9

Heat Conductors

To remove the paradox of classical Fourier theory relating to the instantaneous propagation of thermal disturbances, Cattaneo [59] suggested a generalized Fourier law, which he justified by means of statistical considerations. This constitutive equation relates the heat flux, its time derivative, and the temperature gradient. It is referred to as the *Cattaneo–Maxwell relation*, since Maxwell [254] previously obtained it but immediately eliminated the term involving the time derivative of the heat flux. It leads to a hyperbolic heat equation.

On the basis of Coleman's theory for materials with memory [67], a nonlinear model for rigid heat conductors was developed by Gurtin and Pipkin [191]. In this work, the authors derived a linearization of their theory, corresponding to infinitesimal temperature gradients, which yields a linearized constitutive equation for the heat flux in terms of the history of the temperature gradient. This linear relation is a generalization of the Cattaneo–Maxwell equation.

The Gurtin–Pipkin approach is built into the general theory developed in Chaps. 5 and 7. We refer in particular to the discussion centering on (5.1.8).

More recent work on this topic includes [4, 12, 13].

9.1 Constitutive Equations for Rigid Heat Conductors

A rigid heat conductor with memory effects within the linear theory developed in [191] and considered also in [102] is characterized by the constitutive equation

$$\mathbf{q}(\mathbf{x}, t) = -\int_0^\infty \mathbf{k}(s)\mathbf{g}^t(\mathbf{x}, s)ds, \tag{9.1.1}$$

where $\mathbf{x}$ denotes the position vector, $t \in \mathbb{R}^+$ is the time variable, and $\mathbf{g} = \nabla\theta$ is the temperature gradient, expressed in terms of θ, which denotes the absolute tem-

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perature.* Moreover, $\mathbf{g}^t(\mathbf{x}, s) = \mathbf{g}(\mathbf{x}, t - s)\ \forall s \in \mathbb{R}^{++}$ denotes the past history of the temperature gradient. We consider the heat flux relaxation function $\mathbf{k} : \mathbb{R}^+ \to \mathrm{Sym}$, such that $\mathbf{k} \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ [102, 115, 147] and

$$\lim_{t\to\infty} \mathbf{k}(t) = \mathbf{0}.$$

Referring to the discussion around (5.1.8) and (5.1.9), we introduce the integrated history of $\mathbf{g}$ [191], which is the function $\overline{\mathbf{g}}^t(\mathbf{x}, \cdot) : \mathbb{R}^+ \to \mathbb{R}^3$ defined by

$$\overline{\mathbf{g}}^t(\mathbf{x}, s) = \int_{t-s}^{t} \mathbf{g}(\mathbf{x}, \tau) d\tau. \tag{9.1.2}$$

Note that

$$\frac{\partial}{\partial s}\overline{\mathbf{g}}^t(\mathbf{x}, s) = \mathbf{g}^t(\mathbf{x}, s).$$

The constitutive equation (9.1.1) can be expressed in terms of $\overline{\mathbf{g}}^t$, by means of an integration by parts, yielding

$$\mathbf{q}(\mathbf{x}, t) = \int_0^\infty \mathbf{k}'(s)\overline{\mathbf{g}}^t(\mathbf{x}, s) ds. \tag{9.1.3}$$

The evolution problem for a rigid heat conductor is governed by the energy equation (see (3.3.7))

$$\dot{e}(\mathbf{x}, t) = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t), \tag{9.1.4}$$

where r denotes the external heat supply per unit volume and e is the internal energy per unit volume. This relation is also given by (5.1.2), since the Lagrangian and Eulerian descriptions coincide for rigid bodies. We take the constant mass density ρ to be unity. Equation (5.1.3) can be rewritten as

$$\theta\dot{\eta} \geq \dot{e} + \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g}, \tag{9.1.5}$$

with the aid of (9.1.4), where η is the entropy per unit volume. The internal energy is assumed to be given by the constitutive equation

$$e(\mathbf{x}, t) = \alpha_0\vartheta(\mathbf{x}, t) + \int_0^\infty \alpha'(s)\vartheta^t(\mathbf{x}, s) ds, \quad \vartheta = \theta - \Theta_0, \tag{9.1.6}$$

where $\alpha' \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ and Θ_0 is a reference temperature, uniform in the body. The internal energy relaxation function is given by

$$\alpha(t) = \alpha_0 + \int_0^t \alpha'(\tau) d\tau \qquad \forall\, t \in \mathbb{R}^+, \qquad \alpha_0 \in \mathbb{R}^{++}. \tag{9.1.7}$$

* Consider (7.1.23), neglecting the first two integrals on the right and carrying out an integration by parts in the third integral. In the linear approximation, $\mathbf{d}(t)$, defined in general by (5.1.1), is given by $-\mathbf{g}/\Theta_0^2$, where Θ_0 is defined after (9.1.6). Absorbing the constant $\rho\Theta_0^2$ into the kernel, we see that this relation is the inverted form of (9.1.1).

We introduce the pseudoenergy [102]

$$\psi(\mathbf{x},t) = \Theta_0(e - \Theta_0\eta),$$

which will play the role of the free energy in the present context. It follows from (9.1.5) that

$$\dot{\psi} - \dot{e}\vartheta\frac{\Theta_0}{\theta} + \mathbf{q}\cdot\mathbf{g}\frac{\Theta_0^2}{\theta^2} \le 0. \tag{9.1.8}$$

The approximate theory developed in [102] requires a linearization of the Clausius–Duhem inequality (9.1.8) to the form

$$\dot{\psi}(\mathbf{x},t) \le \dot{e}(\mathbf{x},t)\vartheta(\mathbf{x},t) - \mathbf{q}(\mathbf{x},t)\cdot\mathbf{g}(\mathbf{x},t). \tag{9.1.9}$$

We refer to [102] for a detailed derivation of (9.1.9). This approximate form of the Clausius–Duhem relation will be used in the present chapter. By introducing the internal dissipation function $D(t) \ge 0$, we can write (9.1.9) as an equality

$$\dot{\psi}(\mathbf{x},t) + D(t) = \dot{e}(\mathbf{x},t)\vartheta(\mathbf{x},t) - \mathbf{q}(\mathbf{x},t)\cdot\mathbf{g}(\mathbf{x},t). \tag{9.1.10}$$

A rigid heat conductor, characterized by the constitutive equations (9.1.1) and (9.1.6), is a simple material, and therefore, its behavior can be described by means of states and processes, as described in Chaps. 3 and 4. We shall introduce these concepts in a more systematic way here than was done in Chap. 8.

In the following, the dependence on $\mathbf{x}$ will be understood.

9.1.1 States in Terms of $\vartheta^t(s)$ and $\mathbf{g}^t$

We observe that in the linear theory, the internal energy depends on all the history $\vartheta^t(s) = \vartheta(t-s)\,\forall s \in \mathbb{R}^+$, that is, both on the past history $\vartheta^t(s) = \vartheta(t-s)\,\forall s \in \mathbb{R}^{++}$ and on the present value $\vartheta(t)$, while the present value of the temperature gradient does not have an equivalent role in the constitutive equation for the heat flux. We shall identify the history of any function f up to and including time t, $f^t(s) = f(t-s)\,\forall s \in \mathbb{R}^+$, with the pair $(f(t), f^t)$.

The thermodynamic state at time t and at any fixed point $\mathbf{x}$ of the body, taking into account (9.1.1) and (9.1.6), is

$$\sigma(t) = (\vartheta(t),\ \vartheta^t,\ \mathbf{g}^t), \tag{9.1.11}$$

The set of possible states is denoted by Σ.

