8

Viscoelastic Solids and Fluids

We now consider special cases of the constitutive relations (7.1.13), namely linear viscoelastic solids and fluids with linear memory under isothermal conditions in the present chapter and an approximate version of rigid heat conductors in Chap. 9. Some of the formulas are similar to those derived in the general case, and detailed proofs are omitted or a different version is given. Other formulas are specific to completely linear materials, for example.

More use will be made, for illustrative purposes, of the abstract terminology and notation introduced in Chaps. 3 and 4 in discussing these specific materials than in the general case. This is particularly true for Chap. 9.

Combining constitutive relations for solids and fluids with the equations of motion (1.3.25) yields the dynamical equations describing the time evolution of the material under specified initial and boundary conditions. Questions of the existence, uniqueness, and stability of the solutions of these integro–partial differential equations are considered in Part IV, particularly in Chap. 24. Practical methods for obtaining explicit solutions, particularly in the quasistatic approximation, may be found in older texts such as [65] and especially [167].

8.1 Linear Viscoelastic Solids

In the general form of the theory, we are dealing with finite linear viscoelasticity [73], where the stress is given by a linear memory functional of the strain history and a nonlinear (or linear) function of the current strain. The space $\varGamma$ reduces to Sym and Lin($\varGamma$) to Lin(Sym). Thus, (7.1.21) reduces to

$$\widehat{\mathbf{S}}(t) = \widehat{\mathbf{S}}_e(\mathbf{E}(t)) + \int_0^\infty \mathbb{G}'(u)\mathbf{E}_r^t(u)du, \qquad \mathbb{G}'(u) = \frac{\rho}{\kappa}\mathbb{L}_E'(u). \tag{8.1.1}$$

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For the case of completely linear viscoelasticity, we replace $\widehat{\mathbf{S}}$ by the Cauchy stress tensor $\mathbf{T}$, and (8.1.1) becomes

$$\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t) + \int_0^\infty \mathbb{G}'(u)\mathbf{E}_r^t(u)du, \tag{8.1.2}$$

where the relative strain history $\mathbf{E}_r^t$ is given by

$$\mathbf{E}_r^t(s) = \mathbf{E}^t(s) - \mathbf{E}(t). \tag{8.1.3}$$

The quantity $\mathbb{G}$ is the relaxation function of the viscoelastic material. For a viscoelastic solid, $\mathbb{G}(\infty) = \mathbb{G}_\infty$ is a positive tensor, defined by $(1.4.12)_2$. Thus, we no longer have the condition $(7.1.15)_1$. This property can be retained by using $\mathcal{G}(u) = \mathbb{G}(u) - \mathbb{G}_\infty$. We will not do so, however, for reasons of convention. This means that certain partial integrations are slightly more complicated. We are therefore adopting what was referred to in Remark 7.1.2 as the conventional choice. Note that the assumption (7.2.4) no longer applies. We shall assume that $\mathbb{G}(\cdot) - \mathbb{G}_\infty \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. Relations $(7.1.13)_{2,3}$ become

$$\begin{aligned}
\widehat{\mathbf{S}}(t) &= \widehat{\mathbf{S}}_e(\mathbf{E}(t)) + (\mathbb{G}_0 - \mathbb{G}_\infty)\mathbf{E}(t) + \int_0^\infty \mathbb{G}'(u)\mathbf{E}^t(u)du, \\
&= \widehat{\mathbf{S}}_e(\mathbf{E}(t)) - \mathbb{G}_\infty \mathbf{E}(t) + \int_0^\infty \mathbb{G}(u)\dot{\mathbf{E}}^t(u)du, \qquad \mathbb{G}_0 = \mathbb{G}(0), \\
\dot{\mathbf{E}}^t(u) &= \frac{\partial}{\partial t}\mathbf{E}^t(u) = -\frac{\partial}{\partial u}\mathbf{E}^t(u),
\end{aligned}$$

where we have used (7.1.14) and assumed that $\mathbf{E}^t(\infty) = \mathbf{E}(-\infty) = 0$. Applying (7.2.19) to the subspace Sym of Γ gives

$$\mathbb{G}_0 > \mathbb{G}_\infty \geq \mathbf{0}, \tag{8.1.4}$$

where $\mathbb{G}_\infty$ (or specifically its shear part may vanish for a viscoelastic fluid.

In the completely linear case, these become

$$\mathbf{T}(t) = \mathbb{G}_0 \mathbf{E}(t) + \int_0^\infty \mathbb{G}'(u)\mathbf{E}^t(u)du, \tag{8.1.5}$$

or alternatively,

$$\mathbf{T}(t) = \int_0^\infty \mathbb{G}(u)\dot{\mathbf{E}}^t(u)du. \tag{8.1.6}$$

Equation (8.1.5) is identical to (1.4.11), without the explicit $\mathbf{X}$ dependence. The forms (8.1.2), (8.1.5), and (8.1.6) correspond to (7.1.13).

We have already supposed in Sect. 1.4.3 that $\mathbb{G}'(\cdot) \in L^1(\mathbb{R}^+)$; now we further assume that $\mathbb{G}'(\cdot) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, in accordance with Sect. C.1. The relaxation function is defined by

$$\mathbb{G}(s) = \mathbb{G}_0 + \int_0^s \mathbb{G}'(\xi)d\xi. \tag{8.1.7}$$

Thermodynamics implies the symmetry of $\mathbb{G}_0$ and $\mathbb{G}_\infty$, as we shall see below, but not the symmetry of $\mathbb{G}(s)$, $s \in \mathbb{R}^{++}$. However, we shall assume that $\mathbb{G}(s)$ is a fourth-order symmetric tensor, a special case of (7.1.18).

Remark 8.1.1. In a particular basis, $\mathbb{G}$ has components G_{ijkl}, all subscripts in the range 1–3. The symmetry referred to in the previous paragraph refers to the interchange of $\{ij\}$ and $\{kl\}$, as in (2.4.8). However, we see from (8.1.5), together with the symmetry of $\mathbf{T}$ and $\mathbf{E}$, respectively, that

$$G_{ijkl} = G_{jikl} = G_{ijlk}, \tag{8.1.8}$$

which generalizes (2.4.7) and (2.4.9).

From (1.3.32) in Definition 1.3.10, we see that the stress power per unit volume is $\mathbf{T} \cdot \mathbf{D}$. In the linear approximation, $\mathbf{D}$, given by (1.2.23), reduces to $\dot{\mathbf{E}}$, so that the power now becomes $\mathbf{T} \cdot \dot{\mathbf{E}}$. Thus, the inequality (4.1.7), expressing the dissipation principle, yields in this context

$$\int_0^{d_P} \dot{\mathbf{E}}(t) \cdot \mathbb{G}_0 \mathbf{E}(t) dt + \int_0^{d_P} \dot{\mathbf{E}}(t) \cdot \int_0^{\infty} \mathbb{G}'(s) \mathbf{E}^t(s) ds \geq 0, \tag{8.1.9}$$

which must hold for any cycle, where $\dot{\mathbf{E}}(t)$ is the process with duration d_P.

8.1.1 Thermodynamic Restrictions for Viscoelastic Solids

We now examine thermodynamic restrictions on the relaxation function [118, 120] by an approach equivalent to but more elementary than that developed in Sect. 7.2.1 for the general theory. Let us consider strain-tensor time dependence of the form

$$\mathbf{E}(s) = \mathbf{E}_1 \cos \omega s + \mathbf{E}_2 \sin \omega s, \quad s \leq t, \tag{8.1.10}$$

where $\omega \in \mathbb{R}^{++}$ and $\mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}$. The corresponding process $\tilde{P} \in \Pi$ at time t is given by

$$\tilde{P}(t) = \dot{\mathbf{E}}(t) = -\omega \mathbf{E}_1 \sin \omega t + \omega \mathbf{E}_2 \cos \omega t, \quad t \in [0, d_P), \tag{8.1.11}$$

where $d_P = 2\pi m/\omega$, m being any positive integer. Thus, we obtain a cycle, denoted by $(\tilde{\sigma}(t), \tilde{P})$.

Theorem 8.1.2. *The inequality (8.1.9) holds for any cycle $(\tilde{\sigma}(t), \tilde{P})$ only if the inequality*

$$\begin{aligned} \mathbf{E}_1 \cdot [\mathbb{G}_0^T - \mathbb{G}_0]\mathbf{E}_2 - \int_0^{\infty} [\mathbf{E}_1 \cdot \mathbb{G}'(s)\mathbf{E}_1 + \mathbf{E}_2 \cdot \mathbb{G}'(s)\mathbf{E}_2] \sin \omega s \, ds \\ - \int_0^{\infty} \mathbf{E}_1 \cdot [\mathbb{G}'(s) - \mathbb{G}'^T(s)]\mathbf{E}_2 \cos \omega s \, ds \geq 0 \end{aligned} \tag{8.1.12}$$

holds for every $\omega \in \mathbb{R}^{++}$ and every $\mathbf{E}_1, \mathbf{E}_2 \in$ Sym.

Proof. Substitution of (8.1.10) and (8.1.11) into (8.1.9) gives

$$\int_0^{d_P} (-\omega \mathbf{E}_1 \sin \omega t + \omega \mathbf{E}_2 \cos \omega t) \cdot \mathbb{G}_0(\mathbf{E}_1 \cos \omega t + \mathbf{E}_2 \sin \omega t) dt$$
$$+ \int_0^{d_P} \Big\{ (-\omega \mathbf{E}_1 \sin \omega t + \omega \mathbf{E}_2 \cos \omega t) \int_0^{\infty} \mathbb{G}'(s)[\mathbf{E}_1(\cos \omega t \cos \omega s + \sin \omega t \sin \omega s)$$
$$+ \mathbf{E}_2(\sin \omega t \cos \omega s - \cos \omega t \sin \omega s)] ds \Big\} dt \geq 0,$$

which, after integrating with respect to t, with $d_P = 2\pi m/\omega$ and using (A.2.3), yields (8.1.12). □

Some useful results can be derived by considering in (8.1.12) the limiting cases $\omega \to \infty$ and $\omega \to 0$.

Corollary 8.1.3. *The inequality (8.1.12) implies the symmetry of* $\mathbb{G}_0$*, i.e.,*

$$\mathbb{G}_0 = \mathbb{G}_0^T. \tag{8.1.13}$$

Proof. By virtue of the Riemann–Lebesgue lemma (C.2.13), the integrals in (8.1.12) vanish when we consider the limit $\omega \to \infty$. Hence, the arbitrariness of $\mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}$ gives (8.1.13). □

Corollary 8.1.4. *The inequality (8.1.12) implies the symmetry of* $\mathbb{G}_\infty$*, i.e.,*

$$\mathbb{G}_\infty = \mathbb{G}_\infty^T. \tag{8.1.14}$$

Proof. By virtue of (8.1.13), relation (8.1.12), in the limiting case $\omega \to 0$, gives

$$\mathbf{E}_1 \cdot [\mathbb{G}_\infty^T - \mathbb{G}_\infty]\mathbf{E}_2 \geq 0,$$

and the arbitrariness of $\mathbf{E}_1, \mathbf{E}_2$ leads to (8.1.14). □

By (8.1.13), we have the following result.

Corollary 8.1.5. *The inequality (8.1.12) implies that*

$$\int_0^{\infty} [\mathbf{E}_1 \cdot \mathbb{G}'(s)\mathbf{E}_1 + \mathbf{E}_2 \cdot \mathbb{G}'(s)\mathbf{E}_2] \sin \omega s \, ds$$
$$+ \int_0^{\infty} \mathbf{E}_1 \cdot [\mathbb{G}'(s) - \mathbb{G}'^{\top}(s)]\mathbf{E}_2 \cos \omega s \, ds \leq 0 \tag{8.1.15}$$

for every $\omega \in \mathbb{R}^{++}$ *and every* $\mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}$.

Referring to (C.1.3) and (C.2.2), we put

$$\mathbb{G}'_+(\omega) = \int_0^{\infty} \mathbb{G}'(u) e^{-i\omega u} du = \mathbb{G}'_c(\omega) - i\mathbb{G}'_s(\omega), \tag{8.1.16}$$

where $\mathbb{G}'_c$ and $\mathbb{G}'_s$ denote the Fourier cosine and sine transforms of $\mathbb{G}'$. Explicitly, the sine transform is given by

$$\mathbb{G}'_s(\omega) = \int_0^{\infty} \mathbb{G}'(u) \sin \omega u \, du. \tag{8.1.17}$$

The following important result is a special case of (7.2.12).

Corollary 8.1.6. *The inequality (8.1.12) implies the negative definiteness of* $\mathbb{G}'_s \in \mathrm{Sym}$ *for every* $\omega \in \mathbb{R}^{++}$.

Proof. Putting $\mathbf{E}_1 = \mathbf{E}_2$ in (8.1.15), we obtain $\mathbb{G}'_s(\omega) \leq \mathbf{0}$, $\omega \in \mathbb{R}^+$. Thus, we have

$$\mathbb{G}'_s(\omega) < \mathbf{0}, \qquad \omega \in \mathbb{R}^{++}, \tag{8.1.18}$$

while $\mathbb{G}'_s(0) = \mathbf{0}$. □

If $\mathbb{G}(\tau)$ is assumed to be symmetric for all $\tau \in \mathbb{R}^+$, then condition (8.1.18) implies that (8.1.12) and the dissipation principle (8.1.9) must hold for all histories of the form (8.1.10). More generally, one can show that (8.1.18), for $\mathbb{G}$ symmetric, implies that (8.1.9) holds for any cycle, using histories represented by Fourier series (see Proposition 7.2.2). This procedure is presented in some detail for compressible fluids in Sect. 8.9.3.

The definition of $\mathbb{G}'_s(\omega)$ can be extended to $\mathbb{R}^-$ by the relation $\mathbb{G}'_s(-\omega) = -\mathbb{G}'_s(\omega)$, $\omega \in \mathbb{R}$.

Corollary 8.1.7. *The inequality (8.1.18) implies that (cf.* (7.2.16)*)*

$$\mathbb{G}_0 - \mathbb{G}(s) > \mathbf{0}, \qquad s \in \mathbb{R}^{++}. \tag{8.1.19}$$

Proof. From the inversion formula of the Fourier sine transform $\mathbb{G}'_s(\omega)$, expressed by (see (C.1.6))

$$\mathbb{G}'(s) = \frac{2}{\pi}\int_0^\infty \sin\omega s\,\mathbb{G}'_s(\omega)d\omega,$$

we have, by integrating with respect to s,

$$\mathbb{G}(s) - \mathbb{G}_0 = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega s}{\omega}\mathbb{G}'_s(\omega)d\omega, \tag{8.1.20}$$

which, by virtue of the inequality (8.1.18), provides the desired result. □

For ease in writing let

$$\mathbb{G}'_0 := \mathbb{G}'(0).$$

Corollary 8.1.8. *We have*

$$\mathbb{G}'_0 \leq \mathbf{0} \tag{8.1.21}$$

and

$$\mathbb{G}_0 - \mathbb{G}_\infty \geq \mathbf{0}. \tag{8.1.22}$$

Proof. Relation (8.1.21) can be deduced in the same way as (7.2.17), while (8.1.22) follows from (8.1.19) by taking the limit $s \to \infty$. □

Besides the assumptions that $\mathbb{G}'_0$ exists and is bounded, we now add that it is such that

$$\mathbb{G}'_0 < \mathbf{0}, \tag{8.1.23}$$

which is a special case of (7.2.18). Moreover, since $\mathbb{G}_\infty$ is positive definite, from (8.1.22) it follows that $\mathbb{G}_0$ also has this property.

These results were derived at different times through various approaches. The most pertinent references are now noted.

Coleman [68] proved the symmetry of the instantaneous elastic modulus (8.1.13) from the second law in the form of the Clausius–Duhem inequality. The symmetry of the equilibrium elastic modulus (8.1.14) was obtained by Day [86] via the Clausius inequality. Apart from the inequality being strict, (8.1.18) was first derived by Graffi [169] in the case of isotropic materials by requiring that energy be dissipated in a period of a sinusoidal strain function $\mathbf{E}(t) = \mathbf{E}\sin\omega t$. Accordingly, (8.1.18) may be rightly referred to as *Graffi's inequality*.

The connection between (8.1.18) and energy dissipation is emphasized in [233], where the energy dissipated in one period $[0, d_P]$, $d_P = 2\pi/\omega$, is shown to be

$$\int_0^{d_P} \mathbf{T}(\mathbf{E}^t)\cdot\dot{\mathbf{E}}(t)dt = -\pi\mathbf{E}\cdot\mathbb{G}'_s(\omega)\mathbf{E}. \tag{8.1.24}$$

Incidentally, that is why $-\mathbb{G}'_s(\omega)$ is often referred to as the *loss modulus*. The inequality (8.1.21) for the initial derivative of the relaxation function was proved first by Bowen and Chen [40], by having recourse to discontinuous histories, in the one-dimensional case via the Clausius–Duhem inequality. The same result was proved in [265] with C^∞ histories in the three-dimensional case. The inequality (8.1.22) traces back to Coleman [67, 68].

