1

Introduction to Continuum Mechanics

1.1 Introduction

In this initial chapter, we introduce various fundamentals: description of deformation, definition and interpretation of the strain and stress tensors, balance laws, and general restrictions on constitutive equations. These provide the foundation for later developments.

A number of excellent, indeed hardly to be bettered, presentations of these basic topics exist in the literature, notably in [188, 205, 251, 313, 314]. Several formulations of standard arguments in this chapter and the next are based on those in [188, 251]. Other relevant texts are [281], the recent work [23], and the review [262].

An introduction to some notation and results relating to finite-dimensional vector spaces required in this and later chapters is given in Sect. A.2.

1.2 Kinematics

1.2.1 Continuous Bodies: Deformations—Strain Tensors

We will consider bodies the mass of which is distributed continuously. Moreover, a given body will occupy different regions at different times, but none of these regions will be intrinsically associated with the body. Thus, formally, a *continuous body* $\mathcal{B}$ is a set of material points $\mathbf{X}, \mathbf{Y}, \ldots$ endowed with a structure defined by a class Φ of one-to-one mappings $\varphi : \mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is the three-dimensional Euclidean space, such that

(i) $\varphi(\mathcal{B})$ is a Kellogg regular region;*

* By a Kellogg regular region, we mean a domain of the Euclidean space $\mathcal{E}$ bounded by a union of a finite number of surfaces of class C^1. A more formal definition of a subbody is given in [32, 253, 278] (see also [2]).

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(ii) if $\varphi, \psi \in \Phi$, then the function $\lambda = \varphi \circ \psi^{-1} : \psi(\mathcal{B}) \to \varphi(\mathcal{B}) \in C^1(\psi(\mathcal{B}))$ is called a *deformation* (of class C^1) of $\mathcal{B}$ from $\psi(\mathcal{B})$ to $\varphi(\mathcal{B})$;

(iii) if $\varphi \in \Phi$ and $\lambda : \varphi(\mathcal{B}) \to \mathcal{E}$ is a deformation of class C^1, then the mapping $\lambda \circ \varphi$ is also in Φ.

The functions φ are referred to as localizations of $\mathcal{B}$, and they determine the possible configurations of the body in the space $\mathcal{E}$. A localization provides at any material point $\mathbf{X} \in \mathcal{B}$ the corresponding geometric point $\mathbf{x} = \varphi(\mathbf{X}) \in \mathcal{E}$.

The hypotheses (i)–(iii) introduce a unique structure of a differential variety on $\mathcal{B}$.†

The set Φ of all possible localizations of $\mathcal{B}$ allows us to locate $\mathcal{B}$ in $\mathcal{E}$, as well as to define the internal constraints of material systems. We consider as an example a rigid body for which the class Φ must be defined so that for each pair $\varphi_1, \varphi_2 \in \Phi$, we have

$$d(\varphi_1(\mathbf{X}),\ \varphi_1(\mathbf{Y})) = d(\varphi_2(\mathbf{X}),\ \varphi_2(\mathbf{Y}))$$

for all $\mathbf{X}, \mathbf{Y} \in \mathcal{B}$, where d is the metric of the Euclidean space $\mathcal{E}$.

Moreover, for any continuous body $\mathcal{B}$, it is possible to determine a class $\mathcal{S}$ of subbodies $A, B, C, \ldots$ of $\mathcal{B}$, characterized by the following properties:

(a) $\mathcal{B} \in \mathcal{S}$;

(b) any element $A \in \mathcal{S}$ is such that $\varphi(A)$ is a Kellogg regular region of $\mathcal{E}$, for any $\varphi \in \Phi$.‡

On the class $\mathcal{S}$ of subbodies, it is possible to define a measure that allows us to give a definition of the density and of the mass.

Definition 1.2.1. *The mass is a measure $M : \mathcal{S} \to \mathbb{R}^+$ absolutely continuous with respect to the ordinary volume measure; that is, for each $\varphi \in \Phi$, there is an integrable function $\hat{\rho}_\varphi : \varphi(\mathcal{B}) \to \mathbb{R}^+$, the density of mass, such that the mass relative to A is*

$$M(A) = \int_{\varphi(A)} \hat{\rho}_\varphi(\mathbf{x})\, dv,$$

for all $A \in \mathcal{S}$.

A motion of $\mathcal{B}$ with respect to a fixed observer O is a sufficiently regular function§

$$\tilde{\boldsymbol{\chi}} : \mathcal{B} \times I \to \mathcal{E}, \tag{1.2.1}$$

† In other words, the body $\mathcal{B}$ does not identify itself with a particular configuration, but with the set of all possible configurations it can assume and hence with a differential variety.

‡ The given definition for a subbody is independent of the chosen localization φ. In fact, if ψ is another localization, then the transformation $\lambda = \varphi \circ \psi^{-1} : \psi(\mathcal{B}) \to \varphi(\mathcal{B})$ possesses an inverse of class C^1. Therefore, if $\varphi(A)$ is a regular region, then $\psi(A)$ will be a regular region of $\mathcal{E}$.

§ With respect to each context, the condition of being sufficiently regular may have various senses. For our purposes, the function $\boldsymbol{\chi}$ is assumed to be twice continuously differentiable in the domain of existence.

where $I \subset \mathbb{R}$ is a time interval.

In what follows, we will identify the body $\mathcal{B}$ with one of its particular configurations, namely the reference configuration $\varphi_0(\mathcal{B})$ (see Fig. 1.1). Moreover, the function $\tilde{\chi}$ is such that for each $t \in I$, the new function $\tilde{\chi}_t : \varphi_0(\mathcal{B}) \to \varphi_t(\mathcal{B})$, which represents the deformation of the body $\mathcal{B}$ from $\varphi_0(\mathcal{B})$ to $\varphi_t(\mathcal{B})$, has an inverse, that is, there exists a function

$$\tilde{\chi}_t^{-1} : \varphi_t(\mathcal{B}) \to \varphi_0(\mathcal{B}). \tag{1.2.2}$$

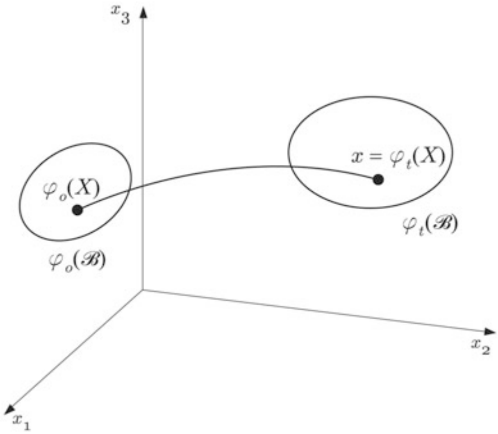

Fig. 1.1. The deformation of a body from $\varphi_0(\mathcal{B})$ to $\varphi_t(\mathcal{B})$

Hence $\tilde{\chi}_t$ is assumed to be one-to-one. This hypothesis expresses the requirement that the body does not penetrate itself. Thus, two distinct points of the configuration $\varphi_0(\mathcal{B})$ must be distinct in all other configurations.

It is possible to write the transformations (1.2.1) and (1.2.2) in the following forms:

$$\begin{aligned} \mathbf{x} &= \tilde{\chi}(\mathbf{X}, t), \\ \mathbf{X} &= \tilde{\chi}^{-1}(\mathbf{x}, t). \end{aligned} \tag{1.2.3}$$

The function defined by $(1.2.3)_1$ represents the position occupied by the particle $\mathbf{X}$ at the instant t, while relation $(1.2.3)_2$ locates the particle $\mathbf{X}$ that occupies the point $\mathbf{x}$ at the instant t. The variables $(\mathbf{X}, t)$ are the *Lagrangian* or *material coordinates*, while $(\mathbf{x}, t)$ are the *Eulerian* or *spatial coordinates*. The relations in (1.2.3) demonstrate that it is possible to express any physical quantity $\mathcal{F}$ in terms of material or spatial coordinates by

$$\tilde{\mathcal{F}}(\mathbf{X}, t) = \tilde{\mathcal{F}}(\tilde{\chi}^{-1}(\mathbf{x}, t),\ t) = \hat{\mathcal{F}}(\mathbf{x}, t). \tag{1.2.4}$$

Definition 1.2.2. *The Lagrangian description is the description of motion in terms of the variables* $(\mathbf{X}, t)$*, while the Eulerian description is that referring to the variables* $(\mathbf{x}, t)$.

As an example we consider the velocity of a particle $\mathbf{X}$ at the instant t, defined as

$$\tilde{\mathbf{v}}(\mathbf{X}, t) = \frac{\partial \tilde{\chi}}{\partial t}(\mathbf{X}, t);$$

on the basis of relation $(1.2.3)_2$, it is possible to express such a quantity in terms of the Eulerian variables as

$$\hat{\mathbf{v}}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\tilde{\chi}^{-1}(\mathbf{x}, t),\ t). \tag{1.2.5}$$

Remark 1.2.3. The time derivative of a quantity $\mathcal{F}$ has different expressions, depending on the description. In fact, by direct differentiation with respect to t of (1.2.4), we obtain

$$\frac{\partial \tilde{\mathcal{F}}}{\partial t} = \frac{\partial \hat{\mathcal{F}}}{\partial t} + \nabla_{\mathbf{x}} \hat{\mathcal{F}} \cdot \mathbf{v}, \tag{1.2.6}$$

where $\nabla_{\mathbf{x}}$ is the spatial gradient operator. The partial derivative on the left is taken holding $\mathbf{X}$ fixed, while in that on the right, $\mathbf{x}$ is fixed.

The derivative $\frac{\partial \tilde{\mathcal{F}}}{\partial t}$ is the *material derivative* (or the total derivative), denoted by

$$\frac{d\hat{\mathcal{F}}}{dt} = \frac{\partial \tilde{\mathcal{F}}}{\partial t}. \tag{1.2.7}$$

If we choose as $\mathcal{F}$ the velocity $\mathbf{v}$, then, by virtue of (1.2.6), we have that the acceleration is given by

$$\mathbf{a} = \frac{\partial}{\partial t}\tilde{\mathbf{v}}(\mathbf{X}, t) = \frac{\partial \hat{\mathbf{v}}}{\partial t}(\mathbf{x}, t) + \nabla_{\mathbf{x}} \hat{\mathbf{v}}(\mathbf{x}, t)\, \mathbf{v}.$$

Definition 1.2.4. *The material gradient of deformation is the tensor*

$$\mathbf{F}(\mathbf{X}, t) = \nabla_{\mathbf{X}} \tilde{\chi}(\mathbf{X}, t), \qquad \text{that is,}\ \ F_{ij} = \frac{\partial \tilde{\chi}_i}{\partial X_j}, \tag{1.2.8}$$

where $\nabla_{\mathbf{X}}$ *is the material gradient operator. The velocity gradient is the tensor*

$$\mathbf{L}(\mathbf{X}, t) = \mathbf{L}(\tilde{\chi}(\mathbf{X}, t),\ t) = \nabla_{\mathbf{x}} \hat{\mathbf{v}}(\mathbf{x}, t). \tag{1.2.9}$$

Remark 1.2.5. If we set $\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial t}$, then

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}. \tag{1.2.10}$$

In fact, we have

$$\dot{\mathbf{F}} = \nabla_{\mathbf{X}} \tilde{\mathbf{v}} = \nabla_{\mathbf{x}} \hat{\mathbf{v}} \nabla_{\mathbf{X}} \tilde{\chi}. \tag{1.2.11}$$

Remark 1.2.6. The requirement that the body does not penetrate itself is expressed by the assumption that

$$\det(\mathbf{F}) = \det(\nabla_{\mathbf{X}} \tilde{\chi}) \neq 0.$$

Furthermore, a deformation with $\det(\nabla_{\mathbf{X}} \tilde{\chi}) < 0$ cannot be reached by a continuous process of deformation starting from the reference configuration, that is, by a continuous one-parameter family $\tilde{\chi}_\sigma$ $(0 \leq \sigma \leq 1)$ of deformations with $\tilde{\chi}_0$ the identity,

$\tilde{\chi}_1 = \tilde{\chi}$, and $\det(\nabla_{\mathbf{X}}\tilde{\chi}_\sigma)$ never zero. Indeed, since $\det(\nabla_{\mathbf{X}}\tilde{\chi}_\sigma)$ is strictly positive at $\sigma = 0$, it must be strictly positive for all σ. Thus, we require that

$$\det \mathbf{F} > \mathbf{0}. \tag{1.2.12}$$

The above discussion motivates the following definition.