The kinetic process of duration $d_P \in \mathbb{R}^+$ is the map, piecewise continuous on the time interval $[0, d_P)$, defined by

$$P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \qquad \forall \tau \in [0, d_P), \tag{9.1.12}$$

where $\dot{\vartheta}_P(\tau)$ is the derivative of the temperature with respect to τ and the temperature gradient $\mathbf{g}_P(\tau)$ is, in particular, defined also for $\tau = 0$, corresponding to the instant when P is applied to the body. The set of all accessible processes for the body is denoted by Π. There exists in Π every type of restriction of a process P, of duration

d_P, to an interval $[\tau_1, \tau_2) \subset [0, d_P)$, denoted by $P_{[\tau_1,\tau_2)} \in \Pi$; if $[\tau_1, \tau_2) \equiv [0, t)$ we shall denote $P_{[0,t)}$ by P_t. The evolution function $\hat{\rho} : \Sigma \times \Pi \to \Sigma$ is defined by the property that $\sigma^f = \hat{\rho}(\sigma^i, P) \in \Sigma$, where $\sigma^i \in \Sigma$ is the initial state and σ^f is the final state obtained by applying the process $P \in \Pi$.

Different choices of state for a heat conductor with memory have been used in [147, 191, 279]. Following [102], we can choose as the thermodynamic state that given by (9.1.11), but where the integrated history $\overline{\mathbf{g}}^t$ takes the place of $\mathbf{g}^t$.

The set of possible states Σ is the set of states $\sigma(t) = (\vartheta(t), \vartheta^t, \mathbf{g}^t)$ such that the corresponding e and $\mathbf{q}$ are both finite, so that

$$\left| \int_0^\infty \alpha'(s)\vartheta^t(s)ds \right| < \infty, \qquad \left| \int_0^\infty \mathbf{k}(s)\mathbf{g}^t(s)ds \right| < \infty. \tag{9.1.13}$$

Let $\sigma(t) = (\vartheta(t), \vartheta^t, \mathbf{g}^t)$ be an initial state of Σ. The evolution function gives a family of states induced by a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ defined for every $\tau \in [0, d_P)$ and applied at the generic time t; in particular, the temperature gradient is the assigned function

$$\mathbf{g}_P : [0, d_P) \to \mathbb{R}^3, \qquad \mathbf{g}_P(\tau) = \mathbf{g}(t + \tau) \qquad \forall \tau \in [0, d_P). \tag{9.1.14}$$

The process P also determines the evolution of temperature according to

$$\vartheta_P : (0, d_P] \to \mathbb{R}, \qquad \vartheta_P(\tau) = \vartheta(t) + \int_0^\tau \dot{\vartheta}_P(\xi)d\xi \qquad \forall \tau \in (0, d_P]; \tag{9.1.15}$$

thus, at each instant $\tau' \equiv t + \tau \leq t + d_P$, the final value of the temperature, yielded by ϑ^t and $\dot{\vartheta}_P$ and denoted by $\vartheta_f(\tau') = (\vartheta_P * \vartheta)(\tau')$, is given by

$$\vartheta_f(t + d_P - s) = (\vartheta_P * \vartheta)(t + d_P - s) = \begin{cases} \vartheta_P(d_P - s), & 0 \leq s < d_P, \\ \vartheta(t + d_P - s), & s \geq d_P, \end{cases} \tag{9.1.16}$$

where the symbol $*$ denotes the continuation of histories with any process. Similarly, the final value of the temperature gradient $\mathbf{g}_f(\tau') = (\mathbf{g}_P * \mathbf{g})(\tau') \, \forall \tau' \equiv t + \tau < t + d_P$ depends on $\mathbf{g}^t$ and $\mathbf{g}_P$ and is expressed by

$$\mathbf{g}_f(t + d_P - s) = (\mathbf{g}_P * \mathbf{g})(t + d_P - s) = \begin{cases} \mathbf{g}_P(d_P - s), & 0 < s \leq d_P, \\ \mathbf{g}(t + d_P - s), & s > d_P, \end{cases} \tag{9.1.17}$$

by virtue of (9.1.14).

Given two histories of the temperature and of the temperature gradient, their static continuations of duration $a \in \mathbb{R}^+$ are defined by

$$\vartheta^{t_a} = \begin{cases} \vartheta^t(s - a), & s > a, \\ \vartheta(t), & s \in [0, a], \end{cases} \qquad \mathbf{g}^{t_a} = \begin{cases} \mathbf{g}^t(s - a), & s > a, \\ \mathbf{g}(t), & s \in [0, a]. \end{cases} \tag{9.1.18}$$

The static continuations applied to (9.1.6) and (9.1.1) yield

$$
\begin{aligned}
e(t+a) &= \alpha(a)\vartheta(t) + \int_0^\infty \alpha'(a+\xi)\vartheta^t(\xi)d\xi, \\
\mathbf{q}(t+a) &= -\mathbf{K}(a)\mathbf{g}(t) - \int_0^\infty \mathbf{k}(a+\xi)\mathbf{g}^t(\xi)d\xi,
\end{aligned}
\tag{9.1.19}
$$

where we have denoted the thermal conductivity tensor by

$$
\mathbf{K}(t) = \int_0^t \mathbf{k}(\xi)d\xi. \tag{9.1.20}
$$

Consequently, by virtue of (9.1.13) and (9.1.19), the set of possible states Σ is characterized by

$$
\Gamma_\alpha = \left\{\vartheta^t : (0,\infty) \to \mathbb{R};\ \left|\int_0^\infty \alpha'(s+\tau)\vartheta^t(s)\,ds\right| < \infty \quad \forall\tau \in \mathbb{R}^+\right\} \tag{9.1.21}
$$

and

$$
\Gamma_k = \left\{\mathbf{g}^t : (0,\infty) \to \mathbb{R}^3;\ \left|\int_0^\infty \mathbf{k}(s+\tau)\mathbf{g}^t(s)\,ds\right| < \infty \quad \forall\tau \in \mathbb{R}^+\right\}, \tag{9.1.22}
$$

where t is a parameter.

In particular, if we consider the constant histories $(\vartheta, \vartheta^\dagger)$, where $\vartheta^t(s) = \vartheta^\dagger = \vartheta\,\forall s \in \mathbb{R}^{++}$, and $\mathbf{g}^t(s) = \mathbf{g}^\dagger = \mathbf{g}\,\forall s \in \mathbb{R}^{++}$, the internal energy (9.1.6) and the heat flux (9.1.1) assume the values

$$
e(t) = \alpha_\infty\vartheta, \qquad \mathbf{q}(t) = -\mathbf{K}_\infty\mathbf{g}, \tag{9.1.23}
$$

where $\alpha_\infty = \lim_{t\to\infty}\alpha(t)$ and $\mathbf{K}_\infty = \lim_{t\to\infty}\mathbf{K}(t)$ are the asymptotic values of α and $\mathbf{K}$ given by (9.1.7) and (9.1.20). These limits are assumed to be finite.

9.1.2 Constitutive Equations in Terms of States and Processes

We now present a decomposition for rigid heat conductors similar to (8.2.8) for viscoelastic solids, (8.9.18), (8.10.5) for fluids but with more detailed discussion in the present case. The constitutive equations (9.1.1) and (9.1.6) define two linear functionals $\tilde{\mathbf{q}} : \Gamma_k \to \mathbb{R}^3$ and $\tilde{e} : \mathbb{R}\times\Gamma_\alpha \to \mathbb{R}$ such that

$$
\tilde{\mathbf{q}}(\mathbf{g}^t) = -\int_0^\infty \mathbf{k}(s)\mathbf{g}^t(s)ds, \quad \tilde{e}(\vartheta(t), \vartheta^t) = \alpha_0\vartheta(t) + \int_0^\infty \alpha'(s)\vartheta^t(s)ds, \tag{9.1.24}
$$

which give the set of heat fluxes and internal energies related to any past history of the temperature gradient $\mathbf{g}^t \in \Gamma_k$ and the temperature $\vartheta^t \in \Gamma_\alpha$.

