Research in Mathematics Education

Series Editors: Jinfa Cai · James A. Middleton

Karen Hollebrands
Robin Anderson
Kevin Oliver *Editors*

Online Learning in Mathematics Education

Springer

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Online Learning in Mathematics Education

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Foreword

Research on instructional design of online learning environments has advanced in concert with technology's improving capability for interaction, assessment, and integrated tool use in learning and instruction. The COVID-19 global pandemic has accelerated the pace of this research.

In many places, in fact, the model for teaching has had to change dramatically to accommodate the constraints and affordances of online environments. Online learning environments magnify a number of logistical, pedagogical, and equity issues addressed in some depth in this volume. Chief among the issues facing mathematics teaching, learning, and policy are the following: (1) logistical hurdles related to obtaining equipment and access; teachers, students (and parents) learning the new technologies; (2) the impact of digital communication in the home environment (including physical, social, and family resources); (3) pedagogical considerations related to meeting virtually; and (4) the changing roles of teachers and students as students are afforded expanded degrees of freedom for and responsibility toward their own learning.

Teachers are at the heart of educational reform and instructional improvement. Helping teachers learn to teach online is not only a timely line of investigation but also a necessity for teachers to teach their students online in ways that are pedagogically sound, mathematically rigorous, motivationally compelling, and equitable for students. This book offers guidance and insights to mathematics educators who are teaching prospective and practicing teachers online. In addition to discussing the design of learning environments for mathematics teacher education, this book discusses the informal and formal ways to help prospective and practicing mathematics teachers develop insight and develop new practices and routines specifically about online teaching.

The primary audience for this book is mathematics teacher educators and mathematics education researchers. We find that the book is insightful and helpful for enhancing curriculum for preparing prospective and practicing teachers to include online instruction. It is also helpful for conducting research on teacher learning that takes place in formal and informal online settings. Importantly, online instruction, it seems, has found a permanent place in schooling. Experiences shared in this book are applicable even when the current COVID-19 pandemic subsides and in-person instruction resumes.

Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared toward highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those in the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission of this book series is:

1. To support the sharing of critical research findings among members of the mathematics education community.
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We are grateful for the support of Melissa James from Springer in developing and publishing this book series, as well as the support for the publication of this volume. And finally, we thank the editors (Karen Hollebrands, Robin Anderson, and Kevin Oliver) and all of the authors and peer reviewers who have contributed to this volume for their insightful and synthetic work!

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Preface

This book brings together research from mathematics education and instructional design to describe the development and impact of online environments on prospective and practicing teachers' learning to teach mathematics. The move to online learning has steadily increased over the past decade. Its most rapid movement occurring in 2020 with most instruction taking place remotely. Thus, the chapters in this book are very timely and can offer guidance and insights to mathematics teacher educators who are teaching prospective and practicing teachers online.

Chapters in this book are organized in three sections that highlight issues related to (1) the design of learning environments for mathematics teacher education; (2) formal experiences for prospective and practicing mathematics teachers; and (3) self-directed, experiential, and practice-based online learning opportunities for mathematics teachers.

Section one of the book focuses on the instructional design of online learning content, activities, courses, and curriculum for training prospective and practicing mathematics teachers. This focus on online learning designs for preparing mathematics teachers is timely as nearly every chapter in this section mentions the recent COVID-19 pandemic as an impetus or rationale for their online programs, with tested innovations that may have been necessitated by this crisis poised to influence how we prepare mathematics teachers in the future under normal conditions. Readers interested in designing similar innovations can take note of how groups of teacher educators in most chapters collaborated to design and enact their online programs.

For readers new to designing online learning experiences, Chap. 1 by **Hodge-Zickerman, York, and Lowenthal** offers a broad orientation to types of online learning environments, learning theories to steer course design, and strategies for engagement through popular online technologies. The authors illustrate with examples how to effectively apply learning theories through related teaching strategies and aligned online learning tools.

Chapters 2 through 4 in this section of the book detail the development of varied online learning activities and materials with an eye toward supporting social interaction among learners. In Chap. 2, **Borba, Engelbrecht, and Llinares** provide informative background on technologies used in mathematics education over the years and definitions of varied learning environments, before focusing on new technologies of participation that allowed their students to communicate and build knowledge via the Internet (e.g., video sharing and analysis, collaborative construction of mind maps). In Chap. 3, **Tran and Nguyen** discuss how the Community of Inquiry theoretical framework and integrated teaching-social-cognitive presences informed activity design in two courses for pre-service mathematics teachers (e.g., social presence supported by Zoom breakout rooms, manipulation of virtual manipulatives, small group work in Google Docs, and synchronous follow-ups to gauge student understanding). In Chap. 4, **Lee, Hudson, Casey, Mojica, and Harrison** present their comprehensive design plan for online curriculum modules to prepare mathematics teachers to teach statistics, involving a cartridge approach supporting re-use across multiple online learning management systems and synchronous/asynchronous modes. Data indicates which module materials and activities were most and least utilized (e.g., videos of classroom practice, forums) with resulting modifications (e.g., PlayPosit interactions layered over video). The authors further describe how math teacher educators were trained to use the modules with implementers leveraged to support new participants as part of a user community.

Finally, Chaps. 5 and 6 in this section of the book focus on rehumanizing and culturally responsive designs that seek to shift classroom power dynamics toward more inclusive learning for those who have been historically excluded. In Chap. 5, **Jessup, Wolfe, and Kalinec-Craig** focus on rehumanizing practices in online settings to develop all learners' identities and provide for mathematics opportunities that may otherwise be limited (e.g., equitable group work, community-building activities). The authors discuss dimensions and teaching practices to ground this work, focusing on the participation dimension and two teaching practices: learning about cultures/identities, and teaching practices for rehumanizing. They also offer a helpful discussion of online practices that may perpetuate inequity and be dehumanizing with suggested alternatives that are more social in nature and align well with the prior three chapters (e.g., use of listening dyads, small group breakouts through synchronous platforms like Zoom). In Chap. 6, **Alarcón, Chauvot, Cutler, and Gronseth** provide a glimpse at how a group of faculty worked collectively to deepen their understanding of culturally responsive teaching of mathematics methods for pre-service teachers (i.e., learning in action with reflective journaling and debriefing of teaching experiences; using Microsoft Teams software to meet online, share documents, and reflect). Continuing with a theme in this section of the book on designs that allow students to co-construct knowledge socially, the authors detail how number talk strategies were designed to be more culturally responsive through online technology (e.g., breakout groups for get-to-know-you activities, Nearpod polling for students to indicate they were ready to share, Jamboard to display thinking, Nearpod collaborative note boards to share answers). In sum, the chapters in this section provide specific examples of mathematics teacher educators designing varied online learning experiences to support prospective and practicing teachers in learning to teach math effectively, drawing on traditional, online-specific, and culturally responsive theories to guide their work.

Formal online experiences for prospective and practicing mathematics teachers to learn how to teach mathematics are the focus of the second section of the book. We begin this section with Chap. 7, authored by **I**, **Martinez**, **and Jackson**, which describes how different frameworks such as culturally relevant pedagogy and culturally sustaining pedagogy, along with iterative design were used to guide the development of an asynchronous, online, course to prepare practicing and prospective K-12 teachers to teach emergent bilinguals. This course is unique in its focus on strategies specific to the teaching of mathematics.

In Chap. 8, **Byeonguk Han and Thanheiser** describe how they responded to the pandemic by transitioning their in-person implementation of addition Number Talks to an asynchronous online environment for prospective elementary teachers. They describe the different strategies teachers used as they engaged online and explain how this practice can be implemented successfully, and perhaps more equitably, in an asynchronous setting.

Transitioning to online professional development for practicing teachers, **Choppin, Amador, Callard, Carson, Gillespie, Kruger, Martin, and Foster** provide illustrations in Chap. 9 of their three-part professional development model to support middle school teachers' implementation of ambitious instructional practices. First, teachers read about instructional practices they may not have experienced as learners, then they observe these practices in action, and finally, they enact these practices with the support of a coach. Delivered online, teachers complete a course that is interspersed with teaching labs and receive video-assisted coaching.

The final chapter of this section, Chap. 10, **Fernández, Llinares, and Rojas**, describes how secondary mathematics teachers developed their noticing skills through the writing of narratives about their teaching during a practicum experience. These narratives were shared with a tutor and their peers in online discussion forums. The authors suggest that the writing-feedback-revision cycles and specific design elements of the online course enabled teachers to improve their abilities to notice mathematical opportunities to build on student thinking (MOSTs) over the course of a semester.

In the final section, a focus is placed on self-directed, experiential, and practice-based online learning opportunities for mathematics teachers. We begin this section with Chap. 11, a theoretical piece on teachers' motivation to engage in online professional development by **Hawk, Bowman, and Xie.** After a review of existing literature on Expected-Value Theory, the authors propose 5 design principles for online professional development to increase teacher motivation. The next three chapters in this section examine self-directed learning experiences for mathematics teachers.

In Chap. 12, **Arzarello, Robutti, and Taranto** present the over 20-year evolution of the Italian Math MOOC UniTo project. Using a theoretical framework of Meta-Didactical Transposition, the authors report on the evolution of course materials and interactions between researchers and teachers. The presentation of meta-didactic praxeologies provides detailed accounts of how course material can be theoretically changed to meet the needs of remote learning. Continuing to discuss self-directed mathematics teacher learning, **Wilhelm and Ruddock** examine social-media facilitated opportunities in Chap. 13. Using social network analysis, the authors examine the two largest mathematics teacher hashtags on Twitter (#MTBoS and #iteachmath). The authors argue that social media connects more mathematics teachers together to informally learn, noting that over 26,500 potential connections could be made in their data set, which provides a large network for potential support and collaboration. Chap. 14 is the final chapter regarding self-directed online mathematics teacher learning. **Miller and Braley** presented an online book club grounded in distributed leadership and communities of practice. The authors provide a description of their protocols and provide an analysis of their implementation during a book study for college-level mathematics instructors. Detail is provided on three macro-activity tasks: (a) launching the book study group, (b) supporting the participants, and (c) supporting the facilitators.

The final two chapters in this section leverage practice-based experiences embedded in online technologies. In Chap. 15, **Milewski, Stevens, Herbst, and Huhn** present the implementation of contingency cards into online professional development. Contingency cards are representations of practice that PD facilitators can call upon to represent realistic situations teachers might face. Drawing on data from an online, practice-based professional development, the authors share how teachers engage in learning experiences using the contingency cards and how these experiences impact a teacher's ability to (1) elicit, (2) select, and (3) respond to students' mathematical contributions. Finally, in Chap. 16, **Bondurant and Amidon** present exploratory work conducted in the Mursion™ environment to prepare pre-service mathematics teachers (PSTs) to use equitable teaching practices through virtual field experiences (VFEs). The authors report on the affordances and constraints of VFEs and their ability to influence PSTs efficacy, skills, and equitable teaching practices of PSTs.

Mathematics teacher educators and mathematics education researchers will find the chapters helpful in preparing prospective and practicing teachers online and conducting research on teacher learning that takes place in formal and informal online settings. While we recognize the pandemic has resulted in the rapid shift to virtual learning, it is likely some practices that have been particularly effective will continue even after in-person instruction resumes.

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Chapter 1
Teaching Mathematics Education Online: Instructional Theories, Strategies, and Technologies

Angie Hodge-Zickerman, Cindy S. York, and Patrick R. Lowenthal

You can't teach math online.
—*Anonymous*

Enrollments in online courses have been increasing for over a decade (Allen and Seaman 2017; NCES 2018). However, despite students' increased interest in taking courses online, many educators have had strong feelings about what can and cannot be taught online in the past (Boz and Adnan 2017; Vivolo 2016). For instance, some have argued that subjects like art (Baker et al. 2016), mathematics (Boz and Adnan 2017), or classes with laboratories (Jeschofnig and Jeschofnig 2011; Reuter 2009) cannot be taught online. At the same time, other research has suggested that some students prefer not to take certain courses, like mathematics, online (Jaggars 2014; Krishnan 2016). However, the COVID-19 pandemic, as well as previous experiences during natural disasters (see Agnew and Hickson 2012; Bozkurt and Sharma 2020), has illustrated that sometimes any and all courses and subjects must be taught either online or remotely in some format for some time. While experienced online mathematics educators have years of experience and strategies to teach mathematics online, most educators (whether at the elementary, secondary, or higher education level) have very little, if any, experience or ideas about how to teach mathematics

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online (Lederman 2019). Given this current state, in this chapter, we provide an overview of instructional theories, strategies, and technologies that can guide instructors new to teaching mathematics education online in a formal online classroom setting.

1.1 Challenges Teaching Mathematics Education Courses Online

Online educators have argued for decades that teaching online is different than teaching face-to-face (Simonson et al. 2011, 2019). This is largely because of the different ways instructors and students communicate and interact online and the challenges this presents (Baran et al. 2011; Jorgensen 2003; Salmon 2012). The typical online course relies solely on asynchronous communication and interaction. This is not without good reason. The asynchronous format enables students to work on their course from any time and any place; it can enable students time to process what they are learning, which can help a variety of different students whether they be introverted, struggling, or second-language learners. This format can also help students interact with instructors and students from all over the world in ways that are not possible in other educational formats. However, despite affordances like these, online educators often struggle with the inherent delay of asynchronous communication and interaction as well as the lack of body language and visual cues with text-based asynchronous communication in particular (Lloyd et al. 2012). Online educators regularly report issues with their inability to see their students in real time to determine what the students do and do not understand (Baran et al. 2011; Conrad 2004; McQuiggan 2012). This can be especially problematic when teaching subjects like mathematics/mathematics education—where many students might struggle and/or have a lot of anxiety. To complicate matters even further, many educators in 2020 found themselves tasked with teaching online mathematics education courses, with little-to-no training on how to teach in blended or online formats (Hodges et al. 2020). Unfortunately, effective online learning requires both intentional and thoughtful design and effective facilitation, which take both time and effort (Hodges et al. 2020). However, we posit that there are some strategies (i.e., both pedagogical and technological) mathematics educators can employ to successfully design and teach online when and if they are tasked to teach in an emergency remote, blended, or online format in the future.

Before describing some of these strategies, it is important to understand the different types of mathematics education courses. Most teacher education programs offer a variety of mathematics teacher education courses—whether that be undergraduate, graduate, or professional development courses, to name a few. While many of the strategies we will discuss apply to all types of mathematics teacher education courses, our focus is on the undergraduate mathematics teacher education courses offered in initial teacher certification/licensure programs that are typically taught in a face-to-face classroom, which usually fall into two categories: mathematics content courses and mathematics methods courses.

Future teachers of all grade levels learn to teach mathematics in mathematics methods courses. In these methods courses, they learn about the interplay of mathematical knowledge and pedagogy. They learn how to implement instructional strategies as well as how a conceptual understanding of the mathematics content can help them teach for deep mathematical understanding. Secondary mathematics teachers are supposed to learn content knowledge in their mathematics content classes (e.g., calculus), but we also know there is often a disconnect from the content they learn in those courses and what they have to teach. Methods courses are typically where students learn to make connections between their mathematics courses and the mathematics they will 1 day teach. The specific content in methods courses vary from institution to institution and are historically not well documented. Most universities do not even use a textbook for their methods courses.

Instructors in face-to-face classrooms model different pedagogies they envision future mathematics teachers adopting. This modeling of pedagogy should continue when instructors transition to online teaching. We contend that modeling active learning strategies, teaching activities, and providing students opportunities to practice-teach using these strategies should happen in all mathematics courses regardless of the learning environment.

1.2 Strategies for Teaching Mathematics Education Courses Online

There are various instructional theories and corresponding instructional strategies and technologies that can support learning mathematics education in an online setting. In the following section, we describe some strategies that mathematics teacher educators can use the next time they teach mathematics education courses online or in some other type of hybrid or remote setting. We, first, discuss common types of online learning environments.

1.2.1 Identify Online Learning Environment

Often educators have no control over the type of learning environment they are required to teach in. For instance, even before COVID-19, administrators often dictated whether a course was taught in a face-to-face, blended, or online format as well as other details about the course whether that be class size, how often it meets, and even the learning platform(s) used (e.g., Blackboard, Canvas, D2L, etc.). This is especially true in the time of COVID-19 when instructors at many universities have been told in what mode they are to teach (which in the case of COVID-19 is largely blended, remote, or fully online). So, while mathematics teacher educators should ideally base their instructional decisions on theory and research, sometimes educators have to work within certain confines, such as the type of learning environment they must teach their course. Hence, in this chapter, we first discuss common learning environments for mathematics education classrooms. Then, we discuss how instructional theories, technologies, and instructional strategies can help instructors effectively teach mathematics education courses online.

There are many different ways to learn online (Hodges et al. 2020; Lowenthal et al. 2009). However, for brevity, we will focus on a few common ways to think about different online learning environments in higher education. We recognize, though, as mentioned earlier that many educators, even long before COVID-19, might have little control over selecting their own online learning environment. For instance, some programs believe that online learning should never require synchronous meetings and therefore might have policies against synchronous requirements. On the other hand, as colleges and universities were forced to close their doors and move all face-to-face courses into some online or remote format due to COVID-19, many faculty were told they had to offer their courses in a "remote" synchronous online learning format, where they could meet online each week virtually, using some type of web conferencing technology (e.g., Zoom, WebEx, Google Meet) at the same day and time as their regular class was originally scheduled to meet. Ideally, an educator would be able to intentionally make an informed decision about which online learning environment is best for them, their students, their instructional theories and strategies, and the course outcomes. We will elaborate on the two main types of online learning environments (i.e., asynchronous and synchronous) and then briefly discuss two less common but still important flexible learning environments that are becoming more common in recent times (i.e., bichronous and HyFlex; see Table 1.1 for a brief overview of each).

Table 1.1 Course classification of online and blended courses

Course classification	Description
Asynchronous online learning	A course where most of the content is delivered online and students can participate in the online course from anywhere and anytime. There are no real-time online or face-to-face meetings
Synchronous online learning	A course where most of the content is delivered online and students can participate in courses from anywhere. There are real-time online meetings, and students log in from anywhere but at the same time to participate in the course
Bichronous (Martin et al. 2020)	The blending of both asynchronous and synchronous online learning, where students can participate in any time, anywhere learning during the asynchronous parts of the course but then participate in real-time activities for the synchronous sessions
HyFlex	HyFlex is designed as a model where the student is given the option to either attend in campus or online. https://edtechbooks.org/hyflex

Adapted from Martin and Oyarzun (2017) and Martin et al. (2020)

Asynchronous Online Learning Environments

Asynchronous online learning environments are essentially online courses that primarily, if not solely, use asynchronous communication and interaction. Thus, its defining feature is that it enables instructors and students the ability to log in at a time convenient to them often within a given time frame (e.g., login 2–3 days in a given week). Online educators often think in terms of communication and interaction. When it comes to interaction, it is common to focus on instructor-student, student-student, and student-content interaction (Moore 1989). So, when educators think about asynchronous online learning environments, they typically think of a learning management system (LMS) like Canvas, Blackboard, or Moodle, the most common way these types of online courses are delivered, and how instructors and students can interact in these LMSs.

An LMS enables students to log in and interact with the content (i.e., student-content interaction), whether that be recorded lectures, reading articles, and so forth, take part in asynchronous threaded discussions in discussion forums, or complete online assessments at a time and place, given course deadlines (in which there are often multiple each week), that is convenient for them. Therefore, unlike synchronous online learning environments, in asynchronous online learning environments, neither students nor instructors have to be online at the same time. This type of environment is often ideal for nontraditional students or graduate students that might have to plan and balance their studies around their multiple priorities and commitments (whether that be a full-time job or a family). However, this type of environment also works best when the course has already been designed and developed in the LMS before the semester begins. Given this, a common critique with teaching online is that it feels like it involves more work than a traditional face-to-face (due in part to needing to have the course set up before a semester begins); others have also complained that teaching asynchronous online courses can feel like the instructor is always on (responding to students questions at all hours of the day and on weekends) (Dunlap 2005; Hogan and McKnight 2007).

When teaching mathematics education courses in an asynchronous learning environment, educators can include learning activities such as threaded discussion boards where students share what they learned from readings and videos or even where they share lesson plan ideas. Or when teaching mathematics content courses for preservice teachers, educators could use recorded lectures to support student learning by illustrating how to think about and solve mathematical problems. These types of approaches allow students to learn at their own pace and at a time that's convenient for them while also learning from others by sharing their ideas using tools like discussion board threads (de Leon and Prudente 2019).

Synchronous Online Learning Environments

Synchronous online learning environments, on the other hand, generally consist of students and instructor(s) who log in at the same time to a live video and/or audio conference session (Finkelstein 2006). This model has its roots in some of the earlier broadcast models of distance education (e.g., education television; see Bates 1988, Kentnor 2015). However, advances in web conferencing technology today make this form of online learning most closely resemble an in-person class where everyone is interacting together in real time, in a shared online environment. In fact, because of the similarities of synchronous online learning to in-person face-to-face learning, when schools, colleges, and universities were forced to close their doors due to the COVID-19 pandemic, many teachers adopted this form of distance or online learning (Lowenthal et al. 2020a, b).

Synchronous online learning environments have many benefits. For instance, they enable instructors and students to meet from any place (assuming everyone has a stable and sufficient internet connection) in a "face-to-face" manner in real time, which can help address some of the communication challenges (i.e., time delays, lack of visual cues) people might experience in asynchronous learning environments (Lowenthal et al. 2020a, b). Online class sessions can also easily be recorded and viewed later, enabling both students and instructors to go back and rewatch/relisten parts they might have missed (or to rewatch parts to solidify learning). Synchronous online learning, though, has some inherent disadvantages such as bandwidth limitations, participants losing internet access and being dropped from the class, difficulties with audio/video, interruptions from participants with technical difficulties, multitasking and distractions, or even simply the lack of participation.

When teaching mathematics education courses in a synchronous learning environment, educators can facilitate the learning environment in much the same manner they facilitate their face-to-face classes. They can problem solve as a whole class or in breakout groups, they can work problems for their students, they can have students present their work to the class, and they can even pose challenges (be it education based or mathematics based) to the class and work on the solutions in real time (Morge 2020).

Flexible Online Learning Environments

Two other increasingly used online learning environments include bichronous and HyFlex learning environments.

Bichronous. Online instructors have been using bichronous online learning environments that incorporate both synchronous and asynchronous environments in their online classes for a number of years now. These environments essentially are a blend of asynchronous and synchronous learning environments (unlike a hybrid environment, which typically refers to online and in-person, without delineating synchronous or asynchronous in the online portion). While instructors who use synchronous online learning environments might use some asynchronous resources (e.g., a learning management system to host readings, assignments, a grade book, etc.), bichronous learning environments place more emphasis on the blending of asynchronous and synchronous communication and interaction, which in turn provides the advantage of the synchronous (real-time) meetings alongside the asynchronous portions (Fadde and Vu 2014). This gives students time to think, work, and prepare for their synchronous meetings (Martínez et al. 2020); yet, in the age of COVID-19, students do not need to step foot on campus for this type of learning environment.

HyFlex. HyFlex is a more recently coined term used to describe a learning environment that gives students the option to attend class and participate in class in a format that works best for them. Like bichronous learning environments, HyFlex environments have been around for some time, though called different things over the years (e.g., blended synchronous learning environments; Conklin et al. 2019). Early on, this type of format appealed to graduate programs that wanted to offer a face-to-face option to local students and a distance option for students living too far from campus to attend in person. Over the past few years, instructors have gained interest in this format because it is designed with all the possibilities available to the students. Typically, with this format, students can attend class in-person face-to-face, asynchronous, synchronous, or a blend of the three in any manner the student chooses. Even before the pandemic, instructors used this type of learning environment to provide students with the most choices (Carbonara 2020).

Although this learning environment provides students with choice and flexibility, it has its own inherent drawbacks. For instance, instructors in this model have less freedom. They are usually in person teaching face-to-face with the students who select the in-person learning environment and on a video conference with students who want to be part of the synchronous classroom, but not be in the classroom. Often, the instructor must also have an asynchronous lesson for students who choose to learn on their own time asynchronously.

1.2.2 Let Instructional Theories, Not Technology, Guide Instruction

Educators new to teaching online often want to begin with tools and technology. However, effective online courses begin and end with sound curricula and instructional design (Lowenthal and Davidson-Shivers 2019). This is evident in the emphasis placed on course design in popular quality assurance models and frameworks like Quality Matters (Baldwin et al. 2018; Baldwin and Trespalacios 2017). The design of any online course should ideally be informed by instructional design theory, learning theories and/or an instructor's philosophical orientation, and finally the course learning objectives, which are all likely influenced and should complement the content of the course. Mathematics education course designers, whether that be instructional designers, veteran online educators, or faculty new to teaching online, should begin asking themselves the following types of questions (Crews et al. 2015; Lewis 2020):

- What learning behaviors and outcomes are intended for the course?
- Is the goal to have students working together to solve problems?
- Should students try a problem first individually and then as a team?

Instructional design theories are intertwined with learning theories and can sometimes be hard to distinguish. However, creating the best learning situation for online students should be the ultimate goal, so keeping in mind both ID theories and learning theories as well as course outcomes is the best way to plan for a good online learning experience. An instructor does not need to select just one ID theory to guide course design, combining theories works too (Lewis 2020).

Cognitive Apprenticeship

In mathematics content courses for teachers (unlike mathematics education methods courses), it is still common for mathematics instructors to use some type of direct instruction (didactic teaching) like a traditional lecture (Stains et al. 2018). The traditional form of direct instruction often used in mathematics courses theoretically aligns with cognitive apprenticeship. Cognitive apprenticeship, York (2013) explained,

> allows a learner to observe processes or methods used by an expert in order to learn an activity. Cognitive apprenticeship differs from traditional apprenticeship in that the activity being learned is less about a physical skill and more about a cognitive skill (p. 35).

When using cognitive apprenticeship for mathematics education, an instructor typically might complete example problems for students, while students take notes. Then, students would show they can complete similar problems by successfully completing assessments such as traditional homework, quizzes, or exams. For mathematics education instructors to demonstrate this to future mathematics teachers in an online environment, the online mathematics instructor can work out problems on a virtual whiteboard or have them pre-worked out on a PowerPoint or video, describing the process as the steps appear to the students. Students can then use the chat feature or microphones, when working in a synchronous environment, for follow-up questions. Essentially, mathematics instructors guided by cognitive apprenticeship are trying to show students what is going on in their head when solving a mathematical problem, or as Collins et al. (1991) explained, “cognitive apprenticeship is a model of instruction that works to make thinking visible” (p. 6). When teaching future mathematics teachers how to do this type of synchronous pedagogy, it is helpful to include lots of practice with many types of technology since technology and applications offered in schools can change rapidly, and future mathematics teachers need to feel comfortable trying different types of synchronous technologies in their online teaching. In addition, the ability to talk aloud while describing what you are writing can be a tricky skill to master. The more future mathematics teachers can practice this in a safe setting, the more comfortable they will be in their future online mathematics classroom. The same goes for the mathematics instructor; they need to be as comfortable as possible while talking through problem-solving in an online synchronous setting.

Individualized/Personalized Instruction

Some mathematics education instructors prefer to have future mathematics teachers try to solve problems themselves before being shown how to solve them in order to demonstrate what future students might be feeling in various math situations. Individualized instruction theory (also known as personalized instruction) emphasizes the importance of having mathematics students with differing instructional needs to work through content at their own level and at their own pace. In some instances, students can even use their own materials (see Keller 1968). In the 1980s, this entailed having mathematics students complete problems in workbooks, level by level. In today's world, this could mean having students individually solving web-based homework problems (e.g., MyLab Math) at their own pace. In terms of student ability, web-based homework (Hodge et al. 2009) can be constructed and personalized to meet students where they are based on their prerequisite knowledge (in alignment with constructivist learning theories). Individualized instruction may be used in mathematics content courses for future teachers such as in mathematics classes for future elementary teachers where databases exist for web-based homework. In terms of student interest, groups can be created in a mathematics methods course based on student interest in a particular grade band (e.g., middle school) or mathematical subject (e.g., geometry). Jigsaw reading where students select articles that interest them to read and then share with their peers is another example where student interest can be used for individualized instruction in a mathematics methods course.

Social Learning

A number of different theories of learning (e.g., social constructivism, situated cognition, activity theory, situated learning, authentic practice, zone of proximal development) argue that learning is inherently a social process (Brown et al. 1989; Carbonara 2013; Lave and Wenger 1991). For instance, social constructivism suggests that individuals construct knowledge and meaning but emphasizes that knowledge and meaning are not constructed in a vacuum but rather are influenced by social contexts, other people, and situations (Lowenthal and Muth 2009). These theories highlight the importance of having future mathematics teachers discuss topics in either synchronous or asynchronous ways to help them understand a topic, construct meaning, and thus learn from others while also leveraging their and others' prior experience. At the same time, the theory of situated cognition places more emphasis on how highly contextualized learning is and how it "takes place within a defined social environment" (Carbonara 2013, p. 287), thus suggesting that people learn best in authentic situations. This theory leads online mathematics education instructors to put future mathematics teachers in groups and in authentic situations (perhaps a simulated mathematics classroom while role-playing different parts), so they can learn from both the situation and from the other future teachers.

Social theories of learning have led educators to place an increased emphasis on developing a learning community or community of practice in the classroom (see Jonassen 1995; Rogoff 1994; Wenger 1998, 2000). The emphasis on community is to help learners frame knowledge and skills with their peers in order to see the different perspectives brought to the subject. In addition, in the case of mathematics education, future mathematics teachers all bring real-world experience to the topics being discussed that can in turn help others learn. In an online mathematics education classroom, this would involve putting students in groups (whether formal or informal) so that the power of group work can be utilized for maximum benefit. It is common for in-person mathematics education classrooms to have students self-select communities in which to study, learn, etc. In order to facilitate this in an online classroom, the instructor can provide an "introduce yourself" discussion forum from which students can then select group members or randomly assign group members as part of an out-of-class study group. Once online students get to know each other, they can typically self-select other peers to work with more easily. Providing them with tools such as breakout rooms, chat rooms, etc. in which they can form a social relationship can help mathematics education students more easily connect.

By grouping future mathematics teachers in their online classrooms, mathematics education instructors can model a variety of opportunities for them to use in their future mathematics classrooms: breaking large tasks into smaller chunks, maximizing time management, making use of peer feedback and assessment, challenging preconceived ideas, facilitating the development of communication skills, increasing achievement using groupthink, enhancing interpersonal skills, increasing opportunities to learn from one another, increasing resources available for students, and bringing together people with different skills/abilities. Groups can be formed in multiple ways as previously discussed and roles assigned in multiple ways. Some examples of different approaches to group responsibilities from Smaldino (2007) are listed in Table 1.2.

Related to social learning, specifically the work of Vygotsky (1978), is the concept of Zone of Proximal Development (ZPD), which is "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (p. 86). ZPD combines both the individual learning abilities with the social aspect by boosting the student to reach the outer edges of what they can learn on their own along with what a peer or instructor can help them learn through scaffolding. By modeling behavior, mathematics education instructors or peers can demonstrate for a future mathematics teacher what success could look like, and thus, the future teacher can more easily

Table 1.2 Approaches to group responsibilities

Collaborative approach	Each member of the group becomes involved with the decisions of the group and volunteers to assume responsibility for tasks or activities
Group task approach	Each member of the group is assigned a task within the group's structure. Tasks are assigned based on the member's skills or knowledge. An attempt is made to equalize the task responsibilities
Group role approach	Each member of the group is assigned a role (e.g., facilitator, recorder). The responsibilities of each role define the tasks or jobs to be done within the group. For each assignment or task, the individual's role will define the way in which that person will complete the job
Role assignment	The group determines what roles or responsibilities are necessary to complete the task or assignment. Once the roles have been defined, members can volunteer or be assigned to a role. Other ways to assign members to roles is by random drawing or voting

reach the desired outcomes (McLeod 2012). In online mathematics education, this could look like grouping students in a breakout room and asking them to try to solve a difficult ill-structured problem (could be a scenario or something otherwise difficult), something that students feel like they cannot solve on their own. But by discussing possibilities, students can see that they can better reach a solution. So, working with scaffolding, a solution can be more easily reached (Xie and Bradshaw 2008).

Inquiry-Based Mathematics Education and Problem-Based Learning

Increasingly, mathematics teacher educators strive to use a variety of active learning strategies in mathematics education courses (Stains et al. 2018). Two popular branches of active learning are inquiry-based learning (Ernst et al. 2017; Mahavier 1997) and inquiry-oriented instruction (Laursen and Rasmussen 2019). For the purposes of this chapter, we will group these practices together as inquiry-based mathematics education (IBME) instructional design principles. Laursen and Rasmussen (2019) discussed the commonalities among different inquiry-based approaches. They conceptualized these as the four pillars of inquiry-based mathematics education; two pillars emphasize student behaviors, and two pillars emphasize instructor behaviors:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices (Laursen and Rasmussen 2019, p. 138).

In online mathematics education courses, this could entail providing future mathematics teachers opportunities to work together to think of different ways their future students could solve the problems at hand. For instance, students could have to come up with different solution paths on their own (without being provided the answer by their instructor). As the students work together in groups, instructors can then ask the students questions about how they arrived at their answers that probe student thinking. Breakout rooms, the share screen feature, and careful selection of mathematical tasks can make moving these IBME teaching strategies from a face-to-face classroom to a remote learning environment possible. The careful selection of tasks is categorized by some as problem-based learning.

In the case of problem-based learning, transitioning from face-to-face to online does not take a great deal of adaptation from a face-to-face classroom. Problem-based learning "situates people within authentic, complex and challenging problems representative of those found within a disciplinary field of practice" (Friesen 2013, p. 246). Therefore, implementing problem-based learning in the online mathematics education classroom is more of an issue of how the problems are delivered/assessed rather than the altering of the problems solved in the classroom (be it virtual or in-person). For example, presenting using threaded discussions, text-based or video-based cases to examine, scenarios to act out, etc. can all be done in an online synchronous or asynchronous format depending on the goals and objectives of the mathematics education lesson.

There are many other instructional theories that can be used to guide online mathematics education. For the purposes of this chapter, we intended to provide an overarching view of common instructional theories and how they could guide online mathematics educators. The next section will describe ways to leverage the technology in an online learning environment to best meet the needs of the aforementioned theories in mathematics education.

1.2.3 Leverage Technology Interventions

There are several technological interventions that can help instructors make any type of learning environment more engaging. While most of these technologies were available before the COVID-19 pandemic, the changing classroom landscape has given instructors more of a need to find technologies that can help them meet their instructional goals (whatever the mode of delivery may be). Some of these interventions are content-specific for mathematics education, while others are content-neutral and could be used in any classroom regardless of the content being taught. We provide examples of each type of intervention (content-specific and content-neutral), knowing that our lists are not exhaustive, to provide online mathematics education instructors with options for interventions that best fit their instructional theories. Some content-specific technology interventions specific to mathematics education include Desmos, Geogebra, mathematics applets, pattern block applets, online free manipulatives such as base 10 blocks, PhET, etc. The types of technology interventions are rapidly increasing as more people are teaching using technology (and many teaching remotely). Some content-neutral technology interventions for teaching mathematics education (or any other subject) are online whiteboards, communication tools, collaboration tools, breakout rooms, graphic organizers, etc.

Grouping Students Online

Online students (and especially adult students) like to have choices when learning online (Oliveira et al. 2011). This likely stems in part from having lives with competing priorities (e.g., multiple professional and family responsibilities). In fact, many students no longer have the same time availability as they did before pre-COVID-19 when their kids were in school or in childcare and their jobs required less work-from-home scenarios. Thus, we contend that online mathematics education instructors should keep this in mind when using groups, allowing group choices, where appropriate, as well as flexible timelines when possible.

Another approach is to design assignments where a student has the choice to complete them individually, in pairs, or in groups. Keeping groups to three or four members will help ensure the workload is divided versus groups that are too big for everyone to get involved. Another option is to break down larger classes into closed discussion groups on the same topic, thus preventing students from seeing the discussions of students in other groups. For example, when creating closed discussion groups for a larger class or 30–40 students, students could be split into groups of 15–20 students; if groups have too few people (~five), the discussions can get stilted and not move forward. Regardless of the group size, it is up to the mathematics education instructor to facilitate the discussion and keep it moving (Lewandowski et al. 2016).

In order to take advantage of the power of cooperative learning (groupthink), mathematics education instructors are increasingly having students work in groups to solve problems (Koçak et al. 2009; Laursen and Rasmussen 2019). Learning management systems with synchronous internal/external breakout rooms is one of the easiest ways to do this online. This allows a larger online classroom (for example, 30 students) to be broken down into more manageable "rooms" for synchronous discussion and collaboration. We typically like to have breakout rooms of three to five students so that everyone gets a chance to talk and participate. Once breakout rooms get too large, some students will become "lurkers" and passively listen to their classmates but not actively participate. Keeping learning objectives in mind when creating the breakout rooms can also guide the size and make up of each group.

Some learning management systems also have a "Groups" feature that allows the instructor to create groups of students (e.g., randomly, manually, or self-enroll), which provides students a space to work with their groups using various LMS tools (e.g., discussion board, whiteboard, email, etc.). One of the benefits of this is that it provides students some standard course-supported ways to communicate. It is easy to be overwhelmed by the mathematics education content, so any way for instructors to reduce the overwhelming feeling of too many communication technology choices for their students helps.

Case Studies Discussions

Case studies can be used in courses for mathematics educators in any course format. Instructors can provide students with scenarios from mathematics classrooms (either content or classroom management scenarios), and students can share their strategies with the class in a variety of formats. Asynchronous discussion boards, for example, could be used as a place where everyone could share their ideas either in smaller groups or the full class and then build on those ideas by reading and interacting with others in the course. Case studies provide mathematics education students with opportunities to think about authentic activities, role-play, learning by doing, and reflect on their future teaching.

Think/Pair/Share

A quick use of synchronous breakout rooms would be to use think/pair/share to work on a mathematics problem or educational question. After a mathematics education instructor poses a question to the full class, the instructor can give the entire class a minute to think about a problem. Then, the students could share their thoughts in pairs for a given amount of time (usually under 5 min). In an online learning environment, a second share could happen when the instructor brings the groups back together and calls on pairs to share with the full class.

Presentation Tools

Student presentations are vital to courses for future mathematics teachers. Future teachers should have opportunities to share their work in a way that models their future mathematics teaching and also allows them to practice communicating mathematics and strategies for teaching mathematics (Natalicio and Pacheco 2000). Affording mathematics education students opportunities to present in an online classroom also helps future teachers get comfortable with both the technology and content they may encounter in their future classrooms. These presentations could be prepared before class starts using pre-recorded video technology (whether that be with a webcam, cell phone, or using screen casting software), or during a synchronous session where they write and speak using interactive whiteboard technology, or a combination of the two using a tool like an interactive PowerPoint presentation.

