if) the necessary condition concerning resultant forces and moments of forces had been deduced. However, we know that Hellenistic scientists were masters in the logical deduction of consequences from "prime principles". Therefore, we can imagine that, as variational principles are usually more easily applicable when the formulation is framed in a Lagrangian description, while fluids are most suitably described in an Eulerian description, Archimedes preferred to develop his fluid mechanics by using some important consequences of the Principle of Virtual Velocities. Of course this is a purely conjectural statement and one cannot rely on it too much.

In any case, Poisson, following the transmitted Archimedean tradition, preferred to base the mechanics of fluids on the postulation making use of the balance of forces. For finding the equations of the motion of a fluid, Poisson uses: i) the principle of equal pressures in all directions and ii) a principle characterizing constitutively fluids, which is formulated in the following vague manner: fluids have the capacity «de se reconstituer toujours semblablement á eux-même autour de chaque point» (i.e. of reconstituting themselves always similarly to themselves in the neighbourhood of every point).

Gabrio Piola, in his work [136], criticizes Poisson's point of view and underlines how it could be difficult to transform unambiguously the second of the listed hypotheses enunciated by Poisson into any kind of formula or, equivalently, that many formulas should be used as its mathematical counterpart.

Moreover, within the volume [47] it is possible to find the translation of Piola's "Riflessioni sulle unità di misura e altre quantità concrete" (Reflections on the Units and on the Measures of the Various Physical Quantities[14]). In this manuscript, while talking about incompressible fluids, Piola states that the deduction of governing equations for incompressible fluids is greatly simplified by making use of the method of Lagrange multipliers:

> «[...] using Lagrange's method, [..] for it is enough to know the constraint equations as a results of internal constraints originating from passive forces; and it is not necessary to imagine how these forces work. Hence the savings of much effort, and the consciousness of greater certainty.»

Then he continue with a criticism of the perspective chosen by "French Geometers":

> «French Geometers of our time, whatever the reason, looked for another way: they wanted to put together the general equations of body motion, doing violence to imagination, so that it could give them a representation of the way of acting of nature in the least, starting with the few data we have around his way of acting at a great extent.»

Piola underlines how French Geometers, by basing their considerations on the concept of pressure, did violence to imagination in "putting together" the general equations of body motion. Piola continues by stating that:

> «So they replaced the study of clear and certain effects, with that of obscure and uncertain causes.»

The reader will recognize that this statement echoes those already quoted by D'Alembert. Then Piola concludes with: i) a clear criticism based on the use of Occam's razor:

> «The effort that accompanies these procedures and, what is more, the little confidence that inspires results obtained in the midst of many weakly reasoned hypotheses, and of a continuous neglecting of supposedly small amounts, in relation to others and others, are arguments that must persuade us to prefer and carefully keep in mind the Italian method.»

[14] This was an unpublished manuscript by Gabrio Piola and we are grateful to the courtesy of Politecnico di Milano, Archivi Storici - Area Servizi Bibliotecari di Ateneo, where the original manuscript is conserved into the "Fondo Gabrio Piola", which allowed for its access.

and ii) a well founded indication for future researches, which echoes the one that can be found in the *Method of Mechanical Theorems* by Archimedes:

> «Let it be applied to all the researches attempted by the French scholars: by largely prevailing in effectiveness over all the analyzes they made, let it make open for us a path towards further discoveries: this is a proposition which I do support for a long time with my voice and with my pen.»

While Piola seems to share with Poisson the opinion that considers the "true physical reality" linked to the existence of atoms (or elementary molecules), their opinion becomes divergent for what concerns the concept of force; in fact, Poisson believed that the law of balance of force could give evolution equations for any mechanical system. Piola relates his deduction of evolution equations for a generalized continuum to a homogenization procedure, based on a micro-model where the considered mechanical system is regarded as constituted by interacting material particles. In a sense, Piola accepts to consider the continuum model as a limit of a discrete system. Piola considers this identification process as based on three steps:

1. the expressions of the Principle of Virtual Work for the micro-model and macro-model, once the kinematics of these models are specified;
2. the identification of a specific micro-motion once given a macro-motion, the micro-motion being considered a meaningful representative of all micro-motions which may be correspondent to the given macro-motion;
3. the identification of macro Virtual Work in terms of the micro Virtual Work functionals, and the correspondent identification of macro constitutive equations in terms of the micro geometry and micro material properties.