If P_τ is a process of duration τ applied at the initial time t, it is defined in the time interval $[t, t+\tau)$; if $(\vartheta(t),\ \vartheta^t,\ \mathbf{g}^t)$ is the initial state, then the final values of e and $\mathbf{q}$ are given by

$$
\begin{aligned}
e(t+\tau) &= \alpha_0 \vartheta(t+\tau) + \int_0^\infty \alpha'(s)\vartheta^{t+\tau}(s)ds, \\
\mathbf{q}(t+\tau) &= -\int_0^\infty \mathbf{k}(s)\mathbf{g}^{t+\tau}(s)ds.
\end{aligned}
\tag{9.1.25}
$$

We consider each of the integrals in (9.1.25) as the sum of two integrals, the first of which is evaluated between 0 and τ and the second between τ and ∞. An integration by parts in the first one, using (9.1.15), yields

$$
\begin{aligned}
e(t+\tau) &= \alpha(\tau)\vartheta(t) + \int_0^\infty \alpha'(\tau+\xi)\vartheta^t(\xi)d\xi + \int_0^\tau \alpha(\tau-\eta)\dot{\vartheta}_{P_\tau}(\eta)d\eta \\
&= \hat{e}(\vartheta(t), \vartheta^t; 0_\tau^\dagger) + \hat{e}(0, 0^\dagger; \dot{\vartheta}_{P_\tau}),
\end{aligned}
\tag{9.1.26}
$$

where $0^\dagger$ is the zero past history for the temperature, given by $0^\dagger(s) = 0\ \forall s \in \mathbf{R}^{++}$, $0_\tau^\dagger$ denotes the zero process with duration τ, $\dot{\vartheta}_{P_\tau}(\eta) = 0_\tau^\dagger(\eta) = 0\ \forall \eta \in [0, \tau)$, and where

$$
\begin{aligned}
\hat{e}(\vartheta(t),\ \vartheta^t; 0_\tau^\dagger) &= \alpha(\tau)\vartheta(t) + \int_0^\infty \alpha'(\tau+\xi)\vartheta^t(\xi)d\xi, \\
\hat{e}(0, 0^\dagger; \dot{\vartheta}_{P_\tau}) &= \int_0^\tau \alpha(\tau-\eta)\dot{\vartheta}_{P_\tau}(\eta)d\eta.
\end{aligned}
\tag{9.1.27}
$$

In $(9.1.26)_2$, we have a superposition of two effects, the first of which depends on the process through $\dot{\vartheta}_{P_\tau}$, while the second is expressed in terms of the initial state through the initial data of the temperature. Explicitly, $\hat{e}(0, 0^\dagger; \dot{\vartheta}_{P_\tau})$ is due to the part of the process, characterized by $\dot{\vartheta}_{P_\tau}$, starting from the initial state with a null temperature history $(\vartheta(t),\ \vartheta^t) = (0, 0^\dagger)$, while $\hat{e}(\vartheta(t),\ \vartheta^t; 0_\tau^\dagger)$ is related to a process with $\dot{\vartheta}_{P_\tau} = 0_\tau^\dagger$, applied to the history $(\vartheta(t),\ \vartheta^t)$. Also, after the same manipulations,

$$
\begin{aligned}
\mathbf{q}(t+\tau) &= -\int_0^\infty \mathbf{k}(\tau+\xi)\mathbf{g}^t(\xi)d\xi - \int_0^\tau \mathbf{k}(\tau-\eta)\mathbf{g}_{P_\tau}(\eta)d\eta \\
&= \hat{\mathbf{q}}(\mathbf{g}^t; \mathbf{0}_\tau^\dagger) + \hat{\mathbf{q}}(\mathbf{0}^\dagger; \mathbf{g}_{P_\tau}),
\end{aligned}
\tag{9.1.28}
$$

where $\mathbf{0}^\dagger$ denotes the zero past history for $\mathbf{g}$, that is, $\mathbf{0}^\dagger(s) = \mathbf{0}\ \forall s \in \mathbb{R}^{++}$, and $\mathbf{0}_\tau^\dagger$ is the zero process of duration τ, i.e., $\mathbf{g}_{P_\tau}(\eta) = \mathbf{0}_\tau^\dagger(\eta) = \mathbf{0}\ \forall \eta \in [0, \tau)$, and where

$$
\begin{aligned}
\hat{\mathbf{q}}(\mathbf{g}^t; \mathbf{0}_\tau^\dagger) &= -\int_0^\infty \mathbf{k}(\tau+\xi)\mathbf{g}^t(\xi)d\xi, \\
\hat{\mathbf{q}}(\mathbf{0}^\dagger; \mathbf{g}_{P_\tau}) &= -\int_0^\tau \mathbf{k}(\tau-\eta)\mathbf{g}_{P_\tau}(\eta)d\eta.
\end{aligned}
\tag{9.1.29}
$$

Thus, also for the heat flux we have a superposition of two effects. The term $\hat{\mathbf{q}}(\mathbf{0}^\dagger; \mathbf{g}_{P_\tau})$ in $(9.1.29)_2$ expresses the heat flux due to the process P_τ characterized by $\mathbf{g}_{P_\tau}$ and applied to the initial state corresponding to a null past history of the temperature gradient $\mathbf{0}^\dagger$, whereas the quantity $\hat{\mathbf{q}}(\mathbf{g}^0; \mathbf{0}_\tau^\dagger)$ is the heat flux obtained by the process $\mathbf{g}_{P_\tau} = \mathbf{0}_\tau^\dagger$ applied to the initial state characterized by the past history $\mathbf{g}^0$.

9.1.3 Equivalent Histories and Minimal States

We now consider the concepts introduced in Definition 4.1.2 and discussed in Sects. 7.4, 8.3 (see also Theorems 8.9.2 and 8.10.2) for rigid heat conductors.

Definition 9.1.1. *Two states $\sigma_j(t) = (\vartheta_i(t), \vartheta_i^t, \mathbf{g}_j^t)$, $j = 1, 2$, of a rigid heat conductor, characterized by the constitutive equations (9.1.24), are equivalent if*

$$\begin{gathered}\tilde{e}(\vartheta_P(\tau), (\vartheta_P * \vartheta_1)^{t+\tau}) = \tilde{e}(\vartheta_P(\tau), (\vartheta_P * \vartheta_2)^{t+\tau}),\\ \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g}_1)^{t+\tau}) = \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g}_2)^{t+\tau})\end{gathered} \tag{9.1.30}$$

for every process $P \in \Pi$ and for every $\tau > 0$.

Thus, the definition of equivalent states ensures the same final value both for the internal energy and for the heat flux, whatever may be their continuations obtained by means of any process, of arbitrary duration, applied to both of them.

Theorem 9.1.2. *Two states $\sigma_j(t) = (\vartheta_j(t), \vartheta_j^t, \mathbf{g}_j^t)$, $j = 1, 2$, are equivalent if and only if*

$$\begin{gathered}\vartheta_1(t) = \vartheta_2(t), \qquad \int_0^\infty \alpha'(\tau+\xi)[\theta_1^t(\xi) - \theta_2^t(\xi)]d\xi = 0,\\ \int_0^\infty \mathbf{k}(\tau+\xi)[\mathbf{g}_1^t(\xi) - \mathbf{g}_2^t(\xi)]d\xi = \mathbf{0}\end{gathered} \tag{9.1.31}$$

for every $\tau > 0$.