Manipulatives

Before COVID-19, most people probably thought of concrete objects when they heard the word “manipulatives.” Manipulatives, by definition, are “objects that a student is instructed to use in a way that teaches or reinforces a lesson”

Table 1.3 Theories, strategies, and technologies

Theory	Strategies (application of theory)	Technologies
Community of practice Social cognition Situated cognition	Collaborative learning Group work Interaction Think/pair/share Cooperative learning	Discussion board Chat room Audio/video conference Breakout rooms Share screen
Authentic activity/practice, situated learning	Learning by doing Authentic activity Practice teaching using active learning strategies Practice integrating technology into teaching	Share screen Co-host option
Individualized instruction	Individualized instruction Differentiated learning Jigsaw Individualized group projects	Videos Online worksheets Online quizzes Online/web-based homework
Cognitive apprenticeship	Direct instruction Advance organizer Presentations Lecture	Whiteboard Presentation software
Inquiry-oriented learning (active learning)	Inquiry-based learning Socratic questioning Scaffolding productive struggle, learning from failure, and perseverance Hands-on, minds-on	Audio/video conference Student presentations Group work-breakout rooms
Problem-based learning	Discovery learning Problem-based learning	Virtual/augmented reality

(Merriam-Webster n.d.). In today's world, most of what can be done with concrete objects can now be done using well-developed online free mathematics manipulatives.

If a mathematics educator wants to teach (or learn) with physical pieces, some manipulatives can also be printed, cut out, and used at home. TACTivities, learning activities that have movable paper pieces, are one example of something that can be used at home to conjure more creative thinking (Hodge-Zickerman et al. 2020) in future mathematics teachers. Having mathematics education instructors model how to use these is an effective way to have future mathematics teachers feel comfortable incorporating them in their classrooms.

Traditional Activities that Translate Easily Online

There are also some activities that are more traditional in nature that work just as well in an online mathematics education environment as in a face-to-face classroom. Mathematics projects, engaging worksheets, individual (or group) quizzes, and homework (either online or paper/pencil submitted as a PDF) are all examples of such traditional activities. One could even argue that the flipped classroom model (Bergmann and Sams 2012) for teaching mathematics educators could translate online easily. In the flipped classroom model, mathematics students typically watch videos before class and then work on harder problems with others during a synchronous class session.

We also urge instructors to think about equity and accessibility. Instructors should have equity in mind and strive to foster classroom that promotes equitable learning in any and all classroom learning environments.

Table 1.3 summarizes the theories, strategies, and technologies we have just discussed.

1.3 Conclusions

There are many ways for mathematics teacher educators to develop high-quality effective mathematics courses for mathematics teacher education students, no matter the mode of delivery. In this chapter, we described common learning environments (asynchronous, synchronous, bichronous, and HyFlex) used in remote, blended, and online instruction. We then provided a brief overview of common instructional theories (cognitive apprenticeship, individualized/personalized instruction, social learning, and inquiry-based and problem-based) that can guide mathematics education and concluded with different ways instructors can leverage technology (content-specific and content-neutral) to enhance the online classroom environment and fit the instructional design choice (grouping students, case studies, think/pair/share, presentations, manipulatives, and so on). Now, the question remains, what do online mathematics education instructors do with all of this information in their classrooms?

We encourage each mathematics education instructor to start somewhere when it comes to creating your online learning environment and using technology. Do not be afraid to try something new. Instructors may not yet know the best instructional theory or technologies to use for their mathematics education classes, and that is okay. As reflective practitioners, instructors have the ability to start somewhere, reflect upon their practice, and then modify their instruction. Teaching should be an interactive process. A course does not need to be perfect the first time (or even the 20th time) it is taught. One piece of advice for new online mathematics education instructors is to avoid trying too many new things at once so as not to overwhelm themselves or their students.

For further information about online mathematics classroom environments, instructional theories, and technologies for teaching mathematics educators, we encourage everyone to read the rest of this book. If mathematics education instructors still do not know the "best" choices to make for their classroom, we also recommend conducting their own research—collect feedback from their students (ask students what they like best and what their suggestions are for the instructor's teaching). Iteratively improve upon practice. Instructors can always change things midsemester or in the following semester. Let the theory and literature not only guide design but also supplement the theory with the instructor's own experiences, students' voices, and advice from peers/colleagues.

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Chapter 2
Using Digital Technology and Blending to Change the Mathematics Classroom and Mathematics Teacher Education

Marcelo C. Borba, Johann Engelbrecht, and Salvador Llinares

The central theme for this chapter is the *transformation and evolution of the mathematics classroom and mathematics teacher education* with the growing integration of the Internet and interactive digital devices into mathematics education. Borba et al. (2016) introduced the idea that digital technology use in mathematics education has been taking place in four distinct phases. Moreover, in this paper, five prominent trends of development were identified within the current fourth phase: mobile technologies, massive open online courses (MOOCs), digital libraries and designing learning objects, collaborative learning using digital technology, and teacher training using blended learning. Engelbrecht et al. (2020) updated Borba et al. (2016) under three main themes: principles of designing professional development opportunities and mathematics teaching contexts; social interaction and construction of knowledge in the digital environment; and tools and resources and how their use is conceptualized in different mathematics teaching contexts, given the emergence of new online mathematics resources and ways of teaching.

In this chapter, following mainly Engelbrecht et al. (2020), Borba et al. (2016), and other surveys and descriptions of the state of the art, such as Borba et al. (2013) and Silverman and Hoyos (2018), we aim to support the building of the domain *blended mathematics education*, by addressing two main strands:

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- Social interaction and construction of knowledge or collaborative learning using digital technology: How technologies in online contexts support social interaction among participants as a medium to support mathematical knowledge construction and teaching competencies.
- Resources, tools, and new learning environments changing the relationships between mathematical knowledge, learners, and teachers. The new tools generate alternative opportunities for learning, linked to new ways of looking for information and shaping students' mathematical experiences.

We discuss these themes, also using examples of *live presentations*, conducted by the first author during the pandemic. These live presentations, called *LIVES*, started with artists, and they moved to sectors such as mathematics education. We will discuss how these new forms of presentation contribute to the changing classroom and how they align with approaches such as MOOCs or the flipped classroom. We also discuss a new agenda in mathematics education.

2.1 Background

At the time of writing this chapter, the entire globe is in the middle of the COVID-19 pandemic. This pandemic has underlined the relevance of online and blended learning programs, repositioned online learning on the educational hierarchy, and accelerated this transformation process of the classroom. The pandemic strongly impacts the worldwide use of digital technology in teaching, in that, on the one hand, the worldwide lockdown increases the rate of change of using digital technology in education but at the same time, it creates an awareness of the need to feel connected to each other. It is clear that we need to rethink the entire model of education and redesign it so that it is more student-centered. This situation is making visible social inequalities revealed by different access to the Internet and other factors throughout the world. We have to think about how adopting new technologies can impact mathematics education in different parts of the world and also rethink what we consider educational success.

Currently, it seems as if digital technology is "deconstructing" the notion of the classroom and the ways through which students and teachers interact with mathematical processes and content. Fixed ideas about what is in and outside of the classroom become different with flipped classrooms, and the roles of students, teachers, and how we communicate mathematics are changing rapidly. What we know as the classroom may change from a physical area with defined boundaries to a virtual environment including various components that will probably be determined by students rather than only by the teacher. Furthermore, the *traditional* ways of communicating mathematics and the knowledge of mathematics teaching are adopting new features.

The new technologies as well as the pandemic also have a serious impact on the nature of mathematics (to be learned) itself; for example, application of procedures is becoming less important, new ways of validation in mathematics are being developed, and growth models are at the order of the day.

It is not only technology but also our target audience, our students, and pre-service teachers that are changing. After the baby boomers and the Generation X groups, new micro-generations evolved in cycles of about 4 years, giving birth to new concerns, new motivations, and new challenges in all aspects of their lives (Morin 2016). These new micro-generations include the Echo-Boomers or Gen Y (1989–1994), Gen Z or the net generation (1994–1998), the post-millennial Generation Z (2002–2006), and the youngest group (2006–2010), the Gen Z Silent Generation. Our current students have grown up in a digital world of computers, the Internet, and social online media such as Instagram, Facebook, Twitter, Google, and other social networks (Selwyn and Stirling 2016). They learn by interacting with other individuals online. They use new communication media that shape how they conceive knowledge and its use. They like to collaborate using the latest technology and visualization opportunities. Today's students prefer seeking their own information rather than being presented with it (Morin 2016), they prefer on-demand access to knowledge, disseminated over the Internet and across different channels. They are in frequent contact with their friends using networks to share and create new knowledge. They can collaborate synchronously and asynchronously to make decisions and elaborate on new proposals.

Our way of teaching has to adapt to meet these challenges with the change in our student population (Dineva et al. 2019). With the development of a more social and connected Web, digital environments evolved, which empower students to conduct social networking, organize social content, and manage social acts by connecting people, resources, and tools. This is done by integrating Internet tools to design environments that are transparent (Borba et al. 2016; Engelbrecht et al. 2020; Tu et al. 2012).

The development of the use of digital technology in mathematics education has been taking place in distinct phases regarding how technology incorporates multimodal media to support communication and collaboration. The first phase commenced with the introduction of Logo as a teaching tool, followed by "content" software such as Cabri or Geometer's Sketchpad. The second phase arrived with new notions such as dragging that allowed students to "experiment mathematics." The relationship revolution arriving with the Internet is referred to as the third phase. This phase prompted us to include collaborative learning using technology as one of the current trends of development. The fourth phase grew out of quantitative change in the Internet and expressions such as Web 2.0 and broadband Internet brought about massive open online courses (MOOCs) enhanced opportunities for collaborative learning and the personalization of the Internet through personal devices. Along with these developments, a move to mobile technology introduces new possibilities in the teaching of mathematics and leads to a further prominent development trend, included in our discussion (Borba et al. 2016). A relationship revolution took place in that communication has moved to two-way communication (Borba et al. 2016; Engelbrecht and Harding 2005a, 2005b; Van de Sande 2011) and enhanced opportunities for collaborative learning. It also brings us the personalization of the Internet through personal devices and through hyper-personalization of learning – supporting the learning needs of individuals – that becomes possible when using adaptive hypermedia (Engelbrecht et al. 2020).

The role of social media is becoming increasingly important and, when used correctly, can move the education process from the traditional "push" approach to a student-centered "pull" approach in which the students become an integral part of many facets of the process (Martinovic et al. 2013; Sánchez-Aguilar and Esparza-Puga 2020; Yerushalmy and Olsher 2020). Furthermore, the use of social media can transform the approaches to mathematics teacher education (Martínez et al. 2020; Hollebrands and Lee 2020). Also, the efficacy of current teacher practices and traditional classrooms are questioned with the introduction of digital resources and tools in the same way that teachers relate to curricular materials (Drijvers et al. 2013; Gueudet and Pepin 2020). Social aspects of the Internet become more and more relevant, and notions such as "humans-with-media" emphasize that if media are changed, the entire knowledge-acquiring process may change (Borba et al. 2018b).

With the vast availability of online resources on specific mathematical content, our students also have to rate the quality of the knowledge disseminated over the Internet and need to be able to select valid resources. Furthermore, mobile digital technologies, such as forums, wikis, Twitter, Instagram, Facebook, and possibilities of collaboration provided by the resources in Google Suite, provide different kinds of learning opportunities, supported in new social interaction spaces. In these interactive technologies, students can collaborate with their peers when they use multimedia and the Internet, allowing new social ways of knowledge construction to emerge (Goos and Geiger 2012; Llinares and Olivero 2008; Joubert et al. 2020; Cendros-Araujo and Gadanidis 2020).

2.2 Blended Approaches

For some time, the traditional online learning environment was viewed with some skepticism in that teachers expected it to be less effective at developing higher cognitive thinking processes than traditional classroom learning (Chaney 2016; Cicconi 2014). With the currently available features, provided by the Internet, however, blended learning allows for including a social learning approach through the presence of other students, teachers, and online resources (Martínez et al. 2020; Goos et al. 2020; Joubert et al. 2020). Researchers suggested that blended learning fits well with Vygotsky's concept of a zone of proximal development despite challenges that arise (Cicconi 2014; Deulen 2013).

Rather than employing new views of pedagogy in teaching and learning in a significant manner (Collis and Van der Wende 2002), blended learning courses often tend to replicate traditional teaching methods and are developed by making minor pedagogical changes with additional resources and supplementary materials (Graham 2006). Borba et al. (2018a) labeled this type "a domesticated use of a medium," in which the new affordance of a new medium is barely explored. By integrating online learning into the system, blended learning expands the learning environment into the virtual world where traditional limitations are removed. Through the online component, differentiation between student needs becomes easier and combines with the social aspect of the actual classroom to create a stronger learning experience. (Engelbrecht et al. 2020).

Properly designed blended learning systems include the important face-to-face interaction that Vygotsky considered to be vital and thus provide all of the benefits of the social aspects of learning (Ting and Chao 2013) and should engage students to give them the opportunity to develop their own opinions, consider new ideas in collaboration with other students online, and try out their own ideas in a relatively anonymous environment (Holley and Oliver 2010).

2.3 Mathematics Teaching and Learning as a Collaborative Process that Is Shaped by Available Technologies Mediating Interactions

Many studies have supported the social constructivist learning theory and claim that it improves student engagement and learning (Grady et al. 2012; Schmidt 2013). In their research on instructional technology, Pepin et al. (2017) reported that studies on the topic are predominantly framed by sociocultural theories underlining the role of discourse in learning. Through the development of information and communication technologies, new forms of discourse have emerged that have the potential to change social relations and the ways through which we come to understand the development of knowledge (Llinares and Olivero 2008; Llinares and Valls 2010; Clay et al. 2012; Cendros-Araujo and Gadanidis 2020; Cooper et al. 2020). The links between participation and construction of meaning are scaffolded by social artifacts such as online collaboration, mind mapping, or sharing narratives in online forums to discuss relevant aspects in mathematics teaching. Students get opportunities to reorganize their knowledge in the course of social interaction through sharing interaction spaces that facilitate asynchronous online discussion and online collaborative and content-focused professional development (Matranga and Silverman 2020).

In an attempt to better understand the links between interaction in online contexts and the construction of knowledge to conceptualize technology-mediated interaction, various theoretical perspectives about learning and knowledge have been used (Clay et al. 2012; Goos and Geiger 2012; Llinares and Valls 2010), and recently, Borba et al. (2018a, 2018b) introduced some new perspectives that consider how newly introduced media reorganize human thinking, favoring connections and group discussion.

Opinions have been raised that the introduction of technology into the learning process has not succeeded in revolutionizing education and the learning process

(Chatti et al. 2010), mainly because most current initiatives take a technology-push approach in which learning content is pushed onto a pre-defined group of learners in a closed environment in a one-size-fits-all, centralized, static, teacher-centered teaching approach. We need a fundamental shift towards a more open and student-pull model for learning – a shift towards a personalized, social, open, dynamic, and student-centered model.

Knowledge construction in collaborative settings takes place through the different forms of discourse that the participants adopt. A variety of collaborative approaches have been developed over years. A *virtual learning environment* (VLE) or *learning management system* (LMS) is a Web-based platform for courses of study, usually used in educational institutions. Some LMSs present resources, activities, and interactions within a course structure; allow participants to be organized into groups; provide for the different stages of assessment; report on participation; and have some integration with other institutional systems. *Personal learning environments* (PLEs) (Engelbrecht et al. 2020) are considered as the latest step in an alternative approach to e-learning. The difference between a PLE and LMS is that an LMS is course-wide (or institution-wide), while a PLE is individual. PLEs often consist of a number of subsystems, such as a desktop application and one or more web-based services. It could even include the LMS used by the institution. A PLE would integrate formal and informal learning, such as using social networks, and use collaboration possibilities, such as small groups or web services, to connect a range of resources and systems in an individual space.

Closely related to the concept of a PLE is the idea of a *personal learning network* (PLN). Whereas PLEs are the tools, artifacts, processes, and physical connections that allow learners to control and manage their learning, PLNs extend this framework to include an informal learning network of people to connect with, for the specific purpose of learning. In a PLN, there is an understanding among participants that the reason they are connecting is for the purpose of active learning (Lalonde 2012).

The creation of rich learning *mash-ups* (often web applications that integrate complementary elements from different sources) currently associated with collaborative learning, resulted from advances in digital media (Engelbrecht et al. 2020). According to Wild et al. (2010), mash-ups are "the frankensteining of software artifacts and data" (p. 3). In these mash-ups, students build their own personal learning environments by not only composing web-based tools into a single-user experience but also getting involved in collaborative activities by sharing their designs with their peers and adapting their designs to reflect their experiences of the learning process. In teaching computer science students, PLEs and PLNs have been extensively implemented. In mathematics education, perhaps too little has been done in implementing these approaches. Harding and Engelbrecht (2015) compared PLN clusters that spontaneously formed among students in two fields of study – mathematics and computer science. Students in a cluster use a number of tools to communicate and learn while using social media, mobile phone technology, and learning management systems, among other platforms for learning purposes.

2.4 Defining New Roles

Technological developments are changing the nature of societies (Borba et al. 2018a, 2018b). Educational processes are being transformed as students incorporate the Internet into the classroom and digital technologies invade the teaching process (Borba 2009). The relationship between humans and media is growing. On the one hand, artifacts shape the human mind, but according to Borba et al. (2018a), the converse process is also taking place: humans shape technology beyond the design of tools and of digital tools. Technology is also seen as having agency in that digital technology is saturated with humanity in its design and in its conception, and humans are transformed by digital technology.

Through interaction, negotiation, and reflection, humans and media produce knowledge. From this perspective, "humans are constituted by technologies that transform and modify their reasoning and, at the same time, these humans are constantly transforming these technologies" (Borba and Villarreal 2005, p. 22). This notion of an intershaping relationship is one of the main components of humans-with-media as a theoretical construct: human beings and media influence and shape each other, contributing to the reorganization of thought and the production of new knowledge and new practices. According to Oechsler and Borba (2020), there can be no production of human knowledge without the influence of media, nor can any media be developed without the influence of humans. Humans reorganize their thinking according to their interaction with media (Souto and Borba 2018).

2.4.1 Producing Videos to Share Mathematics

With the development of digital technology, the roles of teachers and students are changing. The classroom is becoming a new, more open place, with fewer barriers to the rest of society. In recent times, students can access a variety of information and can view, experiment with, and conjecture about information that previously could only be read. With the available technology, students can now create their own learning environment, such as videos and tools to explore topics that interest them.

Video production by students in mathematics classes is not a common practice (Oechsler and Borba 2020). Nonetheless, it has been growing in recent years, such as with video production festivals like the Math Performance Festival in Canada (Borba et al. 2014) and the Festival of Digital Videos and Mathematical Education in Brazil (Domingues and Borba 2018).

Oechsler and Borba (2020) described how the creation of videos with mathematical content may contribute to the process of expanding the classroom and how this activity becomes a teaching and learning tool. They ground their discussion theoretically in social semiotics, a theory that considers the context of production and the negotiations between actors to analyze the meanings produced. Producing videos provides a dynamic in the classroom in which students can become protagonists in the teaching and learning process, giving students an opportunity to express what they have understood. By using videos, students can be exposed to a different kind of mathematics, combining its traditional symbolic language with modes such as image, language, gesture, and music. Producing videos gives students the opportunity to move from a passive receiver of knowledge in the traditional push approach to an active participant, deciding how the content will be explored and how it will be shown to fellow students. Oechsler and Borba (2020) experienced that in the process of producing such a video, students succeeded in developing an understanding of concepts that they themselves struggled with, and in the video, they addressed issues that they found difficult to understand so that other students could understand the subject the way they did, perceiving and analyzing the error.

This reorganization of thinking can lead to new knowledge and understanding (Borba, 2012). Similar to Kress (2010), this reorganization can be seen as a sign of students' learning. Students show their understanding of the content by producing the video. Through this activity, the students themselves became aware of their difficulties and sought ways to resolve the problem. In this way, video production can be considered a teaching and learning tool, through encouraging students' discussion and reflection about content and its exposition to produce meaning.

The boundaries of the conventional classroom are expanded since these videos that were produced can be released outside the classroom, e.g., on social media and video hosting platforms, assisting other students, teachers, and members of society in general in understanding a given subject. Video libraries such as the Mathematics Festival Video Library (https://www.festivalvideomat.com/) are being created with production by teachers and students of different levels. The topology of the classroom and the library is being changed. While in a physical classroom, you could be on YouTube. When at home, you may be in a laboratory at the university at the same time.

2.5 New Ways of Communication through the Internet

The exponential development of interactive digital resources and the increase of connection speed through the Internet have generated new ways of interaction and communication, resulting in new ways of producing knowledge. Regarding mathematics teaching as a social process, involving the connections between the learner and other learners with similar goals, the interaction through the Internet becomes a central element to online teacher education.

In Engelbrecht et al. (2020), two of the three main strands identified from the new contexts of the use of the Internet in mathematics education were the new characteristics of the social interaction and construction of knowledge and how the use of new tools and resources in digital contexts is conceptualized. One characteristic from these strands is the development of new ways of discourse related to new ways of interaction among teacher educators, pre-service teachers, and knowledge for mathematics teaching, as well as between teachers, students, and the mathematics knowledge. This situation cannot separate the features of participation of students and teachers in the new contexts from the scaffold provided by social artifacts such as online collaboration, the new resources available through the Internet, and the emergence of multimodal communication. These aspects underline two relevant issues determining the mathematics teacher development and students' mathematics learning: firstly, how the knowledge is shared, that is to say, through which cognitive process the knowledge is generated, such as producing video, generating mind map, writing descriptions and interpretations of teaching, sharing portfolios in a digital context, and so on and, secondly, using different artifacts and actions to generate knowledge.

The Internet is making us see mathematics teaching and mathematics teacher education as an activity requiring thoughtful deliberation. So, the Internet determines the generation of different forms of knowledge and processes of communication.

2.5.1 Some Examples of New Ways of Communicating and Building Knowledge Using the Internet

We describe some examples of how the Internet is modifying the role of communication processes and the processes of constructing knowledge in mathematics education. We describe instances from sharing discourse on teaching to collectively create mind maps as representations of knowledge, developing new semiotic resources, or generating new artifacts. In this context, different initiatives in teacher education programs are developing new ways of communication among participants (prospective teachers and teachers' educators), using the facilities provided by the Internet (Cendros-Araujo and Gadanidis 2020; Llinares and Valls 2009, 2010; Fernandez et al. 2020).

For example, there are initiatives using the Internet to develop prospective teachers' ability to generate practical arguments about teaching (Llinares and Valls 2009, 2010; Fernandez et al. 2020; Ivars and Fernandez 2018). In these contexts, prospective teachers describe and attend to different types of representations of practice (videos, students' answers, curricular material, etc.), interpreting and labeling what they consider relevant aspects. Current technological resources allow them to communicate and argue with others about the reasons behind their interpretations and decisions. In these types of activities, the Internet supports the communication processes among pre-service mathematics teachers and allows them to describe the events, creating focal points around which generate the process of interpretation. For example, in a distance teacher education program, prospective teachers describe their own teaching using narratives and reason about teaching mathematics (Fernandez et al. 2020). Prospective teachers reconstruct some events from their teaching in their narratives in the form of text, which is shared with other colleagues. In other cases, pre-service mathematics teachers analyze registers of the practice (such as video) as a means to develop processes of thought about teaching, ending in an action or an intention to act (Llinares and Valls 2009, 2010). In these examples, using the Internet and digital resources as scaffolds in the communication, it is assumed that the teaching is not only taking action in the classroom but also noticing and reasoning about features of mathematics teaching. Emphasis is placed on ways of thinking when prospective teachers engage with expert knowledge, to think about practice in a process of making meaning with others through social interaction spaces.

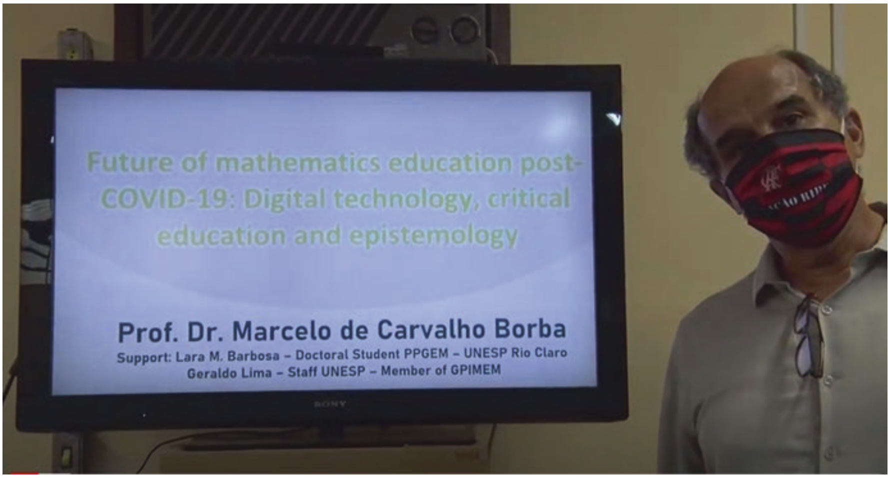
On the other hand, the Internet favors the inclusion of alternative and multimodal methods for online interaction and knowledge construction in mathematics teacher education, extending the semiotic possibilities of conversation and writing (Cendros-Araujo and Gadanidis 2020). Cendros-Araujo and Gadanidis (2020) use the Internet to support the communication processes between prospective teachers while constructing collaborative mind maps, understood as ways of representing knowledge. These authors underline the multimodality and technology, provided by the collaborative construction of mind maps, as a means to consider new forms of representations of knowledge. The collaborative interaction to construct mind maps underlines the extent of semiotic possibilities provided by the Internet. The abundance of multimodal information in an online collaborative context is a new characteristic provided by the use of the Internet. In this case, mind maps are artifacts used to visualize the knowledge, and they are shared with others as tools to construct new knowledge. For Cendros-Araujo and Gadanidis (2020), the knowledge constructed in these mind maps is distinct from the one developed through other online discussion tools such as threaded forums, in the sense that it not only incorporates asynchronous possibilities different from those in classroom settings but also introduces powerful visualizations of collective knowledge (p. 945).

In this case, the communication process supported by the Internet was constructed on multimodal elements connecting multiple ideas. The generated interface allowed participants to use different visualization resources – connectors, shapes and colors, video and images, and chats, to develop a different way of discourse. The use of visual mediators to generate the discourse was complemented by text or writing (written speech) creating a new characteristic of the communication process through the Internet reflecting multiples ways to represent the knowledge.

A third example is provided by considering the digital mathematical performance aimed to explore innovative possibilities for the use of digital technology and the performance arts in mathematics education (Scucuglia 2020). One aspect of the production of these multimodal-artistic narratives is that they can be shared online. The communication of mathematical knowledge adopts a new way of considering aesthetics with the use of the arts and digital media. Scucuglia (2020) describes preservice teachers' mathematical experiences when producing an original mathematical song using virtual instruments.

A fourth example is the *LIVES* activity, which started in Brazil during the 2020 COVID-19 lockdown. Soon after the beginning of the pandemic, a new word was introduced into the Portuguese language spoken in Brazil. The English word *LIVE* was incorporated into everyone's cell phone dictionary. In mid-March 2020, bars and theatres were closed down in Brazil, and despite pressure from business and the government, they stayed closed until October or December, depending on the region in the country. *LIVES* was the word used for famous artists' presentations on the Internet, mostly *YouTube*, *Facebook*, and *Instagram*. For people who were at home, the open-source editions of *LIVES* provided a rich experience of social media.

Soon, educators started using this opportunity, offering *LIVES* presentations on specific subject areas, including the first author of this chapter. Various themes in mathematics education are addressed. In fact, this year, the prestigious annual mathematics education Psychology in Mathematics Education (PME) conference in Thailand was run online. So a *LIVE* may be an online opening talk in a conference, such as the opening talk at PME-44, or it may be an independent event, such as a music show or a lecture at a university.

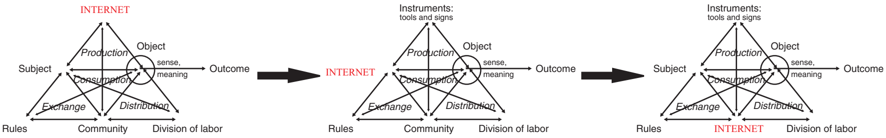


https://www.youtube.com/watch?v=_LEwu-iFuPU

Themes of these LIVE events include relevant topics such as COVID-19, exponential functions, sigmoid curves, normal curves, social inequality, virus transmission, fascism in Brazil, the anti-science perspective, the notion of humans-with-media, and epistemology, and these topics were intertwined in an interdisciplinary manner. Professors and graduate students provided different perspectives by showing, exposing, sharing, or doing mathematics. They employed the agency of broadcast software such as Streamyard and distribution platforms such as YouTube. Most of the *LIVES* are in Portuguese, but this one by Helia Jacinto, Lisbon University, on problem-solving in mathematics education, has subtitles in English.

https://www.youtube.com/watch?v=-TTYxTXnvaA

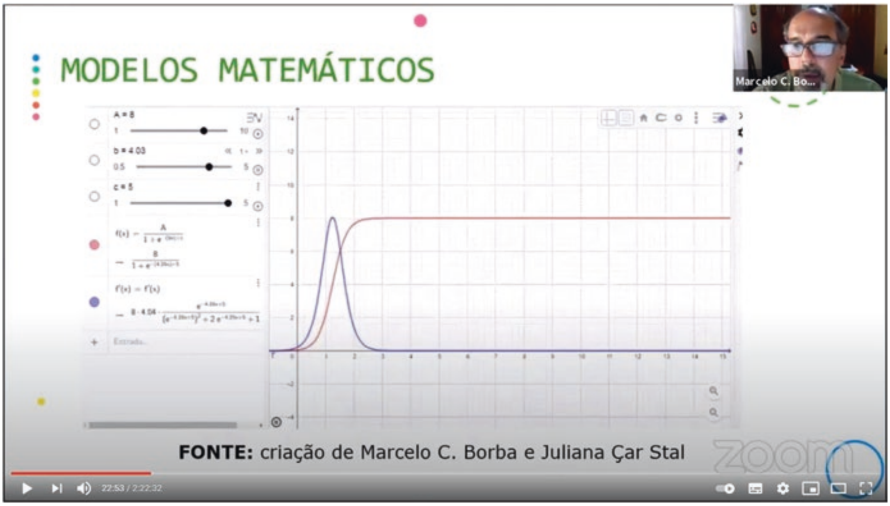
This was unknown territory – no one in the field knew how to do it – and online mathematics distance education had to be reinvented. One example is to organize a talk show where a graduate student would ask questions to the professor or read questions posed by the audience, formed mostly by mathematics teachers, mathematics educators, and mathematicians. In another activity, interviews were combined with GIF pictures in which a different face of the Internet in Activity Systems (Engeström 1999) from the perspective developed by Souto and Borba (2018) was demonstrated, and mathematics education at large was discussed.



SOURCE: Image taken form the text by Sannino & Engeström, 2018. Adaptation and creation of the GIF by Juliana Stal and Marcelo Borba.

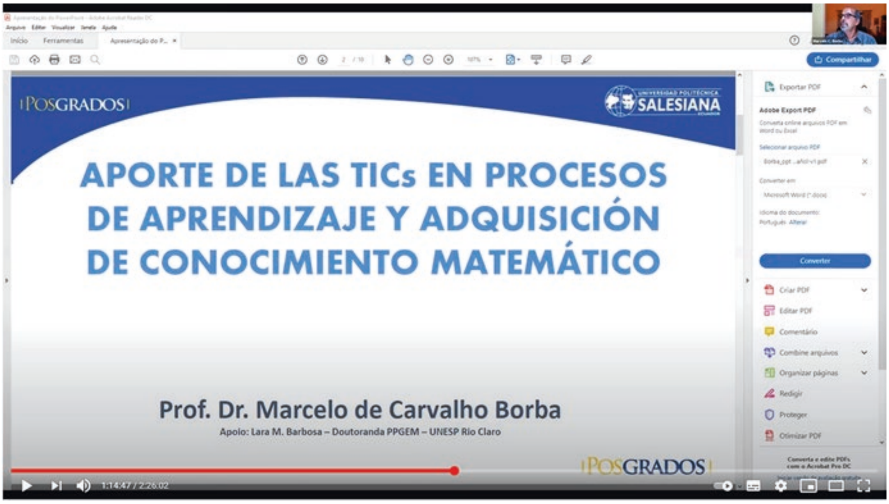
https://igce.rc.unesp.br/#!/pesquisa/gpimem%2D%2D-pesq-em-informatica-outras-midias-e-educacao-matematica/triangulo-de-engestron

Another one of the *LIVES* was a graduate program on science and health education with no mathematics education. "Questions out of the bubble" were asked and the effort to explain mathematical terms such as the sigmoid curve and its derivative, to a public that had a little mathematical background. The *LIVE* presenter was forced to explain in a way that the mathematics could be understood graphically. In interdisciplinary fashion, people could connect the derivative of the sigmoid, a normal curve, to the number of rooms in a hospital. Moreover, with the video app that was used, they could see different sigmoids and different derivatives and imagine different size hospitals.



https://www.youtube.com/watch?v=6FXMBM1bGcM

Regarding the future of the *LIVES* type of activity to date, no actual research was found to have been conducted on the *LIVES* activity yet, but in this chapter, we would like to connect it to the discussions in the studies by Borba et al. (2016) and Engelbrecht et al. (2020). Moreover, we would like to connect *LIVES* video festivals to the notion of expanding the classroom. The notion of the transforming classroom was developed before the pandemic (Borba et al. 2014), but the pandemic scenario provided an excellent opportunity to expand this approach. Furthermore, we should set an agenda of research connecting these *LIVES*, which did not happen only in Brazil (e.g., a webinar in Ecuador, all in Spanish, with participants from Europe and different countries of Latin America).

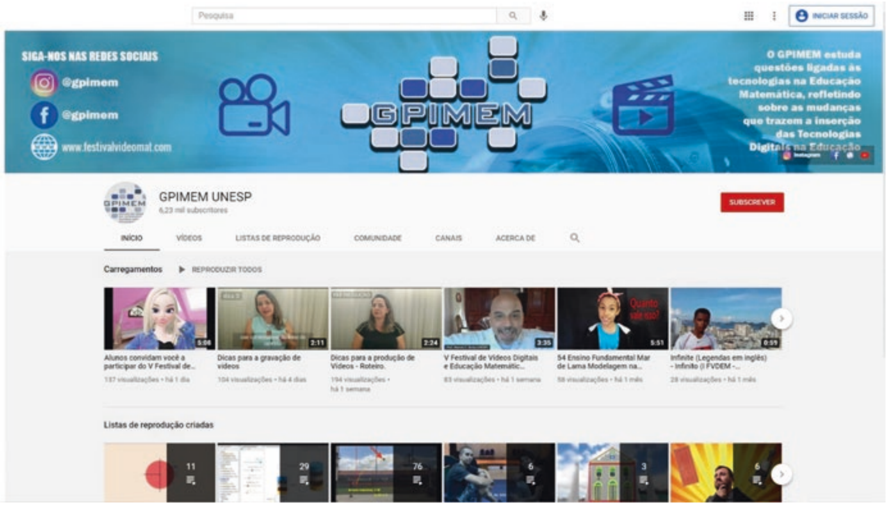


https://www.youtube.com/watch?v=1krf4TAb3QM

We have not developed research on these *LIVES*, but we believe that the model of a classroom that is embedded in four walls is rapidly melting. Inverted classrooms and blended learning are expressions that have signalized that the distinction between face-to-face and online education was changing. We believe that both the pandemic in general and the *LIVES* in particular may be another facet of such a transformation.

More research questions that arose with the *LIVES* activity include how this activity impacts the public image of mathematics. In our sessions, we already employed three languages – English, Portuguese, and Spanish. With our Internet-expanded classrooms, we had sessions in which it was easy to have parents as well as researchers participating. The sessions quickly grew in popularity, and on two occasions, we had a problem in virtually accommodating more than the (allowed) 100 people who entered the room. We have *LIVES* broadcasted to more than 1000 in-service and pre-service teachers, followed by another 1000 visualizations asynchronously. We also need to develop specific research procedures in order to try to understand such a phenomenon.

We often talk about the need for interdisciplinary mathematics education in the literature, but we would like to raise the conjecture that *LIVES*, in its intense multimodality, as a product of humans-with-Internet-videos-PPT, are likely to foster such interdisciplinary facets of mathematics education (see for some examples).



https://www.youtube.com/channel/UCHw13SBPvU-VzPd77V07g0w

Again, more research is needed in order to understand the role of videos and *LIVES* in fostering an interdisciplinary approach, as it happens in an interdisciplinary setting: the Internet!

2.6 Revisiting Theoretical Approaches to Frame New Ways of Communication Using the Internet

Wells (1999) contends that individuals make sense of observed phenomena by experience, information, knowledge building, and understanding. *Experience* is the set of meanings that are construed in the course of participation in the succession of events that make up ones' life trajectory. Information consists of the other people's interpretations of experience, from brochures to authoritative printed works of reference (theoretical information as conceptual tools). But using this information in a new situation depends on the extent to which one can be infused with pre-service mathematics teachers' experiential meaning and deliberatively integrated into their model of the world. *Knowledge building* has to do with how the pre-service primary teacher is engaged in meaning making with others, in an attempt to extend and transform their collective understanding. In this sense, knowledge building typically involves using and progressively improving representational artifacts/conceptual tools. Finally, *understanding* constitutes the interpretative framework in terms of which pre-service teachers make sense of new experiences and which guides effective and responsible action. Understanding can be seen as the culminating moment in a cycle of knowing.

With new interactive digital resources, we assumed that pre-service mathematics teachers construct arguments in interaction with their partners and using multimodal resources in order to construct knowledge about mathematics teaching and to develop the skills needed to learn from practice. Knowledge construction in collaborative settings is based on the assumption that learners engage in discourse activities using multimodal resources of communication. The new digital resources at the disposal of pre-service teachers and the features of new ways of sharing knowledge producing and using new products extend the meaning endowed to technological and conceptual tools.

What we emphasize here is the activity of knowing through using the conceptual tools (the theoretical information provided) as a means of guiding, attending, interpreting, and decision-making and share the knowledge by several types of multimodal media. That is to say, the digital resources and new interactive spaces allow considering the theoretical information as an artifact of teaching culture that serves as a tool for achieving the goals. This is an instance of how an action is mediated by semiotic tools, in this case, the different ways of considering the knowledge in the discourse.