Apparently, the inequality (8.1.19) first appeared in [120], but it is in a sense related to a previous result by Day [86] (cf. also [321]), who showed that as a consequence of dissipativity, the relaxation function satisfies the condition

$$\mathbb{G}_0 - \mathbb{G}_\infty \geq \pm[\mathbb{G}(s) - \mathbb{G}_\infty]. \tag{8.1.25}$$

To show the connection, observe that the limit $s\to\infty$ in the expression (8.1.20) for $\mathbb{G}(s) - \mathbb{G}_0$ gives (cf. (7.2.19))

$$\mathbb{G}_\infty - \mathbb{G}_0 = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}\mathbb{G}'_s(\omega)d\omega. \tag{8.1.26}$$

Consequently,

$$\mathbb{G}(s) - \mathbb{G}_\infty = -\frac{2}{\pi}\int_0^\infty \frac{\cos\omega s}{\omega}\mathbb{G}'_s(\omega)d\omega,$$

and the obvious inequalities

$$\int_0^\infty \frac{1}{\omega}|\mathbf{E}\cdot\mathbb{G}'_s(\omega)\mathbf{E}|d\omega \geq \int_0^\infty \frac{|\cos\omega s|}{\omega}|\mathbf{E}\cdot\mathbb{G}'_s(\omega)\mathbf{E}|d\omega \geq \left|\int_0^\infty \frac{\cos\omega s}{\omega}\mathbf{E}\cdot\mathbb{G}'_s(\omega)\mathbf{E}d\omega\right|$$

for any $\mathbf{E}\in\mathrm{Sym}$ yield (8.1.25).

A relation analogous to (8.1.25) can of course be given for the general theory (Sect. 7.2.2).

Remark 8.1.9. While (8.1.13) and (8.1.14) are enforced by thermodynamics, it is not necessarily the case that $\mathbb{G}(\tau)$ is symmetric for intermediate values of τ. We will, however, assume that (cf. (7.1.18))

$$\mathbb{G}(\tau) = \mathbb{G}^{\top}(\tau), \qquad \tau \in (0, \infty). \tag{8.1.27}$$

Since $\mathbb{G}' \in L^2(\mathbb{R}^+)$, Parseval's formula (C.3.1) allows us to write the constitutive equation (8.1.5) as [104]

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \frac{2}{\pi}\int_0^\infty \mathbb{G}'_s(\omega)\mathbf{E}^t_s(\omega)d\omega \tag{8.1.28}$$

for any $\mathbf{E}^t \in L^2(\mathbb{R}^+)$. This formula is obtained by extending the integral in (8.1.5) to $\mathbb{R}$ and taking the odd extension of $\mathbf{E}^t$, using $(\text{C.1.6})_1$. We can replace $\overline{\mathbb{G}'_+}(\omega)$ by $i\mathbb{G}'_s(\omega)$ because of the oddness of $\mathbf{E}^t_s(\omega)$. Since the integrand is now even, the integration interval can be transformed to $\mathbb{R}^+$.

A more general viewpoint on this kind of manipulation was adopted earlier to yield (7.2.33) and in particular (7.2.34), which corresponds to (8.1.28).

Now it is important to generalize (8.1.28) so that it holds for all $\mathbf{E}^t \in \mathcal{E}$, where $\mathcal{E}$ is the set of histories $\mathbf{E}^t$ such that

$$\left|\int_0^\infty \mathbb{G}'(s)\mathbf{E}^t(s)ds\right| < \infty$$

for a given $\mathbb{G}' \in L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. If we denote by $\mathcal{G}$ the vector space defined by

$$\mathcal{G} = \left\{\mathbb{F} : \mathbb{R}^+ \to \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym}); \mathbb{F} = \alpha\mathbb{G}' + \mathbf{f},\ \forall \mathbf{f} \in \mathbf{C}_0^\infty(\mathbb{R}^+)\right\},$$

then $\mathcal{E} = \mathcal{G}'$, where $\mathcal{G}'$ is the dual of $\mathcal{G}$ or the space of all linear continuous functionals on $\mathcal{G}$. Thus, the elements of $\mathcal{E}$ have a Fourier transform in a distributional sense. Relation (8.1.28) can be carried over into the set $\mathcal{E}$.

8.2 Decomposition of Stress

Consider the constitutive equation of linear viscoelasticity, given by (8.1.2) or (8.1.5). The integrals with $\mathbf{E}^t$ and $\mathbf{E}^t_r$ suggest the introduction of certain functions that will prove useful. These are defined by

$$\begin{aligned}\breve{\mathbf{I}}^t(\tau, \mathbf{E}^t) &:= \mathbb{G}(\tau)\mathbf{E}(t) + \int_0^\infty \mathbb{G}'(s+\tau)\mathbf{E}^t(s)ds \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \mathbf{I}^t(\tau, \mathbf{E}^t_r),\end{aligned} \tag{8.2.1}$$

where (cf. $(7.4.2)_2$)*

* The quantity $\mathbf{I}^t$ was originally defined in the literature as the negative of the functional used here. This change in sign, which is consistent with Sect. 7.4, is introduced here and later so that its relationship with the stress functional is a little more precise.

$$\mathbf{I}^t(\tau, \mathbf{E}_r^t) = \int_0^\infty \mathbb{G}'(s+\tau)\mathbf{E}_r^t(s)ds. \qquad (8.2.2)$$

It is easy to see that $\check{\mathbf{I}}^t$ coincides with the stress resulting from the partly static history (1.4.10), namely $\hat{\mathbf{T}}(\mathbf{E}^{t(\tau)})$, where τ is the duration of the static part.

One can derive from $\check{\mathbf{I}}^t$ both $\mathbf{E}(t)$ and $\mathbf{I}^t$ by virtue of the following relations:

$$\lim_{\tau\to\infty} \check{\mathbf{I}}^t(\tau, \mathbf{E}^t) = \mathbb{G}_\infty \mathbf{E}(t)$$

and

$$\mathbf{I}^t(\tau, \mathbf{E}_r^t) = \check{\mathbf{I}}^t(\tau, \mathbf{E}^t) - \lim_{\tau\to\infty} \mathbf{I}^t(\tau, \mathbf{E}^t).$$

Also, we have

$$\begin{aligned} \mathbf{I}^t(\tau, \mathbf{E}^t) &:= \int_0^\infty \mathbb{G}'(s+\tau)\mathbf{E}^t(s)ds \\ &= -\mathbb{G}(\tau)\mathbf{E}(t) + \check{\mathbf{I}}^t(\tau, \mathbf{E}^t) = -\check{\mathbb{G}}(\tau)\mathbf{E}(t) + \mathbf{I}^t(\tau, \mathbf{E}_r^t), \end{aligned} \qquad (8.2.3)$$

where

$$\check{\mathbb{G}}(\tau) := \mathbb{G}(\tau) - \mathbb{G}_\infty. \qquad (8.2.4)$$

The time derivative of $\mathbf{I}^t(\cdot, \mathbf{E}_r^t)$ with respect to t will be of interest. This is given by

$$\dot{\mathbf{I}}^t(\tau, \mathbf{E}_r^t) = \frac{d}{dt}\mathbf{I}^t(\tau, \mathbf{E}_r^t) = \check{\mathbb{G}}(\tau)\dot{\mathbf{E}}(t) + \mathbf{I}_{(1)}^t(\tau, \mathbf{E}_r^t), \qquad (8.2.5)$$

where

$$\mathbf{I}_{(1)}^t(\tau, \mathbf{E}_r^t) = \frac{d}{d\tau}\mathbf{I}^t(\tau, \mathbf{E}_r^t) = \int_0^\infty \mathbb{G}''(s+\tau)\mathbf{E}_r^t(s)ds. \qquad (8.2.6)$$

In the following we shall also use a simpler notation by writing $\check{\mathbf{I}}^t(\tau)$, $\mathbf{I}^t(\tau)$, and $\tilde{\mathbf{I}}^t(\tau)$ instead of $\check{\mathbf{I}}^t(\tau, \mathbf{E}^t)$, $\mathbf{I}^t(\tau, \mathbf{E}_r^t)$, and $\mathbf{I}^t(\tau, \mathbf{E}^t)$, respectively.

Let $t = 0$ be the initial instant when a process P_τ is applied to the material. The stress will be a function of the initial state σ and of this process. It can be written as follows:

$$\hat{\mathbf{T}}(\sigma, P_\tau) = \mathbb{G}_0\mathbf{E}(\tau) + \int_0^\tau \mathbb{G}'(u)\mathbf{E}^\tau(u)du + \int_\tau^\infty \mathbb{G}'(u)\mathbf{E}^\tau(u)du.$$

An integration by parts and a change of variable give

$$\hat{\mathbf{T}}(\sigma, P_\tau) = \int_0^\tau \mathbb{G}(\tau-u)\dot{\mathbf{E}}(u)du + \mathbb{G}(\tau)\mathbf{E}(0) + \int_0^\infty \mathbb{G}'(\xi+\tau)\mathbf{E}(-\xi)d\xi. \qquad (8.2.7)$$

We can identify the state σ with $(\mathbf{E}(0), \mathbf{E}^0)$, where the history is $\mathbf{E}^0(\xi) = \mathbf{E}(-\xi)$, $\xi \in \mathbb{R}^{++}$, and the process P_τ with $\dot{\mathbf{E}}_\tau^P$, defined as $\dot{\mathbf{E}}(u)$, $u \in [0.\tau)$. Moreover, in (8.2.7) we can distinguish two effects by putting

$$\hat{\mathbf{T}}(\mathbf{0}, \dot{\mathbf{E}}_\tau^P) = \int_0^\tau \mathbb{G}(\tau-u)\dot{\mathbf{E}}_\tau^P(u)du$$

and noting that $\breve{\mathbf{I}}^t$, given by (8.2.1), has the form at $t = 0$

$$\breve{\mathbf{I}}^0(\tau, \mathbf{E}^0) = \mathbb{G}(\tau)\mathbf{E}(0) + \int_0^\infty \mathbb{G}'(\xi + \tau)\mathbf{E}^0(\xi)d\xi.$$

Thus, we can write the stress (8.1.5) in the form

$$\mathbf{T}(\tau) = \hat{\mathbf{T}}(\mathbf{0}, \dot{\mathbf{E}}_\tau^P) + \breve{\mathbf{I}}^0(\tau, \mathbf{E}^0). \tag{8.2.8}$$

Here, we observe that $\hat{\mathbf{T}}(\mathbf{0}, \dot{\mathbf{E}}_\tau^P)$ denotes the stress determined by the initial zero state $\mathbf{0}$ $((\mathbf{E}(0) = 0, \mathbf{E}^0(\xi) = 0,\ \xi \in \mathbb{R}^+)$ and the process $\dot{\mathbf{E}}_\tau^P$, whereas $\breve{\mathbf{I}}^0(\tau, \mathbf{E}^0)$ is the stress determined by the initial state $(\mathbf{E}(0), \mathbf{E}^0)$ and the zero process $\mathbf{0}_\tau^\dagger$, with duration $\tau \in \mathbb{R}^+$; this zero process renders the first term on the right of (8.2.7) zero. Equation (8.2.8) means that $\mathbf{T}(\tau)$ can be viewed as the superposition of the two effects $\hat{\mathbf{T}}(\mathbf{0}, \dot{\mathbf{E}}_\tau^P)$, which involves only the process $\dot{\mathbf{E}}_\tau^P$, and $\mathbf{I}^0(\tau, \mathbf{E}^0)$, which involves only the state $(\mathbf{E}(0), \mathbf{E}^0)$.

8.3 Equivalence and Minimal States

We now suppose that the process is applied at time t, thus acting in the time interval $[t, t + \tau)$, where τ denotes its duration. For a linear viscoelastic solid, in the initial state $\mathbf{E}^t$ at time t, we have

$$\begin{aligned}\mathbf{T}(t + \tau) &= \mathbb{G}_0\mathbf{E}(t + \tau) + \int_0^\tau \mathbb{G}'(u)\mathbf{E}^{t+\tau}(u)du + \int_\tau^\infty \mathbb{G}'(u)\mathbf{E}^{t+\tau}(u)du \\ &= \mathbb{G}(\tau)\mathbf{E}(t) + \int_0^\tau \mathbb{G}(u)\dot{\mathbf{E}}^{t+\tau}(u)du + \int_0^\infty \mathbb{G}'(\tau + \xi)\mathbf{E}^t(\xi)d\xi.\end{aligned} \tag{8.3.1}$$

Definition 4.1.2 of equivalence yields some restrictions on the constitutive equations. The first result [31, 90] is the following theorem (cf. (7.4.3)).

Theorem 8.3.1. *Two histories $\mathbf{E}_1^t$, $\mathbf{E}_2^t$ of $\mathbf{E}$ are equivalent, relative to (8.1.5), if and only if*

$$\mathbf{E}_1^t(0) = \mathbf{E}_2^t(0), \qquad \int_0^\infty \mathbb{G}'(\xi + \tau)\mathbf{E}_1^t(\xi)d\xi = \int_0^\infty \mathbb{G}'(\xi + \tau)\mathbf{E}_2^t(\xi)d\xi \quad \forall \tau \geq 0. \tag{8.3.2}$$

Proof. We have the state $\sigma(t) = \mathbf{E}^t$. The requirement $\hat{\mathbf{T}}(\mathbf{E}_1^t, \dot{\mathbf{E}}_\tau^P) = \hat{\mathbf{T}}(\mathbf{E}_2^t, \dot{\mathbf{E}}_\tau^P)\ \forall \tau \geq 0$, taking into account (8.3.1), yields

$$\begin{aligned}&\mathbb{G}(\tau)\mathbf{E}_1(t) + \int_0^\infty \mathbb{G}'(\xi + t)\mathbf{E}_1^t(\xi)d\xi \\ &\quad = \mathbb{G}(\tau)\mathbf{E}_2(t) + \int_0^\infty \mathbb{G}'(\xi + t)\mathbf{E}_2^t(\xi)d\xi \quad \forall \tau \geq 0.\end{aligned} \tag{8.3.3}$$

Taking $\tau \to \infty$ gives $(8.3.2)_1$. Then $(8.3.2)_2$ follows immediately. □

Some important consequences of this theorem, considered in [123], will now be described.

Corollary 8.3.2. *For a viscoelastic material characterized by (8.1.5), the equivalence conditions for two histories* $\mathbf{E}_1^t$, $\mathbf{E}_2^t$ *can be expressed by*

$$\hat{\mathbf{T}}(\mathbf{E}_1^t, \mathbf{0}_\tau^\dagger) = \hat{\mathbf{T}}(\mathbf{E}_2^t, \mathbf{0}_\tau^\dagger) \qquad \forall\tau \in \mathbf{R}^+, \tag{8.3.4}$$

where $\mathbf{0}_\tau^\dagger$ *is the zero process of duration* $\tau \in \mathbb{R}^+$.

Proof. For any $\tau \in \mathbb{R}^+$ the relation (8.3.4) applied to (8.3.1) gives (8.3.3). □

By virtue of Theorem 8.3.1, we see that equivalent histories are characterized by the pair $(\mathbf{E}^t(0), \tilde{\mathbf{I}}^t)$, where $\tilde{\mathbf{I}}^t(\tau)$ is given by (8.2.3) (note the comment after (8.2.6)). Consequently, the state of a linear viscoelastic solid may be identified with the pair $(\mathbf{E}^t(0), \tilde{\mathbf{I}}^t)$ instead of the whole history $\mathbf{E}^t$. This observation was first made in [176], where the particular case with the kernel $\mathbb{G}'$ given by a sum of exponentials was studied.

Remark 8.3.3. The class σ_R of equivalent histories, by virtue of (8.2.3), can also be represented by the single function given by (8.2.1)

$$\breve{\mathbf{I}}^t(\tau) = \hat{\mathbf{T}}(\mathbf{E}^t, \mathbf{0}_\tau^\dagger) = \tilde{\mathbf{I}}^t(\tau) + \mathbb{G}(\tau)\mathbf{E}(t) \qquad \forall\tau \in \mathbb{R}^+, \tag{8.3.5}$$

where $\mathbf{E}^t$ is any history among those in σ_R, since by definition, the function $\breve{\mathbf{I}}^t(\tau)$ is the same for all $\mathbf{E}^t \in \sigma_R$. Moreover, the knowledge of $\breve{\mathbf{I}}^t$ on $\mathbb{R}^+$ provides

$$\mathbf{E}(t) = \mathbb{G}_\infty^{-1} \lim_{\tau\to\infty} \breve{\mathbf{I}}^t(\tau)$$

and hence also $\tilde{\mathbf{I}}^t$ by (8.3.5).

A minimal state is identified with an equivalence class represented by

$$\sigma_R(t) = (\mathbf{E}(t), \tilde{\mathbf{I}}^t(\cdot)) \tag{8.3.6}$$

or

$$\sigma_R(t) = \breve{\mathbf{I}}^t(\cdot).$$

The description of a state as minimal refers to the fact that it can be characterized by a minimum set of data. Examples are discussed in the next section.

8.4 State and History for Exponential-Type Relaxation Functions

It is of interest to consider materials for which the relaxation function is a linear combination of decaying exponentials, i.e.,

$$\mathbb{G}(\xi) = \mathbb{G}_\infty + \mathbf{\Lambda} \sum_{k=1}^{n} g_k \exp(-\alpha_k \xi),$$
$$\mathbb{G}'(\xi) = -\mathbf{\Lambda} \sum_{k=1}^{n} \alpha_k g_k \exp(-\alpha_k \xi), \tag{8.4.1}$$

where $\mathbf{\Lambda} \in \mathrm{Lin(Sym)}$ is positive definite and the coefficients g_k, α_k are positive, $k = 1, 2, \ldots, n$. We will sometimes refer to these as discrete-spectrum materials.

We will now show that the presence of exponentials allows us to express the state σ in terms of a finite number of quantities instead of the history $\mathbf{E}^t$, which is infinite-dimensional. The description of a state in terms of such quantities can be described as minimal, since it does not contain superfluous variables.