Definition 1.2.7. *By a deformation of* $\mathcal{B}$*, we mean a smooth one-to-one mapping* $\tilde{\chi}$*, which maps* $\mathcal{B}$ *onto a closed region in* $\mathcal{E}$ *and satisfies (1.2.12). The vector*

$$\mathbf{u}(\mathbf{X}, t) = \tilde{\chi}(\mathbf{X}, t) - \mathbf{X}$$

represents the displacement of $\mathbf{X}$*. A deformation with* $\mathbf{F}$ *constant is called homogeneous.*

The geometric significance of the tensor $\mathbf{F}$ becomes clear on observing that

$$\tilde{\chi}(\mathbf{X}',\ t) - \tilde{\chi}(\mathbf{X}, t) = \nabla_{\mathbf{X}}\tilde{\chi}(\mathbf{X}, t)(\mathbf{X}' - \mathbf{X}) + \mathbf{o}\left(\left|\mathbf{X}' - \mathbf{X}\right|\right),$$

for all $\mathbf{X}'$ in a neighborhood of $\mathbf{X}$, so that we can write

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}. \tag{1.2.13}$$

Thus, the tensor $\mathbf{F}$ transforms the small quantity $d\mathbf{X}$ of the configuration $\varphi_0(\mathcal{B})$ into the small displacement $d\mathbf{x}$ of the configuration $\varphi_t(\mathcal{B})$ (see Fig. 1.2). Let

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1.2.14}$$

be the polar decomposition of $\mathbf{F}$ at a given point, where $\mathbf{R}$ represents the rotation tensor, $\mathbf{U}$ is the right stretch tensor, and $\mathbf{V}$ is the left stretch tensor for the deformation $\tilde{\chi}$. Thus, $\mathbf{R}(P)$ measures the local rigid rotation of points near P, while $\mathbf{U}(P)$ and $\mathbf{V}(P)$ measure local stretching from P. The tensors $\mathbf{U}(P)$ and $\mathbf{V}(P)$ are symmetric. Since $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$ involve the square roots of $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$, their computation is often difficult. For this reason we introduce the *right and left Cauchy–Green strain tensors* $\mathbf{C}$ and $\mathbf{B}$, defined by

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \tag{1.2.15}$$

and note that

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T, \qquad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T.$$

In components, we have

$$C_{ij} = \sum_{m=1}^{3} \frac{\partial\tilde{\chi}_m}{\partial X_i}\frac{\partial\tilde{\chi}_m}{\partial X_j}, \quad B_{ij} = \sum_{m=1}^{3} \frac{\partial\tilde{\chi}_i}{\partial X_m}\frac{\partial\tilde{\chi}_j}{\partial X_m}.$$

Since $\mathbf{Cu} \cdot \mathbf{v} = \mathbf{Fu} \cdot \mathbf{Fv}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $\mathbf{Cu} \cdot \mathbf{u} = \mathbf{Fu} \cdot \mathbf{Fu} > 0$ for all $\mathbf{u} \in V \backslash \{\mathbf{0}\}$, it follows that $\mathbf{C}$ is a symmetric and positive definite tensor (Sect. A.2.1).

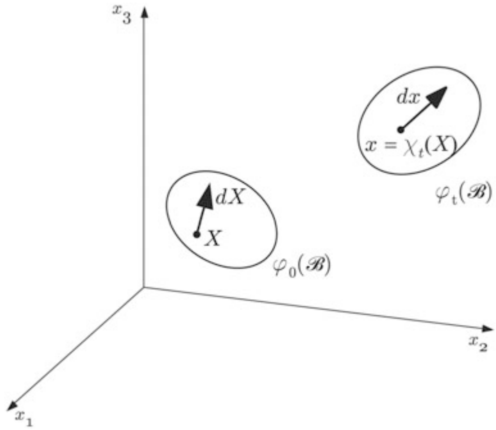

Fig. 1.2. The quantities $d\mathbf{X}$ and $d\mathbf{x}$ related by (1.2.13)

In view of the relation (1.2.12), it follows that $\mathbf{F}$ admits an inverse denoted by $\mathbf{F}^{-1}$, the *spatial gradient of deformation*, given by

$$\mathbf{F}^{-1} = \nabla_{\mathbf{x}} \mathbf{X}, \quad \text{or} \quad F^{-1}_{ij} = \frac{\partial \tilde{\chi}^{-1}_i}{\partial x_j}.$$

With this we can introduce the *right and left Cauchy strain tensors*, $\mathbf{c}$ and $\mathbf{b}$, defined by

$$\mathbf{c} = \left(\mathbf{F}^{-1}\right)^T \mathbf{F}^{-1}, \quad \mathbf{b} = \mathbf{F}^{-1}\left(\mathbf{F}^{-1}\right)^T, \tag{1.2.16}$$

or, in components,

$$c_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}^{-1}_m}{\partial x_i} \frac{\partial \tilde{\chi}^{-1}_m}{\partial x_j}, \quad b_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}^{-1}_i}{\partial x_m} \frac{\partial \tilde{\chi}^{-1}_j}{\partial x_m}.$$

If $d\mathbf{X}$ and $\delta\mathbf{X}$ are two displacement elements related to the point $\mathbf{X}$ that at the instant t are transformed into two displacements $d\mathbf{x}$ and $\delta\mathbf{x}$, respectively, related to the point $\mathbf{x} = \tilde{\chi}(\mathbf{X}, t)$, so that

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \quad \delta\mathbf{x} = \mathbf{F}\delta\mathbf{X}, \tag{1.2.17}$$

then

$$d\mathbf{x} \cdot \delta\mathbf{x} = d\mathbf{X} \cdot \mathbf{F}^T \mathbf{F} \delta\mathbf{X} = d\mathbf{X} \cdot \mathbf{C}\delta\mathbf{X}. \tag{1.2.18}$$

If the continuous body is rigid, then from the relation (1.2.18), we get necessarily $\mathbf{C} = \mathbf{1}$, the unit second-order tensor. When the body is not rigid, we can determine the elongation of the element $d\mathbf{X}$, associated with the tensor $\mathbf{C}$, by

$$|d\mathbf{x}|^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{C} d\mathbf{X}, \tag{1.2.19}$$

so that the relative elongation is

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = 2\mathbf{E} d\mathbf{X} \cdot d\mathbf{X} = 2\mathbf{e} d\mathbf{x} \cdot d\mathbf{x},$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \quad \text{and} \quad \mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{c}) \tag{1.2.20}$$

are *Green's strain tensor* and *Almansi's strain tensor*, respectively. Obviously, for a rigid deformation of the body, we have $\mathbf{E} = \mathbf{0}$ and $\mathbf{e} = \mathbf{0}$. Thus, the tensor $\mathbf{E}$ appears as a measure of Lagrangian deformation, while the tensor $\mathbf{e}$ represents a measure of Eulerian deformation.

In terms of the displacement vector $\mathbf{u}(\mathbf{X}, t) = \tilde{\chi}(\mathbf{X}, t) - \mathbf{X}$ or $\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \tilde{\chi}^{-1}(\mathbf{x}, t)$, the gradients of deformation are

$$\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{u} + \mathbf{1}, \qquad \mathbf{F}^{-1} = \mathbf{1} - \nabla_{\mathbf{x}} \mathbf{u},$$

and hence, from (1.2.20), the strain tensors are

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}\left[\nabla_{\mathbf{X}} \mathbf{u} + (\nabla_{\mathbf{X}} \mathbf{u})^T + (\nabla_{\mathbf{X}} \mathbf{u})^T \nabla_{\mathbf{X}} \mathbf{u}\right], \\ \mathbf{e} &= \frac{1}{2}\left[\nabla_{\mathbf{x}} \mathbf{u} + (\nabla_{\mathbf{x}} \mathbf{u})^T - (\nabla_{\mathbf{x}} \mathbf{u})^T \nabla_{\mathbf{x}} \mathbf{u}\right]. \end{aligned} \tag{1.2.21}$$

The relations in (1.2.21) are known as the *strain–displacement (or geometrical) relations*.

Remark 1.2.8. (**Geometric Significance of the Strain Tensors**) The components E_{11}, E_{22}, and E_{33} of the strain tensor $\mathbf{E}$ characterize the relative elongations in the directions of $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively, while the components E_{ij}, with $i \neq j$, represent a measure of the variation of angles due to the process of deformation.

To see this, we first note that the relation (1.2.19) can be written in the form

$$\frac{|d\mathbf{x}|^2}{|d\mathbf{X}|^2} = \mathbf{N} \cdot \mathbf{C}\mathbf{N},$$

where $\mathbf{N} = \frac{d\mathbf{X}}{|d\mathbf{X}|}$. If we set $\Lambda_{(\mathbf{N})} = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}$, then we have

$$\Lambda_{(\mathbf{N})} = (\mathbf{N} \cdot \mathbf{C}\mathbf{N})^{\frac{1}{2}} = \sqrt{\mathbf{N} \cdot (\mathbf{1} + 2\mathbf{E})}\,\mathbf{N}.$$

We further introduce the unit elongation $E_{(\mathbf{N})}$ in the direction of unit vector $\mathbf{N}$, by

$$E_{(\mathbf{N})} = \Lambda_{(\mathbf{N})} - 1 = \frac{|d\mathbf{x}| - |d\mathbf{X}|}{|d\mathbf{X}|},$$

so that when $\mathbf{N} = \mathbf{i}_1$, for example, then

$$E_{(\mathbf{i}_1)} = \sqrt{1 + 2E_{11}} - 1,$$

and hence E_{11} appears as a measure for the elongation in the direction of $\mathbf{i}_1$.

Let us further consider the vectors $d\mathbf{X}_1 = dX_1\mathbf{i}_1$ and $d\mathbf{X}_2 = dX_2\mathbf{i}_2$, and let $d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1$ and $d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$ be the corresponding vectors in the current configuration. Obviously, we have $d\mathbf{X}_1 \cdot d\mathbf{X}_2 = 0$, that is, the angle Θ_{12} between these vectors is $\frac{\pi}{2}$. On the other hand, the corresponding angle θ_{12} between the vectors $d\mathbf{x}_1$ and $d\mathbf{x}_2$ is given by

$$\cos\theta_{12} = \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|d\mathbf{x}_1|\,|d\mathbf{x}_2|} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}} = \frac{2E_{12}}{\sqrt{(1+2E_{11})(1+2E_{22})}},$$

and hence E_{12} appears as a measure of the variation of the angle Θ_{12} due to the deformation.

We now recall that given a tensor $\mathbf{S} \in \mathrm{Lin}(\mathbb{R}^3)$, the determinant of $\mathbf{S} - \lambda\mathbf{1}$ admits the representation (the Cayley–Hamilton theorem)

$$\det(\mathbf{S} - \lambda\mathbf{1}) = -\lambda^3 + I_1(\mathbf{S})\lambda^2 - I_2(\mathbf{S})\lambda + I_3(\mathbf{S})$$

for every $\lambda \in \mathbb{R}$, where

$$\begin{aligned} I_1(\mathbf{S}) &= tr\mathbf{S} = S_{11} + S_{22} + S_{33}, \\ I_2(\mathbf{S}) &= \frac{1}{2}\left[(tr\mathbf{S})^2 - tr\left(\mathbf{S}^2\right)\right], \\ I_3(\mathbf{S}) &= \det\mathbf{S}. \end{aligned} \tag{1.2.22}$$

We call $I_1(\mathbf{S})$, $I_2(\mathbf{S})$, and $I_3(\mathbf{S})$ the *principal invariants* of $\mathbf{S}$ and observe that they are invariant under changes of reference frames. We also note that any other invariant of $\mathbf{S}$ is a function of its principal invariants. When $\mathbf{S}$ is symmetric, the principal invariants are completely characterized by the spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ of $\mathbf{S}$. Indeed,

$$\begin{aligned} I_1(\mathbf{S}) &= \lambda_1 + \lambda_2 + \lambda_3, \\ I_2(\mathbf{S}) &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ I_3(\mathbf{S}) &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

By substituting $\mathbf{S}$ by $\mathbf{C}$, $\mathbf{c}$, $\mathbf{E}$, or $\mathbf{e}$ in the above relations, we can determine expressions for the principal invariants of these tensors and relationships between them. Thus, from (1.2.20) and (1.2.22), we obtain

$$\begin{aligned} I_1(\mathbf{C}) &= 3 + 2I_1(\mathbf{E}), \quad I_2(\mathbf{C}) = 3 + 4I_1(\mathbf{E}) + 4I_2(\mathbf{E}),\\ I_3(\mathbf{C}) &= 1 + 2I_1(\mathbf{E}) + 4I_2(\mathbf{E}) + 8I_3(\mathbf{E}),\\ I_1(\mathbf{c}) &= 3 - 2I_1(\mathbf{e}), \quad I_2(\mathbf{c}) = 3 - 4I_1(\mathbf{e}) + 4I_2(\mathbf{e}),\\ I_3(\mathbf{c}) &= 1 - 2I_1(\mathbf{e}) + 4I_2(\mathbf{e}) - 8I_3(\mathbf{e}). \end{aligned}$$

Moreover, we observe that the relations (1.2.15), $(1.2.16)_1$, and $(1.2.22)_3$ give

$$I_3(\mathbf{C}) = (\det \mathbf{F})^2, \quad I_3(\mathbf{c}) = \frac{1}{(\det \mathbf{F})^2},$$

and hence

$$I_3(\mathbf{C})I_3(\mathbf{c}) = 1.$$

Definition 1.2.9. *The stretching* **D** *(or velocity of deformation) is*

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \frac{1}{2}\left[\nabla_{\mathbf{x}}\hat{\mathbf{v}} + (\nabla_{\mathbf{x}}\hat{\mathbf{v}})^T\right], \tag{1.2.23}$$

where **L** *is defined by (1.2.9), while the spin* $\boldsymbol{\Omega}$ *is*

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) = \frac{1}{2}\left[\nabla_{\mathbf{x}}\hat{\mathbf{v}} - (\nabla_{\mathbf{x}}\hat{\mathbf{v}})^T\right]. \tag{1.2.24}$$

Thus, the stretching and the spin represent the symmetric and skew parts of the spatial gradient of velocity, respectively. Moreover, we have

$$\mathbf{L} = \mathbf{D} + \boldsymbol{\Omega}. \tag{1.2.25}$$

Note that

$$\begin{aligned} \frac{d}{dt}|d\mathbf{x}|^2 &= \frac{d}{dt}(d\mathbf{x}\cdot d\mathbf{x}) = 2\frac{d}{dt}(d\mathbf{x})\cdot d\mathbf{x}\\ &= 2\frac{d}{dt}(\mathbf{F}d\mathbf{X})\cdot d\mathbf{x} = 2\frac{d}{dt}(\mathbf{F})d\mathbf{X}\cdot d\mathbf{x}, \end{aligned}$$

and hence, in view of relation (1.2.10),

$$\begin{aligned} \frac{d}{dt}|d\mathbf{x}|^2 &= 2\mathbf{LF}d\mathbf{X}\cdot d\mathbf{x} = 2\mathbf{L}d\mathbf{x}\cdot d\mathbf{x} = 2d\mathbf{x}\cdot \mathbf{L}^T d\mathbf{x}\\ &= 2d\mathbf{x}\cdot\left(\frac{\mathbf{L}+\mathbf{L}^T}{2}\right)d\mathbf{x} = 2d\mathbf{x}\cdot\mathbf{D}d\mathbf{x}. \end{aligned} \tag{1.2.26}$$

Thus, the stretching **D** is a measure of the variation per unit time of the arc $(d\mathbf{x})^2$. Therefore, when $\mathbf{D} = \mathbf{0}$, then there is no change in $|d\mathbf{x}|^2$ over time.