We may see new resources as something outside humans. We suggest, as we have hinted before, that the connection between artifacts or media that are considered nonhumans should be investigated more closely. Souto and Borba (2018) and Kaptelinin and Nardi (2006) suggest that artifacts have agency. Santa Ramírez (2016) in a similar fashion claims that *origami* (paper folding) has agency in producing mathematics in the classroom.

In this manner, we believe that collectives of humans-with-Internet produce a different kind of knowledge than collectives of humans-with-paper-and-pencil have developed, in both (what is considered) formal mathematics and school mathematics. The Internet is changing communication and knowledge as it shapes what humans are, and conversely and dialectically, humans are transforming the Internet.

2.7 Conclusions

In this chapter, we built on two papers (Borba et al. 2016; Engelbrecht et al. 2020) that drew a map of current trends of the use of digital technology in mathematics education. We do not know when or whether we will overcome the pandemic. But this pandemic has changed the hierarchy of fields of interest in mathematics education. Using digital technology in mathematics education has outgrown the status of just a topic of interest – it should no longer only be a special interest group in mathematics education. We have never had a specific interest group in "mathematics with paper and pencil"; in the future, we may not have interest groups in digital technology, blended learning, or inverted classroom at ICME, PME, CERME, or other mathematics education conferences, as everyone will have "naturalized" these changes. Today, in 2020, whether your favorite field of interest is ethnomathematics, early-grade teaching, or creativity, digital technology in mathematics education has become relevant as well. In fact, it has become relevant in most fields of interest in mathematics education.

With the emergency remote teaching that was developed during the pandemic, it is not yet clear how much of the earlier research produced by the international community of mathematics education has served as a basis for action. An interesting first research question could be to understand how much use of such existing research teachers and administrators have made.

We are convinced that the changes in the transforming classroom, as was discussed in this chapter, were accelerated by the pandemic. New research is needed to understand the role of home in the classroom of the year 2020. Collectives of humans-with-Internet should be seen as humans-with-Internet-homes. Different homes and different levels of Internet access in different areas of a city or country are part of the collective that generates new knowledge for students and teachers.

A view of technology, condensed by the notion of humans-with-media as basic unity of knowledge production, ends up being relevant to show how different sets of artifacts – computers, the Internet, and home – act in different ways, depending on different social conditions. Gaps and advantages in learning may depend not only on media but also on humans. Besides teachers, parents became even more important as an alternative teacher, a monitor to students confined at home. What about parents who cannot be present to help? We believe there is a rich agenda of investigation regarding access to schooling in the 2020 format. Besides the new questions, we need to develop research protocols in order to develop research in such a "classroom" that involves homes, parents, and the quality of Internet connections.

The *LIVES* type of activity, as discussed earlier, lends itself to be an object of research. They were not recorded talks. If they are seen as videos, they do not fit the classification of different kinds of videos made by Neves et al. (2020). They were not video classes, vlogs, narratives, or parodies. So what is the nature of this kind of presentation? What role have they played in pre-service mathematics education? Again, we will need to develop new research procedures for addressing the *LIVES* and the impact it may have on teachers.

Issues that need to be addressed in future research include (Engelbrecht et al. 2020):

- What technology should students use to support their own mathematical learning? How might they make effective choices with the multitude of options available in informal contexts?
- How can social media tools be combined with the best practices in teaching and contribute effectively to student engagement and the development of deeper mathematical understanding?

- A better understanding is also needed of critical processes or mediating variables, such as structuring of tasks, well- or ill-defined problems, student engagement, teacher scaffolding, and the ways they combine to create online written discourse in meaningful mathematical settings (Resta and Laferrière 2007).
- The *LIVES* type of activity has become a form of communication that, to an extent, resembles MOOCs – it can reach thousands of people synchronously and asynchronously. It can be used in a face-to-face classroom as a video. It also has traces of a conventional online lecture, with interactions in chats in social media. However, we have very little empirical evidence of the success of this activity.

The three authors of this chapter and other colleagues have been developing a synthesis of research on digital technology and mathematics education for the last 6 years. We have developed the notion of a transforming, changing classroom. The pandemic has accelerated such a transformation, and it is not clear, if a cure to COVID-19 is found tomorrow, what part of the 2020 approach will make part of the future blended face-to-face education. We have raised some new research questions, and the need for meta-research regarding how to develop such a research in a way that the research methodology may be seen in a very broad way as an interface of view of knowledge, view of education, and research procedures (Borba and Villarreal 2005).

In the same way that fast internet has been the landmark for the fourth phase, the virus and the intensification of use of technology, the intensification of hybrid practices may be the landmark for this next phase.

So to conclude, traditional rectangular classrooms with four walls will probably continue to be used to teach mathematics in the near future. In this chapter, we gave an outline of how this situation has been changing and to what extent it will change in the future. We have to live with this change and direct the environment with the research that we do.

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Chapter 3
Presence in Online Mathematics Methods Courses: Design Principles Across Institutions

Dung Tran and Giang-Nguyen T. Nguyen

This article addresses the challenge researchers raised about how online learning could provide opportunities for students to interact with other students and the instructor as compared to a face-to-face environment (Paechter and Maier 2010). We exemplify how the Community of Inquiry (CoI) theoretical framework informed our design of learning opportunities for primary mathematics preservice teachers (PSTs) in two courses offered in Australia and the USA. In doing so, we fill the gap of limited research on supporting emerging online communities to engage in particular norms and instructional practices (Matranga et al. 2018).

There are multiple research efforts in online teaching and learning in mathematics (cf. Silverman and Hoyos 2018), especially for mathematics teacher education. These studies focus on different participants including PSTs, in-service teachers, and teacher educators. Matranga et al. (2018) addressed the design of interactive technology to scaffold pedagogical practices for in-service teachers with a focus on developing productive assessment practice in their EnCoMPASS environment. Avineri et al. (2018) elaborated on principles for designing Massive Online Open Courses (MOOCs) for mathematics teachers and their impact on pedagogical practices. Crisan (2018) discussed how she used online video cases to help secondary teachers develop technological pedagogical content knowledge (TPACK) for teaching with technology and to support the teachers to connect research into their

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practices. Chazan et al. (2018) studied mathematics teacher educators' use of the LessonSketch platform to represent and create teaching practices for their teacher candidates.

The current article addresses limited research in online courses for primary PSTs that prepared them to teach mathematics with ambitious goals (NCTM 2000; 2014). We provide an example of educators across institutions collectively developing a common language (Chazan et al. 2018) in course design. This article addresses the question: *How can designers of online courses for mathematics PSTs support interaction for the participants to develop their teaching practices?* This article models designers' intervention and orchestration to promote the exchange between students (Hoyos et al. 2018).

We discuss the social aspect of learning and summarize how the CoI framework was developed for an online environment (Garrison et al. 2000). We elucidate how the framework is applied in two online course settings: synchronous and asynchronous. Finally, we reflect on the transferring process from a face-to-face to an online design and provide suggestions for future research.

3.1 Social Aspect of Mathematics Learning

Sociocultural learning theories (e.g., Vygotsky 1978) advocate for dynamic interaction that encourages learners' participation, collaboration, and community building to construct knowledge. Wenger (1998) argued that when an individual is involved in a community, they engage in shared goals and practices. Students learn through interactions with each other to build their knowledge using common language, tools, and practices. Within this community of practice (Wenger 1998), the educator's role is engineering learning experience through inducting students to collective practices valued in the community.

Learning is considered as participation (Sfard 1998) where the teacher introduces students to mathematical practices such as problem-solving, reasoning, conjecturing, and generalization, or mathematical proficiency (National Research Council [NRC] 2001). Educators must acquaint PSTs with the pedagogical skills needed to help their future students achieve mathematical proficiency.

Researchers (e.g., Magiera and Zawojewski 2011) contended there is a lack of meaningful discussions in mathematics classrooms where students work individually without benefiting from the opportunities to interact with their peers. A general expectation is that students learn independently (e.g., Hoyos 2016) in online settings. Notwithstanding, communication is fundamental to critical thinking and mathematical thinking. In an online environment, it has been more challenging to model the teacher's intervention and interaction's orchestration and to promote the exchange of ideas between students (Wilson 2018). The challenge is more prevalent for mathematics educators who teach mathematics methods courses that aim to prepare PSTs to enact ambitious teaching practices that follow reformed orientations (cf. NCTM 2000; 2014). These PSTs are most likely to teach in face-to-face classrooms in the future, but they learn about mathematics practices in an online environment.

Previous research concerns the deficit interaction in an online environment as compared to face-to-face instruction (Paechter and Maier 2010). However, Borup et al. (2012) proposed that investing in online social interaction for exchanging communicative reflections can outweigh face-to-face communication. The interpersonal aspect of the instructor-student relationship is achievable with the integration of advancing online technologies (Wilson 2018). Kellogg and Edelmann (2015) supported the importance of social interaction in online classes to overcome the potential communicative barrier posed in an online environment. How to take advantage of online technologies to capitalize on the social aspect of learning depends on how the instructor develops the *presence* for the community.

3.2 Community of Inquiry Theoretical Framework: Keeping Presence at the Center

Lipman (2003) popularized the term *community of inquiry* when rethinking educational practice from Dewey's reflective paradigm (1933). Communities of inquiry provide intellectual challenges and the environment for individuals to develop their thinking and learning through collaboration. A connected community is essential to sustained inquiry for participants to exchange ideas, to construct and validate knowledge collectively. Drawing on this foundation, Garrison et al. (2000) constructed a comprehensive framework to describe and guide the study of online learning in higher education, named Community of Inquiry Framework (CoI) and changed to CoI theoretical framework (Garrison and Akyol 2013).

The framework includes three interrelated components of presence: cognitive presence, social presence, and teaching presence. The *cognitive presenc*e refers to the critical inquiry circle that allows participants to construct meaning through sustained communication (Garrison et al. 2000). This component links to opportunities for exploration, integration, finding solutions, and constructing knowledge in the community. The *social presence* refers to "the ability of participants in the CoI to project their personal characteristics into the community, thereby presenting themselves to the other participants as 'real people'" (Garrison et al. 2000, p. 89). Garrison et al. (2000) argued that social presence is crucial to facilitate the cognitive presence in the online learning environment. The two components take place during the academic discourse.

Social presence was first introduced by Short et al. (1976) in a computer-mediated conference as a one-dimensional concept that links to intimacy and immediacy. Since then, multiple efforts to conceptualize and measure social presence in online teaching contexts have taken place, but a comprehensive effort was developed by Kim (2011). Kim synthesized multiple aspects related to social presence discussed in the literature and found the four-factor model: (a) *mutual attention and support*,

(b) *affective connectedness*, (c) *sense of community*, and (d) *open communication* best fit the collected data (Table 3.1). Kim argued that whereas affective connectedness, sense of community, and open communication reflect the social characteristics of students in an educational context, mutual attention and support reflects presence; that is, social presence reflects how people perceive and experience presence, even though they are not physically "there" in online settings.

Adapted from Kim (2011)

The *teaching presence* refers to the role that the instructor plays in the online environment which is defined as "the design, facilitation, and direction of cognitive and social processes for the purpose of realizing personally meaningful and educationally worthwhile learning outcomes" (Anderson et al. 2001, p. 5). When facilitating these aspects, the teacher sets a climate and learning environment for the social presence to happen. When selecting and facilitating learning activities, the teacher ensures the cognitive presence feature.

CoI serves as a framework for designing and researching online education as it showed "sufficient coherence and explanatory power" (Garrison and Akyol 2013, p. 107). The framework has provided a model to develop critical thinking and higher-order learning in the online environment (Garrison and Akyol 2013) and has been adopted in explaining and prescribing the effective conduct of online and blended learning (Arbaugh et al. 2008; Garrison and Arbaugh 2007). Other researchers provided evidence on the potential and success of the framework in creating higher-order learning outcomes in such environments (Akyol and Garrison 2011; Shea and Bidjerano 2009). Also, the framework provided a model for explaining student satisfaction and perceived learning and measuring the quality of online education (Shea and Bidjerano 2009). Other studies focus on (a) conceptualizing and operationalizing, revising, and measuring each of the presence components; (b) connecting among the components (e.g., teaching presence and cognitive presence); and (c) linking between the components and learning (student satisfaction, perceived learning, retention, learning outcomes) (see Garrison and Akyol 2013 for a review). However, not many studies elaborate on the use of the framework in designing online learning experiences. In the present article, we exemplify CoI in two mathematics methods courses. The *cognitive presence* is facilitated by the process

Table 3.1 Social presence

Social presence aspect	Description
Mutual attention and support	Participants become attentive and supportive to other participants and are aware of the others' endeavors to do so
Affective connectedness	Participants feel connected emotionally and socially with others and express intimacy and warmth
Sense of community	Participants share a sense of membership as a group and perceive the usefulness of community support and satisfaction of collective effort and cooperation
Open communication	Participants understand other's views and feel comfort and pleasure in communicating with others

for developing mathematical knowledge for teaching (MKT) (Ball et al. 2008) that prepares PSTs to help their students develop mathematical proficiency. The *social presence* is highlighted with regard to the technology tools available in synchronous and asynchronous settings. These two components are supported by *teaching presence* underscoring how we designed and implemented the online courses.

3.3 Application of CoI in the Mathematics Methods Courses

3.3.1 Course Setting

Course 1 was offered at a public university in Australia for approximately 250 PSTs. The primary PSTs had a series of three mathematics methods courses in their program. Course 1, the first in the series, prepared PSTs with mathematical knowledge and understanding to teach mathematics. The course was designed to develop specialized content knowledge (SCK) with an emphasis on working with learning materials, manipulatives, and representational fluency to help PSTs reflect on meanings for mathematics and develop conceptual understanding, reasoning, and problem-solving. This course offered opportunities for PSTs to develop their understanding as active learners and implicitly reflect on such experiences as future teachers. However, PSTs did not discuss how to implement the tasks in their future classrooms.

Course 1 was a 12-week semester with two 1-h lectures and one 2-h workshop in a small tutorial group (similar to the US recitation section) that has up to 30 students. For consistency among the workshops and due to institutional constraints, the course had a *lecturer in charge*, who delivered the lectures and prepared learning materials, tutorial notes of learning intentions, and the implementation of activities. The re-recorded lectures were created weekly for PSTs to access prior to the synchronous workshops meetings on Zoom. Tutorials took place in synchronous meetings on Zoom with the tutor. This article reflects on the experience of the lecturer in charge and also a tutor in his group, which is referred to as the instructor.

Course 2 was offered at a four-year public institution in the USA in a 16-week semester with approximately 35 students. The course prepared PSTs with pedagogical content knowledge (PCK) for teaching mathematics to help future students develop mathematical proficiency. Canvas was the learning management system with built-in collaboration platforms such as Google Apps. Webex was the online conferencing tool used in the course. The course was designed to help PSTs engage in four pedagogical practices: planning and instruction, assessment, questioning and feedback in mathematics, and differentiated instruction. The PSTs learned about mathematics pedagogical practices by discussing them in specific mathematics topics and envisioning how to apply their understanding of PCK in future teaching.

The content was released on Saturday mornings, and the PSTs completed all work by the following Sunday. Even though the course was asynchronous, the instructor held weekly live sessions to provide the PSTs feedback and address concerns that arise during the week. The live sessions were recorded for later access. Three types of discussion forums in the course include (a) general course questions and answers (CQA), (b) weekly discussion forum (WDF) (instructor assigned), and (c) group discussion (self-selected for the Unit Plan). The WDF has three parts: (a) individual posts, (b) responses to group members, and (3) a discussion summary using a Google Docs Template.

Recent studies shift from considering educational experience as the intersection among the three components (e.g., Garrison et al. 2000) to using teaching presence and social presence to develop cognitive presence (e.g., Shea and Bidjerano 2010). However, in designing the courses, we conceptualized how teaching presence facilitates cognitive presence and social presence. When adopting the social view of learning, we argue that social presence is an important outcome in addition to cognitive presence. Therefore, we use teaching presence to predict and explain the two remaining presences. The three types of presence will be presented separately in the following sections for clarity. However, the cognitive presence was integrated with the social presence when we implemented them in the courses.

3.3.2 Cognitive Presence

Both courses adhere to the *cognitive presence* that helps build a community of practice of mathematics teachers to help their future students develop mathematical proficiency (NRC 2001). NRC provides a comprehensive nature of mathematical learning and doing, which include five intertwined strands:

- *Conceptual understanding*—comprehensions of mathematical concepts, operations, and relations
- *Procedural fluency*—skills in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *Strategic competence*—ability to formulate, represent, and solve problems
- *Adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *Productive disposition*—habitual inclination to see mathematics a sensible, useful, coupled with a belief about one's own efficacy

These mathematical proficiencies indicate what it means to be successful in mathematics learning. To support PSTs' understanding of how to help their future students achieve mathematical proficiency, our courses were designed to provide opportunities for PSTs to engage in mathematical practices (e.g., ACARA 2015; CCSSI 2010) in tandem with school mathematics content taught (Course 1) and to think about ways to provide a similar experience for their future students (Course 2).

Ball et al. (2008) proposed a model of MKT that describes what a mathematics teacher needs to know to teach mathematics effectively. Based on empirical data, they proposed two components of the knowledge including subject matter knowledge and PCK. Situated in the subject matter knowledge component, SCK is the mathematics knowledge that is unique in the profession (Ball et al. 2008). The SCK includes the ability to scale up student errors and difficulties, represent mathematics concepts, and solve problems in different ways. PCK is similar to what Shulman (1986) described, "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Course 1 helps PSTs make sense of mathematics for themselves as active learners (SCK), and Course 2 prepares PSTs with PCK to help future students make sense of mathematics.

Course 1: Synchronous Design for SCK

Course 1 helps PSTs develop SCK by engaging in mathematics differently than what they have experienced in schools. We aimed to (a) develop PSTs' understanding that mathematics is used as tools to model, describe, and predict real-world phenomena (modeling); (b) foster conceptual understanding for PSTs by connecting operations, algorithms to geometric models, and other representations; and (c) help PSTs engage with mathematical (mental and physical) actions observed when mathematicians work on their job such as reasoning, problem-solving, and generalizing (CCSSI 2010). We exemplified SCK in the following examples.

Developing understanding and making reasoning explicit. PSTs often remember the order of operations as a mnemonic, such as BODMAS or PEMDAS (in the US context), and tend to use these to justify why they apply the order without connecting to meanings. In one week, we designed an activity to help PSTs address the issues by linking the rules with geometric representations. The instructor asked the PSTs to distinguish between 3×2 and 3^2, without calculating but using geometric representations. The PSTs were guided to sketch an area model to represent the expressions; the first expression connected to the area of a rectangle (sides 3 and 2) and the second to the area of a square (side 3). This geometric representation helps the PSTs build a connection between a symbolic form and geometric understanding and also addresses a misconception/alternative conception they have related to the power of 2 (squared). The PSTs moved on to work in groups matching geometric representations to symbolic forms (Fig. 3.1), then reflected on the meaning of the order of operations.

During small group discussions, the PSTs were asked to make their reasoning explicit when they matched the cards. Sometimes, they sketched a model if they did not find a card that matched the expression and justified their thinking. They then resumed a whole-class discussion to address any challenges and alternative/misconceptions. Furthermore, the instructor asked the PSTs what happened to the equivalences if the numbers in the card change from whole numbers to fractions (see A11), decimals, and other positive numbers. The discussion helped the PSTs develop algebraic thinking focusing on generalization and operation properties. They were

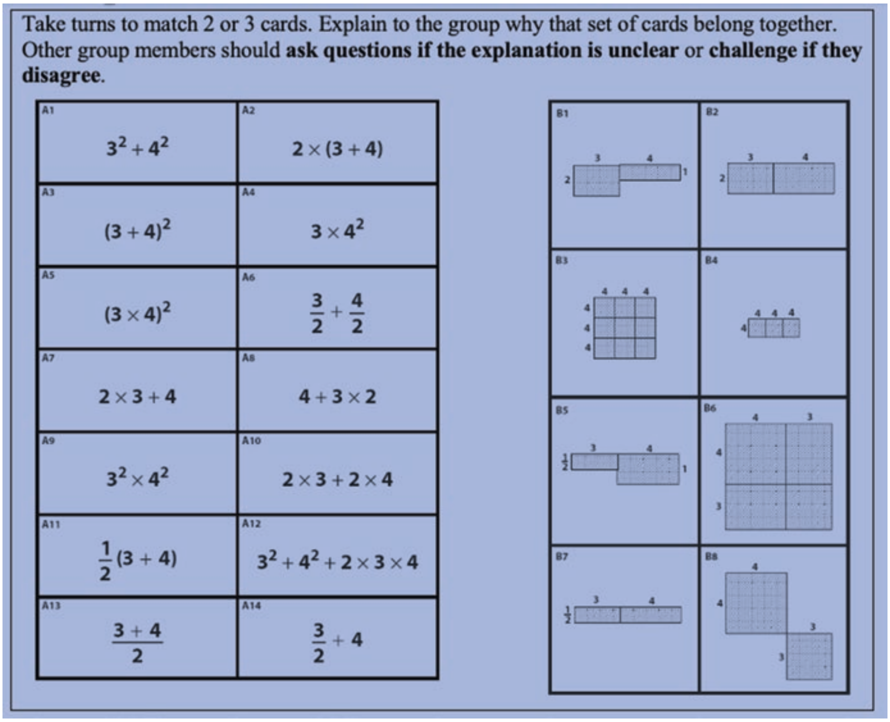

Fig. 3.1 Order of operations task

supported to represent the rule in a verbal form before writing a symbolic form (e.g., distributive property $a * (b + c)$). This task was typical in Course 1 that helped PSTs develop conceptual understanding by focusing on *representational fluency* to help build meaning for rules, formulas, and algorithms and making their reasoning explicit. Although this task was not grounded in a real-life context (horizontal mathematizing), it was worthwhile to help PSTs make sense of abstract concepts and deal with multiple representations (vertical mathematizing) (cf. Gravemeijer 2004).

Engaging in problem-solving and reasoning. Figure 3.2 illustrates a typical example that helps PSTs engage in problem-solving and reasoning. The PSTs worked in pairs to play with a die and record their results. All small group data were aggregated and used to determine if this game was fair. The PSTs analyzed the game from an empirical lens, compared their own data with the aggregated data, and explained the discrepancy by linking it to the sample sizes. The groups were guided to develop a theoretical model of the probability using a tree diagram to confirm or disprove their conjecture. The class ended with a reflection on concepts of fairness in probability, the role of sample size in a probability experiment, and how an empirical probability converges to the theoretical probability as the number of trials increased and their certainty in making decisions based on sample size.

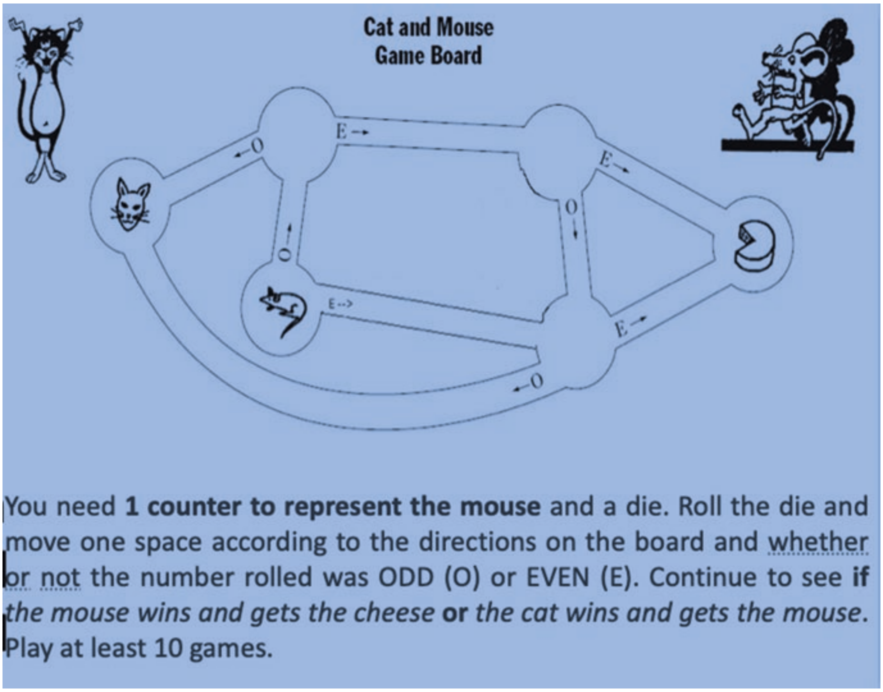

Fig. 3.2 Cat and mouse game

Course 1 helps PSTs experience mathematics differently by focusing on (a) adaptive reasoning (e.g., making justification and explaining the meaning of the order of operations or convincing if the game is fair); (b) strategic competence (e.g., developing strategies to solve problems of deciding if the game is fair); (c) conceptual understanding (e.g., linking numerical, symbolic form to graphical representations or connecting among concepts in probability); and (d) procedural fluency (e.g., deciding which order to use in calculation or using tree diagram in calculating conditional probabilities) (ACARA 2015; NRC 2001). In turn, it built PSTs' SCK that is unique to their profession (Ball et al. 2008). Moreover, interacting with materials to the abstract meaning and connecting ideas helps build their *productive disposition* towards mathematics, not as a collection of rules for memorization.

Course 2: Asynchronous Design for PCK

The PSTs were introduced to pedagogical practices focusing on helping future students develop mathematical proficiency. The instructor aimed to build a community of practice of PSTs to learn the skills and knowledge needed for teaching mathematics and to facilitate ambitious teaching for future students. We exemplify two pedagogical practices as follows.

Designing rich tasks in planning. One aspect of planning is designing and implementing tasks that promote student mathematical proficiencies. Prior to participating in the WDF, the PSTs have already read materials on planning (Reys et al. 2014; Van de Walle et al. 2019), cognitive demand of tasks (Henningsen and Stein 1997), problem-solving (Van de Walle et al. 2019), and meaningful discourse (Wagner and Herbel-Eisenmann 2018). Based on their reading, the PSTs were asked to think about designing, selecting, or adapting a rich task with higher levels of cognitive demands for future students. Figure 3.3 illustrates the activity that the PSTs engaged in task design and implementation.

PSTs solved Task A themselves as learners before thinking about task features. This task offers the potential for higher cognitive demands and features multiple entry-and-exits. The instructor initiated the PSTs into thinking about *assumptions made about the types of numbers used when they came up with such answers.* To further the discussion, the instructor asked the PSTs if they could make a generalization to the solution through the prompt: "*Could one find all possibilities and why*?" The discussion on the solution was extended in the live session when the instructor engaged the PSTs to think about the solution to this equation $x + y = 7$. The task was contrasted with a typical task, adding two whole numbers, which focuses mainly on procedural fluency. By focusing on preparing their future students on mathematical proficiency, PSTs saw the importance of high cognitive demands of tasks to develop (a) *conceptual understanding* (e.g., linking the solution to a graphical display), (b) *adaptive reasoning* (e.g., making their reasoning explicit), and *problem-solving* (e.g., finding a solution plan and checking different domains to solve a problem). Building on this understanding of task features, the PSTs thought about how to implement the task and cater to student diversity. The PSTs were then asked to design a high cognitive demand task for the numbers and operations topic.

Designing assessment to measure understanding and reasoning. One course assignment involved the PSTs in conducting a formative interview with a student. The PSTs had two weeks to prepare for the interview by engaging in instructional resources (Van de Walle et al. 2019). The instructor designed an activity (adapted from Philipp 2005) (Fig. 3.4) to help the PSTs think about (a) choosing assessment

Task A. *Find two numbers that have the sum of 7. Do you think you could find all the possibilities and why?*

Part 1. *Solve the task and discuss*

- *Cognitive demands of the task*
- *Implementation concerns*
  - *Anticipating student conceptions and difficulties when solving the task and how to support students*
  - *Adapt the task to meet the diversity of primary students.*

Part 2. *Design a higher-order cognitive demand task to teach numbers and operations. Convince why your task is a rich task to use.*

Fig. 3.3 Task design and implementation in WDF items that call for reasoning and conceptual understanding and (b) using alternative strategies in assessment.

In Question 1, a student could find the answer by using an algorithm of the division of fractions, “when you divided, you flipped and multiplied.” In contrast, in Part (a) of Question 2, the instructor asked the PSTs to discuss how to build a relationship of $\frac{1}{3}$ and 10 recipes. The instructor prompted the PSTs to think about iterating $\frac{1}{3}$ three times to get one whole (cup of sugar). Moreover, the instructor asked the PSTs to think about how they could use models (rectangles or pattern blocks) to demonstrate their answer and to elicit consideration of alternative strategies in assessment.

In part (b), the PSTs were asked to find ways to demonstrate their conceptual knowledge on the remainder for a division of fraction problems. They struggled when they had to write the answer without a remainder. In solving the problem using the procedure they learned they could come up with $7\frac{1}{2}$. Also, one could provide the answer of 6 recipes with $\frac{1}{2}$ cup of sugar left, which is mathematically correct. The instructor helped further their understanding by encouraging them to think about how many recipes they could make with that $\frac{1}{2}$ cup of sugar left; PSTs came to realize that they can make an additional batch and half of the recipe with $\frac{1}{2}$ cup of sugar. As the PSTs tried to find the answer, they came to realize that they knew how to divide two fractions but could not explain the reason and link a model to the solution. This activity helps PSTs distinguish between questions that measure procedural knowledge focusing on rote memorization with those asked for understanding and reasoning. The PSTs were then asked to design an assessment item to measure conceptual understanding of operations with decimals (Fig. 3.5).

***Part 1**: Discuss how each of the following questions could be used to assess your future students' knowledge of fractions:*

Question 1.

(a) $3\frac{1}{3} \div \frac{1}{3}$

(b) $2\frac{1}{2} \div \frac{1}{3}$

Question 2.

a. A recipe calls for $\frac{1}{3}$ *cup of sugar, and you have* $3\frac{1}{3}$ *cups, how many recipes could you bake?*

b. Now, if you have2 $\frac{1}{2}$ *cups of sugar, how many recipes could you bake if a recipe calls for* $\frac{1}{3}$ *cup of sugar? Do not write your answer as _____ recipes with remainder of _____ cup.*

***Part 2**. Design an item to measure student conceptual understanding of operations with decimals. Explain why your item serves that purpose.*

Fig. 3.4 Designing assessment for conceptual understanding

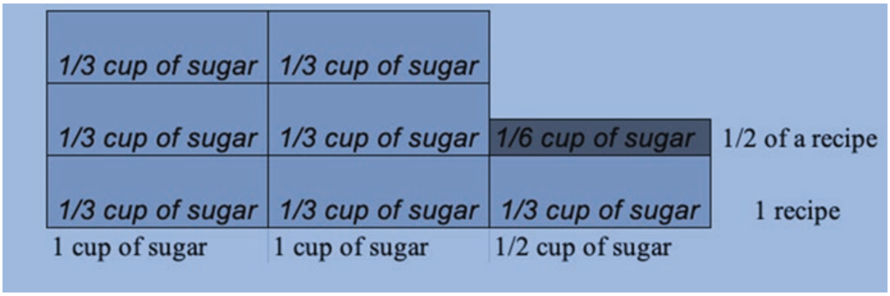

Fig. 3.5 Models for division of factions

The instructor reinforced the use of representations in assessment and discussed questioning and prompting techniques that allow PSTs to assess their student's mathematical proficiency in the live session**.**

3.3.3 Social Presence

Course 1: Breakout Rooms on Zoom and Use of Learning Materials

We discuss how we make artifacts become an instrument for productive learning when the instructor focuses on developing social presence.

Using Zoom for synchronous interaction. Zoom has a unique feature to assign participants into breakout rooms. The breakout rooms helped PSTs overcome their challenges of exposing themselves to a whole class and hesitation to turn on their cameras. PSTs were encouraged to turn on their camera when they interacted with peers, and they felt more comfortable when sharing their ideas in breakout rooms. The instructor decided to keep the groups consistent throughout the semester as the PSTs preferred to stay with the same peers. This design built a *sense of community* for PSTs as they were held accountable for their group work and developed *affective connectedness* when they shared their emotions easily. It also helped with *mutual attention and support* as they were co-present in the same space in the breakout rooms and learned how to listen and attend to others, share their reasoning, and set norms within the small group for *open communication.* This support of social presence helped PSTs understand other's views and feel comfortable when expressing their own view in their small groups. The instructor modeled how to use the whiteboard (Fig. 3.6) with a co-annotating function, which became an artifact to share with the whole class. The breakout rooms and group artifacts (e.g., whiteboard) helped build the social presence within small groups. The instructor also asked the groups to indicate a moderator and report back to the whole class.

The breakout room helps develop social presence, both between the instructor and PSTs and among the PSTs themselves. When the PSTs worked in the breakout rooms, the instructor visited each room with a planned journey: no more than five

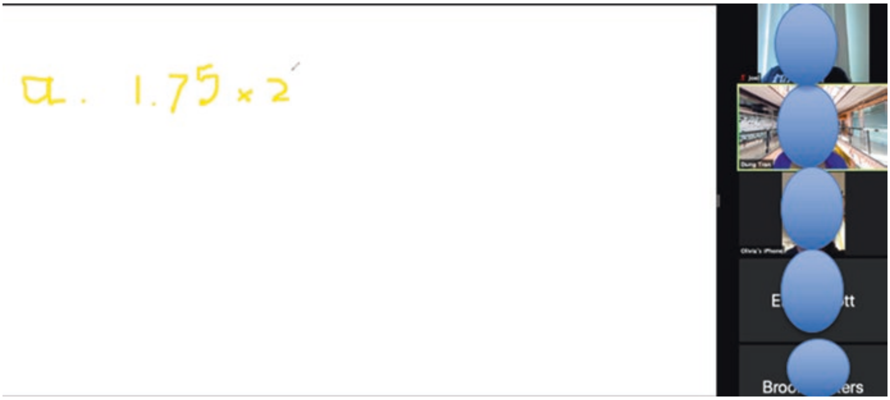

Fig. 3.6 Breakout room with a whiteboard

groups to allow the instructor enough time to visit all groups. The instructor used four types of questions to stimulate mathematical discussion including starter questions, questions to stimulate mathematical thinking, assessment questions, and final discussion questions (Way 2008). In these visits, the instructor planned for sequencing the group responses and challenges based on information discovered when interacting with the PSTs. The class then ended with a consolidation of the content for the week and assigned post-classwork for PSTs.

Rethinking about learning materials for social interaction. Manipulatives become a pedagogical object for PSTs to abstract meaning (Vygotsky 1978). Physical manipulatives and learning materials afford rich learning experiences in a face-to-face classroom. When designing online learning, we strived to create similar experiences for social interaction. For example, PSTs could print out a Tangrams template or buy one available in the market to work on themselves during the workshop. However, in this course, virtual Tangrams were used. The instructor asked the PSTs to work in groups, one controlled a shared screen and other members contributed to it (Fig. 3.7).

Similarly, instead of asking PSTs to cut out the Order of Operations template and sort those out themselves, the instructor requested the PSTs to appoint a facilitator in their breakout rooms and share their screen when opening the template. Each card was labeled for easy referencing in discussions. A PST said, “I am looking at Card A. Can we look for a card in the set to match with this?” Other PSTs then responded to it and provided reasoning for the matched cards. They made annotations on the template and used it to present with the class. In this case, the template facilitated group discussion and was used as a shared artifact. The use of materials was built on the social presence lens that facilitates *mutual attention and support*, *sense of community*, *and open communication* when PSTs did the task in small groups instead of individually.

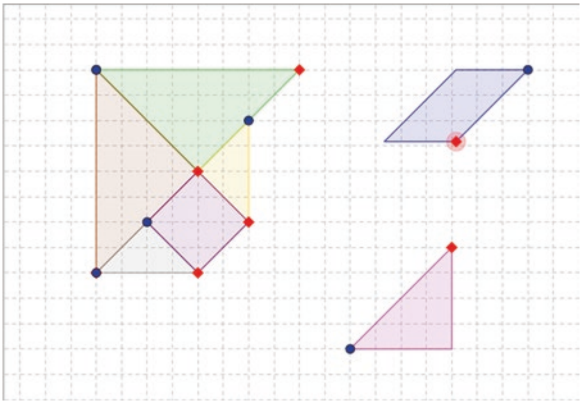

Fig. 3.7 PSTs working together with virtual tangrams

Course 2: Discussion Forums and Live Sessions

We highlight two major features: asynchronous discussion and the built-in collaboration apps (Google Docs) and weekly live sessions that address social presence.

Discussion forums and collaboration tools. Multiple discussion forums were featured including CQA, WDF, "Meet My Classmates and Instructors" Forum, and Unit Plan Workspace. These forums were set up with a protocol for PSTs to express their views and feel comfortable to communicate with others (*open communication*). They served as a platform for PSTs to have *mutual attention and support.* The instructor monitored the forums regularly and responded to PSTs to show *attention and support*. PTSs could send an email, call over the phone, schedule a conference call, or send a text message to the instructor's personal cellular phone, which provides intimacy, immediacy (*affective connectedness*).

The instructor designed forums for small groups. The PSTs worked in a self-selected group to submit the Unit Plan as part of the course assignment. They had a Unit Plan Workspace to interact with their peers. They knew each other through forums, live sessions, and previous courses. They chose whom they worked with, which helped facilitate *affective connectedness* as they felt more emotionally connected. Similarly, in WDF, there were protocols for PSTs to follow. The instructor set the forum where a PST cannot see other members' posts until initial responses were posted to promote the originality of their thoughts. They needed to respond to at least three group members' responses: (a) to someone who had not received any responses, (b) to their original posts (if there was a comment), and (c) to another individual. They used TAG approaches when giving feedback: **tell** something they learned about the task, **ask** something that needed to be clarified, and **give** actionable and effective feedback for improvement. The PSTs were also encouraged to post any concerns about the content in the forum, and if questions were not addressed, they could document those in the discussion summary. They shared responsibilities in the group and felt a *sense of community* as they were a member of the group and found the benefit from their group. The protocols showed to be successful in facilitating *mutual attention and support,* as individual input would be presented and evaluated by others. In this small community, besides working on the discussion assignment, participants could share something personal (*affective connectedness*) and feel more comfortable to share their ideas (*open communication*).

In the self-selected group discussion forums, the PSTs worked on a shared Google Doc with built-in collaboration features as a group artifact (Fig. 3.8). This tool allows interactions among PSTs and the instructor to take place flexibly. Each person in this community could communicate and collaborate indirectly by replying or responding to the threads/comments on the platform. A member could post a comment in the Google Doc for someone to respond to or assign it to someone else. Since each group member was responsible for his or her own work as well as holding each other accountable for the group unit plan, they felt a *sense of community*.

