30

Dynamics of Viscoelastic Fluids

30.1 Introduction

An evolution problem in a bounded domain for viscoelastic fluids of the kind considered in Chaps. 8, 10, and 13 is now presented. Our attention is confined to infinitesimal viscoelasticity for isotropic, homogeneous, and incompressible fluids. Therefore, the constitutive equation for the stress tensor $\mathbf{T}$ is expressed by the hereditary law (8.10.1), which, with an integration by parts, can be rewritten as follows:

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{I} + \int_0^\infty \mu(s)\dot{\mathbf{E}}^t(\mathbf{x}, s)ds. \tag{30.1.1}$$

The stability problem for such fluids was examined under a variety of conditions in many articles (see [215, 216, 300, 301]). We recall, in particular, that Slemrod in [300] showed that if $\mu(s) \in C^2(\mathbb{R}^+)$, $\mu(s) \to 0$ as $s \to \infty$, $\mu(s) > 0$, $\mu'(s) < 0$, and $\mu''(s) \geq 0$, the rest state of these fluids is stable in a suitable "fading memory" norm, and the solution of the linearized boundary initial history value problem converges to the rest state weakly in this norm as $t \to \infty$. The same author proved asymptotic stability by means of the additional assumption that $\int_0^\infty s^2\mu'(s)ds < \infty$.

In [113], it was shown that there exists a strict connection between the thermodynamic restrictions on the relaxation function μ and the existence, uniqueness, and stability theorems relating to the boundary initial history value problem for viscoelastic fluids characterized by the constitutive equation (8.10.1). For this purpose, the strict inequality in condition (8.10.12) of Theorem 8.10.3, $\mu_c(\omega) > 0 \ \forall\omega \in \mathbb{R}$, is important, since the asymptotic stability of the rest state fails when the weaker condition $\mu_c(\omega) \geq 0 \ \forall\omega \in \mathbb{R}$ is assumed, as occurs for a particular family of relaxation functions examined also in [113].

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30.2 An Initial Boundary Value Problem for an Incompressible Viscoelastic Fluid

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain occupied by a linear viscoelastic incompressible fluid, for which the constitutive equation is given by (30.1.1). The linear approximation of the equations of motion on the domain $Q = \Omega \times \mathbb{R}^{++}$ for the initial boundary value problem, characterized by Dirichlet's boundary conditions, yields the following system:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{v}(\mathbf{x},t) &= \nabla \cdot \left[-p(\mathbf{x},t)\mathbf{I} + \int_0^\infty \mu(s)\nabla\mathbf{v}^t(\mathbf{x},s)ds\right] + \mathbf{f}(\mathbf{x},t), \\ \nabla \cdot \mathbf{v}(\mathbf{x},t) &= 0, \\ \mathbf{v}(\mathbf{x},t) &= \mathbf{0} \qquad \forall \mathbf{x} \in \partial\Omega,\ \forall t \in \mathbb{R}^{++}, \\ \mathbf{v}(\mathbf{x},\tau) &= \mathbf{v}_0(\mathbf{x},\tau) \qquad \forall \mathbf{x} \in \Omega,\ \forall \tau \in \mathbb{R}^{-}, \end{aligned} \tag{30.2.1}$$

where the constant mass density ρ is understood and not written, $\mathbf{v}$ and p are the velocity and the pressure fields, $\mathbf{f}$ gives the body forces, and $\mathbf{v}_0$ denotes the history of the velocity up to time $t = 0$. Note that we have used the relation $\nabla \cdot \dot{\mathbf{E}}^t(\mathbf{x},t) = \frac{1}{2}\nabla \cdot [\nabla\mathbf{v}(\mathbf{x},t)]$, which follows from the constraint of incompressibility $\nabla \cdot \mathbf{v}(\mathbf{x},t) = 0$.

The relaxation function μ is assumed to be such that $\mu \in L^1(\mathbb{R}^+)$ and such that it satisfies the following thermodynamic restriction:

$$\mu_c(\omega) = \int_0^\infty \mu(s)\cos\omega s ds > 0 \quad \forall \omega \in \mathbb{R}. \tag{30.2.2}$$

We denote by $L^2_s(\Omega)$ and $H^2_{s0}(\Omega)$ the Hilbert spaces obtained by means of the completion of solenoidal vector fields $\mathbf{v} \in C_0^\infty(\Omega)$ in the $L^2(\Omega)$ and in the $H^1_0(\Omega)$ norms, respectively. Moreover, $L^2_\pi(\Omega)$ is the Hilbert space obtained by virtue of the completion of irrotational vector fields $\mathbf{v} \in C_0^\infty(\Omega)$ in the $L^2(\Omega)$ norm. Thus, we have $L^2(\Omega) = L^2_s(\Omega) \oplus L^2_\pi(\Omega)$.

