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Fractional Derivative Models of Materials with Memory

23.1 Introduction

Materials with constitutive equations expressed in terms of fractional derivatives [47] are of increasing interest in recent years (see [214, 287]). It is well known that such materials can be considered in the class of materials with memory and may describe elastic, fluid, viscoelastic, and electromagnetic materials, but also other kinds of phenomena, such as heat flux models.

The fractional derivative central to the present study is that of Caputo [48, 51]. We consider thermomechanical models with memory within this fractional derivative framework, and compare them with the classical Volterra theory, which is that described and used throughout most of the present work. It emerges that the two viewpoints are formally similar [250]. Indeed, fractional models are those for which the viscoelastic memory kernel (or relaxation function) $G(s)$ is given by

$$G(s) = \frac{k_0}{s^\alpha}\ , \quad \alpha \in (0, 1)\ ,\ k_0 > 0. \tag{23.1.1}$$

However, in contrast to the Volterra theory for fluids with memory, this kernel is not $L^1(0,\infty)$, which implies significant dissimilarity in the behavior of solutions of the dynamical equations, compared with the traditional theory. The differences are more evident for solid materials. Therefore, the fractional and Volterra models provide viewpoints which are not reconcilable. Various of these differences between the two models are noted in [128].

An analysis of the thermodynamic restrictions provides compatibility conditions on the kernels. These conditions, combined with analogies with the Volterra theory, yield certain free energies, which enable the definition of a topology on the history space. A similar analysis can be carried out for the phenomenon of heat propagation with memory.

The derivation of the minimum free energy in this context is presented in particular detail because it requires careful treatment of the factorization problem.

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For solid viscoelastic materials, some experimental observations are particularly in agreement with models using fractional derivatives, because of the power law behavior of the relaxation function given by (23.1.1) (see [50, 167, 202, 280, 293, 296, 303]). The creep function for such models also has a power law form. Such experimental backing has motivated many studies of materials with fading memory given by a fractional derivative, including [26, 48, 140, 200, 214, 249, 261] and in the frequency domain [202, 303] .

Many experimental observations on a variety of materials subject to a constant load show plastic behavior, which can be described by the fractional derivative approach. However, this is not predicted by Volterra models, which under constant load describe elastic materials. Moreover, when the load is removed, fractional models predict recovery of a portion of the deformation, unlike for the case of classical viscous fluids. Thus, the fractional derivative approach allows us to describe materials displaying both elastic and viscous/plastic behavior.

The fractional theory can also be applied to heat diffusion. It seems natural to generalize the Fourier law and the Cattaneo–Maxwell equation, using a fractional derivative instead of the time derivative. This approach allows us to describe a wider range of phenomena and gives a good description of oscillating behavior [1, 223, 285, 299].

In recent decades, the fractional calculus has been widely used, as indicated by the many mathematical volumes dealing with this topic (e.g., Baleanu et al. [28], Caponetto [53], Caputo [46], Diethelm [95], Hilfer [198], Jiao et al. [214], Kilbas et al. [219], Kyriakova [222], Mainardi [248], McBride [259], Miller and Ross [264], Petras [286], Samko et al. [295], Podlubny [287], Sabatier et al. [294], Torres and Malinowska [309], Ying and Chen [327]), by the many meetings dedicated to it and the plethora of articles appearing in mathematical (e.g., Kilbas and Marzan [218], Heinsalu et al. [196], Luchko and Gorenflo [246]) and non-mathematical journals.

The use of derivative of fractional order has also spread into many other fields of science besides mathematics and physics (e.g., Laskin [229], Naber [275], Baleanu et al. [29], Zavada [328], Baleanu et al. [30], Caputo and Fabrizio [55]) such as biology (e.g., Cesarone et al. [62], Caputo and Cametti [52]), economics (e.g., Caputo [54]), demography (e.g., Jumarie [217]), geophysics (e.g., Iaffaldano [207]) and medicine (e.g., El Sahed [99], Magin [247]). However, its somewhat cumbersome mathematical definition and the consequent complications in the solution of fractional order differential equations have led to some difficulties.

23.2 Fractional Derivatives

In this section, the original Caputo fractional derivative is introduced, together with certain new fractional derivative formulae without singularities in the kernel.

23.2.1 The Caputo Fractional Derivative

We begin with an argument which provides motivation for the formula, introduced below [47].

Let $f(z)$ be analytic on an open subset O of the complex plane, where O includes the real axis. Then, an immediate consequence of Cauchy's Integral Formula, (B.1.2) is

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z'-z)^{n+1}} dz' \\ &= \frac{(n-1)!}{2\pi i} \oint_C \frac{f'(z')}{(z'-z)^{n}} dz', \end{aligned}$$

where $f^{(n)}(z)$ is the nth derivative of $f(z)$, while $f'(z) = f^{(1)}(z)$. The contour $C \subset O$ includes the point z. Let us replace n by $\alpha \in \mathbb{R}^+$. We take z to be on the real axis, denoted by t, and the contour to be tightly around the branch cut joining the branch point t to infinity. This step forces the exclusion of integrals with $\alpha \in (1, \infty)$. The cut is assumed to lie along the semi-infinite interval $(-\infty, t)$. The integration variable is changed to τ. We have therefore

$$f^{(\alpha)}(t) = \frac{\Gamma(\alpha)}{2\pi i} \oint_C \frac{f'(\tau)}{|\tau - t|^{\alpha} e^{i\theta}} d\tau, \tag{23.2.1}$$

where θ is the argument of the denominator, which varies as we move around the contour, anti-clockwise. Below the cut, we have $\theta = -\pi\alpha$, while above the cut, it is $\theta = \pi\alpha$. The quantity $\Gamma(\cdot)$ is the Gamma function, given for any $\beta > 0$, by

$$\Gamma(\beta) = \int_0^{\infty} r^{\beta-1} e^{-r}\, dr. \tag{23.2.2}$$

Thus,

$$f^{(\alpha)}(t) = \frac{\Gamma(\alpha)\sin(\pi\alpha)}{\pi} \int_{-\infty}^{t} \frac{f'(\tau)}{|\tau - t|^{\alpha}} d\tau.$$

Using the standard formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \tag{23.2.3}$$

relation (23.2.1) becomes

$$f^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f'(\tau)}{|\tau - t|^{\alpha}} d\tau. \tag{23.2.4}$$

The analyticity property assumed for f can be weakened but must ensure that the derivative and integral in (23.2.4) exist.

The general form of the Caputo α fractional derivative, defined for any $\alpha \in (0, 1)$, is given by

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \tag{23.2.5}$$

where $a \in (-\infty, t)$, $f \in H^1(a, b)$, where $b > t$. We can take $a = -\infty$, since if necessary it is always possible to extend f to the interval $(-\infty, a)$ by the zero function. Thus, we have

$$ {}_{-\infty}^{\ C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau = F(t), \tag{23.2.6} $$

which agrees with the form (23.2.4). We assume that $f(t) \to 0$, as t tends to $-\infty$. Now, (23.2.6) can be written as

$$ F(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}} d\tau, \tag{23.2.7} $$

or, by a change of variable,

$$ F(t) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(t-s) - f(t)}{s^{1+\alpha}} ds, \tag{23.2.8} $$

which are equivalent representations of the Caputo derivative. Thus, the definition (23.2.5) may be rewritten as

$$ \begin{aligned} {}_{-\infty}^{\ C}D_t^{\alpha} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(\tau)}{(t-\tau)^{1+\alpha}} d\tau \\ &= -\frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(t-s) - f(t)}{s^{1+\alpha}} ds. \end{aligned} \tag{23.2.9} $$

Remark 23.2.1. As α tends to zero, we see from (23.2.6) that

$$ {}_{-\infty}^{\ C}D_t^{\alpha} f(t) \to f(t). \tag{23.2.10} $$

If α tends to 1, the integral in (23.2.5) diverges. However, there is a limit which gives a finite value (as must be true, given the derivation of (23.2.4)). Before the limit $\alpha \to 1$ is taken, let us reverse the shrinking of the contour described above, so that the integral exists for all nonnegative powers of the denominator. In particular, as $\alpha \to 1$, we have

$$ {}_{-\infty}^{\ C}D_t^{\alpha} f(t) = f'(t). \tag{23.2.11} $$

23.2.2 Fractional Derivatives Without Singular Kernels

We present a new definition of fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variables, respectively. For the first, operating on time variables, it is natural to use the Laplace transform. For the other representation, related to the spatial variables by a nonlocal fractional derivative, it is more convenient to use the Fourier transform. The interest in this new approach with a regular kernel in terms of spatial variables arose from the perception that there is a class of nonlocal models, which have the ability to describe material heterogeneities and fluctuations on different scales, which cannot be described adequately by classical local theories or by fractional models with singular kernels.

The original definition of fractional derivative appears to be particularly convenient for mechanical and electromagnetic phenomena involving plasticity, fatigue, damage, and electromagnetic hysteresis. When these effects are not present it seems more appropriate to use the new temporal fractional derivative to describe, for example, the behavior of classical viscoelastic materials, thermal media, and electromagnetic systems.

The new nonlocal fractional derivative for spatial variables can describe material heterogeneities and structures with different scales, which cannot be described adequately by classical local theories.

23.2.2.1 A New Fractional Time Derivative

We refer to the usual Caputo fractional time derivative as UFD_t. For order α, it is given by (23.2.5), or in somewhat simplified notation, by

$$D_t^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau \tag{23.2.12}$$

with $\alpha \in [0, 1]$ and $a \in (-\infty, t)$, $f \in H^1(a, b)$, $b > a$. By replacing the kernel $(t - \tau)^{-\alpha}$ with the function $exp(-\frac{\alpha}{1-\alpha}(t-\tau))$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$, we obtain the following definition of a new fractional time derivative, denoted by NFD_t, and given by the formula

$$\mathcal{D}_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau = N(t), \tag{23.2.13}$$

where $M(\alpha)$ is a normalization coefficient with the property that $M(0) = M(1) = 1$. According to the definition (23.2.13), the NFD_t is zero when $f(t)$ is constant, as is the UFD_t. However, in contrast to the UFD_t, the kernel in (23.2.13) does not have a singularity for $t = \tau$.

The NFD_t can also be applied to functions that do not belong to $H^1(a, b)$. Indeed, the definition (23.2.13) can be formulated also for $f \in L^1(-\infty, b)$ and for any $\alpha \in (0, 1)$, which can be seen by expressing (23.2.13) in the form

$$\mathcal{D}_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)^2} \int_{-\infty}^t (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau.$$

If we put

$$\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty] \quad , \quad \alpha = \frac{1}{1+\sigma} \in (0, 1)\,,$$

definition (23.2.13) of NFD_t assumes the form

$$\tilde{\mathcal{D}}_t^{(\sigma)} f(t) = \frac{(1+\sigma)M^*(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau, \tag{23.2.14}$$

where $\sigma \in (0, \infty)$ and $M^*(\sigma)$ is the corresponding normalization term to $M(\alpha)$, and is such that $M^*(0) = M^*(\infty) = 1$.

In the context of mechanical models discussed later, for $a \to -\infty$, the relation (23.2.14) corresponds to a relaxation function given by

$$G(s) = \frac{(1+\sigma)M^*(\sigma)}{\sigma} \exp\left[-\frac{s}{\sigma}\right],$$

instead of that given by (23.1.1) for the Caputo fractional derivative. This describes the well-known Maxwell material (for example [167]), which is a very simple model of a viscoelastic fluid.

For $\alpha \to 1$, we have $\sigma \to 0$. Also,

$$\lim_{\sigma\to 0}\frac{1+\sigma}{\sigma}\exp\left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau).$$

Therefore (see [150] and [197]),

$$\begin{aligned}\lim_{\alpha\to 1}\mathcal{D}_t^{(\alpha)} f(t) &= \lim_{\alpha\to 1}\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \\ &= \lim_{\sigma\to 0}\frac{(1+\sigma)M^*(\sigma)}{\sigma}\int_a^t f'(\tau)\exp\left[-\frac{(t-\tau)}{\sigma}\right]d\tau = f'(t).\end{aligned} \tag{23.2.15}$$

Also, for $\alpha \to 0$ we have $\sigma \to +\infty$. Hence,

$$\begin{aligned}\lim_{\alpha\to 0}\mathcal{D}_t^{(\alpha)} f(t) &= \lim_{\alpha\to 0}\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \\ &= \lim_{\sigma\to +\infty}\frac{(1+\sigma)M^*(\sigma)}{\sigma}\int_a^t f'(\tau)\exp\left[-\frac{(t-\tau)}{\sigma}\right]d\tau = f(t)-f(a).\end{aligned} \tag{23.2.16}$$

If $f(a) = 0$, then

$$\lim_{\alpha\to 0}\mathcal{D}_t^{(\alpha)} f(t) = f(t). \tag{23.2.17}$$

Thus, (23.2.15) and (23.2.17) indicate that for $\alpha = 0$ and 1, the NFD_t behaves as expected in the integer limit.

Simulations comparing the UFD_t and NFD_t for particular choices of $f(\cdot)$ were described in [56]. These suggested that for $\alpha = 0.66$ the classical NFD_t is very similar to the UFD_t. However, for models with α close to 0, we see different behavior. For $\alpha = 0.1$ differences between NFD_t and UFD_t become apparent. In particular the classical UFD_t is more affected by past history than NFD_t, which exhibits rapid stabilization.