From $(8.4.1)_1$, we have

$$\mathbb{G}_0 = \mathbb{G}_\infty + \mathbf{\Lambda} \sum_{k=1}^{n} g_k.$$

Moreover, putting

$$\mathbf{T}_k(t) = \mathbf{\Lambda} g_k \left[\mathbf{E}(t) - \alpha_k \int_0^\infty \exp(-\alpha_k \xi) \mathbf{E}(t - \xi) d\xi \right],$$

the stress tensor, given by (8.1.5), becomes

$$\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t) + \sum_{k=1}^{n} \mathbf{T}_k(t).$$

We can consider the $(n + 1)$-tuple $(\mathbf{E}, \mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n)$, as the state at time t. Alternatively, let

$$\mathbf{E}_k(t) = \int_0^\infty \exp(-\alpha_k \xi) \mathbf{E}(t - \xi) d\xi, \tag{8.4.2}$$

giving

$$\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t) + \mathbf{\Lambda} \sum_{k=1}^{n} g_k [\mathbf{E}(t) - \alpha_k \mathbf{E}_k(t)].$$

Thus, we can also consider the state as the $(n + 1)$-tuple $(\mathbf{E}, \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n)$.

These two modes of description of state are related to but not the same as that discussed in Sect. 8.3. Let us now consider how σ_R might be described for a viscoelastic material with a relaxation function of the form $(8.4.1)_1$.

Let $\sigma_R(t)$ be given by (8.3.6). Using $(8.2.3)_1$ and $(8.4.1)_2$, we obtain

$$\tilde{\mathbf{I}}^t(\tau) = -\mathbf{\Lambda} \sum_{i=1}^{n} \alpha_i g_i \exp(-\alpha_i \tau) \mathbf{E}_i(t),$$

where the quantities $\mathbf{E}_i$ are defined by (8.4.2). The derivatives of the function $\tilde{\mathbf{I}}^t(\tau)$ with respect to τ, at $\tau = 0$, are given by

$$\tilde{\mathbf{I}}^t_p(0) = \mathbf{\Lambda} \sum_{i=1}^{n} g_i(-\alpha_i)^{p+1}\mathbf{E}_i(t), \qquad p = 0, 1, \ldots, n-1.$$

Thus, we obtain a linear system, which can be solved for $\mathbf{E}_1(t), \ldots, \mathbf{E}_n(t)$ in terms of the quantities $\tilde{\mathbf{I}}^t(0), \tilde{\mathbf{I}}^t_1(0), \ldots, \tilde{\mathbf{I}}^t_{(n-1)}(0)$. Accordingly, putting

$$\sigma_R = (\mathbf{E}(t), \tilde{\mathbf{I}}^t(0), \tilde{\mathbf{I}}^t_1(0), \ldots, \tilde{\mathbf{I}}^t_{(n-1)}(0)),$$

the state is $(n+1)$-dimensional.

8.5 Inversion of Constitutive Relations

The inversion of the constitutive equations of linear viscoelasticity has been studied in [41] as a Wiener–Hopf problem. We now describe a more direct approach [123]. Let us consider the constitutive equation (8.1.5), namely

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \mathbb{G}'(s)\mathbf{E}(t-s)ds, \tag{8.5.1}$$

where the domain of $\mathbb{G}'$ is carried over to $\mathbb{R}$ by putting $\mathbb{G}'(s) = \mathbf{0}\ \forall s < 0$. Hence, we also have

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_{-\infty}^\infty \mathbb{G}'(s)\mathbf{E}(t-s)ds.$$

Putting, for formal convenience,

$$\mathbf{H}(t) = \mathbb{G}_0\mathbf{E}(t), \qquad \mathbb{K}(s) = \mathbb{G}'(s)\mathbb{G}_0^{-1},$$

and assuming that $\mathbf{H} \in L^1(\mathbb{R})$, we have

$$\mathbf{T}(t) = \mathbf{H}(t) + \int_{-\infty}^\infty \mathbb{K}(s)\mathbf{H}(t-s)ds \qquad \forall t \in \mathbb{R}.$$

Taking the Fourier transform and applying the convolution theorem (C.3.3) gives

$$\mathbf{T}_F(\omega) = [\mathbf{1} + \mathbb{K}_+(\omega)]\mathbf{H}_F(\omega), \tag{8.5.2}$$

where, referring to (8.1.16),

$$\mathbb{K}_+(\omega) = \mathbb{G}'_+(\omega)\mathbb{G}_0^{-1} = [\mathbb{G}'_c(\omega) - i\mathbb{G}'_s(\omega)]\mathbb{G}_0^{-1}. \tag{8.5.3}$$

It follows that

$$\mathbb{K}_+(0) = [\mathbb{G}_\infty - \mathbb{G}_0]\mathbb{G}_0^{-1} = \mathbb{G}_\infty\mathbb{G}_0^{-1} - \mathbf{1}, \tag{8.5.4}$$

since $\mathbb{G}'_+(0) = \mathbb{G}_\infty - \mathbb{G}_0$; this follows immediately from the definition of $\mathbb{G}'_+(\omega)$, given by $(8.1.16)_1$.

Remark 8.5.1. We see from Proposition C.2.1 that $\mathbb{K}_+(\omega)$ or $[\mathbf{1} + \mathbb{K}_+(\omega)]$ has all its singularities in the upper half-plane and thus is analytic in the lower half-plane. This is equivalent to the requirement that (8.5.1) be a causal relationship.

The quantity $\mathbf{1} + \mathbb{K}_+(\omega)$ is invertible for any $\omega \in \mathbb{R}$. We see this by observing that for real ω,

$$\begin{aligned} \mathrm{Im}[\mathbf{1} + \mathbb{K}_+(\omega)] &= -\mathbb{G}_s'(\omega)\mathbb{G}_0^{-1} \neq \mathbf{0} \quad \forall\omega \neq 0, \\ \mathrm{Re}[\mathbf{1} + \mathbb{K}_+(0)] &= \mathbb{G}_\infty\mathbb{G}_0^{-1} \neq \mathbf{0}. \end{aligned}$$

Accordingly, using (8.5.2), we can express $\mathbf{H}_F(\omega)$ in the form

$$\mathbf{H}_F(\omega) = \mathbf{T}_F(\omega) - [\mathbf{1} + \mathbb{K}_+(\omega)]^{-1}\mathbb{K}_+(\omega)\mathbf{T}_F(\omega).$$

Taking inverse Fourier transforms yields

$$\mathbf{H}(t) = \mathbf{T}(t) + \mathbb{G}_0 \int_{-\infty}^{\infty} \mathbb{J}'(\xi)\mathbf{T}(t-\xi)d\xi,$$

where

$$\mathbb{G}_0\mathbb{J}'(\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} [1 + \mathbb{K}_+(\omega)]^{-1}\mathbb{K}_+(\omega)\exp(i\omega\xi)d\xi. \tag{8.5.5}$$

Applying $\mathbb{G}_0^{-1}$, we have

$$\mathbf{E}(t) = \mathbb{G}_0^{-1}\mathbf{T}(t) + \int_{-\infty}^{\infty} \mathbb{J}'(\xi)\mathbf{T}(t-\xi)d\xi.$$

Now, this relationship must be causal; in other words, we must be able to write it in the form

$$\mathbf{E}(t) = \mathbb{J}_0\mathbf{T}(t) + \int_0^{\infty} \mathbb{J}'(\xi)\mathbf{T}(t-\xi)d\xi, \qquad \mathbb{J}_0 = \mathbb{G}_0^{-1} > \mathbf{0}, \tag{8.5.6}$$

so that

$$\mathbb{J}'(\xi) = \mathbf{0}, \qquad \xi < 0.$$

Remark 8.5.2. It follows from Proposition C.2.1 and (8.5.5) that the zeros of $\mathbf{1} + \mathbb{K}_+(\omega)$ must also be in the upper half-plane [167]. These observations apply where $\mathbf{1} + \mathbb{K}_+(\omega)$ has zeros and isolated singularities. If branch-cut singularities (see Sect. B.1.5), in particular continuous-spectrum singularities, are present, these must be in the upper half-plane for both $\mathbf{1}+\mathbb{K}_+(\omega)$ and its inverse. This is discussed further in Sect. 16.5, in particular in Remark 16.5.1.

As in (8.1.7) for the relaxation function, we define the creep function as

$$\mathbb{J}(s) = \mathbb{J}_0 + \int_0^s \mathbb{J}'(\xi)d\xi, \qquad \mathbb{J}_0 = \mathbb{J}(0). \tag{8.5.7}$$

The inverse Fourier transform of (8.5.5) gives that

$$\mathbb{G}_0^{-1}[\mathbf{1} + \mathbb{K}_+(\omega)]^{-1}\mathbb{K}_+(\omega) = -\int_0^\infty \mathbb{J}'(\xi)e^{-i\omega\xi}\,d\xi. \tag{8.5.8}$$

Let us assume that $\mathbb{J}(\infty)$ is finite. This implies that the material is a solid [167]. Combining (8.5.4) and (8.5.8) at $\omega = 0$, one deduces with the aid of $(8.5.6)_2$ that

$$\mathbb{J}(\infty) = \mathbb{J}_\infty = \mathbb{G}_\infty^{-1} > \mathbf{0}. \tag{8.5.9}$$

Recalling (8.1.22), we see that

$$\mathbb{J}_0 = \mathbb{G}_0^{-1} \leq \mathbb{G}_\infty^{-1} = \mathbb{J}_\infty, \tag{8.5.10}$$

so that the creep function $\mathbb{J}(s)$ is, at least in an overall sense, increasing, while the relaxation function $\mathbb{G}(s)$ is, at least in an overall sense, decreasing.

Indeed, we can come at this result in another way. Let us write (8.5.8) in the form

$$\mathbb{G}_0\mathbb{J}'_+(\omega) = -[\mathbf{1} + \mathbb{K}_+(\omega)]^{-1}\mathbb{K}_+(\omega) = [\mathbf{1} + \mathbb{K}_+(\omega)]^{-1} - \mathbf{1}. \tag{8.5.11}$$

Since, by virtue of (8.1.27), all quantities are symmetric in Lin(Sym), we have that $\overline{\mathbb{K}}_+(\omega) = \mathbb{K}^*(\omega)$, for example. It will be assumed that $\mathbb{K}_+(\omega)$ and $\overline{\mathbb{K}}_+(\omega)$ (or $\mathbb{K}^*_+(\omega)$) commute, so that $\mathbb{K}_+(\omega)$ is a normal transformation (see after (A.2.11)) and that they commute with $\mathbb{G}_0$. These properties will in fact hold under the assumption made in Sect. 7.1.5. Then, from (8.5.11) and (8.5.3),

$$\begin{aligned}[\mathbf{1} + \mathbb{K}_+(\omega)][\mathbf{1} + \overline{\mathbb{K}}_+(\omega)]\mathbb{G}_0\left[\overline{\mathbb{J}}'_+(\omega) - \mathbb{J}'_+(\omega)\right]& \\ = \mathbb{K}_+(\omega) - \overline{\mathbb{K}}_+(\omega) = -2i\mathbb{G}'_s(\omega)\mathbb{G}_0^{-1}&.\end{aligned} \tag{8.5.12}$$

Since $[\mathbf{1} + \mathbb{K}_+(\omega)][\mathbf{1} + \overline{\mathbb{K}}_+(\omega)]\mathbb{G}_0$ is a nonnegative invertible tensor, we deduce from (8.1.18) that

$$\mathbb{J}'_s(\omega) = \frac{1}{2i}\left[\overline{\mathbb{J}}'_+(\omega) - \mathbb{J}'_+(\omega)\right] > 0. \tag{8.5.13}$$

The analogue of Corollary 8.1.7 yields that

$$\mathbb{J}(s) - \mathbb{J}_0 > \mathbf{0}, \quad s \in \mathbb{R}^+,$$

and in particular, (8.5.10) follows.

The general linear relation (7.1.33) can be inverted in a manner similar to that outlined above.

Let $\mathbf{E}$ or $\mathbf{T}$, on $[0, t)$, be regarded as the process; then we can write

$$\mathbf{T}^P(t) = \mathbb{G}_0\mathbf{E}^P(t) + \int_0^t \mathbb{G}'(s)\mathbf{E}^P(t - s)ds + \hat{\mathbf{I}}(t) \tag{8.5.14}$$

and

$$\mathbf{E}^P(t) = \mathbb{J}_0\mathbf{T}^P(t) + \int_0^t \mathbf{J}'(\xi)\mathbf{T}^P(t - \xi)d\xi + \hat{\mathbf{Y}}(t), \tag{8.5.15}$$

where $\mathbf{E}^P$ and $\mathbf{T}^P$ denote the restrictions of $\mathbf{E}$ and $\mathbf{T}$ to $[0,t)$. Hence, we put $\mathbf{E}^P = \mathbf{0}$, $\mathbf{T}^P = \mathbf{0}$ on $(-\infty, 0)$. Moreover, recalling (8.3.1), we see that the two equations (8.5.14) and (8.5.15) hold if

$$\hat{\mathbf{I}}(t) = \begin{cases} \int_0^\infty \mathbb{G}'(t+\xi)\mathbf{E}^0(\xi)d\xi, & \forall t > 0, \\ \mathbf{0}, & \forall t \leq 0, \end{cases} \tag{8.5.16}$$

and

$$\hat{\mathbf{Y}}(t) = \begin{cases} \int_0^\infty \mathbb{J}'(t+\xi)\mathbf{T}^0(\xi)d\xi, & \forall t > 0, \\ \mathbf{0}, & \forall t \leq 0. \end{cases}$$

Let us now consider (8.5.14). The domain of $\mathbb{G}'$ is carried over to $\mathbb{R}$ by putting $\mathbb{G}'(s) = \mathbf{0}\ \forall s < 0$. Again, we put $\mathbf{H}(t) = \mathbb{G}_0\mathbf{E}^P(t)$, $\mathbb{K}(s) = \mathbb{G}'(s)\mathbb{G}_0^{-1}$ and assume that

$$\mathbf{H} \in L^1(\mathbb{R}), \qquad \hat{\mathbf{I}} \in L^1(\mathbb{R}).$$

Hence,

$$\mathbf{T}^P(t) = \mathbf{H}(t) + \int_{-\infty}^{\infty} \mathbb{K}(s)\mathbf{H}(t-s)ds + \hat{\mathbf{I}}(t) \qquad \forall\, t \in \mathbb{R}.$$

After applying a Fourier transform, we can solve for $\mathbf{H}_F$, obtaining

$$\mathbf{H}_F(\omega) = \mathbf{T}_F^P(\omega) - \hat{\mathbf{I}}_F(\omega) - [\mathbf{1} + \mathbb{K}_+(\omega)]^{-1}\mathbb{K}_+(\omega)[\mathbf{T}_F^P(\omega) - \hat{\mathbf{I}}_F(\omega)].$$

Using the inverse Fourier transform, we have

$$\mathbf{E}^P(t) = \mathbb{G}_0^{-1}[\mathbf{T}^P(t) - \hat{\mathbf{I}}(t)] + \int_{-\infty}^{\infty} \mathbb{J}'(\xi)\mathbf{T}^P(t-\xi)d\xi - \int_{-\infty}^{\infty} \mathbb{J}'(\xi)\hat{\mathbf{I}}(t-\xi)d\xi \quad \forall\, t \in \mathbb{R},$$

and hence

$$\begin{aligned} \mathbf{E}^P(t) = \mathbb{G}_0^{-1}\mathbf{T}^P(t) &+ \int_0^t \mathbb{J}'(\xi)\mathbf{T}^P(t-\xi)d\xi - \mathbb{G}_0^{-1}\hat{\mathbf{I}}(t) \\ &- \int_0^\infty \mathbb{J}'(\xi)\hat{\mathbf{I}}(t-\xi)d\xi \quad \forall\, t \in \mathbf{R}. \end{aligned} \tag{8.5.17}$$

Equations (8.5.15) and (8.5.17) are required to provide the same values of $\mathbf{E}^P$ on $\mathbb{R}$. Thus, we find that

$$\hat{\mathbf{Y}}(t) = -\mathbb{J}_0\hat{\mathbf{I}}(t) - \int_0^\infty \mathbb{J}'(\xi)\hat{\mathbf{I}}(t-\xi)d\xi \quad \forall t \in \mathbb{R}.$$

8.6 Linear Viscoelastic Free Energies as Quadratic Functionals

We now give a representation of a free energy in the linear viscoelastic case and examine some of its properties [91, 105, 158].

Relation (7.1.9) reduces to

$$\psi(t) = \widetilde{\phi}(\mathbf{E}(t)) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s) \cdot \mathbb{G}_{12}(s,u)\mathbf{E}_r^t(u)dsdu, \tag{8.6.1}$$

where

$$\mathbb{G}_{12}(s,u) = \frac{\partial^2}{\partial s \partial u}\mathbb{G}(s,u). \tag{8.6.2}$$

By Definition 4.1.6, we must have $\psi \geq 0$ for all states $(\mathbf{E}(t), \mathbf{E}^t)$, in particular for $(\mathbf{0}, \mathbf{E}^t)$, where the history is arbitrary. Thus, $\mathbb{G}_{12}$ must be a nonnegative operator in the sense that the integral in (8.6.17) must be nonnegative for all histories. Therefore, we have

$$\psi(t) \geq \phi(t), \tag{8.6.3}$$

which is a special case of (5.1.25), a property that follows very generally from the second law.

Since the integral in (8.6.1) must exist for finite relative histories, we assume that (cf. (7.1.3)) as $s \to \infty$, the kernel $\mathbb{G}_{12}(s,u)$ goes to zero as

$$\mathbb{G}_{12}(s,u) \sim s^{-1-b}, \quad b > 0, \quad u \in \mathbb{R}^+, \tag{8.6.4}$$

or more strongly; similarly for the limit of large u at fixed s.