We prefer leaving Piola ([40], p. 2) to explain the reasons why such a procedure should be accepted:

> «In my opinion it is not safe enough to found the primordial formulas [of a theory] upon hypotheses which, even being very well-thought, do not receive support if not for a far correspondence with some observed phenomena, correspondence obtained by particularizing general statements, [...] indeed the magisterium of nature [i.e. the experimental evidence] at the very small scale, in which we try to conceive the effect of molecular actions, will perhaps actually be very different from what we can mentally realize by means of the images impressed in our senses when experiencing their effects on a larger scale.»

Here Piola envisions the possibility that micro-physics could be ruled by laws that are very different from those that are valid in macro-physics. In this prudent approach he is really safely and wisely prudent. In fact, Quantum Mechanics is based on laws that are very different from those valid at the macro-scale. The reader will however remark that also Quantum Mechanics has been firmly based on Variational Principles. After having warned about the dangers of extending macro-theories to the micro-level, Piola continues as follows:

> «Even let us assume that this difference be very small: a deviation quite insensitive in the fundamental constituents [of matter] – which one needs to consider as multiplied by millions and by billions before one can reach sensible dimensions – can be the ultimate source of notable errors.»

In these words Piola resumes all the difficulties that would have been met by Statistical Mechanics later on. Subsequently, Piola advocates the efficacy of Lagrangian methods:

> «On the contrary, by using Lagrangian methods, one does not consider in the calculations the actions of internal forces but [only] their effects, which are well-known and are not at all influenced by the incertitude about the effects of prime causes, [so that] no doubt can arise regarding the exactitude of the results.»

The idea of Piola seems clear to us: albeit one is neglecting (by ignorance) the details of micro-motions by identifying the macroscopic expression of Virtual Work as an "average" expression of microscopic Virtual Work, the results «are not influenced by the incertitude about the effects of prime causes». Piola then concludes with some epistemological considerations:

> i) «It is true that our imagination may be less satisfied, as [with Lagrangian methods] we do not allow to it to trace the very fundamental origins of the internal motions in bodies: does it really matter? A very large compensation for this deprivation can be found in the certitude of deductions.»
>
> ii) «It has to be remarked that I do not intend for this reason to proscribe the dictation of modern Physics about the internal constitution of bodies and the molecular interactions;»
>
> iii) «When the equations of equilibrium and motion will be established firmly upon indisputable principles, because one has calculated certain effects [i.e. those contained in the Principle of Virtual Velocities] rather than hypothetical expression of forces, I believe to be licit to try to reconstruct anew these equations by means of [suitable] assumptions about such molecular interactions: and if we manage in this way to get results which are identical to those we already know to be true, I believe that these hypotheses will acquire such a high degree of likeliness which one could never hope to get with other methods.»
>
> iv) «Then the molecular Physics will be encouraged to continue with its deductions, under the condition that, being aware of the aberrations of some bald ancient thinkers, it will always mind to look carefully in the experimental observation those hints [coming by the application of Lagrangian macroscopic methods] which are explicit warnings left there to indicate every eventual deviation.»

When dealing with the particular case of the deduction of the equations of fluids, Piola comes back to the essence of his controversy with Poisson:

> «It is now convenient that we hold to think about the difference between our conclusions and those of Poisson.»

