11

The Minimum Free Energy

Breuer and Onat [42] considered the following question: what is the maximum amount of work recoverable from a body that has undergone a specified strain history? They found that the answer for linear viscoelastic memory materials is provided by the solution of an integral equation of Wiener–Hopf type, which is in fact a special case of the result given in Sect. 5.2. They gave a detailed solution by elementary means for a material with relaxation function in the form of a finite sum of decaying exponentials. The nonuniqueness problem was also explicitly exposed by these authors [43].

Day [84] presented an alternative formulation of the thermodynamics of materials with memory. In [85], he revisited the problem considered in [42] within a more rigorous framework, introducing the concept of a (time) reversible extension and discussing the maximum recoverable work in terms of this concept. An expression for the minimum free energy of a standard linear solid (linear viscoelastic solid with a relaxation function that has only one decaying exponential) was given in [87].

A general expression for the minimum free energy of a linear viscoelastic solid under isothermal conditions was given in [158]. This was for a scalar constitutive relation. A generalization to the full tensor case was presented in [92]. These results were used in the context of the viscoelastic Saint-Venant problem in [93]. Detailed explicit expressions for the minimum free energy and related quantities were given in [92, 158] for discrete-spectrum materials, namely those for which the relaxation function is a sum of exponentials. The minimum free energies of compressible and incompressible viscoelastic fluids were determined in [5, 8, 100], while materials with finite memory were considered in [111]. The maximum recoverable work or equivalently the minimum free energy for rigid heat conductors was considered in [6, 21, 22].

We now derive a general expression for the minimum free energy and the associated rate of dissipation for a material described by the linear memory model. The results discussed above are special cases of this, with the exception of the approximate treatment of rigid heat conductors based on (9.1.9), which corresponds to the results of Sect. 8.6.1, as noted in Sect. 9.3.3.

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11.1 Factorization of Positive Definite Tensors

It will be required to factorize the quantity $\mathbb{H}$, given by (7.2.22), in order to determine an expression for the minimum and other free energies. This was first discussed in [158] for the scalar case, where an appropriate factorization of the one-dimensional counterpart of $\mathbb{H}(\omega)$ was introduced. Such a particular factorization does not apply to fourth-order tensors, so that the extension of the result of [158] to the general case, which was given in [92], is not trivial.

In this section, we show that $\mathbb{H}$ can always be factorized. Use will be made of a result by Gohberg and Kreĭn [156] for tensor-valued functions. Given a nonsingular continuous tensor-valued function $\mathbb{K}(\omega) \in \mathrm{Lin}(\Gamma(\Omega))$, $\omega \in \mathbb{R}$ ($\mathbb{K}$ not connected with the quantity in (7.1.1)), we say that $\mathbb{K}$ has a left [right] factorization if it can be represented in the form

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}_{(-)}(\omega), \qquad [\mathbb{K}(\omega) = \mathbb{K}_{(-)}(\omega)\mathbb{K}_{(+)}(\omega)], \tag{11.1.1}$$

where the tensor functions $\mathbb{K}_{(\pm)}$ admit analytic continuations, analytic in the interior and continuous up to the boundary of the corresponding complex half-planes $\Omega^{\pm}$, and are such that

$$\det \mathbb{K}_{(\pm)}(\zeta) \neq \mathbf{0}, \qquad \zeta \in \Omega^{\pm}.$$

We say that $\mathbb{K}$ belongs to $\mathcal{F}_{m\times m}$, $\mathcal{F}^{+}_{m\times m}$, and $\mathcal{F}^{-}_{m\times m}$, respectively, if there exists a constant tensor $\mathbb{C}_0$ and a tensor function $\mathbb{F}(t)$ such that

$$\begin{gathered}\mathbb{K}(\omega) = \mathbb{C}_0 + \int_{-\infty}^{\infty} \mathbb{F}(t)e^{-i\omega t}\, dt, \\ \mathbb{K}(\omega) = \mathbb{C}_0 + \int_{0}^{\infty} \mathbb{F}(t)e^{-i\omega t}\, dt, \quad \mathbb{K}(\omega) = \mathbb{C}_0 + \int_{-\infty}^{0} \mathbb{F}(t)e^{-i\omega t}\, dt.\end{gathered} \tag{11.1.2}$$

Note that if $\mathbb{K} \in \mathcal{F}^{\pm}_{m\times m}$, it has the analytic properties ascribed to $\mathbb{K}_{(\pm)}$ above. The main result we use is Theorem 8.2 of [156], which can be stated in our context as follows:

Theorem 11.1.1. *(Gohberg–Kreĭn) In order that the nonsingular Hermitian tensor function $\mathbb{K} \in \mathcal{F}_{m\times m}$ possesses a representation of the form*

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}^{*}_{(+)}(\omega), \tag{11.1.3}$$

in which the tensor function $\mathbb{K}_{(+)}$ is in $\mathcal{F}^{+}_{m\times m}$ and satisfies $\det \mathbb{K}_{(+)}(\zeta) \neq 0$ for $\zeta \in \Omega^{+}$, it is necessary and sufficient that $\mathbb{K}(\omega)$ be positive definite for every $\omega \in \mathbb{R}$.

Observe that comparison of (11.1.3) with $(11.1.1)_1$ yields

$$\mathbb{K}_{(-)}(\omega) = \mathbb{K}^{*}_{(+)}(\omega).$$

It follows from the assumption (7.2.20) that

$$\lim_{\omega\to 0} \frac{\mathbb{H}(\omega)}{\omega^2} = \mathbb{H}_0, \tag{11.1.4}$$

where $\mathbb{H}_0$ is symmetric and positive definite. Consider now the tensor

$$\mathbb{K}(\omega) := \frac{\omega_0^2 + \omega^2}{\omega^2}\mathbb{H}(\omega), \tag{11.1.5}$$

where $\omega_0 \neq 0$ is some given frequency. The tensor $\mathbb{K}(\omega)$ is symmetric, real (therefore Hermitian), and positive definite $\forall \omega \in \mathbb{R}$; moreover, it is such that

$$\lim_{\omega\to 0} \mathbb{K}(\omega) = \omega_0^2\mathbb{H}_0, \qquad \lim_{\omega\to\infty} \mathbb{K}(\omega) = \mathbb{H}_\infty.$$

In order to apply Theorem 11.1.1, we have to show that $\mathbb{K} \in \mathcal{F}_{m\times m}$, i.e., that the representation (11.1.2) applies.

Proposition 11.1.2. *If* $\mathbb{L}$ *and* $\mathbb{L}''$ *are tensor functions, integrable on* $[0, \infty)$*, and* $\mathbb{L}'_0$ *is finite, the tensor-valued function* $\mathbb{K}$*, related to* $\mathbb{L}$ *through (7.2.22) and (11.1.5), belongs to* $\mathcal{F}_{m\times m}$*.*

Proof. Observe that from (7.2.22),

$$\mathbb{K}(\omega) = -\left[\omega\mathbb{L}'_s(\omega) + \frac{\omega_0^2}{\omega}\mathbb{L}'_s(\omega)\right]. \tag{11.1.6}$$

Integration by parts of the integral in (7.2.1) and of a corresponding integral for $\mathbb{L}''_+(\omega)$ in terms of $\mathbb{L}''(s)$ yields

$$-\frac{1}{\omega}\mathbb{L}'_s(\omega) = \mathbb{L}_c(\omega), \qquad \omega\mathbb{L}'_s(\omega) = \mathbb{L}'(0) + \mathbb{L}''_c(\omega),$$

so that (11.1.6) becomes

$$\mathbb{K}(\omega) = \left[-\mathbb{L}'(0) - \mathbb{L}''_c(\omega) + \omega_0^2\mathbb{L}_c(\omega)\right]. \tag{11.1.7}$$

Consider now the tensors

$$\mathbb{C}_0 = -\mathbb{L}'(0), \quad \mathbb{F}(t) = \frac{1}{2}\left[-\mathbb{L}''(t) + \omega_0^2\mathbb{L}(t)\right], \quad t \in \mathbb{R}, \tag{11.1.8}$$

where $\mathbb{L}$ and $\mathbb{L}''$ are extended on the real line as even functions, so that from (C.1.5), $\mathbb{L}_F = 2\mathbb{L}_c$ and $\mathbb{L}''_F = 2\mathbb{L}''_c$. Then, in view of (11.1.8), (11.1.7) is equivalent to $(11.1.2)_1$ and the assertion is proved. □

Since $\mathbb{K}(\omega)$ is Hermitian and positive definite for every $\omega \in \mathbb{R}$, it satisfies Theorem 11.1.1. In particular, it has a representation of the form (left factorization)

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}^*_{(+)}(\omega), \tag{11.1.9}$$

with $\mathbb{K}_{(+)}(\omega) \in \mathcal{F}^+_{m\times m}$ and

$$\det \mathbb{K}_{(+)}(\zeta) \neq 0 \qquad \text{for} \quad \zeta \in \Omega^+.$$

Moreover, such a factorization is unique up to a multiplication on the right of $\mathbb{K}_{(+)}$ by a constant unitary tensor.

Similarly, $\mathbb{K}$ has a right factorization of the type

$$\mathbb{K}(\omega) = \mathbb{K}_{(-)}(\omega)\mathbb{K}^*_{(-)}(\omega) \tag{11.1.10}$$

with corresponding properties. In fact, since $\mathbb{K}(\omega)$ is an even function of ω, we can replace ω by $-\omega$ on the right of (11.1.9). Now, $\mathbb{K}_{(+)}(-\omega) \in \mathcal{F}^-_{m\times m}$ with nonzero determinant in $\Omega^{(-)}$, so that $\mathbb{K}_{(-)}(\omega) = \mathbb{K}_{(+)}(-\omega)$.

By virtue of (11.1.5) and (11.1.10), $\mathbb{H}(\omega)$ can be factorized as follows:

$$\mathbb{H}(\omega) = \mathbb{H}_+(\omega)\mathbb{H}_-(\omega), \tag{11.1.11}$$

where

$$\mathbb{H}_+(\omega) = \frac{\omega}{\omega - i\omega_0}\mathbb{K}_{(-)}(\omega), \qquad \mathbb{H}_-(\omega) = \frac{\omega}{\omega + i\omega_0}\mathbb{K}^*_{(-)}(\omega). \tag{11.1.12}$$

Alternatively, the left factorization (11.1.9) may be used, though the right factorization is more convenient in the present context. Representation (11.1.12) gives that

$$\mathbb{H}_\pm(\omega) = \mathbb{H}^*_\mp(\omega). \tag{11.1.13}$$

We have introduced in the present work an assumption that is stronger than those required in Theorem 11.1.1, namely that $\mathbb{H}$ is analytic in an open set including the real axis $\mathbb{R}$. Since $\mathbb{H}_+$ has the singularities of $\mathbb{H}$ in $\Omega^{(+)}$, then $\mathbb{H}_\pm$ will share this property.

In general, $\mathbb{H}_+$ and $\mathbb{H}_-$ do not commute, and various general results can be proved without assuming that they do. However, in order to derive explicit forms for various free energies, we must make an assumption that implies commutativity or, put another way, that $\mathbb{H}_\pm$ are normal tensors. This arises out of an assumption made in Sect. 11.6.

The notation for $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ follows the convention used in [158], i.e., the sign indicates the half-plane in which the singularities of the tensor lie. These factors also have the property that any zeros in their determinant occur also in the indicated half-plane. This latter property will not apply when factorizations leading to free energies other than the minimum are discussed. We adopt, however, the convention that $f_\pm$ has all its singularities in $\Omega^\pm$, respectively. This is in particular the convention adopted in Appendix C for Fourier-transformed quantities.

Recalling (11.1.4), we see that $\mathbb{H}_\pm$ each vanish linearly at the origin.

Note that we require (7.1.18) to ensure that $\mathbb{H}$, and therefore $\mathbb{K}$, is symmetric (i.e., Hermitian for real tensors).

The quantity $\mathbb{H}_\infty$, defined by (7.2.24), is given by

$$\mathbb{H}_\infty = \mathbb{H}_+(\infty)\mathbb{H}_-(\infty) = \mathbb{H}_{+\infty}\mathbb{H}_{-\infty}. \tag{11.1.14}$$

If $\mathbb{H}_{\pm\infty}$ can be chosen to be Hermitian, which is possible at least in the commutative case considered in Sect. 11.6, then they are both equal to the square root of the nonnegative tensor $\mathbb{H}_\infty$.

11.1.1 The Scalar Case

We can derive explicit forms for the factors if H is a scalar function. It is real and nonnegative on $\mathbb{R}$, vanishing quadratically at the origin. It is an even function of ω and therefore a function of ω^2, in view of its analyticity about the origin. Its singularities are as ascribed to $\mathbb{H}$ (and $\mathbb{L}_s'$ before (7.2.22)).