Let us define (cf. (7.1.5))

$$\mathbb{G}(s,u) = \mathbb{G}_\infty + \int_s^\infty \int_u^\infty \mathbb{G}_{12}(s',u')ds'\,du', \quad \mathbb{G}_\infty = \mathbb{G}(\infty,\infty). \tag{8.6.5}$$

Note that we are adopting the conventional choice as specified in Remark 7.1.2. Relations (7.1.7) are replaced by

$$\lim_{s\to\infty} \mathbb{G}(s,u) = \mathbb{G}_\infty, \qquad \lim_{s\to\infty} \frac{\partial}{\partial u}\mathbb{G}(s,u) = \mathbf{0}, \quad u \in \mathbb{R}^+, \tag{8.6.6}$$

with similar limits at large u holding for fixed s.

Remark 8.6.1. In fact, $(7.1.7)_1$ could be retained by using

$$\mathcal{G}(s,u) = \mathbb{G}(s,u) - \mathbb{G}_\infty$$

instead of $\mathbb{G}$.

We impose the conditions

$$\mathbb{G}(s) = \mathbb{G}(0,s) = \mathbb{G}(s,0), \qquad \mathbb{G}(0) = \mathbb{G}_0 = \mathbb{G}(0,0), \tag{8.6.7}$$

where $\mathbb{G}(s)$ is the relaxation function. This ensures the correct constitutive relations, as does $(7.1.14)_3$ in the general case. Relations

$$\mathbb{G}_1(s,0) = \mathbb{G}_2(0,s) = \mathbb{G}'(s), \tag{8.6.8}$$

where $\mathbb{G}'(s)$ is the derivative of the relaxation function $\mathbb{G}(s)$, are an immediate consequence. Note that (8.6.5) gives

$$\mathbb{G}(s,\infty) = \mathbb{G}(\infty,s) = \mathbb{G}_\infty \qquad \forall s \in \mathbb{R}^+, \tag{8.6.9}$$

from which, with (8.6.7), we deduce that

$$\mathbb{G}(\infty) = \mathbb{G}_\infty. \tag{8.6.10}$$

It follows from (8.6.5) and (8.6.4) that $\mathbb{G}_1(s,u)$ and $\mathbb{G}_2(s,u)$ vanish at large s,u, respectively, a property corresponding to (7.1.8).

Equation $(7.1.6)_3$ reduces to

$$\mathbb{G}^\top(s,u) = \mathbb{G}(u,s). \tag{8.6.11}$$

Replacing $\widetilde{\phi}(\mathbf{E}(t))$ by $\phi(t)$ and carrying out two partial integrations, we can write (8.6.1) in the form

$$\psi(t) = \phi(t) - \frac{1}{2}\mathbf{E}(t)\cdot\mathbb{G}_\infty\mathbf{E}(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \dot{\mathbf{E}}^t(s)\cdot\mathbb{G}(s,u)\dot{\mathbf{E}}^t(u)dsdu. \tag{8.6.12}$$

From $(7.1.14)_3$, we have

$$\mathbb{G}(s,0) = \mathbb{G}(0,s) = \mathbb{G}(s). \tag{8.6.13}$$

In the completely linear case,

$$\phi(t) = \frac{1}{2}\mathbf{E}(t)\cdot\mathbb{G}_\infty\mathbf{E}(t). \tag{8.6.14}$$

It is nonnegative by (1.4.13). Thus, (8.6.12) becomes

$$\psi(t) = \frac{1}{2}\int_0^\infty\int_0^\infty \dot{\mathbf{E}}^t(s)\cdot\mathbb{G}(s,u)\dot{\mathbf{E}}^t(u)dsdu.$$

Also, with the aid of $(7.1.14)_5$, the form (7.1.19) becomes

$$\begin{aligned}\psi(t) &= S(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \mathbf{E}^t(s)\cdot\mathbb{G}_{12}(s,u)\mathbf{E}^t(u)dsdu,\\ S(t) &= \phi(t) + \mathbf{E}(t)\cdot[\widehat{\mathbf{S}}(t) - \widehat{\mathbf{S}}_e(t) - (\mathbb{G}_0 - \mathbb{G}_\infty)\mathbf{E}(t)]\\ &\quad + \frac{1}{2}\mathbf{E}(t)\cdot(\mathbb{G}_0 - \mathbb{G}_\infty)\mathbf{E}(t).\end{aligned}$$

In the completely linear case, the form of S is given by (see (7.1.34))

$$S(t) = \mathbf{E}(t)\cdot\mathbf{T}(t) - \frac{1}{2}\mathbf{E}(t)\cdot\mathbb{G}_0\mathbf{E}(t). \tag{8.6.15}$$

These results derive from the general theory for which the equilibrium stress may be nonlinear. It is instructive, however, to work through the completely linear case in some detail.

Relation (5.1.11) reduces to the form

$$\dot{\psi}(t) + D(t) = \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t), \tag{8.6.16}$$

for linear viscoelastic materials under isothermal conditions. The quantity $D(t)$ denotes the internal dissipation function, which must be nonnegative because of thermodynamic considerations.

We consider the quadratic functional form

$$\psi(t) = \frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s) \cdot \mathbb{G}_{12}(s,u)\mathbf{E}_r^t(u)dsdu, \tag{8.6.17}$$

instead of (8.6.1).

The functional (8.6.17) must satisfy (8.6.16), a constraint that will now be explored. The time derivative of (8.6.17), using (8.1.2), (8.1.3) and noting the relations

$$\frac{d}{du}\mathbf{E}_r^t(u) = \frac{d}{du}\mathbf{E}^t(u) = -\frac{d}{dt}\mathbf{E}^t(u) = -\dot{\mathbf{E}}^t(u),$$

gives, with the aid of some integrations by parts,

$$\begin{aligned}
\dot{\psi}(t) &= \dot{\mathbf{E}}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \Big\{[\dot{\mathbf{E}}^t(s) - \dot{\mathbf{E}}(t)] \cdot \mathbb{G}_{12}(s,u)\mathbf{E}_r^t(u) \\
&\quad + \mathbf{E}_r^t(s) \cdot \mathbb{G}_{12}(s,u)[\dot{\mathbf{E}}^t(u) - \dot{\mathbf{E}}(t)]\Big\}dsdu \\
&= \dot{\mathbf{E}}(t) \cdot \left[\mathbb{G}_\infty \mathbf{E}(t) + \int_0^\infty \mathbb{G}'(s)\mathbf{E}_r^t(s)ds\right] \\
&\quad + \frac{1}{2}\int_0^\infty \int_0^\infty \dot{\mathbf{E}}^t(s) \cdot [\mathbb{G}_1(s,u) + \mathbb{G}_2(s,u)]\dot{\mathbf{E}}^t(u)dsdu \\
&= \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \dot{\mathbf{E}}^t(s) \cdot \mathbb{K}(s,u)\dot{\mathbf{E}}^t(u)dsdu \\
&= \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s) \cdot \mathbb{K}_{12}(s,u)\mathbf{E}_r^t(u)dsdu,
\end{aligned} \tag{8.6.18}$$

where $\mathbb{K}_{12}(s,u)$ denotes differentiation with respect to the arguments of[†]

$$\mathbb{K}(s,u) = \mathbb{G}_1(s,u) + \mathbb{G}_2(s,u). \tag{8.6.19}$$

Comparing (8.6.18) with (8.6.16), it follows that

$$\begin{aligned}
D(t) &= -\frac{1}{2}\int_0^\infty \int_0^\infty \dot{\mathbf{E}}^t(s) \cdot \mathbb{K}(s,u)\dot{\mathbf{E}}^t(u)dsdu \\
&= -\frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s) \cdot \mathbb{K}_{12}(s,u)\mathbf{E}_r^t(u)dsdu \geq 0,
\end{aligned} \tag{8.6.20}$$

so that thermodynamics requires that $\mathbb{K}$ and $\mathbb{K}_{12}$ must yield a nonnegative integral. These expressions are special cases of (7.1.25) and (7.1.26).

Note that we can summarize the calculation in (8.6.18) by

[†] Note that the tensor $\mathbb{K}$ here is quite different from the quantity used briefly in the early part of Sect. 7.1.

$$\dot{\widetilde{\psi}}(\mathbf{E}^t, \mathbf{E}(t)) = \frac{\partial}{\partial \mathbf{E}(t)} \widetilde{\psi}(\mathbf{E}^t), \mathbf{E}(t) \cdot \dot{\mathbf{E}}(t) + \partial_t \widetilde{\psi}(\mathbf{E}^t, \mathbf{E}(t)), \tag{8.6.21}$$

where

$$\frac{\partial}{\partial \mathbf{E}(t)} \widetilde{\psi}(\mathbf{E}^t, \mathbf{E}(t)) = \tilde{\mathbf{T}}(\mathbf{E}^t, \mathbf{E}(t)) = \mathbf{T}(t) \tag{8.6.22}$$

and ∂_t indicates differentiation with respect to the t dependence in $\mathbf{E}^t$. We express this as a functional derivative in Sect. 5.1. Relation (8.6.22) follows by comparing (8.6.21) with $(8.6.18)_4$ because the rightmost term in $(8.6.18)_4$ has no dependence on $\dot{\mathbf{E}}(t)$.

Remark 8.6.2. We treat (8.6.22), (8.6.3), and the nonnegative property of D in (8.6.16) as the defining properties of a free energy, referred to as the Graffi conditions in Sect. 5.1.1.

Alternative forms of relations (8.6.17) can be given in terms of histories rather than relative histories. Partial integrations give

$$\begin{aligned}\psi(t) = &\frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) - \frac{1}{2}\mathbf{E}_r^t(\infty) \cdot \mathbb{G}_\infty \mathbf{E}_r^t(\infty) \\ &+ \frac{1}{2}\int_0^\infty \int_0^\infty \dot{\mathbf{E}}^t(s) \cdot \mathbb{G}(s,u)\dot{\mathbf{E}}^t(u) ds du\end{aligned}$$

if $\mathbf{E}(-\infty) = \mathbf{E}^t(\infty)$ is finite. The first two terms on the right cancel if $\mathbf{E}(-\infty)$ is zero.

Also,

$$\begin{aligned}\psi(t) = &\frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(s_1, s_2)\mathbf{E}^t(s_2) ds_1 ds_2 \\ &- \mathbf{E}(t) \cdot \int_0^\infty \int_0^\infty \mathbb{G}_{12}(s_1, s_2)\mathbf{E}^t(s_2) ds_1 ds_2 \\ &+ \frac{1}{2}\mathbf{E}(t) \cdot \int_0^\infty \int_0^\infty \mathbb{G}_{12}(s_1, s_2) ds_1 ds_2 \mathbf{E}(t).\end{aligned}$$

From (8.6.5) and $(8.6.7)_2$ we have

$$\int_0^\infty \int_0^\infty \mathbb{G}_{12}(s_2, s_1) ds_1 ds_2 = \mathbb{G}_0 - \mathbb{G}_\infty.$$

Carrying out an integration in the second integral, we obtain

$$\begin{aligned}\psi(t) = &\frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_0 \mathbf{E}(t) + \mathbf{E}(t) \cdot \int_0^\infty \mathbb{G}'(s_1)\mathbf{E}^t(s_1) ds_1 \\ &+ \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(s_2, s_1)\mathbf{E}^t(s_2) ds_1 ds_2, \\ = &S(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(s_2, s_1)\mathbf{E}^t(s_2) ds_1 ds_2,\end{aligned} \tag{8.6.23}$$

where $S(t)$ is given by (8.6.15). Recalling Remark 7.1.2, we see that this relation is a special case of (7.1.34) if we identify $\mathbb{L}_e$ in that relation with $\mathbb{L}_\infty$. Note that

$$\frac{\partial S(t)}{\partial \mathbf{E}(t)} = \mathbf{T}(t),$$

and (8.6.22) follows immediately.

8.6.1 General Forms of a Free Energy in Terms of Stress

The representation (8.6.17) of a free energy can be given in terms of stress history, rather than strain history, using (8.5.6). Also, consider the following argument. Let us introduce a functional χ with the properties

$$\frac{\partial \chi(t)}{\partial \mathbf{T}(t)} = \mathbf{E}(t) \tag{8.6.24}$$

and

$$\dot{\chi}(t) - D_1(t) = \mathbf{E}(t) \cdot \dot{\mathbf{T}}(t). \tag{8.6.25}$$

Noting the developments from (8.6.16) onward, we see that χ can be represented as

$$\begin{gathered}\chi(t) = \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_\infty \mathbf{T}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{T}_r^t(s) \cdot \mathbb{J}_{12}(s,u)\mathbf{T}_r^t(u)dsdu, \\ \mathbb{J}(s,0) = \mathbb{J}(0,s) = \mathbb{J}(s), \qquad \mathbb{J}_\infty = \mathbb{J}(\infty),\end{gathered} \tag{8.6.26}$$

in terms of the creep function defined by (8.5.5) and (8.5.7) and where $\mathbb{J}(\cdot,\cdot)$ has similar properties to those listed for $\mathbb{G}(\cdot,\cdot)$ in (8.6.6)–(8.6.11). Also, referring to (8.6.19) and (8.6.20), we see that

$$\begin{gathered}D_1(t) = \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{T}_r^t(s) \cdot \mathbb{N}_{12}(s,u)\mathbf{T}_r^t(u)dsdu, \\ \mathbb{N}(s,u) = \mathbb{J}_1(s,u) + \mathbb{J}_2(s,u).\end{gathered} \tag{8.6.27}$$

We can write (8.6.26) as

$$\begin{gathered}\chi(t) = S_1(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{T}^t(s) \cdot \mathbb{J}_{12}(s,u)\mathbf{T}^t(u)dsdu, \\ S_1(t) = \mathbf{T}(t) \cdot \mathbf{E}(t) - \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_0 \mathbf{T}(t), \qquad \mathbb{J}_0 = \mathbb{J}(0,0) = \mathbb{J}(0),\end{gathered} \tag{8.6.28}$$

by analogy with (8.6.23) and (8.6.15). Relation (8.6.25) can be rewritten as

$$\frac{d}{dt}[\mathbf{E}(t) \cdot \mathbf{T}(t) - \chi(t)] + D_1(t) = \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t). \tag{8.6.29}$$

Comparison with (8.6.16) allows us to identify the quantity ψ, defined by the Legendre transformation

$$\psi(t) = \mathbf{E}(t) \cdot \mathbf{T}(t) - \chi(t),$$

as a free energy and $D_1(t)$ as the associated rate of dissipation, denoted conventionally by $D(t)$. We have

$$\frac{\partial \psi(t)}{\partial \mathbf{E}(t)} = \mathbf{T}(t) + \mathbf{E}(t) \cdot \frac{\partial \mathbf{T}(t)}{\partial \mathbf{E}(t)} - \frac{\partial \chi(t)}{\partial \mathbf{T}(t)} \cdot \frac{\partial \mathbf{T}(t)}{\partial \mathbf{E}(t)} = \mathbf{T}(t),$$

by virtue of (8.6.24). From (8.6.26) and (8.6.28), it follows that

$$\begin{aligned}\psi(t) &= \mathbf{E}(t) \cdot \mathbf{T}(t) - \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_\infty \mathbf{T}(t) - \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{T}_r^t(s) \cdot \mathbb{J}_{12}(s,u)\mathbf{T}_r^t(u) ds du \\ &= \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_0 \mathbf{T}(t) - \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{T}^t(s) \cdot \mathbb{J}_{12}(s,u)\mathbf{T}^t(u) ds du.\end{aligned} \tag{8.6.30}$$

Proposition 8.6.3. *The kernel* $\mathbb{J}_{12}$ *must be such that the double integral on the right of* (8.6.30) *is nonpositive and that on the right of* (8.6.27) *is nonnegative.*

Proof. The nonpositivity of the integral on the right of (8.6.30) is sufficient to ensure the non-negativity of $\psi(t)$. We argue that it is also necessary. The property (8.6.3) must always hold, where $\phi(t)$ is the free energy for the special history $\mathbf{E}^t(s) = \mathbf{E}(t)$, $s \le t$. It follows from (8.1.2) that $\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t)$ for this history and indeed $\mathbf{T}(t-u) = \mathbf{T}(t)$, $u \ge 0$, so that the history of stress is also constant in this limit. Thus, we have

$$\begin{aligned}\psi(t) &\ge \frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) = \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{G}_\infty^{-1} \mathbf{T}(t) \\ &= \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_\infty \mathbf{T}(t) \ge \frac{1}{2}\mathbf{T}(t) \cdot \mathbb{J}_0 \mathbf{T}(t),\end{aligned}$$

by virtue of (8.5.9) and (8.5.10). This means that the double integral in (8.6.30) must be nonpositive.

The second law requires that $D_1(t)$ in (8.6.29) be nonnegative, which leads to the claimed property of the integral in (8.6.27). □

8.6.2 The Work Function as a Free Energy

We now consider a particular case of the expression (8.6.17), obtained on supposing that $\mathbb{G}_{12}(s_1, s_2) = \mathbb{G}_{12}(|s_1 - s_2|)$, which clearly obeys (8.6.8). Thus, we introduce the following functional (cf. (7.5.3)):

$$\psi_M(t) = \frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s_1) \cdot \mathbb{G}_{12}(|s_1 - s_2|)\mathbf{E}_r^t(s_2) ds_1 ds_2. \tag{8.6.31}$$

Note that by (8.6.11), we can take $\mathbb{G}_{12}(u)$, $u \in \mathbb{R}^+$, to be symmetric, which was in any case assumed in Sect. 8.1 (see (8.1.27)).