Proof. The equivalence conditions (9.1.30), which are required to be satisfied by the histories, must be evaluated using (9.1.15)–(9.1.17), where the duration of the process d_P is replaced by τ. Consider each integral between zero and infinity as the sum of two integrals, the first between zero and τ and the other between τ and infinity, as we have done in (9.1.26) and (9.1.28). Thus, for every P we have

$$\begin{aligned}\alpha(\tau)[\vartheta_1(t) - \vartheta_2(t)] + \int_\tau^\infty \alpha'(s)[\theta_1^{t+\tau}(s) - \theta_2^{t+\tau}(s)]ds = 0,\\ \int_\tau^\infty \mathbf{k}(s)[\mathbf{g}_1^{t+\tau}(s) - \mathbf{g}_2^{t+\tau}(s)]ds = \mathbf{0}.\end{aligned}$$

The arbitrariness of τ yields

$$\begin{gathered}\vartheta_1(t) = \vartheta_2(t), \qquad \int_0^\infty \alpha'(\tau+\xi)\vartheta_1^t(\xi)d\xi = \int_0^\infty \alpha'(\tau+\xi)\vartheta_2^t(\xi)d\xi,\\ \int_\tau^\infty \mathbf{k}(\tau+\xi)\mathbf{g}_1^t(\xi)d\xi = \int_0^\infty \mathbf{k}(\tau+\xi)\mathbf{g}_2^t(\xi)d\xi,\end{gathered} \tag{9.1.32}$$

for any $\tau > 0$. Using these same relations, the converse also follows. □

We observe that the history $(0,\, \vartheta^t)$ characterized by a zero instantaneous value and a given past history of the temperature and past history of the temperature gradient $\mathbf{g}^t$ is equivalent to the zero history $(0, 0^\dagger)$ of ϑ and the zero past history $\mathbf{0}^\dagger$ of $\mathbf{g}$ if

$$\begin{aligned}
\int_\tau^\infty \alpha'(s)\vartheta^{t+\tau}(s)ds &= \int_0^\infty \alpha'(\tau+\xi)\vartheta^t(\xi)d\xi = 0,\\
\int_\tau^\infty \mathbf{k}(s)\mathbf{g}^{t+\tau}(s)ds &= \int_0^\infty \mathbf{k}(\tau+\xi)\mathbf{g}^t(\xi)d\xi = \mathbf{0}.
\end{aligned} \tag{9.1.33}$$

Thus, from (9.1.32), (9.1.33) it follows that two states $\sigma_j(t) = (\vartheta_j(t),\, \vartheta_j^t,\, \mathbf{g}_j^t)$, $j = 1, 2$, are equivalent in the sense of Definition 9.1.1 if the differences $\vartheta^t = \vartheta_1^t - \vartheta_2^t$ and $\mathbf{g}^t = \mathbf{g}_1^t - \mathbf{g}_2^t$ satisfy (9.1.33) with $\vartheta(t) = \vartheta_1(t) - \vartheta_2(t) = 0$; in other words, two states $\sigma_j(t)$, $j = 1, 2$, are equivalent if the state $\sigma(t) = \sigma_1(t) - \sigma_2(t) = (\vartheta(t),\, \vartheta^t,\, \mathbf{g}^t)$ is equivalent to the zero state $(0, 0^\dagger, \mathbf{0}^\dagger)$.

Furthermore [277], we see that two pairs of histories $(\vartheta_j(t),\, \vartheta_j^t)$, $j = 1, 2$, with $\vartheta_1(t) = \vartheta_2(t)$, and two past histories $\mathbf{g}_j^t$, $j = 1, 2$, whose differences $\vartheta^t = \vartheta_1^t - \vartheta_2^t$ and $\mathbf{g}^t = \mathbf{g}_1^t - \mathbf{g}_2^t$ satisfy the relations (9.1.33), represent the same state $\sigma(t)$. Consequently, this state expresses the "minimum" set of variables that give a univocal relation between the process $P(\cdot) = (\dot{\vartheta}_P(\cdot), \mathbf{g}_P(\cdot))$, defined in $[0, \tau)$, and the internal energy $e(t+\tau) = \tilde{e}(\vartheta_P(\tau), (\vartheta_P * \vartheta)^{t+\tau})$ and the heat flux $\mathbf{q}(t+\tau) = \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g})^{t+\tau})$ for every $\tau > 0$. Finally [90, 176], denoting by Γ_{α_0} and Γ_{k_0} the subsets of the past histories of Γ_α and Γ_k satisfying (9.1.33), respectively, and by $\Gamma_\alpha/\Gamma_{\alpha_0}$ and Γ_k/Γ_{k_0} their usual quotient spaces, the state σ of a rigid heat conductor is characterized as $(\vartheta(t),\, \vartheta^t,\, \mathbf{g}^t) \in \Sigma = \mathbb{R} \times (\Gamma_\alpha/\Gamma_{\alpha_0}) \times (\Gamma_k/\Gamma_{k_0})$.

We define

$$\tilde{I}_\alpha^t(\tau) := \int_0^\infty \alpha'(\tau+\xi)\theta^t(\xi)d\xi, \tag{9.1.34}$$

while for the heat flux we introduce

$$\tilde{\mathbf{I}}_k^t(\tau) := \int_0^\infty \mathbf{k}(\tau+\xi)\mathbf{g}^t(\xi)d\xi. \tag{9.1.35}$$

From Definition 9.1.1 and by virtue of (9.1.31), it follows that equivalent states $(\vartheta_j(t),\, \vartheta_j^t,\, \mathbf{g}_j^t)$, $j = 1, 2$, can be characterized by the triplet $(\vartheta(t), \tilde{I}_\alpha^t, \tilde{\mathbf{I}}_k^t)$, where

$$\begin{aligned}
\vartheta(t) &= \vartheta_1(t) = \vartheta_2(t),\\
\tilde{I}_\alpha^t &= \tilde{I}_{\alpha 1}^t = \tilde{I}_{\alpha 2}^t,\\
\tilde{\mathbf{I}}_k^t &= \tilde{\mathbf{I}}_{k1}^t = \tilde{\mathbf{I}}_{k2}^t.
\end{aligned}$$

The subscripts 1, 2 on $\tilde{I}_\alpha^t$, $\tilde{\mathbf{I}}_k^t$ refer to histories $(\vartheta_1^t,\, \mathbf{g}_1^t)$ and $(\vartheta_2^t,\, \mathbf{g}_2^t)$. Therefore, the minimal state of a rigid heat conductor can be described by $(\vartheta(t), \tilde{I}_\alpha^t, \tilde{\mathbf{I}}_k^t)$.

Let the equivalence relation between states in Σ be denoted by R. The class σ_R of equivalent states can be represented by $\sigma_R = (\vartheta(t), \tilde{I}_\alpha^t, \tilde{\mathbf{I}}_k^t)$ but also by $\sigma_R = (I_\alpha^t, \mathbf{I}_k^t)$ if, taking into account (9.1.34)–(9.1.35) with $(9.1.27)_1$, $(9.1.29)_1$, we introduce

$$I_\alpha^t(\tau) = \tilde{I}_\alpha^t(\tau) + \alpha(\tau)\vartheta(t) = \hat{e}(\vartheta(t),\, \vartheta^t; 0_\tau^\dagger),\ \ \mathbf{I}_k^t(\tau) = \tilde{\mathbf{I}}_k^t(\tau) = -\hat{\mathbf{q}}(\mathbf{g}^t; \mathbf{0}_\tau^\dagger) \quad \forall\tau > 0.$$

We observe that I^t_α and $\mathbf{I}^t_k$ are the same for all $(\theta(t), \theta^t, \mathbf{g}^t) \in \sigma_R$.

For heat conductors with discrete-spectrum relaxation functions, namely those consisting of sums of decaying exponentials, one can show as in Sect. 8.4 that the state is finite-dimensional.