For integer n where $n \geq 1$, and $\alpha \in [0, 1]$ the fractional time derivative $\mathcal{D}_t^{(\alpha+n)} f(t)$ of order $(n+\alpha)$ is defined by

$$\mathcal{D}_t^{(\alpha+n)} f(t) := \mathcal{D}_t^{(\alpha)}(\mathcal{D}_t^{(n)} f(t)). \tag{23.2.18}$$

Theorem 23.2.1. *If the function $f(t)$ is such that*

$$f^{(s)}(a) = 0, \quad s = 1, 2, \ldots, n,$$

then we have

$$\mathcal{D}_t^{(n)}(\mathcal{D}_t^{(\alpha)} f(t)) = \mathcal{D}_t^{(\alpha)}(\mathcal{D}_t^{(n)} f(t)).$$

Proof. Consider the case $n = 1$. From definition (23.2.18) of $\mathcal{D}_t^{(\alpha+1)} f(t)$, we obtain

$$\mathcal{D}_t^{(\alpha)}\left(\mathcal{D}_t^{(1)} f(t)\right) = \frac{M(\alpha)}{1-\alpha}\int_a^t f''(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau.$$

By means of an integration by parts, using the property $f'(a) = 0$, we obtain

$$\begin{aligned}
\mathcal{D}_t^{(\alpha)}\left(\mathcal{D}_t^{(1)} f(t)\right) &= \frac{M(\alpha)}{(1-\alpha)}\int_a^t \left(\frac{d}{d\tau}f'(\tau)\right)\exp-\left[\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau \\
&= \frac{M(\alpha)}{(1-\alpha)}\left[\int_a^t \frac{d}{d\tau}\left\{f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]\right\}d\tau\right. \\
&\quad \left. -\frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right] \\
&= \frac{M(\alpha)}{(1-\alpha)}\left[f'(t) - \frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right].
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{D}_t^{(1)}(\mathcal{D}_t^{(\alpha)} f(t)) &= \frac{d}{dt}\left\{\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right\} \\
&= \frac{M(\alpha)}{1-\alpha}\left[f'(t) - \frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\right].
\end{aligned}$$

It is easy to generalize the proof for any $n > 1$. □

The property asserted in Theorem 23.2.1 is implied by the notation on the left of (23.2.18). Also, note that (23.2.18), together with (23.2.15) and (23.2.16) yield that

$$\lim_{\alpha\to 0}\mathcal{D}_t^{(\alpha+n)} f(t) = f^{(n)}(t), \qquad \lim_{\alpha\to 1}\mathcal{D}_t^{(\alpha+n)} f(t) = f^{(n+1)}(t),$$

which, again, is expected behavior for a non-integer derivative.

In the following, we suppose the function $M(\alpha) = 1$. We can rewrite the definition (23.2.14) in the form

$$\tilde{\mathcal{D}}_t^{(\nu)} f(t) = V(\nu)\int_a^t f'(\tau)\exp[-\nu(t-\tau)]d\tau \tag{23.2.19}$$

obtained from (23.2.13) or (23.2.14) with $\nu = 1/\sigma > 0$, where $V(\nu) = (\nu+1)M^*(1/\nu)$. Then, we have the following theorem.

Theorem 23.2.2. *If the function* $f \in W^{1,1}(a,b)$, *then the integral in (23.2.19) exists for* $t \in [a,b]$ *and* $\tilde{\mathcal{D}}_t^{(\nu)} f(t) \in L^1[a,b]$.

Proof. Let us write

$$\tilde{\mathfrak{D}}_t^{(\nu)} f(t) = V(\nu) \int_a^t f'(\tau) \exp[-\nu(t-\tau)] \mathrm{d}\tau = \\ = V(\nu) \int_{-\infty}^{\infty} p_\nu(t-s) q(s) ds, \tag{23.2.20}$$

where $p_\nu(y) = \exp(-\nu y)$, *when* $0 < y < b-a$, *with* $p_\nu(y) = 0$ *when* $y < 0$ *or* $y > b-a$. *Also,* $q(y) = f'(y)$ *when* $a \le y \le b$. *Finally,* $q(y) = 0$ *when* $y < a$ *or* $y > b$. *Hence, under the hypotheses of the theorem, the functions* p_ν, $q \in L^1(a,b)$. *Then, by the classical results on Lebesgue integrals (see [326]), the integral (23.2.19) exists almost everywhere in* $t \in [a,b]$ *and* $\tilde{\mathfrak{D}}_t^{(\nu)} f(t) \in L^1\ [a,b]$.

The definition (23.2.19) can be generalized by choosing p_ν in (23.2.20) to be any function with the properties assigned to the exponential kernel in Theorem 23.2.2, where $p_\nu(0)$ is a finite number, chosen for convenience to be unity. This latter property ensures that it constitutes a non-singular kernel. It can be shown without difficulty that Theorem 23.2.1 also applies to the generalized definition.

23.2.2.2 Some Results for Given Histories

It is of interest to see the fractional derivatives of elementary functions according to the new definition (23.2.13). We begin with $\sin \omega t$ and $\cos \omega t$. It is convenient to first consider $f(t) = \exp(i\omega t)$, which combined both of these. In fact, we have

$$\mathfrak{D}_t^{(\alpha)} \exp(i\omega t) = \mathfrak{D}_t^{(\alpha)}(\cos \omega t) + i\mathfrak{D}_t^{(\alpha)}(\sin \omega t),$$

and

$$\begin{aligned} \mathfrak{D}_t^{(\alpha)} \exp(i\omega t) &= i\omega E(\alpha) \int_0^t \exp(-\nu(t-s) + i\omega s) ds \\ &= \frac{i\omega E(\alpha)}{i\omega + \nu}[\exp(i\omega t) - \exp(-\nu t)] \\ &= i\omega E(\alpha) \frac{(\cos \omega t + i \sin \omega t - \exp(-\nu t))(\nu - i\omega)}{\nu^2 + \omega^2}, \\ \nu &= \frac{\alpha}{1-\alpha}, \end{aligned}$$

where $E(\alpha) = \dfrac{M(\alpha)}{1-\alpha}$. We have, from these relations that

$$\begin{aligned} \mathfrak{D}_t^{(\alpha)}(\sin \omega t) &= \frac{E(\alpha)\omega}{\nu^2 + \omega^2}[\nu \cos \omega t + \omega \sin \omega t - \nu \exp(-\nu t)] \\ &= \frac{E(\alpha)\omega}{\nu^2 + \omega^2}[\sqrt{\nu^2 + \omega^2} \sin(\omega t + \lambda) - \nu \exp(-\nu t)] \\ &= E(\alpha) \cos \lambda [\sin(\omega t + \lambda) - \sin \lambda \exp(-\nu t)], \end{aligned}$$

where λ is such that

$$\tan\lambda = \frac{\nu}{\omega}, \;\; \sin\lambda = \frac{\nu}{\sqrt{\nu^2+\omega^2}}, \;\; \cos\lambda = \frac{\omega}{\sqrt{\nu^2+\omega^2}}.$$

Thus, the new derivative of $\sin\omega t$ yields a change of phase by amount λ, while the amplitude becomes

$$E(\alpha)\cos\lambda = \frac{\omega E(\alpha)}{\sqrt{\nu^2+\omega^2}}.$$

Also,

$$\mathcal{D}_t^{(\alpha)}(\cos\omega t) = E(\alpha)\cos\lambda[\cos(\omega t+\lambda) - \cos\lambda\exp(-\nu t)],$$

which also exhibits a phase change and the same amplitude variation noted for the case of $\sin\omega t$.

The new derivative, for an exponential history, has the form

$$\begin{aligned}\mathcal{D}_t^{(\alpha)}(\exp\omega t) &= \frac{E(\alpha)\omega}{\nu+\omega}\{\exp(\omega t) - \exp(-\nu t)\}\\ &= \frac{E(\alpha)\omega}{\nu+\omega}\exp(\omega t)\{1 - \exp[-(\omega+\nu t)]\}.\end{aligned}$$

Finally, for a linear history, defined by

$$f(t) = \begin{cases} t, & t \geq 0,\\ 0, & t < 0,\end{cases}$$

we obtain

$$\begin{aligned}\mathcal{D}_t^{(\alpha)} t &= \frac{M(\alpha)}{1-\alpha}\int_0^t \exp(-\nu(t-s))ds\\ &= \frac{M(\alpha)}{\alpha}[1-\exp(-\nu t)], \; 0 < \alpha \leq 1.\end{aligned}$$

23.2.2.3 The Laplace Transform of the NFD$_t$

The Laplace transform of the NFD$_t$, defined by Eq. (23.2.13), will be of interest. We have

$$\begin{aligned} N_L(p) &= \int_0^\infty \exp(-pt)\mathcal{D}_t^\alpha f(t)dt = \int_0^\infty \exp(-pt)N(t)dt,\\ N_L^{(n)}(p) &= \int_0^\infty \exp(-pt)\mathcal{D}_t^{\alpha+n} f(t)(t)dt = \int_0^\infty \exp(-pt)N^{(n)}(t)dt.\end{aligned}$$

One can show that

$$\begin{aligned}&\int_0^\infty \exp(-pt)f'(t)dt = [f_L'](p) = pf_L(p) - f(0),\\ &\int_0^\infty \exp(-pt)\exp\left[-\frac{\alpha t}{1-\alpha}\right]dt = \frac{1-\alpha}{p+\alpha(1-p)}.\end{aligned}$$

Because of the convolution form of $N(t)$ in (23.2.13), we have

$$N_L(p) = M(\alpha)\frac{pf_L(p) - f(0)}{p + \alpha(1-p)}. \tag{23.2.21}$$

Similarly,

$$N_L^{(1)}(p) = M(\alpha)\frac{p^2 f_L(p) - pf(0) - f'(0)}{p + \alpha(1-p)},$$

and, more generally,

$$N_L^{(n)}(p) = M(\alpha)\frac{p^{n+1} f_L(p) - p^n f(0) - p^{n-1} f'(0) \ldots - f^{(n)}(0)}{p + \alpha(1-p)}.$$

23.2.2.4 Fractional Gradient Operator

We introduce a new concept of fractional gradient, which can describe nonlocal dependence in constitutive equations [324, 325].

Let us consider a set $\Omega \in \mathbb{R}^3$ and a scalar function $u(\cdot) : \Omega \to \mathbb{R}$. We define the fractional gradient of order $\alpha \in [0, 1]$ as follows

$$\nabla^{(\alpha)} u(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \tag{23.2.22}$$

with $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega}$. The nonlocal property of this fractional derivative relates to the integration of $\mathbf{y}$ over $\boldsymbol{\Omega}$. A rotationally invariant three-dimensional Normal (Gaussian) distribution has been chosen to describe this nonlocality.

It is easy to prove from definition (23.2.22) that

$$\nabla^{(1)} u(\mathbf{x}) = \nabla u(\mathbf{x}),$$

using the relation

$$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] = \delta(\mathbf{x}-\mathbf{y}).$$

Thus, when $\alpha = 1$, $\nabla^{(\alpha)} u(\mathbf{x})$ loses the nonlocality property. Also, we clearly have

$$\nabla^{(0)} u(\mathbf{x}) = 0.$$

This fractional gradient is easily generalized to the case of a vector $\mathbf{u}(\mathbf{x})$, where the gradient is assumed to exist on Ω. We define the fractional gradient of this vector by

$$\nabla^{(\alpha)} \mathbf{u}(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}.$$

Thus, a material with a nonlocal property may be described by fractional constitutive equations. As an example we consider an elastic nonlocal material, defined by the following constitutive equation between the stress tensor $\mathbf{T}$ and $\nabla^{(\alpha)} \mathbf{u}(\mathbf{x})$

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{A} \nabla^{(\alpha)} \mathbf{u}(\mathbf{x}, t)\,, \ \alpha \in (0, 1]\,,$$

where $\mathbf{A}$ is a fourth order symmetric tensor. The nonlocal property is clear from the detailed form

$$\mathbf{T}(\mathbf{x},t) = \frac{\alpha\mathbf{A}}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla\mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}.$$

Likewise, we can introduce the fractional divergence, defined for a smooth $\mathbf{u}(\cdot) : \Omega \to \mathbb{R}^3$ by

$$\nabla^{(\alpha)} \cdot \mathbf{u}(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot \mathbf{u}(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}. \tag{23.2.23}$$

Theorem 23.2.3. *From definitions (23.2.22) and (23.2.23), we have for any $u(\mathbf{x}) : \Omega \to \mathbb{R}$, such that*

$$\nabla u(\mathbf{x}) \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{23.2.24}$$

the following identity

$$\nabla \cdot \nabla^{(\alpha)} u(\mathbf{x}) = \nabla^{(\alpha)} \cdot \nabla u(\mathbf{x}). \tag{23.2.25}$$

Proof. Using (23.2.22), we obtain

$$\begin{aligned}
\nabla \cdot \nabla^{(\alpha)} u(\mathbf{x}) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \\
&= -\frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla u(\mathbf{y}) \cdot \nabla \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \\
&= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y} \\
&\quad - \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_{\partial\Omega} \nabla u(\mathbf{y}) \cdot \mathbf{n} \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}.
\end{aligned}$$

Hence, for the boundary condition (23.2.24), the identity (23.2.25) is proved, because

$$\nabla^{(\alpha)} \cdot \nabla u(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \int_\Omega \nabla \cdot \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}.$$

23.2.2.5 Fourier Transform of the Fractional Gradient and Divergence

For a smooth function $u(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}$, the Fourier transform of the fractional gradient is defined by

$$(\nabla^{(\alpha)} u)_F(\boldsymbol{\xi}) = \int_{\mathbb{R}^3} \nabla^{(\alpha)} u(\mathbf{x}) \exp\left[-i\boldsymbol{\xi} \cdot \mathbf{x}\right]\, d\mathbf{x}.$$

This quantity is given by

$$\begin{aligned}
(\nabla^\alpha u)_F(\boldsymbol{\xi}) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} \left(\int_{\mathbb{R}^3} \nabla u(\mathbf{y}) \exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right] d\mathbf{y}\right)_F (\boldsymbol{\xi}) \\
&= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}} (\nabla u)_F(\boldsymbol{\xi}) \left(\exp\left[-\frac{\alpha^2\mathbf{x}^2}{(1-\alpha)^2}\right]\right)_F (\boldsymbol{\xi}),
\end{aligned}$$

where the well-known formula for the Fourier transform of a convolution product has been used. From

$$\left(\exp\left[-\frac{\alpha^2\mathbf{x}^2}{(1-\alpha)^2}\right]\right)_F(\boldsymbol{\xi}) = \frac{(1-\alpha)\sqrt{\pi}}{\alpha}\exp\left[-\frac{(1-\alpha)^2\boldsymbol{\xi}^2}{4\alpha^2}\right],$$

we obtain

$$(\nabla^\alpha u)_F(\boldsymbol{\xi}) = \sqrt{\pi^{1-\alpha}}(\nabla u)_F(\boldsymbol{\xi})\exp\left[-\frac{(1-\alpha)^2\boldsymbol{\xi}^2}{4\alpha^2}\right].$$

The Fourier transform of fractional divergence is defined by

$$(\nabla^\alpha \cdot \mathbf{u})_F(\boldsymbol{\xi}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\left(\int_\Omega \nabla\cdot\mathbf{u}(\mathbf{y})\exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right]d\mathbf{y}\right)_F(\boldsymbol{\xi}),$$

from which we have

$$(\nabla^\alpha \cdot \mathbf{u})_F(\boldsymbol{\xi}) = \sqrt{\pi^{1-\alpha}}\,(\nabla\cdot\mathbf{u})_F\,(\boldsymbol{\xi})\exp\left[-\frac{(1-\alpha)^2\boldsymbol{\xi}^2}{4\alpha^2}\right].$$