Theorem 30.2.1. *If the relaxation function μ belongs to $L^1(\mathbb{R}^+)$ and satisfies (30.2.2), the supply $\mathbf{f} \in L^2(\mathbb{R}^+; L^2(\Omega))$, the initial history $\mathbf{v}_0(\cdot,s) \in H^1_{s0}(\Omega)$ for all $s \in \mathbb{R}^-$, and the function $\mathbf{V}_0(\mathbf{x},t) = \nabla \cdot \int_t^\infty \mu(s)\nabla\mathbf{v}_0(\mathbf{x},t-s)ds \in L^2(\mathbb{R}^+; L^2_s(\Omega))$, then the problem (30.2.1) has one and only one solution $\mathbf{v} \in L^2(\mathbb{R}^{++}; H^1_{s0}(\Omega))$.*

Before giving a proof of this theorem, we consider the Laplace-transformed version of the problem and then prove some relevant and useful lemmas.

30.2.1 Transformed Problem

Firstly, recall that the Laplace transform of any smooth function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is defined by $\varphi_L(z) = \int_0^\infty e^{-zs}\varphi(s)ds$, where $z \in \mathbf{C}$ is the Laplace parameter belonging to the complex plane, here denoted by $\mathbf{C}$. This generalizes the quantity given by (C.2.4), where $z \in \mathbb{R}^+$.

Applying a Laplace transform to the dynamical problem (30.2.1) gives

$$\begin{aligned} z\mathbf{v}_L(\mathbf{x},z) &= -\nabla p_L(\mathbf{x},z) + \nabla\cdot[\mu_L(z)\nabla\mathbf{v}_L(\mathbf{x},z)] + \mathbf{F}(\mathbf{x},z) \quad \forall\mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{v}_L(\mathbf{x},z) &= 0 \qquad \forall\mathbf{x}\in\Omega, \\ \mathbf{v}_L(\mathbf{x},z) &= \mathbf{0} \qquad \forall\mathbf{x}\in\partial\Omega, \end{aligned} \tag{30.2.3}$$

where we have put

$$\begin{aligned} \mathbf{F}(\mathbf{x},z) &= \mathbf{f}_L(\mathbf{x},z) + \mathbf{v}_0(\mathbf{x},0) + \int_0^\infty e^{-zs}\nabla\cdot\left[\int_s^\infty \mu(\tau)\nabla\mathbf{v}_0(\mathbf{x},s-\tau)d\tau\right]ds \\ &= \mathbf{f}_L(\mathbf{x},z) + \mathbf{v}_0(\mathbf{x},0) + \mathbf{V}_{0_L}(\mathbf{x},z). \end{aligned}$$

From the hypotheses $\mathbf{f}\in L^2(\mathbb{R}^+;L^2(\Omega))$ and $\mathbf{V}_0\in L^2(\mathbb{R}^+;L^2_s(\Omega))$, it follows that $\mathbf{F}$ is well defined for any complex $z\in\mathbf{C}^+ := \{z\in\mathbf{C};\mathrm{Re}z\geq 0\}$.

Now we give the definition of a weak solution to our problem and consider the variational formulation of the linear differential system (30.2.3).

Definition 30.2.2. *A function $\mathbf{v}_L\in H^1_{s0}(\Omega)$ is said to be a weak solution of (30.2.3) if*

$$\int_\Omega[\mu_L(z)\nabla\mathbf{v}_L(\mathbf{x},z)\cdot\nabla\bar{\mathbf{u}}(\mathbf{x}) + z\mathbf{v}_L(\mathbf{x},\omega)\cdot\bar{\mathbf{u}}(\mathbf{x})]d\mathbf{x} = \int_\Omega \mathbf{F}(\mathbf{x},z)\cdot\bar{\mathbf{u}}(\mathbf{x})d\mathbf{x} \tag{30.2.4}$$

for every complex vector $\mathbf{u}\in H^1_{s0}(\Omega)$.