9.2 Thermodynamic Constraints for Rigid Heat Conductors

We now determine the restrictions imposed on constitutive equation (9.1.1) by thermodynamics. Let us assume that $\alpha'(s)$ in (9.1.6) is zero. Then, integrating (9.1.9) over any cycle of period T, we obtain

$$\oint_0^T \mathbf{q}(t) \cdot \mathbf{g}(t) dt \le 0. \tag{9.2.1}$$

The equality sign occurs if and only if the history of $\mathbf{g}$ in (9.1.1) is zero. Consequently, any cycle characterized by the history

$$\mathbf{g}^t(s) = \mathbf{g}_1 \cos \omega(t-s) + \mathbf{g}_2 \sin \omega(t-s),$$

where $\omega \in \mathbb{R}^{++}$ and $(\mathbf{g}_1, \mathbf{g}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \backslash \{\mathbf{0}, \mathbf{0}\}$, must satisfy (9.2.1) as an inequality, with $\mathbf{q}$ given by (9.1.1); therefore, we must have

$$\int_0^{2\pi/\omega} \int_0^\infty \mathbf{k}(s)[\mathbf{g}_1 \cos \omega(t-s) + \mathbf{g}_2 \sin \omega(t-s)] ds \cdot (\mathbf{g}_1 \cos \omega t + \mathbf{g}_2 \sin \omega t) dt > 0.$$

Integrating with respect to t, we obtain

$$\frac{\pi}{\omega} \int_0^\infty [\mathbf{k}(s)\mathbf{g}_1 \cdot \mathbf{g}_1 + \mathbf{k}(s)\mathbf{g}_2 \cdot \mathbf{g}_2] \cos \omega s \, ds > 0,$$

which, since $\mathbf{g}_1$ and $\mathbf{g}_2$ are arbitrary, yields

$$\mathbf{k}_c(\omega) = \int_0^\infty \mathbf{k}(s) \cos \omega s \, ds > \mathbf{0} \qquad \forall \omega \in \mathbb{R}^{++}, \tag{9.2.2}$$

so that

$$\mathbf{k}'_s(\omega) = -\omega \mathbf{k}_c(\omega) < \mathbf{0} \quad \forall \omega \ne 0, \qquad \mathbf{k}(0) = -\frac{2}{\pi} \int_{-\infty}^\infty \frac{1}{\omega} \mathbf{k}'_s(\omega) d\omega > \mathbf{0} \tag{9.2.3}$$

(see (7.2.19)). Also, by (C.2.17),

$$\lim_{\omega\to\infty} \omega \mathbf{k}'_s(\omega) = -\lim_{\omega\to\infty} \omega^2 \mathbf{k}_c(\omega) = \mathbf{k}'(0) \le \mathbf{0}. \tag{9.2.4}$$

We assume the following stronger conditions:

$$\mathbf{k}_c(0) = \int_0^\infty \mathbf{k}(\xi) d\xi \equiv \mathbf{K}(\infty) > \mathbf{0}, \qquad \mathbf{k}'(0) < \mathbf{0}. \tag{9.2.5}$$

Relation (9.1.20) has been used here. The assumption $(9.2.5)_1$, in particular, yields that the heat flux $(9.1.23)_2$ resulting from a constant past history of the temperature gradient has the opposite sign to that of $\mathbf{g}$.

Analogously, one can show that the thermodynamic restriction on the memory function α' for the internal energy is expressed by [144, 147] (cf. (8.5.13))

$$\omega\alpha'_s(\omega) > 0, \qquad \omega \neq 0. \tag{9.2.6}$$

Under the hypothesis that $\alpha'' \in L^1(\mathbf{R}^+)$ and using the analogue of (8.1.20), we have

$$\alpha''_c(\omega) = \omega\alpha'_s(\omega) - \alpha'(0), \quad \alpha(t) - \alpha_0 = \frac{2}{\pi}\int_0^\infty \frac{\alpha'_s(\omega)}{\omega}[1 - \cos(\omega t)]d\omega > 0. \tag{9.2.7}$$

It follows from $(9.2.7)_2$ that

$$\alpha_\infty - \alpha_0 = \frac{2}{\pi}\int_0^\infty \frac{\alpha'_s(\omega)}{\omega}d\omega > 0. \tag{9.2.8}$$

Also, referring to (9.2.4), we have

$$\lim_{\omega\to\infty} \omega\alpha'_s(\omega) = \alpha'(0) \geq 0.$$

It will be assumed that

$$\alpha'(0) > 0. \tag{9.2.9}$$

9.3 Thermal Work

The linearized form (9.1.9) of the Clausius–Duhem inequality allows us to introduce the thermal power expressed by

$$w(t) = \dot{e}(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t); \tag{9.3.1}$$

hence, the thermal work done on a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ applied for every $\tau \in [0, d_P)$, starting from the initial state $\sigma(t)$ at time t, is expressed by

$$W(\sigma(t), P) = \int_0^{d_P} [\dot{e}(t+\tau)\vartheta_P(\tau) - \mathbf{q}(t+\tau) \cdot \mathbf{g}_P(\tau)]d\tau, \tag{9.3.2}$$

where in particular, $\vartheta_P(\tau)$ is given by (9.1.15).

Remark 9.3.1. We observe that in the first term on the right of (9.3.1), the time derivative is on the dependent field variable e rather than the independent variable ϑ, in contrast to (8.6.32), for example. This results in certain differences between the developments in this chapter and those in most of Chap. 8, dealing with solids and fluids. There are similarities, however, with Sect. 8.6.1.

9.3.1 Integrated Histories for Isotropic Heat Conductors

We consider a homogeneous and isotropic rigid heat conductor for which (9.1.6) and (9.1.3) become

$$\begin{aligned} e(t) &= \alpha_0 \vartheta(t) + \int_0^\infty \alpha'(s)\vartheta^t(s)ds, \\ \mathbf{q}(t) &= \int_0^\infty k'(s)\bar{\mathbf{g}}^t(s)ds, \end{aligned} \tag{9.3.3}$$

where $\bar{\mathbf{g}}^t$ is the integrated history (9.1.2). The relaxation function for the heat flux is the function $k : \mathbb{R}^+ \to \mathbb{R}$ such that $k \in L^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ satisfies the thermodynamic restrictions (9.2.2) and the consequences and assumptions (9.2.3)–(9.2.5). Similarly, $\alpha' : \mathbb{R}^+ \to \mathbb{R}$ obeys (9.2.6)–(9.2.9).

Instead of the definition (9.1.11) for the thermodynamic state of the conductor, we now choose the triplet

$$\sigma(t) = (\vartheta(t), \vartheta^t, \bar{\mathbf{g}}^t),$$

where the history of the temperature up to time t is again expressed by means of the pair $(\vartheta(t), \vartheta^t)$. The process $P : [0, d_P) \to \mathbb{R} \times \mathbb{R}^3$ is still defined by (9.1.12).