Observe that for the choice of kernel in (8.6.31), the quantity $\mathbb{K}$, given by (8.6.19), vanishes, so that D is zero and (8.6.16) reduces to

$$\dot{\psi}_M(t) = \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t). \tag{8.6.32}$$

We can therefore identify ψ_M with the work function

$$\psi_M(t) = \psi_M(t_0) + \int_{t_0}^{t} \mathbf{T}(u) \cdot \dot{\mathbf{E}}(u)du,$$

where t_0 is some fixed time, which may be $-\infty$ if the integral exists (see $(7.5.1)_1$).

It is clear that ψ_M obeys the requirements of a free energy if the memory integral in (8.6.31) is nonnegative. This indeed follows from the fact that (8.6.31) is a special case of (7.5.3) and by recalling the argument leading to (7.5.8). However, it is instructive to outline a more detailed, though equivalent, argument.

There is, however, a problem with categorizing the work function as a free energy, which arises out of Remark 18.2.

Proposition 8.6.4. *Let* $\mathbf{W} \in L^2(\mathbb{R}^+)$ *and* $\mathbb{M} \in L^1(\mathbb{R}^+)$*;* $\mathbf{W}$ *has values in* $\mathrm{Lin}(\mathbb{R}^3)$ *and* $\mathbb{M} \in \mathrm{Lin}(\mathrm{Lin}(\mathbb{R}^3))$*. Also, let* $\mathbb{M}^\top = \mathbb{M}$*. The integral*

$$J = \int_0^\infty \mathbf{W}(t) \cdot \int_0^t \mathbb{M}(t-\tau)\mathbf{W}(\tau)d\tau dt$$

is positive for every nonzero $\mathbf{W}$ *if and only if the Fourier cosine transform* $\mathbb{M}_c$ *is positive definite for* $\omega \in \mathbb{R}^{++}$*.*

Proof. The faltung theorem applied to causal functions (see Sect. C.3) gives that if

$$\mathbf{V}(t) = \int_0^t \mathbb{M}(t-\tau)\mathbf{W}(\tau)d\tau,$$

then $\mathbf{V}_+(\omega) = \mathbb{M}_+(\omega)\mathbf{W}_+(\omega)$. Given two functions $\mathbf{W}$ and $\mathbb{M} \in L^2(\mathbb{R}^+)$ that vanish on $\mathbb{R}^-$, then Parseval's formula (29.2.2) yields

$$\int_0^\infty \mathbf{W}(t) \cdot \mathbf{V}(t)dt = \frac{1}{2\pi}\int_{-\infty}^\infty \overline{\mathbf{W}_+}(\omega) \cdot \mathbf{V}_+(\omega)d\omega.$$

It follows from (C.2.2) that

$$\int_{-\infty}^\infty \overline{\mathbf{W}_+}(\omega) \cdot \mathbf{V}_+(\omega)d\omega = \int_{-\infty}^\infty [\mathbf{W}_c(\omega) \cdot \mathbb{M}_c(\omega)\mathbf{W}_c(\omega) + \mathbf{W}_s(\omega) \cdot \mathbb{M}_c(\omega)\mathbf{W}_s(\omega)]d\omega.$$

The remaining terms vanish due to either the oddness of the integrand or a cancellation that occurs by virtue of the symmetry assumption on $\mathbb{M}$. Hence, we see that $J > 0$ for every nonzero function $\mathbf{W}$ if $\mathbb{M}_c(\omega)$ is positive definite for every $\omega \in \mathbb{R}^+$. Conversely, if $J > 0$ for every nonzero choice of $\mathbf{W}$, then $\mathbf{W}_c(\omega) \cdot \mathbb{M}_c(\omega)\mathbf{W}_c(\omega) + \mathbf{W}_s(\omega) \cdot \mathbb{M}_c(\omega)\mathbf{V}_s(\omega) > 0$ for $\omega > 0$, and hence it follows that $\mathbb{M}_c(\omega)$ is positive definite for every $\omega \in \mathbb{R}^{++}$. □

Relation (7.5.6) becomes, in the current context,

$$\mathbb{G}_{12}(|s_1 - s_2|) = -2\delta(s_1 - s_2)\mathbb{G}'(|s_1 - s_2|) - \mathbb{G}''(|s_1 - s_2|). \tag{8.6.33}$$

Using this result, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \mathbf{W}(s_1) \cdot \mathbb{G}_{12}(|s_1 - s_2|)\mathbf{W}(s_2)ds_1 ds_2 \\
&\quad = -2 \int_0^\infty \int_0^\infty \mathbf{W}(s_1) \cdot \delta(s_1 - s_2)\mathbb{G}'(|s_1 - s_2|)\mathbf{W}(s_2)ds_1 ds_2 \\
&\qquad - \int_0^\infty \int_0^\infty \mathbf{W}(s_1) \cdot \mathbb{G}''(|s_1 - s_2|)\mathbf{W}(s_2)ds_1 ds_2 \\
&\quad = -2 \int_0^\infty \mathbf{W}(s) \cdot \mathbb{G}'_0\mathbf{W}(s)ds - 2\int_0^\infty \int_0^t \mathbf{W}(t) \cdot \mathbb{G}''(t-\tau)\mathbf{W}(\tau)d\tau dt \\
&\quad = -2 \int_0^\infty \mathbf{W}(t) \cdot \int_0^t [\mathbb{G}''(t-\tau) + \mathbb{G}'_0\delta(t-\tau)]\mathbf{W}(\tau)d\tau dt.
\end{aligned} \tag{8.6.34}$$

Hence, by virtue of Proposition 8.6.4, it follows that $\mathbb{G}_{12}$ is a positive definite kernel if and only if the cosine transform of $\mathbb{G}''(t-\tau) + \mathbb{G}'_0\delta(t-\tau)$ is negative definite. Now, from the definition of the Fourier cosine transform $(\text{C.1.3})_4$ we have

$$\begin{aligned}
\mathbb{G}''_c(\omega) &= \int_0^\infty \mathbb{G}''(\xi)\cos\omega\xi d\xi \\
&= \mathbb{G}'(\xi)\cos\omega\xi\,|_0^\infty + \omega\int_0^\infty \mathbb{G}'(\xi)\sin\omega\xi\, d\xi,
\end{aligned}$$

whence

$$\mathbb{G}''_c(\omega) + \mathbb{G}'_0 = \omega\mathbb{G}'_s(\omega) < 0 \quad \forall\omega \in \mathbb{R}^{++}, \tag{8.6.35}$$

by (8.1.18) and the oddness of $\mathbb{G}'_s$. Since

$$\int_0^\infty [\mathbb{G}''(\xi) + \mathbb{G}'(\xi)\delta(\xi)]\cos\omega\xi\, d\xi = \mathbb{G}''_c(\omega) + \mathbb{G}'_0,$$

the quadratic form

$$\int_0^\infty \int_0^\infty \mathbf{W}(s_1) \cdot \mathbb{G}_{12}(|s_1 - s_2|)\mathbf{W}(s_2)ds_1 ds_2$$

is positive definite and hence vanishes only at $\mathbf{W} = \mathbf{0}$. Putting $\mathbf{W}(s) = \mathbf{E}^t_r(s)$, we conclude that the memory integral in (8.6.31) is nonnegative. Thus, ψ_M is minimal at constant histories.

8.7 The Relaxation Property and a Work Function Norm

Let us consider the fading memory property introduced in Sect. 1.4.3 for linear viscoelastic materials. The function $\mathbb{H}$, defined in the general case by (7.2.22), is given in the present context by

$$\mathbb{H}(\omega) = -\omega\mathbb{G}'_s(\omega) = \omega^2\mathbb{G}_c(\omega) \geq \mathbf{0} \quad \forall\omega \in \mathbb{R}, \tag{8.7.1}$$

where $\mathbb{G}'_s$ is defined by (8.1.17). It follows in general from (7.2.23) that $\mathbb{H}$ is bounded on $\mathbb{R}^+$.

From (8.6.34) and (8.6.35) it follows that (8.6.31) can be written as

$$\begin{aligned}\widetilde{\psi}_M(\mathbf{E}^t_r, \mathbf{E}(t)) &= \psi_M(t)\\ &= \frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathbf{E}^t_{r+}}(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}^t_{r+}(\omega) d\omega,\end{aligned} \tag{8.7.2}$$

which is of course a special case of $(7.5.7)_2$. From (7.2.29), we have

$$\mathbf{E}^t_{r+}(\omega) = \mathbf{E}^t_{+}(\omega) - \frac{\mathbf{E}(t)}{i\omega^-} = \mathbf{E}^t_c(\omega) - i\left(\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right). \tag{8.7.3}$$

The subscripts c, s indicate the cosine and sine transforms, defined by $(C.1.3)_{3,4}$. The quantity ω^- may be replaced by ω, since the terms with ω^- in the denominator multiply $\mathbb{H}(\omega)$, which vanishes quadratically at the origin. From (8.7.2) and (8.7.3), using the evenness of $\mathbb{H}$, $\mathbf{E}^t_c$ and the oddness of $\mathbf{E}^t_s$, it follows that

$$\begin{aligned}\widetilde{\psi}_M(\mathbf{E}^t_r, \mathbf{E}(t)) = \psi_M(t) = \frac{1}{2}\mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) + \frac{1}{\pi}\int_0^{\infty} \Bigg[\left(\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right)\\ \cdot \mathbb{H}(\omega)\left(\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right) + \mathbf{E}^t_c(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}^t_c(\omega)\Bigg] d\omega.\end{aligned} \tag{8.7.4}$$

Let us introduce the space $\mathcal{H}$ of histories $(\mathbf{E}^t_r, \mathbf{E}(t))$ on $\mathbb{R}^+$ with values in Sym, defined by

$$\begin{aligned}\mathcal{H} = \Bigg\{(\mathbf{E}^t_r, \mathbf{E}(t)) \in L^2(\mathbb{R}^{++}) \times \text{Sym}; \int_0^{\infty} \Bigg[\left(\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right)\\ \cdot\ \mathbb{H}(\omega)\left(\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right) + \mathbf{E}^t_c(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}^t_c(\omega)\Bigg] d\omega < \infty\Bigg\}.\end{aligned}$$

The space $\mathcal{H}$ becomes a pre-Hilbert space with the inner product

$$\begin{aligned}(\mathbf{E}^t_1, \mathbf{E}^t_2) = \frac{1}{2}\mathbf{E}_1(t) \cdot \mathbb{G}_\infty \mathbf{E}_2(t) + \frac{1}{\pi}\int_0^{\infty} \Bigg[\left(\mathbf{E}^t_{1s}(\omega) - \frac{\mathbf{E}_1(t)}{\omega}\right)\\ \cdot\ \mathbb{H}(\omega)\left(\mathbf{E}^t_{2s}(\omega) - \frac{\mathbf{E}_2(t)}{\omega}\right) + \mathbf{E}^t_{1c}(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}^t_{2c}(\omega)\Bigg] d\omega,\end{aligned}$$

where $\mathbf{E}^t_1$, $\mathbf{E}^t_2$ are elements of $\mathcal{H}$ and the corresponding norm is denoted by $\|\cdot\|_{\mathcal{H}}$.

We observe that

$$\|\mathbf{E}^t\|^2_{\mathcal{H}} = \widetilde{\psi}_M(\mathbf{E}^t_r, \mathbf{E}(t)). \tag{8.7.5}$$

Using (8.1.2) rather than (8.1.5) for the stress functional in linear viscoelasticity, we can write, instead of (8.1.28),

$$\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t) + \frac{2}{\pi}\int_0^{\infty} \mathbb{G}'_s(\omega)\left[\mathbf{E}^t_s(\omega) - \frac{\mathbf{E}(t)}{\omega}\right] d\omega.$$

Indeed, this follows directly from (8.1.28) on noting (8.1.26). This is a continuous functional with respect to the norm $\|\cdot\|_{\mathcal{H}}$ [104].

Let

$$T^{(a)}\mathbf{E}^t = \begin{cases} \mathbf{0} & \forall s \in [0, a], \\ \mathbf{E}^t(s-a) & \forall s \in (a, \infty) \end{cases}$$

be the translated or partly static (see (1.4.10)) history associated with any history $\mathbf{E}^t \in \mathcal{H}$. From the definitions of the Fourier sine and cosine transforms $(C.1.3)_{3,4}$, we obtain

$$(T^{(a)}\mathbf{E}^t)_s(\omega) = \mathbf{E}^t_c(\omega)\sin\omega a + \mathbf{E}^t_s(\omega)\cos\omega a,$$
$$(T^{(a)}\mathbf{E}^t)_c(\omega) = \mathbf{E}^t_c(\omega)\cos\omega a - \mathbf{E}^t_s(\omega)\sin\omega a.$$

Using these relations, we can evaluate $\widetilde{\psi}_M(T^{(a)}\mathbf{E}^t)$, given by (8.7.4), which yields the following expression for the norm, as defined by (8.7.5):

$$\|T^{(a)}\mathbf{E}^t\|^2_{\mathcal{H}} = \int_0^\infty [\mathbf{E}^t_s(\omega)\cdot\mathbb{H}(\omega)\mathbf{E}^t_s(\omega) + \mathbf{E}^t_c(\omega)\cdot\mathbb{H}(\omega)\mathbf{E}^t_c(\omega)]d\omega.$$

Thus, we see that the norm $\|T^{(a)}\mathbf{E}^t\|_{\mathcal{H}}$, defined by (8.7.5), is independent of a, and hence as $a \to \infty$, it does not approach zero, so that

$$\lim_{a\to\infty} \|T^{(a)}\mathbf{E}^t\|_{\mathcal{H}} \neq 0.$$

We observe that in the standard theory of fading memory, the norm $\|\cdot\|$ involves an influence function k [72], and the relaxation property assumes the form

$$\lim_{a\to\infty} \|T^{(a)}\mathbf{E}^t\| = 0$$

for every history $\mathbf{E}^t$ belonging to the corresponding function space. Hence, we might say that with respect to the norm $\|\cdot\|_{\mathcal{H}}$, the relaxation property does not hold for linear viscoelastic solids. However, it has the fading memory property, defined in Sect. 1.4.3, as expressed by $(1.4.12)_1$.

8.8 Viscoelastic Fluids

Fluids are with a symmetry group that is the full unimodular group. Memory effects can be included also for these materials.

A viscoelastic fluid*may remember everything that ever happened to it, yet it cannot recall any one configuration as being physically different from any other except in regard to its mass density*" [313]. Moreover, a "*fluid may have definite memory of all its past experience*, [yet] *it reacts to those experiences only by comparing them with its present configuration*" [312]. In other words, the stress in a fluid is unchanged by a change of the reference configuration. Therefore, the present configuration is used as reference.

We confine our attention to the classical theory of linear viscoelasticity. A constitutive equation of the Boltzmann type yields a hereditary law expressed by a linear relationship between the stress and the infinitesimal strain history. A fluid characterized by such a constitutive equation is a simple material in the sense of the definition given in [75, 103], and therefore and processes can be introduced as in Sect. 3.2. We shall distinguish the cases of compressible and incompressible fluids.

8.9 Compressible Viscoelastic Fluids

Consider a viscoelastic fluid, with a constitutive equation for the stress of the form

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t), \tag{8.9.1}$$

where $p(\mathbf{x}, t)$ is the mass density, $\mathbf{E}_r^t(\mathbf{x}, s)\, \forall s \in \mathbb{R}^{++}$ is the relative strain history, given by

$$\mathbf{E}_r^t(\mathbf{x}, s) = \mathbf{E}^t(\mathbf{x}, s) - \mathbf{E}(\mathbf{x}, t), \tag{8.9.2}$$

while p denotes the pressure, $\mathbf{I}$ is the identity second-order tensor, and $\mathbf{x}$ is the position vector, which will be omitted henceforth. The last term $\tilde{\mathbf{T}}$ is the extra stress given by

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \int_0^\infty \lambda_1(\rho, s) E_r^t(s) ds\, \mathbf{I} + 2 \int_0^\infty \mu_1(\rho, s) \mathbf{E}_r^t(s) ds, \tag{8.9.3}$$

where $E_r^t(s) = \mathrm{tr}(\mathbf{E}_r^t)$ and the memory kernels $\lambda_1(\rho, \cdot)$, $\mu_1(\rho, \cdot)$ belong to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ for any fixed $\rho > 0$.

The state of such a compressible fluid can be described by means of the mass density ρ and the history of $\mathbf{E}_r^t$. Thus

$$\sigma = (\rho, \mathbf{E}_r^t).$$

The process P is expressed by means of a piecewise continuous function $\mathbf{D}^P : [0, d_P) \to \mathrm{Sym}$, defined by $\mathbf{D}^P(\tau) = \dot{\mathbf{E}}_P(\tau)$, the time derivative of the strain tensor over the time interval $[0, d_P)$, d_P being the duration of the process. We refer in this context to the paragraph before (8.1.9). The ensuing states $\sigma(t + \tau) = (\rho(t + \tau), \mathbf{E}_r^{t+\tau})\ \forall \tau \in (0, d_P]$, due to the application of any process $P = \mathbf{D}^P$, are solutions of two differential equations. That determining strain evolution has the form

$$\frac{d}{d\tau} \mathbf{E}_r^{t+\tau}(s) = \mathbf{D}^P(\tau - s) - \mathbf{D}^P(\tau), \quad 0 < s < \tau. \tag{8.9.4}$$

Moreover, referring to the conservation of mass relation $(1.3.2)_2$, we see that since

$$\nabla \cdot \mathbf{v} = \mathrm{tr}\dot{\mathbf{E}} = \mathrm{tr}\mathbf{D} = D = -\dot{\rho}/\rho, \tag{8.9.5}$$

the balance of mass in the evolving system is expressed by the equation

$$\frac{d}{d\tau}\rho(t+\tau) = -\rho(t+\tau)D^P(\tau), \tag{8.9.6}$$

with solution

$$\rho(t+\tau) = \rho(t)e^{-\int_0^\tau D^P(s)ds}, \tag{8.9.7}$$

which specifies the evolution of the density function.