23.2.2.6 Fractional Laplacian

In the study of partial differential equations, the Laplacian is of considerable interest. It is therefore of interest to determine the factional Laplacian. Based on the definitions of fractional gradient and divergence, we suggest a representation of the fractional Laplacian for a smooth function $f(\mathbf{x})\colon \Omega \to \mathbb{R}^3$, such that $\nabla f(\mathbf{x})\cdot\mathbf{n}|_{\partial\Omega} = 0$, of the form

$$(\nabla^2)^\alpha f(\mathbf{x}) = \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\int_\Omega \nabla\cdot\nabla f(\mathbf{y})\exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right]d\mathbf{y}.$$

By the use of Theorem 23.2.3, we have

$$(\nabla^2)^\alpha f(\mathbf{x}) = \nabla\cdot\nabla^\alpha f(\mathbf{x}) = \nabla^\alpha\cdot\nabla f(\mathbf{x}).$$

Suppose that

$$f(\mathbf{x}) = 0 \quad on\ \partial\Omega.$$

Then, we can extend the function $f(\mathbf{x})$ to $\mathbb{R}^3$ by taking it to be zero on $\mathbb{R}^3\backslash\Omega$. This allows us to consider the Fourier transform

$$\begin{aligned}\left((\nabla^2)^\alpha f\right)_F(\boldsymbol{\xi}) &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}\left(\int_{\mathbb{R}^3}\nabla^2 f(\mathbf{y})\exp\left[-\frac{\alpha^2(\mathbf{x}-\mathbf{y})^2}{(1-\alpha)^2}\right]d\mathbf{y}\right)_F(\boldsymbol{\xi})\\ &= \frac{\alpha}{(1-\alpha)\sqrt{\pi^\alpha}}(\nabla\cdot\nabla f)_F(\boldsymbol{\xi})\left(\exp\left[-\frac{\alpha^2\mathbf{x}^2}{(1-\alpha)^2}\right]\right)_F(\boldsymbol{\xi}) \qquad (23.2.26)\\ &= 4\pi\,|\boldsymbol{\xi}|^2\,f_F(\boldsymbol{\xi})\,\sqrt{\pi^{1-\alpha}}\exp\left[-\frac{(1-\alpha)^2\boldsymbol{\xi}^2}{4\alpha^2}\right].\end{aligned}$$

Finally, if $\alpha = 1$ we obtain from (23.2.26)

$$\begin{aligned}\left(\nabla^2 f\right)_F(\boldsymbol{\xi}) &= \lim_{\alpha\to 1} 4\pi\,|\boldsymbol{\xi}|^2\, f_F(\boldsymbol{\xi})\,\sqrt{\pi^{1-\alpha}}\exp\left[-\frac{(1-\alpha)^2\boldsymbol{\xi}^2}{\alpha^2}\right]\\ &= 4\pi\,|\boldsymbol{\xi}|^2\, f_F(\boldsymbol{\xi}).\end{aligned}$$

23.2.2.7 Memory Operators

Fractional derivatives are memory operators which usually represent dissipation of energy [128, 248, 287] or damage [55] in the medium, as in the case of anelastic media or diffusion in porous media. In general, they are in agreement with the Second Law of Thermodynamics [109, 128].

Their validity rests not only on the fact that they represent appropriately a variety of phenomena, but also, in the case of the Caputo derivative, because they have the "elegant and rigorous property" that when the order of differentiation is integer, they coincide with the classic derivative of that order. However, this property is not relevant to the effects they represent in physical phenomena. It may be that using other differential operators, possibly simpler but without this property, one may obtain the same results as for Caputo fractional derivatives.

The effects of the fractional memory formalism for the new fractional derivative (NFD_t), compared with the Caputo derivative (UFD_t) on a linear trend are presented in [56].

A distributed order fractional memory operator may be introduced, which is simpler and easier to handle than the Caputo derivative [326]. It is defined by

$$\begin{aligned}{}_aP_b f(t) &= \int_a^b g(\alpha)\mathcal{D}_t^{(\alpha)} f(t)d\alpha\\ &= \int_a^b g(\alpha)\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau))f'(\tau)d\tau d\alpha,\end{aligned} \tag{23.2.27}$$

where $g(\alpha)$ is a weight function and $0 < a < b < 1$. We now take the Laplace transform of (23.2.27). Following the method of Caputo [45, 46, 56], one may interchange the order of integration of α and t. Thus, we obtain

$$\begin{aligned}({}_aP_b f(t))_L(p) &= \int_0^\infty \int_a^b \left[g(\alpha)\mathcal{D}_t^{(\alpha)} f(t)\right]\exp(-pt)d\alpha dt\\ &= \int_0^\infty \int_a^b \left[g(\alpha)\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau))f'(\tau)d\tau\right]\exp(-pt)d\alpha dt\\ &= \int_a^b \left\{\int_0^\infty \left[\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-\tau))f'(\tau)d\tau\right]\exp(-pt)dt\right\} g(\alpha)d\alpha.\end{aligned}$$

By virtue of the convolution form of the integral over τ, we have, as in (23.2.21),

$$\begin{aligned}({}_aP_b f)_L(p) &= \frac{r(p)F(p)}{p}\int_a^b \frac{g(\alpha)(1-\alpha)}{r(p)+\alpha}d\alpha,\\ F(p) &= pf_L(p) - f(0), \qquad r(p) = \frac{p}{1-p},\end{aligned}$$

which represents the filtering properties of the operator and is simpler than that obtained using the Caputo derivative.

As an example we may consider the simple case $g(\alpha) = 1$, which gives

$$
\begin{aligned}
({}_aP_bf)_L(p) &= \frac{r(p)}{p}F(p)\int_a^b \frac{1-\alpha}{\alpha + r(p)}d\alpha \\
&= \frac{r(p)}{p}F(p)\left[\int_a^b \frac{1}{\alpha + r(p)}d\alpha - \int_a^b \frac{\alpha}{\alpha + r(p)}d\alpha\right].
\end{aligned}
\tag{23.2.28}
$$

This can be written as follow

$$
\begin{aligned}
({}_aP_bf)_L(p) &= \frac{1}{1-p}F(p)\left\{\frac{1}{1-p}\log\frac{b+r(p)}{a+r(p)} - (b-a)\right\} \\
&= F(p)\left\{\frac{1}{(1-p)^2}\log\frac{b+r(p)}{a+r(p)} - \frac{1}{1-p}(b-a)\right\}.
\end{aligned}
$$

23.3 The Fractional Derivative Memory Model

Fractional derivatives and their connection with power law relaxation functions are now discussed. Constraints imposed by thermodynamics are derived, and the Graffi–Volterra free energy for fractional derivative models is introduced.

23.3.1 Power Laws and Fractional Derivatives

Let us assume that the viscoelastic memory kernel or relaxation function $\mathbb{G}(s)$ is given by the power law form (23.1.1)

$$
\mathbb{G}(s) = \frac{\mathbb{C}}{\Gamma(1-\alpha)s^{\alpha}}, \qquad \alpha \in (0,1), \tag{23.3.1}
$$

where $\Gamma(\cdot)$ is the gamma function and $\mathbb{C}$ is a fourth order tensor. Relaxation functions of this type are discussed briefly in [167, page 32], where references to older works are given. We have

$$
\mathbb{G}(\infty) = \mathbb{G}_\infty = \mathbf{0}, \qquad \lim_{s\to 0}\mathbb{G}(s) = \mathbb{G}_0 = \infty. \tag{23.3.2}
$$

The property $\mathbb{G}_\infty = \mathbf{0}$ in the classical Volterra theory corresponds to that for viscoelastic fluids. In the fractional model though the kernel (23.3.1) is not $L^1(0,\infty)$, we will see (Remark 23.3.1 below) that this is true for values of α close to 1. However, for solid viscoelastic materials, some experimental observations are in approximate agreement with predictions based on (23.3.1) [167, 280, 293, 296], notably the property that the loss angle [167] is independent of frequency, as indicated by (23.5.2) below.

It follows from (23.3.1) that

$$\mathbb{G}'(s) = -\frac{\alpha\,\mathbb{C}}{\Gamma(1-\alpha)s^{1+\alpha}}. \tag{23.3.3}$$

The creep function for materials characterized by (23.3.1) or (23.3.3) also has a power law form [167].

Observe from (23.3.1) that there are no dimensional parameters in the theory, other than the overall coefficient $\mathbb{C}$ and the time dimensional quantity s. This implies that various quantities can be simply determined to within a multiplying constant, by means of dimensional analysis.

It is in fact advisable to introduce an extra parameter ξ with the dimension of time. This can usually be absorbed into $\mathbb{C}$ or η, if the parameter α is no being explicitly varied. For solids, we put

$$\mathbb{C} = \mathbb{C}_1\xi^{\alpha}, \tag{23.3.4}$$

so that $\mathbb{C}_1$ has dimensions of stress. For fluids, we have

$$\eta = \eta_1\xi^{\alpha},$$

where η_1 has dimensions of stress.

The use of the Caputo fractional derivative, defined for any $\alpha \in (0, 1)$ by any of the forms (23.2.6)–(23.2.9), is equivalent to adopting (23.3.1) as the relaxation function of the material. This model may be applied to both fluids and solids.

We follow here the classical papers [249] on fractional derivatives in defining the constitutive equation of viscoelasticity by

$$\mathbf{T}(\mathbf{x}, t) = \frac{\mathbb{C}(\mathbf{x})}{\Gamma(1-\alpha)} \int_a^t \frac{\dot{\mathbf{E}}(\mathbf{x}, \tau)}{(t-\tau)^{\alpha}} d\tau. \tag{23.3.5}$$

Let us take $a = -\infty$, since if necessary it is always possible to extend $\mathbf{E}$ to the interval $(-\infty, a)$ by the null tensor. Thus, (23.3.5) can be written as

$$\mathbf{T}(\mathbf{x}, t) = \frac{\alpha\mathbb{C}(\mathbf{x})}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\mathbf{E}(\mathbf{x}, t) - \mathbf{E}(\mathbf{x}, \tau)}{(t-\tau)^{1+\alpha}} d\tau, \tag{23.3.6}$$

or, by a change of variable,

$$\mathbf{T}(\mathbf{x}, t) = -\frac{\alpha\mathbb{C}(\mathbf{x})}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\mathbf{E}_r^t(\mathbf{x}, s)}{s^{1+\alpha}} ds, \tag{23.3.7}$$

which are equivalent representations of the Caputo derivative. Using the notation of Sect. 23.2.1, relations (23.3.5)–(23.3.7) may be put in the form

$$\mathbf{T}(\mathbf{x}, t) = {}^{C}_{-\infty}D_t^{\alpha}[\mathbb{C}(\mathbf{x})\mathbf{E}(\mathbf{x}, t)]. \tag{23.3.8}$$

The constitutive equations (23.3.6) or (23.3.7) allow us to define the domain of definition of these functionals by a fractional Sobolev space, now called a Gagliardo space [142], defined for any $x \in \Omega$,

$$W^{\alpha,1}(-\infty,\infty) = \left\{ \mathbf{E}(t) \in L^1(-\infty,\infty)\,,\ \frac{\mathbf{E}(t)-\mathbf{E}(\tau)}{(t-\tau)^{1+\alpha}} \in L^1((-\infty,t)\times(-\infty,\infty)) \right\},$$

with norm given by

$$||\mathbf{E}||_{W^{\alpha,1}_{(-\infty,\infty)}} = \left(\int_0^\infty |\mathbf{E}(t)|\,dt + \int_{-\infty}^\infty \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{|\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)|}{(t-\tau)^{1+\alpha}}\,d\tau\,dt \right).$$

In this framework, the constitutive equation of an incompressible viscoelastic fluid is entirely analogous to (23.3.5), (23.3.6) or (23.3.7), with the formal difference that instead of the tensor $\mathbf{C}(\mathbf{x})$, we now have a scalar constant η, which is related to the viscosity of the fluid. So, the constitutive functional is given by the stress

$$\mathbf{T}(\mathbf{x},t) = -p(\mathbf{x},t)\mathbf{I} + \mathbf{T}_E(\mathbf{x},t),$$

where p denotes the pressure and $\mathbf{T}_E$ the extra-stress defined by

$$\begin{aligned} \mathbf{T}_E(\mathbf{x},t) &= \frac{2\eta}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\dot{\mathbf{E}}(\mathbf{x},\tau)}{(t-\tau)^\alpha} d\tau \\ &= \frac{\eta}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\nabla\mathbf{v}(\mathbf{x},\tau) + (\nabla\mathbf{v}(\mathbf{x},\tau))^T}{(t-\tau)^\alpha} d\tau, \end{aligned} \tag{23.3.9}$$

where $\mathbf{v}$ is the fluid velocity. Alternative forms are given by

$$\mathbf{T}_E(\mathbf{x},t) = \frac{2\alpha\eta}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)}{(t-\tau)^{1+\alpha}} d\tau \quad = -\frac{2\alpha\eta}{\Gamma(1-\alpha)} \int_0^\infty \frac{\mathbf{E}^t_r(\mathbf{x},s)}{s^{1+\alpha}} ds. \tag{23.3.10}$$

Remark 23.3.1. Note that when α is close to 0, the model well represents a viscoelastic solid. When α is close to 1, we have a viscoelastic fluid. These features of the model are implied by the (23.3.8), in view of the properties of the Caputo fractional derivative given by (23.2.10) and (23.2.11), leading to

$$\mathbf{T}(\mathbf{x},t) = \begin{cases} \mathbb{C}_1(\mathbf{x})\mathbf{E}(\mathbf{x},t) & \text{solids,} \\ \xi\eta_1(\mathbf{x})\dot{\mathbf{E}}(\mathbf{x},t) & \text{fluids,} \end{cases}$$

where we have used (23.3.4).

Remark 23.3.2. Another important feature of a solid is the existence of only one null strain $\mathbf{E}_0$ (or reference configuration) such that the space of histories is a subset of

$$\mathcal{G}^t_S = \left\{ \mathbf{E}^t(\cdot) : [0,\infty) \to Sym(\mathbb{R}^3);\ \mathbf{E}^t \in L^1(0,\infty);\ \lim_{s\to\infty} \mathbf{E}^t(s) = \mathbf{E}_0 \right\},$$

where, from (23.3.5) or (23.3.6), if $\mathbf{E}^t(s) = \mathbf{E}_0$ we have $\mathbf{T}(\mathbf{E}^t(s)) = \mathbf{0}$.