Here, as previously, the bar over a quantity indicates complex conjugate.

If $\mathbf{v}_L$ is a weak solution, then following Teman ([307], Lemma 2.1), we can prove that there exists a scalar field $p_L\in L^2(\Omega)$ such that

$$\begin{aligned} z\mathbf{v}_L(\mathbf{x},z) &= -\nabla p_L(\mathbf{x},z) + \nabla\cdot[\mu_L(z)\nabla\mathbf{v}_L(\mathbf{x},z)] + \mathbf{F}(\mathbf{x},z) \quad \forall\mathbf{x}\in\Omega, \\ \nabla\cdot\mathbf{v}_L(\mathbf{x},z)ds &= 0 \qquad \forall\mathbf{x}\in\Omega, \end{aligned}$$

in the sense of distributions, and moreover, $\mathbf{v}_L(\mathbf{x},z)=\mathbf{0}$ on $\partial\Omega$.

Lemma 30.2.3. *Under the hypotheses of Theorem 30.2.1, the problem (30.2.3) has one and only one weak solution.*

We now observe that the bilinear form

$$a(\mathbf{v},\mathbf{u};z) = \int_\Omega[\mu_L(z)\nabla\mathbf{v}(\mathbf{x})\cdot\nabla\bar{\mathbf{u}}(\mathbf{x}) + z\mathbf{v}(\mathbf{x})\cdot\bar{\mathbf{u}}(\mathbf{x})]d\mathbf{x} \tag{30.2.5}$$

is coercive in $H^1_{s0}(\Omega)$. Thus, by general theorems on elliptic systems (see [134, 307, 310]), the existence and uniqueness of a weak solution to (30.2.4) for every $\mathbf{F}\in L^2(\Omega)$ follow.

To prove Lemma 30.2.3, it must be shown that there exists an $\alpha > 0$, possibly dependent on z, such that

$$|a(\mathbf{v}, \mathbf{v}; z)| \geq \alpha(z)\|\mathbf{v}\|^2_{H^1_{s0}(\Omega)}.$$

We now want to demonstrate that the coerciveness of a can be proved by means of the thermodynamic restriction (30.2.2).

To prove this, recall a property of Fourier integrals.

Proposition 30.2.4. *Let $\varphi_1, \varphi_2 \in L^1(\mathbb{R})$ with $\varphi_1\varphi_2 \in L^1(\mathbb{R})$, and let φ_F denote the Fourier transform of φ. If*

$$\Phi(\omega) = \int_{-\infty}^{\infty} \varphi_{1F}(\tau)\varphi_{2F}(\omega - \tau)d\tau$$

is continuous in ω, then

$$\int_{-\infty}^{\infty} e^{-i\omega s}\varphi_1(s)\varphi_2(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{1F}(\tau)\varphi_{2F}(\omega - \tau)d\tau \quad \forall\omega \in \mathbb{R}. \tag{30.2.6}$$

This is an inverted form of the convolution theorem (C.3.2). It follows immediately from Parseval's formula ([37] and Sect. C.3).

Lemma 30.2.5. *If the relaxation function μ belongs to $L^1(\mathbb{R}^+)$ and satisfies (30.2.2), then*

$$\int_0^{\infty} e^{-\sigma s}\cos\omega s\mu(s)ds > 0 \quad \forall\omega \in \mathbb{R},\ \forall\sigma \in \mathbb{R}^+. \tag{30.2.7}$$

Proof. If $\sigma = 0$, then (30.2.7) holds, since it coincides with (30.2.2). Let $\sigma > 0$, and consider the function [146]

$$\varphi_1(s) = \mu(|s|), \quad \varphi_2(s) = \begin{cases} 0 & \forall s < 0, \\ e^{-\sigma s} & \forall s \geq 0. \end{cases}$$