Piola clearly states that his analysis gives the same results as those obtained by Euler, and argues about the different results obtained by Poisson:

> «Our analysis, confirming the Eulerian theory, would embrace both the fluid in equilibrium and those in motion, so the liquid as the aeriform fluids. On the contrary, Poisson thought to add new terms to the general equations of fluid motion: and, here is, if I have well understood, the thread of his argument. [Poisson] Begins to say that the equations we already had to express the movement of fluids, were derived using the principle of D'Alembert from those ones of the equilibrium, which presuppose the principle of equal pressure in all directions, a principle experimentally recognized [to be] true only for fluids at rest. [Poisson] continues and asserts that the property to press equally in all directions comes from another property that the fluids fulfill, [that is] always to rebuild themselves similarly to themselves around each of their points. Then [Poisson] rightly reflects that this reconstruction requires a bit

> of time to be done: and even if the interval had very short duration, when the fluid is in motion, that reconstruction cannot be at each instant perfect. In the absence of a perfect reconstruction, according to him, the pressure equal in all directions is missing: therefore those equations which originate from such a principle shall be in default.»

One should be surprised of not finding often in the literature any comment about the treatment of the dynamics of perfect fluids proposed by Poisson and the "uncertainties" in his logical argument and final results, that are the consequence of Poisson's preferred postulation scheme. Piola implies that Poisson, while refusing to use the Principle of Virtual Work, is obliged to look for "*ad hoc corrections*" in his deduction process, adjusting and changing nearly at every step the logical flow of his reasoning. The final criticism that Piola formulates about Poisson's deductions deserves to be reported here:

> «But is it true that the principle of equal pressure in all directions is intimately linked with the regular distribution of the molecules, so that it can not exist one without the other? (Poisson. Traité de Mécanique. Tome II p. 506). I doubt it very much, and I think that, here as well, one has gone forward a bit too far into the deductions: and this because the ideas around that quantity which we call the internal pressure of the fluid have not yet completely clarified.».

In fact, the whole oeuvre by Piola was dedicated to the clarification of the concept of internal state of tension of a continuum body.

5.7 Navier, Cauchy, Poisson, and Saint-Venant *versus* Lagrange, Piola, and George Green *or* postulations based on Balance Laws *versus* postulations based on the Principle of Virtual Velocity.

The discussion developed in this section starts from the analysis developed in [16], which is a really valuable secondary source organizing in an original way the primary sources of Theoretical Mechanics and its applications to Structural Mechanics. Benvenuto transfers in the field of history of science the doctrine developed by William of Moerbeke for the "correct" translation of scientific texts. Benvenuto refers, nearly with the words of the original authors, the principal parts of their results and scientific points of view. The choice of the arguments to be discussed is one of Benvenuto's main contribution to science: and we agree nearly completely with him.

Benvenuto considers the scientific personality of George Green (1793-1841) and his role in the development of continuum mechanics. He clearly states that Green was a follower of Lagrangian mechanics. Therefore, it is not surprising that George Green's contributions to mechanical science support and confirm those already obtained by Piola, which have been up to now. Green's contribution to the problem of determining the most general expression for constitutive equations in linear isotropic elasticity can be found in his works published in between 1834 and 1839 (the interested reader is referred to [104]).

This very particular problem is confronted by Green with a clear and firm epistemological point of view, that can be overlapped to that expressed by Piola (see the excerpts quoted previously). Rephrasing Green we can say that:

> «instead of trying to understand the ultimate root of reality, it is much more prudent and convenient to deduce all possible consequences from some fundamental and general principles whose validity seems well-grounded».

The principle that George Green considers generally valid for all conservative systems is easily formulated. The Principle of Virtual Work involves expressions for expended work that are the first variations of some energy functionals. By quoting the true words by George Green (see [103]):

> «The principle selected as the basis of the reasoning contained in the following paper is this : In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But, this function being known, we can immediately apply the general method given in the *Mécanique Analytique*, and which appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other. One of the advantages of this method, of great importance, is, that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are requisite and sufficient for the complete solution of any problem to which it may be applied.»