For a fluid, the set of histories belongs to

$$\mathcal{G}^t_F = \left\{ \mathbf{E}^t(\cdot) : [0,\infty) \to Sym(\mathbb{R}^3);\ \mathbf{E}^t \in L^1(0,\infty) \right\}.$$

The main difference between the Volterra and fractional derivative models, is evident in the study of solid materials, if we examine stress behavior for $t \to \infty$. Indeed, in the context of the Volterra theory, if the system is subject to a constant strain $\mathbf{E}_0$, then stress will tend to $\mathbf{G}_\infty \mathbf{E}_0$. This can be proved by assuming a constant strain $\mathbf{E}_0(\mathbf{x})$ for $t > t_0$, so that the following limit is obtained

$$\begin{aligned}\lim_{t\to\infty}\mathbf{T}(\mathbf{x},t) &= \mathbb{G}_\infty(\mathbf{x})\mathbf{E}_0(\mathbf{x}) \\ &\quad + \lim_{t\to\infty}\int_{t-t_0}^{\infty}\mathbb{G}'(\mathbf{x},s)(\mathbf{E}(\mathbf{x},t-s)-\mathbf{E}_0(\mathbf{x}))ds \\ &= \mathbb{G}_\infty(\mathbf{x})\mathbf{E}_0(\mathbf{x}),\ \ t > t_0.\end{aligned} \tag{23.3.11}$$

On the other hand, in the fractional theory we find that the stress will go to zero. Indeed,

$$\lim_{t\to\infty}\frac{\alpha\mathbb{C}(\mathbf{x})}{\Gamma(1-\alpha)}\int_{-\infty}^{t_0}\frac{\mathbf{E}_0(\mathbf{x})-\mathbf{E}(\mathbf{x},\tau)}{(t-\tau)^{1+\alpha}}d\tau = \mathbf{0},\ \ t > t_0.$$

In this case, the material undergoes a kind of plastic deformation [49].

Remark 23.3.3. It is interesting to observe that Volterra sought to describe properties related to dislocation phenomena in terms of memory effects. However, it is now known that the model with the standard constitutive relation ((8.1.5), for example) is not capable of describing plastic effects produced by dislocations.

23.4 Thermodynamical Constraints and Free Energies

The issue of compatibility of fractional derivative models with thermodynamics is explored in this section. Only isothermal processes will be considered, so that the Second Law of Thermodynamics reduces to the Dissipation Principle

$$\rho(\mathbf{x})\dot{\psi}(\mathbf{x},t) \le \mathbf{T}(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t), \tag{23.4.1}$$

where ψ denotes a free energy and ρ is the mass density. This is equivalent to property P3 given by (16.1.28). We have from (23.4.1) that on any cyclic process of period $T = 2\pi/\omega$ with $\omega \in \mathbb{R}^{++}$,

$$\int_0^T \mathbf{T}(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t)\,dt \ge 0. \tag{23.4.2}$$

In particular, for periodic strain processes of the form

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_1(\mathbf{x})\cos\omega t + \mathbf{E}_2(\mathbf{x})\sin\omega t,$$

it follows from (23.4.2) (see [108, 124]) that for all $\mathbf{E}_1, \mathbf{E}_2 \in Sym(V)$,

$$\begin{aligned}&\int_0^\infty\left(\mathbf{E}_1\cdot\frac{\mathbb{C}}{s^{1+\alpha}}\mathbf{E}_1 + \mathbf{E}_2\cdot\frac{\mathbb{C}}{s^{1+\alpha}}\mathbf{E}_2\right)\sin\omega s\,ds \\ &\quad + \int_0^\infty \mathbf{E}_1\cdot\frac{\mathbb{C}-\mathbb{C}^T}{s^{1+\alpha}}\mathbf{E}_2\cos\omega s\,ds \le 0\,,\quad \text{for all } \omega\in\mathbb{R}^+.\end{aligned} \tag{23.4.3}$$

The second term can vary arbitrarily in sign and magnitude for different choices of $\mathbf{E}_1$ and $\mathbf{E}_2$, so that it can be concluded that the fourth order tensor $\mathbb{C}$ is symmetric. Also, in the first term, the quantity $\mathbf{E} \cdot \mathbb{C}\mathbf{E}$ must have a definite signature for the inequality to be obeyed in a simple manner. We choose $\mathbf{E} \cdot \mathbb{C}\mathbf{E} \geq 0$ so that $\mathbb{C}$ is at least positive semidefinite. Thus, the condition (23.4.3) becomes

$$\int_0^\infty \frac{1}{s^{1+\alpha}} \sin \omega s \, ds \leq 0 \text{ for all } \omega \in \mathbb{R}^+. \tag{23.4.4}$$

Remark 23.4.1. For a fluid defined by (23.3.10), we obtain from the second law the same inequality (23.4.4). Alternatively, if (23.3.9) is used, the equivalent condition emerges

$$\int_0^\infty \frac{1}{s^{\alpha}} \cos \omega s \, ds \geq 0 \ \ for \ all \ \ \omega \in \mathbb{R}. \tag{23.4.5}$$

We observe that (23.4.5) has the form (16.1.12) for the case of a fractional derivative relaxation function. It will be confirmed that this property is actually true in Remark 23.5.1.

23.4.1 The Graffi–Volterra Free Energy

We now consider a particular free energy within fractional derivative theory. Let us first take the case of a solid, described by Eq. (23.3.7). This functional is denoted by ψ^S. Any free energy functional must satisfy the inequality (23.4.1). Thus, we must have

$$\rho(\mathbf{x})\dot{\psi}^S(\mathbf{x}, t) \leq \mathbf{T}(\mathbf{x}, t) \cdot \dot{\mathbf{E}}(\mathbf{x}, t). \tag{23.4.6}$$

For simplicity, let us take the $\mathbb{C}(\mathbf{x})$ to be a scalar quantity $C(\mathbf{x})$. Using the identity

$$\frac{d}{dt}\mathbf{E}(\mathbf{x}, t) = -\frac{d}{ds}\mathbf{E}_r^t(\mathbf{x}, s) - \frac{d}{dt}\mathbf{E}_r^t(\mathbf{x}, s),$$

we have

$$\begin{aligned}
\mathbf{T}(\mathbf{x}, t) \cdot \dot{\mathbf{E}}(\mathbf{x}, t) &= -\frac{\alpha C(\mathbf{x})}{\Gamma(1-\alpha)} \int_0^\infty \frac{\mathbf{E}_r^t(\mathbf{x}, s)}{s^{1+\alpha}} ds \cdot \dot{\mathbf{E}}(\mathbf{x}, t) \\
&= \frac{\alpha C(\mathbf{x})}{\Gamma(1-\alpha)} \int_0^\infty \frac{\mathbf{E}_r^t(\mathbf{x}, s)}{s^{1+\alpha}} \cdot \frac{d}{dt}\mathbf{E}_r^t(\mathbf{x}, s)) ds \\
&\quad + \frac{\alpha C(\mathbf{x})}{\Gamma(1-\alpha)} \int_0^\infty \frac{\mathbf{E}_r^t(\mathbf{x}, s)}{s^{1+\alpha}} \cdot \frac{d}{ds}\mathbf{E}_r^t(\mathbf{x}, s) ds.
\end{aligned} \tag{23.4.7}$$

Let us assume that $\psi^S(\mathbf{x}, t)$ is given by

$$\psi^S(\mathbf{x}, t) = \frac{\alpha C(\mathbf{x})}{2\rho(\mathbf{x})\Gamma(1-\alpha)} \int_0^\infty \frac{|\mathbf{E}_r^t(\mathbf{x}, s)|^2}{s^{1+\alpha}} ds. \tag{23.4.8}$$

This is the Graffi–Volterra free energy for fractional derivative models. It is discussed in a more general context in Sects. 17.3.1 and 10.1.1, where it is shown to be the only free energy that is a single-integral quadratic form. The inequality (23.4.6) is satisfied because by (23.4.7) we obtain

$$\rho(\mathbf{x})\dot{\psi}^S(\mathbf{x},t) = \mathbf{T}(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t) - \frac{\alpha C(\mathbf{x})}{2\Gamma(1-\alpha)}\int_0^\infty \frac{\frac{d}{ds}|\mathbf{E}_r^t(\mathbf{x},s))|^2}{s^{1+\alpha}}ds$$
$$= \mathbf{T}(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t) - \frac{\alpha C(\mathbf{x})(1+\alpha)}{2\Gamma(1-\alpha)}\int_0^\infty \frac{|\mathbf{E}_r^t(\mathbf{x},s)|^2}{s^{2+\alpha}}ds.$$

Hence

$$\rho(\mathbf{x})\dot{\psi}^S(\mathbf{x},t) = \mathbf{T}(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t) - D(\mathbf{x},t), \tag{23.4.9}$$

where $D(\mathbf{x},t)\geq 0$ denotes the rate of dissipation, given by

$$D(\mathbf{x},t) = \frac{\alpha C(\mathbf{x})(1+\alpha)}{2\Gamma(1-\alpha)}\int_0^\infty \frac{|\mathbf{E}_r^t(\mathbf{x},s)|^2}{s^{2+\alpha}}ds. \tag{23.4.10}$$

The set of histories $\mathcal{H}_S^t$ available with this model is defined by

$$\mathcal{H}_S^t = \left\{\mathbf{E}^t : [0,\infty)\to Sym(V);\ \psi^S(\mathbf{E}(t),\mathbf{E}^t(\cdot)) < \infty\right\}.$$

For a viscoelastic fluid, we use the constitutive Eq. (23.3.10), which is more convenient than (23.3.9). This gives

$$\mathbf{T}_E(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t) = \frac{2\alpha\eta}{\Gamma(1-\alpha)}\int_{-\infty}^t \frac{\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)}{(t-\tau)^{1+\alpha}}d\tau\cdot\dot{\mathbf{E}}(\mathbf{x},t)$$
$$= \frac{2\alpha\eta}{\Gamma(1-\alpha)}\int_{-\infty}^t \frac{\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)}{(t-\tau)^{1+\alpha}}\cdot\frac{d}{dt}(\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau))d\tau$$
$$= \frac{\alpha\eta}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_{-\infty}^t \frac{|\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)|^2}{(t-\tau)^{1+\alpha}}d\tau\right.$$
$$\left.-(1+\alpha)\int_{-\infty}^t \frac{|\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)|^2}{(t-\tau)^{2+\alpha}}d\tau\right].$$

The Graffi–Volterra for fluids is given in this case by

$$\psi^F(\mathbf{x},t) = \frac{\alpha\eta}{\rho(\mathbf{x})\Gamma(1-\alpha)}\int_{-\infty}^t \frac{|\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)|^2}{(t-\tau)^{1+\alpha}}d\tau$$

or, using the variable $s = t-\tau$, the equivalent form

$$\psi^F(\mathbf{x},t)) = \frac{\alpha\eta}{\rho(\mathbf{x})\Gamma(1-\alpha)}\int_0^\infty \frac{|(\mathbf{E}_r^t(\mathbf{x},s)|^2}{s^{1+\alpha}}ds.$$

Of course, it has a similar form to (23.4.8) for solids. We may define the set of histories $\mathcal{H}_F^t$ available with this model by

$$\mathcal{H}_F^t = \left\{\mathbf{E}^t : [0,\infty)\to Sym(V);\ \psi^F(\mathbf{E}(t),\mathbf{E}^t(\cdot)) < \infty\right\}.$$

The rate of dissipation $D(\mathbf{x}, t)$ is given by

$$D(\mathbf{x}, t) = \frac{\alpha\eta(1+\alpha)}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{|\mathbf{E}(\mathbf{x}, t) - \mathbf{E}(\mathbf{x}, \tau)|^2}{(t-\tau)^{2+\alpha}} d\tau \geq 0,$$

or

$$D(\mathbf{x}, t) = \frac{\alpha\eta(1+\alpha)}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{|\mathbf{E}_r^t(\mathbf{x}, s)|^2}{s^{2+\alpha}} ds \geq 0.$$

23.5 Frequency-Domain Quantities for Scalar Fractional Derivative Materials

In the next two sections, we deal, for simplicity, with the scalar theory, for which the relaxation function and strain history are scalar quantities. Also, we generally omit the space variable **x**.

23.5.1 Complex Modulus for the Fractional Derivative Model

It follows from (23.3.3) and $(16.1.7)_2$ that the quantity $G'_+(\omega)$ does not exist for fractional derivative forms, while $\widetilde{G}_+(\omega)$, given by $(16.1.7)_1$, is finite. Therefore, the complex modulus, defined by $(16.1.9)_2$, is finite. Recalling (23.3.1) and (23.3.2), we see that it is given by

$$M(\omega) = i\omega \int_0^\infty G(s) e^{-i\omega s} ds = i\omega \frac{k}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} e^{-i\omega s} ds,$$

where the coefficient k corresponds to $\mathbb{C}$ (or C) and η in Sect. 23.3. From dimensional analysis, we can determine that

$$M(\omega) = k\, c\, \omega^\alpha,$$

where c is a dimensionless constant to be determined. Putting $z = i\omega$ and rotating it to a point on the positive real axis, the integral can be evaluated in terms of the Gamma function. Rotating back, we find that ([167, page 33])

$$M(\omega) = i\omega G_+(\omega) = k\, c\, \omega^\alpha \quad \forall \omega \geq 0, \qquad c = \exp\left(i\frac{\alpha\pi}{2}\right).$$

The real and imaginary parts of $M(\omega)$ have the form

$$\begin{aligned} M_1(\omega) &= \omega G_s(\omega) = k\,\omega^\alpha \cos\left(\frac{\alpha\pi}{2}\right), \\ M_2(\omega) &= \omega G_c(\omega) = k\,\omega^\alpha \sin\left(\frac{\alpha\pi}{2}\right) \quad \forall \omega \geq 0. \end{aligned} \tag{23.5.1}$$

Remark 23.5.1. The positivity of $G_c(\omega)$ is clear from (23.5.1), for $\omega \in \mathbb{R}^+$ and therefore for all real ω. This confirms the thermodynamic constraint $(16.1.12)_1$.

The loss angle γ, defined by the relationship

$$\frac{M_2(\omega)}{M_1(\omega)} = \tan\gamma,$$

is given by

$$\gamma = \frac{\alpha\pi}{2}, \tag{23.5.2}$$

which is independent of ω, as noted earlier.