Live sessions. Weekly live sessions took place on Thursdays, Fridays, or Saturdays based on PSTs' preferences from a poll's results. This two-way communication between PSTs and instructor allowed learners to negotiate and structure meaningful knowledge in distance education, much like what occurred in traditional classrooms (Garrison and Shale 1990). While monitoring PSTs learning in WDF, the instructor informally assessed student learning progress and addressed them in the live sessions. In the sessions, PSTs could share what they learned and ask questions through talking or using chat features. These informal, non-compulsory sessions offered intimacy and immediacy in the class.

The instructor often posted a question related to the weekly content for the PSTs to discuss, and their views were presented and acknowledged (*open communication*). When a PST shared intermediate understanding, the instructor did not evaluate the answer but prompted them to elaborate their responses. This helped them feel "safe" and get *attention and support.* The instructor modeled providing constructive feedback in a way that was empathetic to learners' experience in a classroom setting and maintained positive relationships. Together with the forums, live sessions help build a *sense of community* between the PSTs and the instructor.

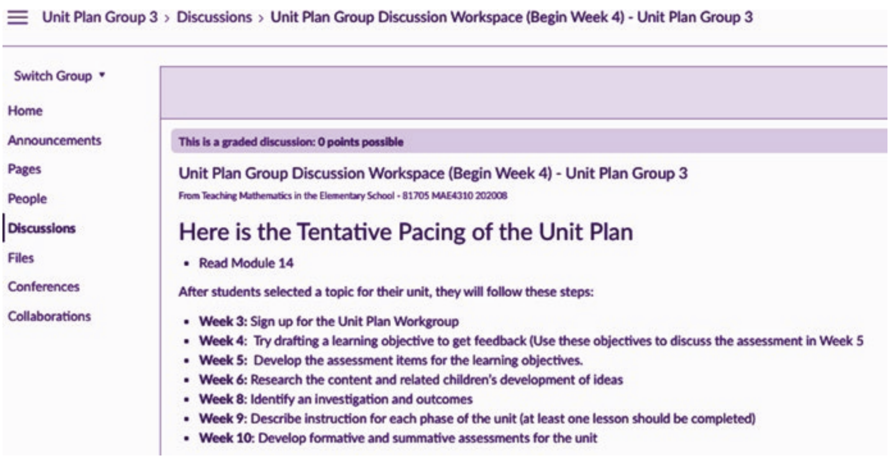

Fig. 3.8 A google doc for collaboration

3.3.4 Teaching Presence

Teaching presence was crucial when developing and maintaining cognitive presence and social presence during the academic discourse.

Teaching Presence in the Synchronous Setting

Teaching presence was crucial when the instructor designed and implemented tasks to scaffold mathematical processes in small-groups and whole-class discussions. It was evident in the tasks chosen paired with learning materials that facilitate the experience with mathematics (*design*). Teaching presence was featured when the instructor decided how to carry out the online class intending to make cognitive presence happen that takes advantage of social presence. The instructor *facilitated* the discussion in breakout rooms, assigned tasks, modeled how the group could proceed, pointed out materials needed, and periodically came into breakout rooms to engage with the discussion to maintain meaningful discourse. Almost all lessons were designed with several planned breakout rooms and the expectation of group report to the whole class at the end. The instructor then summarized the content of each session and asked the PSTs to reflect on their learning for the week (*direction*).

Teaching Presence in the Asynchronous Setting

The instructor *designed* activities to induct PSTs into a community of practice of mathematics teachers who hold ambitious teaching for mathematical proficiency. This is evident in the choice of tasks to engage in the teaching practices. The instructor engaged PSTs in rich tasks that have the potential for rich discussion about teaching practice. The readings selected by the instructor focused on (a) what it means to be proficient in mathematics, (b) using higher levels of cognitive demands for task design and implementation, (c) using Bloom's taxonomy in designing assessments that are coherent, (d) questioning for making student reasoning explicit, and (e) ways to provide feedback to improve student understanding. When implementing the course, the instructor set up norms, discussion protocol, and the timeline for students to learn the content each week. She monitored the forums and held live sessions. These components were strategically planned out to support cognitive presence and social presence. The instructor *facilitated* the learning experience through multiple interventions and provided *direction* to explicitly addressed PSTs' misconceptions in the WDF and recorded live sessions. The instructor also modeled and demonstrated teaching practices during live sessions.

3.4 Discussion and Implications

We, instructors, elaborated on how we used the CoI framework in designing and implementing online mathematics methods courses for primary PSTs. The cognitive presence was featured on helping PSTs experience mathematics as active learners (Course 1) and discussing pedagogical practices to develop mathematical proficiency for future students (Course 2). In this way, we built a community of practice of educators that addresses ambitious teaching aiming at developing comprehensive learning experience for students (NRC 2001). We also highlighted how we created social presence that supports cognitive presence. We capitalized on tools for synchronous discussion that supports a sense of community, open communication, mutual attention and support, and affective connectedness. This included using the breakout rooms function on Zoom and virtual manipulatives and materials for conductive collaboration in Course 1. In Course 2, we designed instructions for PSTs to be cognizant of relations among the members and the degree of proximity and affiliation formed via discussion forums, collaboration tools, and live sessions. Underlying these two components, we showed the role of the instructor, teaching presence, in design and implementation, and direction of learning opportunities for PSTs.

This article offers a unique contrast between two different settings sharing a common goal: preparing PSTs with MKT to teach mathematics. Studying the impact of online professional development on the transfer of knowledge, Herrington et al. (2009) found that teachers succeeded in implementing new pedagogical strategies in their classrooms when they felt supported by their online community of practice. We provided an example of how to support that online community of practice for PSTs. We addressed the gap of supporting online communities that engage in norms and instructional practices to develop mathematical practices for PSTs and prepared them to teach future students. We focused on preservice teachers instead of in-service teachers (cf. Matranga et al. 2018) and addressed a broad range of teaching practices not just assessment practices in their studies. We also elucidated how the communities can be formed in two contrasting settings: synchronous and asynchronous. Matranga et al. showed that the technology mediating interactions in a collaborative environment had more potential to impact teacher instruction than a group of teacher educators facilitating professional development activities. We speculate that the results might hold for PSTs.

Like Crisan (2018), we developed MKT for teachers; however, Crisan's focused was on developing secondary teachers' TPACK that keeps technology at the front of the teaching and used online videos as the main means to achieve this. We emphasized a holistic experience of the preservice teacher preparation with multiple components including the online learning environment and other facilities attached to the environment. The courses share some similarities of design as the MOOC-Ed by Avineri et al. (2018), but they focused on PSTs, and the learning experience is more strictly pacing and universal compared with more personalized.

3.4.1 Reflection on Designing Process

As we have each taught a similar course in a face-to-face setting, we reflected on the design thinking process, especially what maintains and what changes across different settings. The cognitive presence stays the same regarding which types are offered. The focus was on developing MKT to bring PSTs closer to the community of practice of educators who value meaningful learning for students; however, social presence and teaching presence have been operationalized differently in the online environment.

Herbst et al. (2016) maintained that the curriculum creation process in teacher education in an online setting is influenced by the digital nature of technological artifacts. Knowing technology affordance and how to take advantage and leverage tools for building a community of practice is crucial. Moyer et al. (2005) argued that virtual manipulatives create a unique technological representation that is especially dynamic, and children who work with manipulatives will have different mathematical experiences. We support their view and add that it also helps PSTs develop a different experience for their teaching practice. More importantly, with the social presence in mind, instructors need to think about how materials are used to facilitate group dynamics and building.

In these courses, we transfer the sociocultural feature of a face-to-face class into a virtual environment. For example, a common practice in the classroom is students discussing in groups to construct meaning about mathematical concepts or the teaching of mathematics. In the online environment, we made such features take place in breakout rooms with online materials, and the instructor visited the rooms during class time.

In the asynchronous course, the social presence was maintained by facilitating online forums, collaboration tools (e.g., Google Docs), and live sessions. This setting reinforces a sense of community and provides mutual attention and support between the instructor and PSTs as well as among PSTs. It is helpful to distinguish two levels of communities here: the small community within PSTs and the online cohort including the instructor and PSTs learning about teaching practices. By distinguishing the two levels of communities, the instructor can conceptualize the role of the teacher (teaching presence) and how to form social presence differently.

3.4.2 Future Study

We exemplify the design principles and the applicability of the design from an instructor perspective (aiming to describe the presence as a noun). We have not provided data about how PSTs engage in the online environment, that is, how they really experience the presence (as a verb). Anecdotally, we have evidence that designer goals have been met. However, future research can address how the design works; how the effectiveness in two courses are compared, and how they are

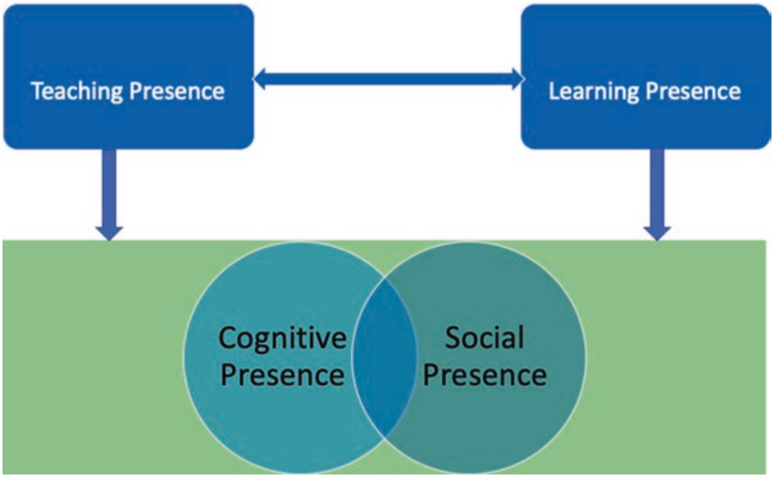

Fig. 3.9 A model for researching online course

compared with a face-to-face setting. Also, in what ways do the forums and virtual manipulatives afford learning opportunities in groups?

As noted, we have not provided data about how PSTs experience the presence designed by the instructors. Future studies could investigate how PSTs experience the two components (cognitive and social presence) in online courses. That is, learning presence needs to be considered, which is arguably an important factor to explain and research in an online environment. Based on this view, we propose a model (Fig. 3.9) to investigate the relationships between the multiple components in an online environment.

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Chapter 4
Online Curriculum Modules for Preparing Teachers to Teach Statistics: Design, Implementation, and Results

Hollylynne S. Lee, Rick Hudson, Stephanie Casey, Gemma Mojica, and Taylor Harrison

Online teacher education and preparing teachers to teach statistics are two areas of growth needed in mathematics teacher education. Undergraduate teacher education programs traditionally include mostly face-to-face courses, and many teacher educators have reported a lack of preparation and readiness to use online modalities in their instruction (Downing and Dyment 2013; Holmes and Prieto-Rodriguez 2018). Recently though, some have used flipped instruction, hybrid courses, and synchronous and asynchronous online courses in mathematics teacher education (e.g., Harrison et al. 2018; Hjalmarson 2015; Starling and Lee 2015), and during the COVID-19 pandemic, most mathematics teacher educators (MTEs) had to quickly convert courses into online or hybrid formats.

While statistics and data science now play a larger role in secondary school curricula, many in-service and preservice teachers are inadequately prepared to teach statistics (e.g., Burrill and Biehler 2011; Lovett and Lee 2017). MTEs, often lacking experience and expertise in *statistics* education and the use of technology for investigating data, need access to teacher education materials focused on preparing

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teachers for teaching statistics (Franklin et al. 2015), along with networking and support in implementing such materials.

In this chapter, we share the evolution and impact of a curriculum project designed to fill a gap in teacher education materials focusing on statistics education and to innovate an approach to assist MTEs in utilizing online instructional methods. We structure the chapter around: (1) design of online modules for statistics teacher education, (2) implementation of modules, and (3) results of research on the effectiveness of the modules.

4.1 Principles to Guide Design of Online Modules

The Enhancing Statistics Teacher Education through E-Modules (ESTEEM) project, funded by the National Science Foundation (DUE 1625713), began in 2016 to develop online modules designed to support prospective secondary mathematics teachers to learn to teach statistics. Mathematics teacher preparation programs vary widely, and statistical content and pedagogy may be introduced in a number of different courses, such as a general mathematics methods course, a course that focuses on teaching and learning statistics, a statistics content course, or courses focused on technology for teaching mathematics. Course modalities also vary greatly across programs, and there is an increased need for resources that support online learning as a result of the COVID-19 pandemic. Consequently, MTEs desire curriculum materials that can easily be embedded into diverse programs and courses and that are easily adaptable to meet the needs of their students. Learning management systems (LMSs) are commonly used by institutions and provide accessibility for multimedia materials such as readings, videos, images, slide decks, and interactivity options such as quizzes with feedback, discussion boards and synchronous communications such as video-based learning environments (e.g., Blackboard Collaborate, ZOOM) and live chat (McBrien et al. 2009; Park 2015). Three fundamental design principles used in ESTEEM are to provide MTEs with online instructional materials that are:

1. Modular and adaptable
2. Easily accessible and integrated into LMSs
3. Interactive in asynchronous or synchronous modalities

The ESTEEM materials' modular approach and its creation of e-modules for import into LMSs meet MTEs' expectations of adaptability and accessibility and can promote interactivity online. As the ESTEEM modules were designed, we recognized that MTEs needed to easily access and share the modules with their students. The ESTEEM modules are designed within the three most commonly used LMSs in the United States - Moodle, Blackboard, and Canvas- as well as a common cartridge package that can be imported into other LMSs. A common cartridge is essentially a format for exchanging content between LMSs, so these systems have a way to interpret the digital learning content and how it is organized (see http://www.imsglobal.

org/activity/common-cartridge). The ESTEEM modules, packaged for portability, can be downloaded from a web portal upon registration (https://go.ncsu.edu/esteem). Sharing modules through LMSs gives greater autonomy to MTEs by allowing them to add, modify, or delete content to meet the needs of their local learning contexts. Our materials are shared under a *Creative Commons Attribution Non-commercial Share-alike* license. We believe that having the flexibility to modify content was important for teacher education since statistics content and pedagogy are commonly addressed in different courses within secondary teacher preparation programs.

Several additional design principles based on research from statistics education, mathematics teacher education, and online teaching and learning guided the development of the modules (Hudson et al. 2018). We highlight each of these below.

4.1.1 Use of a Free Web-Based Tool for Learning with Data

We intentionally chose to use the Common Online Data Analysis Platform (CODAP, https://codap.concord.org) as the primary data analysis tool utilized in the ESTEEM materials because it promotes exploratory data analysis, is based on research regarding how students learn with data, and is free and accessible (Finzer and Damelin 2018; Mojica et al. 2019). Ease of use, no cost, and accessibility via a web browser are also of paramount importance for use in K-12 settings. Mathematics teachers may use tools they learned about during their teacher preparation but strongly prefer tools that are free and accessible (McCulloch et al. 2018). CODAP also has strong visualization capabilities and allows users to dynamically link multiple representations and explore relationships among variables.

4.1.2 Teachers Learn by Doing Data Investigations

Mathematics teacher preparation often emphasizes the importance of developing specialized knowledge for teaching that includes a deeper understanding of mathematics and statistics content (e.g., Groth 2013; Hill et al. 2008). This specialized knowledge can be developed through teachers' engagement with mathematics and statistics tasks as learners themselves. Through such experiences, particularly with technology, teachers have opportunities to revisit concepts they had opportunities to learn in prior experiences in K-12 or college and deepen their understandings in ways that can build specialized knowledge useful for teaching (e.g., Lee and Hollebrands 2011; Wilson et al. 2011). Lee et al. (2014) and Pulis and Lee (2015) have shown that teachers' use of dynamic statistics technology tools to investigate multivariate data enhances their statistical problem-solving skills. We follow recent suggestions (Franklin et al. 2015; Gould and Cetinkaya–Rundel 2014; Hayden 2015) for teachers to have multiple opportunities to investigate data sets that are large, multivariate, and from real sources. Figure 4.1 illustrates CODAP's visualization and linking capabilities using data about roller coasters.

4.1.3 Use Frameworks Common in Mathematics Teacher Education

Since the ESTEEM materials prepare *mathematics* teachers to teach statistics, we felt it was important to incorporate frameworks commonly used in mathematics teacher education. One such framework is the professional noticing of children's mathematical thinking (Jacobs et al. 2010). Professional noticing consists of three related skills: (a) attending to a student's strategies as they reason mathematically or statistically, (b) interpreting the student's strategies, and (c) deciding how to respond to the student based on the student's understanding. The ESTEEM materials aim to develop in teachers all three skills of professional noticing for the specific content area of statistics—an area that has not received much attention in the work on professional noticing in mathematics teacher education to this point.

Another framework we drew upon when developing the ESTEEM materials is the Five Practices model for productive classroom discourse centered around engagement in meaningful tasks (Smith and Stein 2011). The five practices emphasized in this model are *anticipating* students' responses to a task, *monitoring* students' responses to a task, *selecting* specific students to present mathematical ideas, *sequencing* students' responses that will be publicly displayed, and *connecting* between student responses as well as to key ideas. The groundwork for

Fig. 4.1 Exploring multivariate data about roller coasters using CODAP

implementing this model also involves setting instructional goals and selecting an appropriate task. The ESTEEM materials attend to all of these aspects of using statistical investigation tasks with additional attention given to characteristics of an effective statistical task launch.

4.1.4 Incorporate Representations of Practice

Another design principle of the ESTEEM materials was to incorporate representations of the practice of teaching and learning statistics throughout our materials. These allow teachers to have a shared common experience of viewing statistical instruction, critically analyze teachers' practice (e.g., Seago and Mumme 2002; Sowder 2007), examine students' use of technology to support data investigations (Wilson et al. 2011), and develop professional noticing skills. For example, we include videos of secondary classrooms and individual student talk-alouds where teaching and learning of statistics are occurring. When real classroom video is not available, new capabilities in the creation of animated videos, such as complete stop-motion animations, can create engaging learning opportunities for teachers (e.g., Herbst and Chazan 2020; Herbst and Kosko 2014; Laaser and Toloza 2017). Thus, we also captured and used videos from real classrooms to produce several animated videos depicting teachers and students engaged in statistics tasks.

4.1.5 Promote Learning through Multiple Perspectives

Our team of authors bring a wealth of experience in teaching and research in statistics education to inform the design of our materials. We ensure that the brief readings and videos represent syntheses of literature and guidelines from professional organizations (e.g., American Statistical Association, National Council of Teachers of Mathematics) from a variety of perspectives and that all works are cited. The use of representations of practice that include teachers and students bring those perspectives and voices into learning opportunities for teachers using ESTEEM materials. We also take advantage of the multimedia format of our materials to offer video-based conversations with teachers and experts in panel-like discussions where teachers can listen to and consider the perspectives of those with experience in teaching statistics and designing curricula and software tools. In a study examining shifts in teachers' perspectives and practices in teaching statistics, Lee et al. (2020) identified expert panel videos as a critical learning experience for reflection and change about one's own perspectives on teaching statistics. Lastly, the ESTEEM modules are designed to include repeated opportunities for teachers to learn from each others' perspectives and participate within a student-driven virtual learning environment, including interacting with one another through online discussions (Park 2015; Revere and Kovach 2011).

4.2 Online Modules for Preparing to Teach Statistics

4.2.1 Overview of Learning Goals and Opportunities

The ESTEEM materials consist of three interconnected modules and two independent assignments, as shown in Fig. 4.2. Collectively, the modules aim to develop teachers' critical understanding about the differences between mathematics and statistics and key statistical concepts, abilities to use CODAP to investigate real-world phenomena with bigger data, professional noticing of students' thinking about statistics, abilities to make and evaluate data-based claims and arguments themselves and how to navigate data-based discourse in classrooms, and competencies in planning for and leading students in data investigations.

The *Foundation in Statistics Teaching Module* (Module 1) emphasizes the differences between mathematics and statistics as well as how to support students in learning to reason statistically. It includes activities concerning launching statistical tasks, the statistical investigation cycle, and fostering discussions around students' statistical thinking utilizing the Five Practices model (Smith and Stein 2011). Teachers are also introduced to CODAP in this module, using it themselves to analyze data concerning roller coasters as well as analyzing videos of classroom lessons where students are using CODAP. Module 1 provides the minimal learning experiences that teachers need to develop key understandings and strategies for supporting students' engagement in statistical investigations. This is a purposeful design of the ESTEEM materials so that if MTEs only have about 2–3 weeks in a course to devote to preparing teachers to teach statistics, this module will provide a minimal foundation. Module 1 is also a prerequisite to the other two modules, the *Teaching Inferential Reasoning* Module (Module A) and the *Teaching Statistical Association* Module (Module B).

Module A focuses on how questions, modeling processes, simulation tools, and tasks can support students to reason inferentially. Teachers use CODAP's Sampler

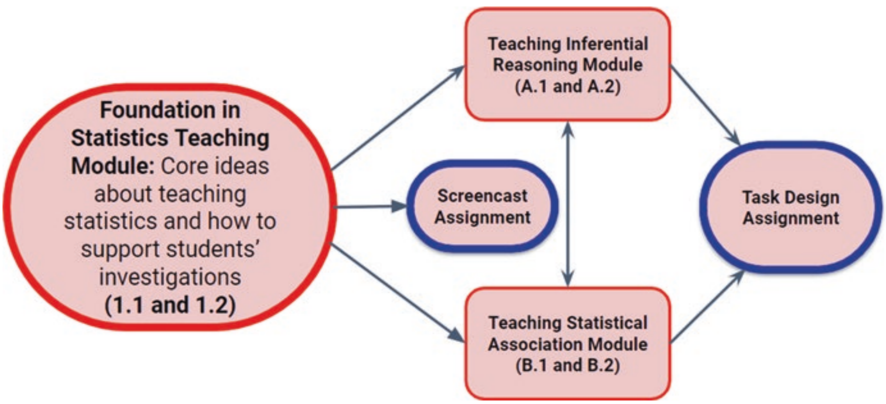


Fig. 4.2 Structure of ESTEEM online modules and assignments

plugin to investigate inferential questions, make claims based on simulated data, and develop an understanding of sampling distributions. Module B focuses on preparing teachers to teach statistical association, including both quantitative and categorical association. The module shares typical conceptions students have when considering whether variables are associated and develops teachers' professional noticing skills regarding students' thinking about the association. Teachers extend their use of CODAP by learning to add attributes to a data set, create two-way binned plots and segmented bar graphs, model data with a least-squares line, and create and analyze residual plots.

4.2.2 Organization of Modules

Each module has a common organizational structure. A module is split into two parts, and each part is expected to correspond to about 5–8 h of learning opportunities for teachers. Each activity in the ESTEEM materials is assigned a three-digit code which designates the module, part, and activity. For example, activity A.1.b signifies that the activity is in Module A, Part 1, Activity b. Although activities are organized sequentially within each part, teacher educators have the flexibility to reorganize and rename activities within their LMS for their course. A table of contents of all ESTEEM activities is provided in Appendix 1.

Each part of a module is divided into three sections: Read and Watch, Engage with Data, and Synthesize and Apply. The Read and Watch materials are further separated into two types of content: Essential Materials and Learn from Practice. Read and Watch: Essential Materials include readings and videos that introduce fundamental concepts about the teaching and learning of statistics (e.g., brief reading about differences between mathematics and statistics, reading and video introducing key aspects of inferential reasoning). The Read and Watch: Learn from Practice materials consist of documentation of the teaching and learning of statistics, such as videos of secondary classrooms engaging in statistical investigations (example in Fig. 4.3), an animation video that demonstrates common student approaches to placing an informal line of best fit (as shown in Fig. 4.4), videos of students creating graphs to display bivariate categorical data, and videos of educators discussing tasks they use to teach inferential reasoning.

The second section of each module part involves Engage with Data activities, which are statistics investigations teachers complete using CODAP. These activities include multimedia components (videos, images) to contextualize the investigation, a CODAP file which includes the data to be analyzed, and handouts in PDF and Word versions that can be downloaded and used directly by teachers. Teachers investigate a variety of statistical questions in these activities, including questions about roller coasters, healthiness of granola bars, carbon emissions (see Appendix 2), and fuel efficiency of vehicles, and engage in model-building and simulating outcomes from contexts such as dice and predicting soccer wins (see Fig. 4.5).

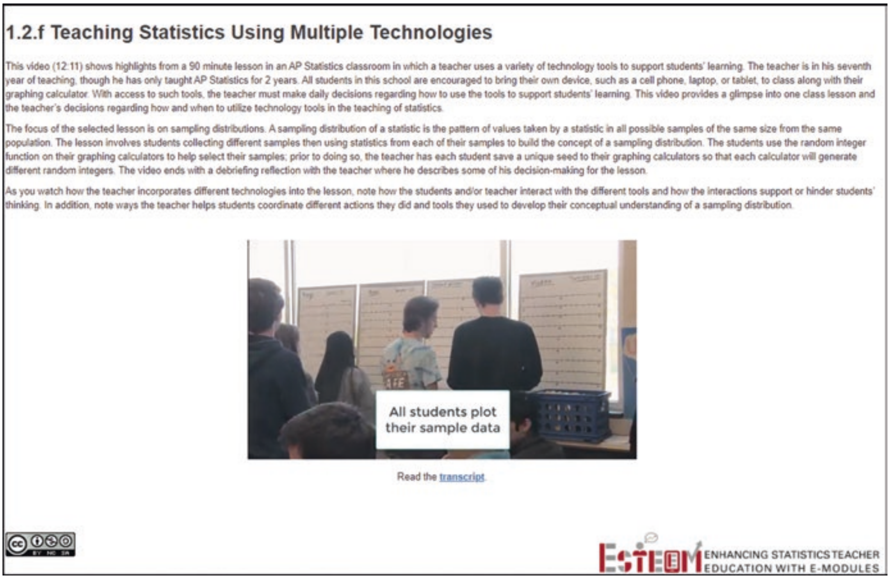


Fig. 4.3 Video lesson on sampling distributions involving multiple technologies (Activity 1.2.f)

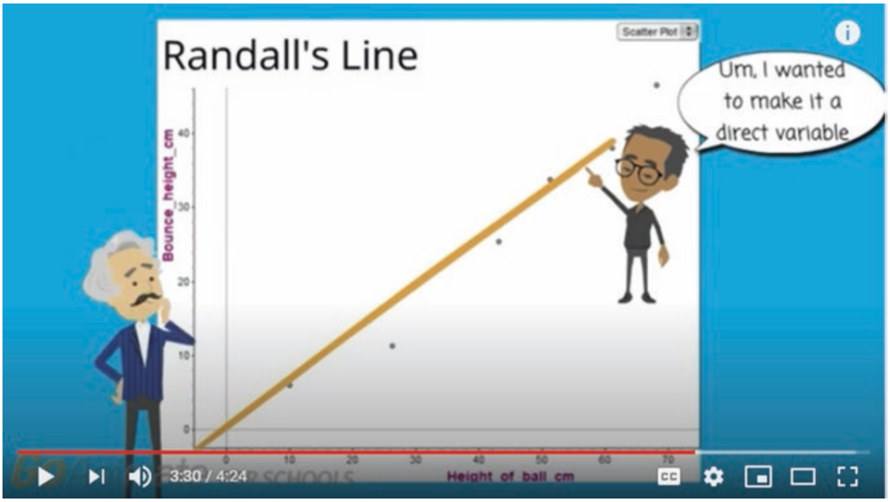


Fig. 4.4 Animation video of a class investigating the placement of an informal line of best fit (Activity B.2.e)

Some Engage with Data activities also focus on pedagogical and technological aspects of data investigations.

Synthesize and Apply activities are the final section of each part of a module. These activities ask teachers to reflect on and apply what they have learned to the

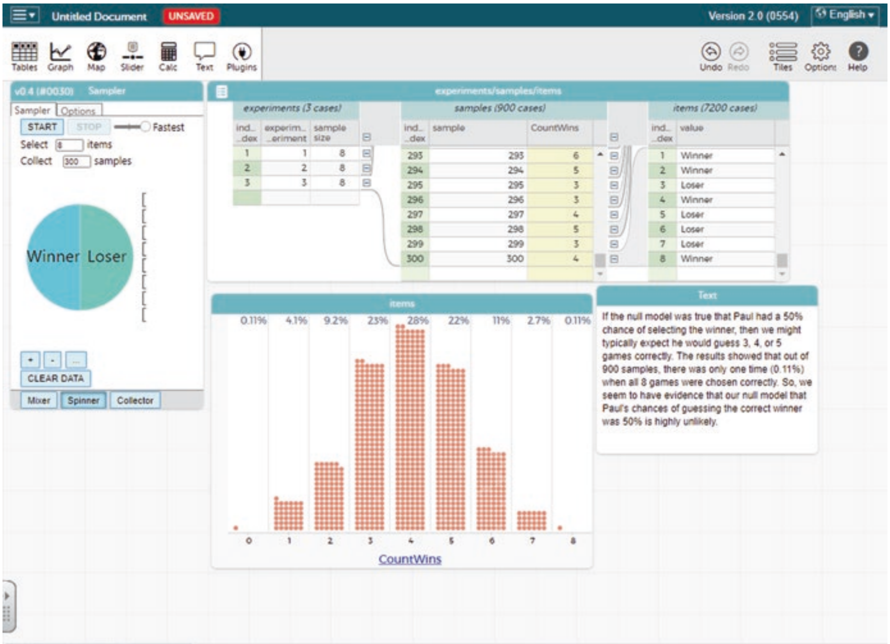


Fig. 4.5 Sample model and simulated data in CODAP (Activity A.2.e)

practice of teaching. Often, these activities explicitly use the aforementioned frameworks commonly used in mathematics teacher education of professional noticing (Jacobs et al. 2010) and the Five Practices model for productive classroom discourse (Smith and Stein 2011). For example, in Activity 1.2.i., teachers read a summary of the Five Practices model and then apply their understanding of the model to analyze a video of a teacher orchestrating a whole class discussion using students' work in CODAP (see Fig. 4.6). Other Synthesize and Apply activities expect teachers to participate in a discussion board to analyze and discuss characteristics of different statistical tasks, design the launch for a statistical investigation, and craft responses to students whose thinking has been shared in videos, all of which align with the noted frameworks.

4.2.3 Assignments to Demonstrate Competency

In addition to the three modules, the ESTEEM materials also include two assignments for teachers to demonstrate their competency in engaging in statistics investigations and planning to teach statistics: the Screencast Assignment and the Task Design Assignment. The Screencast Assignment may be used at any point after the completion of Module 1. In this assignment, teachers record the actions on their

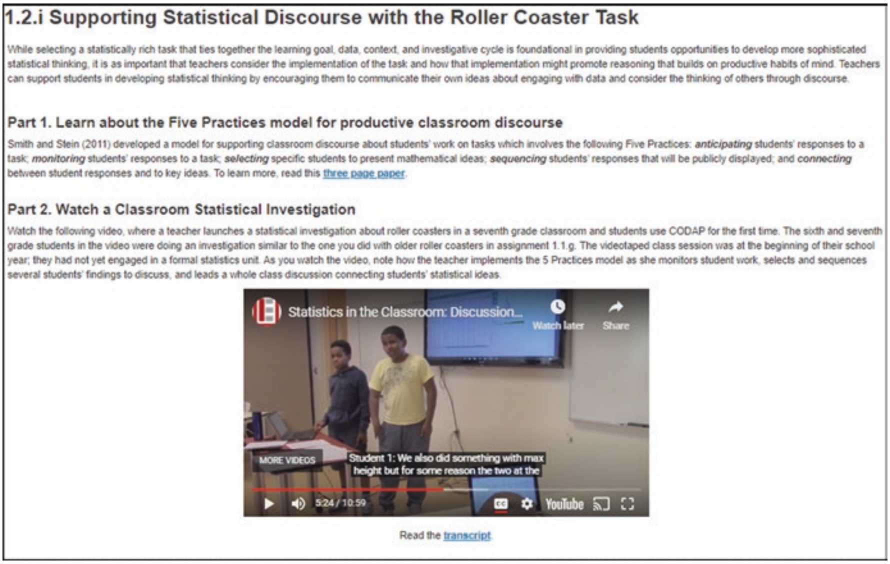


Fig. 4.6 Focus on orchestrating statistical discourse using students' work in CODAP (Activity 1.2.i)

computer screen and talk aloud as they complete a new data investigation in CODAP. These screencasts reveal how the teachers use CODAP and provide an opportunity to communicate their statistical thinking. The Task Design Assignment is intended to be assigned after completing Module 1 *and* at least one of the other two modules. In this assignment, the teachers design a CODAP-based statistical task and create a plan for implementing the task. The assignment consists of six parts: (1) background information including alignment to standards, student learning objectives, and a link to a CODAP file; (2) a plan for how the task will be launched with students; (3) the task as it will be posed to students; (4) anticipated student responses to the task; (5) a description of the intended implementation, including how the teacher intends to scaffold students' thinking and use students' work to discuss the task; and (6) a reflection where the teachers explain the choices they made in developing the task and identify what they learned in developing the task.

4.3 Implementation of Modules

We actively recruited mathematics and statistics teacher educators to participate in professional development workshops and subsequently implement ESTEEM material in their university course(s). Forty-five MTEs participated in at least one workshop during 2018 or 2019. Between the spring of 2018 and summer of 2020, 30 of these MTEs participated in a study in which data was collected. The MTEs taught at 27 institutions across the United States that implemented ESTEEM materials in a variety of undergraduate and graduate courses (48 courses total). These courses included 804 enrolled students, most of whom were preservice mathematics teachers and some in-service teachers or general statistics students. For simplicity in reporting, we refer to students enrolled in these courses as teachers and the postsecondary educators as MTEs. Most MTEs received professional development through a one-day workshop, and some participated in additional online webinars. Our research examines implementation across settings and ways materials impacted MTEs' and teachers' learning about teaching statistics.

4.3.1 Data Collection During Implementation

In 31 courses, 298 teachers took a self-efficacy survey that measured their before and after confidence levels for teaching statistics (Harrell-Williams et al. 2019), and teachers and MTEs completed a post-implementation survey about their learning experiences. The self-efficacy to teach statistics survey was given in a retrospective format so that teachers were only asked to engage in the survey once. The retrospective version of the survey has been shown to have a similar structure, validity, and response trends as the version of the survey given in pre-post format (Harrell-Williams et al. 2020). The post-implementation surveys for MTEs and teachers included Likert scale ratings about impressions of materials, open-ended feedback about most impactful learning experiences, and suggestions for improvement. In addition, MTEs indicated in a post-implementation survey which module activities they implemented and whether teachers in their course engaged in an activity in a face-to-face, online synchronous, or online asynchronous setting. Other data sources included interviews with MTEs ($n = 25$) and a sample of volunteer teachers ($n = 16$), statistics tasks designed by teachers, and other assignments.

4.3.2 Use of Modules and Activities

Of the 48 courses in which MTEs used any part of the ESTEEM modules, we have post-implementation survey results from 46 courses. Even though ESTEEM materials are designed for secondary preservice mathematics teachers, 37% of courses ($n = 17$) included elementary preservice teachers, and 7% of courses ($n = 3$) were undergraduate statistics courses serving different disciplines. LMSs used in courses included Blackboard (39%), Canvas (26%), and Moodle (17%). The move to remote instruction experienced by many universities in the spring and summer of 2020 increased the number of MTEs who used our materials in either a hybrid (35% of courses) or completely online setting (20% of courses), though these modes of instruction still occurred less often than courses taught in face-to-face settings (45% of courses).

When planning for a course, MTEs indicated which modules they intended to use or partially use in 46 different courses. Once MTEs selected which module(s) were appropriate for a course, 65% of the courses implemented 80% or more of the activities in those modules. Only 7% of the courses implemented less than 40% of the activities within their selected modules, and these courses tended to be general statistics courses or content courses for elementary preservice teachers. Both parts of the *Foundation in Statistics Teaching Module* (Module 1) were available to all MTEs (n = 46 courses). Parts 1 of the *Teaching Inferential Reasoning* Module (Module A) and the *Teaching Statistical Association* Module (Module B) were available starting in spring of 2019, with the potential to be used in 30 of the 46 courses. Part 2 of Module A and Module B were available starting in the fall of 2019, with only 17 courses having the potential to implement.

Regardless of the mode of instruction for the course, MTEs may have implemented activities in a face-to-face setting, a synchronous online setting (e.g., video conferencing tools such as Zoom or Google Meet), or an asynchronous online setting (e.g., as an assignment posted in their LMSs to be done without a MTE's interaction). Figure 4.7 shows the number of courses in which each activity across the three modules was used and the modality in which they were used: 46 courses used activities from *Foundations in Statistics Teaching*, 17 courses used *Teaching*

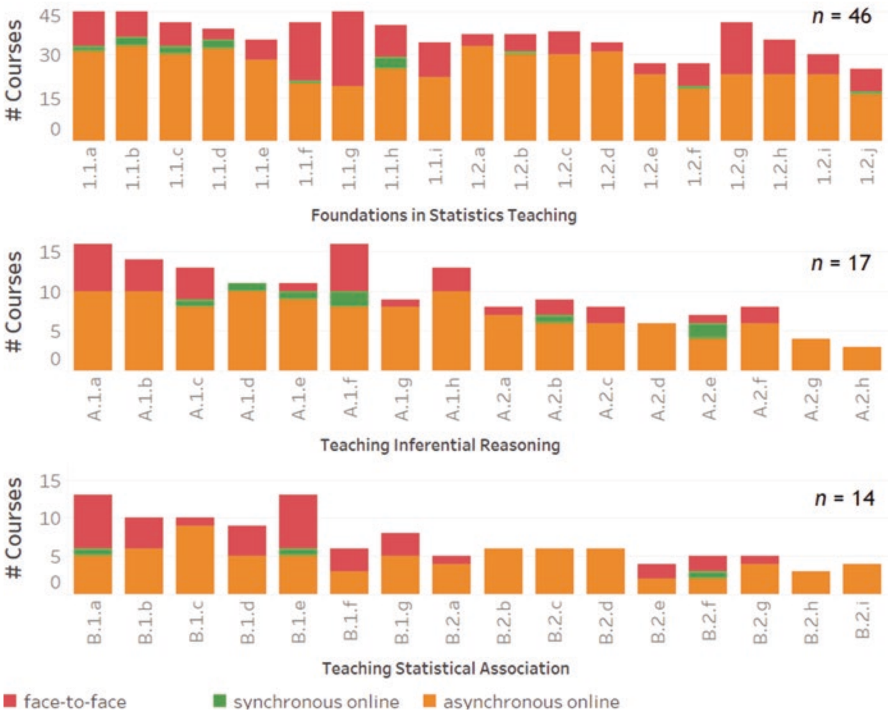


Fig. 4.7 Use and modality of ESTEEM activities implemented from fall of 2018 to summer of 2020

Inferential Reasoning activities, and 14 used activities from *Teaching Statistical Association*. The activity numbers on the horizontal axes in Fig. 4.7 correspond with the activities described in Appendix 1.