Thus, $\varphi_1, \varphi_2, \varphi_1\varphi_2 \in L^1(\mathbb{R})$, and we have

$$\varphi_{1_F}(\tau) = 2\int_0^{\infty} \cos\tau s\mu(s)ds, \quad \varphi_{2_F}(\tau) = \frac{1}{\sigma + i\tau},$$

whence it follows that

$$\Phi(\omega) = \int_{-\infty}^{\infty} \frac{2}{\sigma + i(\omega - \tau)}\int_0^{\infty} \cos\tau s\mu(s)dsd\tau$$

is continuous. From (30.2.6), applied to the functions φ_1 and φ_2, we have

$$\int_0^{\infty} e^{-(\sigma + i\omega)s}\mu(s)ds = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\sigma + i(\omega - \tau)}\int_0^{\infty} \cos\tau s\mu(s)dsd\tau,$$

which has a real part given by

$$\int_0^\infty e^{-\sigma s} \cos \omega s \mu(s) ds = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sigma}{\sigma^2 + (\omega - \tau)^2} \int_0^\infty \cos \tau s \mu(s) ds d\tau,$$

whence (30.2.7) follows by virtue of the conditions $\sigma > 0$ and (30.2.2). □

We can now establish the coerciveness of $a(\mathbf{v}, \mathbf{u}, z)$.

Lemma 30.2.6. *If the relaxation function μ belongs to $L^1(\mathbb{R}^+)$ and satisfies (30.2.2), then the bilinear form a is coercive for every complex number $z \in \mathbf{C}^+$.*

Proof. Since $|a(\mathbf{v}, \mathbf{v}, z)| \geq \text{Re}a(\mathbf{v}, \mathbf{v}, z)$, it is enough to show that

$$\text{Re}a(\mathbf{v}, \mathbf{v}; z) \geq \alpha(z) \|\mathbf{v}\|^2_{H^1_{s0}(\Omega)}$$

for every $z \in \mathbf{C}^+$.

The Laplace transform μ_L, putting $z = \sigma + i\omega$, where $\omega \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, becomes

$$\mu_L(z) = \int_0^\infty e^{-\sigma s} \cos \omega s \mu(s) ds - i \int_0^\infty e^{-\sigma s} \sin \omega s \mu(s) ds,$$

which, substituted into (30.2.5), yields

$$\text{Re}a(\mathbf{v}, \mathbf{v}; z) = \int_0^\infty e^{-\sigma s} \cos \omega s \mu(s) ds \int_\Omega |\nabla \mathbf{v}(\mathbf{x})|^2 d\mathbf{x} + \sigma \int_\Omega |\mathbf{v}(\mathbf{x})|^2 d\mathbf{x}.$$

Hence, by Korn's inequality and the arbitrariness of $\sigma > 0$, we have

$$\text{Re}a(\mathbf{v}, \mathbf{v}; z) \geq C(\Omega) \int_{-\infty}^\infty e^{-\sigma s} \cos \omega s \mu(s) ds \|\mathbf{v}(\mathbf{x})\|^2_{H^1_{s0}(\Omega)},$$

where $C(\Omega)$ is a strictly positive constant that depends on the domain Ω and the integral satisfies (30.2.7). Thus, the proof is complete. □

Since Lemma 30.2.3 yields the existence and uniqueness of the solution of (30.2.4), we can study the properties of this solution $\mathbf{v}_L$.

Consider the Green tensor function $\mathbf{\Gamma} \in H^1_{s0}(\Omega)$, defined as a solution of the problem

$$\int_\Omega [\mu_L(z) \nabla_{\mathbf{x}'} \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z) \nabla_{\mathbf{x}'} \mathbf{u}(\mathbf{x}') + z \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z) \mathbf{u}(\mathbf{x}')] d\mathbf{x}' = \int_\Omega \delta(\mathbf{x} - \mathbf{x}') \mathbf{u}(\mathbf{x}') d\mathbf{x}' \tag{30.2.8}$$

for every $\mathbf{u} \in H^1_{s0}(\Omega)$, where δ is the Dirac delta function.

The solution of (30.2.4) can be written in terms of $\mathbf{\Gamma}$ as follows:

$$\mathbf{v}_L(\mathbf{x}, z) = \int_\Omega \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z) \mathbf{F}(\mathbf{x}', z) d\mathbf{x}'. \tag{30.2.9}$$

We now prove existence, uniqueness, and asymptotic behavior with respect to the parameter z of solutions $\mathbf{\Gamma}$ of Eq. (30.2.8).