The reader will appreciate the Tacitean style (see, e.g., Tacitus' style (as an instrument of thought) in [92]) used by Green. It is possible that one of the reasons for which Green's works had a greater success than Piola's ones, in subsequent literature, is related not only to the fact that Piola used Italian in writing his works while Green English, but also to the difficulties found in reading the complex writing style used by Piola. By rephrasing Benvenuto:

> «The construction by Green is admirable: from a unique principle he deduces all the properties of the constitutive equations for a linear elastic isotropic solid.
>
> Green does not need to introduce further hypotheses on the material particles constituting the considered body, and can avoid all of them. Instead of adding, one after the other, a series of conjectures about the physical reality, the follower of Lagrange manages to account for some general properties that are observed without compromising about unnecessary conjectures about minute details of phenomena.»

As observed again by Benvenuto (who seems not to be aware of the great contribution by Piola to Continuum Mechanics, albeit he cites one time his name in his fundamental historical book) the French champions of the postulation based on balance of forces and molecular mechanics (who were headed by Navier) had to fight a hopeless battle and "had to clash against a physics of solids mainly imaginary". On the other hand, the champions of Lagrangian mechanics (namely Piola, George Green and, later on, Hamilton) could avoid useless and empty discussions about the properties of atoms and use the firm mathematical properties that are common to all conservative phenomena.

It is nowadays clear that French Geometers in contrast with Piola started a controversy against Green and the other British followers of Lagrange. The final result of this scientific controversy could not have a different result than the establishment of the prevalence of variational methods upon the complex postulation scheme started by Navier, Cauchy, Poisson and Saint-Venant. Every effort aimed to found the basic equations of continuum mechanics on molecular microscopic models was abandoned. At that point, a bifurcation occurred: Cauchy, making use of his "tetrahedron argument", managed to formulate a postulation at macro-level which was still based on the balance of force, albeit he had to add the balance of moment of forces as additional postulate. The assumption by Cauchy (see [26]) excluded edge contact forces (see for an in-depth explanation of this fact the papers [52, 45, 51]) and, therefore, his results are limited to continua in which the deformation energy depends on the first gradient of placement only. Instead, and most likely a few years before Cauchy, Piola laid down some more general foundations for Continuum Mechanics.

5.8 Nationalistic Science *or* How Piola's legacy has been blurred because of writing in Italian and counter-posing Italian science to French science.

We believe that an interesting subject to be studied using the scientific method concerns the phenomenon of removal and/or erasure of scientific results in the tradition of one discipline. This kind of phenomena do occur rather often, and surely deserve a careful and deep investigation by using an advanced version of sociological theories. Here, we limit ourselves to remark that it's really surprising to notice that very topical and important contributions to continuum mechanics, as those to be credited to Gabrio Piola, could have been ignored or nearly completely neglected (even by Italian authors) for more than 150 years. A superficial analysis attempting to understand why Gabrio Piola's contributions to mechanical sciences were ignored easily leads to conjecture the following concurring reasons:

i) his works were written in Italian. This was a nationalistic choice, as Piola could surely write fluently at least in French;
ii) the Lagrangian school, which he championed, was rather bitterly countered by the French school, which was, in that historical period, the strongest in the world;
iii) the Italian attitude towards compatriots seems to be rather negative: in general, Italians are rather xenophiles.

The reason why Gabrio Piola wanted to write his works in Italian can be related to his leading cultural and scientific role in the Italian Risorgimento (Resurgence): Piola invested many of his intellectual resources in promoting Italian science, in organizing Pan-Italian scientific conferences, in supporting a unitary vision of Italian science and culture. He wanted to prove that Italian language could be the «vector» of advanced scientific theories, and that Italian scientists were capable to keep up scientists of other nationalities. This conjecture can be proven by observing that Piola's eulogy in memoriam of his "Maestro" Vincenzo Brunacci is concluded by this statement:

> «In the life long efforts that produced Brunacci's works one can recognize a strong commitment "for the advancement of SCIENCES, for the glory of the AUTHOR and for the prestige of ITALY"».