Data investigations using CODAP were highly used in courses (1.1.g, 1.2.g, A.1.f, A.2.e, A.2.f, B.1.e, B.2.f) and often used in face-to-face settings (red in Fig. 4.7) or online synchronous sessions (green). Many MTEs also chose to engage their classes with the videos in 1.1.f and B.1.a during in-person class sessions—videos that were brief and showed students and teachers discussing critical concepts in statistics (investigative cycles and categorical association). However, many activities that included longer videos of expert teacher discussions or videos showing students and teachers engaged in statistics were often used asynchronously (orange bars in Fig. 4.7) through independent work (e.g., 1.1.e, 1.2.d, A.1.d, B.1.c, B.2.d). Even when MTEs were using other activities from a module, some of the discussion board activities were used much less often than other types of activities (e.g., 1.1.i, 1.2.j, A.1.g, A.2.g, B.1.f, B.2.h). However, it is also clear that a few MTEs used the activities from a discussion board to structure an in-class experience or discussion with their teachers, as indicated by the small red bars for those activities.

4.4 Effectiveness of Modules

We used several data sources from ESTEEM material implementations to examine the effectiveness of the materials, from the perspective of MTEs and teachers in their courses.

4.4.1 Feedback from MTEs and Teachers

We collected post-implementation feedback from almost all MTEs in the 48 courses (96% of MTEs, $n = 46$) but had a much lower response rate from teachers (41% of the 804 enrolled, $n = 329$). After examining MTEs and teachers' responses to these post-implementation surveys, ESTEEM materials were found to be effective in several ways. MTEs and teachers had overall positive impressions of materials, where 89% of MTEs and 87% of teachers strongly agreed or agreed that the materials were easy to access and navigate online. In post-implementation MTEs' interviews ($n = 25$), many MTEs commented that it was "very easy" to use ESTEEM materials in their own LMSs. Eighty-five percent of MTEs and 74% of teachers strongly agreed or agreed they felt more prepared to teach statistics after using ESTEEM materials. A majority of MTEs and teachers strongly agreed or agreed that the materials were effective in developing their statistical knowledge and knowledge about teaching statistics, particularly in relation to using CODAP to engage in statistical investigations and to analyze multivariate data (see Table 4.2).

Table 4.2 Ways ESTEEM materials were useful, according to MTEs and teachers

Ways ESTEEM materials were useful	% who strongly agreed or agreed MTEs $n = 46$	 Teachers $n = 326$
Understanding statistical concepts	85	75
Developing my ideas about teaching statistics	85	72
Learning about teaching strategies and approaches for statistics topics	85	76
Using technology to engage in statistical investigations	96	80
CODAP was useful in supporting my analysis of multivariate data	92	82
Illustrating ways in which students may approach statistical tasks	85	78

Both MTEs and teachers overwhelmingly found ESTEEM-designed CODAP materials to be useful resources that they would implement in their own teaching (91% and 82% strongly agreed or agreed, respectively). They indicated that videos showing students and teachers working with statistics tasks and online statistics tools were useful in understanding classroom issues related to teaching and learning statistics (83% and 76% strongly agreed or agreed, respectively), as well as videos of discussion with teachers and statistics education experts (75% and 64% strongly agreed or agreed, respectively). From open-ended comments and interviews, we also learned that teachers highly valued the opportunity to observe authentic learning through multiple video resources (i.e., real classroom videos, animated videos, and teacher reflection videos). Videos showing students and teachers working on statistics tasks were particularly helpful in illustrating how to implement the ESTEEM data activities in classrooms, as well as teachers' pedagogical strategies for exposing students to less familiar contexts in a data investigation (e.g., noticing characteristics of roller coasters while experiencing a point of view video of a roller coaster ride).

We next share results related to specific themes regarding MTEs' and teachers' impressions of the effectiveness of the ESTEEM materials based on responses to open-ended questions in the post-implementation survey and interviews. For MTEs, we share both positive and negative feedback. We briefly describe revisions to the materials based on negative feedback.

MTEs' Impressions of Materials

Overall, MTEs felt ESTEEM materials were of high quality and compared very favorably to other teacher education materials. They indicated that the **materials provide a much-needed emphasis on the key ideas in statistics** and they fills "a dire need" for material with a focus on teaching statistical practices. For many, ESTEEM materials were a missing puzzle piece that supported them in filling existing gaps in their curriculum. One MTE stated,

> I felt like in the past, ... I was really struggling to find resources that would give my teachers experience not just doing the statistics their future students would do, but to think about teaching the statistics after they had done the activity or seen a piece of the activity and to step back and think, "What are the teaching principles behind this?" These materials really helped fill in those gaps for me as far as providing a tool-kit for me and teaching the course.

Others found the materials to be **cohesive, well-thought-out, organized, and well-packaged, enabling smooth integration into existing courses**. Most MTEs reported little difficulty in integrating modules into their LMSs. One MTE elaborated in the following way:

> I could access everything right from that Moodle course. So, that was so nice not to have separate files for all the documents I gave them. Or just all the links could take me to anywhere where I needed to go in the materials themselves. Super helpful. Super timesaving.

While a small number of Blackboard users experienced some technical difficulties, almost all other MTEs expressed positive experiences in using ESTEEM materials in their LMSs.

MTEs also reported that ESTEEM materials **contributed to their own professional growth**. Although many MTEs felt they had a strong understanding of statistical content knowledge, many expressed in interviews that they had not previously felt prepared to teach preservice teachers how to teach statistics. MTEs believed ESTEEM resources better equipped them to prepare teachers to teach statistics.

While MTEs overall reported that the ESTEEM materials were of high quality and useful, some reported challenges in using the materials. Five themes emerged around the following: time constraints, teachers' knowledge, discussion forums, videos, and technology issues. One of the greatest challenges in MTEs implementing the materials was that they felt constrained by time. Some felt that there was too much material to get through in their already packed semesters. Other MTEs indicated that their teachers lacked sufficient statistical content knowledge to engage with the materials and they needed more support in helping teachers develop this knowledge, sometimes indicating that teachers did not get as much from specific resources (e.g., videos) because teachers' content knowledge was not sophisticated enough. An MTE also indicated that teachers' pedagogical knowledge was sometimes limited because they did not have enough field experiences or professional maturity to appreciate some resources (e.g., videos).

Some MTEs reported discussion forums did not facilitate meaningful discourse in their courses. As a result, the ESTEEM team reviewed discussion forum prompts, particularly those which featured representations of practice (e.g., classroom video, video of a student working with technology) and modified prompts to focus the discussion. When able, we added background information to provide more context.

While most MTEs had positive impressions of the video resources in the materials, some indicated that it was sometimes difficult for teachers to take away the message of the video. Other MTEs indicated some videos were too long. Yet others pointed to the vast amount of video resources, indicating there were too many. Rewording discussion prompts around videos as described was also used to address this issue. Additionally, we enhanced longer videos by using PlayPosit, an online tool that allows an instructor to embed questions and interactions in videos to provide a more interactive experience for the learner, along with feedback. This also focused on important ideas from the videos and made them more explicit for teachers.

A small number of MTEs reported technical issues; however, this did not interfere with their use of the materials. Issues typically related to importing materials into MTEs' specific LMSs, like being unable to import without IT assistance, extra assignments being imported that could not be deleted, or issues with their grade books. Others found it cumbersome to select specific materials to import when they did not want all the materials. Almost all issues were reported by Blackboard users. Some MTEs were not able to easily change the format of assignments (e.g., PDF to Word or Google doc). Issues were usually quickly resolved with assistance from a member of our team or IT at their institution. The ESTEEM team updated written instructions for importing materials specific to MTEs' LMSs for Blackboard, Canvas, and Moodle and also created how-to videos for importing materials into an existing course.

MTEs also made a couple of recommendations for implementing the materials. They advocated for the creation of an implementation guide and establishing a community of implementers. While creating a full implementation guide was beyond the scope of the project, the ESTEEM team engaged in several activities to address these needs. The team designed and implemented five free webinars to date to support implementers of the ESTEEM materials. Three were held specifically for MTEs who had already implemented materials, where we focused on needs they identified from a survey. In these webinars, a few implementing MTEs participated and shared their own experiences, providing advice for implementation. Additionally, we included MTEs who had field-tested our materials in a professional development workshop to support new MTEs to offer implementation advice. We also provided implementation advice for specific contexts (i.e., face-to-face, online, hybrid) during this workshop. The workshops and webinars were spaces for establishing and building a community of users.

Teachers' Impressions of Materials

Based on teachers' post-implementation surveys, the vast majority strongly agreed or agreed that all materials were useful for their learning about various aspects of statistics teaching (76%) and increasing their own content knowledge (75%). In responses to open-ended questions from the post-implementation survey and interviews, five themes emerged in relation to what teachers learned from the ESTEEM materials, including their experience exploring data with CODAP. In general, teachers believed that the ESTEEM materials developed their understanding of statistics and provided them with valuable resources for teaching, particularly how to engage students in statistical tasks **using technology as a tool to teach statistics**.

Teachers also noted the importance of **engaging students in the full statistical investigation cycle** and in data collection. They appreciated the power of having their future students explore data through CODAP and overwhelmingly identified CODAP as the resource that had the biggest impact on their thinking, where they identified their role as asking good questions to students and facilitating while students themselves engage in investigations. A preservice teacher expressed the importance of engaging students in the investigative cycle:

> Students are involved when they feel they are engaged in the process. They are willing to look for data, analyze it, and find the results of data if they can relate to it. You want to have a statistics lesson that allows students to participate in all steps of the process.

Teachers also indicated they learned it is **critical to make statistics relevant to students by using real data** and about the power of statistics in solving real-world problems, and some emphasized the **importance of context.** One teacher from 2019 discussed:

> The biggest lesson learned from these materials that will inform my teaching of statistics is to have the students work with real-world data, preferably data that they retrieved themselves. Have them pose a question they are interested in finding an answer to, then have them collect data and analyze it using technology, and finally, interpret what the data means and make a connection to the original question.

Many teachers also noted gaining a deeper **understanding of students' thinking about statistics,** as one described:

> I learned that students, no matter their age, can understand statistical concepts. Statistics is much more than I thought it was before, and I could see that through viewing the videos of students interacting. Students can look at things, make assumptions, and get interested in learning more about the data before them. They can investigate on their own and find out more information without having to [be] walked through the process.

Many teachers reflected on the value of the **Students' Approaches to Statistical Investigations framework**, which explains how students' statistical sophistication develops across three levels in relation to the investigation cycle (first introduced in Activity 1.2.b), as a resource for better understanding how students develop their statistical thinking. One teacher discussed the value of the framework in the following way:

> I really liked the handout we got that broke down the different levels of thinking students can have with the different aspects of statistics. Posing questions, collecting data, analyzing data, and making conclusions. I learned that students, independent of the grade level, can all be at different levels of statistical thinking, and it is important to always start at their level and build them up from there.

And for some teachers, they learned **how statistical thinking differs from mathematical thinking** and about common student misconceptions, as one MTE stated,

> We had some very good discussions about how math and [statistics] are related [and] the differences between math and [statistics]. I don't think that's something a lot of them had considered closely before. So the discussion that we had related to our reading; that was really critical to fostering their thinking about how those are different and how they should be taught differently.

4.4.2 Impact on Teachers' Self-Efficacy to Teach Statistics

Teachers' self-efficacy to teach statistics increased following the use of the ESTEEM materials. In interviews, many teachers described themselves as "math" people rather than "statistics" people before engaging with the ESTEEM materials. The ESTEEM materials provided a safe context to learn and explore, and teachers felt they experienced **increased confidence with statistics** as a result of their engagement with the ESTEEM materials, helping them learn to both appreciate and apply statistics. Other teachers expressed more comfort in conducting statistical investigations with large data sets as a result of ESTEEM's Engage with Data experiences.

Results from the self-efficacy to teach statistics (SETS) survey also indicated that teachers' confidence to teach statistics significantly increased after engaging with the ESTEEM materials. The SETS survey asks teachers to rate their confidence to teach students 44 different, specific, statistical topics, ranging from less sophisticated topics to more sophisticated topics. Responses on the survey were on a scale from 1 (not at all confident) to 6 (completely confident). Collectively, 298 preservice and in-service teachers in 31 courses using the ESTEEM materials completed the retrospective SETS survey, typically within 1–8 weeks after completing the ESTEEM materials in the course. Across all topics, teachers' mean confidence increased by 1.22 points, from a mean of 2.85 (before) to 4.07 (after), which represents a large gain in confidence to teach statistics.

4.4.3 Designing Tasks with Data Investigations

Courses targeted at preparing teachers often implemented the Task Design Assignment. We examined products of this assignment, including 73 CODAP-enhanced data investigation tasks and accompanying task "launches" (Casey et al. 2020a, b; Hudson et al. 2020). In the task launches, the majority of teachers oriented students to the context of the data (68%) and prompted students to make a personal connection with the data's context (53%). Few (25%) of the task launches, however, motivated a need for a driving statistical question that would guide engagement in the task. Analysis of the tasks themselves showed that most tasks incorporated three key aspects for teaching statistics espoused in ESTEEM materials: analysis of large, multivariate, real data sets; connection to the data's context throughout the task, including through task presentation and prompting of students to connect their work to the data's context; and engagement in multiple phases of the statistical investigation cycle. The tasks in general often lacked a driving statistical question, which resulted in tasks that consisted of a series of disconnected prompts without a clear statistical purpose. In addition, many tasks did not honor differences between mathematics and statistics, asking students to prove things based on their analysis or to make predictions that are "exact solutions" rather than estimates with a margin of error.

4.5 Discussion and Implications

Online modes of instruction have become increasingly available in mathematics teacher preparation programs. Graduate programs designed for initial licensure in secondary mathematics and for supporting in-service teachers in developing their mathematics teaching practices were some of the first to utilize synchronous and asynchronous course modalities. Our project was primarily targeting undergraduate mathematics teacher education courses. We were innovating solutions for two gaps in undergraduate secondary mathematics teacher preparation: use of online modalities for instruction and access to high-quality statistics teacher preparation materials. We discuss successes, struggles, and implications for each of these innovations.

4.5.1 Flexibility in Online Instructional Modules

It is clear that MTEs found great value in the ability to easily import ESTEEM materials into their courses and integrate them with other materials in their LMSs. MTEs made different choices about which activities to use in their classes that matched the needs of their students and the goals of their course or program. In accord with findings from Holmes and Prieto-Rodriguez (2018), ESTEEM-using MTEs appreciated the flexibility to rearrange materials within their LMSs. They appreciated that they did not have to go to different sites themselves to find codes to embed videos, upload PDFs, or find CODAP documents, and their teachers did not have to leave their course LMSs to engage with materials, except when opening CODAP in a new browser tab.

Although the ESTEEM modules were designed to provide accessibility and interactivity in online asynchronous settings (Holmes and Prieto-Rodriguez 2018; Park 2015), MTEs found ways to modify the materials into face-to-face or online synchronous activities and also used many online activities as asynchronous homework that students accessed through the LMSs outside of the in-person setting. Since March 2020 and an abrupt shift to online instruction at many institutions, we have seen a greater interest in our materials and use of them in online or hybrid settings. Very few courses which used ESTEEM were completely online (20%), and our data indicate that very few activities were used in synchronous sessions online (small green bars in Fig. 4.7). In the next phase of our project, we will be expanding support for online instruction and are curious to investigate how MTEs engage teachers in completely online courses, synchronously or asynchronously. For example, we know that various features of video-based synchronous learning environments are highly effective in promoting student engagement: polling, emoticons, hand raising, chats, shared whiteboards, and screen sharing (McBrien et al. 2009; Starling and Lee 2015). We have not yet collected data on how features of synchronous learning environments may impact MTEs and teachers' learning experiences with the ESTEEM materials.

Publishers of college textbooks often give MTEs ways to create links between an instructor's course site in LMSs and an external site housed with the publisher containing electronic material (applets, videos, readings, data sets) and assessments to support instruction with an adopted textbook. Accessibility of material in an organized, structured manner within LMSs is highly valued by future teachers and MTEs (Holmes and Prieto-Rodriguez 2018). When materials are distributed in LMSs in a way that MTEs then have direct control to reorder, rename, modify, or supplement, it can empower them to actually learn more about how features of LMSs and the organization of materials could support their work as teacher educators. We encourage more teacher education curriculum projects to consider the development of online materials with Creative Commons licensing and distribution in a ready-to-use format that can be easily integrated and modified by MTEs in LMSs to fit the unique needs of a course.

4.5.2 Constraints in Using LMSs Distribution

Our project aimed to use the common cartridge standard for working with digital learning content. Our approach to packaging materials in a way that can be used across different LMSs resulted in a necessary constraint in the types of elements (or activities) that could be used. We were limited to using only those elements that are supported across LMSs: pages, assignments, quizzes, and discussion forums. Even though we included a few instances of an interactive video using a third-party tool like PlayPosit, the use of such a tool had limited benefit for an instructor, who could not view their teachers' responses to video prompts. Even when using these elements, the import from the common cartridge did not appear well organized in all LMSs. For example, when the common cartridge was imported into Blackboard, the import included empty folders and unorganized pages. Additionally, importing into different versions of the same LMSs was sometimes an issue. Our team needed to have access and the skills to build materials directly in three LMSs (Moodle, Blackboard, and Canvas). Thus, any revisions to materials created a domino effect of changes that had to be made across all three LMS exports.

Although MTEs appreciated the access of materials in LMSs and the flexibility to modify materials as needed, they also cited issues with importing modules into their LMSs. For example, when a Blackboard user who only planned to implement the Foundation module imported material into a course, the import included all three modules and both assignments. Although we made a design decision to package all three modules together, this decision created additional work for MTEs who did not intend to implement all modules. An MTE would need to hide or delete unused materials and delete unneeded assignments that were auto-added to an

LMSs' gradebook. Instead, in future iterations of our materials and as a suggestion for others developing online materials, MTEs should be able to select which materials they wish to export and import into their courses. This would require more careful thought in an MTE's planning process for a course but less work in structuring, organizing, and integrating materials in their LMSs.

4.5.3 Improving Preparation for Teaching Statistics

MTEs integrated ESTEEM materials into a variety of undergraduate and some graduate courses that contribute to teachers' preparedness to teach statistics. MTEs used materials in courses such as Teaching Mathematics with Technology (typically serving teachers for grades 6–12), Statistics for Teachers, Secondary Mathematics Methods (9–12 or 6–12), Elementary Mathematics Methods (K-5 or K-8), and Introductory Statistics. Teacher preparation programs vary greatly across institutions, and MTEs report it was easy to find courses in their program where the modules fit well, at least partially, for the statistical and pedagogical goals of a course.

Multiple MTEs who used the ESTEEM materials stated that oftentimes, their teachers did not have the prerequisite statistics content knowledge needed to meaningfully engage in ESTEEM activities. We purposefully designed ESTEEM materials to focus on developing teachers' pedagogical knowledge for teaching statistics since there are few secondary mathematics teacher education materials that do so and the development of this knowledge is crucial for teacher development. Based on feedback from MTEs that used the ESTEEM materials and this design decision, MTEs may need to provide additional learning opportunities for teachers to ensure they have the statistical knowledge needed to engage in ESTEEM activities.

Teachers that engaged with ESTEEM materials built their confidence and reported feeling better about their statistical understandings, comfort in exploring multivariate data on their own with CODAP, and knowing ways to engage students in key statistical practices and data investigations using technology. MTEs also felt better about their ability to prepare teachers to teach statistics, particularly related to the use of CODAP and understanding how to promote statistical discourse. Thus, the ESTEEM materials can be used to help MTEs meet the recommendations in the Statistical Education for Teachers report (Franklin et al. 2015). Both MTEs and teachers had positive experiences with CODAP and viewing videos of its use in classrooms that appeared to impact their perspective and comfort with investigating large multivariate data sets using linked representations. Disaggregating feedback data based on the learning activities MTEs and teachers used is an important next step in further research to unpack relationships between ESTEEM activities utilized in a course and reported impressions and growth in confidence.

Though teachers were able to demonstrate an ability to design tasks that can give students appropriate experiences in the age of the "data deluge" (Gould and Cetinkaya–Rundel 2014) by using large real data sets and connecting statistical thinking to the context of data, they still struggled with keeping a focus on statistical practices, investigations, and uncertainty in claims. Instead, some still focused on procedures and exact answers and seemed to not be able to put into action what they had an opportunity to learn about the differences between mathematics and statistics. What we don't know yet, and will be the focus of future research, is which ESTEEM activities seem to be connected with stronger task design. For example, we noticed that while most teachers had several opportunities to engage with statistics tasks themselves as learners, several activities in our modules that are specifically targeted at developing pedagogical strategies for designing worthwhile statistics tasks were used *less often* than other activities that focus on content (e.g., see Fig. 4.7, 1.2.j, A.2.d, A.2.g, B.2.h). Some courses focused more on content, so these differences may be a result of the goals of a course. This impels us to dig into our data further and perhaps design a comparative study of tasks designed by teachers who did experience those pedagogy-focused activities and those who did not.

MTEs can readily integrate ESTEEM materials into a variety of courses to help ensure future teachers are prepared for teaching statistics. We have learned that teachers are gaining better understanding of the nature of statistics by experiencing data investigations with CODAP and critically examining videos of students and teachers engaged with statistics tasks. The ESTEEM videos are some of the few available that include a focus on teaching secondary statistics with exploratory technology tools. Teacher educators, however, may need further assistance in ensuring that their choice of activities to implement may matter and impact what teachers are ready to do as statistics teachers. For mathematics teacher education programs that have multiple courses, there may also need to be concerted effort to weave the set of ESTEEM materials into more than one course in a way that can build teachers' sophistication in statistical understandings and pedagogy across more than one semester.

Acknowledgements The ESTEEM materials are freely available upon registration at https://go.ncsu.edu/esteem. Materials are distributed under Creative Commons Attribution, Non-commercial, Share-alike licensing. The ESTEEM project is supported by the National Science Foundation under Grant No. DUE 1625713 awarded to North Carolina State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation. The ESTEEM materials and research in this article would not have been possible without the significant contributions made by Bill Finzer, Christina Azmy, Heather Barker, Adam Eide, Allison Black-Meier, and Callie Edwards.

Appendix 1: ESTEEM Table of Contents

Full Annotated Table of Contents is available at https://fi-esteem.s3.amazonaws.com/lmsbackups/annotated_table_of_contents.pdf

Module 1: Foundation in Statistics Teaching

The focus of this module is the core ideas about teaching statistics and how to support students' investigations. It is highly recommended that teachers complete this module before being introduced to materials from either of the other two modules (Module A or Module B). Module 1 is organized into two parts.

Module 1.1 What is statistics and how should we teach it?

This module focuses on learning to engage in statistical investigations using technology (CODAP) and developing knowledge and skills for planning lessons for teaching statistics. The emphasis is on understanding how statistics is different from other areas in the mathematics curriculum and how students can develop statistical ways of reasoning.

Read and Watch

Essential Materials

1.1.a How is Statistics Different from Mathematics? (Reading)
1.1.b Statistical Investigations and Habits of Mind (Video, Reading, Diagram)
1.1.c Considering the Importance of Teaching Statistics (Video, Reading)
1.1.d Quiz on Read & Watch Material (Quiz: 6 multiple-choice questions & 2 free-response questions)

Learn from Practice

1.1.e Teaching Statistics in the Mathematics Curriculum (Video)
1.1.f Statistical Investigation Cycle in a Classroom (Video)

Engage with Data

1.1.g Investigating Older Roller Coasters in the US (Video, CODAP Data Investigation)

Synthesize and Apply

1.1.h Discuss Learning Statistics through Investigations with Real Data (Discussion Forum)
1.1.i Using an Online Data Analysis Tool (Discussion Forum)

Module 1.2 What is statistics and how should we teach it?

In this module, teachers will learn about a framework that can guide them in supporting students' statistical reasoning, including designing tasks and making sense of students' work. Teachers will also engage in a statistical investigation from the previous module, with a larger data set.

Read and Watch

Essential Materials

1.2.a Supports for Learning to Do Statistical Investigations (Readings, 2 Videos)
1.2.b A Guiding Framework for Teaching Statistics (Reading, 2 Videos)
1.2.c Tasks as Opportunities for Statistical Learning (Table, Video)
1.2.d Read & Watch Quiz (Quiz: 8 multiple choice questions)

Learn from Practice

1.2.e Expert Teacher Interview on Tools & Resources (Video)
1.2.f Teaching Statistics Using Multiple Technologies (Video)

Engage with Data

1.2.g Investigating More Roller Coasters (CODAP Data Investigation)
1.2.h Examining Students' Work on the Roller Coaster Task (Video, Discussion Forum)

Synthesize and Apply

1.2.i Supporting Statistical Discourse with the Roller Coaster Task (Reading, Video, Reflection Paper)
1.2.j Analyze Tasks and Discuss (Discussion Forum)

Module A: Teaching Inferential Reasoning

This module focuses on ideas around supporting the development of students' inferential reasoning. Teachers should be familiar with the ideas in Module 1 prior to being introduced to this module. Module A is organized into two parts.

Module A.1 Promoting and supporting inferential reasoning

The first part of module A focuses on developing essential understandings of how to support inferential reasoning. Materials will help teachers consider why it is important to teach students to reason inferentially and how questions, modeling processes, simulation tools, and tasks can support students' development of inferential reasoning. **Read and Watch** *Essential Materials* A.1.a What is Inferential Reasoning? (Reading, 2 Reflection Questions A.1.b Promoting Key Aspects of Inferential Reasoning (Video, Reading) A.1.c Using Models to Build Inferential Reasoning (Reading, CODAP Sampler Activity, Reading, Video) *Learn from Practice* A.1.d Considering the Importance of Inferential Reasoning (Interactive Video) A.1.e Anchoring Inference in a Cycle of Investigation (Video) **Engage with Data** A.1.f Investigating Fairness of Dice (CODAP Data Investigation) **Synthesize and Apply** A.1.g Comparing Use of Models in Tasks for Inferential Reasoning (Discussion Forum) A.1.h Analyzing Students' Work on Schoolopoly (2 Interactive Animation Videos, Discussion Forum)
Module A.2 Using models and repeated samples to develop inferential reasoning
The second part of module A focuses on how models and repeated sampling can be used to support inferential reasoning. Materials will delve deeper into critical understandings related to sampling distributions and how learning experiences can assist students in developing inferential reasoning. **Read and Watch** *Essential Materials* A.2.a Critical Role of Samples, Sampling, and Sampling Distributions (Reading, Video A.2.b Attention to Sampling Variability and Sample Size (Reading, CODAP Activity, Reading) *Learn from Practice* A.2.c Using Repeated Sampling to Introduce Sampling Distributions (2 Videos, 4 Reflection Questions) A.2.d Statistics Tasks to Promote Inferential Reasoning (Video) **Engage with Data** A.2.e Investigating the Success of Paul the Octopus (CODAP Data Investigation) A.2.f Investigating Carbon Dioxide Emissions in Vehicles (CODAP Data Investigation) **Synthesize and Apply** A.2.g Discussing Launching a Task to Support Inferential Reasoning (Discussion Forum) A.2.h Applying Modeling and Simulation to a Probability Comparison Task (Written Response)
Module B: Teaching Statistical Association
This module focuses on ideas around the teaching and learning of statistical association. Teachers should be familiar with the ideas in Module 1 prior to being introduced to this module. Module B is organized into two parts.
Module B.1 Statistical association of categorical variables

The first part of module B focuses on association of categorical variables. Materials assist teachers in developing critical understandings related to graphs and measures used to describe association between categorical variables and how learning experiences can assist students in developing reasoning about association.

Read and Watch

Essential Materials

B.1.a Investigating Categorical Variables in CODAP (Reading, 2 Videos)

B.1.b Common Student Approaches when Analyzing Bivariate Categorical Data (Reading, Videos)

B.1.c Quiz on Read & Watch material (Quiz: 7 multiple choice and 1 open-ended questions)

Learn from Practice

B.1.d Student-created Graphs of Bivariate Categorical Data (Video, Discussion Forum)

Engage with Data

B.1.e Investigating Data about Granola Bars (CODAP Data Investigation)

Synthesize and Apply

B.1.f Discuss Representations of Bivariate Categorical Data (Discussion Forum)

B.1.g Students' Reasoning about a Segmented Bar Graph (Activity; 5 Videos, Written Response)

Module B.2 Statistical association of quantitative variables

The focus of the second part of module B is association of quantitative variables. Materials assist teachers in developing critical understandings related to graphs and measures used to describe association between quantitative variables and how learning experiences can assist students in developing reasoning about association.

Read and Watch

Essential Materials

B.2.a Introducing Students to the Topic of Statistical Association (Activity, Video)

B.2.b Measures of Association and Lines of Best Fit (Video)

B.2.c Distinguishing Between Correlation and Causation (Reading)

B.2.d Quiz on Read and Watch Materials (Quiz: 6 multiple-choice and 1 open-ended questions)

Learn from Practice

B.2.e Considering Student Approaches to Placing the Informal Line of Best Fit (Activity, 2 Animated Videos)

Engage with Data

B.2.f Investigating Data about Vehicles (CODAP Data Investigation)

B.2.g Teaching Statistics with CODAP (Video)

Synthesize and Apply

B.2.h Discuss Differences between Mathematics and Statistics in the Study of Association (Discussion Forum)

B.2.i Investigating Data from the Census at School Random Sampler (readings, CODAP Data Investigation)

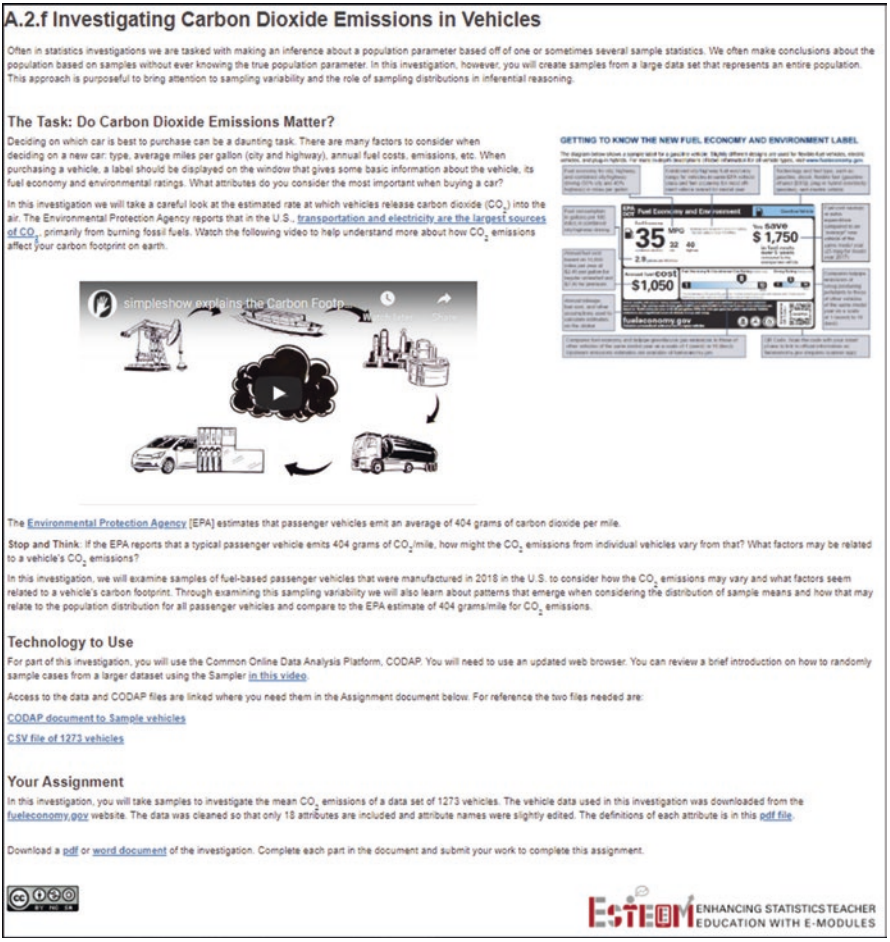
Screencast Assignment

This assignment allows teachers to illustrate their ability to conduct a statistical investigation with larger, multivariate data sets in CODAP. Teachers record themselves with screencast software while conducting a statistical investigation and explain their thinking throughout the process. This assignment can be done after any of the modules.

Task Design Assignment

The purpose of this assignment is to design a task that illustrates how one can develop students' statistical thinking utilizing CODAP as a tool. This assignment can be completed ***after*** *Module A and/or Module B.*

Appendix 2: Sample Multimedia Page from Teaching Inferential Reasoning Module



A.2.f Investigating Carbon Dioxide Emissions in Vehicles

Often in statistics investigations we are tasked with making an inference about a population parameter based off of one or sometimes several sample statistics. We often make conclusions about the population based on samples without ever knowing the true population parameter. In this investigation, however, you will create samples from a large data set that represents an entire population. This approach is purposeful to bring attention to sampling variability and the role of sampling distributions in inferential reasoning.

The Task: Do Carbon Dioxide Emissions Matter?

Deciding on which car is best to purchase can be a daunting task. There are many factors to consider when deciding on a new car: type, average miles per gallon (city and highway), annual fuel costs, emissions, etc. When purchasing a vehicle, a label should be displayed on the window that gives some basic information about the vehicle, its fuel economy and environmental ratings. What attributes do you consider the most important when buying a car?

In this investigation we will take a careful look at the estimated rate at which vehicles release carbon dioxide (CO_2) into the air. The Environmental Protection Agency reports that in the U.S., transportation and electricity are the largest sources of CO_2, primarily from burning fossil fuels. Watch the following video to help understand more about how CO_2 emissions affect your carbon footprint on earth.

The Environmental Protection Agency [EPA] estimates that passenger vehicles emit an average of 404 grams of carbon dioxide per mile.

Stop and Think: If the EPA reports that a typical passenger vehicle emits 404 grams of CO_2/mile, how might the CO_2 emissions from individual vehicles vary from that? What factors may be related to a vehicle's CO_2 emissions?

In this investigation, we will examine samples of fuel-based passenger vehicles that were manufactured in 2018 in the U.S. to consider how the CO_2 emissions may vary and what factors seem related to a vehicle's carbon footprint. Through examining this sampling variability we will also learn about patterns that emerge when considering the distribution of sample means and how that may relate to the population distribution for all passenger vehicles and compare to the EPA estimate of 404 grams/mile for CO_2 emissions.

Technology to Use

For part of this investigation, you will use the Common Online Data Analysis Platform, CODAP. You will need to use an updated web browser. You can review a brief introduction on how to randomly sample cases from a larger dataset using the Sampler in this video.

Access to the data and CODAP files are linked where you need them in the Assignment document below. For reference the two files needed are:

CODAP document to Sample vehicles

CSV file of 1273 vehicles

Your Assignment

In this investigation, you will take samples to investigate the mean CO_2 emissions of a data set of 1273 vehicles. The vehicle data used in this investigation was downloaded from the fueleconomy.gov website. The data was cleaned so that only 18 attributes are included and attribute names were slightly edited. The definitions of each attribute is in this pdf file.

Download a pdf or word document of the investigation. Complete each part in the document and submit your work to complete this assignment.

ESTEEM ENHANCING STATISTICS TEACHER EDUCATION WITH E-MODULES

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Chapter 5
Rehumanizing Mathematics Education and Building Community for Online Learning

Naomi A. Jessup, Jennifer A. Wolfe, and Crystal Kalinec-Craig

In April 2020, a small group of mathematics teacher educators (MTEs) met to discuss navigating online teaching and its implications for our practice as a result of the COVID-19 pandemic. We soon recognized tensions between our teaching style in a physical classroom, and throughout the summer, we held a series of informal online discussions about future teaching aspirations and plans for the fall. Through our sharing of strategies and online tools, we built a sense of community and rehumanized the meaning of collaboration and working remotely. Each meeting began with an opportunity to share our troubles and joys. These sharing sessions would bring us closer to what we knew to be a humanizing classroom that promoted safety and encouraged participation in whatever way felt comfortable.

The COVID-19 pandemic forced many MTEs to take an alternative approach to their teaching. Before the pandemic, many MTEs taught in physical classrooms with teacher candidates (TCs) and employed rehumanizing practices (Gutiérrez, 2018) that afforded opportunities to work and listen to each other. Implementing rehumanizing practices in collective community afford TCs opportunities to learn from and with one another on how to actively work towards dismantling hierarchies of perceived competence (which impacts students' participation and how they are

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positioned in the community), enact pedagogical practices that shift authority from the teacher to the students, and help one another and their students see mathematics as a "living practice" (p. 5). As Gutiérrez (2018) argues, "When students can see mathematics as full of not just culture and history, but power dynamics, debates, divergent answers, and rule breaking, it highlights the human element…" (p. 5). In moving to the online environment, what does it mean to implement and employ rehumanizing pedagogical practices? As MTEs, how do we actively attend to issues of power, positioning, agency, and equity in the online learning environment? What does it look like to shift resources and create online space towards amplifying voices of the historically excluded peoples of the global majority?

Long-standing practices like equitable group work and engaging in community-building activities look different in online spaces. For those MTEs who felt unprepared to teach in an online-only environment with many technology platforms to learn, the pandemic transition also meant rethinking what a "rehumanizing classroom" would look like online (e.g., revisiting attendance, assessment, grading, and participation policies that perpetuate inequities online). MTEs would need to consider ways of "building community" using various digital platforms. Furthermore, the transition to online teaching highlighted long-standing inequities that existed before the pandemic (e.g., issues with internet connectivity, uncertain changes in university and school expectations for participation, and negotiations when working from home with other family members and obligations; see Barrett-Fox, 2020).

Our chapter draws upon illustrative examples that conceptualize and problematize mathematics instructional practices for online spaces that are rehumanizing and build a sense of community. We focus on these practices for TCs within the formal learning space of methods and content courses. We acknowledge the work of rehumanizing mathematics courses means developing practices and measures that feel humane for TCs given that structures, procedures, and practices can be experienced as dehumanizing (Gutiérrez, 2018). The work of rehumanizing spaces online requires we see students and ourselves as whole beings and that we navigate our collaborative and collective learning spaces with grace, empathy, compassion, and in community. MTEs must reexamine and constantly reflect upon our positionality, our intentionality, our beliefs, and our community if we are to do the work dismantling and disrupting oppressive structures (Jessup et al., 2020b). In the next two sections, we describe the rehumanizing mathematics education and considerations of online mathematics instructional design.