We denote by Π the set of all processes $P = \mathbf{D}^P$ with finite duration. For the set Σ of states we give a definition characterized by the boundedness of the stress, putting

$$\Sigma = \left\{\sigma = (\rho, \mathbf{E}_r^t);\ |\mathbf{T}(\rho, \mathbf{E}_r^{t+\tau_{(c)}})| < \infty \quad \forall\tau \in \mathbb{R}^+\right\}, \tag{8.9.8}$$

where t is a parameter. Here we have used the partly static history (cf. (1.4.10))

$$\mathbf{E}_r^{t+\tau_{(c)}} = \begin{cases} \mathbf{E}_r^t(s-\tau) & \forall s \in [\tau, \infty), \\ \mathbf{0} & \forall s \in (0, \tau). \end{cases} \tag{8.9.9}$$

The extra stress, given by (8.9.3), obeys the equation

$$\lim_{\tau\to\infty} \tilde{\mathbf{T}}(\rho, \mathbf{E}_r^{t+\tau_{(c)}}) = \tilde{\mathbf{T}}(\rho, \mathbf{0}^\dagger) = \mathbf{0}, \tag{8.9.10}$$

because of the fading memory property (Sect. 1.4.3).

The concept of equivalence of states, as defined in general by Definition 4.1.2, can be introduced for fluids as follows.

Definition 8.9.1. *Two states* $\sigma_j(t) = (\rho_j(t), \mathbf{E}_{r_j}^t)$ $(j = 1, 2)$ *are equivalent if they give the same stress,*

$$\mathbf{T}(\rho_1(t+\tau), \mathbf{E}_{r_1}^{t+\tau}) = \mathbf{T}(\rho_2(t+\tau), \mathbf{E}_{r_2}^{t+\tau}) \quad \forall\tau \in (0, d_P], \tag{8.9.11}$$

for any process $\mathbf{D}^P : [0, d_P) \to \mathrm{Sym}$.

The equivalence class induced by such a definition comprises the minimal states σ_R of the fluid.

8.9.1 A Particular Class of Compressible Fluids

We consider the particular class of viscoelastic fluids characterized by the following kernels:

$$\lambda_1(\rho, s) = \rho\lambda'(s), \qquad \mu_1(\rho, s) = \rho\mu'(s),$$

such that $\lambda', \mu' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. Also $\lambda, \mu \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, where

$$\lambda(t) = -\int_t^\infty \lambda'(s)ds, \qquad \mu(t) = -\int_t^\infty \mu'(s)ds. \tag{8.9.12}$$

Thus, in the constitutive equation (8.9.1), the extra stress $\tilde{\mathbf{T}}$, given by (8.9.3), assumes the form

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \rho \mathbf{V}(\mathbf{E}_r^t) = \rho \int_0^\infty \kappa'(s) E_r^t(s) ds\, \mathbf{I} + 2\rho \int_0^\infty \mu'(s) \breve{\mathbf{E}}_r^t(s) ds, \tag{8.9.13}$$

where $\breve{\mathbf{E}}_r^t = \mathbf{E}_r^t - \frac{1}{3} E_r^t \mathbf{I}$ is the trace-free part of $\mathbf{E}_r^t$ and

$$\kappa'(s) = \lambda'(s) + \frac{2}{3}\mu'(s).$$

We introduce a compact representation for $\tilde{\mathbf{T}}$, putting

$$\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t) = \rho \mathbf{V}(\mathbf{E}_r^t) = \rho \int_0^\infty \mathbb{G}'(s) \mathbf{E}_r^t(s), \tag{8.9.14}$$

where $\mathbb{G}'(s)$ is a fourth-order tensor-valued function $\mathbb{G}' : \mathbb{R}^+ \to \mathrm{Lin(Sym)}$, with representation in $\mathrm{Lin}(\mathbb{R}^6)$ as a diagonal matrix. The nonvanishing diagonal elements are $\kappa'(s)$ and $2\mu'(s)$. The stress tensor, given by (8.9.1), can be written as

$$\mathbf{T}(\rho, \mathbf{E}_r^t) = -p(\rho)\mathbf{I} + \rho \int_0^\infty \mathbb{G}'(s) \mathbf{E}_r^t(s) ds. \tag{8.9.15}$$

For such materials, under the hypothesis that any finite density ρ yields a finite pressure $p(\rho)$, the space (8.9.8) of possible states can be defined as $\Sigma = \mathbb{R}^+ \times \Gamma$, where

$$\Gamma = \left\{ \mathbf{E}_r^t ; \left| \int_0^\infty \mathbb{G}'(s+\tau) \mathbf{E}_r^t(s) ds \right| < \infty \quad \forall \tau \in \mathbb{R}^+ \right\}. \tag{8.9.16}$$

Moreover, the space of minimal states σ_R, denoted by Σ_R, can be characterized as follows [100] (cf. Theorem 8.3.1).

Theorem 8.9.2. *For a viscoelastic fluid of type (8.9.15), two states $\sigma_1 = (\rho_1, \mathbf{E}_{r_1}^t)$ and $\sigma_2 = (\rho_2, \mathbf{E}_{r_2}^t)$ are equivalent in the sense of Definition 8.9.1 if and only if*

$$\rho_1(t) = \rho_2(t), \quad \int_0^\infty \mu'(s+\tau) \breve{\mathbf{E}}_r^t(s) ds = \mathbf{0}, \\ \int_0^\infty \kappa'(s+\tau) E_r^t(s) ds = 0 \quad \forall \tau \in \mathbb{R}^+, \tag{8.9.17}$$

where $\mathbf{E}_r^t = \mathbf{E}_{r_1}^t - \mathbf{E}_{r_2}^t$.

Proof. If (8.9.17) are satisfied, then (8.9.11) follows immediately. Conversely, if (8.9.11) holds for any process, then the expression (8.9.15) for time $t + \tau$ yields

$$\begin{aligned} \mathbf{T}(\rho, \mathbf{E}_r^{t+\tau}) &= -p(t+\tau)\mathbf{I} + \rho(t+\tau) \int_0^\infty \mathbb{G}'(s) \mathbf{E}_r^{t+\tau}(s) ds \\ &= -p(t+\tau)\mathbf{I} + \rho(t+\tau) \int_0^\infty \mathbb{G}'(u+\tau) \mathbf{E}_r^t(u) du \\ &\quad + \rho(t+\tau) \int_{-\tau}^0 \mathbb{G}'(u+\tau) \mathbf{E}_r^t(u) du. \end{aligned} \tag{8.9.18}$$

The quantity $\mathbf{E}_r^t$ in the last integral is determined by the process through (8.9.4). Then, from (8.9.11),

$$
\begin{aligned}
&-[p(\rho_1(t+\tau)) - p(\rho_2(t+\tau))]\mathbf{I} + \rho_1(t+\tau)\int_0^\infty \mathbb{G}'(u+\tau)\mathbf{E}_{r_1}^t(u)du \\
&\quad - \rho_2(t+\tau)\int_0^\infty \mathbb{G}'(u+\tau)\mathbf{E}_{r_2}^t(u)du \\
&\quad + [\rho_1(t+\tau)) - \rho_2(t+\tau)]\int_{-\tau}^0 \mathbb{G}'(u+\tau)\mathbf{E}_r^t(u)du = \mathbf{0}
\end{aligned}
\tag{8.9.19}
$$

is satisfied for any process. The scalar part of the process, which determines $\rho_1(t+\tau)$ and $\rho_2(t+\tau)$, is specified by (8.9.7); moreover, for any fixed scalar part D^P, the trace-free part can be changed arbitrarily, affecting only the last integral in (8.9.19). Thus, from (8.9.19) we obtain

$$
\rho_1(t+\tau) = \rho_2(t+\tau) \qquad \forall \tau \in \mathbb{R}^+, \tag{8.9.20}
$$

which, using (8.9.7), yields

$$
\rho_1(t) = \rho_2(t).
$$

These results give the other two conditions in (8.9.17), on using (8.9.20) in (8.9.19).□

This theorem allows us to state that the minimal state of a linear viscoelastic fluid is an element of

$$
\Sigma_R = \mathbb{R}^+ \times (\Gamma/\Gamma_0),
$$

where Γ/Γ_0 denotes the usual quotient space, where Γ_0 is the set of the histories $\mathbf{E}_r^t \in \Gamma$ satisfying $(8.9.17)_{2,3}$.

A process can be considered as a function $P : \Sigma \to \Sigma$; thus, it maps the initial state $\sigma^i \in \Sigma$ into the final state $P\sigma^i = \sigma^f \in \Sigma$, and the differential equations (8.9.4)–(8.9.6) govern the evolution. Recalling (8.9.16), we can also consider $P : \Gamma \to \Gamma$, such that any initial relative strain history $\gamma^i \in \Gamma$ is associated with $P\gamma^i = \gamma^f \in \Gamma$; then only (8.9.4) governs such an evolution.

8.9.2 Representation of Free Energies for Compressible Fluids

Under the hypothesis of isothermal processes, the dissipation principle (4.1.7) and (5.1.10) (using only the mechanical components and replacing $\widehat{\mathbf{S}}$ by the Cauchy stress tensor in this linearized theory; see also (3.4.9)) yield in the present context that starting from a state σ, the work done on any cycle is nonnegative, i.e.,

$$
W(\sigma, P) = \oint_0^{d_P} \frac{1}{\rho}\mathbf{T}(\rho, \mathbf{E}_r^t) \cdot \mathbf{D}(t)dt \geq 0, \tag{8.9.21}
$$

where we omit the superscript on $\mathbf{D}^P$. For a fluid characterized by the constitutive equation (8.9.15), we obtain two terms in (8.9.21), the first of which vanishes on a cycle, by virtue of (8.9.5), since

$$-\oint_0^{d_P} \frac{p(\rho)}{\rho}\mathbf{I}\cdot\mathbf{D}(t)dt = -\oint_0^{d_P} \frac{p(\rho)}{\rho}D(t)dt = \oint_0^{d_P} \frac{p(\rho)}{\rho^2}\dot{\rho}dt = 0.$$

This follows on observing that the quantity

$$\phi(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\xi^2}p(\xi)d\xi, \tag{8.9.22}$$

ρ_0 being the equilibrium mass density, is a single-valued function of ρ, with the consequence that $\phi(\rho_0)$ vanishes. Thus, (8.9.21) reduces to

$$W(\sigma, P) = \oint_0^{d_P} \frac{1}{\rho}\tilde{\mathbf{T}}(\rho, \mathbf{E}_r^t)\cdot\mathbf{D}(t)dt \geq 0. \tag{8.9.23}$$

We observe that the zero state is $\sigma^\dagger = (\rho_0, \mathbf{0}^\dagger)$, where $\mathbf{0}^\dagger$ is the zero relative history.

Taking into account Definitions 4.1.6 of a free energy and 4.2.1 of a minimum free energy, we prove an important property of the free energy of a fluid characterized by (8.9.15) [100].

Theorem 8.9.3. *For materials described by (8.9.15), every free energy may be written as the sum of two terms*

$$\psi(\sigma) = \phi(\rho) + \varphi(\gamma),$$

where ϕ is given by (8.9.22) *and $\varphi : \mathcal{S}_\Gamma \to \mathbb{R}$ is defined on a set $\mathcal{S}_\Gamma$ that is Γ-invariant (namely, if $\gamma \in \mathcal{S}_\Gamma$, then $P\gamma \in \mathcal{S}_\Gamma$ for every $P \in \Pi$) and satisfies*

$$\varphi(\gamma_2) - \varphi(\gamma_1) \leq \int_0^{d_P} \mathbf{V}(\mathbf{E}_r^t)\cdot\mathbf{D}(t)dt, \tag{8.9.24}$$

where $P\gamma_1 = \gamma_2$. Moreover, if $\psi(\sigma^\dagger) = 0$, then

$$\varphi(\mathbf{0}^\dagger) = 0.$$

Proof. We recall from Definition 4.1.6 that the domain of a free energy must be invariant under the action of any process $P \in \Pi$. Let $\sigma(t) = (\rho(t), \gamma(t)) \in \mathcal{S}$ be a state and P a process with duration d_P. Then, $P\sigma(t) \in \mathcal{S}$ if $(\rho(t+d_P), \gamma(t+d_P)) \in \mathcal{S}$, where $\rho(t+d_P)$ is the solution (8.9.7) of (8.9.6) with $\tau = d_P$ and $\gamma(t+d_P) = P\gamma(t)$, the solution of (8.9.4). Therefore, $\mathcal{S}$ is invariant if and only if $\mathcal{S} = \mathbb{R}^+ \times \mathcal{S}_\Gamma$, where $\mathcal{S}_\Gamma$ is Γ-invariant.

Moreover, let P be a process of duration d_P such that $P\sigma_1 = \sigma_2$, where σ_1 is the state at time $t = 0$. The inequality defining a free energy (see (4.1.8)) yields

$$\begin{aligned}\psi(\sigma_2) - \psi(\sigma_1) &\leq \int_0^{d_P} \frac{1}{\rho(t)}\mathbf{T}(t)\cdot\mathbf{D}(t)dt = -\int_0^{d_P} \frac{p(\rho(t))}{\rho(t)}D(t)dt \\ &+ \int_0^{d_P}\left[\int_0^\infty \kappa'(s)E_r^t(s)ds\, D(t) + 2\int_0^\infty \mu'(s)\check{\mathbf{E}}_r^t(s)\cdot\check{\mathbf{D}}(t)ds\right]dt,\end{aligned} \tag{8.9.25}$$

since

$$\breve{\mathbf{E}}^t_r(s)\cdot\mathbf{D}(t)=\breve{\mathbf{E}}^t_r(s)\cdot\breve{\mathbf{D}},$$

where $\breve{\mathbf{D}}$ is the trace-free part of $\mathbf{D}$. Using (8.9.5), the first term of the right-hand side of (8.9.25) becomes

$$-\int_0^{d_P}\frac{p(\rho(t))}{\rho(t)}D(t)dt=\int_{\rho_1}^{\rho_2}\frac{p(\rho)}{\rho^2}d\rho=\phi(\rho_2)-\phi(\rho_1) \tag{8.9.26}$$

with $\phi(\rho)$ given by (8.9.22). Substituting, we obtain

$$\begin{aligned}\psi(\sigma_2)-\psi(\sigma_1)&\le\phi(\rho_2)-\phi(\rho_1)\\&+\int_0^{d_P}\left[\int_0^\infty\kappa'(s)E^t_r(s)ds\,D(t)+2\int_0^\infty\mu'(s)\breve{\mathbf{E}}^t_r(s)\cdot\breve{\mathbf{D}}(t)ds\right]dt.\end{aligned}$$

From this relation it follows that the function $\varphi=\psi-\phi$ satisfies

$$\varphi(\gamma_2)-\varphi(\gamma_1)\le\int_0^{d_P}\left[\int_0^\infty\kappa'(s)E^t_r(s)ds\,D(t)+2\int_0^\infty\mu'(s)\breve{\mathbf{E}}^t_r(s)\cdot\breve{\mathbf{D}}(t)ds\right]dt,$$

which is (8.9.24).

Finally, since

$$\psi(\sigma^\dagger)=\phi(\rho_0)+\varphi(\mathbf{0}^\dagger),\qquad\phi(\rho_0)=0,$$

then $\psi(\sigma^\dagger)=0$ if and only if $\varphi(\mathbf{0}^\dagger)=0$. □

Therefore, the right-hand side of (8.9.24) gives the work done by starting from $\gamma\in\Gamma$,

$$W(\gamma,P)=\int_0^{d_P}\frac{1}{\rho}\tilde{\mathbf{T}}(\rho,\mathbf{E}^t_r)\cdot\mathbf{D}(t)dt=\int_0^{d_P}\mathbf{V}(\mathbf{E}^t_r)\cdot\mathbf{D}(t)dt. \tag{8.9.27}$$

We define

$$\mathcal{W}_\Gamma(\gamma)=\{W(\gamma,P);\ P\in\Pi\}.$$

It is easy to prove that the minimum free energy is given by

$$\psi_m(\sigma)=\phi(\rho)+\varphi_m(\gamma),$$

where, recalling (4.2.2),

$$\varphi_m(\gamma)=-\inf\mathcal{W}_\Gamma(\gamma). \tag{8.9.28}$$

The right-hand side is the able work.

A corresponding result can be proved for the general quadratic model discussed in Chap. 7 (and therefore for all materials considered in Part III), where instead of (8.9.15), we have $(7.1.13)_1$, and the generalization of (8.9.26) follows from $(7.1.14)_2$.

The discussion in Sect. 8.6 on free energies as quadratic functionals goes through for fluids also but with $\phi(t)$, defined by (8.6.14), replaced by $\phi(\rho)$), which is given by (8.9.22).

8.9.3 Thermodynamic Restrictions for Compressible Fluids

A procedure similar to that developed for viscoelastic solids in Theorem 8.1.2 and corollaries will now be used to derive the restrictions imposed by the dissipation principle on the constitutive equation (8.9.15).