Lemma 30.2.7. *Under the hypotheses of Theorem 30.2.1, there exists a unique solution* $\mathbf{\Gamma}$ *of (30.2.8) such that*

(i) $\mathbf{\Gamma}(x, \cdot; z) \in H^1_{s0}(\Omega)$ *for every* $z \in \mathbf{C}^+$,
(ii) $\mathbf{\Gamma}(x, x'; \cdot)$ *is continuous in* $\mathbf{C}^+$, *and*
(iii) for $z \in \mathbf{C}^+$,

$$\lim_{z\to\infty} z^{1-\alpha}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z) = \mathbf{0}, \qquad \alpha > 0,$$
$$\lim_{z\to\infty} \int_\Omega z\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)\mathbf{u}(\mathbf{x}')d\mathbf{x}' = \mathbf{u}(\mathbf{x}), \tag{30.2.10}$$

in the sense of distributions.

Proof. The first property holds by virtue of the coerciveness of the bilinear form $a(\mathbf{v}, \mathbf{v}; z)$, since δ is in the dual space $H^{-1}(\Omega)$. The second property follows from the continuity of the bilinear form $a(\mathbf{v}, \mathbf{u}; \cdot)$ with respect to the third argument (see [310], Lemma 44.1). Finally, note that from (30.2.8), we have

$$\int_\Omega z^{1-\alpha}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)\left[\mathbf{u}(\mathbf{x}') - z^{-1}\mu_L(z)\nabla_{\mathbf{x}'} \cdot \nabla_{\mathbf{x}'}\mathbf{u}(\mathbf{x}')\right] d\mathbf{x}' = z^{-\alpha}\mathbf{u}(\mathbf{x})$$

for every $\alpha > 0$ and $\mathbf{u} \in C_0^\infty(\Omega)$. Hence, since μ_L is a bounded function of z, the limit as $z \to \infty$ yields (30.2.10). □

A representation for $\nabla_\mathbf{x}\mathbf{v}_L$ can be given in terms of the Green function. We denote by $\nabla_\mathbf{x}\mathbf{\Gamma}$ the third-order tensor function such that

$$\int_\Omega \{\mu_L(z)\nabla_{\mathbf{x}'}[\nabla_\mathbf{x}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)]\nabla_{\mathbf{x}'}\mathbf{u}(\mathbf{x}') + z[\nabla_\mathbf{x}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)]\mathbf{u}(\mathbf{x}')\}d\mathbf{x}'$$
$$= \int_\Omega [\delta(\mathbf{x} - \mathbf{x}')\mathbf{I}]_\mathbf{x}\mathbf{u}(\mathbf{x}')d\mathbf{x}' \tag{30.2.11}$$

for every $\mathbf{u} \in H^1_{s0}(\Omega)$. In terms of $\nabla_\mathbf{x}\mathbf{\Gamma}$,

$$\nabla_\mathbf{x}\mathbf{v}_L(\mathbf{x}, z) = \int_\Omega \nabla_\mathbf{x}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)\mathbf{F}(\mathbf{x}', z)d\mathbf{x}'.$$

Using the proof of Lemma 30.2.7 and replacing $\mathbf{\Gamma}$ with $\nabla_\mathbf{x}\mathbf{\Gamma}$, we obtain the following result.

Lemma 30.2.8. *Under the hypotheses of Theorem 30.2.1, there exists a unique solution* $\nabla_\mathbf{x}\mathbf{\Gamma}$ *of (30.2.11) such that*

(i) $\nabla_\mathbf{x}\mathbf{\Gamma}(\mathbf{x}, \cdot; z) \in L^2_s(\Omega)$ *for every* $z \in \mathbf{C}^+$ *(see [310], Lemma 23.2),*
(ii) $\nabla_\mathbf{x}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; \cdot)$ *is continuous on* $\mathbf{C}^+$,

(iii) for $\alpha > 0, z \in \mathbf{C}^+$,

$$\lim_{z\to\infty} z^{1-\alpha}\nabla_{\mathbf{x}}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z) = \mathbf{0}$$

in the sense of distributions.

These results allow us to prove existence, uniqueness, and stability of the solution to (30.2.1).

A proof of Theorem 30.2.1 is now outlined.