5.1 Rehumanizing Mathematics Education

Rehumanizing mathematics education is not static. Rather, it requires action that "reflects an ongoing process and requires constant vigilance to maintain and to evolve with contexts" (Gutiérrez, 2018, p. 3). Rehumanizing requires actions to disrupt and dismantle inequities, while simultaneously confirming that attempts are experienced humanely by the receiver. In mathematics, rehumanizing centers the humanity of students engaged in learning mathematics while attending to power, status, and agentic practices that affirm positive identity development (Morales & DiNapoli, 2018). Rehumanizing mathematics is communal and disrupts structural aspects of learning environments that stifle the mathematical brilliance and opportunities for those who have experienced violence and techniques of silence.

Mathematics education needs a process of rehumanizing given the historical accounts that mathematics acts as a gatekeeper both structurally and pedagogically across social markers (e.g., race, class, gender, ability, and language) (see Gholson, 2016; Leyva, 2017; Martin, 2013; Stinson 2007, 2008; Turner et al., 2013; Yeh et al., 2020). Mathematics and mathematics education maintains a pervasiveness of anti-Blackness (Martin, 2019; McGee, 2013), deficit-narratives (Adiredja, 2019), and hierarchical structures (Featherstone et al., 2011; Louie, 2020) that shapes whose cultural capital is valued. Gutiérrez (2018) describes eight dimensions to rehumanizing mathematics for students who are Black, Latinx, and Indigenous which include (1) participation/positioning, (2) cultures/histories, (3) windows/mirrors, (4) living practice, (5) creation, (6) broadening mathematics, (7) body/emotions, and (8) ownership. Along those eight dimensions, Goffney (2018) synthesizes five teaching practices that support the eight dimensions for teachers and students (see Goffney, 2018 for complete list). Within the dimension of participation, we focus on two of those practices to highlight online learning designs in formal spaces that involve (a) the work of MTEs to learn about and embrace cultures and identities of their students and (b) the process of rehumanizing mathematics that uses a variety of teaching practices (e.g., co-constructing community and mathematics knowledge, sharing authority between MTEs and TCs, and broadening types of participation).

Rehumanizing mathematics means centering the student as a complete human being that desires a sense of belonging, community, affirmation, and acceptance (Goffney, 2018; Id-deen, 2017) regardless of the age of the learner and platform of learning. Yet, the design of mathematics learning and pedagogical practices “convince people they are no longer mathematical” (Gutiérrez, 2018, p. 2). If rehumanizing mathematics means centering students as whole beings, then this should apply to TCs online mathematics learning experiences. In mathematics methods and content courses, dehumanizing practices occur when TCs’ are required to adhere to a learning environment that is built on assimilation, policing of bodies and behaviors, and deficit narratives that create dehumanizing experiences (see Bullock & Meiners, 2019). The design, policies, and practices used in mathematics courses should be created in rehumanizing ways so that TCs are not treated as interchangeable with limited attention to who they are. In our chapter, we focus on rehumanizing participation and positioning during online learning of formal learning spaces. We include ways to reimagine community building across digital spaces and platforms and interrogate the carceral nature of policies and practices.

5.2 Online Mathematics Instructional Design

We center the intersectional identities of TCs in the design of rehumanizing online mathematics learning experiences in formal spaces. In a review of research on online learning, Aparicio et al. (2016) note that online studies emphasize three main areas: students, technology usage, and services provided by e-learning systems. When K-12 and higher education students are the focus of these studies, researchers sought to understand students' interactions, success and satisfaction of online learning courses and modules, and cultural differences (Baran, 2014). Few studies considered students' experiences in connection to students' ability to bring their complete selves into the learning space (Corey & Bower, 2005) and how students' identities were considered in the design of the learning. In addition, online learning studies suggest that the design of courses and modules should consider the pedagogical model of learning (e.g., open learning, distributed learning, learning communities) and instructional strategies that support collaboration, content learning, and assessment (Aparicio et al., 2016; Corey & Bower, 2005). We would add that designing for online learning interrogates the traditional norms for students and teachers who might normally expect physical behaviors that regulate students' ways of participating and identities as learners (e.g., maintaining eye contact, participation by raising hands or speaking up in class).

We suggest an expansion of instructional designs and teaching approaches that reflect rehumanizing pedagogies given concerns for digital equity that create a digital divide across all courses in every program. Digital equity includes access to Internet connections, software, digital tools, and resources. Yet, once colleges and universities moved to online learning, and several students moved back home or aspects of campus had limited hours (e.g., computer lab and library), access to technology became constrained and challenging, causing a digital divide. The digital divide describes disparities that exist relative to students' access to the Internet and computers based on their socioeconomic status, race, cultural identifiers, and gender (Gorski 2009). The digital divide exposes the stratification of social capital where access to preferred types of technology (e.g., devices and tools) has status, is privileged, and positions those without at the bottom of the social spectrum (Harris, 2015). Therefore, in the instructional design of mathematics content and methods courses, we question how technology is privileged and the stratification of access.

The abrupt shift in moving most mathematics education courses from face-to-face instruction to online-only occurred in a matter of weeks and days for MTEs in response to the COVID-19 pandemic. We acknowledge that teaching mathematics instruction through online methods is not new while also considering the research about online learning is limited (Trenholm et al., 2016). Our goal is to push MTEs to interrogate how the overall design of our courses can foster community building, expand approaches for engaging in formal learning spaces, and disrupt norms that can be experienced as carceral and dehumanizing. In particular, MTEs have even more responsibilities (compared to our colleagues who do not teach as a part of a certification program) as those who should model practices that are rehumanizing and encourage future teachers to take up similar practices in their own emerging philosophies. Prior to presenting our thoughts about the notion of rehumanizing mathematics (teacher) education, we first present our own positionality and backgrounds for context.

5.3 Our Positionality and Context

5.3.1 Positionality

The three authors come from different institutions across the United States and reflect diverse demographics and our experiences as MTEs. Jessup identifies as a Black mother scholar who was previously a district-level K-8 mathematics instructional coach and former elementary teacher in most metropolitan areas in the Southeast United States. At Georgia State University, she brings her experience supporting and working alongside teachers and students at the district and local levels to expose the systematic practices within mathematics education spaces that cause harm. Jessup believes that mathematics education programs should help TCs examine the multiple layers of educational inequities and develop mechanisms that disrupt racialized and stereotypical views, and low expectations for students, particularly those whom society has pushed to the margins while also attending to their mathematical thinking (Louie et al., 2021). She believes that classrooms can provide opportunities for students to thrive and make sense of the world through mathematics regardless of their racial, ethnic, and linguistic identities (Jessup et al., 2020a).

Wolfe identifies as a multiracial, Thai Asian American, cis hetero woman at The University of Arizona in the department of mathematics. She has been teaching mathematics for over 20 years. As an MTE, she views her role as both a facilitator and learner in a collective community seeking to learn from and with her students by leveraging strengths and brilliance through collaboration. As an educator-in-progress, she grounds herself around two central questions: (1) How do I collectively promote and value students' participation in mathematics discourse that position them as mathematical competent? and (2) How do I cultivate a learning community where students develop robust positive mathematics identities and experience a sense of belongingness and inclusion? (Wolfe, 2021).

Kalinec-Craig identifies as a White woman who was previously a middle and high school mathematics teacher in five states in the United States and Germany. As an MTE now at the University of Texas at San Antonio (UTSA), she believes that mathematics classrooms are racialized spaces that have perpetuated years of inequalities upon students who have the potential to succeed and rise above dehumanizing practices in traditional classrooms. She believes that mathematics classrooms (and teacher education programs) can be rehumanized so that students, especially those who are Black, Brown, and/or emerging multilingual who have experienced the vast majority of the inequalities in schools and ways of knowing and being are honored to build community and are encouraged to exercise their Torres' Rights as Learners (Kalinec-Craig, 2017; Torres, 2020).

5.3.2 Our Context

Our three teacher preparation programs reflect a variety of demographics and programmatic structures. Each of our institutions is situated in highly dense cities that engage in rigorous research activity. GSU and UTSA serve a majority of Black students and Chicanx students (e.g., students of Mexican descent) from the local community, whereas UA is a Hispanic Service Institution and balances out-of-state students with in-state. At UTSA, 60% of teacher candidates transfer from the local community colleges. All three institutions have high populations of first-generation students, and the majority of undergraduate courses are designed for face-to-face learning. Many of those students live off campus or at home, and some support their families or have families of their own. So, moving the majority of instruction online amid the COVID-19 was not the desired option.

We collectively teach mathematics methods and content courses to elementary education majors and secondary math majors, and all three of our courses vary but have some commonalities. Given the toll of the COVID-19 pandemic and the overall disruption of our "normal" way of living, we designed our courses to provide flexibility in formats (i.e., asynchronously, synchronously, or both). Prior to the start of class, we provided TCs with a survey to inquire about digital resources and supports that are accessible to them, perceptions of what has worked or has not worked for them online, and to obtain a general gauge of their social-emotional well-being. Results of the survey were used in designing the formats of our classes as a means of acknowledging and taking concrete actions to address issues of digital equity within our contexts. For example, Wolfe's students meet synchronously, and course materials are posted within 24 h of the class. Given the survey results, Jessup's courses were designed for asynchronous sessions with optional synchronous sessions that occurred every other week or every 2 weeks. TCs participate in weekly modules which include short, prerecorded lectures for each assignment within the module, and synchronous sessions include major assignments within each module. Synchronous session discussions are provided immediately after each class. Whereas, Kalinec-Craig has asynchronous videos that present educational learning theories about children's mathematical thinking; the videos are followed up with synchronous meetings. The authors have different contexts and online modalities for learning, but the instructional designs across all courses center on rehumanizing the classroom space.

Our courses meet via Zoom where students collaborate using a variety of tools from the Google Suite products (e.g., Slides, Docs, Jamboard, etc.), Nearpod, Flipgrid, Desmos, Geogebra, and Padlet. TCs engage in weekly self-assessments of their progress which include reflections on course readings and experiences through multiple forms of expression including, but not limited to, writing, video, audio, oral presentation, and drawings/visuals. In addition, TCs have the choice and flexibility to show and share their knowledge. Some course activities include learning to create and analyze culturally relevant mathematics tasks and lessons (Aguirre & del Rosario Zavala, 2013), teacher noticing (Louie et al., 2021), enacting high-leverage teaching practices (Ball & Forzani, 2009), using and critiquing practices for orchestrating mathematical discussions (Smith & Sherin, 2019; Smith et al., 2020), engaging in "rough draft math" (Jansen, 2020), and group worthy tasks (Cohen & Lotan, 2014; Horn, 2012) that emphasize the Torres' Rights of the Learner.

5.4 Rehumanizing Instructional Designs in Mathematics Education: A Metaphor of Community

MTEs belong to a community that prepares future teachers. Our classrooms, which serve as another example (and an emerging cadre) of our community, may look very different given our context, locale, goals of the course and program, etc., especially in the time of the COVID-19 pandemic. But one thing that may be consistent is how we seek to build, participate, and maintain that community for TCs and MTEs to come. A community thrives when there exists trust, safety, and respect for each other; the same is true of a community of learners in mathematics teacher education courses. The following sections unpack and interrogate this metaphor of a community for MTEs and TCs both in a physical space and in this new online-only format, in response to the pandemic.

5.4.1 Building a Community Online

Building community is an integral part of rehumanizing mathematics learning. "As teachers build community in the classroom, they should also focus on creating a safe space where students feel like they are part of a community of learners" (Milner et al., 2019, p. 86). Creating safe, trusting, and healing spaces for learning is a process contingent upon how students are positioned, how and whose ideas are taken up, how power and intellectual authority are distributed, and how teachers draw upon, value, and integrate the diverse cultural practices and linguistic richness of students and their local communities. The development of powerful and healing relationships between and among students and teachers is central to cultivating a collegial learning environment, where students feel a sense of belonging and inclusion and where everyone (including the teacher) is positioned as sense makers and generous listeners. In what follows, we describe some practices we have used in our courses to build community online. We should note that these practices are not unique to the online environment but are adaptable across multiple modalities (live synchronous, asynchronous, hybrid, and face-to-face instruction). In particular, we will focus on three areas for building community in teacher preparation, (1) reexamining institutional policies and practices and our beliefs about mathematics teaching and learning for equity, (2) exploring identity towards developing positive mathematical identities, and (3) learning to listen and listening to learn.

Getting Started: Reexamining and Interrogating Institutional Policies

Building community begins even before students log on to the virtual learning space. As MTEs, we must reexamine, interrogate, and problematize the inequitable policies, practices, and aspects of institutional culture that target and harm (whether or not intentional) our Black, Indigenous, Latinx, and peoples of the global majority. If we are to design online learning spaces centered on rehumanizing, we must first recognize this work "requires the redistribution of material, cultural, and social access and opportunity...by changing inequitable policies, eliminating oppressive aspects of institutional culture, and examining how practices and programs might advantage some students over others" (DuBose & Gorski, 2020, slide 26). How are we designing our online classroom spaces where we are actively redistributing power and resources? How are we changing, and in many cases, eliminating inequitable classroom policies and practices, such as enacting carceral pedagogies (Berkshire, 2020), that seek to control the bodies of Black and Brown students? We must begin transforming the online learning conditions that marginalize and oppress students (DuBose & Gorski, 2020) by focusing on "students over policies" and "fixing injustice not people." How are we changing online classroom policies and practices that seek to help students "communicate to learn rather than to perform" (Jansen, 2020, p. 2). Do our actions and course design for online instruction align with a strengths-based anti-racist perspective of what research says about effective mathematics teaching and learning experiences? How are we building trusting and actively caring relationships with our TCs? How are we making space for TCs to collaborate? In what follows, we describe some practices we have enacted to address these reflective questions and rehumanize our online learning spaces.

Before the beginning of the first class session and assuming access to digital resources, TCs create an introductory slide along with the MTE and use the slide to begin working towards building communities that attend to students' identities. For the first day of class and thereafter, we engage in some meditative breathing followed by check-in on how TCs are feeling and/or celebrations to share; all can be done using collaboration tools within Google Slides, Google Jamboard, chats, discussion boards, emojis, gifs, reactions, and other online platforms for sharing ideas (see Wills, 2020; Wolfe & Amidon, 2020-present; Yu, 2020). Additionally, we engage in follow-up conversations based on those check-ins. For example, in synchronous spaces, when we notice someone's image that seems melancholy, we send an in-the-moment private message to follow-up. In the event collaborative group work occurs later, we provide the choice to opt in and opt out. This type of humanizing responsiveness provides a way to extend and express empathy, grace, and care to and for our students. We seek to center our students' identities, values, and diverse needs by providing opportunities for choice and honoring and valuing the choices they make for what makes sense to them at that moment. Checking in with our students is a nonnegotiable in the online space as we realize that our students may be feeling the effects of trauma to some degree, whether it is the COVID-19 pandemic, racial or economic pandemic, or feelings of isolation and loneliness. As Milner et al. (2019) argue:

> ...Teachers should take the social context of a particular place into consideration when managing the daily classroom life for a particular group of students (Milner, 2010). Opportunities and resources are not equal and certainly not equitable across educational settings. (p. 85)

We use these check-ins to guide our moment-to-moment instructional decisions in the online space towards rehumanizing our classrooms and supporting our students' social-emotional needs.

Exploring and Developing Positive Mathematical Identities

As MTEs, we should seek to cultivate online learning spaces that work to develop TCs' positive mathematical identities. "All math teachers are 'identity workers,' regardless of whether they consider themselves as such or not" (Gutiérrez, 2013, p. 11). Anti-racist teaching and rehumanizing work begin with an exploration of our own identities and how those identities are positioned and influenced through our interactions in this world, both outside and within the classroom. "Because a mathematics teacher identity is at least partly, in teacher's experiences as a mathematics learner, we must explore how those experiences may have been shaped, in turn, by race, class, gender, and language" (Aguirre et al., 2013, p. 28). Students are assigned the first two chapters of Aguirre et al. (2013) *The Impact of Identity in K-8 Mathematics: Rethinking Equity-Based Practices*. After reflecting on these readings, the MTE and TCs create a digital story (Chao, 2014) Google Slides /Microsoft Powerpoint files (filled with images, photographs, visuals, texts, gifs, videos, etc.) that best represent their experiences and journeys in learning mathematics.

Learning to Listen and Listening to Learn

Mathematics teaching involves listening. We learn from and with one another in the community through carefully and generously listening to one another's mathematical insights and remaining present in the dialogue. As Jansen (2020) explains, "Listening to our students is not only productive for their learning but also powerful for developing their identities as learners" (p. 17). Furthermore, Hintz and colleagues (2018) found that "...teachers who listen pedagogically are not listening to measure the child, but rather to measure the environment. They constantly ask themselves if the environment is supporting and hindering the child's ability to learn as a mathematical sense maker and sociocultural, affect well-being" (p. 5). Thus, developing the skill of listening is essential for the development of TCs engagement in equitable teaching practices. To help our TCs further develop their listening skills, we engaged in constructivist listening dyad protocols (Burkhalter et al., 2020), where the students watched a NY times video *A Conversation With Native Americans on Race* (https://www.youtube.com/watch?v=siMal6QVblE). We then engaged in a learning and listening protocol using a modification of Hintz et al.'s (2018) work on pedagogical listening and Jansen's (2020) rough draft math. Then, we engage in another round of the protocol with the students sharing their digital storytelling slides. Through digital storytelling, TCs were beginning to learn more about their peers and their journeys to becoming who they are in the moment. Identity work is further developed through additional readings and webinars (e.g., *NCTM 100 Days of Professional Learning* webinars, Deborah Ball's AERA 2018 presidential address on anti-Black racism and discretionary spaces of mathematics teaching). Through this work, our students begin critically analyzing their own identities, those of others, and how the classroom environment plays a role in when, how, and if identities are honored, valued, and used to meet the community's collective needs.

MTEs actively rehumanize their mathematics courses when we listen and learn to meet our students' needs. In what follows, we provide an account of action taken in response to a TCs' concern around online digital storytelling. Through digital storytelling, MTEs and TCs alike have opportunities to learn from and with each other about one another's multifaceted intersectional identities. Consequently, MTEs are better equipped to attend to TCs development of positive mathematical identities in the online learning environment. At the start of the semester, one of the authors meets with each of their students one-to-one to get to know them, their interests, aspirations, concerns, challenges, successes, communities, ways of being, and knowing, in efforts to begin to build trusting, robust, positive relationships for learning in the community. During the conversation, a TC who identifies as nonbinary transmasculine and uses they/them pronouns raised concerns about the digital storytelling task. The task had initially been designed where TCs would share photographs that represented their mathematics identities. The TC felt comfortable sharing their discomfort in sharing photographs that were before their transition. This author had the best intention of creating an activity to build community and positive mathematical identities through digital storytelling. However, by listening to the TC's experience, the author realized that in limiting the digital storytelling to just photographs, unintentional harm occurred. The assignment was immediately modified to include a variety of representations (e.g., visuals, gifs, sketches, videos). The author acknowledged the harm, apologized to the TC, and thanked them for speaking up and calling them in (Ross, 2019). Additionally, the author expressed appreciation for helping them become a better MTE for those in the community.

5.4.2 *Participating with an Online Community of Learners*

Online classrooms in mathematics spaces need to be places that honor students' identities and create a safe, trusting, and healing space that interrogates who has intellectual authority and for what purpose does this authority serve (e.g., to elevate everyone's ideas). The notion of participation in our classrooms has changed, some in subtle ways whereas others are more drastic. In face-to-face learning, mathematics teacher education classrooms are ones with desks, tables, dry erase boards, and other tools that could be shared amongst the TCs. However, for a majority of MTEs who were thrust into teaching online, considerations for participating in a mathematics method or content course meant a substantial change in how MTEs considered the following questions: (1) How can we disrupt assumptions regarding TCs comfort with digital tools for participation? (2) What are the parallels in participation practices in the physical and online classroom space? (3) How can we encourage participation so that we build and maintain our community of learners, especially while in an online-only format? and (4) What are the online-only learning practices that still perpetuate more inequities? The following paragraphs will unpack these questions and pose more.

Disrupt Assumptions of Participation

Online-only learning is relatively unfamiliar to most TCs, and we, as MTEs, should avoid making assumptions about the TCs' technology familiarity or fluidity, given the vast number of digital tools available to support their learning. Instead, we intentionally spend the first few classes becoming familiar with the tools of the online platform used for collaboration, ways of sharing mathematical reasoning and problem-solving strategies, and accessing various features of our learning management system (e.g., D2L, Canvas, iCollege, Google Classroom). For example, TCs explore digital collaboration tools such as Padlet (https://padlet.com) to become familiar with the course syllabus and expectations through a digital scavenger hunt. Later in the course, Padlet is used to collaboratively curate digital resources for mathematics teaching and learning that are accessible to the entire community. Other MTEs might consider having TCs create a video using Flipgrid (https://flipgrid.com) to take the class on a guided tutorial of using Google Classroom or Google Jamboard as a means of familiarizing the community with the new platform. Flipgrid can also be used to have TCs illustrate their solution strategies using virtual manipulatives to solve mathematics problems in a variety of ways. MTEs might consider having students create a Bitmoji virtual classroom (Catlin, 2020) and then use Flipgrid to describe the different aspects of the Bitmoji classroom as a representation of what is important to the TC in their lives and experiences. By creating a virtual Bitmoji classroom, TCs have the opportunity to express their intersectional identities within the space while also hyperlinking important learning resources (e.g., virtual mathematics, e-books, google classroom, websites, etc.).

How TCs choose to express their identities and what they post on the walls of their virtual classroom also sends messages about who can do mathematics, whose mathematics we value, and what it means to do mathematics; representation not only matters, but it is also essential (Berry et al., 2014; Jessup, 2020). This sample of interactive tasks within the instructional design of the course gives students an opportunity to practice learning how to collaborate in the online spaces (e.g., see Wolfe & Amidon, 2020-present).

Participation in Face-to-Face Versus Online-Only Format

Over decades of educational research, there are well-documented examples of dehumanizing practices in the physical classroom related to participation (White et al., 2016) that can show up in online spaces. These practices are well-identified in mathematics spaces: students seated in rows with little room or autonomy to interact as a small group; students seated facing the teacher with rare opportunities to speak out of turn to ask for assistance or to challenge a statement; students are considered distracted, lazy, or defiant if they do not have their eyes affixed to the teacher or board (Lemov, 2015; Teach Like a Champion, 2019). Again, these types of carceral participation practices are steeped in the teachers' need to control bodies and ways of communicating and are easily visible within a physical classroom but can persist in online spaces.

One eerie parallel is present in online learning spaces that mimic the abovementioned dehumanizing practices shared in the popular book, *Teach Like a Champion* (Lemov, 2015, 2020). Lemov (2020) outlines the need to mandate requiring "cameras on" to ensure learner participation during class, which translates into a means of regulating students' behavior. As MTEs, when we demand TCs have their microphones and cameras on at all times regardless of the work environment, we risk creating a violent, intrusive, and shaming space for those with challenging circumstances (e.g., a baby who is crying, a messy room, and/or an inconsistent Internet connection). Furthermore, when MTEs engage in Lemov's practices, or what we describe as "encouraging participation with a dehumanizing demand," those MTEs do not create a safe and brave space for TCs to participate further harming their mathematical identity development. As Jackson (2020) writes, participation with a dehumanizing demand can also risk engaging in "video classism" (para. 1) that are intrusive, shaming, and judge TCs' homes and lives.

MTEs who employ dehumanizing practices like those mentioned above may not have considered the importance of establishing a community of learners that seeks and maintains a sense of trust and safety; the MTEs did not attempt to embrace TCs as whole human beings who have a multitude of social identities. Yet, when MTEs embrace a rehumanizing stance, traditional notions of social interaction and participation within mathematics spaces are dismantled. MTEs who embrace vocal interruptions by their TCs can clarify in-the-moment issues as a means of not letting them feel perceived as being defiant or rude (Hintz et al., 2018; Kalinec-Craig, 2017; Torres, 2020). When MTEs begin to interrogate their practice so that they emphasize the goal of building a community of learners, they open more ways for TCs to participate so that their whole selves are recognized and valued.

Complex Instruction in the Physical and Online-Only Format

Complex Instruction (CI) continues to be one body of research that encourages MTEs to take strength-based approaches to students' thinking and ways of collaborating with each other so that everyone plays a role in the learning process (Cohen & Lotan, 2014; Featherstone et al., 2011). MTEs who employ CI attend to multiple issues regarding participation in the class using groupworthy tasks. Groupworthy tasks encourage approaching tasks with multiple entry points as a part of a collaboration and with attention to issues of status. As Wood et al. (2019) describe the intersection of participation and status in CI classrooms:

> Students often choose to participate, and are allowed to participate, in mathematics to the degree that they are seen (and see themselves) as smart. Students' perceived mathematical skill is intertwined with their social, peer, and academic standing—their status. "Higher status" students are seen as smarter (by their peers, by their teachers, and even by themselves) and participate more often, whereas "lower status" students often get sidelined. (p. 219)

Therefore, in a physical classroom, mathematics teachers and MTEs may have more information about how their students are participating and engaging in the mathematical task with visual, audio, and gesturing cues. With online-only formats, the information received with respect to the norms for student participation in a CI classroom has dramatically changed.

But teaching mathematics content and methods courses in online-only formats disrupts several norms and practices that once defined a rehumanizing classroom. Learning mathematics online means there are no more physical tables, moveable desks, hands-on mathematics manipulatives, and long whiteboards that everyone can see others' mathematical thinking; TCs' desks may now be a bed, couch, and/or the inside of a car because that is the only quiet place in the home. Gone are the typical opportunities for physically turning and talking to each other; "Turning to talk" in an online-only platform is complex and dependent upon whether the classroom is held (a)synchronously. "Breakout Rooms" are one way to mimic the idea of small group work where students can have more opportunities to share their thinking and have equalized status amongst their peers, as opposed to just the handful of students who would unmute their microphones and reply to the teacher's questions.

Interestingly, the norm of multiple students talking at the same time, which might be seen as a sign of a spirited debate in a face-to-face class that uses CI, has now become an online-only taboo; there are now (un)spoken rules of "one-person-at-a-time-to-talk," and students are encouraged to constantly "mute the mics." The mere phrase of "mute the mics" is a painful reminder that online-only learning has already begun to shape what participation looks like in ways that dehumanize students and limit their opportunity to participate and gain equal status amongst themselves. If this online-only space sounds inhumane, lonely, and individualistic and counter to the research of CI–you are right, it is. But it does not have to be.

Encouraging Participation While Building and Maintaining a Community of Learners

As a part of writing this chapter, we kept in mind the notion of carceral pedagogies for mathematics learning online (as described in the section named "Getting Started: Reexamining and Interrogating Institutional Policies") and sought to recognize the ways in which we created the safe space for participation and collaboration; one tool has been collaborative platforms that are accessible for all. When once Powerpoint was the leader in presentation software, Google Slides helps to present ideas with the opportunity of collective ownership (e.g., students can share their thoughts by directly editing the slides). Instant editing to the presentation can be one powerful signal to students that the ownership of the slide deck is no longer *only* that of the teacher, but is owned by the community of learners, no matter where the class is located or how the students choose to participate. This ownership and sharing of ideas in the online classroom through an editable slide deck can also provide opportunities for TCs to build *representational competence* (Huinker, 2015) by creating and connecting multiple representations-visual, symbolic, contextual, physical, and verbal in solving mathematical problems.

The use of collaborative platforms that are editable by teachers and students can signal a subtle, yet powerful shift in the norm for who can participate and in what way. Suppose a student accidentally makes a change on the Google Slide deck? Fortunately, there is a functionality to revert to the old history, and/or other students can step in and assist to retrieve the information. Imagine what might have happened in a traditional classroom where the students could edit the presentation slides *while* the teacher was talking and presenting? One could imagine traditional teachers would cry anarchy and chaos while protesting such a disrespectful interference with their lesson; however, in online-only teaching that embraces a rehumanizing stance of a community of learners would give students editable access to the slides as a way to encourage participation and to maintain a running record of the class' thinking. Through editing, revising, and reflecting on their mathematical work in the online community, TCs begin to view that "participating during mathematics class is an opportunity to continue learning, not an obligation to perform what we already know" (Jansen, 2020, p. 2). The goal of our chapter is to *(re)orient* what we expected for participation before the pandemic and how we can trouble our assumptions and learning moving forward.

Lingering Inequities with Participation in Online-Only Formats

As we ourselves write this chapter in an online-only environment, we have not taken a naive stance towards online learning and rehumanizing practices that create safe spaces for students to participate. On the contrary, we still recognize realities of institutionalized racism, White supremacy, and unequal status that still persist in our online-only classrooms. Teaching to a screen full of blank squares that represent students in a Zoom Room can be jarring, even if students participate on Google

Slides, enter their thoughts in a chat box, and/or unmute their microphones to speak. In a few months of online-only courses, some MTEs like ourselves still have not seen the faces nor heard the voices of our students, *even while* the students say they feel comfortable participating in class and feel a part of the community of learners that the MTE has established and nurtured. We as MTEs need to resist the urge of normalized carceral pedagogies that demand participation by asking students to turn on their cameras and microphones. As MTEs, we must broaden our notions of what it means to participate online, recognize that we are a guest in our TCs' homes, and understand the many reasons why a TC may choose to have their cameras off and mics muted.

In the age of online-only learning, MTEs might need to balance at least three things: what participation used to look like before the pandemic (e.g., seeing faces, hearing voices, listening in to chatter), what participation looks like during online-only format, and what kind of classroom will continue to encourage a trusting, safe, healing space for my students to participate. For example, when we return to physical classrooms, what will we learn from the "mics off, cameras on" norm? Will we as MTEs encourage lively participation where students can talk over each other? Will MTEs create multiple spaces for students to communicate their thinking without only looking for raised hands or maintaining eye contact (e.g., writing down ideas on paper, recording thinking on a voice memo, typing ideas in Google Slides that everyone can edit in real time)? Will MTEs create humanizing spaces for students who have children and may have issues with childcare (e.g., classrooms that welcome our students' children, recording classroom sessions so that those who could not attend can still participate asynchronously)? In that sense, we likely will have to relinquish much of our pre-pandemic expectations and return to trust that our students are present, actively listening, and feel safe to share in ways that honor themselves as whole beings and rooted in the lives they live. One takeaway from what we have learned about participation in an online-only format and what it might look like when we return to physical classrooms is to not make assumptions that you know why students are participating based on what you are seeing and hearing from students.

5.5 Conclusion

For online learning, our chapter described rehumanizing mathematics instructional practices and ways of building community in the midst of the COVID-19 pandemic and afterwards when these inequities will likely persist. We considered participation in online spaces of mathematics courses that included assessing students' access to technology through surveys, examining what counts as participation, engaging in collaborative learning with CI, and disrupting harmful norms and deficit thinking. Instructors who represent various content areas (e.g., physics, mathematics, foreign language) should engage in rehumanizing practices, but *teacher educators* have a deeper responsibility for modeling these practices.

As MTEs, we draw attention to the distinction between the experiences and goals for the facilitation of mathematics teaching and learning across mathematics content and methods courses. As MTEs, the pedagogical moves we make towards rehumanizing practices in our content courses and methods courses have similar and different purposes. In content courses, MTEs can help students to see their ideas valued from the perspective of a learner of mathematics. In methods, MTEs are simultaneously focused upon having TCs value their own contributions and the contributions of their peers both from the perspective of a learner of mathematics but also as a future teacher of mathematics. Therefore, when MTEs model for future teachers the rehumanizing practices in their courses, there may be more potential for future teachers to experience a sense of belonging as a student and thereby employ similar practices with their future students, thereby continuing the promise of rehumanizing education.

As MTEs, we acknowledge the work on rehumanizing our mathematics courses was undergirded by notions of responsive reflexibility and anti-racist social and emotional learning practices (Simmons, 2019). As such, we can consider more questions about our practice as MTEs. How can we, as MTEs design online courses that practice responsive flexibility, communicate to students their presence matters, and provide space for social-emotional learning? How can we incorporate social-emotional learning practices that involve creating tasks and assignments that allow for multiple modalities for students to demonstrate learning? As aforementioned, mathematics serves as a gatekeeper for students at all levels of schooling, and MTEs should be concerned about how the design of mathematics courses, structures, policies, and practices communicate the importance of a product over people and product over process.

Throughout our chapter, we posed several questions to help guide MTEs in examining their structural and pedagogical approaches to online mathematics teaching. As MTEs engage in online learning, it will be important to maintain focus on the humanity of our students and of the process of learning new and complex ideas. As online learning continues, that does not mean issues of inequity and the balance of responsibilities end; we will still have children and families to care for, as well as social and financial responsibilities. Therefore, it would behoove us as a collective field to return to our syllabi and critically examine what we mean by rehumanizing mathematics learning in the online environment.

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Chapter 6
An Interdisciplinary Approach to Collaborative Professional Development for Teacher Educators: Number Talks as Culturally Responsive Online Teaching

Jeannette D. Alarcón, Jennifer B. Chauvot, Carrie S. Cutler, and Susie L. Gronseth

As the 2020 COVID-19 pandemic raged amid a summer of social unrest, educators moved their work from traditional, in-person teaching to remote online teaching. Like teacher educators across the globe (e.g., Ferdig et al. 2020; Kidd and Murray 2020), the authors of this chapter faced considerable uncertainty in the redesign of course assignments and delivery methods associated with the transition to online instruction during this challenging time. Specifically, the problems of practice for teacher educators may be viewed as twofold. First, teacher educators have generally thrived in traditional face-to-face settings and have understandably resisted moving to online spaces (Martin et al. 2019; Mills et al. 2009). We understand how our pedagogy works in face-to-face environments but are less certain about how to translate practices, maintain intention, and engage students in an online environment (Cutri et al. 2019). Second, considerations for practice go beyond our own teaching, as they also encompass equipping preservice teachers (PSTs) for how to manage a virtual instructional environment in which we, ourselves, may still be learning (Downing and Dyment 2013).

As the authors worked toward designing meaningful online courses, we also shared a reinvigorated awareness regarding persistent social and educational inequities, particularly in terms of equitable access to quality and appropriate tools for engaging diverse online learners successfully (Carrillo and Flores 2020). This awareness led to a recommitment to developing our understanding of culturally responsive teaching (CRT) as an available tool for several aims. First, it enabled us to direct attention to design moves that would facilitate equitable access in the course online learning environment. Second, we endeavored to model teaching strategies that exemplified shared power and authority in mathematical

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sense-making. Finally, we aimed to integrate instructional practices that would, in turn, equip our PSTs to be able to take up CRT practices. In this conceptual chapter, we share how planning together shaped a professional learning experience that resulted in greater attention to CRT in mathematics methods for PSTs and a deeper understanding of the connection between CRT and equity in education.

Collaborative planning for the teacher preparation coursework played an integral role in helping us, as teacher educators, understand CRT as a pathway to actualizing equity and justice in education (e.g., Gay 2013; Gutstein and Peterson 2013; Hammond 2015). Our aim for this work was to more fully integrate equity issues and CRT practices in the mathematics methods courses. To do this, we realized the importance of identifying the ways that each of us thought about and understood CRT.

When we formed the group, each member described their understanding of CRT within the context of teacher education and each of our disciplines. As the facilitator, Alarcón noted that there was little mention of the critical frameworks underpinning the various CRT descriptions. In fact, one reason this work is relevant for current teacher preparation programs lies in the fact that most faculty have adopted the term *culturally responsive teaching* to describe their pedagogy, yet few actualize much of a departure from traditional instructional and assessment practices. Features of critical pedagogy such as *dialogue, deconstructing power dynamics,* and *strategies for co-constructing knowledge* (Darder et al. 2017) were useful frameworks as the group established a focus for collaborative professional learning aimed at developing critical consciousness to build upon CRT practices. As such, the authors explored problems of practice together while intentionally moving toward what we describe as *critically conscious CRT* in online mathematics teacher preparation. The authors formed an interdisciplinary collegial professional development collective (referred to henceforth as "the Collective") to share ideas for course policies and practices, read and discuss related research and writings on equity in education, collaboratively write together, and ultimately support each other through planning and enacting CRT practices while teaching fully online courses during Fall 2020 amidst the challenges of the COVID-19 pandemic.

This is a timely issue and approach for teacher educator professional development for two reasons. First, the COVID-19 interim changes to teacher preparation programs are poised to persist for some time (Ellis et al. 2020). Second, we remain committed to designing coursework that helps PSTs experience and understand CRT grounded in critical pedagogy and contributes to the realization of equity in education. While there may be a return to face-to-face course offerings in the near future, the substantial investment in resources to support remote learning at this time prompts conversations at universities regarding possible expansions of fully online program offerings. The authors, thus, believe that in addition to learning to adapt current practices for an online environment in meaningful and student-responsive ways, critical questions about equity and access in teacher education are worthwhile to pursue.

6.1 Context

The Collective included three distinct areas of expertise in teacher education—mathematics education, instructional technology, and critical pedagogy for elementary education. The collegial journey began during the summer of 2020, as two members of the Collective (Alarcón and Chauvot) explored how to raise critical awareness as a component of fostering a culturally responsive learning environment with PSTs. We drew insights from Gay's (2013) work explicating critical pedagogy as a cornerstone for CRT and from Hammond's (2015) descriptions of rigorous CRT practices to inform our thinking about strategies we could employ in the methods courses so that PSTs could experience CRT. Given our student population and positioning of the university, we believe that greater attention to critically conscious CRT in teacher preparation will not only enhance PSTs' future practices for working with diverse student populations, it will also provide a pathway for thinking about how White teacher educators approach their own work with PSTs of color and international students.

Our university is an urban research university of about 47,000 students and is identified as a Hispanic-Serving Institution and Asian American, Native American, Pacific Islander-Serving Institution by the U.S. Department of Education. The teacher preparation program has an enrollment of about 1,100 students which offers 18 certifications, including elementary, middle grades, and secondary teaching in a variety of content areas. At the time of this writing, nearly 58% are seeking certification for early childhood through sixth grade (EC-6) (n = 444) or bilingual EC-6 (n = 106) certification. Almost 8% (n = 83) are seeking grades 4–8 mathematics certification. Student racial identification across the program includes Hispanic: 67%; White: 29%; Asian: 15%; Black: 9%; International: 2%; Multiple racial identity: 2%; and Unknown: 2%.

In consideration of the program's context, conversations within the Collective involved discussions about the importance of centralizing CRT as an outgrowth of the multicultural education movement spurred by the Civil Rights movement. In doing so, the group attended particularly to issues of equity and justice criteria for embodying CRT (Gay 2013). Though each of us came to the Collective with varying understandings of CRT as a concept, we strove to support each other in our development of a critically conscious understanding of CRT for the online environment in mathematics teacher preparation.