Another excerpt of the same eulogy is also meaningful in this context. In fact, Piola writes that

> «It seemed as if the Spirit of Italy, who was in great sufferance because in that time the most brilliant star of all mathematical sciences, the illustrious Lagrangia, had left the Nation, that Spirit wanted to have the rise of another star, which being born on the banks of the river Arno [he refers to Brunacci], was bound to become the successor of the first one.»

It is very important to remark that Piola refers to Lagrange using the original Italian version of his name, Lagrangia, and to Italy as a unique Nation, by evoking its "Spirit". In the eulogy for Brunacci, Piola focuses also some problems in which the Italian school of mechanics managed to give important contributions:

> «I will content myself to indicate here three Memoirs where he [i.e. Brunacci] examines the doctrine of capillary attraction of Monsieur Laplace, comparing it with that of Pessuti and where, with his usual frankness which is originated by his being persuaded of how well-founded was his case, he proves with his firm reasoning, whatever it is said by the French geometers, some propositions which are of great praise for the mentioned Italian geometer.».

The Nationalism of Piola is revealed by the bold statement «whatever is said by the French geometers», in which he pours his courageous pride of being Italian. It has also to be remarked that Brunacci, Pessutti and Piola already engaged themselves in the study of capillary phenomena and that, some centuries later, the Lagrangian French school headed by Pierre Casal [25] and Paul Germain [88, 90] recovered the Italian spirit, as represented by Lagrange's Principle of Virtual Velocities, to firmly found a continuum model for capillary fluids that cannot be framed into the postulation scheme preferred by Cauchy and Navier.

We can affirm that the finality of every work written by Piola and the principal aim of his scientific activity has been to prove that every mechanical theory can be founded by using the Principle of Virtual Work and that, when one is faced with the problem of postulating a novel model, this principle is the best guidance. Piola was surely the first scientist who, using Lagrange's postulation scheme, defined precisely, for a generic continuum, the dual in work of the gradient of virtual displacement in the referential description. This mathematical object conceived by Piola will then be framed later on in the modern theory of distributions (as defined by Schwartz). In fact, once Tensor Calculus will have been developed by Ricci and Levi-Civita, this dual in work will be called the *Piola stress tensor*[15].

Because of the neglect reserved to Piola's work and the dominance of Cauchy postulation scheme, the greatest part of Piola's most original results (in particular his studies about continua whose deformation energy depends on higher gradients of the strain) are, even nowadays, not known to the great majority of scholars.

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[15] While, most likely, Kirchhoff studied this tensor later than Piola, Truesdell named it after him because, most likely, he read Müller and Timpe [130, p. 23], according to whom Kirchhoff was the first to formulate continuum mechanics based on the integral balance of forces and moment of forces (on this point see also the introduction of [80, p. 301])

5.9 The formulation of N-th Gradient Continuum Mechanics by Piola: an ignored result that is still topical after more than 150 years

When inventing a theory for describing some phenomena, one must start from specifying its kinematics.

The chosen space of configurations gives the mathematical model of the space of states of the physical system which is studied. Once the set of admissible configurations is fixed, then the concept of motion can be easily introduced: it is a function defined in a time interval which maps any time instant into the configuration assumed in that time instant. The most important epistemological question that is debated since the times of Archytas concerns the problem of finding some equations and/or algorithm for calculating, under specified external interactions, initial and boundary conditions, the predicted motion for the studied physical system. The most effective meta-theory that has been proposed up to now builds the "dynamics" (that is the part of the model which predicts the system's motion) of any model following what we could call Lagrange-Hamilton-Rayleigh (LHR-)scheme (see e.g. [44, 7, 48, 53, 41, 31, 36, 17, 23, 72, 21]).

In the LHR-scheme, once fixed the space of configurations and the set of admissible motions, the predicted motion is found by formulating the Principle of Virtual Work. In the case of Generalized Continuum Mechanics, the already formulated version of this principle by Lagrange can be made more specific, by using some concepts of functional analysis (for a discussion of this specific point see [88, 90, 89, 91]).