We define the function

$$K(\omega) = \log[H(\omega)T(\omega)], \qquad T(\omega) = \frac{\omega^2 + \omega_0^2}{H_\infty \omega^2},$$

where ω_0 may be chosen arbitrarily on $\mathbb{R}$. Then, K is a well-defined analytic function on $\mathbb{R}$, vanishing like ω^{-2} for large values of ω. Consider the quantity

$$M(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K(\omega)}{\omega - z} d\omega, \quad z \in \Omega\backslash R,$$

which goes to zero like z^{-1} at large z (see (B.2.13)). For $z \in \Omega^{(+)}$, $M(z) = M_-(z)$ is a function analytic in $\Omega^{(+)}$, while for $z \in \Omega^{(-)}$, $M(z) = M_+(z)$, which is analytic in $\Omega^{(-)}$ (Sect. B.2.1). The Plemelj formulas (B.2.14) take the form

$$M_-(\omega) = \frac{1}{2}K(\omega) + \frac{1}{2\pi i}P\int_{-\infty}^{\infty} \frac{K(\omega')}{\omega' - \omega} d\omega',$$
$$M_+(\omega) = -\frac{1}{2}K(\omega) + \frac{1}{2\pi i}P\int_{-\infty}^{\infty} \frac{K(\omega')}{\omega' - \omega} d\omega',$$

giving that

$$M_-(\omega) - M_+(\omega) = K(\omega), \quad \omega \in R,$$

where $M_+(\omega)\,(M_-(\omega))$ are the limiting values of $M_+(z)\,(M_-(z))$ as z approaches the real axis from below (above). Then, if we put

$$\begin{aligned} H_+(\omega) &= \frac{\omega h_\infty}{\omega - i\omega_0} e^{-M_+(\omega)}, \\ H_-(\omega) &= \frac{\omega h_\infty}{\omega + i\omega_0} e^{+M_-(\omega)}, \qquad h_\infty = H_\infty^{1/2}, \end{aligned} \tag{11.1.15}$$

it follows that $H_+(z)$ is analytic and free of zeros in $\Omega^{(-)}$; similarly for $H_-(z)$ in $\Omega^{(+)}$. Also,

$$H_+(\omega)H_-(\omega) = H(\omega). \tag{11.1.16}$$

Noting that $\overline{M_\pm}(\omega) = M_\pm(-\omega) = -M_\mp(\omega)$, we see that

$$\begin{aligned} H_\pm(\omega) &= H_\mp(-\omega) = \overline{H_\mp}(\omega), \\ H(\omega) &= |H_\pm(\omega)|^2, \quad \omega \in \mathbb{R}. \end{aligned} \tag{11.1.17}$$

According to the general result on the uniqueness of the factorization, noted above, a scalar factorization should be unique up to multiplication by a phase factor $e^{i\alpha}$, where α is a constant. Relation $(11.1.17)_2$ reduces this arbitrariness to multiplication by a factor ± 1.

It follows from (11.1.15) that

$$H_{\pm}(\infty) = h_{\infty} = H_{\infty}^{1/2}, \tag{11.1.18}$$

so that $H_{\pm}(\infty)$ are real and equal.

11.2 Derivation of the Form of the Minimum Free Energy

We shall be seeking to find the continuation $\mathbf{\Lambda}^t(u)$, $u \in \mathbb{R}^{--}$, that maximizes the recoverable work (Theorem 4.2.3). For this purpose, our attention will be confined to the family of continuations that vanish at large u, since it can be shown [92, 145] that the maximum recoverable work obtained by searching in this set is equal to that obtained by a wider search in the general set of bounded recoverable works.

The set of continuations for which

$$W(\infty) = \int_{-\infty}^{\infty} \mathbf{\Sigma}(u) \cdot \dot{\mathbf{\Lambda}}(u) du \tag{11.2.1}$$

exists and $\mathbf{\Lambda}(\infty)$ vanishes, we label $\mathcal{C}_0$. Using (7.5.5), we can write (11.2.1) as

$$\begin{aligned} W(\infty) &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{L}_{12}(|s-u|)\mathbf{\Lambda}(u) \cdot \mathbf{\Lambda}(s) du ds \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{L}_{12}(|s-u|)\mathbf{\Lambda}^t(u) \cdot \mathbf{\Lambda}^t(s) du ds, \end{aligned} \tag{11.2.2}$$

where the latter form is obtained by changes of integration variables. The superscript t is now an arbitrary parameter, which we interpret again as the current time. Applying the convolution theorem and Parseval's formula to (7.5.5) for $t = \infty$, we obtain, as in Sect. 7.5,

$$\begin{aligned} W(\infty) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{\Lambda}_F^t}(\omega) \cdot \mathbb{H}(\omega) \mathbf{\Lambda}_F^t(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{\Lambda}_{rF}^t}(\omega) \cdot \mathbb{H}(\omega) \mathbf{\Lambda}_{rF}^t(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\overline{\mathbf{\Lambda}_{r+}^t}(\omega) + \overline{\mathbf{\Lambda}_{r-}^t}(\omega)\right] \cdot \mathbb{H}(\omega) \left[\mathbf{\Lambda}_{r+}^t(\omega) + \mathbf{\Lambda}_{r-}^t(\omega)\right] d\omega \end{aligned} \tag{11.2.3}$$

by virtue of (C.1.4). We have used the notation of Sect. 7.2.3. Relations $(11.2.3)_1$ and $(11.2.3)_2$ are equal by virtue of (7.2.31), (C.2.19), and the fact that $\mathbb{H}(\omega)$ vanishes for $\omega = 0$. For continuations in $\mathcal{C}_0$, the recoverable work from the state at time t (see (5.2.3)) is given by

$$W_R(t) = -\int_t^{\infty} \mathbf{\Sigma}(u) \cdot \dot{\mathbf{\Lambda}}(u) du = W(t) - W(\infty). \tag{11.2.4}$$

To obtain the minimum free energy, we seek to maximize $W_R(t)$. Since $W(t)$ is a given quantity, this is equivalent to minimizing $W(\infty)$.

We now give three derivations of the form of the minimum free energy. The first uses a variational technique developed in [92, 158]. Also, a quite different and simplified version of this approach is presented. The third approach is based on the solution of (5.2.8) in the linear memory case, where it reduces to a linear Wiener–Hopf equation.

11.2.1 A Variational Approach

Let $\mathbf{\Lambda}_o^t$ be the optimal future continuation (so that $\dot{\mathbf{\Lambda}}_o$ is the optimal process) and

$$\mathbf{\Lambda}_{ro}^t(s) = \mathbf{\Lambda}_o^t(s) - \mathbf{\Lambda}(t), \quad s \in \mathbb{R}^-. \tag{11.2.5}$$

Let $\mathbf{\Lambda}_m^t$ denote the Fourier transform of $\mathbf{\Lambda}_{ro}^t$, so that $\mathbf{\Lambda}_m^t(\omega) = \mathbf{\Lambda}_{ro-}^t(\omega)$. Put

$$\mathbf{\Lambda}_{r-}^t(\omega) = \mathbf{\Lambda}_m^t(\omega) + \mathbf{k}_-(\omega), \tag{11.2.6}$$

where $\mathbf{k}_-(\omega)$ is arbitrary apart from the fact that it must have the same analytic properties as $\mathbf{\Lambda}_{r-}^t(\omega)$, i.e., $\mathbf{k}_-(z)$ must be analytic in Ω^+, and vanish like z^{-1} at large z. Then, varying $\mathbf{k}_-$, we find that $W(\infty)$ will be minimized by $\mathbf{\Lambda}_m^t$ if

$$\int_{-\infty}^{\infty} Re\left\{\mathbb{H}(\omega)\left[\mathbf{\Lambda}_{r+}^t(\omega) + \mathbf{\Lambda}_m^t(\omega)\right] \cdot \overline{\mathbf{k}_-}(\omega)\right\} d\omega = 0.$$

The restriction to the real part of the integral may be removed, since the imaginary part vanishes by virtue of the symmetric range of integration and (C.1.7). Using the factorization (11.1.11), we can rewrite this condition in the form

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathbb{H}(\omega)\left[\mathbf{\Lambda}_{r+}^t(\omega) + \mathbf{\Lambda}_m^t(\omega)\right] \cdot \overline{\mathbf{k}_-}(\omega) d\omega \\ &\quad = \int_{-\infty}^{\infty} \mathbb{H}_+(\omega)\left[\mathbb{H}_-(\omega)\mathbf{\Lambda}_{r+}^t(\omega) + \mathbb{H}_-(\omega)\mathbf{\Lambda}_m^t(\omega)\right] \cdot \overline{\mathbf{k}_-}(\omega) d\omega = 0.\end{aligned} \tag{11.2.7}$$

Consider now the quantity $\mathbb{H}_-(\omega)\mathbf{\Lambda}_{r+}^t(\omega)$, the components of which are continuous, indeed analytic on $\mathbb{R}$, by virtue of the analyticity properties of $\mathbb{H}_-(\omega)$ and $\mathbf{\Lambda}_+^t(\omega)$. The Plemelj formula $(B.2.15)_2$ gives that

$$\mathbf{P}^t(\omega) = \mathbb{H}_-(\omega)\mathbf{\Lambda}_{r+}^t(\omega) = \mathbf{p}_-^t(\omega) - \mathbf{p}_+^t(\omega), \tag{11.2.8}$$

where

$$\mathbf{p}^t(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{P}^t(\omega')}{\omega' - z} d\omega', \quad \mathbf{p}_\pm^t(\omega) := \lim_{\alpha \to 0^\mp} \mathbf{p}^t(\omega + i\alpha). \tag{11.2.9}$$

Moreover, $\mathbf{p}^t(z) = \mathbf{p}_+^t(z)$ is analytic for $z \in \Omega^{(-)}$, and $\mathbf{p}^t(z) = \mathbf{p}_-^t(z)$ is analytic for $z \in \Omega^{(+)}$. Both are analytic on the real axis by virtue of the argument leading up to Remark B.2.2. We write them in the form

$$\mathbf{p}^t_{\pm}(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{P}^t(\omega')}{\omega' - \omega^{\mp}}d\omega', \tag{11.2.10}$$

where the notation $\omega^{\pm}$ of (C.2.10) and (C.2.11) has been used. If we can determine explicit formulas for $\mathbf{p}^t_{\pm}(\omega)$, $\omega \in \mathbb{R}$ (or $\omega \in \Omega^{(\mp)}$), then they can be analytically continued into $\Omega^{(\pm)}$, respectively, defined everywhere except at singularities, unless a blocking branch cut prevents this (Sect. B.1). Examples will be given later. Using (11.2.8) in (11.2.7), we obtain

$$\int_{-\infty}^{\infty}\mathbb{H}_{+}(\omega)\left[\mathbf{p}^t_{-}(\omega) - \mathbf{p}^t_{+}(\omega) + \mathbb{H}_{-}(\omega)\mathbf{\Lambda}^t_m(\omega)\right]\cdot\overline{\mathbf{k}_{-}}(\omega)d\omega = 0. \tag{11.2.11}$$

Note that the integral

$$\int_{-\infty}^{\infty}\mathbb{H}_{+}(\omega)\mathbf{p}^t_{+}(\omega)\cdot\overline{\mathbf{k}_{-}}(\omega)d\omega$$

vanishes identically by virtue of (B.1.14), because the integrand is analytic on $\Omega^{(-)}$, by Remark B.1.2, and vanishes like z^{-2} at large z. Therefore, (11.2.11) becomes

$$\int_{-\infty}^{\infty}\mathbb{H}_{+}(\omega)\left[\mathbf{p}^t_{-}(\omega) + \mathbb{H}_{-}(\omega)\mathbf{\Lambda}^t_m(\omega)\right]\cdot\overline{\mathbf{k}_{-}}(\omega)d\omega = 0. \tag{11.2.12}$$

This will be true for arbitrary $\overline{\mathbf{k}_{-}}(\omega)$ only if the expression in brackets is a function that is analytic in $\Omega^{(-)}$. However, $\mathbf{\Lambda}^t_m(\omega)$ must be analytic in Ω^{+}. Remembering that $\mathbf{p}^t_{-}(\omega)$ and $\mathbb{H}_{-}(\omega)$ are also analytic in Ω^{+}, we see that the expression in brackets must be analytic in both the upper and the lower half-planes and on the real axis. Thus, it is analytic over the entire complex plane. Now, $\mathbf{p}^t_{-}(\omega)$ clearly vanishes like ω^{-1} at infinity, as also must $\mathbf{\Lambda}^t_m(\omega)$ by (C.2.16) if $\mathbf{\Lambda}^t_o$ is to be nonzero and finite at $s = 0$. Therefore, the function in brackets is analytic everywhere, zero at infinity, and consequently must vanish everywhere by Liouville's theorem (Sect. B.1.3). Thus,

$$\mathbf{p}^t_{-}(\omega) + \mathbb{H}_{-}(\omega)\mathbf{\Lambda}^t_m(\omega) = \mathbf{0} \qquad \forall\omega \in \mathbb{R}, \tag{11.2.13}$$

whence

$$\mathbf{\Lambda}^t_m(\omega) = -[\mathbb{H}_{-}(\omega)]^{-1}\,\mathbf{p}^t_{-}(\omega). \tag{11.2.14}$$

Using this relation and (11.1.11) in $(11.2.3)_3$, we find that the optimal value of $W(\infty)$ is

$$W_{opt}(\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_{+}(\omega)|^2\,d\omega. \tag{11.2.15}$$

Note that from (11.1.11), $(7.5.7)_2$, and (11.2.8),

$$\begin{aligned} W(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_{-}(\omega) - \mathbf{p}^t_{+}(\omega)|^2\,d\omega \\ &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|\mathbf{p}^t_{-}(\omega)|^2 + |\mathbf{p}^t_{+}(\omega)|^2\right]d\omega, \end{aligned} \tag{11.2.16}$$

since the cross terms vanish by Proposition B.1.3. Thus, from (11.2.4), (11.2.15), and (11.2.16), we have

$$\psi_m(t) = \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_-^t(\omega)|^2 \, d\omega. \tag{11.2.17}$$

Using (11.2.14), we can write this as

$$\begin{aligned}\psi_m(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\boldsymbol{\Lambda}_m^t}(\omega)\cdot \mathbb{H}(\omega)\boldsymbol{\Lambda}_m^t(\omega)d\omega \\ &= \phi(t) + \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{L}_{12}(|s-u|)\boldsymbol{\Lambda}_{ro}^t(u)\cdot\boldsymbol{\Lambda}_{ro}^t(s)duds \\ &= \phi(t) + \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{L}(|s-u|)\dot{\boldsymbol{\Lambda}}_o^t(u)\cdot\dot{\boldsymbol{\Lambda}}_o^t(s)duds \\ &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\dot{\boldsymbol{\Lambda}}_m^t}(\omega)\cdot \frac{\mathbb{H}(\omega)}{\omega^2}\dot{\boldsymbol{\Lambda}}_m^t(\omega)d\omega,\end{aligned} \tag{11.2.18}$$

where $\boldsymbol{\Lambda}_{ro}^t$ is defined by (11.2.5). This last form can be seen to be a special case of (5.2.11).