6.2 Focus

The chapter delves into the collective knowledge construction via an interdisciplinary collegial professional development approach and highlights how the work of the Collective heightened critical consciousness related to CRT. The work was conducted as part of course redesign efforts for the preparation of future mathematics teachers and within the context of a synchronous online course environment. In order to address one focus of this book, online instructional design and mathematics education, we identified a pedagogical strategy used in our mathematics education methods courses in the Fall 2020 semester–number talks. When the group convened in the summer of 2020, conversations mainly revolved around how to design mathematics methods courses so that they would attend to issues of equity and justice while also facilitating mathematics pedagogy learning. Simultaneously, the Collective wrestled with strategies for effective online instruction. As we supported each other's learning around CRT and online instruction, we realized the ways in which the features of a number talk could be used to model CRT. In doing so, we engaged in conversations about teaching the practice in the synchronous online setting while also challenging each other to foster tenets of CRT within the number talks teaching and debrief.

The conversations served to move each member of the Collective from their initial conceptual understandings to a more robust facility with the varied ways attending to CRT could be accomplished. Thus, this chapter relays our learning in action and how this approach to professional learning for teacher educators can be used as an effective intervention to support critically conscious CRT that maintains its roots in the multicultural education movement (Banks 1993). Working together to design number talks for the online learning environment, keeping reflective journals, and debriefing teaching experiences throughout the semester allowed the Collective to deepen understanding of what critically conscious CRT could look like for elementary and middle grades mathematics methods courses.

6.3 Number Talks

For the number talks strategy specifically, we found substantial alignment between its features and tenets of CRT, including increasing awareness of power and authority in classrooms and engaging students as mathematics sense-makers (Sun et al. 2018). Number talks in K-12 settings are brief, regular routines that teachers incorporate in ongoing instruction to support and build ownership in mathematical sense-making and mental computation (Parrish 2011). The key features of a number talk include the teacher's purposeful selection of a computation item (or string of related items), wait time to allow for individual student thinking about the computation item, students' private indication to the teacher when they are ready to share their strategies, public sharing of students' answers without evaluation, and public sharing of students' processes and strategies. Similar to the K-12 environment, number talks within mathematics methods courses provide rich opportunities for PSTs to engage in mathematical sense-making. At the same time, the number talks practice illustrates defined and productive teacher-student and student-student participation structures that ultimately support inclusive whole-class discussions around the varied strategies for thinking through mathematics. Because of this, number talks are often used as a hands-on activity for PSTs and mathematics teacher educators to collaboratively plan, carry out, and debrief children's mathematical thinking (AMTE 2017) when conducted during a student-teaching experience. Through number talks debriefs, teacher educators can engage in dialogue with future mathematics teachers about ways to foster flexible mathematical thinking and enact norms for meaningful mathematical discourse with a focus on equity (Bonner 2021).

6.4 Instructional Design Approach

The redesign decisions involved in the shift of course activities to online were constructed collegially within the Collective. In this way, the mathematics educators met with the instructional technology and critical pedagogy colleagues to share and discuss ideas. We then drafted materials for the lesson to bring back to the group. Finally, we strategized solutions to challenges as they emerged in the move of the teaching of number talks to a synchronous online format. This process exemplifies reciprocity as each teacher educator drew from her own expertise to make contributions. The Collective utilized a Microsoft Teams virtual space to facilitate the collaboration, including sharing documents, course material artifacts, and journal reflections and meeting through web conference to view and iterate on the developing instructional designs. We also communicated regularly via email to discuss questions around criticality and associated course policy.

To articulate the theoretical framing of this approach, the Collective drew inspiration from the Technological Pedagogical Content Knowledge (TPACK) model (Mishra and Koehler 2006). As illustrated in Fig. 6.1, the approach accounts for the integration of content knowledge (teaching of mathematics methods), pedagogical knowledge (enacting equity-focused CRT practices), and technological knowledge (synchronous online instructional technologies) involved in the planning and

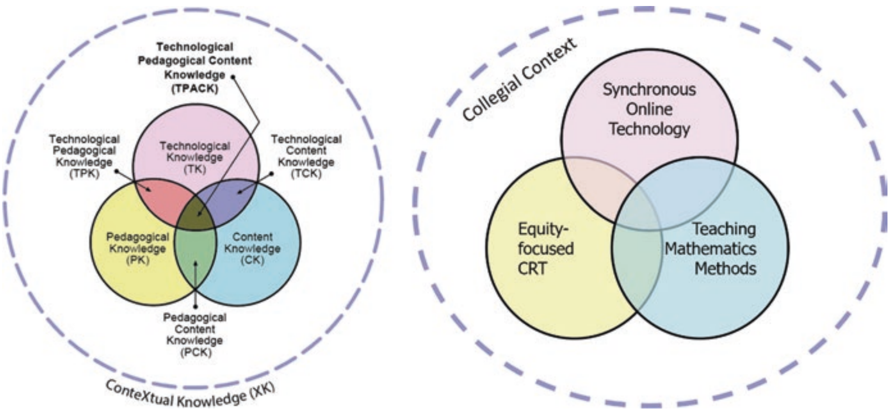

Fig. 6.1 Instructional design approach for reimagining number talks for synchronous online instruction. (*Note:* TPACK image reproduced by permission of the publisher, © 2012 by tpack.org)

implementation process. The design work is situated within the collegial context of the Collective, comprised of teacher educators from differing expertise areas who contributed to and supported the unfolding design.

Through this approach, we grappled with how the elements of the number talks might be enacted in the synchronous online format in ways that embodied tenets of criticality, specifically reflecting upon how shared power, equity, varied learner backgrounds, and responsiveness might be facilitated. We first landed on amplifying the transparency of the teacher educators' thinking so that the PSTs might engage in the activity from a simulated student perspective while also glimpsing the facilitation of the activity from an instructor's perspective. We came to find that planning for the number talks instruction cultivated multiple approaches and space for honoring a variety of sense-making processes. We strategized about the various ways that the activity could be conducted synchronously online and the associated mathematics content represented multimodally through text, imagery, video, and real-time virtual discussion. Finally, we considered aspects of student and instructor social presence and how these could be fostered through synchronous interactions via chat, poll, voice, and on-camera gesturing, which would be necessary for the PSTs to discretely indicate the number of strategies that they were prepared to share during the number talk enactment. Instructional technologies were explored for these purposes, including features of Blackboard, Microsoft Teams, Google Jamboard, and Nearpod.

6.5 Our Collective: Interdisciplinary Collegial Professional Development as Intervention

We present the approach taken through the interdisciplinary collegial professional development collective as a viable intervention for advancing teacher education course design and instructor professional learning. We call our work as a Collective an intervention approach because our design, decisions, and curricular implementation have been cultivated as a result of our participation in the group. Our Collective originated from a shared desire to support one another in improving outcomes for students during the stressful, and often confusing, period of the COVID-19 pandemic. As we thoughtfully planned how to teach our coursework in a virtual classroom the upcoming semester, we came together to work through syllabi- and assignment-level moves that would support CRT in our redesign. We first turned our attention to adjusting expectations for attendance, use of cameras during synchronous class sessions, and dealing with late work so that our policies in these areas would empower and support our learners, rather than be punitive. We then worked to frame the number talks assignment to prompt PSTs to view mathematics from an equity perspective. Next, we searched for ways to leverage PSTs' online synchronous learning about number talks into a co-constructive exercise. Finally, the Collective sought to (re)commit to CRT as a guide for decision-making within a collegial yet vulnerable space for professional learning. In this section, we explore applications of this approach in the realm of mathematics teacher education methods. In later sections of the chapter, we offer broad considerations for teacher education more generally.

The two mathematics methods faculty in the Collective had differing experiences with the number talks strategy. In prior iterations of her mathematics methods courses, Cutler had employed the number talks strategy using the norms of a traditional face-to-face classroom format. Chauvot had not enacted number talks, per se, in her teaching of middle grades methods courses, but she was motivated by the collegial environment established in the Collective to include the practice of number talks as part of her curriculum. It was after a group conversation in which Cutler had described the breadth of computation appropriate for a number talk that Chauvot realized this strategy could be applicable in the middle grades. The Collective began to consider ways that number talks might serve as a useful tool for addressing some of the gaps in mathematical knowledge that their PSTs typically had.

As we worked through preparations for synchronous online instruction, the Collective drew upon the pedagogical strategies, synchronous technology tools, and collegial design thinking to recreate the number talks lesson as a virtual learning experience. In Cutler's case, a key focus was on modifications needed for attending to community building in novel ways (e.g., emailing students who contributed to the success of class discussions, capitalizing on bonding moments like snapping and sharing Zoom selfies, and using breakout groups for get-to-know-you activities). Simultaneously, Chauvot, who had some prior experience teaching mathematics methods in an online environment, offered her expertise with technology tools that could aid its goals. In addition, Chauvot had taught this group of students in a face-to-face environment during the Fall 2019 semester. The instructional technology and critical pedagogy colleagues contributed to these Collective conversations by offering ideas for maintaining student privacy for initial responses, elevating student voice equitably through the sharing of their thinking in visual virtual ways, and facilitating meaningful debriefs to obtain student perspectives about their online learning experiences as well as the number talk strategy itself. Table 6.1 highlights the ways in which resulting enactments of the number talks by each mathematics teacher educator demonstrated more intentional connectedness with CRT.

6.6 Enactments of Number Talks as Outcomes

In this section, the mathematics teacher educators describe their approaches to teaching number talks in the online environment while considering how to draw attention to CRT practices. The descriptions serve as examples of outcomes from the Collective's ongoing collegial professional development.

Table 6.1 Strategic connections to CRT with the synchronous online number talks

	Traditional process	Technology-enriched enactment (Chauvot)	Modified face-to-face enactment (Cutler)	Connections to CRT
Wait time and private signal to the instructor	Students quietly hold their thumb against the chest, adding a finger for each additional strategy they find.	Preservice teachers (PSTs) took a Nearpod poll to indicate readiness. The instructor used the "show student names" feature.	PSTs used the chat feature to private message the instructor to indicate readiness.	Private signal removes social pressure to answer quickly and promotes divergent thinking.
Sharing of answers	The teacher accepts and records all answers on board.	PSTs contributed to a Collaborate Board in Nearpod. All saw "sticky notes" of answers with student names.	The instructor recorded all answers on paper and projected via a document camera.	All answers are given equal value in the class discussion.
Supports for equitable participation	Students participate in a think-pair-share structure where they first think independently then turn and talk to a peer.	The instructor monitored PSTs' contributions to the poll and Collaborate Board.	The instructor placed PSTs briefly in breakout groups with 2–3 peers to facilitate a think-pair-share format.	Teacher values divergent sense making and co-construction of knowledge.
Recording student thinking	The teacher provides clear, concise scribing on whiteboard or chart paper to capture students' strategies.	PSTs used Google Jamboard to explain and display their thinking. The instructor called for volunteers for as long as time permitted.	The instructor scribed PSTs' answers and strategies on a sheet of paper and projected via a document camera.	Teacher demonstrates that students' ideas are the currency of the mathematics classroom, honoring shared responsibility and power for making sense of mathematics.
Orienting students to one another	The teacher encourages students to listen attentively while others explain strategies and use hand signals to indicate if a peer's strategy matches their own.	Instructor modeled discourse moves by calling on volunteers to re-voice or comment on a classmate's solution.	Instructor asked students to turn on cameras and use hand signals to indicate if a peer's strategy matched their own.	Students recognize that there are divergent ways to solve a problem.

6.6.1 Modified Face-to-Face Enactment (Cutler)

Cutler designed a lesson using the 5E lesson format (*Engage, Explore, Explain, Elaborate, Evaluate*) to introduce PSTs to number talks, supporting a course-long goal of promoting inquiry-based learning in mathematics. Hence, the number talk lesson allowed Cutler to model a pedagogy where the power to build mathematical knowledge and knowledge for teaching did not lie solely with the instructor, a belief that aligns with CRT. An overview of the lesson will be described (follow this link for the chapter Appendix that includes complete lesson plans: https://www.carriecutler.com/research-and-scholarship).

To *Engage,* PSTs worked asynchronously and individually during the week preceding the Zoom class meeting to complete two number talk experiences. First, PSTs were offered choice for how they studied number talks by choosing to view a 30-min video or read a journal article. Next, PSTs watched a video clip of a kindergarten class engaging in a number talk. They used a T-chart to organize what they *Noticed* and *Wondered* during the video, experiencing an inquiry-based, equity-focused teaching strategy wherein all learners could contribute to the group discussion. Completing the *Engage* portion of the 5E lesson prior to the Zoom class meeting built PSTs' background knowledge for number talks and provided a common experience for the whole class to discuss during the Zoom class meeting.

During the synchronous Zoom class meeting, PSTs participated in the *Explore, Explain,* and *Elaborate* portions of the lesson. For the *Explore,* PSTs worked in Zoom breakout groups to apply their Noticing and Wondering from the *Engage* portion of the lesson and create a definition of a number talk. Cutler joined each breakout group briefly to ask questions that promoted understanding of the purpose and structure of number talks. The *Explain* portion brought the PSTs together for a whole-group discussion of the breakout groups' definitions of a number talk. Cutler used prepared PowerPoint slides to guide a discussion of how number talks support computational fluency and mental math.

The *Elaborate* portion of the lesson advanced PSTs' understanding of number talks with three additional video examples of number talks conducted with elementary school children. Cutler told the PSTs that they would be planning and carrying out a number talk as part of a course assignment; therefore, they should watch the videos with an eye to classroom norms. During the first video, the PSTs took notes on the teacher's role and the students' role during a number talk and discussed their ideas with the whole group. Between videos, Cutler guided discussion about steps to implementing number talks, including the teacher's role as facilitator, scribe, and questioner. A full list of the discussion points can be found in the Appendix. The *Evaluate* portion of the lesson challenged PSTs to work individually to prepare and conduct a number talk in their field placement classrooms. The assignment guidelines were provided to the students in the course syllabus and can be found in the Appendix.

6.6.2 Technology-Enriched Enactment (Chauvot)

Chauvot had not previously used number talks specifically in her teaching of grades 4–8 mathematics methods courses. For the past decade, most of her teaching had been online at the graduate level with in-service mathematics and science teachers. Fall 2020 was her first experience teaching undergraduate students in an online environment. The specific details of her enactment are also in the Appendix.

As Cutler shared about her number talk lesson plans with the Collective, Chauvot challenged herself to expand upon the technological tools that could facilitate the features of the number talk. Furthermore, since her goals for the activity centered on teaching about a pedagogical practice, she intentionally chose to center her number talk on a mathematical task the PSTs had explored in an earlier course in the program.

Earlier in the semester, her class had read and discussed Kalinec-Craig and Robles (2020) work-around *Rights of the Learner (RotL)*, where it is argued that "[t]eachers who commit to the RotL also empower more students to see themselves as valuable contributors in the classroom" (p. 469). The reading explains that the four rights are that learners have the right "to be confused; to claim a mistake; to speak, listen, and be heard; and to write, do, and represent what makes sense to them" (p. 469). The class considered an additional reading in preparation for the number talk synchronous session, Herbel-Eisenmann and Shah (2019), which focused on implicit bias, teacher questioning, and discourse moves. They engaged in a synchronous Nearpod lesson that used the Poll feature, the Matching Pairs feature, and the Collaborate Board feature to review and discuss the main ideas of the reading.

In her enactment, Chauvot focused on the main number talk features (e.g., choosing a task that had multiple approaches and multiple correct answers, private signaling to the teacher, public sharing of strategies, and discourse moves). Within the implementation of the activity, she chose to toggle back and forth between the roles of mathematics teacher and mathematics teacher educator. She "partially" modeled each implementation phase of the number talk as a mathematics teacher might, and then switched to her teacher educator role to direct explicit attention to the feature that she was aiming to illustrate. Chauvot's corresponding debrief activity consisted of a jigsaw-like homework assignment wherein groups of students were responsible for familiarizing themselves with different portions of content and collaboratively prepare a presentation for the whole class. For a complete list of readings and presentation guidelines, see the Appendix.

6.7 Deepening Criticality Via Distributed Expertise

We believe that the strength in this interdisciplinary professional development project came from the distributed expertise present in the Collective. The Collective draws from a critical friend framework to engage in facilitated dialogue around considering how to address issues of equity, access, and opportunity in an online learning format (Fahey and Ippolito 2015). Fahey and Ippolito's work attends to the importance of educators engaging in purposeful dialogue with each other as professional development.

Engaging in this type of dialogic professional learning required establishing a trusting environment. We did this by intentionally defining our goals for participating in the Collective. Establishing group norms and meeting structures aided trust-building because they provided parameters that we could rely upon during uncomfortable or challenging learning moments. The Collective used reflective dialogue to surface the ways that our own investments in traditional student and teacher roles led to enacted punitive or controlling mechanisms within current and past course design and implementation. We came to understand the varied ways that these can often be contradictory to the goals of CRT. We then shifted our focus toward our teaching practices within the methods courses. Alarcón and Chauvot frequently discussed pathways for attending to issues of equity, access, and diversity in the methods courses and how to enact these effectively when students viewed course content as taken-for-granted knowledge.

Number talks provided the Collective with a concrete example of an evidence-based practice that could be leveraged toward CRT aims through design decisions that resulted in building PST capacity for listening across differences and coming to mutual understanding (Gay 2018). The Collective worked to embolden their knowledge and skill in CRT to then apply it to the reenvisioning of the number talks lessons in synchronous online delivery. In order to do this, the group engaged in a variety of activities that went well beyond simply planning for the number talk highlighted in this chapter. Drawing on Alarcón's expertise with integrating CRT within content-based methods courses, we established group commitments to create the space of vulnerability required for openly discussing misconceptions and new learning. We reflectively journaled during planning phases so that we could keep track of pressing questions and discuss them during our meetings. Also, to practice facilitating difficult dialogue, we engaged in a book study. While these activities were not directly linked to implementing number talks, they were integral to establishing our professional learning relationships with each other. As the discussions turned more pointedly toward course planning, Gronseth's instructional technology knowledge became a useful resource for prompting the group to consider differing ways that we could attend to equity in online environments, particularly regarding our expectations and scaffolds for student participation.

During the Fall 2020 semester, the Collective noticed how our approaches to addressing issues that arose during our synchronous online teaching were being impacted by the group's ongoing dialogue about CRT. For example, instead of trying to devise ways to coax student compliance, we began to consider the potential barriers students were facing during the semester. This shift was notable because it revealed a deeper connection between student well-being and course expectations. We began to connect via email in addition to our meetings to brainstorm about these types of issues together. The email exchanges often evidenced how our understandings were moving beyond thinking about CRT as a set of instructional strategies to considering how CRT could help expose the more deeply entrenched systems of inequity present within the structures of teacher preparation programs (e.g., expectations for use of video cameras, participation in synchronous meetings, and completing pre-work). The following example illustrates one of these learning moments.

In an email to the Collective, Cutler expressed concern about evidence that many of the PSTs in her course did not complete the number talks prework before attending the synchronous meeting in which number talks would be the focus. She felt that this lessened the experience and the level to which her students could engage in dialogue about their understandings of number talks. In the email exchange that followed, the Collective brainstormed possible solutions. We noticed during this exchange that none of us suggested implementing an accountability measure such as a reading quiz or completion grade. Instead, our discussion revolved around the myriad reasons that students may not have completed the work (i.e., confusion, unfamiliarity with the format) and the purposes behind assigning the prework in the first place. Ultimately, both mathematics educators decided to use an anonymous poll at the beginning of the synchronous session to gauge students' level of preparation.

We see this as a small but significant shift toward an embodiment of CRT in a few ways. First, because we had framed CRT through a critical pedagogy lens, we were able to recognize and consider the PSTs' perspectives as valid. Next, the Collective structure evidenced the power of dialogue for co-constructing new ways of understanding taken-for-granted practices such as when an instructor might leverage her power to force compliance via punitive punishment. Finally, the Collective realized the complexities involved in (re)committing to CRT as praxis and created a space for vulnerable professional learning. The authors do not claim to have engendered a full realization of critically conscious CRT in mathematics methods courses but are continuing to do the reflective work necessary for making informed changes in our practices and programming. In the next section, we discuss implications for teacher education more broadly in terms of peer-to-peer professional learning about CRT from a critical perspective for mathematics methods teaching.

6.8 Implications for Teacher Education

6.8.1 Creating Spaces for Professional Learning

From what we have learned through this work, we advocate for the creation of such professional learning spaces wherein teacher educators from varying disciplines may come together to learn more about the critically conscious CRT as applied to our specializations. Doing so holds the potential for leveraging existing practices used in teacher education to further expand aims of increasing equitable opportunities in classrooms. Taking up a more critically conscious form of CRT helps teacher educators learn to bring important discussions about power dynamics and knowledge construction in education into teacher preparation courses. Thus, the aim of this chapter was to share viable approaches for such discussions while also drawing attention to the group's strategic and emergent design steps. Engaging in significant course revision can result in teaching conundrums that are difficult to answer in isolation; and as such, the support of the interdisciplinary collegial professional development collective was key to our learning. Cutler noted that support from the Collective mitigated feelings of isolation caused by the pandemic. Chauvot felt that she was gaining a more nuanced understanding of CRT because she was seeing the different ways each member of the Collective was enacting CRT in their courses. Alarcón gained deeper insight into engendering CRT in mathematics education methods courses, while Gronseth felt her involvement in the Collective enabled her to innovate novel applications of online learning technologies in ways that were more human-centered and inclusive.

6.8.2 Mathematics Methods Instruction

Because of the rapidly developing needs associated with pandemic-related impacts, the authors experienced at the time of this work increased instances of having to develop expectations for the courses in real time. In a sense, discussions about CRT that took place during course planning created deeper understandings regarding the significance of strategically designed flexibility for meeting diverse learner needs. Further, we were forced to reconsider routines and procedures beyond building pedagogical content knowledge for mathematics. Such rethinking included reflecting upon our own motives to adjust our expectations for what we thought engagement would look and sound like during synchronous class sessions. For example, as planning began, most of the group could justify requesting that students turn on their cameras during synchronous meetings. However, as we continued to unpack this practice, the stance shifted toward PST autonomy when deciding when/if to use their cameras.

Another implication involves the debrief structuring for the number talk strategy in the synchronous online learning environment. The Collective reflected on how the number talk debriefs in a traditional face-to-face mathematics teacher education course could aim to centralize the revealing of the steps that teachers take to plan and implement this pedagogical practice. For the online environment, the Collective attended to the additional consideration of managing time and engagement in differing ways. In doing so, the Collective focused on equitable strategies for engaging PSTs in a debrief that allowed for multiple representations of understanding, such as via a Jamboard. Through a centering of critical consciousness in the course redesign efforts, the potential for using the debrief as a space for addressing equity, power, access, and multiple perspectives with PSTs was revealed.

6.9 Conclusion

Our work in the Collective spurred conversations around power and authority in teacher preparation methods virtual classrooms that led us to rethink course policies, assignments, and approaches to online instructional delivery. We remain engaged in the co-construction of a knowledge base that contributes to the notion of critically conscious CRT for mathematics teacher education integrated with meaningful instructional design approaches. Developing course revisions is ongoing as we continue to learn more about embodying CRT in methods courses for PSTs. Once the Fall 2020 semester ended, the Collective continued to meet to discuss how we could use our new knowledge to plan for number talks in the Spring 2021 semester. Currently, we are engaged in a study of using number talks in a simulated environment as a way to elicit student thinking and then debrief with PSTs about decision-making.

The iterative nature of teacher educators' work facilitates continual reflection and dialogue around issues of equity, access, and knowledge building as we reimagine instruction in an online learning environment. As mentioned early in the chapter, we see evidence that the expectations for online teaching accelerated during the COVID-19 pandemic will remain with us for some time. As such, we feel it is important for teacher educators to develop skills for integrating CRT principles with instructional design for online learning. Each semester provides a fresh opportunity to make purposeful adjustments and strengthen the ways we address issues of power, authority, equity, and access in mathematics methods courses. The hope in sharing the Collective's work is to inspire critical dialogue among teacher educators who are working to engage in revision, improvement, and collaboration in their own teacher preparation programs.

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Chapter 7
Culturally Sustaining Pedagogy for Emergent Bilinguals in a Teacher Education Online Course

Ji-Yeong I, Ricardo Martinez, and Christa Jackson

7.1 Introduction

Despite the rapid increase of the population of emergent bilinguals[1] (EBs; a.k.a. English language learners; ELLs) and the importance of providing teachers with content-specific training to teach EBs (Vomvoridi-Ivanovic and Chval 2014), teacher preparation programs in the United States have yet to adequately prepare teachers to effectively work with linguistically and culturally diverse students (Education Commission of the State 2019). A sufficient body of research (e.g., Fernandes 2012; I 2019) have shown that when preservice teachers received EB-focused interventions, they demonstrated an increase of effective teaching practices and asset-based perspectives towards EBs. Research supports the need for teacher preparation programs to offer a course that provides preservice teachers

[1] We chose to use Emergent Bilinguals (EBs), instead of ELLs or ELs, because it conveys an intellectual image of bilingualism and shines the linguistic asset that these students already have. Moreover, this term emphasizes our educational goal is not making EBs mere English speakers, but proficient bi/multilinguals by sustaining their native language.

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with rich opportunities to learn and experience EB-focused teaching strategies. Responding to the necessity of holistically equipped mathematics teachers to effectively teach EBs, we designed a 100% asynchronous online course that provides various learning opportunities for preservice and in-service teachers to learn research-based, EB-focused teaching approaches to teach mathematics. Although many online and offline courses have been offered that include strategies for teaching EBs, most are not content-specific. Consequently, mathematics teachers struggle when teaching EBs because the role of language and culture in mathematics is often not visible.

The online course we developed satisfies the needs of mathematics teachers because it focuses on mathematics, EBs, and teacher education in tandem. Since the online course *Teaching Math to English Language Learners* was first offered in 2017 through Canvas.net, a massive open online course system, it has been well received by both mathematics teachers and English-as-a-second-language teachers in and outside the United States, and the student enrollment has significantly increased each semester. In this chapter, we describe how the theoretical perspectives of culturally relevant and culturally sustaining pedagogy guided the online course design and how these perspectives further transformed the revised online course curriculum.

7.2 From Culturally Relevant Pedagogy to Culturally Sustaining Pedagogy

In this section, we describe the two theoretical frameworks of the online course design, Culturally Relevant Pedagogy and Culturally Sustaining Pedagogy, and the ways they support teaching mathematics to EBs.

Culturally Relevant Pedagogy, theorized by Gloria Ladson-Billings (1995), focused on three tenets: (1) students' academic achievement; (2) students' cultural competence (maintaining their cultural history while gaining access to the dominant culture); and (3) students' developing an understanding and critique of societal norms, which requires critical reflection in guiding action. Researchers (e.g., Aguirre and del Rosario Zavala 2013; Ahn et al. 2015) have argued Culturally Relevant Pedagogy is an effective teaching approach for EBs' mathematical learning. For instance, Ahn et al. (2015) examined the culturally relevant teaching practices in mathematics classrooms of culturally and linguistically diverse groups in the United States and Japan. Their findings reveal the teaching practices in both the United States and Japan share similar pedagogical methods, such as using everyday language prior to academic language, integrating literacy throughout instruction, and cultivating critical thinking through mathematical discussions. In addition, Aguirre and del Rosario Zavala (2013) argue culturally responsive mathematics teaching is essential to improve students' learning in mathematics. In their study, they introduced a framework that teachers could use to evaluate their mathematics lessons to ensure they are meeting the needs of the diverse students in their classrooms. The framework consisted of six categories: (1) cognitive demand, (2) depth of knowledge and student understanding, (3) mathematical discourse, (4) power and participation, (5) academic language support for ELL[EBs], and (6) cultural/community-based funds of knowledge. This framework was further developed by I et al. (2019) to investigate elementary teachers' perspectives and practices with EBs in South Korea and I and Son (2019) to examine mathematics preservice teachers' lesson planning for EBs.

Culturally Sustaining Pedagogy extends Culturally Relevant Pedagogy by explicitly emphasizing pluralistic societies. Paris (2012) asserted,

> The term *culturally sustaining* requires that our pedagogies be more than responsive of or relevant to the cultural experiences and practices of young people—it requires that they support young people in sustaining the cultural and linguistic competence of their communities while simultaneously offering access to dominant cultural competence (p. 95).

Culturally Sustaining Pedagogy views students' cultural and linguistic assets as the center of the classroom environment. Hence, teachers should focus on building a multicultural and multilingual class environment rather than intermittently adding a few cultural components to a dominantly English-monolingual mainstream class. In culturally sustaining classrooms, all students are valuable members who make meaningful contributions.

Culturally Sustaining Pedagogy is also theoretically grounded in Funds of Knowledge and Third Space. Funds of Knowledge is "the historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being" (González et al. 2005, p. 133). For EBs, actively integrating Funds of Knowledge into instruction and classroom environments is crucial because they experience significant linguistic and cultural differences between home and school. Similarly, Third Space is a "hybrid space created when classroom members bring together elements of school culture and home culture to create something new" (Carlone and Johnson 2012, p. 155). For EBs, an English-only class or school can be the first space. Their home, where their first language is dominantly used, can be the second space. Subsequently, a Third Space can be created by combining and interacting with both first and second spaces. A classroom built as a Third Space allows both the teacher and students to creatively explore Funds of Knowledge from multiple cultures (González et al. 2005). Jobe and Coles-Ritchie (2016) applied the concept of Third Space to teaching EBs and explained Third Space is beneficial to both EBs and non-EBs for not only their learning but also their overall social experience. It is in this Third Space that EBs can comfortably keep their identities, and non-EBs can learn to engage within the space without dominance. Thus, Third Space does not exclude anyone. Instead, it values all students' learning.

To create a Third Space for EBs, enacting translanguaging approaches is essential. Translanguaging is a natural way bi/multilingual use language in which two or more languages interact to perform the best meaning-making within their language system (García and Lin 2017). For example, fluent Spanish-English bilinguals may mix two languages in one sentence to deliver the intended meaning more efficiently and effectively, although using only one language in the sentence is not difficult for them. Using two languages in one sentence does not mean a lack of language acquisition but fluency in two languages. Consequently, teachers do not need to insist that EBs use only one language when speaking or writing. If teachers force EBs to use only one language, it denies EBs of their whole language system and rejects part of their bi/multilingual identities and linguistic assets (I et al. 2020a, b). Translanguaging approaches do not reduce language demand but provide a rich and safe language environment. By valuing and enacting translanguaging, teachers value not only students' culture and language but also their identities as bi/multilingual. It is worth noting translanguaging is an effective way to enact Culturally Sustaining Pedagogy. Guadalupe Valdés noted in the foreward of the book, *The Translanguaging Classroom: Leveraging Student Bilingualism for Learning* (García et al. 2017), "[translanguaging is] by far the most compelling example proposed to date of a culturally sustaining pedagogy" (p. vii). Guadalupe also explained, "It explicitly takes the position [of Culturally Sustaining Pedagogy] that past scholarship on language has misunderstood the nature of bilingualism and bilingual practices. It insists that students be invited to foster, maintain, and develop their complex repertoires" (p. vii).

Culturally Sustaining Pedagogy actively supports EBs, who need to learn English while ideally maintaining their native language and heritage culture. EBs and immigrant students who live in two different worlds (school and home) often hide their true identities at school, because they know parts of their identities are not accepted or valued. Culturally Sustaining Pedagogy creates spaces that foster linguistic and cultural openness by viewing students' cultural identities as assets to be incorporated into the classroom (Paris 2012), where EBs do not have to hide or detach from their identities.

In sum, Culturally Sustaining Pedagogy builds off of Culturally Relevant Pedagogy by explicitly centering language as a resource and the relationship between the historical-cultural world and the action of students in dreaming and creating the future. Culturally Sustaining Pedagogy also aligns with recent research in mathematics education to rehumanize mathematics (Gutiérrez 2018) by explicitly paying attention to how EBs do and do not participate when learning mathematics. In the subsequent sections, we describe how Culturally Sustaining Pedagogy was foundational in designing our online course.

7.3 Online Course Development

In this section, we briefly describe our journey of course development, the revision, and how prior research and theoretical frameworks informed this process (see Table 7.1). The theoretical underpinnings of the online course began with an examination of the work with epistemic roots in bilingual and multicultural education within mathematics education with respect to EBs (see Aguirre and Bunch 2012;

Table 7.1 Guiding research categorized by topics

Topic	Informed course design	Core literatures
Guiding principles for EBs	Initial design based on research and guiding principles	Celedon-Pattichis and Ramirez (2012), Moschkovich (2010)
EB-focused strategy/ pedagogy	Initial design connecting to practice in mathematics	Chval and Chávez (2012)
Language	Initial design borrowed from the SIOP model the importance of a language goal	Echevarria et al. (2004)
	Both designs emphasized the importance of language demands in mathematics education	Aguirre and Bunch (2012)
Bilingual education	Initial design shifted from ELL to EB	García and Kleifgen (2010)
	Redesign dove deeper into bilingual education by reflecting on the language and culture of teaching	Nieto (2009)
Culturally sustaining/ relevant pedagogy	Initial design highly influenced by culturally relevant pedagogy	Ladson-Billings (1995)
	Redesign expanded to culturally sustaining pedagogy	Paris (2012)
Third space	Redesign related to culturally sustaining pedagogy	Gutiérrez (2008)
Funds of knowledge	Initial design mentioned funds of knowledge which were deeply explored in the redesign as it relates to culturally sustaining pedagogy	González et al. (2005), Civil (1994)
Translanguaging	Redesign pulled from bilingual education in viewing multiple languages as a resource	García et al. (2017), Fu et al. (2019)
Community cultural wealth	Redesign added more asset-based framing from the classroom to the community and what we can learn in mathematics education from the community and why we should.	Yosso (2005), Civil (2014)

Chval and Chávez 2012; Celedon-Pattichis and Ramirez 2012; Moschkovich 2010). Unpacking multiple asset-based theories, Culturally Sustaining Pedagogy acts as an anchor to bring these ideas to the forefront of the course.

The need for a course focusing on mathematics and EBs emerged as we conducted a needs analysis (Peterson 2003), where we noticed no such class was offered in the United States or online through the Canvas network. We leveraged our expertise in EB-focused mathematics education, online learning, and equity to design a research-based online course that brings theories into practice.

We initially developed the online course with two primary goals: research-based and practitioner-friendly. We did not want to provide general principles for teachers and leave them to unwittingly apply them with their EBs during mathematics lessons. Consequently, we purposefully chose to situate the modules in Culturally Relevant Pedagogy (Ladson-Billings 1995) and incorporated the Culturally Responsive Mathematics Teaching Lesson Analysis Tool (CRMT-LA Tool, Aguirre and del Rosario Zavala 2013). The online course was designed to (1) provide teachers with research-based strategies for teaching mathematics to EBs (Celedon-Pattichis and Ramirez 2012) and (2) support teachers to shift their perspectives to more asset-based views towards EBs (Moschkovich 2010) in the teaching and learning of mathematics. The initial online course consisted of six modules (see Fig. 7.1) and used the textbook, *Beyond Good Teaching* (Celedon-Pattichis and Ramirez 2012), which provided research-based teaching approaches to support EBs during mathematics lessons.

Within each online module, we included readings, videos, discussion forums, and an exit assignment, which were designed to be applicable to the course takers' classrooms. All the videos lasted between 3 and 10 min to keep watchers' attention and included reflective discussion prompts that were embedded in each video using Vizia (vizia.co) so that teachers can type their responses to the prompts within the interactive videos. We used a few relevant YouTube videos but mostly created our own videos based on the PowerPoint slides and the instructors' verbal explanations. Separate from the Vizia discussion prompts, we included two online discussion forums per module, designed where the teachers submitted their initial posts before viewing other teachers' posts. We formatted the discussion forums in this way so that all teachers were able to share their genuine ideas without getting influenced by the dominant responses posted by other teachers.

After each semester the course was offered, the development team analyzed the impact of the course on teacher learning. Despite teachers' positive shift in their perspective towards EBs' mathematical learning (I et al. 2020a), we realized some teachers were either hesitant or resistant to include EBs' cultural knowledge and language in their classes. The results of both pre-and post-surveys indicated a clear pattern of resistance to sharing EBs' culture and language in class. For example, we asked the teachers to *explain why you will have EBs share their home culture with the entire class,* four of the nine in-service teachers said, "It depends on the students." Another in-service teacher stated, "Kids are given the opportunity to share, but some of them do not wish, too. Some are fleeing areas of trauma and don't want to discuss it or are embarrassed about their history." Unfortunately, it appears this teacher assumed EBs' history or home culture as embarrassing and failed to mention how they would encourage EBs to share their culture. Several teachers were afraid to have EBs share their home culture because they assumed it consisted of "trauma" or an "embarrassing history." Unfortunately, this reifies a deficit view towards EBs' cultures and histories. To compound the issue, some EBs may have inferior views of their own culture because of how society views them. Furthermore, EBs may be more afraid of their peers' negative reactions and bullies than sharing their culture with others. Hence, it was necessary to refine and revise the online course, by adding content that specifically focused on valuing EBs' culture,

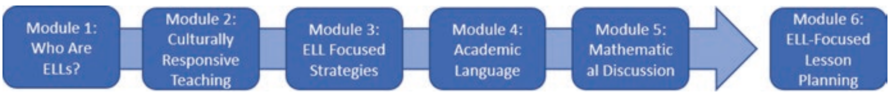

Fig. 7.1 Initial course design

language, and identities in mathematics classrooms and emphasizing the importance of the teacher's role that helps EBs reclaim their cultural/ethnical identity within the classroom with a positive image.

In the course redesign, we strengthened the theoretical framework to include Culturally Sustaining Pedagogy (Paris 2012) as a theoretical lens because it explicitly positions students' cultures and languages as the center of a class and points out how current schooling is damaging students' multiple identities. We also built on the premises of Funds of Knowledge (Civil 1994; Moll et al. 1992) and Third Space (Gutiérrez 2008). While the background of Culturally Sustaining Pedagogy is introduced in Module 2 of the revised online course, the philosophy of Culturally Sustaining Pedagogy is embedded throughout each module (see Fig. 7.2). For example, Module 3 introduces research-based teaching strategies for EBs that are aligned with Culturally Sustaining Pedagogy. Accordingly, the modules on translanguaging (García and Lin 2017) and Community Cultural Wealth (Yosso 2005) align to Culturally Sustaining Pedagogy because each centers EBs by integrating their common language practices and utilizing the totality of their social and cultural resources.