Relations (9.1.15)–(9.1.16), which express the continuation $(\vartheta_P * \vartheta)^{t+d_P}$, also remain applicable, together with the set of possible states defined by (9.1.21). However, the presence of the integrated history of the temperature gradient in the state means that we must replace (9.1.17) with the continuation $(\mathbf{g}_P * \bar{\mathbf{g}})^{t+d_P}$ defined by

$$\bar{\mathbf{g}}(t + d_P - s) = (\mathbf{g}_P * \bar{\mathbf{g}})^{t+d_P}(s) = \begin{cases} \displaystyle\int_{d_P - s}^{d_P} \mathbf{g}_P(\xi)d\xi = \bar{\mathbf{g}}_P^{d_P}(s), & \forall s \in [0, d_P), \\ \bar{\mathbf{g}}_P^{d_P}(d_P) + \bar{\mathbf{g}}^t(s - d_P), & \forall s \geq d_P. \end{cases} \tag{9.3.4}$$

The integrated history of $\mathbf{g}$ corresponding to a static continuation of a specified past history $\mathbf{g}^t$, defined in (9.1.18), is given by

$$\bar{\mathbf{g}}^{t+a}(s) = \begin{cases} \displaystyle\int_{a-s}^{a} \mathbf{g}(t)d\xi = s\mathbf{g}(t) \quad \forall s \in [0, a], \\ \displaystyle\int_0^a \mathbf{g}(t)d\xi + \int_{t-(s-a)}^{t} \mathbf{g}(\xi)d\xi = a\mathbf{g}(t) + \int_0^{s-a} \mathbf{g}^t(\rho)d\rho \quad \forall s > a; \end{cases}$$

thus, we obtain the expression (9.1.19) modified as follows:

$$\mathbf{q}(t + a) = -K(a)\mathbf{g}(t) + \int_0^\infty k'(\xi + a)\bar{\mathbf{g}}^t(\xi)d\xi,$$

where the thermal conductivity K is given by the scalar form of (9.1.20). Consequently, the function space (9.1.22) must be replaced by

$$\bar{\Gamma}_k = \left\{ \bar{\mathbf{g}}^t : \mathbb{R}^+ \to \mathbf{R}^3; \left| \int_0^\infty k'(\xi + \tau)\bar{\mathbf{g}}^t(\xi)d\xi \right| < \infty \; \forall \tau \geq 0 \right\}, \tag{9.3.5}$$

where t is a parameter.

Let P_τ be the restriction of a process applied at time t to the state $\sigma(t) = (\vartheta(t),\ \vartheta^t, \bar{\mathbf{g}}^t)$. By using (9.1.16) and (9.3.4), where d_P is replaced by τ, we have

$$\begin{aligned} e(t+\tau) &= \alpha_0\vartheta_P(\tau) + \int_0^\tau \alpha'(s)\vartheta_P^\tau(s)ds + \int_\tau^\infty \alpha'(s)\vartheta(t+\tau-s)ds, \\ \mathbf{q}(t+\tau) &= \int_0^\tau k'(s)\bar{\mathbf{g}}_P^\tau(s)ds + \int_\tau^\infty k'(s)[\bar{\mathbf{g}}_P^\tau(\tau) + \bar{\mathbf{g}}^t(s-\tau)]ds. \end{aligned} \tag{9.3.6}$$

The constitutive equations (9.3.3) can be expressed in the general form

$$e(\sigma(t)) = \tilde{e}(\vartheta(t),\ \vartheta^t), \qquad \mathbf{q}(\sigma(t)) = \hat{\mathbf{q}}(\bar{\mathbf{g}}^t).$$

The equivalence relation introduced in the state space Σ by means of Definition 9.1.1 can now be given as follows.

Definition 9.3.2. *Two states $\sigma_j = (\vartheta_j,\ \vartheta_j^t, \bar{\mathbf{g}}_j^t) \in \Sigma$, $j = 1, 2$, of a rigid heat conductor characterized by the constitutive equations (9.3.3) are equivalent if for every process $P \in \Pi$ and for every $\tau > 0$,*

$$\tilde{e}(\rho(\sigma_1, P_{[0,\tau)})) = \tilde{e}(\rho(\sigma_2, P_{[0,\tau)})),\ \ \hat{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_1)^{t+\tau}) = \hat{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_2)^{t+\tau}). \tag{9.3.7}$$

The following result is the analogue of Theorem 9.1.2 and can be proved similarly.

Theorem 9.3.3. *For a heat conductor characterized by the constitutive equations (9.3.3), two states $\sigma_j = (\vartheta_j,\ \vartheta_j^t, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, are equivalent if and only if*

$$\begin{aligned} \vartheta_1(t) = \vartheta_2(t), \qquad &\int_0^\infty \alpha'(\tau+\rho)\left[\vartheta_1^t(\rho) - \vartheta_2^t(\rho)\right] d\rho = 0, \\ \int_0^\infty k'(\tau+\rho)&\left[\bar{\mathbf{g}}_1^t(\rho) - \bar{\mathbf{g}}_2^t(\rho)\right] d\rho = \mathbf{0} \end{aligned} \tag{9.3.8}$$

for every $\tau > 0$.

Consequently, a state $\sigma(t) = (\vartheta(t),\ \vartheta^t, \bar{\mathbf{g}}^t)$ is equivalent to the zero state $\sigma_0(t) = (0, 0^\dagger, \bar{\mathbf{0}}^\dagger)$, where in particular $\bar{\mathbf{0}}^\dagger(s) = \bar{\mathbf{g}}^t(s) = \mathbf{0}\ \forall s \in \mathbb{R}^+$ is the zero integrated history of $\mathbf{g}$, if

$$\begin{aligned} \vartheta(t) = 0, \qquad \int_\tau^\infty \alpha'(s)\vartheta^{t+\tau}(s)ds &= \int_0^\infty \alpha'(\tau+\xi)\vartheta^t(\xi)d\xi = 0, \\ \int_\tau^\infty k'(s)\bar{\mathbf{g}}^t(s-\tau)ds &= \int_0^\infty k'(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi = \mathbf{0}. \end{aligned}$$

Thus, two equivalent states σ_j, $j = 1, 2$, are such that their difference $\sigma_1(t) - \sigma_2(t) = (\vartheta_1(t) - \vartheta_2(t),\ \vartheta_1^t - \vartheta_2^t, \bar{\mathbf{g}}_1^t - \bar{\mathbf{g}}_2^t)$ is a state equivalent to the zero state, $\sigma_0(t) = (0, 0^\dagger, \bar{\mathbf{0}}^\dagger)$.

9.3.2 Finite Work Processes and *w*-Equivalence for States

The thermal work done during the application of a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))\ \forall \tau \in [0, d_P)$, starting from the initial state $\sigma(t) = (\vartheta(t),\ \vartheta^t, \bar{\mathbf{g}}^t)$ at time t, is given by (9.3.2). To evaluate it we must consider the derivative of the internal energy through $(9.3.6)_1$ and take account of the heat flux in the form $(9.3.6)_2$. From $(9.3.6)_1$, by differentiating with respect to τ and integrating by parts, we have

$$\dot{e}(t+\tau) = \alpha_0 \dot{\vartheta}_P(\tau) + \alpha'(0)\vartheta_P(\tau) + \int_0^\tau \alpha''(s)\vartheta_P(\tau - s)ds\ + \int_\tau^\infty \alpha''(s)\,\vartheta(t+\tau-s)ds. \tag{9.3.9}$$

To derive the expression for the work due only to a process P of duration $d_P < \infty$, applied at time $t = 0$, we suppose that the initial state is $\sigma_0(0) = (0, 0^\dagger, \bar{\mathbf{0}}^\dagger)$. Denoting the ensuing fields by $(\vartheta_0,\ \vartheta_0^t, \bar{\mathbf{g}}_0^t)$, (9.1.15)–(9.1.16) with (9.3.4) yield

$$\begin{aligned}
\vartheta_0(t) &= \int_0^t \dot{\vartheta}_P(s)ds, \\
\vartheta_0^t(s) = (\vartheta_P * 0^\dagger)^t(s) &= \begin{cases} \displaystyle\int_0^{t-s} \dot{\vartheta}_P(\eta)d\eta & \forall s \in (0, t], \\ 0 & \forall s > t, \end{cases} \\
\bar{\mathbf{g}}_0^t(s) = (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t(s) &= \begin{cases} \bar{\mathbf{g}}_0^t(s) & \forall s \in [0, t), \\ \bar{\mathbf{g}}_0^t(t) & \forall s \ge t. \end{cases}
\end{aligned} \tag{9.3.10}$$