Theorem 1.2.10. *A necessary and sufficient condition for a motion to be locally rigid is* $\mathbf{D} = \mathbf{0}$.

Proof. From Taylor's formula, the velocity in a neighborhood of the point $\mathbf{x}_0$ is

$$\hat{\mathbf{v}}(\mathbf{x}, t) = \hat{\mathbf{v}}(\mathbf{x}_0, t) + \nabla_{\mathbf{x}}\hat{\mathbf{v}}(\mathbf{x}_0, t)(\mathbf{x} - \mathbf{x}_0) + \mathbf{o}\,(|\mathbf{x} - \mathbf{x}_0|),$$

so that in view of the relation (1.2.25), we obtain

$$\hat{\mathbf{v}}(\mathbf{x}, t) = \hat{\mathbf{v}}(\mathbf{x}_0, t) + \mathbf{D}(\mathbf{x}_0, t)(\mathbf{x} - \mathbf{x}_0) + \boldsymbol{\Omega}(\mathbf{x}_0, t)(\mathbf{x} - \mathbf{x}_0) + \mathbf{o}\,(|\mathbf{x} - \mathbf{x}_0|).$$

Since $\boldsymbol{\Omega}$ is a skew-symmetric tensor, it follows that it is possible to associate with it the vector $\boldsymbol{\omega} = \Omega_{32}\mathbf{i}_1 + \Omega_{13}\mathbf{i}_2 + \Omega_{21}\mathbf{i}_3$, known as the *vorticity vector*, such that

$$\boldsymbol{\Omega}(\mathbf{x} - \mathbf{x}_0) = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_0). \tag{1.2.27}$$

Therefore, in a neighborhood of the point $\mathbf{x}_0$, neglecting terms of order higher than 1 in $|\mathbf{x} - \mathbf{x}_0|$, we have

$$\hat{\mathbf{v}}(\mathbf{x}, t) = \hat{\mathbf{v}}(\mathbf{x}_0, t) + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_0) + \mathbf{D}(\mathbf{x} - \mathbf{x}_0). \tag{1.2.28}$$

Thus, when $\mathbf{D} = \mathbf{0}$, the velocity is a composition of a translation and a rotation, which is a rigid motion.

Conversely, when the motion is rigid, (1.2.28) implies that $\mathbf{D} = \mathbf{0}$. □

Remark 1.2.11. In general, as can be seen from (1.2.28), the motion is a superposed rigid motion on an instantaneous extension.

From (1.2.24) and (1.2.27), we have

$$\boldsymbol{\omega} = \Omega_{32}\mathbf{i}_1 + \Omega_{13}\mathbf{i}_2 + \Omega_{21}\mathbf{i}_3 = \frac{1}{2}\left[\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{i}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{i}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{i}_3\right],$$

and hence

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla_{\mathbf{x}} \times \mathbf{v}).$$

If $\boldsymbol{\omega} = \mathbf{0}$, then we say that the motion is *irrotational*, and the velocity field has no vortices. In this case there exists a scalar field such that $\mathbf{v} = \nabla_{\mathbf{x}}\varphi$, as stated by the following theorem.

Theorem 1.2.12. *Let D be a given simply connected volume in $\mathbb{R}^3$ and $\mathbf{v} : D \to \mathbb{R}^3$ a function of class $C^1(D)$ that satisfies*

$$\nabla_{\mathbf{x}} \times \mathbf{v} = \mathbf{0} \quad \text{in } D.$$

Then there exists a function $\varphi : D \to \mathbb{R}$ such that

$$\mathbf{v} = \nabla_{\mathbf{x}}\varphi \quad \text{in } D.$$

Proof. Let S be an arbitrary surface contained in D, and let C be its relative boundary curve. Under appropriate regularity assumptions upon S and C, we can use Stokes's formula

$$\int_S (\nabla_{\mathbf{x}} \times \mathbf{v}) \cdot \mathbf{n} d\sigma = \int_C \mathbf{v} \cdot d\mathbf{x} = \int_C v_1 dx_1 + v_2 dx_2 + v_3 dx_3,$$

where, since $\nabla_{\mathbf{x}} \times \mathbf{v} = \mathbf{0}$, the differential form $v_1 dx_1 + v_2 dx_2 + v_3 dx_3$ is a total differential. Therefore, there exists a potential φ such that

$$d\varphi = v_1 dx_1 + v_2 dx_2 + v_3 dx_3,$$

which yields $\mathbf{v} = \nabla_{\mathbf{x}} \varphi$. □

Let us consider the transformation between the configuration $\varphi_0(\mathcal{B})$ and the configuration $\varphi_t(\mathcal{B})$ given by

$$\tilde{\chi}_t : \varphi_0(\mathcal{B}) \to \varphi_t(\mathcal{B}).$$

The Jacobian of the transformation,

$$J(\mathbf{X}, t) = \det\left(\frac{\partial \tilde{\chi}_t}{\partial \mathbf{X}}\right) = \det \mathbf{F}(\mathbf{X}, t),$$

is a measure of volume change due to the deformation. If we denote by dv_0 a volume element in the configuration $\varphi_0(\mathcal{B})$ and by dv_t the corresponding volume element in the configuration $\varphi_t(\mathcal{B})$, then we have

$$dv_t = J dv_0. \tag{1.2.29}$$

Theorem 1.2.13. *The time derivative of the Jacobian is given by*

$$\frac{dJ}{dt} = \dot{J} = J \operatorname{div}_{\mathbf{x}} \mathbf{v}, \tag{1.2.30}$$

where $\operatorname{div}_{\mathbf{x}}$ *is the spatial divergence operator.*

Proof. Direct differentiation with respect to t of the relation $J = \det \mathbf{F}$ gives

$$\dot{J} = \frac{dJ}{dt} = \dot{\mathbf{F}} \cdot \mathbf{A},$$

where the tensor $\mathbf{A}$ has components $A_{hm} = J\left(\mathbf{F}^{-1}\right)_{mh}$. Therefore, using relation (1.2.10), we obtain

$$\dot{J} = L_{hk} F_{km} A_{hm} = J L_{hk} F_{km} \left(F^{-1}\right)_{mh} = J L_{hk} \delta_{kh} = J L_{hh},$$

which is relation (1.2.30). □

Remark 1.2.14. It is understood that these italic subscripts range over 1, 2, and 3. Moreover, we use the convention of summation over repeated subscripts, unless stated otherwise.

Definition 1.2.15. *A deformation* $\mathbf{x} = \tilde{\chi}(\mathbf{X}, t)$ *is isochoric (volume-preserving) if given any subbody* B *of* $\varphi_0(\mathcal{B})$*, we have* $\mathrm{vol}(\tilde{\chi}(B)) = \mathrm{vol}(B)$.

An immediate consequence of this definition is the following result.

Proposition 1.2.16. *A deformation is isochoric if and only if* $\det \mathbf{F} = 1$.

Remark 1.2.17. From relation (1.2.19), we deduce that

$$\frac{d}{dt}|d\mathbf{x}|^2 = d\mathbf{X} \cdot \dot{\mathbf{C}} d\mathbf{X},$$

so that comparing with relation (1.2.26) and using (1.2.17), we obtain

$$\dot{\mathbf{C}} = 2\mathbf{F}^T \mathbf{D} \mathbf{F}.$$

Moreover, $(1.2.20)_1$ gives

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{D} \mathbf{F}.$$

1.2.2 Small Deformations: The Saint-Venant Compatibility Conditions

We now study the behavior of the various kinematic fields when the displacement vector is of the form $\mathbf{u}_\epsilon = \varepsilon \mathbf{u}$, where ε is a parameter such that ε^p is negligible if $p \geq 2$, while $\mathbf{u}$ is a vector independent of ε. The theory corresponding to such small displacements is known as the *infinitesimal or linear theory of deformation*. In such a theory, we have

$$x_i = X_i + u_{\epsilon i},$$

and the partial derivatives of the displacement vector with respect to the spatial coordinates coincide with the partial derivatives of the same vector with respect to the material coordinates. In fact, we have, for example,

$$\frac{\partial u_{\epsilon i}}{\partial X_1} = \frac{\partial u_{\epsilon i}}{\partial x_j} \frac{\partial x_j}{\partial X_1} = \frac{\partial u_{\epsilon i}}{\partial x_j} \left(\delta_{1j} + \frac{\partial u_{\epsilon j}}{\partial X_1} \right) = \frac{\partial u_{\epsilon i}}{\partial x_1} + O\left(\varepsilon^2\right), \quad \text{etc.}$$

On the basis of relations of this type and from (1.2.21), we deduce that the Lagrangian and Eulerian strain tensors $\mathbf{E}$ and $\mathbf{e}$ coincide with the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right), \tag{1.2.31}$$

where $\nabla \mathbf{u} = \nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{X}} \mathbf{u}$. In component form, we have

$$\varepsilon_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right). \tag{1.2.32}$$

Theorem 1.2.18. (**Saint-Venant's Compatibility Conditions**) *The infinitesimal strain tensor* ε_{ij} *corresponding to a displacement vector field* $\mathbf{u}$ *of class* C^3 *satisfies the following compatibility equations:*

$$\begin{aligned}&\varepsilon_{ii,jj} + \varepsilon_{jj,ii} = 2\varepsilon_{ij,ij} \quad (i \neq j, \textit{ not summed}),\\&\varepsilon_{ij,rr} + \varepsilon_{rr,ij} = \varepsilon_{jr,ir} + \varepsilon_{ir,jr} \quad (i \neq j \neq r \neq i, \textit{ not summed}).\end{aligned} \tag{1.2.33}$$

Moreover, if B_0 is a simply connected region in $\mathbb{R}^3$ and ε_{ij} is a symmetric tensor of class C^2 defined on B_0 satisfying the conditions described by (1.2.33), then there exists a displacement vector field **u** *such that its corresponding strain tensor calculated by means of relation (1.2.32) coincides with ε_{ij}. Such a displacement vector field is given by*

$$u_i(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \left(\varepsilon_{ij} + \omega^*_{ij}\right) d\xi_j + \omega^0_{ij}\left(x_j - x^0_j\right) + u^0_i, \tag{1.2.34}$$

where

$$\omega^*_{ij}(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \left(\varepsilon_{ik,j} - \varepsilon_{kj,i}\right) d\xi_k,$$

while $\omega^0_{ij} = -\omega^0_{ji}$ and u^0_i are arbitrary constants. Also, the integrals are independent of the curve connecting the points $\mathbf{x}_0$ and $\mathbf{x}$.

Proof. We first note that the relations in (1.2.33) are identically satisfied for ε_{ij} given by (1.2.32).

In order to prove the second part of the theorem, we introduce the skew-symmetric tensor

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right),$$

which, when coupled with (1.2.32), gives

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij}.$$

Furthermore,

$$du_i = u_{i,j} dx_j = \left(\varepsilon_{ij} + \omega_{ij}\right) dx_j \tag{1.2.35}$$

is an exact differential in B_0 (that is, $u_{i,jk} = u_{i,kj}$) if and only if

$$\varepsilon_{ij,k} + \omega_{ij,k} = \varepsilon_{ik,j} + \omega_{ik,j}. \tag{1.2.36}$$

By a cyclic permutation of the indices i, j, and k in (1.2.36), we obtain

$$\varepsilon_{jk,i} + \omega_{jk,i} = \varepsilon_{ji,k} + \omega_{ji,k} \tag{1.2.37}$$

and

$$\varepsilon_{ki,j} + \omega_{ki,j} = \varepsilon_{kj,i} + \omega_{kj,i}. \tag{1.2.38}$$

If we now add (1.2.36) and (1.2.37) and from the result subtract (1.2.38), taking into account the relations $\varepsilon_{ij} = \varepsilon_{ji}$ and $\omega_{ij} = -\omega_{ji}$, then we obtain

$$\omega_{ij,k} = \varepsilon_{ik,j} - \varepsilon_{kj,i}.$$

Furthermore, $d\omega_{ij} = \omega_{ij,k} dx_k = \left(\varepsilon_{ik,j} - \varepsilon_{kj,i}\right) dx_k$ is an exact differential in B_0 (that is, $\omega_{ij,kl} = \omega_{ij,lk}$) if and only if

$$\varepsilon_{ik,jl} - \varepsilon_{kj,il} = \varepsilon_{il,jk} - \varepsilon_{lj,ik},$$

or

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} - \varepsilon_{jk,il} = 0. \tag{1.2.39}$$

It is easy to verify that the relations in (1.2.39) are equivalent to those given by (1.2.33), and from these conditions, it follows that $d\omega_{ij} = \omega_{ij,k} dx_k = \left(\varepsilon_{ik,j} - \varepsilon_{kj,i}\right) dx_k$ is an exact differential, giving

$$\omega_{ij}(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \left(\varepsilon_{ik,j} - \varepsilon_{kj,i}\right) d\xi_k + \omega_{ij}^0, \tag{1.2.40}$$

where $\omega_{ij}^0 = -\omega_{ji}^0$ are arbitrary constants. We note that the above integral is independent of the curve connecting the points $\mathbf{x}_0$ and $\mathbf{x}$.

At this stage we observe that the necessary and sufficient conditions for the integrability of the differential form (1.2.35) are satisfied and hence

$$u_i = \int_{\mathbf{x}_0}^{\mathbf{x}} \left(\varepsilon_{ij} + \omega_{ij}\right) d\xi_j + u_i^0, \tag{1.2.41}$$

where u_i^0 are arbitrary constants.

Finally, we substitute $\omega_{ij}(\cdot)$ given by (1.2.40) into (1.2.41) to obtain the relation (1.2.36). □

Remark 1.2.19. From the above analysis, we can deduce that $\varepsilon_{ij} = 0$ if and only if $\mathbf{u}$ is an infinitesimal rigid displacement $\mathbf{u}^*$, given by

$$u_i^* = a_i + e_{ijk} x_j b_k,$$

where $a_i = u_i^0 - \omega_{ij}^0 x_j^0$ and $b_i = e_{ijk}\omega_{jk}^0$ are arbitrary constants.