Proof. By hypothesis, $\mathbf{f} \in L^2(\mathbb{R}^+; L^2(\Omega))$ and

$$\mathbf{V}_0(\mathbf{x}, t) = \nabla \cdot \int_t^\infty \mu(s)\nabla\mathbf{v}_0(\mathbf{x}, t-s)ds$$

belongs to $L^2(\mathbb{R}^+; L^2_s(\Omega))$; therefore, we obtain

$$\begin{aligned}\lim_{z\to\infty}\mathbf{F}(\mathbf{x}, z) &= \lim_{z\to\infty}\left[\mathbf{f}_L(\mathbf{x}, z) + \mathbf{v}_0(\mathbf{x}, 0) + \int_0^\infty e^{-zs}\nabla \cdot \int_s^\infty \mu(\tau)\nabla\mathbf{v}_0(\mathbf{x}, s-\tau)d\tau ds\right] \\ &= \mathbf{v}(\mathbf{x}, 0).\end{aligned}$$

Moreover, (30.2.9) and property (*iii*) of $\mathbf{\Gamma}$, for $\alpha > 0$, give

$$\begin{aligned}\lim_{z\to\infty} z^{1-\alpha}\mathbf{v}_L(\mathbf{x}, z) &= \lim_{z\to\infty}\int_\Omega z^{1-\alpha}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)\mathbf{F}(\mathbf{x}', z)d\mathbf{x}' \\ &= \lim_{z\to\infty}\int_\Omega z^{1-\alpha}\mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; z)\mathbf{v}(\mathbf{x}', 0)d\mathbf{x}' = \mathbf{0}.\end{aligned} \tag{30.2.12}$$

Let $z = i\omega$ and $0 < \alpha < \frac{1}{2}$; from (30.2.12), it follows that $\mathbf{v}_L(\mathbf{x}, i\omega)$ is in L^2 with respect to ω; we can view it as the Fourier transform of the function

$$\check{\mathbf{v}}(\mathbf{x}, t) = \begin{cases}\mathbf{0}, & t < 0, \\ \mathbf{v}(\mathbf{x}, t), & t \geq 0.\end{cases}$$

Using Parseval's formula, we have

$$\frac{1}{2\pi}\int_{-\infty}^\infty\int_\Omega |\mathbf{v}_L(\mathbf{x}, i\omega)|^2 d\mathbf{x}d\omega = \int_{-\infty}^\infty\int_\Omega |\check{\mathbf{v}}(\mathbf{x}, t)|^2 d\mathbf{x}dt = \int_0^\infty\int_\Omega |\mathbf{v}(\mathbf{x}, t)|^2 d\mathbf{x}dt \tag{30.2.13}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^\infty\int_\Omega |\nabla_{\mathbf{x}}\mathbf{v}_L(\mathbf{x}, i\omega)|^2 d\mathbf{x}\omega &= \int_{-\infty}^\infty\int_\Omega |\nabla_{\mathbf{x}}\check{\mathbf{v}}(\mathbf{x}, t)|^2 d\mathbf{x}dt \\ &= \int_0^\infty\int_\Omega |\nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x}, t)|^2 d\mathbf{x}dt.\end{aligned} \tag{30.2.14}$$

From (30.2.13) and (30.2.14), it follows that

$$\int_0^\infty \int_\Omega \left[|\nabla \mathbf{v}(\mathbf{x},t)|^2 + |\mathbf{v}(\mathbf{x},t)|^2 \right] d\mathbf{x}dt < \infty.$$

Hence, we obtain the required result $\mathbf{v} \in L^2(\mathbb{R}^{++}; H^1_{s0}(\Omega))$. □

30.2.2 Counterexamples to Asymptotic Stability

We consider, for the sake of simplicity, one-dimensional evolution problems.

Suppose that an incompressible viscoelastic fluid occupies the strip $0 < x < l$, $(x, y) \in \mathbb{R}^2$, between two fixed plates. Let the supply $\mathbf{f}$ and the initial history $\mathbf{v}_0$ have the form $\mathbf{f} = f(x,t)\mathbf{k}$, $\mathbf{v}_0 = v_0(x,\tau)\mathbf{k}$ $\forall \tau \in \mathbb{R}^-$; then, $\mathbf{v} = v(x,t)\mathbf{k}$ must be a solution of the scalar boundary initial history value problem

$$\begin{aligned} v_t(x,t) &= \int_0^\infty \mu(s) v_{xx}(x, t-s) ds + f(x,t) \quad \forall x \in (0,l),\ \forall t \in \mathbb{R}^{++}, \\ v(0,t) &= v(l,t) = 0 \qquad \forall t \in \mathbb{R}^{++}, \\ v(x,\tau) &= v_0(x,\tau) \qquad \forall x \in (0,l),\ \forall \tau \in \mathbb{R}^-. \end{aligned} \tag{30.2.15}$$

Since Theorem 30.2.1 can be applied to domains that are bounded in some direction (see, for example, [307], Theorem 2.1), if the relaxation function μ satisfies (30.2.2), then the problem (30.2.15) has one and only one solution $v \in L^2(\mathbb{R}^{++}; H^1_{s0}(0,l))$, for any $f \in L^2(\mathbb{R}^+; L^2(0,l))$ and v_0 such that $\int_t^\infty \mu(s) \mathbf{v}_{0_{xx}}(\mathbf{x}, t - s)ds \in L^2(\mathbb{R}^+; L^2_s(0,l))$.