It is apparent from (23.5.1) that the only singularity in $G_+(\omega)$ is a branch cut from the origin to infinity. Thus, minimal states for power law materials are singletons, in accordance with Proposition 16.5.2. The branch cut must lie in Ω^+. Apart from this constraint, we can choose it to be in any direction. Each choice yields a different function. We choose it to be along the positive imaginary axis. An expression for the minimum free energy of this material is derived in Chap. 14. Thus, we can reach the negative real axis, without crossing singularities, by a rotation $e^{-i\pi}$ of ω. This gives

$$M(-\omega) = -i\omega G_+(-\omega) = k|\omega|^{\alpha} \exp\left(-i\frac{\alpha\pi}{2}\right) = \overline{M}(\omega), \quad \forall\omega \geq 0, \tag{23.5.3}$$

which has a branch cut along the negative imaginary axis.

From $(16.1.21)_4$, we see that the frequency-domain version of the constitutive relation has the form

$$\begin{aligned} T(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{M}(\omega)E^t_{r+}(\omega)d\omega \\ &= \frac{k}{2\pi}\int_0^{\infty} e^{-i\frac{\alpha\pi}{2}}\omega^{\alpha}E^t_{r+}(\omega)d\omega + \frac{k}{2\pi}\int_{-\infty}^{0} e^{i\frac{\alpha\pi}{2}}(-\omega)^{\alpha}E^t_{r+}(\omega)d\omega \\ &= -\frac{k\sin(\alpha\pi)}{\pi}\int_0^{\infty} r^{\alpha}E^t_{r+}(-ir)dr. \end{aligned} \tag{23.5.4}$$

The last forms are obtained by moving the contour to closely surround the cut on the negative imaginary axis. The first term of $(23.5.4)_2$ becomes the integral over $[0, -i\infty)$ on the right side of the negative imaginary axis, while in the second term becomes the integral over $(-i\infty, 0]$ on the left side. Relation (16.1.17) has been invoked in writing the last equation.

Using (16.1.13), $(23.5.1)_2$ and (23.5.3), we deduce that the function H is defined over $\mathbb{R}$ by

$$H(\omega) = a|\omega|^{\alpha+1}, \quad \forall\omega \in \mathbb{R}, \qquad \text{where} \quad a = k\sin\left(\frac{\alpha\pi}{2}\right). \tag{23.5.5}$$

Using $(16.1.23)_1$, we can write $T(t)$ in the form

$$
\begin{aligned}
T(t) &= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{H(\omega)}{\omega}E_{r+}^t(\omega)d\omega \\
&= \frac{a}{\pi i}\int_{0}^{\infty}\omega^{\alpha}E_{r+}^t(\omega)d\omega - \frac{a}{\pi i}\int_{-\infty}^{0}(-\omega)^{\alpha}E_{r+}^t(\omega)d\omega \\
&= -k\frac{\sin(\alpha\pi)}{\pi}\int_{0}^{\infty}r^{\alpha}E_{r+}^t(-ir)dr = k\frac{\sin(\alpha\pi)}{\pi}\int_{0}^{\infty}r^{\alpha-1}\dot{E}_{+}^t(-ir)dr.
\end{aligned}
\tag{23.5.6}
$$

The penultimate form is obtained by transforming the integrals according to the changes described in relation to (23.5.4), while the final form uses (16.1.17).

23.5.2 The Work Function for Fractional Derivative Materials

Relation (17.3.19) for the work function becomes

$$
W(t) = \frac{k}{2\Gamma(1-\alpha)}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\dot{E}^t(s)\dot{E}^t(u)}{|s-u|^{\alpha}}duds,
$$

which involves an integrable singularity. Using (23.5.5), relations (17.3.19) become

$$
\begin{aligned}
W(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(\omega)}{\omega^2}|\dot{E}_{+}^t(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)|E_{r+}^t(\omega)|^2d\omega \\
&= \frac{a}{2\pi}\int_{-\infty}^{\infty}|\omega|^{\alpha-1}|\dot{E}_{+}^t(\omega)|^2\,d\omega = \frac{a}{2\pi}\int_{-\infty}^{\infty}|\omega|^{\alpha+1}|E_{r+}^t(\omega)|^2d\omega.
\end{aligned}
\tag{23.5.7}
$$

The basic property $\dot{W}(t) = T(t)\dot{E}(t)$ can be shown using $(23.5.6)_1$, (23.5.7), $(16.1.18)_2$ and the evenness of $H(\omega)$.

23.6 The Minimum Free Energy for Fractional Derivative Models

We now derive the form for the minimum free energy and the corresponding rate of dissipation for fractional derivative materials. These are for general histories. Simple explicit formulae are also given for sinusoidal and exponential histories. The derivations are for the scalar case.

23.6.1 General Form of the Minimum Free Energy

The factors $H_{\pm}(\omega)$ of $H(\omega)$ have the form

$$
\begin{aligned}
H_{+}(\omega) &= \sqrt{a}\,\omega^{\eta}e^{i\lambda(\omega)}, \\
H_{-}(\omega) &= \sqrt{a}\,\omega^{\eta}e^{-i\lambda(\omega)}, \qquad \eta = \frac{\alpha+1}{2} \in (\frac{1}{2}, 1),
\end{aligned}
\tag{23.6.1}
$$

for $\omega > 0$. The phase $\lambda(\omega)$ remains to be determined. On the complex plane, $H_{\pm}(\omega)$ are analytic continuations of these quantities, except at their singularities, which are now described.

The singularities of $H_+(\omega)$ are chosen to be along the positive imaginary axis, similarly to $G_+(\omega)$ as described before (23.5.3). Then $H_-(\omega)$ is the complex conjugate of this function with singularities consisting of a branch cut along the negative imaginary axis.

If we had chosen different straight line branch cuts, the resulting factors would differ from those we determine here by a constant phase factor. This can be seen from (23.6.1) by considering a transformed complex plane with typical point $\omega_1 = \omega\exp(i\delta)$, where δ is a constant. It is relevant that $\lambda(\omega)$ is later shown to be independent of ω. The phase factor δ would not affect the resulting formula for the minimum free energy.

The branch cuts described are simple consequences of the factors ω^η in (23.6.1). The product of $H_\pm(\omega)$ will lead to branch cuts in $H(\omega)$. The factors of $H(\omega)$ obey the following relationships, for general complex ω:

$$\overline{H_\pm(\omega)} = \overline{H}_\pm(\overline{\omega}) = H_\mp(\overline{\omega}) = H_\pm(-\overline{\omega}),$$

from which it follows that $H_+(\omega)$ ($H_-(\omega)$) must be real on the negative (positive) imaginary axis; indeed, this property applies to any point on the imaginary axis where these factors exist. Thus,

$$\lambda(re^{i\frac{\pi}{2}}) = \lambda(re^{-i\frac{\pi}{2}}) = \frac{\pi\eta}{2}. \tag{23.6.2}$$

The discontinuity in $H_-(\omega)$ across the branch cut along the imaginary axis is given by the discontinuity in

$$H_-(\omega) = \frac{H(\omega)}{H_+(\omega)}, \qquad H_+(re^{-i\frac{\pi}{2}}) = \sqrt{a}r^\eta.$$

Thus, the discontinuity is determined by $H(\omega)$, divided by a real quantity of the form $H_+(r\exp(-i\frac{\pi}{2}))$. Therefore, the singularities in $H_-(\omega)$ are determined by those in $H(\omega)$. Similar observations apply to the singularities of $H_+(\omega)$.

Proposition 23.6.1. *The phase $\lambda(\omega)$ is independent of ω.*

Proof. The phase factors $e^{\pm i\lambda(\omega)}$ are continued analytically to the whole complex plane. By the argument just outlined, they cannot contribute new singularities over and above those determined by $H(\omega)$, which is independent of $\lambda(\omega)$. Thus, we conclude that the phase factors will yield no singularities and must therefore be entire functions. However, this means either that they contribute essential singularities at infinity, which must be excluded in the same way as singularities on the finite plane, or they are constant. Therefore, the quantity λ is independent of ω. □

The factorization (16.1.14) clearly allows us to replace $H_\pm(\omega)$ by $-H_\pm(\omega)$.

For $\omega \in \mathbb{R}^{++}$, we have

$$\begin{aligned} H_+(\omega) &= \sqrt{a}\,\omega^\eta\, e^{i\lambda}, \\ H_-(\omega) &= \sqrt{a}\,\omega^\eta\, e^{-i\lambda}, \end{aligned} \tag{23.6.3}$$

where, from (23.6.2),

$$\lambda = \frac{\pi\eta}{2}. \tag{23.6.4}$$

We put

$$\omega = re^{i\theta}$$

so that $\theta = \arg(\omega)$. The behavior of $H_+(\omega)$ as ω approaches the positive imaginary axis from the first and second quadrants, respectively, are given by

$$H_+(\omega) = \begin{cases} \sqrt{a}\, r^\eta\, e^{i\frac{\pi(\alpha+1)}{4} + i\lambda}, & \text{1st quadrant}, \quad \theta = \frac{\pi}{2}, \\ \sqrt{a}\, r^\eta\, e^{-3i\frac{\pi(\alpha+1)}{4} + i\lambda}, & \text{2nd quadrant}, \quad \theta = -\frac{3\pi}{2}. \end{cases} \tag{23.6.5}$$

Similarly, the behavior of $H_-(\omega)$ as it approaches the negative imaginary axis from the fourth and third quadrants, respectively, are given by

$$H_-(\omega) = \begin{cases} \sqrt{a}\, r^\eta\, e^{-i\frac{\pi(\alpha+1)}{4} - i\lambda}, & \text{4th quadrant}, \quad \theta = -\frac{\pi}{2}, \\ \sqrt{a}\, r^\eta\, e^{3i\frac{\pi(\alpha+1)}{4} - i\lambda}, & \text{3rd quadrant}, \quad \theta = \frac{3\pi}{2}. \end{cases} \tag{23.6.6}$$

In the light of (23.6.4), the limiting values (23.6.5) and (23.6.6) reduce to

$$H_+(\omega) = \begin{cases} \sqrt{a}\, r^\eta\, e^{2i\lambda}, & \text{1st quadrant}, \quad \theta = \frac{\pi}{2}, \\ \sqrt{a}\, r^\eta\, e^{-2i\lambda}, & \text{2nd quadrant}, \quad \theta = -\frac{3\pi}{2}, \end{cases}$$

and

$$H_-(\omega) = \begin{cases} \sqrt{a}\, r^\eta\, e^{-2i\lambda}, & \text{4th quadrant}, \quad \theta = -\frac{\pi}{2}, \\ \sqrt{a}\, r^\eta\, e^{2i\lambda}, & \text{3rd quadrant}, \quad \theta = \frac{3\pi}{2}. \end{cases}$$

Let us use (23.6.3) to extend $H_\pm(\omega)$ to $\omega \in \mathbb{R}$. We take $\theta = -\pi$ to obtain $H_+(\omega)$, and $\theta = \pi$ for $H_-(\omega)$. This gives, for $\omega < 0$,

$$\begin{aligned} H_+(\omega) &= \sqrt{a}\, r^\eta\, e^{-i\eta\pi + i\lambda} = \sqrt{a}|\omega|^\eta e^{-i\lambda}, \\ H_-(\omega) &= \sqrt{a}\, r^\eta\, e^{i\eta\pi - i\lambda} = \sqrt{a}|\omega|^\eta e^{i\lambda}. \end{aligned} \tag{23.6.7}$$

These formulae are consistent with (22.1.8). Note that on multiplying the factors in (23.6.3) and (23.6.7) together, we obtain (23.5.5) for $\omega \in \mathbb{R}$.

The quantity $p_-^t(\omega)$ has the form

$$\begin{aligned} p_-^t(\omega) = \frac{\sqrt{a}}{2\pi i} e^{-i\lambda} \int_0^\infty \frac{\omega_1^\eta E_{r+}^t(\omega_1)}{\omega_1 - \omega^+} d\omega_1 \\ + \frac{\sqrt{a}}{2\pi i} e^{i\lambda} \int_{-\infty}^0 \frac{(-\omega_1)^\eta E_{r+}^t(\omega_1)}{\omega_1 - \omega^+} d\omega_1, \end{aligned}$$

on using (23.6.3) and (23.6.7). The singularities of $E^t_{r+}(\omega)$ are in $\mathbb{C}^{(+)}$. Recalling the position of the cut in $H_-(\omega)$, we see that the integrations over the real axis can be moved into the lower half-plane to closely surround the branch cut, yielding

$$\begin{aligned} p^t_-(\omega) &= \frac{\sqrt{a}}{2\pi i}\int_0^\infty \frac{r_1^\eta E^t_{r+}(-ir_1)}{r_1 - i\omega}\left(e^{-2i\lambda} - e^{2i\lambda}\right)dr_1 \\ &= -\frac{\sqrt{a}\sin 2\lambda}{\pi}\int_0^\infty \frac{r_1^\eta E^t_{r+}(-ir_1)}{r_1 - i\omega}dr_1, \end{aligned}$$

where $E^t_{r+}(-ir_1)$ is a real quantity given by

$$E^t_{r+}(-ir_1) = \int_0^\infty E^t_r(u)e^{-r_1 u}du. \tag{23.6.8}$$

Using

$$\psi_m(t) = \frac{1}{2\pi}\int_{-\infty}^\infty |p^t_-(\omega)|^2 d\omega$$

and Cauchy's integral formula, we obtain

$$\begin{aligned} \psi_m(t) &= \kappa\int_0^\infty\int_0^\infty \frac{(r_1 r_2)^\eta E^t_{r+}(-ir_1)E^t_{r+}(-ir_2)}{r_1 + r_2}dr_1 dr_2 \\ &= \kappa\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(r_1 r_2)^\eta E^t_r(u)E^t_r(v)e^{-r_1 u - r_2 v}}{r_1 + r_2}dr_1 dr_2 du dv, \\ \kappa &= \frac{a}{\pi^2}\sin^2\left[\frac{\pi}{2}(\alpha + 1)\right] = \frac{k}{\pi^2}\sin\frac{\alpha\pi}{2}\cos^2\frac{\alpha\pi}{2}. \end{aligned} \tag{23.6.9}$$

Using (16.1.17), we can also write (23.6.9) in the form

$$\begin{aligned} \psi_m(t) &= \kappa\int_0^\infty\int_0^\infty \frac{(r_1 r_2)^{\eta - 1}\dot{E}^t_+(-ir_1)\dot{E}^t_+(-ir_2)}{r_1 + r_2}dr_1 dr_2 \\ &= \kappa\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(r_1 r_2)^{\eta - 1}\dot{E}^t(u)\dot{E}^t(v)e^{-r_1 u - r_2 v}}{r_1 + r_2}dr_1 dr_2 du dv. \end{aligned} \tag{23.6.10}$$

The quantity η is defined by $(23.6.1)_3$.