Remark 1.2.20. The relation (1.2.34) can be rewritten as

$$u_i = \int_{\mathbf{x}_0}^{\mathbf{x}} \left(\varepsilon_{ij} + \omega_{ij}^*\right) d\xi_j + u_i^*,$$

so that the displacement vector field is determined uniquely by ε_{ij} up to an infinitesimal rigid displacement.

1.2.3 Transformation of Areas and Volumes: Transport Theorems

We first discuss how the area and volume elements change as a result of a given deformation. To this end, let us consider the vectors $d\mathbf{X}_1 = dX_1\mathbf{i}_1$, $d\mathbf{X}_2 = dX_2\mathbf{i}_2$, and $d\mathbf{X}_3 = dX_3\mathbf{i}_3$, which, with the deformation $\mathbf{x} = \tilde{\chi}(\mathbf{X}, t)$, become $d\mathbf{x}_1 = \frac{\partial \mathbf{x}_1}{\partial X_1} dX_1$, $d\mathbf{x}_2 = \frac{\partial \mathbf{x}_2}{\partial X_2} dX_2$, and $d\mathbf{x}_3 = \frac{\partial \mathbf{x}_3}{\partial X_3} dX_3$, respectively. Let $d\mathbf{A}_3$ be the area vector associated with the rectangle determined by the vectors $d\mathbf{X}_1$ and $d\mathbf{X}_2$, and let $d\boldsymbol{\sigma}_3$ be the corresponding area vector associated with the parallelogram determined by the vectors $d\mathbf{x}_1$ and $d\mathbf{x}_2$, that is,

$$dA_3 = dX_1 \times dX_2, \quad d\sigma_3 = d\mathbf{x}_1 \times d\mathbf{x}_2.$$

Obviously, we have

$$d\boldsymbol{\sigma}_3 = \frac{\partial \mathbf{x}_1}{\partial X_1} \times \frac{\partial \mathbf{x}_2}{\partial X_2} dA_3 = e_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \mathbf{i}_k dA_3. \tag{1.2.42}$$

Since

$$J = \det \mathbf{F} = e_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3},$$

we can rewrite the relation (1.2.42) in the form

$$d\boldsymbol{\sigma}_3 = J \frac{\partial X_3}{\partial x_j} \mathbf{i}_j dA_3. \tag{1.2.43}$$

A general area element $d\mathbf{A}$ will have components on all three axes. By an analogous procedure, one obtains

$$d\boldsymbol{\sigma}_1 = J \frac{\partial X_1}{\partial x_j} \mathbf{i}_j dA_1, \quad d\boldsymbol{\sigma}_2 = J \frac{\partial X_2}{\partial x_j} \mathbf{i}_j dA_2. \tag{1.2.44}$$

If we now set

$$d\mathbf{a} = d\boldsymbol{\sigma}_1 + d\boldsymbol{\sigma}_2 + d\boldsymbol{\sigma}_3,$$

then, by (1.2.43) and (1.2.44),

$$d\mathbf{a} = J \frac{\partial X_k}{\partial x_j} dA_k \mathbf{i}_j. \tag{1.2.45}$$

Thus, putting

$$d\mathbf{a} = da_j \mathbf{i}_j,$$

it follows from (1.2.45) that

$$da_j = J \frac{\partial X_k}{\partial x_j} dA_k, \tag{1.2.46}$$

a relation that expresses the change of an area element due to the given deformation.

On the other hand, the volume element dv_t of the parallelepiped, determined by the vectors $d\mathbf{x}_1$, $d\mathbf{x}_2$, and $d\mathbf{x}_3$, is

$$dv_t = d\mathbf{x}_1 \times d\mathbf{x}_2 \cdot d\mathbf{x}_3 = J dX_1 dX_2 dX_3 = J dv_0. \tag{1.2.47}$$

It can be shown that for small deformations, in the limit of a linear theory, relation (1.2.47) gives

$$\frac{dv_t - dv_0}{dv_0} = \varepsilon_{ii} = \operatorname{tr} \boldsymbol{\varepsilon} = I_1(\boldsymbol{\varepsilon}),$$

so that $I_1(\boldsymbol{\varepsilon})$ represents the variation of volume per unit undeformed volume.

Let $\mathbf{x} = \tilde{\boldsymbol{\chi}}(\mathbf{X}, t)$ be a motion of the body $\mathcal{B}$. For any subbody A of $\mathcal{B}$, we write $\varphi_t(A) = \tilde{\chi}(A, t)$ for the region of space occupied by A at time t. Then the volume of $\varphi_t(A)$ is

$$\mathrm{vol}(\varphi_t(A)) = \int_{\varphi_t(A)} dv_t,$$

so that using a change of variables in this volume integral, we can write

$$\mathrm{vol}(\varphi_t(A)) = \int_{\varphi_0(A)} J dv_0.$$

Thus, by virtue of relation (1.2.30), we have

$$\frac{d}{dt}[\mathrm{vol}(\varphi_t(A))] = \int_{\varphi_0(A)} \dot{J} dv_0 = \int_{\varphi_t(A)} \dot{J} J^{-1} dv_t = \int_{\varphi_t(A)} \mathrm{div}_{\mathbf{x}} \mathbf{v} dv_t.$$

This relation allows us to formulate the following results.

Theorem 1.2.21. (**Transport of Volume**) *For any subbody A of $\mathcal{B}$ and time t, denoting by $\mathbf{n}$ the outward unit normal vector on the boundary $\partial\varphi_t(A)$ of $\varphi_t(A)$, we have*

$$\frac{d}{dt}[\mathrm{vol}(\varphi_t(A))] = \int_{\varphi_0(A)} \dot{J} dv_0 = \int_{\varphi_t(A)} \mathrm{div}_{\mathbf{x}} \mathbf{v} dv_t = \int_{\partial\varphi_t(A)} \mathbf{v}\cdot\mathbf{n} da_t.$$

Since A is arbitrary, it follows from the third integral that $\mathrm{div}_{\mathbf{x}}\mathbf{v}$ *represents the rate of change of volume per unit volume in the current configuration.*

Theorem 1.2.22. (**Characterization of Isochoric Motions**) *The following assertions are equivalent: (a) $\mathbf{x} = \tilde{\boldsymbol{\chi}}(\mathbf{X}, t)$ is isochoric, (b) $\dot{J} = 0$, (c) $\mathrm{div}_{\mathbf{x}}\mathbf{v} = 0$, and (d) $\int_{\partial\varphi_t(A)} \mathbf{v}\cdot\mathbf{n} da_t = 0$ for every subbody A and any time t.*

We can now establish the following general result.

Theorem 1.2.23. (**Reynold's Transport Theorem**) *Let $\mathfrak{F}$ be a smooth spatial field, and assume that $\mathfrak{F}$ is either scalar-valued or vector-valued. Then for any subbody A and time t, we have*

$$\begin{aligned} \frac{d}{dt}\int_{\varphi_t(A)} \hat{\mathfrak{F}}(\mathbf{x}, t)\, dv_t &= \int_{\varphi_t(A)} \left[\frac{d}{dt}\hat{\mathfrak{F}}(\mathbf{x}, t) + \hat{\mathfrak{F}}\mathrm{div}_{\mathbf{x}}\mathbf{v}\right] dv_t \\ &= \int_{\varphi_t(A)} \frac{\partial\hat{\mathfrak{F}}}{\partial t}(\mathbf{x}, t)\, dv_t + \int_{\partial\varphi_t(A)} \hat{\mathfrak{F}}\mathbf{v}\cdot\mathbf{n} da_t. \end{aligned} \tag{1.2.48}$$

Proof. For the transformation $\mathbf{x} = \tilde{\boldsymbol{\chi}}(\mathbf{X}, t)$, since $dv_t = J dv_0$, we have

$$\int_{\varphi_t(A)} \hat{\mathfrak{F}}(\mathbf{x}, t)\, dv_t = \int_{\varphi_0(A)} \tilde{\mathfrak{F}}(\mathbf{X}, t) J(\mathbf{X}, t)\, dv_0,$$

and hence

$$\begin{aligned} \frac{d}{dt}\int_{\varphi_t(A)} \hat{\mathfrak{F}}(\mathbf{x}, t)\, dv_t &= \int_{\varphi_0(A)} \frac{\partial}{\partial t}[\tilde{\mathfrak{F}}(\mathbf{X}, t) J(\mathbf{X}, t)] dv_0 \\ &= \int_{\varphi_0(A)} \left[\frac{\partial}{\partial t}\tilde{\mathfrak{F}}(\mathbf{X}, t) J(\mathbf{X}, t) + \tilde{\mathfrak{F}}(\mathbf{X}, t)\frac{\partial}{\partial t} J(\mathbf{X}, t)\right] dv_0, \end{aligned}$$

so that using (1.2.7) and (1.2.30), we have

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(A)} \hat{\mathfrak{F}}(\mathbf{x},t)\,dv_t &= \int_{\varphi_0(A)} \left[\frac{\partial}{\partial t}\tilde{\mathfrak{F}}(\mathbf{X},t) + \tilde{\mathfrak{F}}(\mathbf{X},t)\mathrm{div}_{\mathbf{x}}\mathbf{v}\right] J(\mathbf{X},t)\,dv_0 \\ &= \int_{\varphi_t(A)} \left[\frac{d}{dt}\hat{\mathfrak{F}}(\mathbf{x},t) + \hat{\mathfrak{F}}(\mathbf{x},t)\mathrm{div}_{\mathbf{x}}\mathbf{v}\right] dv_t,\end{aligned} \tag{1.2.49}$$

which is $(1.2.48)_1$. Relation $(1.2.48)_2$ follows from $(1.2.49)_1$ by using (1.2.6) and applying the divergence theorem. □

Remark 1.2.24. We note that

$$\int_{\varphi_t(A)} \frac{\partial\hat{\mathfrak{F}}}{\partial t}(\mathbf{x},t)\,dv_t = \left[\frac{d}{d\tau}\int_{\varphi_t(A)} \hat{\mathfrak{F}}(\mathbf{x},\tau)\,dv_t\right]_{\tau=t}.$$

Thus, $(1.2.48)_2$ asserts that the rate at which the integral of $\mathfrak{F}$ over $\varphi_t(A)$ is changing is equal to the rate computed as if $\varphi_t(A)$ were fixed in its current position plus the rate at which $\mathfrak{F}$ is carried out of this region across its boundary.

1.3 Principles of Continuum Mechanics

1.3.1 Principle of Conservation of Mass

Given a deformation $\mathbf{x} = \tilde{\chi}(\mathbf{X},t)$ of the body $\mathcal{B}$, we will write $\rho(\mathbf{x},t) = \rho_{\tilde{\chi}(\cdot,t)}(\mathbf{x})$ for the density at the position $\mathbf{x} \in \tilde{\chi}(\mathcal{B},t)$.

- **Principle of conservation of mass:** *The mass of any subbody A of $\mathcal{B}$ is conserved in time, so that we have*

$$\int_{\varphi_0(A)} \rho(\mathbf{X},0)\,dv_0 = \int_{\varphi_t(A)} \rho(\mathbf{x},t)\,dv_t. \tag{1.3.1}$$

In what follows, we will denote by $\rho_0(\mathbf{X})$ the reference mass density $\rho(\mathbf{X},0)$. Relation (1.3.1) expresses the principle of conservation of mass in integral form. We wish to establish a local form of this principle.

Theorem 1.3.1. *The local version of the principle of conservation of mass takes one of the following forms:*

$$\begin{aligned}\rho_0 &= \rho J, \\ \dot{\rho} + \rho\,\mathrm{div}_{\mathbf{x}}\mathbf{v} &= 0, \\ \frac{\partial\rho}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}) &= 0.\end{aligned} \tag{1.3.2}$$

Proof. If we change the variable of integration on the right-hand side of relation (1.3.1) from **x** to **X**, we arrive at

$$\int_{\varphi_0(A)} \rho_0(\mathbf{X})\, dv_0 = \int_{\varphi_0(A)} \rho(\tilde{\chi}(\mathbf{X},t),\ t) J dv_0,$$

so that

$$\int_{\varphi_0(A)} [\rho(\tilde{\chi}(\mathbf{X},t),\ t)J - \rho_0(\mathbf{X})] dv_0 = 0, \tag{1.3.3}$$

for every subbody A of the body $\mathcal{B}$. We deduce from (1.3.3) the local form of the principle of conservation of mass expressed by $(1.3.2)_1$.

Furthermore, by differentiation of $(1.3.2)_1$ with respect to the time variable, we obtain

$$\rho \dot{J} + \dot{\rho} J = 0,$$

which with the aid of (1.2.30) yields $(1.3.2)_2$. Next, by (1.2.6), we have

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho,$$

so that $(1.3.2)_2$, combined with this relation, implies $(1.3.2)_3$. □

Remark 1.3.2. The local form of the conservation of mass expressed by $(1.3.2)_1$ is referred to as the *continuity equation in Lagrangian form*, while $(1.3.2)_2$ is the *continuity equation in spatial form.*

By virtue of the above forms of the principle of conservation of mass, Reynold's transport theorem takes a simplified form.