For this purpose, taking into account (8.9.13) and (8.9.12), we can express the constitutive equation in the form

$$\begin{aligned}\mathbf{T}(t) &= -p(\rho(t))\mathbf{I} + \rho(t)\int_0^\infty \kappa(s)\dot{E}^t(s)ds\,\mathbf{I} + 2\rho(t)\int_0^\infty \mu(s)\frac{d}{dt}\breve{\mathbf{E}}^t(s)ds \\ &= -p(\rho(t))\mathbf{I} + \rho(t)\int_0^\infty \kappa(s)D^t(s)ds\,\mathbf{I} + 2\rho(t)\int_0^\infty \mu(s)\breve{\mathbf{D}}^t(s)ds,\end{aligned} \tag{8.9.29}$$

with the aid of two integrations by parts.

Theorem 8.9.4. *The constitutive equation (8.9.29) complies with the dissipation principle if and only if*

$$\kappa_c(\omega) > 0, \qquad \mu_c(\omega) > 0 \quad \forall\omega \in \mathbb{R}^{++}. \tag{8.9.30}$$

Proof. The expression for the work given by (8.9.21), on substituting (8.9.29), has two terms, the first of which, as we have already observed to derive (8.9.23), vanishes, by virtue of the balance of mass (8.9.5) and because the integral is evaluated along a cycle. To discuss the other term, we consider the periodic function

$$\mathbf{D}(t) = \mathbf{D}_1 \cos\omega t + \mathbf{D}_2 \sin\omega t \quad \forall\omega \in \mathbb{R}^{++},$$

where $\mathbf{D}_1, \mathbf{D}_2 \in \mathrm{Sym}$, and assume that the duration of the process P is $2\pi/\omega$ times some positive integer. As t runs over $[0, d_P)$ we obtain a cycle, since $\mathbf{D}(0) = \mathbf{D}(d_P)$ and, by virtue of (8.9.7), where the integral on $[0, d_P)$ vanishes, $\rho(0) = \rho(d_P)$.

We can put $\mathbf{D}_1 = \breve{\mathbf{D}}_1 + \frac{1}{3}D_1\mathbf{I}$, with $D_1 = \mathrm{tr}\mathbf{D}_1$; similarly for $\mathbf{D}_2$. Therefore, (8.9.21) reduces to

$$\begin{aligned}\oint_0^{d_P} \Big\{ \int_0^\infty & \kappa(s)\Big(D_1^2 \cos\omega(t-s)\cos\omega t + D_2^2 \sin\omega(t-s)\sin\omega t \\ &+ D_1D_2[\cos\omega(t-s)\sin\omega t + \sin\omega(t-s)\cos\omega t]\Big)\,ds \\ &+ 2\int_0^\infty \mu(s)(\breve{\mathbf{D}}_1\cdot\breve{\mathbf{D}}_1\cos\omega(t-s)\cos\omega t + \breve{\mathbf{D}}_2\cdot\breve{\mathbf{D}}_2\sin\omega(t-s)\sin\omega t \\ &+ \breve{\mathbf{D}}_1\cdot\breve{\mathbf{D}}_2[\cos\omega(t-s)\sin\omega t + \sin\omega(t-s)\cos\omega t]\Big)\,ds\Big\}\,dt > 0,\end{aligned}$$

and hence, by integrating with respect to t, we have

$$\kappa_c(\omega)\left(D_1^2 + D_2^2\right) + 2\mu_c(\omega)\left(\breve{\mathbf{D}}_1\cdot\breve{\mathbf{D}}_1 + \breve{\mathbf{D}}_2\cdot\breve{\mathbf{D}}_2\right) > 0 \quad \forall\omega \in \mathbb{R}^{++}$$

for any nonzero $\mathbf{D}_1$ and $\mathbf{D}_2$. Thus, the results (8.9.30) follow.

To show that (8.9.30) is a sufficient condition for the validity of (8.9.21), we note that $(\sigma(0), P)$ is a cycle if and only if $\mathbf{D}$ is periodic in $[0, d_P)$ with vanishing mean value in the period. However, since any periodic history can be expressed through a Fourier series, we can write

$$\mathbf{D}^t(s) = \sum_{h=1}^{\infty} \Big\{ \breve{\mathbf{A}}_h \cos h\omega(t-s) + \breve{\mathbf{B}}_h \sin h\omega(t-s) + \frac{1}{3}[A_h \cos h\omega(t-s) + B_h \sin h\omega(t-s)]\mathbf{I} \Big\},$$

where $\mathbf{A}_h, \mathbf{B}_h \in \mathrm{Sym}$ and $\omega = 2\pi/d_P$.

The work done on a cycle is expressed by

$$\begin{aligned} W(\gamma, P) &= \oint_0^{d_P} \int_0^{\infty} \kappa(s) \sum_{h,k=1}^{\infty} [A_h \cos h\omega(t-s) + B_h \sin h\omega(t-s)] \\ &\quad \times (A_k \cos k\omega t + B_k \sin k\omega t) ds dt \\ &\quad + 2 \oint_0^{d_P} \int_0^{\infty} \mu(s) \sum_{h,k=1}^{\infty} \left[\breve{\mathbf{A}}_h \cos h\omega(t-s) + \breve{\mathbf{B}}_h \sin h\omega(t-s)\right] \\ &\quad \times (\breve{\mathbf{A}}_k \cos k\omega t + \breve{\mathbf{B}}_k \sin k\omega t) ds dt \\ &= \frac{\pi}{\omega} \sum_{k=1}^{\infty} \left[\kappa_c(k\omega)\left(A_k^2 + B_k^2\right) + 2\mu_c(k\omega)\left(\breve{\mathbf{A}}_k^2 + \breve{\mathbf{B}}_k^2\right)\right] > 0, \end{aligned}$$

because of (8.9.30); thus, the work on any nontrivial cycle satisfies the dissipation principle. □

For viscoelastic solids, the negative definiteness of the half-range sine transform of $\mathbb{G}(s)$ is required by thermodynamics, as specified by (8.1.18). More generally, we refer to (7.2.12). For viscoelastic fluids, the dissipation principle yields the positive definiteness of the half-range cosine transforms of $\kappa(s)$ and $\mu(s)$. However, from (8.9.30), taking account of the relation $f_s'(\omega) = -\omega f_c(\omega)$ (see (7.2.13)), we also have

$$\frac{1}{\omega}\kappa_s'(\omega) < 0, \qquad \frac{1}{\omega}\mu_s'(\omega) < 0 \quad \forall \omega \in \mathbb{R}^{++}.$$

Hence, it follows that

$$\mathbb{G}_c(\omega) = -\frac{1}{\omega}\mathbb{G}_s'(\omega) > \mathbf{0} \quad \forall \omega \in \mathbb{R}^{++}, \tag{8.9.31}$$

in terms of the kernel introduced in (8.9.14). Moreover, $\mathbb{G}_c$ vanishes like ω^{-2} for large ω, since by $(C.2.17)_1$, we have

$$\lim_{\omega\to\infty} \omega^2 \mathbb{G}_c(\omega) = -\mathbb{G}'(0),$$

where $\mathbb{G}'(0)$ is negative definite by virtue of the same argument as that leading to (8.1.21).

8.10 Incompressible Viscoelastic Fluids

For compressible fluids, the pressure p is a scalar function of the density ρ, on which also the extra stress $\tilde{\mathbf{T}}$ depends. However, for incompressible fluids the density ρ does not depend on time, and therefore, it is omitted from the constitutive equations for the stress tensor. Consequently, the pressure p is no longer a function of ρ, but now becomes an indeterminate function of time.

Incompressible viscoelastic fluids are the simplest materials with memory considered in this work, in that they are characterized by a single scalar memory kernel. Some of the results presented correspond to properties demonstrated earlier for solids and compressible fluids. Indeed, we shall abbreviate or omit certain demonstrations because closely analogous proofs have been given earlier. Others derived in this section have corresponding analogues for solids and compressible fluids that were not discussed earlier. They are conveniently included here because of the simplicity of the model. More recent work on this topic includes [11].

The constitutive equation for the stress tensor of a linear viscoelastic incompressible fluid, supposed for simplicity homogeneous, is given by

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{I} + 2\int_0^\infty \mu'(s)\mathbf{E}_r^t(\mathbf{x}, s)ds, \tag{8.10.1}$$

where the material function $\mu' \in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$ yields the shear relaxation function

$$\mu(s) = -\int_0^\infty \mu'(\tau)d\tau \quad \forall s \in \mathbf{R}^+. \tag{8.10.2}$$

A fluid so characterized is a simple material, with state determined only by the relative strain history, i.e.,

$$\sigma(t) = \mathbf{E}_r^t(s) \qquad \forall s \in \mathbb{R}^{++}.$$

The set of all states of the fluid is denoted by Σ. A process is a piecewise continuous map $P : [0, d_P) \to \mathrm{Sym}$ defined by

$$P(\tau) = \dot{\mathbf{E}}^P(\tau) \qquad \forall \tau \in [0, d_P). \tag{8.10.3}$$

We denote by Π the set of all processes.

It is useful, as in the compressible case, to introduce the extra stress

$$\begin{aligned}\tilde{\mathbf{T}}(\mathbf{E}_r^t) = \mathbf{T}(t) + p(t)\mathbf{I} &= 2\int_0^\infty \mu'(s)\mathbf{E}_r^t(s)ds \\ &= 2\int_0^\infty \mu(s)\dot{\mathbf{E}}^t(s)ds = 2\int_0^\infty \mu(s)\mathbf{D}(t-s)ds,\end{aligned} \tag{8.10.4}$$

which expresses the part of the stress due only to the relative strain history. We have used the relation, valid in the linear approximation, that $\mathbf{D}(t) = \dot{\mathbf{E}}(t)$, which has been invoked earlier for both solids and fluids. Note that (8.10.1) reduces to the linear version of the Navier–Stokes equation (2.2.15), if $\mu(s)$ is given by $\mu_0\delta(s)$, in terms of the delta function. This is the short-memory limit discussed in Sect. 7.1.6.

Referring to (8.9.9) and (8.9.10), we note that similar properties hold for incompressible fluids,

$$\lim_{\tau\to\infty} \tilde{\mathbf{T}}(\mathbf{E}_r^{t+\tau_{(c)}}) = \lim_{\tau\to\infty} 2\int_\tau^\infty \mu'(s)\mathbf{E}_r^{t+\tau_{(c)}}(s)ds = \tilde{\mathbf{T}}(\mathbf{0}^\dagger) = \mathbf{0}.$$

For a partly static history (see (1.4.10) and (8.9.9)), we have

$$\tilde{\mathbf{T}}(t+\tau) = \mathbf{T}(t+\tau) + p(t+\tau)\mathbf{I} = 2\int_0^\infty \mu'(\tau+\xi)\mathbf{E}_r^t(\xi)d\xi.$$

This expression suggests the following definition of the space of states:

$$\Sigma = \left\{\mathbf{E}_r^t : \mathbb{R}^{++} \to \text{Sym}; \left|\int_0^\infty \mu'(\tau+\xi)\mathbf{E}_r^t(\xi)d\xi\right| < \infty \quad \forall\tau\in\mathbb{R}^+\right\},$$

where t is a parameter.

The process $P \in \Pi$ is applied to a given $\sigma(t) = \mathbf{E}_r^t(s)\,\forall s \in \mathbb{R}^{++}$ and $\tau \in [0, d_P)$ as in (8.10.3). The extra stress is given by

$$\tilde{\mathbf{T}}(t+\tau) = \mathbf{T}(t+\tau) + p(t+\tau)\mathbf{I} = 2\int_0^\tau \mu(s)\dot{\mathbf{E}}^{t+\tau}(s)ds + \mathbf{I}^t(\tau, \mathbf{E}_r^t), \qquad (8.10.5)$$

where (cf. (8.2.2); the footnote relating to that equation applies here also)

$$\mathbf{I}^t(\tau, \mathbf{E}_r^t) = 2\int_0^\infty \mu'(\xi+\tau)\mathbf{E}_r^t(\xi)d\xi, \qquad \tau \ge 0. \qquad (8.10.6)$$

Note that

$$\mathbf{I}^t(\tau, \mathbf{E}_r^t) = 2\int_{-\infty}^0 \mu'(\tau-u)\mathbf{E}_r^t(-u)du, \qquad \tau \ge 0. \qquad (8.10.7)$$

The definition of $\mathbf{I}^t$ gives the extra stress due to a partly static relative strain history. The decomposition in $(8.10.5)_2$ is similar to that in (8.9.18) for compressible fluids and (8.2.8) for solids.

Definition 8.9.1 reduces here to the following statement.

Definition 8.10.1. *Two states $\sigma_j(t) = \mathbf{E}_{r_j}^t$ $(j = 1, 2)$ are said to be equivalent if for every process P of duration d_P, the subsequent states $\sigma_j(t+\tau) = \mathbf{E}_{r_j}^{t+\tau}$ $(j = 1, 2)$ satisfy*

$$\tilde{\mathbf{T}}(\mathbf{E}_{r_1}^{t+\tau}) = \tilde{\mathbf{T}}(\mathbf{E}_{r_2}^{t+\tau}) \qquad \forall\tau \in (0, d_P]. \qquad (8.10.8)$$

Therefore, two equivalent states are indistinguishable, since they give the same subsequent extra stress. Thus, we can introduce an equivalence relation R in Σ, the quotient space Σ_R of which has as elements the equivalence classes σ_R, each of which is a set of equivalent states. These are the minimal states of the material.

Theorem 8.10.2. *For a viscoelastic fluid of type (8.10.1), two states* $\sigma_j(t) = \mathbf{E}^t_{r_j}$ $(j = 1, 2)$ *are equivalent if and only if*

$$\int_0^\infty \mu'(\xi+\tau)\left[\mathbf{E}^t_{r_1}(\xi) - \mathbf{E}^t_{r_2}(\xi)\right] d\xi = \mathbf{0} \qquad \forall \tau > 0. \tag{8.10.9}$$

Proof. The proof, simpler than that of Theorem 8.9.2 for compressible fluids, follows at once by considering the expression (8.10.5) for the extra stress and the arbitrariness of P and τ. □

The equivalent forms (8.10.8) and (8.10.9) can be expressed also in terms of the function $\mathbf{I}^t$ given by (8.10.6). Two states, i.e., two relative strain histories, are equivalent if and only if

$$\mathbf{I}^t(\tau, \mathbf{E}^t_{r_1}) = \mathbf{I}^t(\tau, \mathbf{E}^t_{r_2}) \qquad \forall \tau > 0. \tag{8.10.10}$$

Consequently, the function $\mathbf{I}^t$ represents an equivalence class or minimal state of Σ_R.

8.10.1 Thermodynamic Restrictions for Incompressible Viscoelastic Fluids

Putting $\rho = 1$ and $\mathbf{D}(t) = \dot{\mathbf{E}}(t)$ in (8.9.21), we see that the work on a path γ performed by going from an initial state σ to a final state $\hat{\rho}(\sigma, P)$ by means of a process P is given by

$$W(\sigma, P) = \int_\gamma \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) dt. \tag{8.10.11}$$

The path γ is not necessarily a cycle in general. The dissipation principle is expressed by

$$W(\sigma, P) = \oint \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t) dt \geq 0,$$

where the integral is evaluated on any cycle (σ, P) and the equality sign refers only to reversible processes [297].

The analytical restrictions imposed by thermodynamics on the constitutive equation (8.10.1) have been derived in [113] and are stated by the following theorem.

Theorem 8.10.3. *The constitutive equation (8.10.1) is compatible with the dissipation principle if and only if*

$$\mu_c(\omega) > 0 \qquad \forall \omega \in \mathbb{R}, \tag{8.10.12}$$

where $\mu \in L^1(\mathbb{R}^+)$ *and* $\int_0^\infty \mu(s) ds \neq 0$.

This follows by a simplified version of the proof of Theorem 8.9.4, where κ is put equal to zero. The extension to $\mathbb{R}$ follows from the evenness of μ_c and the final assumption on the integral of μ. This extension also applies to Theorem 8.9.4 under the same assumption.

By continuity, we have

$$\int_0^\infty \mu(s)ds = \mu_c(0) > 0.$$

Moreover, since $\mu' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, its Fourier transform is given by

$$\mu'_+(\omega) = \mu'_c(\omega) - i\mu'_s(\omega) \qquad \forall \omega \in \mathbb{R}$$

and belongs to $L^2(\mathbb{R})$. Since

$$\mu'_s(\omega) = -\omega\mu_c(\omega), \tag{8.10.13}$$

it follows from (8.10.1) that μ'_s vanishes linearly at the origin. Also, (8.10.12) implies that

$$\omega\mu'_s(\omega) < 0 \qquad \forall \omega \neq 0. \tag{8.10.14}$$

Moreover, from the inverse Fourier transform of $\mu_c(\omega)$ given by

$$\mu(s) = \frac{2}{\pi}\int_0^\infty \mu_c(\omega)\cos(\omega s)d\omega,$$

we obtain that

$$\mu(0) = \frac{2}{\pi}\int_0^\infty \mu_c(\omega)d\omega = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{\mu'_s(\omega)}{\omega}d\omega > 0.$$

From (C.2.17), we see that

$$\mu'(0) = -\lim_{\omega\to\infty} \omega^2\mu_c(\omega) \leq 0 \tag{8.10.15}$$

and that the asymptotic behavior of $\mu'_c(\omega)$ and $\mu'_s(\omega)$ is given by

$$\mu'_s(\omega) \sim \frac{\mu'(0)}{\omega}, \qquad \mu'_c(\omega) \sim -\frac{\mu''(0)}{\omega^2},$$

where it is assumed that $0 \neq |\mu'(0)| < \infty$. If $\mu'' \in L^1(\mathbb{R}^+)$, we obtain

$$\omega\mu'_s(\omega) = \mu'(0) + \mu''_c(\omega). \tag{8.10.16}$$

8.10.2 The Mechanical Work

Firstly, we consider the work done on the material by the relative strain history up to time t, when the final state is $\sigma(t) = \mathbf{E}^t_r$. Referring to (8.10.11), one has

$$\tilde{W}(\mathbf{E}^t_r) = \int_{-\infty}^t \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)d\tau = 2\int_{-\infty}^t\int_0^\infty \mu'(s)\mathbf{E}^\tau_r(s)ds\cdot\dot{\mathbf{E}}(\tau)d\tau, \tag{8.10.17}$$

by virtue of the expression (8.10.1) for the stress tensor and the constraint of incompressibility, $\mathbf{I}\cdot\dot{\mathbf{E}} = 0$. We shall be concerned with relative strain histories that yield finite work, i.e., histories such that $\tilde{W}(\mathbf{E}_r^t) < \infty$. This work function can be expressed in various forms. We refer to Sect. 8.6.2 for analogous results applicable to a viscoelastic solid. Similar relations can be obtained for compressible fluids. In the present case, where the memory kernel is one scalar functional, the manipulations are particularly simple.