From (5.1.13), we have

$$\dot{\psi}_m(t) + D_m(t) = \boldsymbol{\Sigma}(t)\cdot\dot{\boldsymbol{\Lambda}}(t), \tag{11.2.19}$$

where D_m is the rate of dissipation corresponding to the minimum free energy and must be nonnegative by the second law. Let us assume that the material is undisturbed in the distant past. Integrating (11.2.19) up to time t gives a special case of (5.1.34):

$$\psi_m(t) + \mathcal{D}_m(t) = W(t), \tag{11.2.20}$$

where

$$\mathcal{D}_m(t) = \int_{-\infty}^{t} D_m(s)ds$$

is the total dissipation up to time t, corresponding to the minimum free energy. Since ψ_m is less than or equal to any other free energy, it follows from (11.2.20) that $\mathcal{D}_m(t)$ is the largest estimate of dissipation in the material element. We have, from (11.2.15)–(11.2.17),

$$\mathcal{D}_m(t) = W(t) - \psi_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_+^t(\omega)|^2 \, d\omega = W_{opt}(\infty) \geq 0. \tag{11.2.21}$$

Also,

$$D_m(t) = \dot{\mathcal{D}}_m(t).$$

In order to give an explicit expression for D_m, we note certain properties of $\mathbf{p}_\pm^t$. From (11.2.10) and $(7.2.30)_2$, it follows that

$$\begin{aligned}\frac{d}{dt}\mathbf{p}_+^t(\omega) &= -i\omega\mathbf{p}_+^t(\omega) - \mathbf{K}(t), \\ \frac{d}{dt}\mathbf{p}_-^t(\omega) &= -i\omega\mathbf{p}_-^t(\omega) - \mathbf{K}(t) - \frac{\mathbb{H}_-(\omega)\dot{\boldsymbol{\Lambda}}(t)}{i\omega}, \\ \mathbf{K}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}_-(\omega)\boldsymbol{\Lambda}_{r+}^t(\omega)d\omega.\end{aligned} \tag{11.2.22}$$

The quantity $\mathbf{K}$ is in fact real if (11.6.3) below holds, which is true for commuting factors. The relation

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega')}{i\omega'^-(\omega'-\omega^+)}d\omega' = \frac{\mathbb{H}_-(\omega)}{i\omega} \tag{11.2.23}$$

has been used. This follows by remembering that $\mathbb{H}_-(\omega)$ vanishes linearly at the origin and by closing the contour on $\Omega^{(+)}$, on which half-plane $\mathbb{H}_-$ is analytic. If $(\omega' - \omega^-)$ occurs in the denominator, the integral vanishes. Furthermore,

$$\begin{aligned}\lim_{|\omega|\to\infty}\omega\mathbf{p}^t_\pm(\omega) &= i\mathbf{K}(t),\\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{p}^t_\pm(\omega)d\omega &= \mp\frac{1}{2}\mathbf{K}(t).\end{aligned} \tag{11.2.24}$$

The first relation follows from (11.2.10) and the second from (B.1.13) and the first relation, remembering the analyticity properties of $\mathbf{p}^t_\pm$. Differentiating (11.2.21) with respect to t, we find the explicitly nonnegative form for the rate of dissipation:

$$D_m(t) = |\mathbf{K}(t)|^2, \tag{11.2.25}$$

where $\mathbf{K}$ is given by $(11.2.22)_3$.

Remark 11.2.1. The following, simpler, derivation of (11.2.14) and (11.2.15) (yielding (11.2.17)) can be given. Let us write $(11.2.3)_3$ as

$$W(\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_-(\omega) - \mathbf{p}^t_+(\omega) + \mathbb{H}_-(\omega)\boldsymbol{\Lambda}^t_{r-}(\omega)|^2\,d\omega.$$

Putting

$$\mathbf{p}^t_{1-}(\omega) = \mathbf{p}^t_-(\omega) + \mathbb{H}_-(\omega)\boldsymbol{\Lambda}^t_{r-}(\omega),$$

where $\mathbf{p}^t_{1-}$ is analytic on $\Omega^{(+)}$, we have

$$W(\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_{1-}(\omega) - \mathbf{p}^t_+(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}[|\mathbf{p}^t_{1-}(\omega)|^2 + |\mathbf{p}^t_+(\omega)|^2]\,d\omega,$$

by Proposition B.1.3. Only $\mathbf{p}^t_{1-}$ depends on $\boldsymbol{\Lambda}^t_{r-}$, so that the minimum must be given by the condition

$$\mathbf{p}^t_{1-}(\omega) = \mathbf{0},$$

which is (11.2.14). Relation (11.2.15) follows immediately.

11.2.2 The Wiener–Hopf Method

The first-order variation of $(11.2.2)_2$ due to $\boldsymbol{\Lambda}^t(u) \to \boldsymbol{\Lambda}^t(u) + \delta\boldsymbol{\Lambda}^t(u)$ has the form

$$\delta W(\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{L}_{12}(|s-u|)\boldsymbol{\Lambda}^t(u)\cdot\delta\boldsymbol{\Lambda}^t(s)duds,$$

where the symmetry of $\mathbb{L}_{12}(|s-u|)$ has been used. We vary only the future continuation, so that

$$\delta\mathbf{\Lambda}^t(s) = \mathbf{0}, \qquad s \in \mathbb{R}^+.$$

The resulting $\delta W(\infty)$ is put equal to zero, yielding the optimization condition

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial s \partial u}\mathbb{L}(|s-u|)\mathbf{\Lambda}^t(u)du = \mathbf{0}, \qquad s \in \mathbb{R}^-.$$

Removing the derivative with respect to s gives a constant on the right-hand side, which can be shown to be zero by observing that from $(7.1.15)_2$,

$$\lim_{s\to-\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial u}\mathbb{L}(|s-u|)\mathbf{\Lambda}^t(u)du = \mathbf{0}.$$

Thus, we obtain the Wiener–Hopf equation

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial}{\partial u}\mathbb{L}(|s-u|)\mathbf{\Lambda}^t(u)du &= \int_{-\infty}^{\infty} \frac{\partial}{\partial u}\mathbb{L}(|s-u|)\mathbf{\Lambda}_r^t(u)du \\ = -\int_{-\infty}^{\infty} \frac{\partial}{\partial s}\mathbb{L}(|s-u|)\mathbf{\Lambda}_r^t(u)du &= -\mathbf{R}(s), \qquad \mathbf{R}(s) = \mathbf{0} \quad \forall s \in \mathbb{R}^-.\end{aligned} \tag{11.2.26}$$

Relation $(7.1.15)_1$ allows $\mathbf{\Lambda}^t(u)$ to be replaced by $\mathbf{\Lambda}_r^t(u)$. This is an integral equation for the optimal continuation $\mathbf{\Lambda}_{ro}^t$, defined by (11.2.5).* The quantity $\mathbf{R}$ on $\mathbb{R}^+$ is for the moment undetermined. Taking Fourier transforms of (11.2.26) and multiplying across by ω, we obtain, with the aid of the convolution theorem (C.3.3) together with (7.2.22) and (7.2.25),

$$2i\mathbb{H}(\omega)\left[\mathbf{\Lambda}_{r+}^t(\omega) + \mathbf{\Lambda}_m^t(\omega)\right] = \omega\mathbf{R}_+(\omega), \tag{11.2.27}$$

where $\mathbf{\Lambda}_m^t(\omega)$ is the Fourier transform of $\mathbf{\Lambda}_{ro}^t$ on $\mathbb{R}^-$ and is the quantity we wish to determine. It is analytic on $\Omega^{(+)}$ and by assumption also on $\mathbb{R}$ (Sect. C.2). Similarly, $\mathbf{R}_+(\omega)$ is analytic on $\Omega^{(-)}$ and by assumption also on $\mathbb{R}$.

The factorization (11.1.11) is now used. We multiply (11.2.27) by $[2i\mathbb{H}_+(\omega)]^{-1}$ to obtain

$$\mathbb{H}_-(\omega)\left[\mathbf{\Lambda}_{r+}^t(\omega) + \mathbf{\Lambda}_m^t(\omega)\right] = \frac{\omega}{2i}[\mathbb{H}_+(\omega)]^{-1}\mathbf{R}_+(\omega). \tag{11.2.28}$$

Substituting (11.2.8) into (11.2.28), we obtain

$$\mathbf{A}(\omega) = \mathbf{p}_-^t(\omega) + \mathbb{H}_-(\omega)\mathbf{\Lambda}_m^t(\omega) = \mathbf{p}_+^t(\omega) + \frac{\omega}{2i}[\mathbb{H}_+(\omega)]^{-1}\,\mathbf{R}_+(\omega). \tag{11.2.29}$$

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* Carrying out a partial integration in (11.2.26), we have the form

$$\int_{-\infty}^{\infty} \mathbb{L}(|s-u|)\dot{\mathbf{\Lambda}}^t(u)du = \mathbf{R}(s).$$

This is a special case of (5.2.8), as can be seen by splitting the integral at $u = s$, changing the integration variable, and recalling $(7.1.17)_4$.

The function $\mathbf{A}(\omega)$ is analytic on Ω^- by virtue of the first relation and analytic on Ω^+ by virtue of the second. It is therefore analytic over the entire complex plane. By Liouville's theorem, it must be a polynomial. However, for $|\omega| \to \infty$, $\mathbf{A}(\omega) \to 0$ like $1/\omega$, on applying the argument presented after (11.2.12). Hence, it must vanish everywhere, so that

$$\mathbb{H}_-(\omega)\mathbf{\Lambda}_m^t(\omega) + \mathbf{p}_-^t(\omega) = \mathbf{0},$$

which is (11.2.13). The right-hand side of (11.2.29) also vanishes, which yields a relationship for $\mathbf{R}_+$.

The above solution can be extended to a more general set of histories. We write (11.2.26) in the form

$$\begin{aligned} \int_{-\infty}^{0} \frac{\partial}{\partial s}\mathbb{L}(|s-u|)\mathbf{\Lambda}_r^t(u)du - \mathbf{J}^t(s) = \mathbf{R}(s), \quad s \in \mathbb{R}, \\ \mathbf{J}^t(s) = -\int_0^{\infty} \frac{\partial}{\partial s}\mathbb{L}(|s-u|)\mathbf{\Lambda}_r^t(u)du. \end{aligned} \tag{11.2.30}$$

Observe that

$$\mathbf{J}^t(s) = \mathbf{I}^t(-s, \mathbf{\Lambda}_r^t), \quad s \in \mathbb{R}^-, \tag{11.2.31}$$

where $\mathbf{I}^t(\cdot, \mathbf{\Lambda}_r^t)$ is defined by (7.4.2). Let us assume that $\mathbf{J}^t \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Denoting the Fourier transform of $\mathbf{J}^t$ by $\mathbf{J}_F^t \in L^2(\mathbb{R})$, we obtain, instead of (11.2.27),

$$2i\mathbb{H}(\omega)\mathbf{\Lambda}_m^t(\omega) - \omega\mathbf{J}_F^t(\omega) = \omega\mathbf{R}_+(\omega).$$

The argument proceeds as outlined above but where $\mathbf{P}^t(\omega)$ in (11.2.8) is now defined by

$$\mathbf{P}^t(\omega) = -\frac{\omega}{2i}[\mathbb{H}_+(\omega)]^{-1}\,\mathbf{J}_F^t(\omega) = \mathbf{p}_-^t(\omega) - \mathbf{p}_+^t(\omega). \tag{11.2.32}$$

It is assumed that $\mathbf{J}_F^t$ is analytic on $\mathbb{R}$. The Fourier transform on $\mathbb{R}$ of a function that is continuous at $t = 0$ behaves like ω^{-2} at large frequencies, by virtue of (C.2.16) (putting $m = 0$ in (C.2.18)), so that $\mathbf{J}_F^t$ has this property. Thus, $\mathbf{P}^t(\omega) \sim \omega^{-1}$ at large ω, as required for the convergence of the integral in (11.2.9).