Theorem 1.3.3. *Let $\mathcal{F}$ be a smooth spatial field, either scalar-valued or vector-valued. Then, for any subbody A of $\mathcal{B}$ and time t, we have*

$$\frac{d}{dt} \int_{\varphi_t(A)} \hat{\mathcal{F}}(\mathbf{x},t)\rho(\mathbf{x},t)\, dv_t = \int_{\varphi_t(A)} \frac{d}{dt}\left[\hat{\mathcal{F}}(\mathbf{x},t)\right] \rho(\mathbf{x},t)\, dv_t. \tag{1.3.4}$$

Thus, to differentiate the integral

$$\int_{\varphi_t(A)} \mathcal{F} \rho dv_t$$

with respect to time, we simply differentiate under the integral sign, treating the mass measure *ρdv_t as a constant.*

Proof. We replace $\mathcal{F}$ by $\mathcal{F}_\rho$ in Reynold's transport relation (1.2.48) and then use the form $(1.3.2)_2$ of the principle of conservation of mass to obtain (1.3.4). □

1.3.2 Momentum Balance Principles

Let $\mathbf{x} = \tilde{\boldsymbol{\chi}}(\mathbf{X}, t)$ be a motion of the body $\mathcal{B}$, and let A be a subbody of $\mathcal{B}$. Then the linear momentum $\mathbf{Q}(A,\ t)$ and the angular momentum $\mathbf{K}_0(A,\ t)$ (about the origin) of A at time t are given by

$$\mathbf{Q}(A,\ t) = \int_{\varphi_t(A)} \mathbf{v}\rho dv_t \tag{1.3.5}$$

and

$$\mathbf{K}_0(A,\ t) = \int_{\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}\rho dv_t. \tag{1.3.6}$$

In view of the rule (1.3.4), we obtain, from (1.3.5) and (1.3.6),

$$\frac{d}{dt}\mathbf{Q}(A,\ t) = \int_{\varphi_t(A)} \dot{\mathbf{v}}\rho dv_t \tag{1.3.7}$$

and

$$\frac{d}{dt}\mathbf{K}_0(A,\ t) = \int_{\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times \dot{\mathbf{v}}\rho dv_t. \tag{1.3.8}$$

During a given motion, the mechanical interactions between parts of a body or between a body and its environment are described by forces. In what follows, we will be concerned with three types of force: (i) contact forces between parts of a body, (ii) contact forces exerted on the boundary of a body by its environment, and (iii) body forces exerted on the interior points of a body by the environment.

The environment can exert forces on interior points of $\mathcal{B}$, a classical example being the force field due to gravity. Such forces are determined by a prescribed vector field $\mathbf{b}$ on the trajectory $\mathcal{T}$ of the motion, so that $\mathbf{b}(\mathbf{x}, t)$ gives the force, per unit mass, exerted by the environment on $\mathbf{x}$ at time t. Thus, for any subbody A of $\mathcal{B}$, the integral

$$\int_{\varphi_t(A)} \mathbf{b}(\mathbf{x}, t)\rho dv_t$$

gives that part of the environmental force on A acting at a distance at time t (not due to contact).

Let us now consider the contact forces. To this end we use *Cauchy's hypothesis* concerning the form of the contact forces: *Assume the existence of a surface force density* $\mathbf{t} = \mathbf{t}(\mathbf{x}, t;\ \mathbf{n})$ *defined for every* $(\mathbf{x}, t)$ *in the trajectory* $\mathcal{T}$ *of the motion and for each unit vector* $\mathbf{n}$. To make this hypothesis more precise, we consider an oriented surface $\mathcal{S}$ in $\varphi_t(\mathcal{B})$ with positive unit normal $\mathbf{n}$ at $\mathbf{x}$. Then $\mathbf{t}(\mathbf{x}, t;\ \mathbf{n})$ represents the force, per unit area, exerted across $\mathcal{S}$ upon the material on the negative side of $\mathcal{S}$ by the material on the positive side. To determine the contact force between two subbodies A and C at time t, one integrates $\mathbf{t}$ over the surface of contact $\mathcal{S}_t = \varphi_t(A) \cap \varphi_t(C)$. Thus, denoting by $\mathbf{n}_x$ the outward unit normal to $\partial\varphi_t(A)$ at $\mathbf{x}$,

$$\int_{\mathcal{S}_t} \mathbf{t}(\mathbf{x}, t;\ \mathbf{n}_\mathbf{x})da_t = \int_{\mathcal{S}_t} \mathbf{t}(\mathbf{n})da_t \tag{1.3.9}$$

gives the force exerted on A by C at time t. Such a contact force depends on the intrinsic structure of the material and is therefore unknown in general.

For points on the boundary of $\varphi_t(\mathcal{B})$, $\mathbf{t}(\mathbf{x}, t;\ \mathbf{n})$, with $\mathbf{n}$ the outward unit normal to $\partial\varphi_t(\mathcal{B})$ at $\mathbf{x}$, gives the surface force, per unit area, applied to the body by the environment. This force is referred to as the *surface traction*, and it is usually known.

The above discussion motivates the following definition.

Definition 1.3.4. *By a system of forces for $\mathcal{B}$ during a motion (with trajectory $\mathcal{T}$), we mean a pair $(\mathbf{t}, \mathbf{b})$ of functions $\mathbf{t} : \mathcal{T} \times \mathcal{N} \to V$, $\mathbf{b} : \mathcal{T} \to V$, where $\mathcal{N}$ is the set of all unit vectors and V is the vector space $\mathbb{R}^3$, so that (i) $\mathbf{t}(\mathbf{x}, t;\ \mathbf{n})$ is a smooth function of $\mathbf{x}$ on $\varphi_t(\mathcal{B})$, for each $\mathbf{n} \in \mathcal{N}$ and $t \geq 0$ and (ii) $\mathbf{b}(\mathbf{x}, t)$ is a continuous function of $\mathbf{x}$ on $\varphi_t(\mathcal{B})$, for each $t \geq 0$. We refer to $\mathbf{t}$ as the surface force and $\mathbf{b}$ as the body force. The force $\mathbf{F}(A,\ t)$ and the moment $\mathbf{\Omega}_0(A,\ t)$ (about the origin) on a subbody A at time t are defined by*

$$\mathbf{F}(A,\ t) = \int_{\varphi_t(A)} \mathbf{b}\rho dv_t + \int_{\partial\varphi_t(A)} \mathbf{t}(\mathbf{n}) da_t \tag{1.3.10}$$

and

$$\mathbf{\Omega}_0(A,\ t) = \int_{\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b}\rho dv_t + \int_{\partial\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t}(\mathbf{n}) da_t. \tag{1.3.11}$$

- **The balance law of linear momentum:** *The time derivative of the linear momentum of every subbody A of $\mathcal{B}$ at time t is equal to the force $\mathbf{F}(A,\ t)$ acting on that subbody at time t, so that*

$$\frac{d}{dt}\mathbf{Q}(A,\ t) = \mathbf{F}(A,\ t). \tag{1.3.12}$$

- **The balance law of angular momentum:** *The time derivative of the angular momentum $\mathbf{K}_0(A,\ t)$ of every subbody A of the body at time t is equal to the moment $\mathbf{\Omega}_0(A,\ t)$ acting on that subbody at time t, that is,*

$$\frac{d}{dt}\mathbf{K}_0(A,\ t) = \mathbf{\Omega}_0(A,\ t). \tag{1.3.13}$$

Remark 1.3.5. We assume that there exists a *laboratory frame of reference* in which Newton's second law, (1.3.12), holds to a good approximation and refer to this and all frames of reference traveling at constant velocities relative to it as *inertial frames*. These are all connected by *Galilean transformations*, and Newton's second law applies equally in all of them.

In view of relations (1.3.7)–(1.3.11), the laws (1.3.12) and (1.3.13) of momentum balance can be written as follows:

$$\int_{\varphi_t(A)} \dot{\mathbf{v}}\rho dv_t = \int_{\varphi_t(A)} \mathbf{b}\rho dv_t + \int_{\partial\varphi_t(A)} \mathbf{t}(\mathbf{n}) da_t \tag{1.3.14}$$

and

$$\int_{\varphi_t(A)} (\mathbf{x}-\mathbf{x}_0)\times\dot{\mathbf{v}}\rho dv_t = \int_{\varphi_t(A)} (\mathbf{x}-\mathbf{x}_0)\times\mathbf{b}\rho dv_t + \int_{\partial\varphi_t(A)} (\mathbf{x}-\mathbf{x}_0)\times\mathbf{t}(\mathbf{n})da_t. \tag{1.3.15}$$

Lemma 1.3.6. (**Newton's Law of Action and Reaction**) *For each* $\mathbf{x}\in\varphi_t(\mathcal{B})$ *and for each unit vector* $\mathbf{n}\in\mathcal{N}$, *it follows that*

$$\mathbf{t}(\mathbf{x},t;\ \mathbf{n}) = -\mathbf{t}(\mathbf{x},t;\ -\mathbf{n}), \tag{1.3.16}$$

for fixed time t.

Proof. Let us denote by R_δ a volume centered at $\mathbf{x}$, with rectangular sides. It has dimensions $\delta\times\delta\times\delta^2$, and $\mathbf{n}$ is the unit normal to the $\delta\times\delta$ faces. Let Σ_δ^+ be the face with the outward unit normal $\mathbf{n}$ and Σ_δ^- the face with the outward unit normal $-\mathbf{n}$. Furthermore, we set $\partial R_\delta = \Sigma_\delta^+\cup\Sigma_\delta^-\cup\Sigma$. Obviously, we have

$$\text{vol}(R_\delta) = \delta^4, \qquad \text{Area}(\Sigma_\delta^+) = \text{Area}(\Sigma_\delta^-) = \delta^2, \qquad \text{Area}(\Sigma) = 4\delta^3. \tag{1.3.17}$$

We further note that R_δ is contained in the interior of $\varphi_t(\mathcal{B})$ for all sufficiently small δ, say $\delta\le\delta_0$.

We now apply (1.3.14) to the subbody A that occupies the region $\varphi_t(A)\equiv R_\delta$. Since $\mathbf{b}(\mathbf{x},t)$, $\rho(\mathbf{x},t)$, and $\dot{\mathbf{v}}(\mathbf{x},t)$ are continuous in $\mathbf{x}$, it follows that the function $\mathbf{b}_*(\mathbf{x},t)$ defined by $\mathbf{b}_* = \rho(\mathbf{b}-\dot{\mathbf{v}})$ is bounded on R_{δ_0} for t fixed, and hence

$$\kappa(t) = \sup_{\mathbf{x}\in R_{\delta_0}} |\mathbf{b}_*(\mathbf{x},t)| < \infty. \tag{1.3.18}$$

For convenience, we fix the time t and suppress it as an argument in most of what follows.

From (1.3.14), we deduce

$$\left|\int_{\partial R_\delta} \mathbf{t}(\mathbf{n})da_t\right| \le \kappa(t)\text{vol}\,(R_\delta), \tag{1.3.19}$$

so that on the basis of relations (1.3.17) and (1.3.18), we obtain

$$\frac{1}{\delta^2}\int_{\partial R_\delta} \mathbf{t}(\mathbf{n})da_t \to \mathbf{0} \text{ when } \delta\to 0. \tag{1.3.20}$$

But

$$\int_{\partial R_\delta} \mathbf{t}(\mathbf{n})da_t = \int_{\Sigma_\delta^+} \mathbf{t}(+\mathbf{n})da_t + \int_{\Sigma_\delta^-} \mathbf{t}(-\mathbf{n})da_t + \int_{\Sigma} \mathbf{t}(\mathbf{n})da_t. \tag{1.3.21}$$

Since $\mathbf{t}(\mathbf{x};\mathbf{n})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{n}\in\mathcal{N}$, we have, using (1.3.17),

$$\frac{1}{\delta^2}\int_{\Sigma} \mathbf{t}(\mathbf{n})da_t \to \mathbf{0} \text{ when } \delta\to 0, \tag{1.3.22}$$

$$\frac{1}{\delta^2}\int_{\Sigma_\delta^+} \mathbf{t}(+\mathbf{n})da_t \to \mathbf{t}(\mathbf{x};+\mathbf{n}), \qquad \frac{1}{\delta^2}\int_{\Sigma_\delta^-} \mathbf{t}(-\mathbf{n})da_t \to \mathbf{t}(\mathbf{x};-\mathbf{n}), \tag{1.3.23}$$

when $\delta \to 0$. Thus, relations (1.3.20)–(1.3.23) give

$$\mathbf{t}(\mathbf{x}; +\mathbf{n}) + \mathbf{t}(\mathbf{x}; -\mathbf{n}) = \mathbf{0},$$

which is (1.3.16). □

Theorem 1.3.7. **(Cauchy's Theorem for the Existence of Stress)** *Let* $(\mathbf{t}, \mathbf{b})$ *be a system of forces for* $\mathcal{B}$ *during a motion. Then, a necessary and sufficient condition that the momentum balance laws be satisfied is that there exists a spatial tensor field* $\mathbf{T}$ *(the* Cauchy stress*) such that*

(i) for each unit vector $\mathbf{n} \in \mathcal{N}$,

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}; \tag{1.3.24}$$

(ii) $\mathbf{T}$ *is symmetric;*
(iii) $\mathbf{T}$ *satisfies the equation of motion*

$$\rho\dot{\mathbf{v}} = \mathrm{div}_{\mathbf{x}}\mathbf{T} + \rho\mathbf{b}. \tag{1.3.25}$$

Proof. Necessity. Assume that the momentum balance laws (1.3.12) and (1.3.13) are satisfied. We first note that (1.3.16) holds. Furthermore, we choose $\mathbf{x}$ to belong to the interior of $\varphi_t(\mathcal{B})$ and let $\delta > 0$. Consider the tetrahedron $\mathcal{T}_\delta$ with the following properties: the faces of $\mathcal{T}_\delta$ are $\mathcal{S}_\delta$, $\mathcal{S}_{1\delta}$, $\mathcal{S}_{2\delta}$, and $\mathcal{S}_{3\delta}$, where $\mathbf{n}$ and $-\mathbf{i}_j$ are the outward unit normal vectors to $\partial\mathcal{T}_\delta$ on $\mathcal{S}_\delta$ and $\mathcal{S}_{j\delta}$, $j = 1, 2, 3$, respectively; the vertex opposite to $\mathcal{S}_\delta$ is $\mathbf{x}$; the distance from $\mathbf{x}$ to $\mathcal{S}_\delta$ is δ (Fig. 1.3). Clearly, $\mathcal{S}_\delta$ is contained in the interior of $\varphi_t(\mathcal{B})$ for all sufficiently small choices of δ, say $\delta \le \delta_0$. Thus, we can apply (1.3.14) to the subbody A that occupies the region $\mathcal{T}_\delta$ at time t, and since $\mathbf{b}_* = \rho(\mathbf{b} - \dot{\mathbf{v}})$ is bounded on $\mathcal{T}_{\delta_0}$, we can conclude that

$$\left| \int_{\partial\mathcal{T}_\delta} \mathbf{t}(\mathbf{n}) da_t \right| \le \kappa_1(t)\mathrm{vol}(\mathcal{T}_\delta), \qquad \text{for all } \delta \le \delta_0, \tag{1.3.26}$$

where $\kappa_1(t)$ is finite and independent of δ.