Let $W(\sigma_0(0), P)$ be the work obtained by applying $P(t) = (\dot{\vartheta}_P(t), \mathbf{g}_P(t))\,\forall\ t \in [0, d_P)$ to the zero state $\sigma_0(0)$, at time $t = 0$. By evaluating directly from $(9.3.3)_1$ or from (9.3.9), we have

$$\dot{e}(t) = \alpha_0 \dot{\vartheta}_0(t) + \alpha'(0)\vartheta_0(t) + \int_0^t \alpha''(s)\vartheta^t(s)ds$$

and, from $(9.3.3)_2$ with $(9.3.10)_3$,

$$-\mathbf{q}(t)\cdot \mathbf{g}_P(t) = -\left[\int_0^t k'(s)\bar{\mathbf{g}}_0^t(s)ds + \int_t^\infty k'(s)\bar{\mathbf{g}}_0^t(t)ds\right]\cdot \mathbf{g}_0(t) = \int_0^t k(s)\mathbf{g}_0^t(s)ds\ \cdot\ \mathbf{g}_0(t).$$

We see that this work is given by

$$\begin{aligned}
\tilde{W}(0, 0^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) &= \frac{1}{2}\alpha_0\vartheta_0^2(d_P) + \alpha'(0)\int_0^{d_P} \vartheta_0^2(t)dt \\
&+ \int_0^{d_P}\int_0^t \alpha''(s)\vartheta_0^t(s)ds\vartheta_0(t)dt + \int_0^{d_P}\int_0^t k(s)\mathbf{g}_0^t(t)ds\ \cdot\ \mathbf{g}_0(t)dt.
\end{aligned} \tag{9.3.11}$$

Definition 9.3.4. *A process P of duration d_P is a finite work process if*

$$W(\sigma_0(0), P) < \infty.$$

Lemma 9.3.5. *The work done by any finite process is positive.*

Proof. In fact, by assuming that the integrands in (9.3.11) are equal to zero for any $t > d_P$, we can extend the integrals to $\mathbb{R}^+$ and apply Parseval's formula (C.3.1) to obtain

$$\begin{aligned}
W(\sigma_0(0), P) &= \frac{1}{2}\alpha_0\vartheta_0^2(d_P) + \frac{\alpha'(0)}{2\pi}\int_{-\infty}^{\infty}|\vartheta_{0+}(\omega)|^2 d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha''_+(\omega)|\vartheta_{0+}(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}k_+(\omega)|\mathbf{g}_{0+}(\omega)|^2 d\omega \\
&= \frac{1}{2}\alpha_0\vartheta_0^2(d_P) + \frac{1}{2\pi}\int_{-\infty}^{\infty}[\alpha'(0) + \alpha''_c(\omega)]|\vartheta_+(\omega)|^2 d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}k_c(\omega)|\mathbf{g}_{0+}(\omega)|^2 d\omega \\
&= \frac{1}{2}\alpha_0\vartheta_0^2(d_P) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\omega\alpha'_s(\omega)|\vartheta_{0+}(\omega)|^2 + k_c(\omega)|\mathbf{g}_{0+}(\omega)|^2\right\}d\omega > 0,
\end{aligned}$$

by virtue of the oddness of the sine Fourier transform together with $(9.2.7)_1$, (9.2.6), and the scalar form of $(9.2.3)_1$. □

Hence, to characterize the set of finite work processes we consider the following sets [145]:

$$\begin{aligned}
\tilde{H}_\alpha(\mathbb{R}^+, \mathbb{R}) &= \left\{\vartheta : \mathbb{R}^+ \to \mathbb{R};\ \int_{-\infty}^{\infty}\omega\alpha'_s(\omega)|\vartheta_{P+}(\omega)|^2 d\omega < \infty\right\}, \\
\tilde{H}_k(\mathbb{R}^+, \mathbb{R}^3) &= \left\{\mathbf{g} : \mathbb{R}^+ \to \mathbb{R}^3;\ \int_{-\infty}^{\infty}k_c(\omega)|\mathbf{g}_+(\omega)|^2 d\omega < \infty\right\}.
\end{aligned} \tag{9.3.12}$$

With the completion with respect to the norm corresponding to the inner product $(\vartheta_1, \vartheta_2)_\alpha = \int_{-\infty}^{\infty}\omega\alpha'_s(\omega)\vartheta_{1+}(\omega)\overline{\vartheta_{2+}(\omega)}d\omega$, we have another Hilbert space $H_\alpha(\mathbb{R}^+, \mathbb{R})$, besides $H_k(\mathbb{R}^+, \mathbb{R}^3)$.

Let $\sigma(t) = (\vartheta(t),\ \vartheta^t, \overline{\mathbf{g}}^t)$ be the initial state of the body at time $t > 0$, where $\vartheta^t \in \Gamma_\alpha$ and $\overline{\mathbf{g}}^t \in \overline{\Gamma}_k$, the spaces Γ_α and $\overline{\Gamma}_k$ being defined by (9.1.21) and (9.3.5), are possible histories that yield finite work during any process, as defined by (9.3.12). Any of these processes $P = (\dot{\vartheta}_P, \mathbf{g}_P)$ with a finite duration $d_P < \infty$ may be extended to $\mathbb{R}^+$ by putting $P(\tau) = (0, \mathbf{0})\ \forall\tau \geq d_P$. The expression (9.3.2) for the work, taking into account (9.3.9) for $\dot{e}(t+\tau)$ and $(9.3.6)_2$ for $\mathbf{q}(t+\tau)$, with some integrations, becomes

$$\begin{aligned}
W(\sigma(t), P) &= \tilde{W}(\vartheta(t),\ \vartheta^t, \overline{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) \\
&= \frac{1}{2}\alpha_0\left[\vartheta_P^2(d_P) - \vartheta_P^2(0)\right] + \alpha'(0)\int_0^{\infty}\vartheta_P^2(\tau)d\tau \\
&\quad + \int_0^{\infty}\left[\frac{1}{2}\int_0^{\infty}\alpha''(|\tau-\eta|)\vartheta_P(\eta)d\eta + I^t_{(\alpha)}(\tau, \vartheta^t)\right]\vartheta_P(\tau)d\tau \\
&\quad + \int_0^{\infty}\left[\frac{1}{2}\int_0^{\infty}k(|\tau-\eta|)\mathbf{g}_P(\eta)d\eta + \mathbf{I}^t_{(k)}(\tau, \overline{\mathbf{g}}^t)\right]\cdot\mathbf{g}_P(\tau)d\tau,
\end{aligned} \tag{9.3.13}$$

where

$$\begin{aligned} I^t_{(\alpha)}(\tau,\vartheta^t) &= \int_0^\infty \alpha''(\tau+\xi)\vartheta^t(\xi)d\xi, \\ \mathbf{I}^t_{(k)}(\tau,\overline{\mathbf{g}}^t) &= -\int_0^\infty k'(\tau+\xi)\overline{\mathbf{g}}^t(\xi)d\xi, \ \tau \ge 0. \end{aligned} \tag{9.3.14}$$

A definition of equivalence of states is now given in terms of the work function, which we must compare with Definition 9.3.2.

Definition 9.3.6. *Two states $\sigma_j(t) = (\vartheta_j(t), \vartheta^t_j, \overline{\mathbf{g}}^t_j)$, $j = 1,2$, are said to be w-equivalent if they satisfy*

$$W(\sigma_1(t),P) = W(\sigma_2(t),P) \tag{9.3.15}$$

for every process $P : [0,\tau) \to \mathbb{R}\times\mathbb{R}^3$ and for every $\tau > 0$.

Theorem 9.3.7. *Two states are equivalent in the sense of Definition 9.3.2 if and only if they are w-equivalent.*

Proof. Two states $\sigma_j(t) = (\vartheta_j(t), \vartheta^t_j, \overline{\mathbf{g}}^t_j)$, $j = 1,2$, equivalent in the sense of Definition 9.3.2, satisfy (9.3.7) for every process $P_{[0,\tau)}$ and for every $\tau > 0$. Hence, it follows that we have the same derivative with respect to τ of $(9.3.7)_1$, which appears in the expression (9.3.2) for the work, as well as the same heat flux. Thus, the work done by the same process applied to both $\sigma_j(t)$, $j = 1,2$, coincide and (9.3.15) holds.