We want to show that there exist relaxation functions that comply with weaker thermodynamic requirements but do not allow asymptotic stability of the solution of the problem (30.2.15) under the hypotheses of Theorem 30.2.1.

For this purpose, let us assume a nonnegative relaxation function μ with properties

(P_1) μ is a positive decreasing function belonging to $H^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\mu(s) < \frac{k}{(1+s)^{2+\varepsilon}}, \qquad k, \varepsilon > 0;$$

(P_2) μ satisfies the "weak formulation" of the second law of thermodynamics for isothermal processes; that is,

$$\int_0^\infty \mu(s) \cos \omega s ds \geq 0 \qquad \forall \omega \in \mathbb{R},$$

and there exists at least one frequency $\omega^* \neq 0$ such that

$$\mu_c(\omega^*) = \int_0^\infty \mu(s) \cos \omega^* s ds = 0. \tag{30.2.16}$$

Theorem 30.2.9. *Let μ be a relaxation function that has properties (P_1) and (P_2). Then there exists a critical length l^* for the strip such that for $l = l^*$, $f = 0$, and the initial history $v_0(x, \tau) = \sin \frac{\pi}{l^*}(c_1 \cos \omega^* \tau + c_2 \sin \omega^* \tau)$, the problem (30.2.15) has a unique periodic solution (not belonging to $L^2(\mathbb{R}^{++}; H^1_{s0}(0, l^*))$,*

$$v(x,t) = \sin \frac{\pi x}{l^*}(c_1 \cos \omega^* t + c_2 \sin \omega^* t).$$

Before giving the proof of this theorem, we make an observation and prove an auxiliary lemma.

Remark 30.2.10. Observe that if (P_1) holds, then (see [148]) it is possible to prove that the function

$$V_0(x,t) = \int_t^\infty \mu(s) v_{0_{xx}}(x, t-s) ds$$

belongs to $L^2(\mathbb{R}^+; L^2_s(0,l))$ for any $v_0 \in L^\infty(\mathbb{R}^-; H^1_{s0}(0,l))$.

Lemma 30.2.11. *Let μ be a relaxation function that has properties (P_1) and (P_2), let $\omega^* \neq 0$ be a frequency satisfying (30.2.16), and let μ_+ be the half-range Fourier transform of μ (see* (C.1.3)*). Then there exists a critical length l^* such that the problem*

$$\begin{aligned} i\omega^* v_F(x,\omega^*) - \mu_+(\omega^*) v_{xxF}(x,\omega^*) &= 0 \qquad \forall x \in (0,l), \\ v_F(0,\omega^*) = v_F(l,\omega^*) &= 0 \end{aligned} \tag{30.2.17}$$

has infinitely many complex-valued solutions for $l = l^*$.

Proof. Substituting (30.2.16) into (30.2.17), we see that both the real and the imaginary parts of the solution v of (30.2.17) must satisfy the following:

$$\begin{aligned} \omega^* u(x,\omega^*) + \mu_s(\omega^*) u_{xx}(x,\omega^*) &= 0 \qquad \forall x \in (0,l), \\ u(0,\omega^*) = u(l,\omega^*) &= 0. \end{aligned} \tag{30.2.18}$$