Let $\mathcal{A}(\delta)$ denote the area of $\mathcal{S}_\delta$. Then $\mathrm{vol}(\mathcal{T}_\delta) = \frac{1}{3}\delta\mathcal{A}(\delta)$, and hence we can conclude from (1.3.26) that

$$\frac{1}{\mathcal{A}(\delta)} \int_{\partial\mathcal{T}_\delta} \mathbf{t}(\mathbf{n}) da_t \to \mathbf{0} \quad \text{as } \delta \to 0. \tag{1.3.27}$$

But

$$\int_{\partial\mathcal{T}_\delta} \mathbf{t}(\mathbf{n}) da_t = \int_{\mathcal{S}_\delta} \mathbf{t}(\mathbf{n}) da_t + \sum_{j=1}^{3} \int_{\mathcal{S}_{j\delta}} \mathbf{t}(-\mathbf{i}_j) da_t, \tag{1.3.28}$$

and since $\mathbf{t}(\mathbf{x}; \mathbf{n})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{n} = n_j\mathbf{i}_j \in \mathcal{N}$ and Area $\left(\mathcal{S}_{j\delta}\right) = \mathcal{A}(\delta)n_j$, we have

$$\frac{1}{\mathcal{A}(\delta)} \int_{\mathcal{S}_\delta} \mathbf{t}(\mathbf{n}) da_t \to \mathbf{t}(\mathbf{x}; \mathbf{n}) \quad \text{as } \delta \to 0, \tag{1.3.29}$$

$$\frac{1}{\mathcal{A}(\delta)} \int_{\mathcal{S}_{j\delta}} \mathbf{t}(-\mathbf{i}_j) da_t \to \mathbf{t}(\mathbf{x}; -\mathbf{i}_j)n_j \quad \text{as } \delta \to 0 \quad (\text{not summed on } j).$$

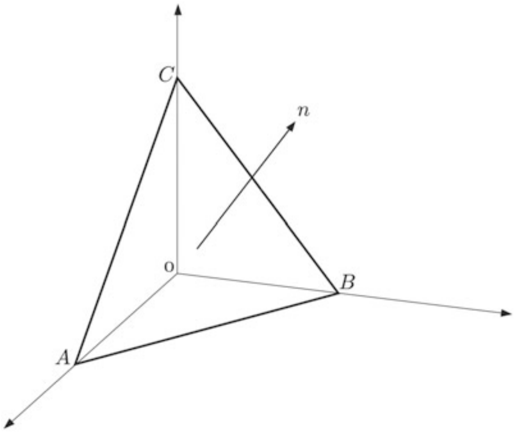

Fig. 1.3. The stress tetrahedron

Combining (1.3.27)–(1.3.29) with Newton's law of action and reaction (1.3.16), we conclude that

$$\mathbf{t}(\mathbf{x};\mathbf{n}) = \mathbf{t}(\mathbf{x};\mathbf{i}_j)n_j, \tag{1.3.30}$$

so $\mathbf{t}(\mathbf{x};\mathbf{n})$ is a linear function of the components of $\mathbf{n}$, or

$$\mathbf{t}(\mathbf{x};\mathbf{n}) = \mathbf{T}(\mathbf{x})\mathbf{n}. \tag{1.3.31}$$

We write this in components as

$$t_i(\mathbf{x};\ \mathbf{n}) = T_{ij}(\mathbf{x})n_j,$$

where $\mathbf{T}$, given by

$$\left(T_{ij}\right) = \begin{pmatrix} t_1\ (\mathbf{i}_1)\ t_1\ (\mathbf{i}_2)\ t_1\ (\mathbf{i}_3) \\ t_2\ (\mathbf{i}_1)\ t_2\ (\mathbf{i}_2)\ t_2\ (\mathbf{i}_3) \\ t_3\ (\mathbf{i}_1)\ t_3\ (\mathbf{i}_2)\ t_3\ (\mathbf{i}_3) \end{pmatrix},$$

is the *Cauchy stress tensor*.

Using (1.3.31), the balance of linear momentum (1.3.14) takes the form

$$\int_{\varphi_t(A)} \dot{\mathbf{v}}\rho dv_t = \int_{\varphi_t(A)} \mathbf{b}\rho dv_t + \int_{\partial\varphi_t(A)} \mathbf{T}\mathbf{n} da_t,$$

or equivalently, on applying the divergence theorem,

$$\int_{\varphi_t(A)} (\dot{\mathbf{v}}\rho - \mathbf{b}\rho - div_{\mathbf{x}}\mathbf{T})\, dv_t = \mathbf{0}.$$

This last relation can hold for every subbody A of the body and time t only if the equation of motion (1.3.25) is satisfied.

To complete the proof of necessity, we have only to establish the symmetry of the Cauchy stress $\mathbf{T}$. In fact, if we substitute (1.3.31) into (1.3.15), we obtain

$$\int_{\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times (\dot{\mathbf{v}}\rho - \mathbf{b}\rho)\, dv_t - \int_{\partial\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times \ (\mathbf{Tn})\, da_t = \mathbf{0},$$

or, using the divergence theorem,

$$\int_{\varphi_t(A)} (\mathbf{x} - \mathbf{x}_0) \times (\dot{\mathbf{v}}\rho - \mathbf{b}\rho - div_{\mathbf{x}}\mathbf{T})\, dv_t - \int_{\varphi_t(A)} e_{ijk} T_{jk} \mathbf{i}_i dv_t = \mathbf{0},$$

and hence, with the aid of the equation of motion (1.3.25), we deduce that

$$\int_{\varphi_t(A)} e_{ijk} T_{jk} dv_t = 0.$$

This relation can hold for every subbody A and time t only if

$$e_{ijk} T_{jk} = 0, \qquad \text{that is,} \quad T_{23} = T_{32}, \quad T_{31} = T_{13}, \quad T_{12} = T_{21}.$$

Sufficiency. Assume that there exists a symmetric spatial tensor field $\mathbf{T}$ consistent with the relations (1.3.24) and (1.3.25). Then it is an easy task to prove that the momentum balance laws (1.3.14) and (1.3.15) hold, and the proof is complete. □

Remark 1.3.8. Actually, one can see that the points (i) and (iii) are equivalent to balance of linear momentum, while, granted (1.3.14), the symmetry of the Cauchy stress $\mathbf{T}$ is equivalent to balance of angular momentum.

Definition 1.3.9. *If*

$$\mathbf{Tn} = \sigma\mathbf{n}, \qquad |\mathbf{n}| = 1,$$

then σ is a principal stress and $\mathbf{n}$ *is a principal direction, so that principal stresses and principal directions are eigenvalues and eigenvectors of* $\mathbf{T}$.

Since $\mathbf{T}$ is symmetric, it follows that there exist three mutually perpendicular principal directions and three corresponding principal stresses.

In general, the surface force $\mathbf{t} = \mathbf{Tn}$ can be decomposed into the sum

$$\mathbf{t} = T_0\mathbf{n} + \mathbf{t}_0,$$

where $T_0\mathbf{n}$ is the *normal force* and $\mathbf{t}_0$ is the *shearing force* perpendicular to $\mathbf{n}$. Obviously, we have

$$T_0 = \mathbf{n} \cdot \mathbf{Tn}, \qquad \mathbf{t}_0 = \mathbf{t} - (\mathbf{n} \cdot \mathbf{Tn})\mathbf{n} = (\mathbf{1} - \mathbf{n} \otimes \mathbf{n})\mathbf{Tn},$$

where $\otimes$ denotes the tensor product of two vectors.[¶] Clearly, $\mathbf{n}$ is a principal direction if and only if the corresponding shearing force vanishes. The normal component of the surface force is then a principal stress.

We now outline some important states of stress.

First consider a fluid at rest. It is incapable of exerting shearing forces, so that $\mathbf{Tn}$ is parallel to $\mathbf{n}$ for each unit vector $\mathbf{n}$, and hence every such unit vector is an eigenvector of $\mathbf{T}$. Thus, we have

$$\mathbf{T} = -p\mathbf{1},$$

where p is a scalar quantity referred to as the *pressure* of the fluid. The force per unit area on any surface in the fluid with unit normal $\mathbf{n}$ is $-p\mathbf{n}$.

Other states of stress are *pure tension* (or *compression*), where the tensile stress σ in the direction $\boldsymbol{\nu}$, with $|\boldsymbol{\nu}| = 1$, is defined by

$$\mathbf{T} = \sigma\,(\boldsymbol{\nu} \otimes \boldsymbol{\nu}),$$

and *pure shear* with shear stress τ relative to the direction pair $(\mathbf{k}, \mathbf{n})$, where $\mathbf{k}$ and $\mathbf{n}$ are orthogonal unit vectors, given by

$$\mathbf{T} = \tau(\mathbf{k} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{k}).$$

1.3.3 Consequences of Momentum Balance Laws

Definition 1.3.10. *For every subbody A of a continuous body, we define the kinetic energy of A at time t by*

$$\frac{1}{2}\int_{\varphi_t(A)} \rho\mathbf{v}^2 dv_t;$$

we further define the stress power of A at time t by

$$\int_{\varphi_t(A)} \mathbf{T}\cdot\mathbf{D}\,dv_t, \tag{1.3.32}$$

where $\mathbf{D}$ is the stretching, defined by (1.2.23).

Theorem 1.3.11. *The power expended on any subbody A at time t by the surface and body forces is equal to the rate of change of kinetic energy plus the stress power, so that*

$$\int_{\varphi_t(A)} \rho\mathbf{b}\cdot\mathbf{v}\,dv_t + \int_{\partial\varphi_t(A)} \mathbf{t}(\mathbf{n})\cdot\mathbf{v}\,da_t = \frac{d}{dt}\int_{\varphi_t(A)} \frac{1}{2}\rho\mathbf{v}^2 dv_t + \int_{\varphi_t(A)} \mathbf{T}\cdot\mathbf{D}\,dv_t. \tag{1.3.33}$$

Proof. Since $\mathbf{T}$ is symmetric, then with the aid of (1.2.23), we can write

$$\mathbf{T}\cdot\nabla_{\mathbf{x}}\mathbf{v} = T_{ij}\frac{\partial v_i}{\partial x_j} = T_{ji}\frac{\partial v_j}{\partial x_i} = T_{ij}\frac{\partial v_j}{\partial x_i} = T_{ij}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \mathbf{T}\cdot\mathbf{D}. \tag{1.3.34}$$

[¶] This is defined by the requirement that $(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are given vectors and $\mathbf{c}$ is an arbitrary vector.

Furthermore, invoking (1.3.4), we have

$$\frac{d}{dt}\int_{\varphi_t(A)} \frac{1}{2}\rho \mathbf{v}^2 dv_t = \int_{\varphi_t(A)} \rho \mathbf{v}\cdot\dot{\mathbf{v}} dv_t. \tag{1.3.35}$$

Now, using (1.3.24) and the divergence theorem, we obtain

$$\begin{aligned}\int_{\varphi_t(A)} \rho\mathbf{b}\cdot\mathbf{v}dv_t + \int_{\partial\varphi_t(A)} \mathbf{t}(\mathbf{n})\cdot\mathbf{v}da_t &= \int_{\varphi_t(A)} \rho\mathbf{b}\cdot\mathbf{v}dv_t + \int_{\partial\varphi_t(A)} \mathbf{Tn}\cdot\mathbf{v}da_t \\ &= \int_{\varphi_t(A)} \rho\mathbf{b}\cdot\mathbf{v}dv_t + \int_{\partial\varphi_t(A)} \mathbf{n}\cdot\mathbf{T}^T\mathbf{v}da_t \\ &= \int_{\varphi_t(A)} [(\rho\mathbf{b} + div_{\mathbf{x}}\mathbf{T})\cdot\mathbf{v} + \mathbf{T}\cdot\nabla_{\mathbf{x}}\mathbf{v}]dv_t.\end{aligned} \tag{1.3.36}$$

Combining (1.3.34) with (1.3.36) and using the equation of motion (1.3.25), we obtain the relation (1.3.33). □

Definition 1.3.12. *Given a motion of the material, we refer to the list* $\{\mathbf{v},\rho,\mathbf{T}\}$ *as a flow. The flow is steady if* $\varphi_t(\mathcal{B}) = \varphi_0(\mathcal{B})$ *for all* t *and*

$$\frac{\partial\mathbf{v}}{\partial t} = \mathbf{0}, \quad \frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial\mathbf{T}}{\partial t} = \mathbf{0}. \tag{1.3.37}$$

In this case $\varphi_0(\mathcal{B})$ *is called the flow region. A flow is potential if the velocity is the gradient of a potential, that is, if there exists a function* ϕ *with the property that*

$$\mathbf{v} = \nabla_{\mathbf{x}}\phi. \tag{1.3.38}$$

Finally, a flow is irrotational if $\mathrm{curl}_{\mathbf{x}}\mathbf{v} = \mathbf{0}$.