The Principle of Virtual Work for Generalized Continua

Let us start by postulating the existence of three functionals, each defined on the Cartesian product of the space of admissible motions and the space of variations of admissible motions, respectively called work of internal interactions, work of external interactions and work of inertial interactions. Let us assume that they are linear and continuous with respect to the variations of admissible motions. The work of internal interactions and the work of external interactions are both decomposed into a conservative and non-conservative part. The conservative part is the functional derivative of the mechanical energy of the system with respect to the variations of motions. The non-conservative part is built in terms of the so-called Rayleigh dissipative functional depending on the variations of admissible motions and on the time derivatives of the variations of admissible motions. The non-conservative part of internal and external interactions works are obtained by calculating their functional derivative with respect to the time derivatives of the variations of the admissible motions.

The Principle of Virtual Work states that the predicted motion can be characterized as that motion for which the sum of internal, external and inertial interaction work linear functionals vanish for every admissible variation of motion. The reader is referred for instance to ([6, 53]) for a more technical presentation of the principle that, however, we believe has been presented clearly enough in the previous sentences for a reader who is familiar with the basic ideas of functional analysis. The Principle of Virtual Work reduces any continuum mechanics theory to the formulation of some clear conjectures: the choice of the space of configurations and the choice of conservative, non-conservative and inertia work functionals. There is no *ad hoc* adaptation of the hypotheses while developing the theory, there are not lacking terms that have to be added to the evolution equations *a posteriori* for avoiding logical incongruences and, paraphrasing Lagrange, the mathematical deduction process flows smoothly starting from the initial assumptions to the most detailed predictions. Of course, these predictions must be in agreement with experimental evidence: otherwise, some of the initially postulated expressions for the functionals must be modified accordingly.

The Principle of Virtual Work is very conveniently placed at the basis of continuum mechanics, as it is also the mathematical basis of the analysis of so-called weak solutions for mechanical problems. Weak statements of the boundary-value problems of continuum mechanics and mechanics of structures are unavoidable for the development of finite element techniques, Rayleigh-Ritz and Galerkin-type approximated solutions. Making use of the virtual work principle formulation, based on a clear understanding of the mechanical phenomenology, some problems were profitably studied within the framework of the surface elasticity of energetic boundaries or of the Steigmann-Ogden surface elasticity (see for instance [111, 112, 27, 28, 3, 4, 65, 64, 164, 131]). There are many other fields of modern mechanics that may exploit, or have already exploited, the modeling efficacy of the Principle of Virtual Work. The list could be very long and we limit ourselves here to list some works that have had some influence on our own research efforts

- in the generalization to bio-mechanical and bone growth phenomena of classical mathematical methods used in continuum mechanics [162]
- in the study of large deformations of beams and lattices of beams [165, 167, 70, 83, 81]
- in generalized continuum mechanics [77, 158]
- generalized shell theory [66, 63, 152].

The Principle of Virtual Work, as clearly understood by Piola, has many important consequences, one of which needs to be recalled explicitly here. As the equality to zero of the sum of the internal work and external work functionals must be assured, for instance in mechanical equilibrium configurations, for any admissible variation of placement, it is clear that, given a class of continua characterized by a specific class of internal work functionals, NOT ALL EXTERNAL INTERACTIONS can be applied. Let us make this explicit with an example: if one assumes that the internal work functional is the functional derivative of a deformation energy depending on the Eulerian mass density only (this is the case of Eulerian perfect fluids) then the external interactions which can involve the considered fluid cannot include shear surface contact forces. This fact is well-known and accepted by all mechanicians.

However, the followers of postulations based on the balance of force do not seem to consider this circumstance. Indeed, contact interactions at the external boundaries of continua are NOT determined by "experimental evidence", as sometimes has been claimed, and independently from the postulated form of the internal work functionals. The choice of admissible contact external interactions is implicit in the postulated form of the internal work functional.