This formulation is valid for histories $\mathbf{\Lambda}^t$ that do not have a Fourier transform but where $\mathbf{J}^t$ exists and has a Fourier transform [110, 145].

The quantity $\mathbf{\Lambda}_m^t$ is the Fourier transform of the optimal future continuation $\mathbf{\Lambda}_{ro}^t$ introduced in (11.2.6). Consider (11.2.30) for $s < 0$. We differentiate this relation, multiplying by $\mathbf{\Lambda}_{ro}^t$, and integrate over $\mathbb{R}^-$ to obtain

$$\begin{aligned} \int_{-\infty}^{0}\int_{-\infty}^{0} \mathbf{\Lambda}_{ro}^t(s)\cdot\mathbb{L}_{12}(|s-u|)\mathbf{\Lambda}_{ro}^t(u)duds &= \int_{-\infty}^{0} \mathbf{J}^t(s)\cdot\mathbf{\Lambda}_{ro}^t(s)ds \\ &= \int_{-\infty}^{0} \mathbf{I}(-s, \mathbf{\Lambda}_r^t)\cdot\mathbf{\Lambda}_{ro}^t(s)ds. \end{aligned}$$

Relation $(11.2.18)_2$ gives that

$$\psi(t) = \phi(t) + \frac{1}{2}\int_{-\infty}^{0} \mathbf{I}(-s, \mathbf{\Lambda}_r^t)\cdot\mathbf{\Lambda}_{ro}^t(s)ds$$

and, from (11.2.17),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}_-^t(\omega)|^2 d\omega = \frac{1}{2}\int_{-\infty}^{0}\mathbf{I}(-s,\boldsymbol{\Lambda}_r^t)\cdot\boldsymbol{\Lambda}_{ro}^t(s)ds. \tag{11.2.33}$$

11.2.3 Histories Rather Than Relative Histories

In early work on the minimum and other free energies [92, 110, 158], histories, rather than relative histories, were used. By essentially identical arguments to those above, one can show the following result. Let

$$\mathbf{Q}^t(\omega) := \mathbb{H}_-(\omega)\boldsymbol{\Lambda}_+^t(\omega) = \mathbf{q}_-^t(\omega) - \mathbf{q}_+^t(\omega),$$

where

$$\mathbf{q}_\pm^t(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{Q}^t(\omega')}{\omega'-\omega^\mp}d\omega'.$$

Then,

$$\psi(t) = S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{q}_-^t(\omega)|^2 d\omega, \tag{11.2.34}$$

where $S(t)$ is given by $(7.1.19)_2$, or, in the completely linear case, (7.1.34). The free energies in Sect. 16.4 can also be expressed as functionals of histories rather than relative histories. The situation for free energies in dielectrics is in a sense reversed, as we see from Sect. 22.3. There are two disadvantages to this approach, one noted earlier, namely that $S(t)$ is not a nonnegative quantity. The other is that $\boldsymbol{\Lambda}_+^t(\omega)$ behaves like ω^{-1} at large ω, while $\boldsymbol{\Lambda}_{r+}^t(\omega)$ behaves like ω^{-2}.

11.2.4 Confirmation That ψ_m Is a Free Energy

We now ascertain that ψ_m has the characteristic properties of a free energy.

Proposition 11.2.2. *The functional $\psi_m(t)$, given by* (11.2.17)*, obeys the Graffi conditions, given by P1–P4 in Sect. 5.1.1 or* (5.1.30)–(5.1.33).

Proof. Property P2 follows from the fact that $\boldsymbol{\Lambda}_{r+}^t$, and therefore $\mathbf{P}^t$, defined by (11.2.8), vanishes for a static history. Property P3 is immediately apparent, while P4 follows from the fact that D_m, given by (11.2.25), is nonnegative. Property P1 can be proved as follows. Using (7.2.29), we can write

$$\mathbf{p}_-^t(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega_1)\left[\boldsymbol{\Lambda}_+^t(\omega_1) - \frac{\boldsymbol{\Lambda}(t)}{i\omega_1^-}\right]}{\omega_1-\omega^+}d\omega_1, \tag{11.2.35}$$

giving

$$\frac{\partial\mathbf{p}_-^t(\omega)}{\partial\boldsymbol{\Lambda}(t)} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega_1)}{i\omega_1^-(\omega_1-\omega^+)}d\omega_1 = -\frac{\mathbb{H}_-(\omega)}{i\omega}, \tag{11.2.36}$$

where we have used (11.2.23). Also,

$$\frac{\partial \overline{\mathbf{p}_-^t(\omega)}}{\partial \mathbf{\Lambda}(t)} = \frac{\mathbb{H}_+(\omega)^\top}{i\omega}, \tag{11.2.37}$$

by virtue of (11.1.13). Thus,

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{\Lambda}(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{p}_-^t}(\omega) \cdot \mathbf{p}_-^t(\omega) d\omega &= \frac{\partial}{\partial \mathbf{\Lambda}(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{p}_-^t}(\omega)^\top \mathbf{p}_-^t(\omega) d\omega \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega} \mathbf{p}_-^t(\omega) d\omega - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{\frac{\mathbb{H}_+(\omega)}{\omega} \mathbf{p}_-^t(\omega)} d\omega \\
&= \mathrm{Re}\left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega} \mathbf{p}_-^t(\omega) d\omega \right\}.
\end{aligned}$$

Recall that $\mathbb{H}_-$ vanishes linearly at the origin. Also, using (11.2.8), the frequency integral in (7.2.34), which must be real, can be written as

$$\begin{aligned}
-\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\mathbb{H}(\omega)}{\omega} \mathbf{\Lambda}_{r+}^t(\omega) d\omega &= -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega} [\mathbf{p}_-^t(\omega) - \mathbf{p}_+^t(\omega)] d\omega \\
&= -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega} \mathbf{p}_-^t(\omega) d\omega,
\end{aligned} \tag{11.2.38}$$

because the term involving $\mathbf{p}_+^t$ vanishes by Cauchy's theorem. Since the last quantity is real, P1 follows. □

It is shown in [92] that ψ_m is a free energy also under the definition of Coleman and Owen [75, 76]. This was for linear isothermal systems, but the proof applies to the present, more general, case.

11.2.5 Double Frequency Integral Form

We can write (11.2.17) in a more explicit form by carrying out the integration over ω. The following relationships are required. Firstly, we have

$$\begin{aligned}
\overline{\mathbb{H}_-(\omega_1)\mathbf{\Lambda}_r^t(\omega_1)} \cdot \mathbb{H}_-(\omega_2)\mathbf{\Lambda}_r^t(\omega_2) &= \overline{\mathbf{\Lambda}_r^t(\omega_1)} \cdot \mathbb{H}_-^*(\omega_1)\mathbb{H}_-(\omega_2)\mathbf{\Lambda}_r^t(\omega_2) \\
&= \overline{\mathbf{\Lambda}_r^t(\omega_1)} \cdot \mathbb{H}_+(\omega_1)\mathbb{H}_-(\omega_2)\mathbf{\Lambda}_r^t(\omega_2),
\end{aligned}$$

where (11.1.13), (A.2.7), and (A.2.8) have been used. Also, recalling (C.2.10) and (C.2.11), we can write for real ω_1, ω_2, and ω,

$$\begin{aligned}
\overline{\omega^-} &= \overline{\lim_{\alpha \to 0^+} (\omega - i\alpha)} = \omega^+, \\
\omega_1 - \omega^- &= \lim_{\alpha \to 0^+} (\omega_1 - \omega + i\alpha) = \omega_1^+ - \omega, \\
\omega_2 - \omega^+ &= \lim_{\alpha \to 0^+} (\omega_2 - \omega - i\alpha) = \omega_2^- - \omega,
\end{aligned}$$

where the limits are taken after any integrations are carried out. Finally,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{(\omega_1^+ - \omega)(\omega_2^- - \omega)} = \frac{1}{\omega_1^+ - \omega_2^-},$$

by closing the contour on either $\Omega^{(+)}$ or $\Omega^{(-)}$.

Using these results, we can write the expression (11.2.17) for $\psi_m(t)$, where $\mathbf{p}_-^t(\omega)$ is given by (11.2.10) or (11.2.35), in the form

$$\begin{aligned}\psi_m(t) &= \phi(t) + \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{\mathbf{\Lambda}_{r+}^t}(\omega_1)\cdot\mathbb{M}_m(\omega_1,\omega_2)\mathbf{\Lambda}_{r+}^t(\omega_2)}{\omega_1^+ - \omega_2^-}d\omega_1 d\omega_2,\\ \mathbb{M}_m(\omega_1,\omega_2) &= \mathbb{H}_+(\omega_1)\mathbb{H}_-(\omega_2).\end{aligned} \tag{11.2.39}$$

The notation in the denominator of the integral in $(11.2.39)_1$ means that if we integrate first over ω_1, it becomes $(\omega_1 - \omega_2^-)$, or if over ω_2 first, then it is $(\omega_1^+ - \omega_2)$. Also, using $(11.2.22)_3$, $D_m(t)$, given by (11.2.25), can be expressed as

$$D_m(t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\mathbf{\Lambda}_{r+}^t}(\omega_1)\cdot\mathbb{M}_m(\omega_1,\omega_2)\mathbf{\Lambda}_{r+}^t(\omega_2)d\omega_1 d\omega_2.$$

Let us write the double integral in (11.2.39) as

$$\begin{aligned}P_-(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}_-^t(\omega)|^2 d\omega = \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^t(\omega_1,\omega_2)}{\omega_1^+ - \omega_2^-}d\omega_1 d\omega_2,\\ A^t(\omega_1,\omega_2) &= \overline{\mathbf{\Lambda}_{r+}^t}(\omega_1)\cdot\mathbb{M}_m(\omega_1,\omega_2)\mathbf{\Lambda}_{r+}^t(\omega_2).\end{aligned} \tag{11.2.40}$$

In the same way, we obtain

$$P_+(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}_+^t(\omega)|^2 d\omega = -\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^t(\omega_1,\omega_2)}{\omega_1^- - \omega_2^+}d\omega_1 d\omega_2. \tag{11.2.41}$$

From (11.2.21), we see that this is the total dissipation up to time t. One can show that

$$\begin{aligned}R_-(t) &= \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{B^t(\omega_1,\omega_2)}{\omega_1^+ - \omega_2^-}d\omega_1 d\omega_2 = 0,\\ B^t(\omega_1,\omega_2) &= \overline{\mathbf{\Lambda}_{r+}^t}(\omega_1)\cdot\mathbb{M}_m(\omega_2,\omega_1)\mathbf{\Lambda}_{r+}^t(\omega_2),\end{aligned} \tag{11.2.42}$$

by integrating over ω_2 for example and closing the contour on $\Omega^{(-)}$, since $\mathbb{H}_+$ and $\mathbf{\Lambda}_{r+}^t$ have no singularity in the lower half-plane. Furthermore, using the same procedure, one obtains

$$\begin{aligned}R_+(t) &= -\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{B^t(\omega_1,\omega_2)}{\omega_1^- - \omega_2^+}d\omega_1 d\omega_2\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathbf{\Lambda}_{r+}^t}(\omega)\cdot\mathbb{H}(\omega)\mathbf{\Lambda}_{r+}^t(\omega)d\omega\\ &= P_-(t) + P_+(t),\end{aligned} \tag{11.2.43}$$

by virtue of $(7.5.7)_2$ and (11.2.16). Relation $(11.2.42)_1$ allows us to write (11.2.40) in the explicitly convergent form

$$P_-(t) = \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^t(\omega_1,\omega_2) - B^t(\omega_1,\omega_2)}{\omega_1 - \omega_2}d\omega_1 d\omega_2, \tag{11.2.44}$$

which is convenient for numerical evaluation. We can replace the $(\omega_1 - \omega_2)$ in the denominator by $(\omega_1^+ - \omega_2^-)$, which gives (11.2.40), or by $(\omega_1^- - \omega_2^+)$, which gives the same result by way of (11.2.43) and (11.2.41). Relation (11.2.44) implies that the kernel

$$\mathbb{D}(\omega_1,\omega_2) = i\frac{[\mathbb{H}_+(\omega_1)\mathbb{H}_-(\omega_2) - \mathbb{H}_+(\omega_2)\mathbb{H}_-(\omega_1)]}{\omega_1 - \omega_2}$$

is nonnegative in the operator sense, i.e., it must yield a nonnegative value for the integral, for all histories. Using very localized choices of $\mathbf{\Lambda}_+^t(\omega)$, we deduce that the "diagonal elements" of $\mathbb{D}(\omega_1,\omega_2)$ are nonnegative, as in Remark 7.1.3. Using a prime to denote differentiation, we can write these as

$$\mathbb{D}(\omega) = i[\mathbb{H}_+'(\omega)\mathbb{H}_-(\omega) - \mathbb{H}_+(\omega)\mathbb{H}_-'(\omega)] \geq \mathbf{0}, \qquad \omega \in \mathbb{R}. \tag{11.2.45}$$

Proposition 11.2.3. *We have*

$$\begin{aligned}\int_{-\infty}^{t} P_+(u)du &= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^t(\omega_1,\omega_2)}{(\omega_1^- - \omega_2^+)^2}d\omega_1 d\omega_2 \\ &= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{A^t(\omega_1,\omega_2)}{(\omega_1^+ - \omega_2^-)^2} + \frac{B^t(\omega_1,\omega_2)}{(\omega_1^- - \omega_2^+)^2}\right]d\omega_1 d\omega_2 \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathbf{\Lambda}_{r+}^t}(\omega)\cdot\mathbb{D}(\omega)\mathbf{\Lambda}_{r+}^t(\omega)d\omega,\end{aligned} \tag{11.2.46}$$

where $\mathbb{D}(\cdot)$ *is defined by* (11.2.45) *and the integral on the left is assumed to exist for all finite values of* t.