Theorem 1.3.13. (Bernoulli's Theorem) *Consider a flow* $\{\mathbf{v},\rho,\mathbf{T}\}$, *where the stress tensor is given by a pressure* $-p\mathbf{1}$ *and the body force is conservative with potential energy* $\mathcal{V}$. *We have the following:*

(i) If the flow is potential, then

$$\nabla_{\mathbf{x}}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + \mathcal{V}\right) + \frac{1}{\rho}\nabla_{\mathbf{x}}p = \mathbf{0}. \tag{1.3.39}$$

(ii) If the flow is steady, then

$$\mathbf{v}\cdot\nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{v}^2 + \mathcal{V}\right) + \frac{1}{\rho}\mathbf{v}\cdot\nabla_{\mathbf{x}}p = 0. \tag{1.3.40}$$

(iii) If the flow is steady and irrotational, then

$$\nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{v}^2 + \mathcal{V}\right) + \frac{1}{\rho}\nabla_{\mathbf{x}}p = \mathbf{0}. \tag{1.3.41}$$

Proof. Since we have $\mathbf{T} = -p\mathbf{1}$, it follows that $\text{div}_{\mathbf{x}}\mathbf{T} = -\nabla_{\mathbf{x}} p$, and therefore the equation of motion (1.3.25) takes the form

$$\rho\dot{\mathbf{v}} = -\nabla_{\mathbf{x}} p + \rho\mathbf{b}. \tag{1.3.42}$$

Since the body force is conservative, we have $\mathbf{b} = -\nabla_{\mathbf{x}}\mathcal{V}$. Moreover, for a potential flow, we have, with the aid of (1.2.7),

$$\dot{\mathbf{v}} = \frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla_{\mathbf{x}}(\mathbf{v}^2) = \nabla_{\mathbf{x}}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}^2\right), \tag{1.3.43}$$

while for a steady flow,

$$\mathbf{v}\cdot\dot{\mathbf{v}} = \mathbf{v}\cdot\nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{v}^2\right), \tag{1.3.44}$$

and for a steady irrotational flow,

$$\dot{\mathbf{v}} = \nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{v}^2\right). \tag{1.3.45}$$

Relations (1.3.43)–(1.3.45), when combined with (1.3.42), yield the desired results (1.3.39)–(1.3.41). □

1.3.4 The Piola–Kirchhoff Stresses

The Cauchy stress tensor $\mathbf{T}$ measures the contact force per unit area in the deformed configuration, and it is convenient, especially for fluids whose current configuration is supposed known in advance. For many other problems of interest—especially those involving solids—it is convenient to work with a stress tensor that gives the force measured per unit area in the reference configuration. This is because in such problems the current configuration is not known in advance. To establish the form of this tensor, we have to formulate the momentum balance laws relative to the reference configuration $\varphi_0(\mathcal{B})$.

Note that by virtue of the mass balance law $(1.3.2)_1$, we can rewrite the momentum balance laws (1.3.14) and (1.3.15) in the following forms:

$$\int_{\varphi_0(A)} \dot{\mathbf{v}}\rho_0 dv_0 = \int_{\varphi_0(A)} \mathbf{b}\rho_0 dv_0 + \int_{\partial\varphi_0(A)} \mathbf{s}(\mathbf{N}) da_0 \tag{1.3.46}$$

and

$$\int_{\varphi_0(A)} (\mathbf{x}-\mathbf{x}_0)\times\dot{\mathbf{v}}\rho_0 dv_0 = \int_{\varphi_0(A)} (\mathbf{x}-\mathbf{x}_0)\times\mathbf{b}\rho_0 dv_0 + \int_{\partial\varphi_0(A)} (\mathbf{x}-\mathbf{x}_0)\times\mathbf{s}(\mathbf{N}) da_0, \tag{1.3.47}$$

where $\mathbf{s}(\mathbf{N})$ represents the force vector acting on the surface $\partial\varphi_t(A)$ but measured per unit area of the surface $\partial\varphi_0(A)$ in the reference configuration, the outward unit normal of which is denoted by $\mathbf{N}$. We have

$$\mathbf{s}(\mathbf{N})da_0 = \mathbf{t}(\mathbf{n})da_t,$$

so that using (1.2.46) and (1.3.30), where $da_i = n_i da_t$, $dA_j = N_j da_0$, with $\mathbf{n} = n_i \mathbf{i}_i$, $\mathbf{N} = N_i \mathbf{i}_i$, we obtain that

$$\mathbf{s}(\mathbf{N}) da_0 = \mathbf{t}(\mathbf{i}_j) n_j da_t = \mathbf{t}(\mathbf{i}_j) da_j = \mathbf{t}(\mathbf{i}_j) J \frac{\partial X_i}{\partial x_j} dA_i = \mathbf{t}(\mathbf{i}_j) J \frac{\partial X_i}{\partial x_j} N_i da_0.$$

We can write this relation in the form

$$\mathbf{s}(\mathbf{N}) = \mathbf{S}(\mathbf{X})\mathbf{N}, \tag{1.3.48}$$

where $\mathbf{S}$ is a tensor given by

$$\mathbf{S} = J\mathbf{T}\left(\mathbf{F}^{-1}\right)^T, \tag{1.3.49}$$

known as the *first Piola–Kirchhoff stress*. In terms of components, this relation is

$$S_{ij} = JT_{ik} \frac{\partial X_j}{\partial x_k} \quad \Leftrightarrow \quad T_{ij} = \frac{1}{J} \frac{\partial x_j}{\partial X_k} S_{ik}.$$

Note that (1.3.48) is of the same form as (1.3.24). If we use (1.3.48) in the balance laws (1.3.46) and (1.3.47), then the following result is obtained.

Proposition 1.3.14. *The first Piola–Kirchhoff stress tensor satisfies the field equations*

$$\rho_0 \ddot{\mathbf{x}} = \mathrm{Div}_{\mathbf{X}} \mathbf{S} + \rho_0 \mathbf{b} \tag{1.3.50}$$

and

$$\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T. \tag{1.3.51}$$

Here, the operator $\mathrm{Div}_{\mathbf{X}}$ is evaluated with respect to the material point $\mathbf{X}$ in the reference configuration.

Remark 1.3.15. It is important to note that by (1.3.51), $\mathbf{S}$ generally is not symmetric. If we introduce the second Piola–Kirchhoff stress tensor $\widehat{\mathbf{S}}$, defined by

$$\widehat{\mathbf{S}} = \mathbf{F}^{-1}\mathbf{S}, \quad \text{or, in components,} \; \widehat{S}_{ij} = \frac{\partial X_i}{\partial x_k} S_{kj}, \tag{1.3.52}$$

then from (1.3.51), it follows that $\widehat{\mathbf{S}}$ is a symmetric tensor.

It is related to the Cauchy stress tensor by

$$\widehat{\mathbf{S}} = J\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^{\top}. \tag{1.3.53}$$

We further have the following alternative version of the relation (1.3.33).

Theorem 1.3.16. (**Theorem of Power Expended**) *For every subbody A of the body, we have*

$$\int_{\varphi_0(A)} \rho_0 \mathbf{b} \cdot \dot{\mathbf{x}} dv_0 + \int_{\partial\varphi_0(A)} \mathbf{s}(\mathbf{N}) \cdot \dot{\mathbf{x}} da_0 = \frac{d}{dt} \int_{\varphi_0(A)} \frac{1}{2} \rho_0 \dot{\mathbf{x}}^2 dv_0 + \int_{\varphi_0(A)} \mathbf{S} \cdot \dot{\mathbf{F}} dv_0. \tag{1.3.54}$$

Proof. Let us take the inner product of the equation of motion (1.3.50) with $\dot{\mathbf{x}}$ and integrate over $\varphi_0(A)$. Using the divergence theorem and relation (1.2.11), we obtain (1.3.54). □

Note that using (A.2.4), we have

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}^\top) = \mathrm{tr}(\widehat{\mathbf{S}}\dot{\mathbf{F}}^\top \mathbf{F}) = \mathrm{tr}(\mathbf{S}^\top \dot{\mathbf{F}}) = \mathrm{tr}(\widehat{\mathbf{S}}\mathbf{F}^\top \dot{\mathbf{F}}).$$

Thus, by virtue of (1.2.15) and (1.2.20),

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \frac{1}{2}\mathrm{tr}(\widehat{\mathbf{S}}\dot{\mathbf{C}}) = \mathrm{tr}(\widehat{\mathbf{S}}\dot{\mathbf{E}}) = \widehat{\mathbf{S}} \cdot \dot{\mathbf{E}}.$$

1.4 Constitutive Equations

The mass and momentum balance principles apply to all bodies in nature and do not distinguish between different types of materials, in that they do not depend on the intrinsic structure of the material. Two different bodies of the same size and shape subjected to the same deformation will generally not have the same resulting stress distribution. For example, two thin wires of the same length and diameter, one of steel and one of copper, will require different forces to produce the same elongation. Therefore, the balance principles are insufficient to fully characterize behavior, and some additional hypotheses are required for a complete description of the behavior of a continuous body. Such supplementary hypotheses are known as *constitutive equations* and serve to distinguish different types of material behavior.

Constitutive equations also serve the purpose of providing a well-posed mathematical model for describing the deformation of a continuous body. In fact, supposing that the mass density of the body in the reference configuration is known and the body force field has been assigned, we have four differential equations (one is the continuity equation and the other three are the equations of motion) for the unknown set of functions defining, for example, the components of the displacement vector field, the mass density in the current configuration, and the components of the stress tensor. Clearly, the mathematical problem is underdetermined.

The possibility of dependence of constitutive quantities on not only the current values of field variables but also their past history is fundamental to the present work.

The importance of such memory properties in the study of the behavior of materials was first described by Cauchy in 1828 [60]. In this work, he observed that for solid bodies that are not quite elastic, "*les pressions ou tensions ne dépendent pas seulement du changement de form que le corps éprouve en passant de l'état naturel à un nouvel état, mais aussi des états intermédiaires et du temps pendant lequel le changement de form s'effectue*" [see [313] on page 56 (1960 edition)].

We now introduce the concept of objective tensors and the principle of material objectivity, which imposes constraints on the possible forms of constitutive equations. The remaining chapters of Part I and all of Parts II and III deal largely with properties of various specific constitutive equations, in most cases involving linear memory functionals, and of energy functionals associated with them.

1.4.1 Objectivity

Inertial frames are defined in Remark 1.3.5. However, we wish to consider more general frames of reference. Let $(\mathbf{x}, t)$ be the spatial coordinates in an inertial frame. Consider the frame of reference with coordinates $(\mathbf{x}',\ t')$ given by

$$\mathbf{x}' = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \qquad t' = t + t_0, \tag{1.4.1}$$

where t_0 is a constant. The reference description coordinates $\mathbf{X}$ are unchanged. The quantity $\mathbf{Q}(t)$ is an orthogonal matrix, so that

$$\mathbf{Q}^\top(t)\mathbf{Q}(t) = \mathbf{1}.$$

Thus $\mathbf{Q}$ is a time-dependent rotation and $\mathbf{c}$ is a time-dependent translation. We take $\det \mathbf{Q} = 1$.

Relation (1.4.1) is a Euclidean transformation. In a Galilean transformation (Remark 1.3.5), $\mathbf{Q}$ is time-independent and $\mathbf{c}(t) = \mathbf{c}_0 + \mathbf{V}t$, the quantity $\mathbf{V}$ being the relative velocity of the origins of the two frames under consideration, while $\mathbf{c}_0$ is a fixed vector.

Tensor quantities transform in a well-defined manner under (1.4.1) for $\mathbf{Q}$ time-independent. A subset of these quantities have the same transformation properties even if $\mathbf{Q}$ is time-dependent. Tensors in this subset will be referred to as objective tensors. In particular, if ϕ is an objective scalar, $\mathbf{a}$ an objective vector, and $\mathbf{B}$ an objective second-order tensor, then

$$\begin{aligned} &\phi'(\mathbf{X}, t') = \phi(\mathbf{X}, t), \quad \mathbf{a}'(\mathbf{X}, t') = \mathbf{Q}(t)\mathbf{a}(\mathbf{X}, t), \\ &\mathbf{B}'(\mathbf{X}, t') = \mathbf{Q}(t)\mathbf{B}(\mathbf{X}, t)\mathbf{Q}^\top(t). \end{aligned} \tag{1.4.2}$$

Various physical quantities are assumed to be objective tensors. These assumptions are linked to the principle of material objectivity discussed in Sect. 1.4.2. Thermodynamic quantities introduced later such as the internal and free energies, the entropy, and the temperature are taken to be objective scalars, while the heat flux is assumed to be an objective vector. The Piola–Kirchhoff heat flux, defined analogously to the Piola–Kirchhoff stress tensor, is an objective scalar, by virtue of the device introduced in (1.4.5) below. The Cauchy stress tensor is assumed to transform as a second-order objective tensor under a change of observer, so that

$$\mathbf{T}' = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t). \tag{1.4.3}$$

The particle velocity $\mathbf{v}$, given by (1.2.5), is not objective, nor is the kinetic energy density.

We note that the second-order tensor $\mathbf{F}$ is a transformation from the material to the spatial description. It acts like an objective vector in that under (1.4.1),

$$\mathbf{F}'(t) = \mathbf{Q}(t)\mathbf{F}(t). \tag{1.4.4}$$

Proposition 1.4.1. *The second Piola–Kirchhoff tensor* $\widehat{\mathbf{S}}$*, defined by* (1.3.52) *or* (1.3.53)*, is an objective scalar; all its components have this property.*

Proof. This follows from (1.4.3), (1.4.4), and the observation that $J = \det \mathbf{F} = \det(\mathbf{QF})$ is an objective scalar. □

Observe that

$$\dot{\mathbf{F}}'(t) = \mathbf{Q}(t)\dot{\mathbf{F}}(t) + \boldsymbol{\Sigma}(t)\mathbf{F}'(t),$$

where $\boldsymbol{\Sigma}$ is the spin tensor, defined as

$$\boldsymbol{\Sigma}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^\top(t).$$

Then by (1.2.10),

$$\mathbf{L}'(t) = \dot{\mathbf{F}}'(t)[\mathbf{F}'(t)]^{-1} = \mathbf{Q}(t)\mathbf{L}(t)\mathbf{Q}^\top(t) + \boldsymbol{\Sigma}(t).$$

We see from (1.4.4) that $\mathbf{C}$, given by $(1.2.15)_1$, and $\mathbf{E}$, given by $(1.2.20)_1$, are unaffected by the transformation (1.4.1). Thus, we have

$$\mathbf{C}' = \mathbf{C}, \qquad \mathbf{E}' = \mathbf{E},$$

and therefore, since, from $(1.2.15)_1$, $\mathbf{U} = \mathbf{C}^{1/2}$, we have

$$\mathbf{U}' = \mathbf{U}.$$

Thus, all these quantities are objective scalars, as shown for $\widehat{\mathbf{S}}$ in Proposition 1.4.1.