On the other hand, let two states $\sigma_j(t)$, $j = 1,2$, be w-equivalent. Then for any P with arbitrary duration d_P, taking account of (9.3.13) and (9.1.15), we obtain

$$\begin{aligned} &\alpha_0 \int_0^{d_P} \dot\vartheta_P(\tau)d\tau[\vartheta_1(t)-\vartheta_2(t)] + \alpha'(0)[\vartheta_1(t)-\vartheta_2(t)]\int_0^{d_P}\{[\vartheta_1(t)+\vartheta_2(t)] \\ &\quad + 2\int_0^\tau \dot\vartheta_P(\xi)d\xi\Big\}d\tau + \frac{1}{2}\int_0^\infty\int_0^\infty \alpha''(|\tau-\eta|)[\vartheta_1(t)-\vartheta_2(t)]\{[\vartheta_1(t)+\vartheta_2(t)] \\ &\quad + 2\Big[\int_0^\tau \dot\vartheta_P(\rho)d\rho + \int_0^\eta \dot\vartheta_P(\xi)d\xi\Big]\Big\}d\eta d\tau \\ &= -\int_0^{d_P}\Big\{\Big[I^t_{(\alpha)}(\tau,\vartheta^t_1)\vartheta_1(t) - I^t_{(\alpha)}(\tau,\vartheta^t_2)\vartheta_2(t)\Big] \\ &\quad + \Big[I^t_{(\alpha)}(\tau,\vartheta^t_1) - I^t_{(\alpha)}(\tau,\vartheta^t_2)\Big]\int_0^\tau \dot\vartheta_P(\xi)d\xi\Big\}d\tau \\ &\quad - \int_0^\infty \Big[\mathbf{I}^t_{(k)}(\tau,\overline{\mathbf{g}}^t_1) - \mathbf{I}^t_{(k)}(\tau,\overline{\mathbf{g}}^t_2)\Big]\cdot \mathbf{g}_P(\tau)d\tau, \end{aligned}$$

where the integrals with $k(|\tau-\eta|)$ cancel, since they have the same $\mathbf{g}_P$. Since in this relation $\dot\vartheta_P$ and d_P, as well as $\mathbf{g}_P$, are arbitrary, it follows that

$$\vartheta_1(t) = \vartheta_2(t), \quad I^t_{(\alpha)}(\tau,\vartheta^t_1) = I^t_{(\alpha)}(\tau,\vartheta^t_2), \quad \mathbf{I}^t_{(k)}(\tau,\overline{\mathbf{g}}^t_1) = \mathbf{I}^t_{(k)}(\tau,\overline{\mathbf{g}}^t_2). \tag{9.3.16}$$

The first of these conditions coincides with $(9.3.8)_1$, while the third, by virtue of $(9.3.14)_2$, yields $(9.3.8)_3$; the second equality, using $(9.3.14)_1$, yields

$$I^t_{(\alpha)}(\tau,\vartheta^t_1) - I^t_{(\alpha)}(\tau,\vartheta^t_2) = \int_0^\infty \alpha''(\tau+\xi)\left[\vartheta^t_1(\xi) - \vartheta^t_2(\xi)\right] d\xi$$
$$= \frac{d}{d\tau}\int_0^\infty \alpha'(\tau+\xi)\left[\vartheta^t_1(\xi) - \vartheta^t_2(\xi)\right] d\xi = 0.$$

Hence, the function

$$f(\tau) \equiv \int_0^\infty \alpha'(\tau+\xi)\left[\vartheta^t_1(\xi) - \vartheta^t_2(\xi)\right] d\xi$$

is equal to the constant c_1, which can be evaluated by means of

$$c_1 = \lim_{\tau\to\infty} f(\tau) = 0.$$

Thus, (9.3.16) and (9.3.8) coincide. □

9.3.3 Free Energies as Quadratic Functionals for Rigid Heat Conductors

We can express free energies obeying (9.1.10) as quadratic functionals of the independent quantities $\mathbf{g}^t$ and $(\vartheta(t), \vartheta^t)$, respectively, based on the constitutive relations (9.1.1) and (9.1.6), using a formalism analogous to that in Sect. 8.6. This yields a free energy

$$\psi = \psi_e + \psi_{\mathbf{g}},$$

where ψ_e is a quadratic functional of temperature and $\psi_{\mathbf{g}}$ a similar functional of the temperature gradient. Noting Remark 9.3.1, we see that an analogy with the formalism sketched out in Sect. 8.6.1 is the appropriate one for ψ_e.

Let us write the special case of (9.3.1) in which the contribution from the temperature gradient is neglected:

$$w(t) = \vartheta(t)\dot{e}(t). \tag{9.3.17}$$

The analogue of (8.6.30) in this context is

$$\begin{aligned} \psi^{(e)}(t) &= \frac{1}{2}\alpha_0\vartheta^2(t) - \frac{1}{2}\int_0^\infty\int_0^\infty \alpha_{12}(s,u)\vartheta^t(s)\vartheta^t(u)dsdu, \\ \alpha_{12}(s,u) &= \frac{\partial^2}{\partial s\partial u}\alpha(s,u), \qquad \alpha_0 = \alpha(0,0), \end{aligned} \tag{9.3.18}$$

where we must choose $\alpha(\cdot,\cdot)$ so that the integral in $(9.3.18)_1$ exists and is nonpositive for all finite relative histories. Thus, the equivalent of condition (8.6.4) must apply. Putting

$$\alpha(s,u) = \alpha_\infty + \int_s^\infty\int_u^\infty \alpha_{12}(s',u')ds'\,du',$$

we have

$$\alpha(s,\infty) = \alpha(\infty,s) = \alpha_\infty.$$

Also, $(8.6.26)_2$ becomes

$$\alpha(s, 0) = \alpha(0, s) = \alpha(s), \quad s \in \mathbb{R},$$

so that $\alpha_\infty = \alpha(\infty)$.

From (8.6.27), the rate of dissipation is given by

$$D(t) = \frac{1}{2} \int_0^\infty \int_0^\infty [\alpha_{121}(s, u) + \alpha_{122}(s, u)] \vartheta_r^t(s) \vartheta_r^t(u) ds du. \tag{9.3.19}$$

This involves the further constraint on α that the kernel in (9.3.19) must be such that the integral is nonnegative for all relative histories of the internal energy.

By differentiation of $(9.3.18)_1$ and use of $(9.3.3)_1$, we have (cf. (8.6.29))

$$\dot{\psi}^{(e)}(t) + D(t) = \vartheta \dot{e}(t).$$

9.3.4 The Work Function

The work function or maximum free energy (or upper bound on free energies) is obtained from (9.3.18) by putting $\alpha(s, u) = \alpha(|s - u|)$, so that

$$\psi_M^{(e)}(t) = \frac{1}{2} \alpha_\infty \vartheta^2(t) - \frac{1}{2} \int_0^\infty \int_0^\infty \alpha_{12}(|s_1 - s_2|) \vartheta^t(s_1) \vartheta^t(s_2) ds_1 ds_2.$$

Applying (8.10.20), we see that this agrees with the relevant terms (9.3.11) if $\vartheta^t(s)$ vanishes for $s > d_P$, using an argument similar to that leading to (7.5.2). Clearly $D(t)$, given by (9.3.19), vanishes in this case and

$$\dot{\psi}_M^{(e)}(t) = \dot{e}(t) \vartheta(t) = w(t),$$

from (9.3.17).

Recall, however, that there is a problem with categorizing the work function as a free energy, arising out of Remark 18.2.