The minimum free energy may be written in the form (see (11.9.14))

$$\psi_m(t) = \frac{1}{2}\int_0^\infty\int_0^\infty \dot{E}^t(s)G_m(s,u)\dot{E}^t(u)ds du,$$

where the equilibrium term vanishes, since $G_\infty = 0$. We must have

$$G(u) = G(0,u) = G(u,0) = \frac{k}{\Gamma(1-\alpha)u^\alpha}, \tag{23.6.11}$$

where $G(u)$ is the relaxation function. Thus, we have

$$G_m(u,v) = 2\kappa \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta - 1} e^{-r_1 u - r_2 v}}{r_1 + r_2} dr_1 dr_2.$$

Recalling (23.3.2), we see that the properties (17.3.7) are valid in this case. Relation (23.6.11) is confirmed since

$$\begin{aligned}
2\kappa \int_0^\infty \int_0^\infty & \frac{(r_1 r_2)^{\eta - 1} e^{-r_2 v}}{r_1 + r_2} dr_1 dr_2 \\
&= 2\kappa \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta - 1} e^{-r_1 v}}{r_1 + r_2} dr_1 dr_2 \\
&= \frac{k \sin \alpha\pi}{\pi} \int_0^\infty r_1^{\alpha - 1} e^{-r_1 v} dr_1 \\
&= \frac{k \sin \alpha\pi \Gamma(\alpha)}{\pi v^\alpha} = \frac{k}{\Gamma(1-\alpha) v^\alpha},
\end{aligned}$$

by virtue of the formula, for complex z ([168, page 285]),

$$\int_0^\infty \frac{r^{\mu - 1}}{r + z} dr = \frac{\pi z^{\mu - 1}}{\sin \pi\mu}, \qquad |\arg z| < \pi, \qquad 0 < \mu < 1, \tag{23.6.12}$$

together with the relations (23.2.2) and (23.2.3). We now seek to determine the rate of dissipation corresponding to the minimum free energy. Relation $(16.1.18)_2$ yields that

$$\frac{d}{dt} E^t_{r+}(-ir) = -r E^t_{r+}(-ir) - \frac{\dot{E}(t)}{r},$$

so that, from (23.6.9), we have

$$\begin{aligned}
\dot{\psi}_m(t) = -\kappa \left| \int_0^\infty r_1^\eta E^t_{r+}(-ir_1) dr_1 \right|^2 \\
- 2\kappa \dot{E}(t) \int_0^\infty \int_0^\infty \frac{r_1^\eta E^t_{r+}(-ir_1) r_2^{\eta - 1}}{r_1 + r_2} dr_1 dr_2.
\end{aligned} \tag{23.6.13}$$

Using (23.6.12) and the expression $(23.6.9)_3$ for κ, the last term of (23.6.13) becomes

$$-\frac{k \sin \alpha\pi}{\pi} \int_0^\infty r_1^\alpha E^t_{r+}(-ir_1) dr_1 \dot{E}(t) = T(t) \dot{E}(t),$$

by virtue of (23.5.6). Consequently, recalling (16.1.28), it follows that the rate of dissipation is given by the negative of the first term on the right-hand side of (23.6.13), where we have used (16.1.17). These relations can be deduced also from (16.4.12). The quantity $D_m(t)$ can be written in the form (17.3.11) with

$$K_m(s,u) = -\frac{2\kappa \Gamma^2(\eta)}{(su)^\eta},$$

$$\begin{aligned}D_m(t) &= \kappa\left|\int_0^\infty r^\eta E^t_{r+}(-ir)dr\right|^2 = \kappa\left|\int_0^\infty r^{\eta-1}\dot{E}^t_+(-ir)dr\right|^2\\ &= \kappa\left|\int_0^\infty\int_0^\infty r^{\eta-1}e^{-ru}\dot{E}^t(u)drdu\right|^2 \qquad (23.6.14)\\ &= \kappa\Gamma^2(\eta)\left|\int_0^\infty \frac{\dot{E}^t(u)}{u^\eta}du\right|^2 \geq 0.\end{aligned}$$

23.6.2 The Minimum Free Energy for Simple Histories

Free energies and rates of dissipation for sinusoidal and increasing exponential histories are discussed in [15, 16]. Sinusoidal histories are useful in many practical contexts, though the total dissipation and the work function, defined by (16.1.30), are infinite. Increasing exponential histories provide a simple example where all quantities are finite. Also, the algebra involved is similar to, though simpler than, the sinusoidal case.

23.6.2.1 Sinusoidal Histories

Formulae relating to general materials for sinusoidal histories are presented in [15, 16] and earlier papers. Also, for exponential histories, similar general results are introduced in [16]. Here, we consider the specific cases (23.6.10) and (23.6.14) directly for the relevant forms of the strain history.

Consider a history and current value $(E^t, E(t))$ defined by

$$E(t) = E_0 e^{i\omega_0 t} + \overline{E}_0 e^{-i\omega_0 t}, \quad E^t(s) = E(t-s), \qquad (23.6.15)$$

where E_0 is an amplitude and $\overline{E}_0$ its complex conjugate. The quantities E^t_+ and $\dot{E}^t_+$ have the form

$$\begin{aligned}E^t_+(\omega) &= E_0\frac{e^{i\omega_0 t}}{i(\omega+\omega_0)} + \overline{E}_0\frac{e^{-i\omega_0 t}}{i(\omega-\omega_0)},\\ \dot{E}^t_+(\omega) &= \omega_0 E_0\frac{e^{i\omega_0 t}}{\omega+\omega_0} - \omega_0\overline{E}_0\frac{e^{-i\omega_0 t}}{\omega-\omega_0}.\end{aligned} \qquad (23.6.16)$$

From $(17.6.8)_2$, we find that

$$\dot{E}^t_+(-ir) = \omega_0 E_0\frac{e^{i\omega_0 t}}{\omega_0 - ir} + \omega_0\overline{E}_0\frac{e^{-i\omega_0 t}}{\omega_0+ir}, \qquad (23.6.17)$$

where r is real. The final form of (23.5.6), together with (23.6.12) and (23.6.17) give

$$T(t) = k\omega_0^\alpha\left[e^{i\pi\alpha/2}E_0e^{i\omega_0 t} + e^{-i\pi\alpha/2}\overline{E}_0e^{-i\omega_0 t}\right]. \qquad (23.6.18)$$

Any real algebraic quadratic form in $E(t)$ or real functional quadratic form in $E^t(s)$ can be written in the form (16.11.3), denoted by V. Recalling (17.6.12) we introduce the abbreviated notation

$$V = \{A, B\}. \tag{23.6.19}$$

Thus, we can write $\psi_m(t)$, given by (23.6.10), as

$$\psi_m(t) = \{A, B\},$$

where, by dimensional arguments, it follows that

$$A = k\rho\,\omega_0^{\alpha}, \qquad B = k\chi\,\omega_0^{\alpha}. \tag{23.6.20}$$

The dimensionless quantities ρ and χ are to be determined. We find, using (23.6.17), that

$$A = -\kappa\omega_0^2 \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta-1}}{(r_1 + i\omega_0)(r_2 + i\omega_0)(r_1 + r_2)} dr_1 dr_2,$$
$$B = 2\kappa\omega_0^2 Re \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta-1}}{(r_1 + i\omega_0)(r_2 - i\omega_0)(r_1 + r_2)} dr_1 dr_2.$$

The following integrals, together with (23.6.12), will be required in the calculations below. For complex y, z, we have

$$\begin{aligned} &\int_0^\infty \frac{r^{\mu-1}}{(r+y)(r+z)} dr = \frac{\pi}{\sin\mu\pi}\left[\frac{y^{\mu-1} - z^{\mu-1}}{z-y}\right], \quad |\arg y|, |\arg z| < \pi, \\ &\int_0^\infty \frac{r^{\mu-1}}{(r+z)^2} dr = -\frac{\pi(\mu-1)}{\sin\mu\pi} z^{\mu-2}, \quad 0<\mu<2, \quad |\arg z| < \pi. \end{aligned} \tag{23.6.21}$$

Relation $(23.6.21)_1$ is given in [168, page 289], while $(23.6.21)_2$ is a special case of a result in [168, page 285]. The latter can also be obtained by differentiating (23.6.12) with respect to z.

With the aid of $(23.6.21)_1$, we find that

$$A = -\frac{\kappa\omega_0^2\pi}{\sin\eta\pi} \int_0^\infty \frac{r_2^{\eta-1}\left[(i\omega_0)^{\eta-1} - r_2^{\eta-1}\right]}{r_2^2 + \omega_0^2} dr_2,$$
$$B = 2\frac{\kappa\omega_0^2\pi}{\sin\eta\pi} Re \int_0^\infty \frac{r_2^{\eta-1}\left[(i\omega_0)^{\eta-1} - r_2^{\eta-1}\right]}{(r_2 - i\omega_0)^2} dr_2.$$

These integrals can also be evaluated using (23.6.21). With the use of results derivable from $(23.6.21)_1$, we find that ρ in $(23.6.20)_1$ has the form

$$\begin{aligned} \rho &= -\frac{1}{4}\sin\alpha\pi\left[1 - \frac{1}{\sin\frac{\pi\alpha}{2}} - i\frac{1 - \sin\frac{\alpha\pi}{2}}{\cos\frac{\alpha\pi}{2}}\right] \\ &= \frac{1}{2}\left(1 - \sin\frac{\pi\alpha}{2}\right)e^{i\frac{\pi\alpha}{2}}. \end{aligned}$$

Also, using $(23.6.21)_2$, one can show that the quantity χ in $(23.6.20)_2$ is given by

$$\chi = k(1-\alpha)\cos\frac{\alpha\pi}{2} \geq 0.$$

Observe that

$$\dot{\psi}_m(t) = \{2\,i\,\omega_0\,A, 0\} = \left\{2\,i\,k\,\rho\,\omega_0^{\alpha+1}, 0\right\}.$$

The rate of dissipation given by (23.6.14)$_2$ is now considered. Let us put

$$\int_0^\infty r_1^{\eta-1}\dot{E}_+^t(-ir_1)dr_1 = K_0 e^{i\omega_0 t} + \overline{K}_0 e^{-i\omega_0 t}.$$

Then

$$D_m(t) = \kappa\left\{K_0^2, 2|K_0|^2\right\} = k\omega_0^{\alpha+1}\{D_1, D_2\}, \tag{23.6.22}$$

in the notation (23.6.19), where from (23.6.17) and (23.6.12),

$$K_0 = i\omega_0\int_0^\infty \frac{r^{\eta-1}}{r+i\omega_0}dr = (i\omega_0)^\eta\frac{\pi}{\sin\eta\pi}.$$

In the rightmost form of (23.6.22), D_1 and D_2 are dimensionless constants, which are now determined. The coefficient in this term emerges from dimensional analysis. Relation (23.6.22)$_1$ becomes

$$D_m(t) = \frac{\kappa\pi^2\omega_0^{2\eta}}{\sin^2\eta\pi}\left\{e^{i\pi\eta}, 2\right\} = k\sin\frac{\alpha\pi}{2}\omega_0^{\alpha+1}\left\{\exp\left(i\pi\frac{\alpha+1}{2}\right), 2\right\},$$

which gives

$$D_1 = \sin\frac{\alpha\pi}{2}\exp\left[i\frac{\pi(\alpha+1)}{2}\right], \qquad D_2 = 2\sin\frac{\alpha\pi}{2}.$$

From (23.6.15) and (23.6.18), we see that the rate of input of mechanical energy is given by

$$T(t)\dot{E}(t) = k\omega_0^{\alpha+1}\{W_1, W_2\},$$
$$W_1 = \exp\left[i\frac{\pi(\alpha+1)}{2}\right], \qquad W_2 = 2\sin\frac{\alpha\pi}{2}.$$

By virtue of (16.1.28), we must have

$$2i\rho + D_1 = W_1, \qquad D_2 = W_2,$$

which are easily confirmed. Relation (16.1.29) involves divergent quantities, namely $\mathcal{D}(t)$ and $W(t)$, for sinusoidal histories [15, 16].

23.6.2.2 Exponential History

Consider a history and current value $(E^t, E(t))$ given by

$$E(t) = E_1 e^{\gamma t}, \qquad E^t(s) = E(t-s), \quad s\in\mathbb{R}^+, \tag{23.6.23}$$

where E_1 is a constant amplitude. Some of the formulae for this history may be obtained from the sinusoidal case by simple substitutions [16]. However, we present direct derivations here. Instead of (23.6.16), we have

$$E_+^t(\omega) = \frac{E_1 e^{\gamma t}}{(\gamma + i\omega)} = \frac{E(t)}{(\gamma + i\omega)},$$
$$\dot{E}_+^t(\omega) = \frac{\gamma E(t)}{\gamma + i\omega}, \qquad \dot{E}_+^t(-ir) = \frac{\gamma E(t)}{\gamma + r}.$$

The stress function becomes

$$T(t) = kE(t)\gamma^{\alpha}, \tag{23.6.24}$$

on using (23.6.12). The minimum free energy $\psi_m(t)$, given by $(23.6.10)_1$, has the form

$$\psi_m(t) = E_1^2 e^{2\gamma t} A_\gamma = E^2(t) A_\gamma, \tag{23.6.25}$$

where A_γ is given by

$$\begin{aligned} A_\gamma &= \kappa\gamma^2 \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta - 1}}{(r_1 + \gamma)(r_2 + \gamma)(r_1 + r_2)} dr_1 dr_2, \\ &= \frac{\kappa\gamma^2\pi}{\sin\eta\pi} \int_0^\infty \frac{r_2^{\eta - 1}\left[\gamma^{\eta - 1} - r_2^{\eta - 1}\right]}{(r_2 + \gamma)(r_2 - \gamma)} dr_2, \\ &= \frac{1}{2} k\gamma^{\alpha} (1 - \sin\frac{\alpha\pi}{2}) \geq 0, \end{aligned}$$

with the aid of (23.6.21) and the integral ([168, page 289])

$$\int_0^\infty \frac{r^{\mu - 1}}{(r + \gamma)(r - \gamma)} dr = -\frac{\pi\gamma^{\mu - 2}}{2\sin\mu\pi}(1 + \cos\mu\pi), \qquad \gamma > 0, \quad 0 < Re\mu < 2.$$

This formula is in fact in a similar category to (23.6.21) but where the parameters are in a different range. We conclude that

$$\dot{\psi}_m(t) = k E_1^2 \gamma^{\alpha + 1} e^{2\gamma t} \left(1 - \sin\frac{\alpha\pi}{2}\right).$$

It follows from $(23.6.9)_3$, (23.6.12), and (23.6.14) that

$$D_m(t) = k E_1^2 \gamma^{\alpha + 1} e^{2\gamma t} \sin\frac{\alpha\pi}{2}. \tag{23.6.26}$$

Finally, from (23.6.23) and (23.6.24) we find that

$$T(t)\dot{E}(t) = k E_1^2 \gamma^{\alpha + 1} e^{2\gamma t}, \tag{23.6.27}$$

and (16.1.28) is obeyed. Indeed, we can also consider (16.1.29) in the case of exponential histories, since there are no convergence difficulties. We write $\psi_m(t)$, given by (23.6.25), and the integrated forms of (23.6.26) and (23.6.27), as

$$\psi_m(t) = F_m W(t), \qquad F_m = 1 - \sin\frac{\alpha\pi}{2}, \qquad W(t) = \frac{1}{2} k E_1^2 \gamma^\alpha e^{2\gamma t},$$
$$\mathcal{D}_m(t) = S_m W(t), \qquad S_m = \sin\frac{\alpha\pi}{2}.$$

Thus, (16.1.29) is also clearly obeyed.