Proof. Relation $(11.2.46)_1$ follows immediately, by time differentiation, using (7.2.30) and the relations

$$\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega_2)}{\omega_2^-(\omega_1^- - \omega_2^+)^2}d\omega_2 = \int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega_1)}{\omega_1^+(\omega_1^- - \omega_2^+)^2}d\omega_1 = \mathbf{0},$$

which follow from the fact that $\mathbb{H}_\pm$ vanish linearly at the origin and Cauchy's theorem, closing the first integral on $\Omega^{(+)}$ and the second on $\Omega^{(-)}$. There can be no term independent of t, since the integral on the left of (11.2.46) vanishes as $t \to -\infty$. Equation $(11.2.46)_2$ can be verified similarly, on noting a cancellation between the derivatives of the single and double integral terms. Relations such as

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega_2)}{i\omega_2^-(\omega_1^+ - \omega_2^-)^2}d\omega_2 &= \frac{d}{d\omega_1}\left[\frac{\mathbb{H}_-(\omega_1)}{i\omega_1}\right], \\ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega_1)}{i\omega_1^+(\omega_1^+ - \omega_2^-)^2}d\omega_1 &= -\frac{d}{d\omega_2}\left[\frac{\mathbb{H}_+(\omega_2)}{i\omega_2}\right],\end{aligned}$$

are required. These follow from (B.1.3). The minus sign in the second relation is a consequence of the fact that the contour must be completed on $\Omega^{(-)}$ (Sect. B.1.1). □

The quantity $\mathbb{D}(\cdot)$ occurs in (11.7.10) in the context of the minimum free energy for sinusoidal histories.

11.3 Characterization of the Minimal State in the Frequency Domain

In this section, we show that the quantity $\mathbf{p}_-^t$, defined by (11.2.10), and occurring in $\psi_m(t)$, given by (11.2.17), is a function of the minimal state. Two histories $\mathbf{\Lambda}_1^t$ and $\mathbf{\Lambda}_2^t$ are equivalent if their difference $\mathbf{\Lambda}^t = \mathbf{\Lambda}_1^t - \mathbf{\Lambda}_2^t$ satisfies (7.4.3). Relation $(7.4.3)_2$ can be written in the form

$$\mathbf{F}^t(\tau) := \int_\tau^\infty \mathbb{L}'(s)\mathbf{\Lambda}^{t+\tau}(s)ds = \mathbf{0}, \qquad \forall\tau \geq 0. \tag{11.3.1}$$

We adopt the assumptions made before (7.4.5), so that $(7.4.3)_1$ and (7.4.4) are replaced by

$$\mathbf{\Lambda}_1(t+\tau) = \mathbf{\Lambda}_2(t+\tau), \quad \tau \geq 0. \tag{11.3.2}$$

Condition (11.3.2) gives that $\mathbf{\Lambda}^{t+\tau}(s)$ is equal to zero for $\tau \geq s$. Let us identify $\mathbb{L}'$ with its odd extension on $\mathbb{R}$, so that (7.2.25) applies. Then, $\mathbf{F}^t(\tau)$ can be rewritten in terms of Fourier transforms:

$$\mathbf{F}^t(\tau) = \int_{-\infty}^\infty \mathbb{L}'(s)\mathbf{\Lambda}^{t+\tau}(s)ds = \frac{1}{2\pi}\int_{-\infty}^\infty \overline{\mathbb{L}'_F}(\omega)\mathbf{\Lambda}_+^{t+\tau}(\omega)d\omega$$
$$= -\frac{i}{\pi}\int_{-\infty}^\infty \mathbb{L}'_s(\omega)\mathbf{\Lambda}_+^{t+\tau}(\omega)d\omega.$$

Moreover, note that

$$\mathbf{\Lambda}_+^{t+\tau}(\omega) = \int_0^\infty \mathbf{\Lambda}^{t+\tau}(s)e^{-i\omega s}ds = \int_\tau^\infty \mathbf{\Lambda}^t(s-\tau)e^{-i\omega(s-\tau)}ds\, e^{-i\omega\tau} = \mathbf{\Lambda}_+^t(\omega)e^{-i\omega\tau},$$

which yields

$$\mathbf{F}^t(\tau) = \frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}(\omega)}{\omega}\mathbf{\Lambda}_+^t(\omega)e^{-i\omega\tau}\,d\omega. \tag{11.3.3}$$

This will be taken as the definition of $\mathbf{F}^t(\tau)$ for $\tau \in \mathbb{R}$.

Remembering the factorization of $\mathbb{H}(\omega)$ given by (11.1.11), (11.3.3) can be rewritten as

$$\mathbf{F}^t(\tau) = \frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}_+(\omega)}{\omega}\mathbb{H}_-(\omega)\mathbf{\Lambda}_+^t(\omega)e^{-i\omega\tau}\,d\omega, \tag{11.3.4}$$

and the substitution of (11.2.8) into (11.3.4) yields

$$\mathbf{F}^t(\tau) = \frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}_+(\omega)}{\omega}\left[\mathbf{p}_-^t(\omega) - \mathbf{p}_+^t(\omega)\right]e^{-i\omega\tau}\,d\omega. \tag{11.3.5}$$

Observe that $\mathbb{H}_+(z)$, $\mathbf{p}_+^t(z)$, and $e^{-iz\tau}$ $(\tau > 0)$ are analytic functions in the lower half-plane $z \in \Omega^{(-)}$, their product converging strongly to zero at infinity, so that by Cauchy's theorem, (11.3.5) reduces to (cf. (11.2.38))

$$\mathbf{F}^t(\tau) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}_-^t(\omega)e^{-i\omega\tau}\,d\omega. \tag{11.3.6}$$

This form of $\mathbf{F}^t$ allows us to prove the following theorem characterizing a minimal state in the frequency domain.

Theorem 11.3.1. *For every material with linear memory and a symmetric (required for factorization) relaxation function* $\mathbb{L}$*, a given history* $\mathbf{\Lambda}^t$ *is equivalent to the zero history* $\mathbf{0}^\dagger$ *if and only if the* $\mathbf{p}_-^t$ *related to* $\mathbf{\Lambda}^t$ *by* (11.2.10) *with* (11.2.8) *is such that*

$$\mathbf{p}_-^t(\omega) = \mathbf{0}, \qquad \forall\omega\in\mathbb{R}.$$

Observe that the theorem in effect states that

$$\mathbf{F}^t(\tau) = \mathbf{0}\ \forall\tau\geq 0 \qquad \Longleftrightarrow \qquad \mathbf{p}_-^t(\omega) = \mathbf{0}\ \forall\omega\in\mathbb{R}. \tag{11.3.7}$$

Proof. The statement relating to the left-pointing arrow of (11.3.7) follows trivially from (11.3.6). In order to prove the statement relating to the right arrow, let us invert the Fourier transform in (11.3.6) to obtain

$$\begin{aligned}\mathbf{f}^t(\omega) &= \frac{i}{\pi}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}_-^t(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{F}^t(\tau)e^{iw\tau}\,d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{0}\mathbf{F}^t(\tau)e^{iw\tau}d\tau = \frac{1}{2\pi}\int_{0}^{\infty}\mathbf{F}^t(-u)e^{-iwu}\,du.\end{aligned} \tag{11.3.8}$$

It follows from Proposition C.2.1 that $\mathbf{f}^t$ (the Fourier transform of $\mathbf{F}^t(-u)$, a function that is zero on $\mathbb{R}^{--}$) is analytic in $\Omega^{(-)}$. The zeros in rows or columns of $\mathbb{H}_+$ (zeros of $\det\mathbb{H}_+$) cannot cancel singularities of $\mathbf{p}_-^t$, since all such zeros are in $\Omega^{(+)}$. Also, any branch-cut singularity of $\mathbb{H}_+$ is in $\Omega^{(+)}$, and those of $\mathbf{p}_-^t$ are in $\Omega^{(-)}$, so there can be no neutralization of such singularities. Thus $\mathbf{p}_-^t$ must be analytic in $\Omega^{(-)}$ and therefore in Ω. It goes to zero at infinity and must therefore be zero everywhere by Liouville's theorem (Sect. B.1.3).

□

Note that Theorem 11.3.1 is in effect saying that if a history is equivalent to the zero history, then the minimum free energy vanishes, or (7.4.8) holds. We now show that this implies the following result.

Proposition 11.3.2. *If two states are equivalent, then* (7.4.6) *holds, so that the minimum free energy, given by* (11.2.17)*, is a functional of the minimal state. Furthermore,* (7.4.8) *is true.*

Proof. Let $(\mathbf{\Lambda}_1^t,\mathbf{\Lambda}(t))$ and $(\mathbf{\Lambda}_2^t,\mathbf{\Lambda}(t))$ be equivalent states. Then the difference of their histories is equivalent to the zero history, as argued before (7.4.5). Also, let $\mathbf{p}_-^t(\omega,\mathbf{\Lambda}_1^t)$ indicate this quantity defined by (11.2.10) for a history $\mathbf{\Lambda}_1^t$ and similarly for $\mathbf{p}_-^t(\omega,\mathbf{\Lambda}_2^t)$. Then,

$$\mathbf{p}_-^t(\cdot,\mathbf{\Lambda}_1^t) - \mathbf{p}_-^t(\cdot,\mathbf{\Lambda}_2^t) = \mathbf{p}_-^t(\cdot,\mathbf{\Lambda}_1^t-\mathbf{\Lambda}_2^t) = \mathbf{0}$$

by the linearity of $\mathbf{p}_-^t$ and Theorem 11.3.1. Thus, (7.4.6) and (7.4.8) follow immediately. Also, (7.4.12) must be true, since (7.4.6) and (7.4.8) hold. □

The observations around (7.4.10) and (7.4.11) are relevant in the present context.

11.4 The Space of States and Processes

We recall from Definition 4.2.1 of the minimum free energy that it must be defined on the entire space of states. From (11.2.17), this is the space of relative histories and current values $(\mathbf{\Lambda}_r^t, \mathbf{\Lambda}(t))$ such that $\mathbf{p}_-^t$, defined by (11.2.10) with (11.2.8), belongs to $L^2(\mathbb{R})$. As we have seen in Sect. 11.2.2, the Fourier transform $\mathbf{\Lambda}_{r+}^t$ need not exist. However, $\mathbf{P}^t$, given by (11.2.32) in terms of $\mathbf{J}_F^t$, must be finite for $\omega \in \mathbb{R}$.

Recalling $(11.2.18)_{3,4}$, we define the space of processes to be [145]

$$\begin{aligned}\mathcal{H}_\Gamma(\mathbb{R}^-) = \Bigg\{\boldsymbol{\phi} : \mathbb{R}^- \mapsto \Gamma; \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \boldsymbol{\phi}(s)\cdot \mathbb{L}(|s-u|)\boldsymbol{\phi}(u)dsdu \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\boldsymbol{\phi}_-}(\omega)\cdot \frac{\mathbb{H}(\omega)}{\omega^2}\boldsymbol{\phi}_-(\omega)d\omega < \infty\Bigg\}.\end{aligned} \tag{11.4.1}$$

The dual of this space is defined as

$$\mathcal{H}'_\Gamma(\mathbb{R}^-) = \left\{\mathbf{f} : \mathbb{R}^- \mapsto \Gamma; \left|\int_{-\infty}^{0} \mathbf{f}(u)\boldsymbol{\phi}(u)du\right| < \infty \forall \boldsymbol{\phi} \in \mathcal{H}_\Gamma(\mathbb{R}^-)\right\}.$$

The space of histories is defined as those for which

$$\mathbf{J}^t \in \mathcal{H}'_\Gamma(\mathbb{R}^-), \tag{11.4.2}$$

where $\mathbf{J}^t(\cdot)$ is related to $\mathbf{I}(\cdot, \mathbf{\Lambda}_r^t)$ by (11.2.31).

We now prove the following result of Gentili [145].

Proposition 11.4.1. *Given the relative history* $\mathbf{\Lambda}_r^t : \mathbb{R}^+ \mapsto \Gamma$, *then* $\mathbf{J}^t \in \mathcal{H}'_\Gamma(\mathbb{R}^-)$ *if and only if* $\mathbf{p}_-^t \in L^2(\mathbb{R})$.