Now, following what was done by Piola, we are ready to define N-th gradient continua. These continua have been completely characterized in [52, 45]. We recall here the main results that can be found in the fundamental works by Gabrio Piola and in the aforementioned works, that try to complete his scientific program. First gradient continua are characterized by the validity of the so-called Cauchy postulate, plus Cauchy's implicit assumptions about contact interactions. These implicit assumptions can be stated as simply as follows: contact interactions are expending work only on variations of placement (and therefore they are not expending work, for instance, on the surface normal derivatives of variations of placement) and they are only concentrated on contact surfaces (and therefore there are not, for instance, contact forces per unit line or concentrated on points). Noll's Theorem, which proves the Cauchy postulate, is based on the same assumptions and therefore does not increase really the generality of Cauchy's treatment.

As already proven by Piola, first gradient continua verify Cauchy's so-called postulate, as in this class of continua contact interactions are concentrated on surfaces and are depending on the shape of the contact surface only via its normal. The vice-versa is proven in [50]. More generally, in N-th gradient continua, as envisaged already by Piola (see [42, 39]) the structure of contact forces is (much) more complex and still to be explored. In second (and higher) gradient continua, one can have contact forces concentrated on lines, also. In third (and higher) gradient continua, one can have forces concentrated on points of the contact surface between bodies. However the presence of concentrated forces on points and lines are not the only non-standard (that is: not included in Cauchy continuum mechanics) features of contact interactions in N-th gradient continua.

In fact, as already observed by Germain and fully exploited in [50, 52], internal and external work functionals can be regarded as a particular kind of distributions in the sense of Schwartz. Now, some general theorems by Schwartz (see [153]) prove that, in general, distributions concentrated on embedded manifolds involve not only the values of test functions, but also the values of all derivatives normal to the embedded manifolds of the test functions.

Therefore, as expected, in N-th gradient continua one can have up to N-forces concentrated on contact surfaces, up to (N-1)-forces on contact lines and up to (N-2)-forces on contact points. To be more precise, we will recall that, following Germain, 1-forces are those vectors that expend work on variations of displacements (therefore are the well-known forces), and N-forces are those vectors that expend work on (N-1)-normal derivatives of the variations of displacements. The expression of contact interactions in terms of the many stress tensors needed to describe the state of stress in N-th gradient continua can be found in [52], where the representation theorem for contact forces in terms of the stress tensor and of the normal to the contact surface is generalized.

5.10 Research perspectives as suggested by the lesson given by History of Mechanics

As we have already discussed in a previous section, unfortunately, there is not (yet!) a meta-theory telling us how to build new theories. However, relying on the Latin expression *Historia magistra vitae*, we can exploit the experience gathered in the past (partially described above) efforts made to advance the scientific understanding of reality. We underline that new theories are not only demanded in physics, engineering sciences or the other so-called *hard* sciences. In fact, any intellectual activity of the human being should tend to produce predictive knowledge, as also envisaged by [8]. Indeed, the absence of a predictive knowledge tends to produce in the human being a feeling of impotence, sometimes mixed with marvel. As Giambattista Vico already pondered in his masterpiece (for a detailed description of the personality of Vico we refer to [32])

> «The marvel is daughter of ignorance»[16]
> Giambattista Vico, Scienza Nuova (1725, New Science) (libro I, II, 35; p. 45)

It is suggestive, as also imagined in [38], to conjecture that a kind of minimum principle holds also in social sciences, so that the hope to find equilibrium configurations in social groups, and to model the evolution from an equilibrium to another one, will be, one day, realized somehow similarly to what has been done in mechanics. Such generalizations to social sciences have been already demanded by eminent philosophers of science. To quote again Giambattista Vico:

> «Things, outside their natural state, neither tend to remain nor last.»[17]
> Giambattista Vico, Scienza Nuova (1725, New Science) (libro I, II, 8; p. 39)

As remarked by Edmund Wilson [57]:

> «Vico had read Francis Bacon, and had decided that it ought to be possible to the study of human history methods similar to those proposed by Bacon for the study of the natural world.»