23.6.2.3 The Physical Free Energy

The physical free energy, discussed in several different contexts in the present work is defined by the property that its associated rate of dissipation is the true rate for the material. We tentatively identify the physical free energy of a fractional derivative material, as the minimum free energy by virtue of the following argument.

Two functionals have been identified as being associated with such materials, the Graffi–Volterra and the minimum free energies. The Graffi–Volterra functional is a degenerate version of the full two variable quadratic form discussed in earlier chapters. As such, it cannot be included as a free energy with deeper physical meaning, though it is a simple functional with the correct positivity properties and therefore very useful as a mathematical tool.

Invoking the property P4 introduced in (18.2.1), we see that the work function cannot be a valid free energy. Also, it has degenerate features somewhat similar to the Graffi–Volterra functional. Therefore, the minimum free energy, given by the elementary explicit expression (23.6.10), is the only non-degenerate free energy functional associated with the material. It must therefore be identified as the physical free energy of this material. Furthermore, the physical rate of dissipation has the form (23.6.14).

23.7 Application to Viscoelastic Systems

We now consider the dynamical equations for viscoelastic solids and fluids within the framework of fractional derivative models. An energy theorem is proved in both of these cases.

23.7.1 Viscoelastic Solids

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain of a linear viscoelastic solid, whose constitutive equation is given by the fractional model with constitutive relation given by (23.3.7). The initial boundary value problem is defined by the differential system in the domain $Q = \Omega \times (0, T)$ by

$$\begin{aligned} \rho_0(\mathbf{x}) \frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} &= \nabla \cdot \mathbf{T}(\mathbf{x}, t) + \rho_0(\mathbf{x}) \mathbf{f}(\mathbf{x}, t) \\ &= -\frac{\alpha}{\Gamma(1-\alpha)} \nabla \cdot \left[\mathbb{C}(\mathbf{x}) \int_0^\infty \frac{\mathbf{E}_r^t(\mathbf{x}, s)}{s^{1+\alpha}} ds \right] \\ &\qquad + \rho_0(\mathbf{x}) \mathbf{f}(\mathbf{x}, t), \end{aligned} \tag{23.7.1}$$

where $\rho_0(\mathbf{x})$ denotes the mass density, $\mathbf{u}(\mathbf{x}, t)$ the displacement such that $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and $\mathbf{f}(\mathbf{x}, t)$ the body forces. The initial conditions are

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \,, \quad \left.\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right|_{t=0} = \mathbf{v}_0(\mathbf{x})$$

along with the boundary conditions

$$\mathbf{u}(\mathbf{x}, t)|_{\partial\Omega} = \mathbf{u}^0(\mathbf{x}), \tag{23.7.2}$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ are given functions.

Using the definition of fractional derivative given in (23.3.8), Eq. (23.7.1) can be rewritten in the form

$$\rho_0(\mathbf{x})\frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} = \nabla \cdot \left[{}^{C}_{-\infty}D^{\alpha}_{t}\mathbb{C}(\mathbf{x})\mathbf{E}(\mathbf{x}, t)\right] + \rho_0(\mathbf{x})\mathbf{f}(\mathbf{x}, t). \tag{23.7.3}$$

Now, our purpose is to obtain an energy theorem for the problem (23.7.1) and (23.7.2). To this end, we multiply (23.7.3) by the first time derivative of $\mathbf{u}(\mathbf{x}, t)$. Then, after an integration on $Q = \Omega \times (0, T)$, we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \rho_0(\mathbf{x})\frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} d\mathbf{x}\, dt \\ &= \int_0^T \int_\Omega \left\{(\nabla \cdot \left[{}^{C}_{-\infty}D^{\alpha}_{t}\mathbb{C}(\mathbf{x})\mathbf{E}(\mathbf{x}, t)\right] \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \rho_0(\mathbf{x})\mathbf{f}(\mathbf{x}, t) \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right\} d\mathbf{x}\, dt,\end{aligned} \tag{23.7.4}$$

where $d\mathbf{x}$ is the three-dimensional space volume element. Hence, using the divergence theorem and the boundary condition (23.7.2), it follows from (23.7.4) that

$$\begin{aligned}&\int_0^T \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega \rho_0(\mathbf{x})\left(\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right)^2 d\mathbf{x}\, dt \\ &= \int_0^T \int_\Omega \left[-\mathbb{C}(\mathbf{x})\left({}^{C}_{-\infty}D^{\alpha}_{t}\mathbf{E}(\mathbf{x}, t)\right) \cdot \frac{\partial \mathbf{E}(\mathbf{x}, t)}{\partial t} + \rho_0(\mathbf{x})\mathbf{f}(\mathbf{x}, t) \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right] d\mathbf{x}\, dt \\ &= \int_0^T \int_\Omega \left[-\mathbf{T}(\mathbf{x}, t) \cdot \dot{\mathbf{E}}(\mathbf{x}, t) + \rho_0(\mathbf{x})\mathbf{f}(\mathbf{x}, t) \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right] d\mathbf{x}\, dt.\end{aligned}$$

Then, from (23.4.6) or (23.4.9), we obtain

$$\begin{aligned}&\int_0^T \frac{\partial}{\partial t}\frac{1}{2}\int_\Omega \rho_0(\mathbf{x})\left[\left(\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right)^2 + \Psi^S(\mathbf{x}, t)\right] d\mathbf{x}\, dt \\ &\le \int_0^T \int_\Omega \rho_0(\mathbf{x})\mathbf{f}(\mathbf{x}, t) \cdot \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} d\mathbf{x}\, dt,\end{aligned}$$

where $\Psi^S(\mathbf{x}, t)$ is the Graffi–Volterra free energy functional for solids, given by (23.4.8). Finally, carrying out the time integration, we find that

$$\frac{1}{2}\int_{\Omega}\rho_0(\mathbf{x})\left[\left(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}\right)^2 + \Psi^S(\mathbf{x},t)\right]d\mathbf{x}$$
$$\leq \frac{1}{2}\int_{\Omega}\rho_0(\mathbf{x})\left(\mathbf{v}_0(\mathbf{x})^2 + \Psi^S(\mathbf{x},0)\right)dx + \int_0^T\int_{\Omega}\rho_0(\mathbf{x})\mathbf{f}(\mathbf{x},t)\cdot\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}d\mathbf{x}\,dt.$$

It is easily checked that the same inequalities hold for any free energy for the system. This in effect means any positive functional obeying (23.4.6) or (23.4.9). A similar observation applies to the case of fluids, which is now discussed.

23.7.2 Viscoelastic Fluids

The initial boundary value problem for a viscoelastic incompressible fluid described by the velocity $\mathbf{v}(\mathbf{x},t)$, the pressure $p(\mathbf{x},t)$ and the constant density ρ_0, is defined by the differential system

$$\begin{aligned}\rho_0\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} &= -\nabla p + \nabla\cdot\mathbf{T}_E(\mathbf{x},t) + \rho_0\mathbf{f}(\mathbf{x},t)\\ &= -\nabla p + \frac{2\eta}{\Gamma(1-\alpha)}\nabla\cdot\int_{-\infty}^{t}\frac{\mathbf{E}(\mathbf{x},t)-\mathbf{E}(\mathbf{x},\tau)}{(t-\tau)^{1+\alpha}}d\tau + \rho_0\mathbf{f}(\mathbf{x},t)\\ &= -\nabla p + \frac{\eta}{\Gamma(1-\alpha)}\nabla\cdot\int_{-\infty}^{t}\frac{\nabla\mathbf{v}(\mathbf{x},\tau)}{(t-\tau)^{\alpha}}d\tau + \rho_0\mathbf{f}(\mathbf{x},t),\\ &\nabla\cdot\mathbf{v}(\mathbf{x},t) = 0,\end{aligned} \tag{23.7.5}$$

with initial and boundary conditions

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \ , \qquad \mathbf{v}(\mathbf{x},t)|_{\partial\Omega} = \mathbf{0}. \tag{23.7.6}$$

We again seek an energy theorem. It follows from $(23.7.5)_3$ and (23.7.6), together with a standard step involving integration by parts, that

$$\begin{aligned}&\int_0^T\frac{d}{dt}\int_{\Omega}\rho_0\,\mathbf{v}^2(\mathbf{x},t)d\mathbf{x}dt\\ &= \int_0^T\int_{\Omega}\left[\nabla\mathbf{T}_E(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t) + \rho_0\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\right]\,d\mathbf{x}\,dt\\ &= \int_0^T\int_{\Omega}\left[\mathbf{T}_E(\mathbf{x},t)\cdot\dot{\mathbf{E}}(\mathbf{x},t) + \rho_0\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\right]\,d\mathbf{x}\,dt.\end{aligned}$$

Using the inequality (23.4.6), we obtain

$$\int_0^T\frac{d}{dt}\int_{\Omega}\left[\rho_0\,\mathbf{v}^2(\mathbf{x},t) + \Psi^F(\mathbf{x},t)\right]d\mathbf{x}dt \leq \int_0^T\int_{\Omega}\rho_0\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)d\mathbf{x}dt.$$

Hence, we have

$$\int_{\Omega}\left[\rho_0\,\mathbf{v}^2(\mathbf{x},t) + \Psi^F(\mathbf{x},t))\right]d\mathbf{x}$$
$$\leq \int_{\Omega}\left[\rho_0\,\mathbf{v}^2(\mathbf{x},0) + \Psi^F(\mathbf{x},0))\right]d\mathbf{x} + \int_0^T\int_{\Omega}\rho_0\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)d\mathbf{x}dt.$$

23.8 Application to Rigid Heat Conductors

In this section, we use the fractional method to describe the behavior of heat conductors. The literature on this topic, including [1, 141, 299], is motivated by experimental findings such as those reported in [223, 285].

The classical Fourier constitutive equation is given by

$$\mathbf{q}(t) = -k_c \mathbf{g}(t),$$

where $\mathbf{q}$ is the heat flux , $\mathbf{g}$ the temperature gradient and $k_c > 0$ is the thermal conductivity, which may be space dependent. In order to obtain a finite wave propagation speed, Cattaneo [59] proposed the modified constitutive relation

$$-\gamma\dot{\mathbf{q}}(t) = \mathbf{q}(t) + k_c \mathbf{g}(t), \quad \gamma > 0.$$

This is the Cattaneo–Maxwell or, for brevity the Cattaneo equation. It can be put in the form of an integral equation, describing a material with thermal memory

$$\begin{aligned} \mathbf{q}(t) &= -\int_{-\infty}^{t} k(t-s)\mathbf{g}(s)ds = -\int_{0}^{\infty} k(s)\mathbf{g}^t(s)ds, \\ \mathbf{g}^t(s) &= \mathbf{g}(t-s), \qquad k(s) = k_0 e^{-\lambda s}, \quad \lambda = \frac{1}{\gamma}, \quad k_c = \gamma k_0. \end{aligned} \tag{23.8.1}$$

Relation (23.8.1) can be expressed in terms of the NFD$_t$, given by (23.2.13) or (23.2.14), where

$$\sigma = \gamma, \qquad k_0 = \frac{1+\sigma}{\sigma} M^*(\sigma), \qquad \alpha = \frac{1}{1+\sigma}.$$

Thus, $(23.8.1)_1$ can be written as

$$\mathbf{q}(t) = -\mathcal{D}_t^{(\alpha)} \overline{\mathbf{g}}(t), \tag{23.8.2}$$

where

$$\overline{\mathbf{g}}(t) = \int_0^t \mathbf{g}(u)du, \qquad \dot{\overline{\mathbf{g}}}(t) = \mathbf{g}(t).$$

23.8.1 UFD$_t$ Fractional Cattaneo Equation

Let us replace the NFD$_t$ in (23.8.2) by the UFD$_t$ as defined by (23.2.12) or (23.2.6). Thus, we put

$$k(s) = \frac{k_0}{\Gamma(1-\alpha)} \frac{1}{s^\alpha},$$

giving

$$\begin{aligned} \mathbf{q}(t) = -k_0 D_t^{(\alpha)} \overline{\mathbf{g}}(t) &= -\frac{k_0}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{\dot{\overline{\mathbf{g}}}(\tau)}{(t-\tau)^\alpha} d\tau \\ &= \frac{\alpha k_0}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{\overline{\mathbf{g}}(\tau) - \overline{\mathbf{g}}(t)}{(t-\tau)^{1+\alpha}} d\tau = \frac{\alpha k_0}{\Gamma(1-\alpha)} \int_0^\infty \frac{\overline{\mathbf{g}}_r^t(s)}{s^{1+\alpha}} ds, \end{aligned}$$

where, recalling (9.1.2), we introduce $\overline{\mathbf{g}}_r^t(s)$, defined as

$$\overline{\mathbf{g}}_r^t(s) = \overline{\mathbf{g}}(t-s) - \overline{\mathbf{g}}(t) = -\int_{t-s}^{t} \mathbf{g}(u)du,$$

to emphasize an analogy between heat flow and viscoelasticity whereby $-\overline{\mathbf{g}}(t)$, $\mathbf{q}(t)$ correspond to $\mathbf{E}(t)$, $\mathbf{T}(t)$. It can be concluded from this and (23.4.8) that the functional

$$\psi(t) = \frac{\alpha k_0}{2\Gamma(1-\alpha)} \int_0^\infty \frac{\overline{\mathbf{g}}_r^t(s) \cdot \overline{\mathbf{g}}_r^t(s)}{s^{1+\alpha}} ds$$

is the Graffi-Volterra free energy for fractional Cattaneo models, with corresponding rate of dissipation deduced from (23.4.10) to be

$$D(t) = \frac{\alpha(1+\alpha)k_1}{2\Gamma(1-\alpha)} \int_0^\infty \frac{\overline{\mathbf{g}}_r^t(s) \cdot \overline{\mathbf{g}}_r^t(s)}{s^{2+\alpha}} ds.$$