Proof. For any $\boldsymbol{\phi} \in \mathcal{H}_\Gamma(\mathbb{R}^-)$,

$$\begin{aligned}\int_{-\infty}^{0} \mathbf{J}^t(s)\cdot\boldsymbol{\phi}(s)ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{J}_F^t(\omega)\cdot\overline{\boldsymbol{\phi}_-}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega}{2i}[\mathbb{H}_+(\omega)]^{-1}\mathbf{J}_F^t(\omega)\cdot\frac{2i}{\omega}\mathbb{H}_+^\top(\omega)\overline{\boldsymbol{\phi}_-}(\omega)d\omega \\ &= -\frac{i}{\pi}\int_{-\infty}^{\infty} \mathbf{P}^t(\omega)\cdot\frac{1}{\omega}\overline{\mathbb{H}_-(\omega)\boldsymbol{\phi}_-(\omega)}d\omega,\end{aligned}$$

where (11.2.32) has been used. The $\mathbf{p}_+^t$ term in $\mathbf{P}^t$ yields zero by Cauchy's theorem and Remark B.1.2, so we have

$$\begin{aligned}\int_{-\infty}^{0} \mathbf{J}^t(s)\cdot\boldsymbol{\phi}(s)ds &= -\frac{i}{\pi}\int_{-\infty}^{\infty} \mathbf{p}_-^t(\omega)\cdot\frac{1}{\omega}\overline{\mathbb{H}_-(\omega)\boldsymbol{\phi}_-(\omega)}d\omega \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_-^t(\omega)|^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2}|\mathbb{H}_-(\omega)\boldsymbol{\phi}_-(\omega)|^2 d\omega.\end{aligned}$$

If $\mathbf{p}_-^t \in L^2(\mathbb{R})$, the first term is bounded. Also, the second term is bounded by (11.4.1). It follows that (11.4.2) holds. If $\mathbf{J}^t \in \mathcal{H}'_\Gamma(\mathbb{R}^-)$, then from (11.2.33), we see that $\mathbf{p}_-^t \in L^2(\mathbb{R})$. □

11.5 Limiting Properties of the Optimal Future Continuation

We draw attention to certain properties of the optimal future continuation. From (11.2.14) and $(11.2.24)_1$, it follows that

$$\mathbf{\Lambda}_m^t(\omega) \underset{\omega\to\infty}{\longrightarrow} \frac{\mathbb{H}_{\frac{1}{2}}^{-1}\mathbf{K}(t)}{i\omega}, \tag{11.5.1}$$

where

$$\mathbb{H}_{\frac{1}{2}} = \mathbb{H}_-(\infty)\ (= \mathbb{H}_+(\infty)).$$

The last relation holds for commutative factors, by virtue of (11.6.2) below. Thus, by (C.2.16), the relative optimal continuation, given by (11.2.5), has the form at $s = 0$

$$\mathbf{\Lambda}_{ro}^t(0) = -\mathbb{H}_{\frac{1}{2}}^{-1}\mathbf{K}(t), \tag{11.5.2}$$

and the optimal continuation $\mathbf{\Lambda}_o^t(0)$ is given by

$$\mathbf{\Lambda}_o^t(0) = \mathbf{\Lambda}(t) - \mathbb{H}_{\frac{1}{2}}^{-1}\mathbf{K}(t). \tag{11.5.3}$$

Therefore, the optimal continuation involves a sudden discontinuity at time t, the magnitude of which is related to the rate of dissipation, as we see from (11.2.25).

Also, putting

$$\mathbb{H}_1(\omega) = \frac{1}{\omega}\mathbb{H}_-(\omega), \tag{11.5.4}$$

we have, with the aid of (11.2.10),

$$\mathbf{\Lambda}_m^t(\omega) \approx -\frac{[\mathbb{H}_1(0)]^{-1}}{2\pi i\omega^+}\int_{-\infty}^{\infty}\mathbb{H}_1(\omega')\mathbf{\Lambda}_{r+}^t(\omega')d\omega'$$

as $\omega \to 0$. The quantity ω in the denominator is replaced by ω^+, since the singularities of $\mathbf{\Lambda}_m^t(\omega)$ must be in $\Omega^{(-)}$. Then, with the aid of (C.2.11), we have

$$\mathbf{\Lambda}_{ro}^t(-\infty) = \frac{[\mathbb{H}_1(0)]^{-1}}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}_1(\omega')\mathbf{\Lambda}_{r+}^t(\omega')d\omega' \tag{11.5.5}$$

and

$$\mathbf{\Lambda}_o^t(-\infty) = \mathbf{\Lambda}_{ro}^t(-\infty) + \mathbf{\Lambda}(t).$$

This quantity is in general nonzero.

11.6 Time-Independent Eigenspaces

We now make the assumption (7.1.36), which immediately yields

$$\mathbb{L}'(t) = \sum_{k=1}^{m} L_k'(t)\mathbb{B}_k, \quad t \in \mathbb{R}.$$

Taking Fourier transforms gives

$$\mathbb{L}'_+(\omega) = \sum_{k=1}^{m} L'_{k+}(\omega)\mathbb{B}_k, \quad \omega \in \mathbb{R}.$$

This relation can also be written at $-\omega$. Adding and subtracting give that $\mathbb{L}'_c$ and $\mathbb{L}'_s$ can also be represented in this manner, and furthermore,

$$\mathbb{H}(\omega) = -\omega\mathbb{L}'_s(\omega) = \sum_{k=1}^{m} H_k(\omega)\mathbb{B}^k,$$
$$H_k(\omega) > 0, \quad \omega \in \mathbb{R}\setminus\{0\}, \quad k = 1, \ldots, m.$$

Using the technique of Sect. 11.1.1, the quantities H_k can be factorized into H_{k+} and H_{k-}, where from (11.1.17),

$$H_{k\pm}(\omega) = H_{k\mp}(-\omega) = \overline{H_{k\mp}}(\omega); \quad H_k(\omega) = |H_{k\pm}(\omega)|^2. \tag{11.6.1}$$

We put

$$H_k(\infty) = H_{k\infty}, \quad k = 1, 2, \ldots, m.$$

Since the $\{\mathbb{B}^k\}$ are orthonormal projectors, it follows that

$$\mathbb{H}(\omega) = \sum_{k=1}^{m} H_{k+}(\omega)H_{k-}(\omega)\mathbb{B}^k = \mathbb{H}_+(\omega)\mathbb{H}_-(\omega),$$

where

$$\mathbb{H}_\pm(\omega) = \sum_{k=1}^{m} H_{k\pm}(\omega)\mathbb{B}^k = \mathbb{H}_\pm^\top(\omega).$$

The last relation follows from the symmetry of the $\mathbb{B}^k$. Thus, $\mathbb{H}_\pm$ are symmetric for all frequencies. We see that the factors $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ commute, so that they are normal transformations; see the comment relating to (A.2.11). Recalling (11.1.18) and the comment after (11.1.14), we see that

$$\mathbb{H}_\pm(\infty) = \mathbb{H}_\infty^{1/2}. \tag{11.6.2}$$

The quantities $\mathbb{H}_\pm$ also commute when evaluated at different frequencies, by virtue of (A.2.11). It follows that products of these factors at the same or different frequencies are symmetric. From (11.6.1), we have[†]

$$\mathbb{H}_\pm(\omega) = \mathbb{H}_\mp(-\omega) = \overline{\mathbb{H}_\mp}(\omega), \tag{11.6.3}$$

[†] These relations allow us to show that

$$\overline{\mathbf{p}^t_\pm}(\omega) = \mathbf{p}^t_\pm(-\omega), \quad \omega \in \mathbb{R},$$

with the aid of (11.2.10) and (C.1.7).

which also hold for complex ω, where the rightmost term involves the complex conjugate of the functional form, leaving ω unchanged, or

$$\overline{\mathbb{H}_{\mp}}(\omega) = \overline{\mathbb{H}_{\mp}(\overline{\omega})}.$$

These relations are consistent with but more detailed than (11.1.11). They reduce the nonuniqueness of the factorization to an arbitrariness of sign on each eigenspace. Also, $\mathbb{M}_m$, given by $(11.2.39)_2$, can be expanded on this basis:

$$\mathbb{M}_m(\omega_1, \omega_2) = \sum_{k=1}^{m} M_k^{(m)}(\omega_1, \omega_2)\mathbb{B}^k,$$
$$M_k^{(m)}(\omega_1, \omega_2) = H_{k+}(\omega_1)H_{k-}(\omega_2).$$

In the basis $\{\mathbf{B}_k\}$ (defined after (7.1.36)) and $\{\mathbb{B}_k\}$, the individual components of each of the relevant quantities obey the relationships that hold in the scalar case. We can expand any member of Γ in this basis; in particular,

$$\mathbf{\Lambda}(t) = \sum_{k=1}^{m} \Lambda_k(t)\mathbf{B}^k, \qquad \mathbf{\Lambda}^t(s) = \sum_{k=1}^{m} \Lambda_k^t(s)\mathbf{B}^k,$$
$$\mathbf{\Lambda}_r^t(s) = \sum_{k=1}^{m} \Lambda_{kr}^t(s)\mathbf{B}^k, \qquad \mathbf{\Lambda}_{r\pm}^t(\omega) = \sum_{k=1}^{m} \Lambda_{kr\pm}^t(\omega)\mathbf{B}^k.$$

Scalar quadratic forms, such as the memory-dependent part of a free energy or a rate of dissipation, are given by the sum of contributions to this quantity from each eigenspace. In particular,

$$\psi_m(t) - \phi(t) = \sum_{k=1}^{m} \psi_{mk}^{(k)}(t),$$

where $\psi_{mk}^{(k)}$ is the minimum free energy relating to the scalar problem for H_k. Such relations follow readily from the orthonormality and time (frequency) independence of the basis.

In the particular example discussed in Sect. 7.1.5, we can write any free energy (not just the minimum free energy) in the form

$$\psi(t) - \phi(t) = 5\psi_S(t) + \psi_{B1}(t) + \psi_{B2}(t) + 3\psi_q(t),$$

where $\psi_S(t)$, $\psi_{B1}(t)$, $\psi_{B2}(t)$, and $\psi_q(t)$ are the memory-dependent parts of the free energies corresponding to the scalar problems with relaxation function derivatives G_S', G_1', G_2', and V_m' and constants $G_{S\infty}, G_1$, and G_2. The coefficient 5 reflects the degeneracy or symmetry of the five-dimensional representation of the rotation group corresponding to shear deformation in a mechanically isotropic material, while the coefficient 3 results from thermal isotropy. For completely linear materials, we can include the equilibrium terms explicitly in these formulas.

11.7 The Minimum Free Energy for Sinusoidal Histories

Consider a history and current value $(\mathbf{\Lambda}^t, \mathbf{\Lambda}(t))$ defined by

$$\mathbf{\Lambda}(t) = \mathbf{C}e^{i\omega_- t} + \overline{\mathbf{C}}e^{-i\omega_+ t}, \quad \mathbf{\Lambda}^t(s) = \mathbf{\Lambda}(t-s), \tag{11.7.1}$$

where $\mathbf{C}$ is an amplitude in Γ and $\overline{\mathbf{C}}$ is its complex conjugate. Furthermore,

$$\omega_- = \omega_0 - i\eta, \quad \omega_+ = \overline{\omega_-}, \quad \omega_0 \in \mathbb{R}, \quad \eta \in \mathbb{R}^{++}.$$

The quantity η is introduced to ensure finite results in certain quantities. The quantity $\mathbf{\Lambda}^t_+$ has the form

$$\mathbf{\Lambda}^t_+(\omega) = \mathbf{C}\frac{e^{i\omega_- t}}{i(\omega + \omega_-)} + \overline{\mathbf{C}}\frac{e^{-i\omega_+ t}}{i(\omega - \omega_+)},$$

and the Fourier transform of the relative history $\mathbf{\Lambda}^t_r(s) = \mathbf{\Lambda}^t(s) - \mathbf{\Lambda}(t)$, namely $\mathbf{\Lambda}^t_{r+}(\omega)$, is given by

$$\mathbf{\Lambda}^t_{r+}(\omega) = \mathbf{\Lambda}^t_+(\omega) - \frac{\mathbf{\Lambda}(t)}{i\omega^-} = -\mathbf{C}\frac{\omega_-}{\omega^-}\frac{e^{i\omega_- t}}{i(\omega + \omega_-)} + \overline{\mathbf{C}}\frac{\omega_+}{\omega^-}\frac{e^{-i\omega_+ t}}{i(\omega - \omega_+)}. \tag{11.7.2}$$

From (7.1.17) and $(7.1.14)_5$, the generalized stress has the form

$$\begin{aligned} \mathbf{\Sigma}(t) &= \mathbf{\Sigma}_0(t) + \mathbf{\Sigma}_h(t), \\ \mathbf{\Sigma}_h(t) &= \int_0^\infty \mathbb{L}'(s)\mathbf{\Lambda}^t(s)\,ds \\ &= \mathbb{L}'_+(\omega_-)\mathbf{C}e^{i\omega_- t} + \mathbb{L}'_+(-\omega_+)\overline{\mathbf{C}}e^{-i\omega_+ t}, \\ \mathbf{\Sigma}_0(t) &= \mathbf{\Sigma}_e(t) + \mathbb{L}_0\mathbf{\Lambda}(t). \end{aligned} \tag{11.7.3}$$

The expression for $\mathbf{\Sigma}_h(t)$ reduces to that given in $(7.2.6)_3$ as $\eta \to 0$.