Integrating by parts in (8.10.17) (see (8.10.4)) and using the definition of $\mathbf{E}_r^t$, given by (8.9.2), we have

$$\begin{aligned}\tilde{W}(\mathbf{E}_r^t) &= 2\int_{-\infty}^{t}\int_{0}^{\infty}\mu(s)\dot{\mathbf{E}}(\tau-s)ds\cdot\dot{\mathbf{E}}(\tau)d\tau\\ &= \int_{-\infty}^{t}\int_{-\infty}^{t}\mu(|\rho-u|)\dot{\mathbf{E}}(u)\cdot\dot{\mathbf{E}}(\rho)dud\rho,\end{aligned}$$

where a change of variables has been implemented. Changing variables again, we have

$$\tilde{W}(\mathbf{E}_r^t) = \int_{0}^{\infty}\int_{0}^{\infty}\mu(|\eta-\xi|)\dot{\mathbf{E}}^t(\xi)\cdot\dot{\mathbf{E}}^t(\eta)d\xi d\eta, \tag{8.10.18}$$

which, with two integrations by parts, yields

$$\tilde{W}(\mathbf{E}_r^t) = \int_{0}^{\infty}\int_{0}^{\infty}\mu_{12}(|\eta-\xi|)\mathbf{E}_r^t(\xi)\cdot\mathbf{E}_r^t(\eta)d\xi d\eta, \tag{8.10.19}$$

where (cf. (8.6.33))

$$\mu_{12}(|\eta-\xi|) = \frac{\partial^2}{\partial\eta\partial\xi}\mu(|\eta-\xi|) = -\mu''(|\eta-\xi|) - 2\delta(\eta-\xi)\mu'(|\eta-\xi|). \tag{8.10.20}$$

The following result corresponds to that proved in Proposition 8.6.4. The manipulations are particularly simple in the present case.

Lemma 8.10.4. *The work done on the material by the relative strain history,* $\mathbf{E}_r^t(s)$ $\forall s\in\mathbb{R}^{++}$*, is a nonnegative quantity.*

Proof. Consider the expression (8.10.19) for $\tilde{W}(\mathbf{E}_r^t)$. By substituting (8.10.20), we obtain

$$\tilde{W}(\mathbf{E}_r^t) = -\int_{0}^{\infty}\int_{0}^{\infty}\mu''(|\eta-\xi|)\mathbf{E}_r^t(\xi)\cdot\mathbf{E}_r^t(\eta)d\xi d\eta - 2\mu'(0)\int_{0}^{\infty}\mathbf{E}_r^t(s)\cdot\mathbf{E}_r^t(s)ds.$$

Applying the convolution theorem and Parseval's formula (Sect. C.3), together with (C.1.5), gives

$$\begin{aligned}\tilde{W}(\mathbf{E}_r^t) &= -\frac{1}{\pi}\int_{-\infty}^{\infty}[\mu_c''(\omega)+\mu'(0)]\mathbf{E}_{r+}^t(\omega)\cdot\overline{\mathbf{E}_{r+}^t}(\omega)d\omega\\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\omega\mu_s'(\omega)\mathbf{E}_{r+}^t(\omega)\cdot\overline{\mathbf{E}_{r+}^t}(\omega)d\omega,\end{aligned} \tag{8.10.21}$$

because of (8.10.16). This expression is nonnegative by virtue of (8.10.14). □

Now we consider the work done by the process $P(\tau) = \dot{\mathbf{E}}_P(\tau)\,\forall\tau \in [0, d_P)$, applied at time t when $\sigma(t) = \mathbf{E}_r^t$ is the initial state. It is a function of the state σ and the process P, given by (8.10.11), which, taking account of (8.10.5) and (8.10.6), yields

$$\begin{aligned} W(\sigma, P) &= \int_t^{t+d} \mathbf{T}(\xi) \cdot \dot{\mathbf{E}}(\xi) d\xi = \int_0^d \mathbf{T}(t+\tau) \cdot \dot{\mathbf{E}}_P(\tau) d\tau \\ &= \int_0^d \left[2\int_0^\tau \mu(\tau - \xi)\dot{\mathbf{E}}_P(\xi) d\xi + \mathbf{I}^t(\tau, \mathbf{E}_r^t) \right] \cdot \dot{\mathbf{E}}_P(\tau) d\tau, \end{aligned} \tag{8.10.22}$$

putting $\dot{\mathbf{E}}_P(\tau) = \dot{\mathbf{E}}(t+\tau)$.

The process P, defined for any $\tau \in [0, d_P)$, d_P being its finite duration, can be extended to $\mathbb{R}^+$ by means of the trivial extension $P(\tau) = \mathbf{0}\,\forall\tau \in [d_P, \infty)$.

Firstly, we consider the case that such a process P is applied at time $t = 0$, when the initial state is $\sigma(0) = \mathbf{E}_r^0(s)$. Putting $t = 0$ and replacing τ by t, we have

$$\begin{aligned} \widetilde{W}(\mathbf{E}_r^0, \dot{\mathbf{E}}) &= \int_0^\infty \left[2\int_0^t \mu(t-\tau)\dot{\mathbf{E}}(\tau) d\tau + \mathbf{I}^0(t, \mathbf{E}_r^0) \right] \cdot \dot{\mathbf{E}}(t) dt \\ &= \int_0^\infty \left[\int_0^\infty \mu(|t-\tau|)\dot{\mathbf{E}}(\tau) d\tau + \mathbf{I}^0(t, \mathbf{E}_r^0) \right] \cdot \dot{\mathbf{E}}(t) dt, \end{aligned}$$

where $\dot{\mathbf{E}}_P$ is now denoted by $\dot{\mathbf{E}}(t)\,(t > 0)$.

The work due only to the process can be evaluated by supposing that the initial relative strain history is $\mathbf{E}_r^0(s) = \mathbf{0}^\dagger(s) = \mathbf{0}\,\forall s \in \mathbb{R}^{++}$; thus, the last relation yields

$$\widetilde{W}(\mathbf{0}^\dagger, \dot{\mathbf{E}}) = \int_0^\infty \int_0^\infty \mu(|t-\tau|)\dot{\mathbf{E}}(\tau) \cdot \dot{\mathbf{E}}(t) d\tau dt. \tag{8.10.23}$$

We now give a definition introduced by Gentili [145] for viscoelastic solids.

Definition 8.10.5. *A process P, of any duration, is said to be a finite work process if*

$$\widetilde{W}(\mathbf{0}^\dagger, \dot{\mathbf{E}}_P) < \infty.$$

This work is a positive quantity as the following lemma states [8].

Lemma 8.10.6. *The work in Definition 8.10.5 satisfies the inequality*

$$\widetilde{W}(\mathbf{0}^\dagger, \dot{\mathbf{E}}_P) > 0.$$

Proof. Using the same procedure as in (8.10.21), one can show that relation (8.10.23), the expression for the work done by P, applied at time $t = 0$ to the null relative strain history $\mathbf{0}^\dagger$, can be expressed in the form

$$\widetilde{W}(\mathbf{0}^\dagger, \dot{\mathbf{E}}_P) = \frac{1}{\pi}\int_{-\infty}^\infty \mu_c(\omega)\dot{\mathbf{E}}_{P+}(\omega) \cdot \overline{\dot{\mathbf{E}}_{\mathbf{P}+}}(\omega) d\omega > 0. \tag{8.10.24}$$

The inequality follows by virtue of (8.10.12). □

We can characterize the set of finite work processes by means of [145],

$$\tilde{H}_\mu(\mathbb{R}^+) = \left\{ \mathbf{g} : \mathbb{R}^+ \to \mathrm{Sym};\ \int_{-\infty}^{\infty} \mu_c(\omega)\mathbf{g}_+(\omega) \cdot \overline{\mathbf{g}_+}(\omega) d\omega < \infty \right\},$$

the completion of which, with the norm induced by the inner product

$$(\mathbf{g}_1, \mathbf{g}_2)_\mu = \int_{-\infty}^{\infty} \mu_c(\omega)\mathbf{g}_{1+}(\omega) \cdot \overline{\mathbf{g}_{2+}}(\omega) d\omega,$$

yields the Hilbert space $H_\mu(\mathbb{R}^+)$ of processes.

Let us now consider the general case in which P is applied at any time $t > 0$. The expression for the work done by the process P, again extended to $\mathbb{R}^+$ by means of its trivial extension, becomes, using (8.10.22),

$$\begin{aligned} W(\mathbf{I}^t, \dot{\mathbf{E}}_P) &= \widetilde{W}(\mathbf{E}_r^t, \dot{\mathbf{E}}_P) \\ &= \int_0^{\infty} \left[\int_0^{\infty} \mu(|\tau - \xi|)\dot{\mathbf{E}}_P(\xi) d\xi + \mathbf{I}^t(\tau, \mathbf{E}_r^t) \right] \cdot \dot{\mathbf{E}}_P(\tau) d\tau, \end{aligned} \tag{8.10.25}$$

where (σ, P) has been replaced by $(\mathbf{I}^t, \dot{\mathbf{E}}_P)$ or $(\mathbf{E}_r^t, \dot{\mathbf{E}}_P)$. This becomes, in the frequency domain (see (8.10.24)),

$$W(\mathbf{I}^t, \dot{\mathbf{E}}_P) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mu_c(\omega)\dot{\mathbf{E}}_{P+}(\omega) \cdot \overline{\dot{\mathbf{E}}}_{P+}(\omega) d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{I}_+^t(\omega) \cdot \overline{\dot{\mathbf{E}}}_{P+}(\omega) d\omega,$$

where $\mathbf{I}_+^t(\omega)$ denotes the Fourier transform of $\mathbf{I}^t(\tau, \mathbf{E}_r^t)$, defined by (8.10.6) on $\mathbb{R}^+$. The states σ are now expressed by means of $\mathbf{I}^t$, which belong to the dual of $H_\mu(\mathbb{R}^+)$, i.e.,

$$H'_\mu(\mathbb{R}^+) = \left\{ \mathbf{I}^t : \mathbb{R}^+ \to \mathrm{Sym};\ \int_{-\infty}^{\infty} \mathbf{I}_+^t(\omega) \cdot \overline{\dot{\mathbf{E}}_{\mathbf{P}+}}(\omega) d\omega < \infty\ \forall \dot{\mathbf{E}}_P \in H_\mu(\mathbb{R}^+) \right\}.$$

The definition of equivalence for two states, that is, for two relative strain histories, can be expressed in terms of the work, as for viscoelastic solids [145]. A similar result can be given for compressible fluids.

Definition 8.10.7. *Two states* $\sigma_j(t) = \mathbf{E}_{r_j}^t$ $(j = 1, 2)$ *are said to be w-equivalent if for every* $P : [0, \tau) \to \mathrm{Sym}$ *and for every* $\tau > 0$*, they satisfy*

$$\tilde{W}(\mathbf{E}_{r_1}^t, \dot{\mathbf{E}}_P) = \tilde{W}(\mathbf{E}_{r_2}^t, \dot{\mathbf{E}}_P). \tag{8.10.26}$$

The two definitions of equivalence coincide by virtue of the following lemma.

Lemma 8.10.8. *For every fluid characterized by the constitutive equation (8.10.1), two states are w-equivalent if and only if they are equivalent in the sense of Definition 8.10.1.*

Proof. Let us consider the expression (8.10.22) for the work corresponding to two states $\sigma_j(t) = \mathbf{E}^t_{r_j}$ $(j = 1, 2)$. If these two states are equivalent, then the two expressions for the work for any P coincide, i.e., they are w-equivalent. On the other hand, if (8.10.26) is satisfied for any P of any duration, then by virtue of (8.10.25), relation (8.10.10) holds and hence also (8.10.9) and (8.10.8). Thus, two w-equivalent states are equivalent also in the sense of Definition 8.10.1. □

The general representation of a free energy, already studied for viscoelastic solids in Sect. 8.6, can be considered for fluids, as noted at the end of Sect. 8.9.2. For incompressible fluids, this representation takes a particularly simple form by putting $\mathbb{G}_\infty = 0$ in (8.6.17) and replacing $\mathbb{G}_{12}(s, u)$ by a scalar kernel.

8.10.3 Maximum Free Energy for Incompressible Fluids

We seek to show that the mechanical work as given by (8.10.18) or (8.10.19) is the maximum free energy for incompressible fluids, just as (8.6.31) has this property in the case of viscoelastic solids. Let us put

$$\psi_M(t) = \int_0^\infty \int_0^\infty \mu_{12}(|s-u|)\mathbf{E}^t_r(s) \cdot \mathbf{E}^t_r(u)dsdu.$$

We rewrite this expression as (cf. (8.6.23))

$$\psi_M(t) = \mathbf{E}(t) \cdot \left[\mu_0\mathbf{E}(t) + 2\int_0^\infty \mu'(u)\mathbf{E}^t(u)du\right] + \int_0^\infty \int_0^\infty \mu_{12}(|s-u|)\mathbf{E}^t(s) \cdot \mathbf{E}^t(u)dsdu,$$

using (8.9.2) and noting that

$$\int_0^\infty \int_0^\infty \mu_{12}(|s-u|)dsdu = \mu(0,0) = \mu(0) = \mu_0.$$

Hence, recalling Remark 8.6.2, we observe that the property of a free energy expressed by $\partial\psi_M/\partial\mathbf{E} = \tilde{\mathbf{T}}$ follows. Moreover, differentiating with respect to t, we obtain

$$\dot{\psi}_M(t) = 2\left[\mu_0\mathbf{E}(t) + \int_0^\infty \mu'(u)\mathbf{E}^t(u)du\right] \cdot \dot{\mathbf{E}}(t) + 2\mathbf{E}(t) \cdot \int_0^\infty \mu'(u)\dot{\mathbf{E}}^t(u)du + 2\int_0^\infty \int_0^\infty \mu_{12}(|s-u|)\dot{\mathbf{E}}^t(s) \cdot \mathbf{E}^t(u)dsdu.$$

The last integral, with an integration by parts, gives two terms, one of which vanishes on account of the oddness of

$$\frac{\partial\mu}{\partial s}(|s-u|) = sign(s-u)\mu'(|s-u|),$$

while the second one cancels the third term in the expression of $\dot{\psi}_M$. Thus, taking into account the constitutive equation (8.10.1), it follows that

$$\dot{\psi}_M(t) = \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t), \tag{8.10.27}$$

by virtue of the constraint of incompressibility. Therefore, (8.6.16) is satisfied without dissipation, that is, with

$$D(t) = 0.$$

Integrating (8.6.16) over all past time, under the assumption that the integrals exist (as in (5.1.34)), it is easy to see that $\psi_M(t) \geq \psi(t)$, where ψ is any other free energy. It is of interest to present a demonstration that ψ_M is the maximum free energy, which avoids infinite integrals.

For this purpose, we consider an arbitrary process P, applied at time $t = 0$ to the zero state $\sigma^\dagger = \mathbf{E}_r^0(s) = \mathbf{0}\ \forall s \in \mathbb{R}^{++}$, and we denote by $\sigma(t) = \mathbf{E}_r^t = \hat{\rho}(\mathbf{0}^\dagger, P_{[0,t)})$ the final state, belonging to

$$\mathcal{D}_{\psi_M} = \left\{\mathbf{E}_r^t;\ \psi_M(\mathbf{E}_r^t) < \infty\right\},$$

the set of all relative histories that yield a finite $\psi_M(\sigma(t)) = \psi_M(\mathbf{E}_r^t)$. An integration of (8.10.27) on the time interval $(0, t)$ gives

$$\psi_M(\sigma(t)) = \int_0^t \mathbf{T}(s) \cdot \dot{\mathbf{E}}(s) ds,$$

since $\psi_M(\sigma^\dagger) = 0$. Furthermore, any other free energy $\psi(\sigma(t))$, with $\sigma(t) = \hat{\rho}(\mathbf{0}^\dagger, P_{[0,t)})$, must satisfy

$$\psi(\sigma(t)) \leq \int_0^t \mathbf{T}(s) \cdot \dot{\mathbf{E}}(s) ds,$$

because of (4.1.8) and since $\psi(\sigma^\dagger) = 0$. These last two relations yield the inequality

$$\psi_M(\sigma) \geq \psi(\sigma),$$

where $\sigma(t) = \mathbf{E}_r^t \in \mathcal{D}_{\psi_M}$ is an arbitrary final state, obtained by means of any process P. This inequality holds for any free energy functional ψ. Consequently, ψ_M is the maximum free energy for incompressible viscoelastic fluids. As noted previously, however, there is a problem with categorizing the work function as a free energy, which arises out of Remark 18.2.