To address less ambitious research perspectives, we list here some of the possible fields of research (those in which we feel to be more expert than we are in mathematical sociology) which wait for innovative ideas and models and which can exploit the insight given by the Principle of Virtual Work.

- mathematically singular models arising in the theory of metamaterials. Using a particular class of Sobolev's spaces called anisotropic Sobolev spaces weak solutions were analysed for gradient incomplete strain gradient elasticity

[16] «La maraviglia è figliuola dell'ignoranza».

[17] «Le cose fuori del loro stato naturale né vi si adagiano né vi durano.»

[62, 61, 59, 67];
- formulation of numerical codes based on a formulation of the Principle of Virtual Work based on the mechanical peculiarities of considered mechanical systems [60, 30, 68, 29, 82, 132, 101, 106, 105, 79, 55, 78]. It is worth to remark that some applications of Rayleigh-Ritz techniques to solution of problems within the modeling of various electromagnetic and mechanical couplings were presented in [35, 121, 118, 119, 120];
- rate dependent and rate independent dissipative behaviors in materials with micro-structure [151, 150, 97, 33, 34, 142, 141, 168, 140, 159];
- discrete formulations for the description of elastic materials with micro-structure [93, 171, 14, 172, 173, 11, 22];
- direct or homogenized continuum modeling for the description of mechanical meta-materials [13, 12, 95, 94, 149, 96, 161, 160, 15].

This Chapter must end with some further considerations about the reasons for which the Principle of Virtual Work has found so many opposers in the community of continuum mechanics. We believe that the main reason can be very easily found in the true nature of the Principle, whose formulation needs sophisticated mathematical concepts and tools, as we have seen in the previous sections. In the continuum mechanics formulation, complex concepts from functional analysis, differential geometry of embedded manifolds and theory of distributions must be mastered in order to be capable of capturing the true mathematical essence of the principle and to be able to apply it to "practical" cases.

The Principle of Virtual Work has been systematically used by physicists to guide their researches. It is suitable here to attract the attention of the reader to an interesting quote from a famous astrophysicist (from "I am Neil deGrasse Tyson" – Reddit AMA Session held on November 13, 2011):

> «There are street artists. Street musicians. Street actors. But there are no street physicists. A little known secret is that a physicist is one of the most employable people in the marketplace – a physicist is a trained problem solver. How many times have you heard a person in a workplace say, "I wasn't trained for this!" That's an impossible reaction from a physicist, who would say, instead, "Cool. A problem I've never seen before. Let's see how I can figure out how to solve it!" Oh, and, have fun along the way.»

Unfortunately, there are many mechanicians who refused to adopt the right attitude towards this problem and preferred to look for simplifications and/or shortcuts. The main shortcut that was attempted to overcome the mathematical difficulties implied by the use of the Principle of Virtual Work, that is the use of the law of balance of force, did not even manage to handle the case of solid continua. In passing from Euler's fluids to first gradient solids it was indeed necessary to add the (extra) balance of moment of forces. Then, for every generalization it was necessary to add, one after the other, a series of extra balances of "something". This "something" was chosen to be what was necessary to get the lacking equations. An extra set of constitutive equations had to be introduced to this balance, with some *ad hoc* compatibility or physical consistency demands. Indeed, the entropy inequality was artificially introduced in mechanics exactly to handle this situation, that was simply caused by the absence of an internal energy functional postulated at the beginning, which was instead painfully recovered only *a posteriori*.

Only the school of theoretical physicists, guided by Feynman and Landau, did keep away from this useless effort of avoiding variational principles and remained faithful to the ancient lesson of Archytas. Physicists are continuing the long tradition of looking for the most appropriate model to describe phenomena.

> *«When scientifically investigating the natural world, the only thing worse than a blind believer is a seeing denier.»*
>
> Neil deGrasse Tyson, Death by black hole: And other cosmic quandaries (WW Norton & Company, 2007).

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