The work $W(t)$ done on the material to achieve the state $(\mathbf{\Lambda}^t, \mathbf{\Lambda}(t))$ is given by $(7.5.1)_1$, which in this context becomes

$$\begin{aligned} W(t) = \phi(t) &+ \frac{1}{2}\mathbf{\Lambda}(t)\cdot\mathbb{L}_0\mathbf{\Lambda}(t) \\ &+ \frac{1}{2}\left[\mathbf{C}\cdot\mathbb{L}'_+(\omega_-)\mathbf{C}e^{2i\omega_- t} + \overline{\mathbf{C}}\cdot\mathbb{L}'_+(-\omega_+)\overline{\mathbf{C}}e^{-2i\omega_+ t}\right] \\ &+ \overline{\mathbf{C}}\cdot[\omega_-\mathbb{L}'_+(-\omega_+) - \omega_+\mathbb{L}'_+(\omega_-)]\mathbf{C}\frac{e^{i(\omega_- - \omega_+)t}}{(\omega_- - \omega_+)}, \end{aligned} \tag{11.7.4}$$

where the symmetry of $\mathbb{L}_+$ has been used. This quantity diverges as $\eta \to 0$, as would be expected on physical grounds. Taking the limit $\eta \to 0$ in the terms that are convergent, we can write this in the form

$$\begin{aligned} W(t) = \phi(t) &+ \frac{1}{2}\mathbf{\Lambda}(t)\cdot\mathbb{L}_0\mathbf{\Lambda}(t) \\ &+ \frac{1}{2}\left[\mathbf{C}\cdot\mathbb{L}'_+(\omega_0)\mathbf{C}e^{2i\omega_0 t} + \overline{\mathbf{C}}\cdot\mathbb{L}'_+(-\omega_0)\overline{\mathbf{C}}e^{-2i\omega_0 t}\right] \\ &+ \overline{\mathbf{C}}\cdot\left[\mathbb{L}'_c(\omega_0) - \omega_0\frac{d}{d\omega_0}\mathbb{L}'_c(\omega_0) - 2\omega_0 t\mathbb{L}'_s(\omega_0)\right]\mathbf{C} \\ &- \mathbf{C}\cdot\mathbb{L}'_s(\omega_0)\mathbf{C}\frac{\omega_0}{\eta}, \end{aligned} \tag{11.7.5}$$

on using (7.2.1). The divergence is associated with $\mathbb{L}'_s$, which is physically reasonable.

We shall require the relation

$$\overline{\mathbb{H}_\pm(\omega)} = \mathbb{H}_\pm(-\overline{\omega}) \tag{11.7.6}$$

for complex ω, which follows from (11.6.3). The minimum free energy $\psi_m(t)$ is given by (11.2.17). Using (11.7.2), we evaluate the integral in (11.2.9) by closing the contour on $\Omega^{(+)}$ to obtain

$$\mathbf{p}^t_+(\omega) = -\left[\frac{e^{i\omega_- t}}{i(\omega+\omega_-)}\mathbb{H}_-(-\omega_-)\mathbf{C} + \frac{e^{-i\omega_+ t}}{i(\omega-\omega_+)}\mathbb{H}_-(\omega_+)\overline{\mathbf{C}}\right] \tag{11.7.7}$$

and

$$\mathbf{p}^t_-(\omega) = \mathbb{H}_-(\omega)\boldsymbol{\Lambda}^t_{r+}(\omega) + \mathbf{p}^t_+(\omega).$$

The expression for $\psi_m(t)$ can be obtained from $(11.2.21)_2$ combined with (11.7.4). From (11.7.7), we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_+(\omega)|^2 d\omega = &-\frac{ie^{2i\omega_- t}}{2\omega_-}\mathbf{C}\cdot\mathbb{H}^2_-(-\omega_-)\mathbf{C} + \frac{ie^{-2i\omega_+ t}}{2\omega_+}\overline{\mathbf{C}}\cdot\mathbb{H}^2_-(\omega_+)\overline{\mathbf{C}}\\ &-\frac{2ie^{i(\omega_- - \omega_+)t}}{(\omega_- - \omega_+)}\overline{\mathbf{C}}\cdot\mathbb{H}_-(\omega_+)\mathbb{H}_-(-\omega_-)\mathbf{C},\end{aligned} \tag{11.7.8}$$

where (11.7.6) has been used. It will be observed that the last term diverges in the limit $\eta \to 0$. The quantity given by (11.7.8) in the limit $\eta \to 0$ is in fact the total dissipation over the history, given by (11.2.21), so this divergence is an expression of a physically obvious fact. Its derivative is the rate of dissipation.

Taking the limit $\eta \to 0$ in the convergent terms, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_+(\omega)|^2 d\omega = &-\frac{ie^{2i\omega_0 t}}{2\omega_0}\mathbf{C}\cdot\mathbb{H}^2_-(-\omega_0)\mathbf{C} + \frac{ie^{-2i\omega_0 t}}{2\omega_0}\overline{\mathbf{C}}\cdot\mathbb{H}^2_-(\omega_0)\overline{\mathbf{C}}\\ &-\overline{\mathbf{C}}\cdot[2t\mathbb{H}(\omega_0)+\mathbb{D}(\omega_0)]\mathbf{C} + \frac{1}{\eta}\overline{\mathbf{C}}\cdot\mathbb{H}(\omega_0)\mathbf{C},\end{aligned}$$

where $\mathbb{D}$ is defined by (11.2.45).

From (11.7.4), (11.7.8), and $(11.2.21)_2$, taking the limit $\eta \to 0$, we obtain

$$\begin{aligned}\psi_m(t) = &\,\phi(t) + \frac{1}{2}\boldsymbol{\Lambda}(t)\cdot\mathbb{L}_0\boldsymbol{\Lambda}(t)\\ &+\mathbf{C}\cdot\mathbb{B}_1(\omega_0)\mathbf{C}e^{2i\omega_0 t} + \overline{\mathbf{C}}\cdot\overline{\mathbb{B}}_1(\omega_0)\overline{\mathbf{C}}e^{-2i\omega_0 t} + \overline{\mathbf{C}}\cdot\mathbb{B}_2(\omega_0)\mathbf{C},\end{aligned} \tag{11.7.9}$$

where

$$\begin{aligned}\mathbb{B}_1(\omega_0) &= \frac{1}{2}\left[\mathbb{L}'_+(\omega_0) + \frac{i}{\omega_0}\mathbb{H}^2_-(-\omega_0)\right],\\ \mathbb{B}_2(\omega_0) &= \mathbb{L}'_c(\omega_0) - \omega_0\frac{d}{d\omega_0}\mathbb{L}'_c(\omega_0) + \mathbb{D}(\omega_0).\end{aligned} \tag{11.7.10}$$

The divergent terms and those proportional to t cancel. The most interesting contribution to ψ_m is the rightmost term of (11.7.9), which gives the average over a time cycle

$$(\psi_m)_{av} = \overline{\mathbf{C}} \cdot \mathbb{B}_2(\omega_0)\mathbf{C}.$$

We can express this quantity in terms of $\mathbb{H}$ and its factors, using (7.2.26). Note that $\mathbb{B}_2$ must be a nonnegative quantity in general for all $\omega \in \mathbb{R}$. We recall from (11.2.45) that $\mathbb{D}$ is nonnegative for all $\omega \in \mathbb{R}$.

The rate of dissipation is given by (11.2.25) and $(11.2.22)_3$. Using (11.7.2) and closing on Ω^+, we find that

$$\mathbf{K}(t) = \mathbb{H}_-(-\omega_0)\mathbf{C}e^{i\omega_0 t} + \mathbb{H}_-(\omega_0)\overline{\mathbf{C}}e^{-i\omega_0 t}$$

on taking $\eta \to 0$. Therefore,

$$D(t) = \mathbf{C} \cdot \mathbb{H}_-^2(-\omega_0)\mathbf{C}e^{2i\omega_0 t} + \overline{\mathbf{C}} \cdot \mathbb{H}_-^2(\omega_0)\overline{\mathbf{C}}e^{-2i\omega_0 t} + 2\overline{\mathbf{C}} \cdot \mathbb{H}(\omega)\mathbf{C}. \qquad (11.7.11)$$

One may check that (11.2.19) holds, using (11.7.1), (11.7.3) in the limit $\eta \to 0$, (11.7.9), and (11.7.11).

Similar results apply to the free energies and dissipations in Sect. 16.4.

11.8 Example: Viscoelastic Materials

We write out the main results of the present chapter for perhaps the most important special case, namely the isothermal theory, where temperature variation over time and space is neglected. Only the mechanical equations are relevant.‡

Recall (8.1.16), (8.1.18), and the fact that $\mathbb{G}'_s$ is an odd function of ω. Following (7.2.22) and (11.1.11), let us define $\mathbb{H} \in \mathrm{Lin(Sym)}$ by

$$\mathbb{H}(\omega) = -\omega\mathbb{G}'_s(\omega) = \mathbb{H}_+(\omega)\mathbb{H}_-(\omega) \geq \mathbf{0}, \quad \omega \in \mathbb{R}, \qquad (11.8.1)$$

where

$$\mathbb{H}(\infty) = -\mathbb{G}'(0). \qquad (11.8.2)$$

We have kept the same notation for simplicity. The work function, giving the amount of mechanical work required to achieve the state $(\mathbf{E}^t, \mathbf{E}(t))$, has the form (cf. (7.5.1), (7.5.3), and (7.5.7))

‡ The results of Sect. 13.1.1 overlap to some degree with those in this section. The former results are, however, derived specifically for completely linear viscoelastic solids (as discussed in Sects. 8.1–8.7), using a somewhat different but equivalent methodology to that developed in the present chapter.

$$\begin{aligned}
\widetilde{W}(\mathbf{E}^t, \mathbf{E}(t)) &= W(t) = \phi(-\infty) + \int_{-\infty}^{t} \widehat{\mathbf{S}}(u) \cdot \dot{\mathbf{E}}(u) du \\
&= \phi(t) + \frac{1}{2} \int_0^\infty \int_0^\infty \mathbb{G}_{12}(|s-u|) \mathbf{E}_r^t(u) \cdot \mathbf{E}_r^t(s) du ds \\
&= \phi(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathbf{E}_{r+}^t}(\omega) \cdot \mathbb{H}(\omega) \mathbf{E}_{r+}^t(\omega) d\omega, \\
\mathbf{E}_{r+}^t(\omega) &= \mathbf{E}_+^t(\omega) - \frac{\mathbf{E}(t)}{i\omega^-},
\end{aligned} \tag{11.8.3}$$

where the constitutive equation (8.1.1) has been used. Again, keeping some notation from (11.2.8),

$$\mathbb{H}_-(\omega)\mathbf{E}_{r+}^t(\omega) = \mathbf{p}_-^t(\omega) - \mathbf{p}_+^t(\omega),$$

where

$$\mathbf{p}_\pm^t(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathbb{H}_-(\omega')\mathbf{E}_{r+}^t(\omega')}{\omega' - \omega^\mp} d\omega', \tag{11.8.4}$$

which is the analogue of (11.2.10). We note that

$$\mathbf{E}_{r+}^t(\omega) \sim \omega^{-2} \tag{11.8.5}$$

at large ω and write the analogue of (7.2.30) as

$$\frac{d}{dt}\mathbf{E}_+^t(\omega) = -i\omega \mathbf{E}_{r+}^t(\omega), \qquad \frac{d}{dt}\mathbf{E}_{r+}^t(\omega) = -i\omega \mathbf{E}_{r+}^t(\omega) - \frac{\dot{\mathbf{E}}(t)}{i\omega^-}.$$

The Fourier transform of the relative optimal future continuation has the form

$$\mathbf{E}_m^t(\omega) = -[\mathbb{H}_-(\omega)]^{-1} \, \mathbf{p}_-^t(\omega), \tag{11.8.6}$$

which is a special case of (11.2.14). Its time-domain version $\mathbf{E}_{ro}^t$ is given by

$$\mathbf{E}_{ro}^t(u) = \mathbf{E}_o^t(u) - \mathbf{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}_m^t(\omega) e^{i\omega u} \, d\omega. \tag{11.8.7}$$

The work function, given by (11.8.3), retains the form of (11.2.16),

$$W(t) = \phi(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[|\mathbf{p}_-^t(\omega)|^2 + |\mathbf{p}_+^t(\omega)|^2 \right] d\omega, \tag{11.8.8}$$

but with $\mathbf{p}_\pm^t$ defined by (11.8.4), while the minimum free energy (11.2.17) retains the form

$$\psi_m(t) = \phi(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathbf{p}_-^t(\omega)|^2 d\omega. \tag{11.8.9}$$