Indeed, the minimum free energy for isotropic materials of this kind can be written down immediately from (23.6.9) and (23.6.10), replacing $\dot{E}^t(u)\dot{E}^t(v)$ by $\mathbf{g}^t(u) \cdot \mathbf{g}^t(v)$. Thus, we obtain

$$\psi_m(t) = \kappa \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{(r_1 r_2)^{\eta - 1} \mathbf{g}^t(u) \cdot \mathbf{g}^t(v) e^{-r_1 u - r_2 v}}{r_1 + r_2} dr_1 dr_2 du dv,$$

$$\kappa = \frac{k_0}{\pi^2} \sin\frac{\alpha\pi}{2} \cos^2\frac{\alpha\pi}{2}.$$

The quantity η is defined by (23.6.1). The coefficient k_1 replaces $\mathbb{C}$ in Sect. 23.3 or k in Sects. 23.5 and 23.6. Similarly, the corresponding rate of dissipation is deduced from $(23.6.14)_3$ by replacing $\dot{E}^t(u)$ with $-\mathbf{g}^t(u)$, yielding

$$D_m(t) = \kappa\Gamma^2(\eta) \left| \int_0^\infty \frac{\mathbf{g}^t(u)}{u^\eta} du \right|^2 \geq 0.$$

23.8.2 The NFD$_t$ Model

Analogous results can be derived for the NFD$_t$ model given by (23.8.2), which is of course the Cattaneo equation. Free energies for this relationship are those for a simple memory function $k(s)$ described in (23.8.1). This has one decaying exponential and goes to zero at large times. The Graffi-Volterra free energy corresponding to this form is given by

$$\psi(t) = \frac{1}{2}\lambda k_0 \int_0^\infty e^{-\lambda s} \overline{\mathbf{g}}_r^t(s) \cdot \overline{\mathbf{g}}_r^t(s) ds,$$

while the corresponding rate of dissipation is

$$D(t) = \lambda^2 k_0 \int_0^\infty e^{-\lambda s} \overline{\mathbf{g}}_r^t(s) \cdot \overline{\mathbf{g}}_r^t(s) ds.$$

Also, for a relaxation function consisting of one decaying exponential, the minimum free energy has the form

$$\psi_D(t) = \frac{1}{2}\lambda^2 k_0 \left| \int_0^\infty e^{-\lambda s} \overline{\mathbf{g}}_r^t(s) ds \right|^2,$$

$$D_D(t) = \lambda^3 k_0 \left| \int_0^\infty e^{-\lambda s} \overline{\mathbf{g}}_r^t(s) ds \right|^2.$$

These are a special case of the Day free energy and rate of dissipation where the relaxation function goes to zero at large times.

23.9 Application to Electromagnetic Systems

We now explore some properties of two electromagnetic bodies, which are characterized by constitutive equations expressed as fractional models. The form of a particular free energy will be derived in both cases. These are examples or generalizations of the Graffi–Volterra free energy discussed in Sect. 23.4.1. This section is based on [19].

23.9.1 Visco-Ferromagnetic Materials

Let us consider a visco-ferromagnetic material characterized by the following constitutive equation

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mathbf{C}(\mathbf{x})}{\Gamma(1-\alpha)} \int_a^t \frac{\mathbf{H}_\tau(\mathbf{x}, \tau)}{(t-\tau)^\alpha} d\tau, \quad \mathbf{H}_\tau(\mathbf{x}, \tau) = \frac{\partial}{\partial \tau}\mathbf{H}(\mathbf{x}, \tau), \tag{23.9.1}$$

where the magnetic induction $\mathbf{B}(\mathbf{x}, t)$ and the magnetic field are $\mathbf{H}(\mathbf{x}, t)$ are defined for any point $\mathbf{x} \in \Omega$, the smooth bounded domain occupied by the material; moreover, the quantity $\mathbf{C}(\mathbf{x})$ is a second-order positive tensor, defined for any point $\mathbf{x} \in \Omega$.

This equation is expressed in terms of the $\alpha-$Caputo fractional derivative (23.2.5); see also [47]

$${}_C D_t^\alpha \mathbf{H}(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\mathbf{H}_\tau(\mathbf{x}, \tau)}{(t-\tau)^\alpha} d\tau.$$

Using this definition, Eq. (23.9.1) assumes the following form

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{C}(\mathbf{x})\, {}_C D_t^\alpha \mathbf{H}(\mathbf{x}, t).$$

Taking $a = -\infty$ and carrying a time integration by parts, (23.9.1) assumes the more useful form

$$\mathbf{B}(\mathbf{x}, t) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \mathbf{C}(\mathbf{x}) \frac{\mathbf{H}^{(r)}(\mathbf{x}, \tau)}{(t-\tau)^{\alpha+1}} d\tau, \mathbf{H}^{(r)}(\mathbf{x}, \tau) = \mathbf{H}(\mathbf{x}, \tau) - \mathbf{H}(\mathbf{x}, t),$$

where $\mathbf{H}^{(r)}$ denotes the relative history of the magnetic field.

The dissipation law states that, for any simple electromagnetic material, there exists at least one state functional, denoted by $\psi(\mathbf{x}, t$, and referred to as a free energy, which satisfies the following fundamental requirement:

$$\frac{\partial}{\partial t}\psi(\mathbf{x}, t) \leq \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{H}_t(\mathbf{x}, t). \tag{23.9.2}$$

By introducing the rate of dissipation $\mathcal{D}(\mathbf{x}, t) \geq 0$, the dissipation law can be written as

$$\frac{\partial}{\partial t}\psi(\mathbf{x}, t) + \mathcal{D}(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{H}_t(\mathbf{x}, t). \tag{23.9.3}$$

Taking account of (23.9.1), we can apply an integration by parts to the scalar product on the right-hand side of both Eqs. (23.9.2) and (23.9.3); the expression so derived allows us to identify the following functional as a free energy:

$$\psi(t) = \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^{\infty} \mathbf{C}(\mathbf{x}) \frac{\mathbf{H}^{(r)}(\mathbf{x}, t-s) \cdot \mathbf{H}^{(r)}(\mathbf{x}, t-s)}{s^{\alpha+1}} ds.$$

This is a particular example of the *Graffi-Volterra* free energy. The related rate of dissipation is given by

$$\mathcal{D}(\mathbf{x}, t) = \frac{\alpha(\alpha+1)}{2\Gamma(1-\alpha)} \int_0^{\infty} \mathbf{C}(\mathbf{x}) \frac{\mathbf{H}^{(r)}(\mathbf{x}, t-s) \cdot \mathbf{H}^{(r)}(\mathbf{x}, t-s)}{s^{\alpha+2}} ds.$$

23.9.2 Nonlocal Visco-Ferromagnetic Materials

The behavior of nonlocal visco-ferromagnetic materials can be described by means of a constitutive equation expressed in terms of the fractional operator $\mathcal{M}_{\mathbf{x}}^{\beta}$ of order $\beta \in (\frac{1}{2}, 1)$, acting on a function $\mathbf{f}(\mathbf{x})$, introduced in [56] and expressed by

$$\mathcal{M}_{\mathbf{x}}^{\beta}\mathbf{F}(\mathbf{x}) = \frac{\beta\pi^{-\frac{\beta}{2}}}{1-\beta} \int_{\Omega} \mathbf{f}(\mathbf{y}) e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2} d\mathbf{y},$$

where $\mathbf{x}, \mathbf{y} \in \Omega$.

By replacing $\mathbf{f}(\mathbf{x})$ with $\mathbf{C}(\mathbf{x}, \mathbf{y})\mathbf{H}(\mathbf{x}, \tau)$, we can introduce the following new constitutive equation for the ferromagnetic induction

$$\begin{aligned} \mathbf{B}(\mathbf{x}, t) &= \mathcal{M}_{\mathbf{x}}^{\beta} D_t^{\alpha} \left[\mathbf{C}(\mathbf{x}, \mathbf{y})\mathbf{H}(\mathbf{x}, t)\right] \\ &= \frac{\beta\pi^{-\frac{\beta}{2}}}{(1-\beta)\Gamma(1-\alpha)} \int_{\Omega} \int_a^t \frac{1}{(t-\tau)^{\alpha}} \mathbf{C}(\mathbf{x}, \mathbf{y}) \mathbf{H}_{\tau}(\mathbf{y}, \tau) e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2} d\tau d\mathbf{y}, \end{aligned} \tag{23.9.4}$$

where the scalar α, assumed to be in the interval $\left(0, \frac{1}{2}\right)$, is the degree of the Caputo fractional derivative, denoted by D_t^{α}, while the second-order tensor $\mathbf{C}(\mathbf{x}, \mathbf{y})$ is taken to be symmetric in $\mathbf{x}$ and $\mathbf{y}$ and positively defined.

We note that, for nonsimple materials characterized by (23.9.4), the magnetic induction $\mathbf{B}(\mathbf{x}, t)$ at any fixed point $\mathbf{x} \in \Omega$ depends on the values of the magnetic field $\mathbf{H}(\mathbf{y}, t)\ \forall \mathbf{y} \in \Omega$.

The total internal power can be obtained by integrating over Ω the scalar product in the right-hand side of both Eqs. (23.9.2) and (23.9.3); thus, by virtue of (23.9.4), it is given by

$$\int_\Omega \mathbf{B}(\mathbf{x},t)\cdot\mathbf{H}_t(\mathbf{x},t)d\mathbf{x} = \int_\Omega \mathcal{M}_{\mathbf{x}}^\beta D_t^\alpha \left[\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{H}(\mathbf{x},t)\right]\cdot\mathbf{H}_t(\mathbf{x},t)d\mathbf{x} \tag{23.9.5}$$

$$= \frac{\beta\pi^{-\frac{\beta}{2}}}{(1-\beta)\Gamma(1-\alpha)}\int_\Omega\left[\int_\Omega\int_a^t \mathbf{C}(\mathbf{x},\mathbf{y})\frac{\mathbf{H}_\tau(\mathbf{y},\tau)}{(t-\tau)^\alpha}e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2}d\tau d\mathbf{y}\right]\cdot\mathbf{H}_t(\mathbf{x},t)d\mathbf{x}$$

and the global free energy has the form:

$$\psi(\Omega,t)\equiv\int_\Omega\psi(\mathbf{x},t)d\mathbf{x}. \tag{23.9.6}$$

If we assume $\mathbf{H}(\cdot,\tau)=\mathbf{0}$ for $\tau\leq a$, the inequality (23.9.2), by using (23.9.5) and (23.9.6), becomes

$$\dot\psi(\Omega,t)\equiv\frac{\partial}{\partial t}\int_\Omega\psi(\mathbf{x},t)d\mathbf{x}\leq\int_\Omega\mathbf{B}(\mathbf{x},t)\cdot\mathbf{H}_t(\mathbf{x},t)d\mathbf{x}$$
$$=\frac{\beta\pi^{-\frac{\beta}{2}}}{(1-\beta)\Gamma(1-\alpha)}\int_\Omega\int_\Omega\int_{-\infty}^t\mathbf{C}(\mathbf{x},\mathbf{y})\frac{[\mathbf{H}(\mathbf{y},\tau)-\mathbf{H}(\mathbf{y},t)]_\tau}{(t-\tau)^\alpha}e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2}d\tau d\mathbf{y}$$
$$\cdot\frac{\partial}{\partial t}\left[\mathbf{H}(\mathbf{x},t)-\mathbf{H}(\mathbf{x},\tau)\right]d\mathbf{x},$$

whence, with a time integration by parts, it follows that

$$\frac{\partial}{\partial t}\int_\Omega\psi(\mathbf{x},t)d\mathbf{x}\leq\int_\Omega\mathbf{B}(\mathbf{x},t)\cdot\mathbf{H}_t(\mathbf{x},t)d\mathbf{x}$$
$$=\frac{\alpha\beta\pi^{-\frac{\beta}{2}}}{(1-\beta)\Gamma(1-\alpha)}\int_\Omega\int_\Omega\int_{-\infty}^t\mathbf{C}(\mathbf{x},\mathbf{y})\frac{\mathbf{H}(\mathbf{y},t)-\mathbf{H}(\mathbf{y},\tau)}{(t-\tau)^{\alpha+1}}$$
$$\cdot\frac{\partial}{\partial t}\left[\mathbf{H}(\mathbf{x},t)-\mathbf{H}(\mathbf{x},\tau)\right]e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2}d\tau d\mathbf{y}d\mathbf{x}.$$

This inequality is satisfied by the following functionals:

$$\psi(\Omega,t)=\frac{\alpha\beta\pi^{-\frac{\beta}{2}}}{2(1-\beta)\Gamma(1-\alpha)}\int_\Omega\int_\Omega\int_{-\infty}^t\frac{1}{(t-\tau)^{\alpha+1}}\left[\mathbf{H}(\mathbf{y},t)-\mathbf{H}(\mathbf{y},\tau)\right]$$
$$\cdot\mathbf{C}(\mathbf{x},\mathbf{y})\left[\mathbf{H}(\mathbf{x},t)-\mathbf{H}(\mathbf{x},\tau)\right]e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2}d\tau d\mathbf{y}d\mathbf{x},$$

which gives the global free energy $\psi(\Omega,t)$, and

$$\mathcal{D}(\Omega,t)=\frac{\alpha(\alpha+1)\beta\pi^{-\frac{\beta}{2}}}{2(1-\beta)\Gamma(1-\alpha)}\int_\Omega\int_\Omega\int_{-\infty}^t\frac{1}{(t-\tau)^{\alpha+2}}\left[\mathbf{H}(\mathbf{y},t)-\mathbf{H}(\mathbf{y},\tau)\right]$$
$$\cdot\mathbf{C}(\mathbf{x},\mathbf{y})\left[\mathbf{H}(\mathbf{x},t)-\mathbf{H}(\mathbf{x},\tau)\right]e^{-\frac{\beta^2}{1-\beta^2}(\mathbf{x}-\mathbf{y})^2}d\tau d\mathbf{y}d\mathbf{x},$$

which is the expression for the related global rate of dissipation $\mathcal{D}(\Omega,t)$.