Note that if $\mathbf{a}$ is an objective vector and λ is an objective scalar, then by virtue of (1.4.4),

$$\mathbf{a}_s = \lambda \mathbf{F}^{-1}\mathbf{a} \tag{1.4.5}$$

is an objective scalar.

1.4.2 Principle of Material Objectivity

This principle [313], also termed the *principle of material frame indifference*, postulates that the intrinsic properties of a material, as expressed in its constitutive relations, do not depend on the observer frame. More recent discussions of the topic may be found in particular in [188, 195, 238].

For example, consider the simple case of a spring extended by an applied force [313]. Material frame indifference, in this case, is the statement that the spring constant is the same for all observers in all frames of reference given by (1.4.1).

Expressed more formally, it is the statement that the constitutive equations describing the response of a material must hold in all frames related by (1.4.1).

This principle is accepted as valid for most conditions, though breakdowns have been predicted, notably within the framework of rational extended thermodynamics [269]. An observation on page 258 of [195] is of interest in this context.

This principle is imposed by expressing constitutive relations in terms of objective tensors. For simple materials [195, 238, 313], these relations in general involve functionals of the history of $\mathbf{F}$ and thermodynamic variables that are taken to be objective scalars either by assumption or by construction as in (1.4.5).

Explicit dependence on time is excluded by indifference to the origin of time as expressed by $(1.4.1)_2$. Note, however, that if a material is aging ([167], for example), explicit time dependence can occur.

Explicit dependence on $\mathbf{X}$ will occur if the material is inhomogeneous. The property of inhomogeneity is generally assumed in the present work, though we will often omit explicit inclusion of the space dependence.

Thus, we write a general constitutive relation as

$$\begin{aligned} \mathbf{A}(\mathbf{X}, t) &= \hat{\mathbf{A}}(\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0), \\ \mathbf{F}^t(\mathbf{X}, s) &= \mathbf{F}(\mathbf{X}, t-s), \qquad \mathbf{\Theta}^t(\mathbf{X}, s) = \mathbf{\Theta}(\mathbf{X}, t-s), \quad \forall s \geq 0, \end{aligned} \tag{1.4.6}$$

where $\mathbf{\Theta}$ represents a list of objective scalar variables and $\mathbf{A}$ is a scalar, vector, or higher order objective tensor. In the primed frame of reference, using only $(1.4.1)_1$, this becomes

$$\begin{aligned} \mathbf{A}'(\mathbf{X}, t') &= \hat{\mathbf{A}}'(\mathbf{F}'(\mathbf{X}, t-s),\ \mathbf{\Theta}'(\mathbf{X}, t-s); s \geq 0) \\ &= \hat{\mathbf{A}}'(\mathbf{Q}\,(t-s)\mathbf{F}(\mathbf{X}, t-s),\ \mathbf{\Theta}'(\mathbf{X}, t-s); s \geq 0), \end{aligned}$$

where $(1.4.2)_1$ and (1.4.4) have been used. The principle of material objectivity can be stated as follows: the functional $\hat{\mathbf{A}}'$ is the same functional as $\hat{\mathbf{A}}$ for all frames of reference, or

$$\begin{aligned} &\hat{\mathbf{A}}'(\mathbf{Q}(t-s)\mathbf{F}(\mathbf{X}, t-s), \mathbf{\Theta}'(\mathbf{X}, t-s); s \geq 0) \\ &\qquad = \hat{\mathbf{A}}(\mathbf{Q}\,(t-s)\mathbf{F}(\mathbf{X}, t-s),\ \mathbf{\Theta}'(\mathbf{X}, t-s); s \geq 0), \end{aligned}$$

for all choices of the independent field variables. Thus, for $\hat{\mathbf{A}} = \phi,\ \mathbf{a}, \mathbf{B}$, transforming as specified by $(1.4.2)_2$, we have the conditions

$$\begin{aligned} \hat{\phi}(\mathbf{Q}(t-s)\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0) &= \hat{\phi}(\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0), \\ \hat{\mathbf{a}}(\mathbf{Q}(t-s)\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0) &= \mathbf{Q}(t)\hat{\mathbf{a}}(\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0), \\ \hat{\mathbf{B}}(\mathbf{Q}(t-s)\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0) &= \mathbf{Q}(t)\hat{\mathbf{B}}(\mathbf{F}^t(\mathbf{X}, s),\ \mathbf{\Theta}^t(\mathbf{X}, s); s \geq 0)\mathbf{Q}^\top(t), \end{aligned} \tag{1.4.7}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and orthogonal tensors $\mathbf{Q}$, where the notation $(1.4.6)_{2,3}$ has been used.

The implications of the principle of material objectivity for the possible forms of constitutive equations are considerable [195, 238, 313], as shown by the following example.

Proposition 1.4.2. *Let* $\mathbf{F}(t-s) = \mathbf{R}(t-s)\mathbf{U}(t-s)$ *be the polar decomposition of* $\mathbf{F}$. *If an objective scalar obeys the principle of material objectivity* $(1.4.7)_1$*, then it can be expressed in terms of the current value and history of* $\mathbf{U}$ *and* $\mathbf{\Theta}$*; that is to say, it will not depend on* $\mathbf{R}$. *Since by* (1.2.15)*, we have* $\mathbf{U} = \sqrt{\mathbf{C}}$*, it follows that the scalar is a function of the current value and a functional of the history of* $\mathbf{C} = \mathbf{F}^\top\mathbf{F}$.

Conversely, if it has this property, then $(1.4.7)_1$ *holds.*

Proof. The first assertion follows since we can always choose $\mathbf{Q}(t-s) = \mathbf{R}^\top(t-s)$ in $(1.4.7)_1$. The converse is immediate. □

Remark 1.4.3. Note that by virtue of (1.2.20), we can replace $\mathbf{C}(t-s)$ by the Green strain tensor $\mathbf{E}(t-s)$.

We must emphasize that Proposition 1.4.2 does not refer to the possible dependence on $\mathbf{F}$ of thermodynamic quantities in $\mathbf{\Theta}$ included to make them objective scalars, using (1.4.5).

Consider a particular class of bodies, the behavior of which depends on the history of the deformation gradient up to time t, $\mathbf{F}(\mathbf{X}, t-s) \forall s \in \mathbb{R}^+$. These materials are such that the stress tensor is given by the functional

$$\mathbf{T}(\mathbf{X}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t),\ \mathbf{F}^t(\mathbf{X})), \tag{1.4.8}$$

where $\mathbf{F}(\mathbf{X}, t)$ is the current value and $\mathbf{F}^t(\mathbf{X}, s) = \mathbf{F}(\mathbf{X}, t-s)\ \forall s \in \mathbb{R}^{++}$ denotes the past history of $\mathbf{F}$. The functional $\hat{\mathbf{T}}$ may also depend on objective scalars as denoted by $\mathbf{\Theta}$ above.

The requirement of material frame indifference, as stated by $(1.4.7)_3$, yields that the functional $\hat{\mathbf{T}}$ must obey the relation (omitting $\mathbf{X}$)

$$\begin{aligned} &\hat{\mathbf{T}}(\mathbf{Q}(t)\mathbf{F}(t),\ \mathbf{Q}(t-s)\mathbf{F}(t-s)),\ s \in \mathbb{R}^{++}) \\ &\qquad = \mathbf{Q}(t)\hat{\mathbf{T}}(\mathbf{F}(t),\ \mathbf{F}(t-s),\ s \in \mathbb{R}^{++})\mathbf{Q}^\top(t), \end{aligned} \tag{1.4.9}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and orthogonal tensors $\mathbf{Q}$.

Proposition 1.4.4. *Property* (1.4.9) *is equivalent to the requirement that*

$$\hat{\mathbf{T}}(\mathbf{F}(t),\ \mathbf{F}(t-s)), s \in \mathbb{R}^{++}) = \mathbf{F}(t)\tilde{\mathbf{T}}(\mathbf{C}(t),\ \mathbf{C}(t-s),\ s \in \mathbb{R}^{++})\mathbf{F}^\top(t),$$

where $\mathbf{C}$ *is the right Cauchy–Green tensor, defined by* (1.2.15). *The dependence of* $\tilde{\mathbf{T}}$ *on* $\mathbf{C}$ *is not restricted by the property of material frame indifference. Note that from* (1.3.52)*,* $\tilde{\mathbf{T}}$ *is related to the second Piola–Kirchhoff stress tensor by*

$$J\tilde{\mathbf{T}} = \widehat{\mathbf{S}}.$$

Proof. This follows immediately from Propositions 1.4.1 and 1.4.2. □

1.4.3 Fading Memory

We shall consider materials for which the property of fading memory holds. This property is expressible through the (Volterra) dissipation behavior of hereditary action [318], which states "*the modulus of the variation of the quantity* [given by (1.4.8)], *when* $\mathbf{F}^t$ *varies in any way … in the interval* $(-\infty,\ t_1)$ *(with* $t_1 < t$*) can be made as small as we please by taking the interval* (t_1, t) *sufficiently large.*"[‖]

‖ In the Coleman and Noll theory [73], the fading memory property is given by the continuity of (1.4.8) with respect to the norm

$$||\mathbf{F}^t||^2 = \int_0^\infty h(s)|\mathbf{F}^t(s)|^2 ds,$$

where the map $h \in L^1(\mathbb{R}^+)$ is a suitable positive decreasing function.

A more precise definition of the property of fading memory at a material point $\mathbf{X} \in \mathcal{B}$ can be given by considering the set $\mathcal{D}$ of the histories that make up the domain of definition of the functional (1.4.8).

For this purpose, we suppose the set $\mathcal{D}$ has the following properties:

1. $\mathcal{D} = \text{Lin} \times \mathcal{D}_r$, where $\mathcal{D}_r$ is a set of past histories that contains the space $L^\infty(\mathbb{R}^+)$.
2. The partly static history $\mathbf{F}^{t(\tau)}$, associated with $\mathbf{F}^t(\mathbf{X})$, is defined by

$$\mathbf{F}^{t(\tau)}(\mathbf{X}, s) := \begin{cases} \mathbf{F}(\mathbf{X}, t) & \forall s \in [0,\ \tau), \\ \mathbf{F}^t(\mathbf{X}, s - \tau) & \forall s \in [\tau,\ \infty), \end{cases} \tag{1.4.10}$$

where τ is the duration of the static part of the history. If $\mathbf{F}^t(\mathbf{X}) \in \mathcal{D}$, then $\mathbf{F}^{t(\tau)}$ belongs to $\mathcal{D}$.

Definition 1.4.5. *A viscoelastic material is characterized by the constitutive equation (1.4.8), where* $\mathbf{F}^t \in \mathcal{D}$*, and there exists a constitutive equation* $\mathbf{T}(\mathbf{X}, t) = \tilde{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t))$ *of an elastic material such that*

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}(\mathbf{F}^{t(\tau)}(\mathbf{X})) = \tilde{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t)).$$

Moreover, $\hat{\mathbf{T}}(\mathbf{F}^{t(\tau)}(\mathbf{X})) - \tilde{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t))$ *is a function of* τ*, which belongs to* $L^2(\mathbb{R}^+)$*.*

This definition includes an expression of the fading memory property. Consider its application to the simplest case, namely a linear constitutive relation defining a linear viscoelastic material. Such linear relations will be systematically derived and discussed in Part III. For a linear viscoelastic body, we have

$$\mathbf{T}(\mathbf{X}, t) = \mathbb{G}_0(\mathbf{X})\mathbf{E}(\mathbf{X}, t) + \int_0^\infty \mathbb{G}'(\mathbf{X}, s)\mathbf{E}^t(\mathbf{X}, s)ds, \tag{1.4.11}$$

where $\mathbf{E} \in \text{Sym}$ is the strain tensor.** The infinitesimal approximation to this quantity, as given by (1.2.31), is generally, though not necessarily, used in this context. The quantities $\mathbb{G}_0$ and $\mathbb{G}'$ are fourth-order tensors in Lin(Sym). The domain $\mathcal{D}$ consists of the set of pairs $(\mathbf{E}(t),\ \mathbf{E}^t)$ such that $\mathbf{E}(t) \in \text{Sym}$ and $\mathbb{G}'\mathbf{E}^t \in L^1(\mathbb{R}^+)$.

In the linear theory, $\mathcal{D}$ includes constant histories by property 1. It follows that the kernel $\mathbb{G}'$ belongs to $L^1(\mathbf{R}^+)$. Then if $\mathbb{G}' \in L^1(\mathbb{R}^+)$, we conclude that $\mathbb{G}'\mathbf{E}^{t(\tau)} \in L^1(\mathbb{R}^+)$, where $\mathbf{E}^{t(\tau)}$ is the partly static history associated with $\mathbf{E}^t$. Hence,

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}(\mathbf{E}^{t(\tau)}) = \mathbb{G}_\infty \mathbf{E}(t), \qquad \mathbb{G}_\infty = \mathbb{G}_0 + \int_0^\infty \mathbb{G}'(s)ds. \tag{1.4.12}$$

We observe that (1.4.11) represents a viscoelastic material with the fading memory property, according to Definition 1.4.5, because the right-hand side of $(1.4.12)_1$ is the stress associated with an elastic material. For the same reason, the tensor $\mathbb{G}_\infty$ must be a positive definite tensor in the case of a solid, though it may vanish for a liquid. Thus, we have

$$\mathbb{G}_\infty \geq \mathbf{0}. \tag{1.4.13}$$

** This follows from the principle of material frame indifference as expressed through Proposition 1.4.4.