The relations in (11.2.18) become

$$\begin{aligned}\psi_m(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathbf{E}^t_m}(\omega)\cdot\mathbb{H}(\omega)\mathbf{E}^t_m(\omega)d\omega \\ &= \phi(t) + \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{G}_{12}(|s-u|)\mathbf{E}^t_{ro}(u)\cdot\mathbf{E}^t_{ro}(s)duds \\ &= \phi(t) + \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{G}(|s-u|)\dot{\mathbf{E}}^t_{o}(u)\cdot\dot{\mathbf{E}}^t_{o}(s)duds \\ &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\dot{\mathbf{E}}^t_m}(\omega)\cdot\frac{\mathbb{H}(\omega)}{\omega^2}\dot{\mathbf{E}}^t_m(\omega)d\omega,\end{aligned} \tag{11.8.10}$$

where $\mathbf{E}^t_{ro}$ is defined by (11.8.7). Relation (11.2.19) reduces to

$$\dot{\psi}_m(t) + D_m(t) = \widehat{\mathbf{S}}(t)\cdot\dot{\mathbf{E}}(t), \tag{11.8.11}$$

where D_m is the rate of mechanical dissipation corresponding to the minimum free energy and must be nonnegative by virtue of the second law. Let us assume that the material is undisturbed in the distant past. Integrating (11.8.11) up to time t gives a special case of (11.2.20):

$$\psi_m(t) + \mathcal{D}_m(t) = W(t),$$

where

$$\mathcal{D}_m(t) = \int_{-\infty}^{t} D_m(s)ds$$

is the total mechanical dissipation up to time t, corresponding to the minimum free energy. We have, from (11.8.8) and (11.8.9) (see (11.2.21)),

$$\mathcal{D}_m(t) = W(t) - \psi_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}^t_+(\omega)|^2\, d\omega \geq 0. \tag{11.8.12}$$

The rate of dissipation has the form (11.2.25) or

$$D_m(t) = |\mathbf{K}(t)|^2, \tag{11.8.13}$$

where $\mathbf{K}$ is given by a special case of $(11.2.22)_3$:

$$\mathbf{K}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega)d\omega. \tag{11.8.14}$$

Certain properties of the optimal future continuation are derived in Sect. 11.5. We summarize them here for the isothermal case. Relation (11.5.1) becomes

$$\mathbf{E}^t_m(\omega) \underset{\omega\to\infty}{\longrightarrow} \frac{\mathbb{H}^{-1}_{\frac{1}{2}}\mathbf{K}(t)}{i\omega},$$

while (11.5.2) and (11.5.3) take the form

$$\mathbf{E}^t_{ro}(0) = -\mathbb{H}^{-1}_{\frac{1}{2}}\mathbf{K}(t), \qquad \mathbf{E}^t_{o}(0) = \mathbf{E}(t) - \mathbb{H}^{-1}_{\frac{1}{2}}\mathbf{K}(t). \tag{11.8.15}$$

At low frequencies and large times, we have, from (11.5.4) and (11.5.5),

$$\mathbf{E}^t_{ro}(-\infty) = \frac{\mathbb{H}_1(0)^{-1}}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}_1(\omega')\mathbf{E}^t_{r+}(\omega')d\omega'$$

and

$$\mathbf{E}^t_o(-\infty) = \mathbf{E}^t_{ro}(-\infty) + \mathbf{E}(t). \tag{11.8.16}$$

If we apply the assumptions of Sect. 11.6, that the eigenspaces of $\mathbb{G}$ are time-independent, any problem on each of the individual eigenspaces is in effect a scalar problem, like the one dealt with in the next section, where, however, the subscript indicating the eigenspace is omitted.

11.9 Explicit Forms of the Minimum Free Energy for Discrete-Spectrum Materials

We now consider the general results of earlier sections for a particular class of response functions, namely discrete-spectrum scalar models. Here, as above and in later sections dealing with the scalar case, we shall continue to use notation often associated with viscoelastic materials. We replace $\mathbf{E}(t)$ by $E(t)$ and $\mathbb{G}(s)$ by $G(s)$. However, it must be emphasized that the results apply to any time-independent eigenspace of $\mathbb{G}$ (see the comment at the end of Sect. 11.8) or indeed of $\mathbb{L}$, as introduced in Sect. 11.6.

Let the relaxation function $G(t)$ have the form

$$G(t) = G_\infty + \sum_{i=1}^{n} G_i e^{-\alpha_i t}, \quad G_\infty \geq 0, \tag{11.9.1}$$

where n is a positive integer, the inverse decay times α_i, $i = 1, 2, \ldots, n$, are positive, and the coefficients G_i are also generally assumed to be positive. We arrange that $\alpha_1 < \alpha_2 < \alpha_3 \ldots$ It follows that

$$G'(t) = \sum_{i=1}^{n} g_i e^{-\alpha_i t}, \qquad g_i = -\alpha_i G_i < 0,$$

and

$$G'_+(\omega) = \sum_{i=1}^{n}\frac{g_i}{\alpha_i + i\omega}, \qquad G'_c(\omega) = \sum_{i=1}^{n}\frac{\alpha_i g_i}{\alpha_i^2 + \omega^2}, \qquad G'_s(\omega) = \omega\sum_{i=1}^{n}\frac{g_i}{\alpha_i^2 + \omega^2},$$

recalling (8.1.16). Thus, from $(11.8.1)_1$,

$$H(\omega) = -\omega^2\sum_{i=1}^{n}\frac{g_i}{\alpha_i^2 + \omega^2} \geq 0, \tag{11.9.2}$$

and (11.8.2) can easily be checked. Observe that $f(z) = H(\omega)$, $z = -\omega^2$, has simple poles at α_i^2, $i = 1, 2, \ldots, n$. It will therefore have zeros at γ_i^2, $i = 2, 3, \ldots, n$, where

$$\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < \gamma_3^2 \cdots , \tag{11.9.3}$$

by virtue of Remark B.1.1. It will have no more than one zero between each pole because $H(\infty)$ is a finite constant. The function $f(z)$ also vanishes at $\gamma_1 = 0$. Therefore, H is a rational function of the form

$$H(\omega) = H_\infty \prod_{i=1}^{n} \left\{ \frac{\gamma_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\}, \tag{11.9.4}$$

and either by inspection or by applying the general formula (11.1.15), one can show that

$$\begin{aligned} H_+(\omega) &= h_\infty \prod_{i=1}^{n} \left\{ \frac{\omega - i\gamma_i}{\omega - i\alpha_i} \right\}, \\ H_-(\omega) &= h_\infty \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \qquad h_\infty = [H_\infty]^{1/2}. \end{aligned} \tag{11.9.5}$$

By considering the residue at each pole, we find that

$$\begin{aligned} H_-(\omega) &= h_\infty \left[1 + i \sum_{i=1}^{n} \frac{R_i}{\omega + i\alpha_i} \right], \quad H_+(\omega) = \overline{H_-}(\omega), \\ R_i &= (\gamma_i - \alpha_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left\{ \frac{\gamma_j - \alpha_i}{\alpha_j - \alpha_i} \right\}. \end{aligned} \tag{11.9.6}$$

It follows from (11.9.6) and the fact that H_- vanishes at $\omega = 0$ that

$$\sum_{i=1}^{n} \frac{R_i}{\alpha_i} = -1. \tag{11.9.7}$$

Therefore, we can also write $H_-(\omega)$ in the form

$$H_-(\omega) = -h_\infty \omega \sum_{i=1}^{n} \frac{R_i}{\alpha_i(\omega + i\alpha_i)}. \tag{11.9.8}$$

The quantity $p_-^t(\omega)$, defined by the scalar version of (11.8.4), may be evaluated by closing on $\Omega^{(-)}$, giving, with the aid of $(B.1.15)_3$,

$$p_-^t(\omega) = ih_\infty \sum_{i=1}^{n} \frac{R_i E_{r+}^t(-i\alpha_i)}{\omega + i\alpha_i}. \tag{11.9.9}$$

The quantities $E_{r+}^t(-i\alpha_i)$ are real. Also,

$$\begin{aligned} p_+^t(\omega) &= p_-^t(\omega) - H_-(\omega) E_{r+}^t(\omega) \\ &= ih_\infty \sum_{i=1}^{n} R_i \frac{[E_{r+}^t(-i\alpha_i) - E_{r+}^t(\omega)]}{\omega + i\alpha_i} - h_\infty E_{r+}^t(\omega), \end{aligned}$$

which has singularities at those of $E^t_{r+}(\omega)$ in $\Omega^{(+)}$ but none in $\Omega^{(-)}$. These explicit relations for $p^t_\pm$ allow their analytic continuation to the whole complex plane, excluding singular points, as discussed in Sect. B.1.2.

From (11.8.6), (11.9.5), and (11.9.9),

$$E^t_m(\omega) = -i \sum_{i=1}^{n} J_i(\omega) R_i E^t_{r+}(-i\alpha_i),$$
$$J_i(\omega) = \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n}(\omega + i\alpha_j)}{\prod_{j=1}^{n}(\omega + i\gamma_j)} = \sum_{l=1}^{n} \frac{Q_{il}}{\omega + i\gamma_l}, \tag{11.9.10}$$

where

$$Q_{il} = \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n}(\gamma_l - \alpha_j)}{\prod_{\substack{j=1 \\ j\neq l}}^{n}(\gamma_l - \gamma_j)},$$

so that

$$E^t_m(\omega) = -i \sum_{l=1}^{n} \frac{B^t_l}{\omega + i\gamma_l}, \qquad B^t_l = \sum_{i=1}^{n} R_i Q_{il} E^t_{r+}(-i\alpha_i). \tag{11.9.11}$$

We conclude that the relative optimal deformation as defined in (11.8.7) has the form

$$E^t_{ro}(s) = -\sum_{l=1}^{n} B^t_l e^{\gamma_l s} = -B^t_1 - \sum_{l=2}^{n} B^t_l e^{\gamma_l s}, \quad s < 0,$$

since $\gamma_1 = 0$. It follows that

$$E^t_{ro}(-\infty) = -B^t_1,$$

which is a special case of (11.5.5), as may be seen by using (11.9.8) to determine the form of $H_1(\omega)$ (defined by the scalar version of (11.5.4)) and $(11.9.5)_2$ to write $H_1(0)$. By considering $\omega J_i(\omega)$ for large ω, it can be deduced that

$$\sum_{l=1}^{n} Q_{il} = 1,$$

so that

$$E^t_{ro}(0) = -\sum_{l=1}^{n} B^t_l = -\sum_{i=1}^{n} R_i E^t_{r+}(-i\alpha_i). \tag{11.9.12}$$

Relation (11.9.12) follows from $(11.9.11)_2$ and (C.2.16). From (11.8.14) and (11.9.6), we have

$$K(t) = h_\infty \left[\sum_{i=1}^{n} R_i E^t_{r+}(-i\alpha_i) \right], \tag{11.9.13}$$

since the constant term in H_- yields zero by (B.1.15) and (11.8.5). Observe that (11.9.12) and (11.9.13) agree with (11.8.15).

We deduce from (11.9.9) and (11.8.9) that

$$\begin{aligned}\psi_m(t) &= \phi(t) + H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{\alpha_i + \alpha_j} E^t_{r+}(-i\alpha_i) E^t_{r+}(-i\alpha_j) \\ &= \phi(t) + \frac{1}{2}\int_0^\infty ds_1 \int_0^\infty ds_2 E^t_r(s_1) G_{12}(s_1, s_2) E^t_r(s_2),\end{aligned} \tag{11.9.14}$$

where the reality of $E^t_{r+}(-i\alpha_i)$ has been used. The kernel G_{12} is given by

$$G_{12}(s_1, s_2) = 2H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{\alpha_i + \alpha_j} e^{-\alpha_i s_1 - \alpha_j s_2}.$$

The solution of this partial differential equation under conditions (8.6.6) is

$$G(s_1, s_2) = G_\infty + 2H_\infty \sum_{i,j=1}^{n} \frac{R_i R_j}{(\alpha_i + \alpha_j)\alpha_i \alpha_j} e^{-\alpha_i s_1 - \alpha_j s_2}.$$

The relation (8.6.13) reducing in the scalar case to $G(0, s) = G(s)$, where the latter quantity is given by (11.9.1), can be confirmed with the aid of the identity

$$\sum_{j=1}^{n} \frac{R_j}{(\alpha_i + \alpha_j)\alpha_j} = -\frac{g_i}{2R_i H_\infty}, \tag{11.9.15}$$

which follows from (11.9.7) and the identity

$$\sum_{j=1}^{n} \frac{R_j}{\alpha_i + \alpha_j} = -1 + \frac{\alpha_i g_i}{2R_i H_\infty},$$

which in turn can be deduced by comparing the product $H_+(\omega)H_-(\omega)$ given by (11.9.6) near poles of $H_+(\omega)$ or $H_-(\omega)$ with $H(\omega)$ given by (11.9.2).

It is shown in [158] that (11.9.14) agrees with the expression by Breuer and Onat [42] for the maximum recoverable work. Noting (11.8.13) and (11.9.13), we see that

$$D_m(t) = H_\infty \left[\sum_{i=1}^{n} R_i E^t_{r+}(-i\alpha_i)\right]^2 = H_\infty \left[\int_0^\infty \sum_{i=1}^{n} R_i e^{-\alpha_i s} E^t_r(s) ds\right]^2. \tag{11.9.16}$$

For $n = 1$,

$$\psi_m(t) = \phi(t) + \frac{1}{2} H_\infty \alpha |E^t_{r+}(-i\alpha)|^2 = \phi(t) + \frac{1}{2} H_\infty \alpha \left[\int_0^\infty E^t_r(s) e^{-\alpha s} ds\right]^2, \tag{11.9.17}$$

which can be shown to agree with the result of Day [87]; see also (10.2.15). Finally, (11.9.16) becomes

$$D_m(t) = H_\infty |\alpha E^t_{r+}(-i\alpha)|^2 = H_\infty \left[\alpha \int_0^\infty e^{-\alpha s} E^t_r(s) ds\right]^2. \tag{11.9.18}$$