Integrating by parts, we obtain

$$\begin{aligned} \frac{1}{\omega^*}\mu_s(\omega^*) &= \frac{1}{(\omega^*)^2}\left[\mu(0) + \int_0^\infty \mu'(s) \cos \omega^* s ds\right] \\ &= \frac{1}{(\omega^*)^2}\int_0^\infty \mu'(s)(\cos \omega^* s - 1) ds > 0, \end{aligned}$$

because the hypothesis (P_1) yields $\mu'(s)(\cos \omega^* s - 1) \geq 0 \ \forall s \in \mathbb{R}^+$. Note that if $\mu'(s)(\cos \omega^* s - 1) = 0 \ \forall s \in \mathbb{R}^+$, then $\mu(s) = 0 \ \forall s \in \mathbb{R}^+$. Putting

$$l = l^* = \sqrt{\frac{\mu_s(\omega^*)}{\omega^*}}\pi, \tag{30.2.19}$$

then the function $u^*(x) = c \sin \frac{\pi x}{l^*},\ c \in \mathbb{R}$, is a solution of (30.2.18), since $\frac{\omega^*}{\mu_s(\omega^*)}$ is an eigenvalue of $-\Delta$. The connection between the two problems (30.2.17) and (30.2.18) allows us to conclude that any function $v_F(x, \omega^*) = (a_1 + ia_2) sin \frac{\pi x}{l^*},\ a_1, a_2 \in \mathbb{R}$, is a solution of (30.2.17). □

We can now give the proof of Theorem 30.2.9.

Proof. Let l^* be given by (30.2.19); then using Lemma 30.2.11, the unique solution of the problem

$$\begin{aligned}
&v_t(x,t) = \int_0^\infty \mu(s) v_{xx}(x, t-s)ds \quad \forall x \in (0, l^*),\ \forall t \in \mathbb{R}^{++},\\
&v(0,t) = v(l^*, t) = 0 \qquad \forall t \in \mathbb{R}^{++},\\
&v(x,\tau) = \sin \frac{\pi x}{l^*}(c_1 \cos \omega^* \tau + c_2 \sin \omega^* \tau) \qquad \forall x \in (0, l^*),\ \forall \tau \in \mathbb{R}^-,
\end{aligned}$$

which agrees with the prescribed initial history value, is

$$v(x,t) = \sin \frac{\pi x}{l^*}(c_1 \cos \omega^* t + c_2 \sin \omega^* t).$$

In fact, it is easy to show that the two quantities

$$v_t(x,t) = \omega^* \sin \frac{\pi x}{l^*}(-c_1 \sin \omega^* t + c_2 \cos \omega^* t)$$

and

$$\int_0^\infty \mu(s) v_{xx}(x, t-s)ds = \left(\frac{\pi}{l^*}\right)^2 \sin \frac{\pi x}{l^*} \mu_s(\omega^*)(-c_1 \sin \omega^* t + c_2 \cos \omega^* t)$$

coincide by virtue of (30.2.19). □

Finally, we exhibit a family of nonnegative relaxation functions that comply with requirements (P_1) and (P_2). Consider the function

$$\mu(s) = \left(\frac{s^2}{\beta} - \frac{\alpha - 3}{\beta^2}s + \frac{\alpha^2 - 8\alpha + 24}{8\beta^3}\right)e^{-\beta s}$$

with two parameters α and β, which are assumed to be such that $\alpha \in \left(0, 2 + \sqrt{2}\right)$, $\beta > 0$. This satisfies (P_1); furthermore, taking into account that

$$\int_0^\infty e^{-\beta s} \cos \omega s ds = \frac{\beta}{\beta^2 + \omega^2}, \quad \int_0^\infty e^{-\beta s} s \cos \omega s ds = \frac{\beta^2 - \omega^2}{(\beta^2 + \omega^2)^2},$$

and

$$\int_0^\infty e^{-\beta s} s^2 \cos \omega s ds = 2\beta \frac{\beta^2 - 3\omega^2}{(\beta^2 + \omega^2)^3},$$

its Fourier cosine transform is given by

$$\mu_c(\omega) = \int_0^\infty \mu(s) \cos \omega s ds = \frac{\alpha^2}{8\beta^3(\beta^2 + \omega^2)^3} \left(\omega^2 - \frac{8 - \alpha}{\alpha} \beta^2 \right) \geq 0.$$

Hence, for $\omega^* = \beta \sqrt{\frac{8-\alpha}{\alpha}}$, we have $\mu_c(\omega^*) = 0$, so that (P_2) also holds.

An analogous family was first introduced by Fabrizio and Morro in [119] to show that there exist relaxation functions for linear viscoelastic solids that do not allow the quasistatic problem to have a unique solution in the space of sinusoidal (in time) strain histories. Moreover, the same family was used later on by Giorgi and Lazzari [148], though with a different choice of parameters, to give counterexamples to the asymptotic stability of the rest state for initial boundary value problems relating to linear viscoelastic solid materials.