When EBs feel their culture and language are not respected or valued they will be less likely to speak and lose key practice speaking in both (or multiple) languages. To counter deficit views of culture and language and in turn position language as a resource, the online course capitalizes on translanguaging pedagogy (García et al. 2017) to honor the multiplicity of language in connecting to the wide range of students' identities. Translanguaging paired with Funds of Knowledge (González et al. 2005), which are historically developed aspects of students' household identities, becomes a catalyst for rehumanizing EBs in the mathematics classroom. The situated nature of Funds of Knowledge transcends classroom learning by connecting to household knowledge construction. Additionally, utilizing students' Funds of Knowledge tells EBs and all students that they and their culture are welcome in the class.

Special attention is given to the role of dialogue in the classroom in creating a problem-posing education (Freire 1996) that views students as capable of contributing to the knowledge construction of the classroom. A problem-posing education encourages dialogue between students and between students and teachers that connects learning to the real world. Furthermore, humanizing education for EBs through problem-posing begins with a classroom space where students are welcomed to ask questions and reply to the questions in any language.

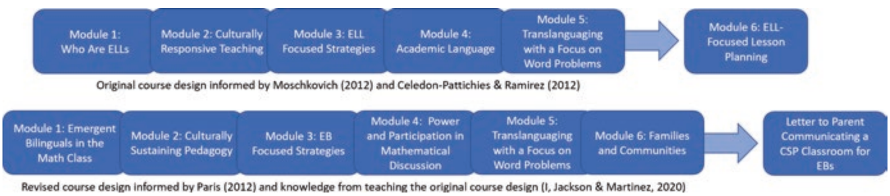

Fig. 7.2 Revised online course design

For EBs, asking a question and being in dialogue with others are opportunities to practice language that simultaneously can be leveraged to build mathematical knowledge. However, dialogue limited to the classroom creates a divide between learning that occurred during normal mathematics class time and learning that occurred in the community. Culture and community are important for EBs with respect to their everyday being, where Funds of Knowledge should not be viewed as a direct flow from the household to the classroom. Viewing community through Community Cultural Wealth (Yosso 2005) further highlights the knowledge that students come from and in turn, how classroom-constructed knowledge should return to the community. A culturally sustaining classroom for EBs can inherently bridge theory to practice, classroom to community, history to future, and mathematics to language in centering the voices of EBs in the teaching and learning of mathematics.

7.4 Practices of Culturally Sustaining Pedagogy

For this online course, we designed the content in a meaningful way for teachers. Rather than providing only theoretical and general principles, we included ample opportunities to explore research-based principles and strategies in actual classrooms with real EBs throughout the online course. In this section, we describe several examples of how the Culturally Sustaining Pedagogy framework was implemented in practical assignments within the online course.

7.4.1 Mathematical Modeling

Culturally Sustaining Pedagogy emphasizes how important it is to have high expectations for all students and integrate their Funds of Knowledge with connections to their community and world. Celedon-Pattichis and Ramirez (2012) also assured that EBs must have challenging mathematics regardless of their English proficiency. Based on the research-based recommendations, we included mathematical modeling as one of the strategies that provide ample learning opportunities for EBs. We introduced the 5 Act Task, which we modified from the 3 Act Task (Meyer 2011) as an example of mathematical modeling that integrates real-world contexts students are familiar with (I et al. 2020a, b). The initial design of the 3 Act Task has 3 steps: (1) share a conflicting story and students pose a problem to resolve the conflict, (2) look for information and make assumptions, and (3) solve the problem and do extension. The student-driven nature of the 3 Act Task has the potential to increase students' cognitive demand, engagement, and ownership of knowledge. However, the 3 Act Task may not engage EBs because the given story may include unknown language or unfamiliar contexts. To increase EBs' access to the 3 Act Task, we added two components to the 3 Act Task: Act 0 and Act 4 with emphasis on a non-linear interactive modeling cycle. Act 0 provides a set-up stage (I and de Araujo 2019) where teachers assess and scaffold EBs' understanding of the language and context to be embedded in Act 1. Act 0 benefits all students by helping them better comprehend the problem situation, but it is essential for EBs because unknown words or cultural biases may prevent them from finding mathematical entry points. Act 0 is an opportunity to tap into students' Funds of Knowledge in creating a positive learning space for EBs. Act 4 is designed to overcome another common challenge of Ebs – discourse. In Act 4, students discuss and present their reasoning and solution pathways, which is a crucial step to deepen mathematical understanding while developing language. If EBs cannot participate in the discussion due to language barriers, they lose the crucial learning opportunity. Hence, it is important for teachers to support and encourage EBs to share their mathematical ideas and reasoning in a safe setting in Act 4. Act 4 can then connect to the same Funds of Knowledge that were cultivated in Act 0. Both Act 0 and Act 4 are welcoming beacons for EBs by creating a complete learning experience that honors their culture, language, and community. In addition, to reduce EBs' anxiety of speaking English and maximize their learning, teachers can apply various strategies such as presenting in groups/pairs, using visuals and gestures, and providing sentence frames. In Table 7.2, we describe each Act in the 5 Act Task.

After we provided explanations and examples of the 5 Act Task in the online course, the teachers were asked to review the 3 Act Task and modify it to 5 Act Task for EBs. Through this assignment, the course takers explore various 3 Act Task available online and use the strategies they gained in the module to modify the 3 Act Task to 5 Act Task that would be more effective for EBs.

Table 7.2 5 Act Task modified for engaging EBs in mathematical modeling

	Description (teachers' & learners' view)	Purpose
Act 0	Assess and scaffold EBs' understanding and context within act 1	To support EBs by providing background knowledge of the problem's language, context, and mathematics
Act 1	A story with conflict via multimedia/visuals/physical movements and problem-posing based on the story	To engage EBs in a relevant real-life story towards an authentic understanding of the mathematical situation and to empower EBs by having them pose their own problem
Act 2	Student-driven collection of information to solve the problem they posed	To give EBs agency to reflect on the story and develop their own assumptions to solve it
Act 3	Model construction with solution(s)	To provide EBs with an opportunity to visualize/model their mathematical process to analyze and improve their decisions in real life.
Act 4	Collective communication of solutions/thinking process	To support EBs as they develop English proficiency while also building mathematical competency

Adapted from I et al. (2020a, b)

7.4.2 Translanguaging

For EBs, translanguaging pedagogy removes arrears caused by English-only instruction that has been harmful to EBs (I et al. 2020a, b). With emphasizing translanguaging as a pedagogical model grounded on pluralist theory, Fu et al. (2019) propose three key tenets for translanguaging practice that can be used in any setting with both emergent and experienced/proficient bi/multilingual students.

1. Individuals have a single, unified linguistic repertoire.
2. Teachers are co-learners in their classrooms, willing to learn from students, their languages, and their cultures, rather than functioning as the sole possessors of knowledge, "the expert" or the only language instructors in classrooms.
3. Translanguaging practice is purposefully and systematically incorporated in both instructional planning and practices. (p. 99).

In the online course, translanguaging is explained through a video, blog article, and video-recorded PowerPoint as well as a reading assignment (I and Martinez 2020). The three key tenets of Fu et al. are also introduced in the reading assignment. To deepen the teachers' understanding of translanguaging, an exit assignment within the translanguaging module asks teachers to propose three translanguaging strategies to teach EBs mathematics and explain why the strategies use translanguaging. In addition, the module exit assignment requires teachers to analyze a sample dialogue in English and Spanish between a teacher and a group of EB students using a translanguaging lens. These assignments provide teachers an opportunity to reflect on how translanguaging can be implemented in mathematics classes with EBs.

7.4.3 Community Cultural Wealth

Yosso (2005) challenges traditional interpretations of cultural capital and conceptualizes an alternative concept, called Community Cultural Wealth. In the traditional interpretation, cultural capital was used to explain why academic and social outcomes of people of color are significantly lower than those of White people. Using critical race theory as a lens, Yosso expanded the view of cultural capital to focus on resistance and the potential already present in the cultures of communities of color. Community Cultural Wealth is a model that acknowledges culture does in fact influence how societies function and how people learn. Yosso (2005) stated, "Community Cultural Wealth involves a commitment to conduct research, teach, and develop schools that serve a larger purpose of struggling toward social and racial justice (p. 82)." We define *wealth* as the total extension of an individual's accumulated assets and resources. The six forms of Community Cultural Wealth include (1) social capital, (2) navigational capital, (3) aspirational capital, (4) linguistic capital, (5) resistance capital, and (6) familial capital.

Although all six capitals are applicable to EBs, linguistic capital impacts their learning and lives the most. "Linguistic capital includes the intellectual and social skills attained through communication experiences in more than one language and/or style" (Yosso 2005, p. 78) and reflects EBs' bi/multilingual system of learning. The linguistic capital also values bilingual education and the social skills needed for the real world that EBs often develop through translating for their parents. Moreover, when considering parents as a main resource of culture for students, familial capital is crucial for EBs because valuing parents' contribution to and involvement in education means their heritage and language are valued by teachers and schools. Community Cultural Wealth allows teachers to gain a deeper understanding of the assets EBs bring to the classroom and the importance of communicating to parents that they are also assets to the classroom.

We designed an assignment where teachers explore Community Cultural Wealth by writing a letter to parents of EBs. In the letter, the teachers are required to explicitly explain their teaching philosophy, which includes how they will support EBs and build a safe multilingual and multicultural environment using their knowledge of Culturally Sustaining Pedagogy and EB-focused strategies. By asking the teachers to incorporate what they learned from the previous modules, the parent letter assignment serves as a summative assessment as well as a useful resource for their future teaching.

7.5 Implications

The theoretical perspectives of Culturally Sustaining Pedagogy, Translanguaging, and Community Cultural Wealth serve as a catalyst for teachers, mathematics teacher educators, and policymakers in teaching EBs. The online course is the first of its kind that prepares both preservice and in-service teachers through research-based principles and strategies to teach mathematics to EBs by simultaneously focusing on mathematics, pedagogy, and EBs. If we want to construct a safe environment for emergent and proficient/experienced bi/multilingual, we must ensure we have knowledge and understanding of EBs as learners of mathematics, translanguaging approaches with EBs, and an asset perspective of ways and how EBs contribute to the mathematics classroom.

Implications to practice are multifaceted in how the online course can impact preservice and in-service teachers, teacher leaders, and policymakers. At the crux of improving, mathematical learning for EBs is the need to center who EBs are, the assets they bring to the classroom, and the reality of EBs being marginalized in our society. Table 7.3 offers entry points on how we can collectively improve the teaching and learning of mathematics for EBs through an online course format.

Moreover, this innovative online course provided an opportunity for enrollees from all over the world to interact and share their educational insights and experiences that enriched the discussion and expanded the focus of mathematics teaching EBs globally. Teachers from other countries added pluralities and cultural diversity

Table 7.3 Implications for practices

Stakeholder	Online course implications
Preservice teachers	• Creates a learning community among multiple iterations of the course that can allow future teachers a space to continue to grow once they become teachers • Creates a positive classroom environment that honors EBs' identities and directly impacts all students. Even if a classroom does not have EBs, all students benefit from culturally sustaining pedagogy
In-service teachers	• Provides an opportunity to reflect on lesson planning by connecting the importance of language goals to strategies that yield rich mathematical discussion • Helps teachers overcome individual biases that keep them from tapping into EBs' culture
Teacher educators	• Provides essential tools to design an effective mathematics methods course, especially about how to teach EBs. Professional development providers can incorporate aspects of this online course when working with teachers • Allows teachers to reflect on who EBs are to better serve EBs.
Administrators	• Provides an opportunity to dismantle the myth that EBs' parents do not care about schooling • Sees the connections between language and mathematics, which positively shifts how principals traditionally see mathematics learning
District policymakers	• Shows how language and mathematics can be learned together placing in question pull-out programs and other policies that may be harmful to EBs • Acts in the best interest of EBs
Researchers	• Introduces researchers to multiple theories that are ripe for future research in mathematics education • Transforms how we interact with both EBs and their community

to this online course that provided U.S. teachers with a unique learning environment. For example, teachers in the United States and other places in the world interacted within this online space, shared the linguistic diversity of the EBs they teach, and collectively developed their instructional practice that is being transformed through their participation in the course. Our online course design extends the effective use of online learning in teacher education in general and mathematics teacher learning to teach EBs in particular. Specifically, this course can be a useful example of developing online content that enhances teachers' understanding of equity in mathematics education.

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Chapter 8
Number Talks in Asynchronous Online Classrooms for More Equitable Participation and as Formative Assessment of Student Thinking

Simon Byeonguk Han and Eva Thanheiser

8.1 Introduction

This chapter focuses on implementing number talks (NTs) in the context of an asynchronous mathematics content course for prospective elementary teachers (PTs). With the move to teaching mathematics asynchronously online, it is important to find ways for all students to share their thinking/strategies and engage with each other's thinking and strategies. In addition, we as instructors need to have a way to access all of our students' thinking.

To address this move, we developed and implemented an asynchronous online version of NTs utilizing the *Post Before You See Other Responses* function of the online learning management system (LMS) to allow all PTs to participate in each NT by sharing their solution before seeing their classmates' solutions. Once each PT shared their own strategy, they were able to see other posted strategies. They were asked to find other PTs' solutions that used the same strategy as they had used. They were also asked to find other PTs' solutions that used different strategies. As such PTs were asked to compare their own solution to other PTs' solutions.

Our main research questions were:

1. RQ1: Can NTs be successfully executed in an online asynchronous classroom?
2. RQ2: What strategies emerge when PTs engage in online asynchronous NTs?
3. RQ3: What do PTs say they learn when engaging in online asynchronous NTs?

We define asynchronous NTs as successful if: (a) the NTs elicit the target strategies among other strategies, (b) most of the PTs share their own strategies, (c) PTs

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engage in discussing each other's solution attending to similarities and differences in strategies, (d) PTs are learning that various sensemaking strategies exist for adding numbers, and (e) PTs value various strategies.

8.2 Literature Review

8.2.1 What Is a Number Talk (NT)?

Number talks are short (5–15 min) whole-class discussions in the mathematics classrooms (Parrish 2011, 2014; Gerstenschlager and Strayer 2019). There are five key components of NTs: (a) classroom environment and community, (b) classroom discussions, (c) the teacher's role, (d) the role of mental math, and (e) purposeful computation problems (Parrish 2014). The teacher poses a purposeful computation problem to explore students' strategies, to elicit specific strategies, and/or to make connections between different strategies (for an example of addition strategies see Table 8.1). Students are given a few minutes to engage in private-think time to mentally solve the problem. In-person classrooms utilize signs to indicate whether a student has solved the problem (a thumbs up on the chest) and how many strategies the student came up with (additional fingers for additional strategies). The teacher waits until most students show a thumbs up. This wait time indicates everyone is expected to participate. Those students, who already solved the problem and are waiting, are encouraged to find additional ways to solve the problem. Once most of the students have raised at least their thumb, the teacher collects the answers to the

Table 8.1 Eight common strategies for addition (see Parrish 2014, pp. 59–61, pp. 171–174)

Category	Example strategy
Counting all	9 + 6; [1, 2, 3, 4, 5, 6, 7, 8, 9], [10, 11, 12, 13, 14, 15]
Counting up	9 + 6; [9], 10, 11, 12, 13, 14, 15
Breaking each number into its place value	127 + 139; (100 + 20 + 7) + (100 + 30 + 9) 100 + 100 = 200; 20 + 30 = 50; 7 + 9 = 16 200 + 50 + 16 = 266
Making landmark or friendly numbers	127 + 139; 127 + (139 + 1) = 127 + 140 = 267; 267–1 = 266
Doubles/near doubles	127 + 139; (125 + 2) + (125 + 14) = 250 + 2 + 14 = 266
Making tens	127 + 139; (120 + 6 + 1) + (130 + 9) =120 + 130 + 6 + (1 + 9) = 120 + 130 + 10 + 6 = 266
Compensation	127 + 139; (127–1) + (139 + 1) = 126 + 140 = 266
Adding up in chunks	127 + 139; 127 + (100 + 30 + 3 + 6) = 227 + 30 + 3 + 6 = 257 + 3 + 6 = 260 + 6 = 266

problem, both correct and incorrect. Then, students share their strategies with the class (the teacher writes students' strategies on the board), and the teacher facilitates the classroom discourse (Parrish 2014).

8.2.2 Research on Number Talks

NTs serve various goals. (1) NTs engage students in mental math, strengthen their number sense (Okamoto 2015) and computation skills (O'Nan 2003; Parrish 2011, 2014; Johnson and Partlo 2014; Okamoto 2015), and can play an important role in developing accuracy, flexibility, and efficiency for computation (O'Nan 2003). (2) NTs allow students to make sense of the underlying reason for computational algorithms by connecting standard algorithms to meaning-making strategies (Johnson and Partlo 2014). In NTs, students are asked to share their mental strategies and to justify their thinking. A focus on justification leads to better math understanding (Parrish 2011; Staples et al. 2012). (3) NTs can provide more equitable mathematics classrooms (Sun et al. 2018) in that every participant gets to share their thinking and everyone gets to see everyone else's thinking. NTs can help students to develop ownership of their mathematics learning (Parrish 2014) in that they recognize what they can make sense of and what they need help with. (4) NTs serve to establish a classroom community in which all student thinking is valued, and students are given time to complete their thinking. The role of private reasoning time is well-documented in the literature (Anthony and Walshaw 2009; Kelemanik et al. 2016; Staples 2007). To make sense of mathematical tasks, students need to be given time to access the task and process their own thinking. Giving private reasoning time also allows greater access as everyone is given a chance to think. Establishing a classroom culture of mutual respect is essential for creating a safe environment for effective NTs. Various solution strategies are elicited allowing various forms of participation. This includes sharing incorrect responses, unfinished solution strategies, etc. One of the main advantages of NTs is that several solution strategies are presented on the board which communicates that there is more than one way to solve a problem. In addition, NTs allow comparing across strategies to determine how they are similar and different. Comparing across strategies (e.g., Durkin et al. 2017) and engaging in critiquing and argumentation (e.g., Yackel and Cobb 1996) can be particularly productive. This allows for more opportunities to make meaning and make connections (Jacobs and Spangler 2017) and thus, develop understanding.

However, despite the wide use of NTs among teachers, schools, professional development, and teacher education, the efficacy of NTs has not yet been studied at a rigorous research level (Matney et al. 2020). Matney et al. conducted an extensive literature search on the efficacy of the instructional practice of NTs from blind-peer-reviewed publications in mathematics education (from 2000 to August 2019). Among 576 articles from the original search, only one article satisfied their criteria; Murata et al. (2017) studied how first graders' strategy development was supported by math-talk. In different work, Lustgarten and Matney (2019) showed that PTs showed statistically significant improvement in the number of strategies they used after engaging in 11 NTs throughout a course.

8.2.3 Asynchronous Collaboration in Learning

Asynchronous online learning can be collaborative. Mallet (2008) showed that students used both illustrative and corrective collaborations when asked to compare answers, find errors, discuss strategies, etc. Students in that setting reported that those activities helped them to learn and clearly related to what they were expected to learn from the course. Asynchronous online learning can lead to improved levels of noticing students' mathematical thinking (Fernandez et al. 2012) and support the development of mathematical knowledge for teaching (MKT) (Clay et al. 2012; Silverman and Clay 2009). Unlike the traditional instructional environment, Online Asynchronous Collaboration (OAC) (Clay et al. 2012) slows the pace of learning, makes all participants' ideas public and permanent, increases access to others' various thinking and understandings, and enables participants to return to their and others' previous thoughts to reinforce their understanding.

8.2.4 Preservice Elementary School Teachers' Conception of Whole Number and Operation

PTs typically enter the mathematics content courses for teachers being able to perform addition and subtraction algorithms but unable to explain why they work (Browning et al. 2014; Thanheiser et al. 2014a, b; Thanheiser 2009, 2010, 2018). This struggle is due to the fact that PTs typically are taught how to solve problems (fluency in the algorithms) but do not connect the digits in the number to their values. So, they may see 123 as a 1 next to a 2 next to a 3 (a *concatenated digits conception*) rather than as 100, 10 tens, or 100 ones combined with 2 tens, or 20 ones, combined with 3 ones (a *reference-units* or *groups-of-ones conception*). This restricts their ability to explain regrouping meaningfully and leaves them with a view of math that is not based on sensemaking. NTs can be used as one way to allow PTs to rediscover sensemaking and using numbers meaningfully.

8.3 Method

8.3.1 Participants

Participants in the study were 23 PTs who were enrolled in the first mathematics content course for elementary teachers in the United States (taught online asynchronously). This course is the first in a sequence of three courses which are a prerequisite to enter the teacher education program. This course was taught in the Spring of 2020 and as such, was moved to remote learning rather suddenly. Neither the participants nor the teacher had purposefully signed up for an asynchronous online course.

8.3.2 Data

We asked PTs to solve the NT problems without writing anything down. Then, by utilizing the function: *Users must start a thread before they can read and reply to other threads in each topic* in an Online Discussion Forum (in our case D2L). We asked PTs to share how they solved the problem before being able to see anyone else's response. Once posted, PTs were able to see everyone else's solutions, and we asked them to look for other solutions that were (a) similar to their own and (b) different from their own and to use the comment function to explain how they were similar and/or different (see Fig. 8.1).

Data to answer RQ1 (Can NTs be successfully executed in an online asynchronous classroom?) and RQ2 (What strategies emerge when PTs engage in online asynchronous NTs?) consisted of each PT's initial posting to each NT, describing their strategy to solve that NT problem (in detail what they did and how they did it) before seeing anyone else's strategy. For the NT problems, the timing of the NTs, and participation, see Table 8.2.

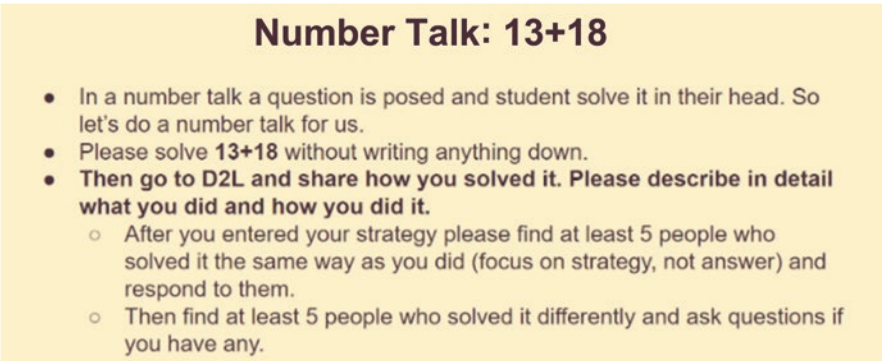

Fig. 8.1 Number talk prompt

Table 8.2 Number talk problems, the timing of the number talks, and participation

	First NT	Second NT	Third NT	Fourth NT
Timing	Week 4	Week 5	Week 6	Week 7
NT/reflection	13 + 18	99 + 98	13 + 35 + 17	124 + 126
Goal/ rationale	Introduce NT to the PTs and observe whether they use the standard algorithm in their head, the breaking into place values strategy, or some other strategy	Observe whether the PTs use the same strategy as they used in NT1 or a different strategy. In particular, we were interested in whether PTs make 100 from 99 or 98 (or both) for the calculation (compensation or making landmark or friendly numbers)	Observe whether the PTs use the same strategy as they used in NT1 or NT2 or a different strategy. In addition, we were interested to see whether they would add the 13 and the 17 before adding the 35	Observe whether the PTs use the same strategy as they used in NT1, NT2, NT3, or a different strategy. In particular, we were interested to see if the PTs would utilize doubles/ near doubles strategy
Researchers' anticipated strategy	Breaking each number into its place value, adding up in chunks, standard algorithm	Breaking each number into its place value, making landmark or friendly numbers, compensation, standard algorithm	Breaking each number into its place value, making tens, compensation, standard algorithm	Breaking each number into its place value, doubles/near doubles, standard algorithm
Number of responses	23: PT8 did not participate; PT9 provided two strategies	24: PT14 provided two strategies	22: PT6 did not participate	23

Table 8.3 Reflections on number talks, timing, and participation

	Reflection 1	Reflection 2
Timing	Week 6	Week 10
NT/reflection	Asked PTs to share what they were learning from engaging in NTs	What were your takeaways from NTs? What were your takeaways from comparing your strategies to others? What were your difficulties with NTs?
Number of reflections	23	21 (PT6, PT13 did not participate)

Data to answer RQ 1 and RQ 3 (What do PTs say they learn when engaging in online asynchronous NTs?) consisted of PT's responses to reflection questions in two surveys. For timing, survey questions, and participation, see Table 8.3.

8.3.3 Analysis

We analyzed the PTs' responses using Parrish's (2014) strategies (see Table 8.1) and Thanheiser's (2009, 2010) framework. We simplified Thanheiser's (2009) framework and utilized a combination of *Reference Unit* or *Groups of Ones* if PTs showed that they drew on the meaning of the digits such as 100 ones, 10 tens, or 1 hundred for the 1 in 123 and *Concatenated* if they seemed to treat digits as concatenated single digits, such as considering the meaning of the digit as 1 for the 1 in 123. We first categorized the PTs' strategies based on each lens for each and examined how many PTs used different strategies across the four NTs.

For example, in NT1 (13 + 18), 1 PT responded, *"When solving an addition problem, like this one (13 + 18), I always separate the tens from the ones. So I know I have two tens here 10 + 10 = 20 and then we have 8 + 3 = 11. Then it's simply 11 + 20 = 31."* We highlighted 10 + 10 = 20, 3 + 8 = 11, and 11 + 20 = 31, and matched it to *Breaking Each Number Into Its Place Value* because both the 13 and the 18 were broken into their respective place values and then added by place value. We matched this response as *Reference Units or Groups of Ones* because the digits were clearly attributed to their place value. Another PT said, *"To solve this problem, I picked the number closest to a base-ten (18). I added two to bring it up to 20. Then added 10 to take care of the tens place in 13. Then added another 1 to include the remaining ones place and bring it to 31."* We highlighted "added two to bring up (18) to 20," "added 10," and "added another 1" in the response and matched it to *Compensation* and *Reference Units or Groups of One* respectively. We analyzed all individual responses in this way and reviewed whether our sorting represented the entire data set.

For the PTs' reflection, we started with carefully reading through all the responses and highlighted the main idea in each response. Using a Thematic Analysis (Braun and Clarke 2006), we sorted all the data into themed categories, discussed the themes to come to an agreement on them, and sorted the data into the themes. For instance, 1 PT responded to the prompt "What were your takeaways from NTs?" with, *"*[NTs]helped me realize that there are *so many ways that people can view and solve* a problem and *all of them are valid* [highlighted for coding]*,"* we categorized this response as fitting into the theme *"learned that there are multiple valid ways to solve problems."* Another PT's response to the same question was "The number talk, I think, pushed me to *not always rely on paper and pencil*. I had to *keep track of what I was doing,* and in the beginning, it was easy, but as weeks passed, I started to feel challenged which I really enjoyed [highlighted for coding]*"* was coded as *"it helped me improve my mental math."*

8.4 Results

We begin by discussing the NTs individually, then look across all four NTs, and finally, consider the PTs' reflections.

8.4.1 Number Talk 1: 13 + 18

In Week 4 of the course, we introduced the PTs to NTs and asked them to solve 13 + 18 mentally. The goal of the first NT was to see whether the PTs would be able to make sense of adding 13 + 18 using a strategy other than the standard algorithm. The main strategy we expected to see was *Breaking Each Number Into Its Place Value,* however, we also anticipated that other strategies could show up. We asked the PTs to describe in detail what they did and how they did it. Twenty-two PTs participated in NT1, and 1 PT provided two strategies. We observed three of the eight common addition strategies (see Table 8.1), namely *Breaking Each Number Into Its Place Value (16 PTs used this strategy - the numbers in parentheses indicate the number of PTs who used each strategy), Compensation (1), and Adding Up In Chunks (2),* and the use of the *Standard Algorithm for Addition (3).* The PT who provided two strategies used *Breaking Each Number Into Its Place Value* twice with different subcategories (see Table 8.4).

Sixteen PTs used the *Breaking Each Number Into Its Place Value* strategy. Within these 16 PTs, we identified four subcategories, namely: (a) *adding the ones first, then the tens* (4), (b) *adding the ones first, then the tens, but treating the tens as ones* (5)*,* (c) *adding the tens first, then the ones* (6), (d) *adding the tens first, then adding in chunks* (1). Identifying the subcategories was helpful as we were discussing whether the two strategies were the same or not. The PT who provided two strategies used (b) and (c). We also identified two subcategories in the *Adding up in Chunks* strategy; (a) add the ones first, (b) add the tens first. Thus, in total, there were eight different strategies for the 23 responses (see Table 8.4).

Once we identified the strategies PTs used, we applied Thanheiser's (2009, 2010) framework to categorize the strategy as indicating an understanding of the meaning of the individual digits or not. Of the strategies in Table 8.3, *Standard Algorithm* and one sub-strategy of the *Breaking Each Number Into Its Place Value* seemed to indicate that PTs considered the digits as representing ones rather than ones and tens, namely (b) *adding the ones first, then the tens, but treating the tens as ones.* As such, only 8 of the 23 PTs clearly indicated that they drew on a concatenated digits conception.

Table 8.4 Categorization of the strategies from the number talk 1

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Breaking each number into its place value (adding the ones first, then the tens)	3 + 8 = 11 10 + 10 = 20 11 + 20 = 31	4	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the ones first, then the tens, but treating the tens as ones)	3 + 8 = 11 1 + 1 + 1 = 3 31	5	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Breaking each number into its place value (adding the tens first, then the ones)	10 + 10 = 20 8 + 3 = 11 20 + 11 = 31	6	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the tens first, then adding in chunks)	10 + 10 = 20 20 + 8 = 28 28 + 3 = 31	1	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
Standard algorithm for addition	1 18 + 13 31	3	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Compensation	18 + 2 = 20 20 + 10 = 30 30 + 1 = 31	1	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
Adding up in chunks (add the ones first)	3 + 18 = 21 21 + 10 = 31	1	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
Adding up in chunks (add the tens first)	10 + 18 = 28 28 + 3 = 31	2	*Reference units or groups of ones* (looks at the 10 as 10 ones or 1 ten)
	Total	23	

8.4.2 Number Talk 2: 99 + 98

In the second week of NT, we asked PTs to solve 99 + 98 mentally. The goal for this NT was to see whether PTs would use a *Compensation* or *Making Landmark or Friendly Numbers* strategy. The *Compensation* strategy had been part of the first NT so at least some PTs had been exposed to it.

As with NT 1, we asked the PTs to describe in detail what they did and how they did it. We had 24 responses, as 1 PT responded twice (see Table 8.5). This PT provided two different strategies, *Breaking Each Number Into Its Place Value* and *Making Landmark or Friendly Numbers*. As expected, 14 PTs drew on *Compensation* (9) and *Making Landmark or Friendly Numbers* (5). The remaining 10 PTs drew on

Table 8.5 Categorization of the strategies from the number talk 2

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Making landmark or friendly numbers	100 + 100 = 200 200 – 3 = 197	9	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Compensation	98 = 97 + 1 99 + 1 = 100 100 + 97 = 197	5	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the tens first, then the ones)	90 + 90 = 180 9 + 8 = 17 180 + 17 = 197	5	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the ones first, then the tens, but treating the tens as ones)	9 + 8 = 17 9 + 9 = 18 180 + 17 = 197	2	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Breaking each number into its place value (adding the ones first, then the tens, but treating the tens as ones)	8 + 9 = 17 1 + 9 = 10 10 + 9 = 19 197	2	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Standard algorithm for addition	1 99 + 98 197	1	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
	Total	24	

the *Breaking Each Number Into Its Place Value* (9) or the *Standard Algorithm for Addition* (1). Since the target strategy in this NT relies on focusing on the underlying quantity, as expected, more PTs drew on a strategy that indicated that they understood the quantities for each digit (*Reference Units or Groups of One*). Only 5 PTs indicated that they did not draw on the quantities for their calculations (*Concatenated*).

8.4.3 Number Talk 3: 13 + 35 + 17

In the third week of NTs, we asked PTs to solve 13 + 35 + 17 mentally. The goal for this NT was to see whether PTs would use a *Making Tens or Compensation* strategy to add 13 and 17 first. The *Compensation* strategy was a part of the goals of the second NT, thus PTs had been exposed to this strategy. On the other hand, *Making Tens strategy* did not appear in the previous two NTs. Therefore, we were curious whether PTs would use a new strategy that never appeared before.

Table 8.6 Categorization of the strategies from the number talk 3

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Compensation	17 – 2 = 15 2 + 13 = 15 15 + 15 = 30 30 + 35 = 65	3	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making landmark or friendly numbers	10 + 30 + 20 = 60 3 + 5 = 8 60 + 8 = 68 68 – 3 = 65	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making tens	3 + 17 = 20 10 + 20 = 30 30 + 30 = 60 60 + 5 = 65	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making tens	17 + 13 = 30 30 + 35 = 65	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making tens	17 + 13 = 30 30 + 30 = 60 60 + 5 = 65	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making tens	7 + 3 = 10 13 + 17 = 30 30 + 35 = 65	4	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Making tens	17 + 13 = 30 3 + 3 = 6 60 + 5 = 65	2	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Making tens	7 + 3 = 10 1 + 1 + 1 = 3 30 + 35 = 65	1	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
Breaking each number into its place value (adding the tens first, then the ones)	10 + 30 + 10 = 50 7 + 3 = 10 10 + 5 = 15 50 + 15 = 65	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the tens first, then the ones)	10 + 10 = 20 3 + 7 = 10 20 + 10 = 30 30 + 30 = 60 60 + 5 = 65	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value (adding the tens first, then the ones)	13 + 35 = 48 10 + 40 = 50 8 + 7 = 15 15 + 50 = 65	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)

(continued)

Table 8.6 (continued)

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Breaking each number into its place value (adding the tens first, then the ones but treating the tens as ones)	13 + 35; 3 + 5 = 8 1 + 3 = 4 48 + 17 8 + 7 = 15 1 + 4 + 1 = 6 65	1	*Concatenated* (at least sometimes it is not clear that the 1 in the tens place is 10 or 1 ten)
	Total	22	

As with NT 1 and NT 2 we asked the PTs to describe in detail what they did and how they did it. In this NT, we had 22 responses (see Table 8.6). Fifteen PTs used *Compensation* (3) or *Making Tens* (12) strategy as we expected. The remaining 7 PTs used *Breaking Each Number Into Its Place Value* (6) and *Making landmark or friendly numbers* (1). Since the target strategies in this NT rely on focusing on the underlying quantity, as expected, more PTs drew on a strategy that indicated that they understood the quantities for each digit (*Reference Units or Groups of One*). Only 4 PTs indicated that they did not draw on the quantities for their calculations (*Concatenated*).

8.4.4 Number Talk 4: 124 + 126

In the fourth NT, we asked PTs to solve 124 + 126 mentally. The goal of the fourth NT was to see whether PTs would use *Doubles/Near Doubles* strategy. Similar to the third NT, PTs had never been exposed to the *Doubles/Near Doubles* strategy. So, we were wondering whether we could observe the *Doubles/Near Doubles* strategy. As with the previous NTs, we asked the PTs to describe in detail what they did and how they did it. Among 23 PTs, 10 PTs relied on *Doubles/Near Doubles* strategy. The remaining 13 PTs utilized *Breaking Each Number Into Its Place Value* (12) or the *Standard Algorithm for Addition* (1) strategies. Since the fourth NT has three-digit addition, we observed slightly different subcategories of *Breaking Each Number Into Its Place Value* from previous NTs namely: (a) *adding the ones first, then the tens, and the hundreds* (4), (b), *adding the ones first, then the tens, and the hundreds, but treating the tens and hundreds as ones* (2), (c) *adding the ones and tens first, then the hundreds* (1), (d) *adding the hundreds first, then the ones and the tens* (2), (e) *adding the hundreds first, then the tens and the ones at once* (1), (f) *adding the hundreds first, then the tens, then the ones* (2). Although more PTs relied on *Breaking Each Number Into Its Place Value* strategy than NT 2 and NT 3, fewer PTs (2) indicated that they considered the digits in tens and hundreds of places as representing ones (*Concatenated*) than the previous NTs (see Table 8.7).

Table 8.7 Categorization of the strategies from number talk 4

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Doubles/near doubles	126 – 1 = 125 124 + 1 = 125 100 + 100 = 200 25 + 25 = 50 200 + 50 = 250	5	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Doubles/near doubles	120 + 120 = 240 4 + 6 = 10 240 + 10 = 250	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Doubles/near doubles	130 + 130 = 260 260 – (4 + 6) = 250	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Doubles/near doubles	4 + 6 = 10 120 + 120 = 240 240 + 10 + 240	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the hundreds first, then the tens, then the ones)	100 + 100 = 200 20 + 20 = 40 4 + 6 = 10 200 + 40 + 6 = 246	2	*Reference Units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the hundreds first, then the tens and the ones at once)	100 + 100 = 200 24 + 26 = 50 200 + 50 = 250	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the hundreds first, then the ones, then the tens)	100 + 100 = 200 4 + 6 = 10 10 + 20 + 20 = 50 200 + 50 = 250	2	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the ones and the tens first, then the hundreds)	24 + 26 = 50 100 + 100 = 200 200 + 50 + 250	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the ones first, then the tens and the hundreds)	4 + 6 = 10 20 + 20 = 40 100 + 100 = 200 40 + 10 = 50 200 + 50 = 250	1	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the ones first, then the tens and the hundreds)	4 + 6 = 10 20 + 20 = 40 40 + 10 = 50 100 + 100 = 200 200 + 50 = 250	3	*Reference units or groups of one* (looks at the 10 as 10 ones or 1 ten)
Breaking each number into its place value(adding the ones first, then the tens and the hundreds but treating the tens and hundreds as ones)	4 + 6 = 10 2 + 2 + 1 = 5 1 + 1 = 2 250	2	*Concatenated*

(continued)

Table 8.7 (continued)

Strategies for addition (Parrish 2014)	Observed strategy	Frequency	Conceptions of digits (Thanheiser 2009)
Standard algorithm for addition	1 124 126 250	1	*Concatenated*
	Total	23	

Table 8.8 Prospective elementary teachers' (PTs') use of strategies in each number talk (NT)

Strategies for addition (Parrish 2014)	NT1 13 + 18	NT2 99 + 98	NT3 13 + 35 + 17	NT4 124 + 126	Total
Breaking each number into its place value	16	9	6	12	43
Compensation	1	5	3		9
Adding up in chunks	3				3
Making landmark or friendly numbers		9	1		10
Doubles/near doubles				10	10
Making tens			12		12
Standard algorithm for addition	3	1		1	5
Total	23	24	22	23	92

Note. In NT 1 and 3, 1 PT did not participate. In NT 1 and 2, 1 PT provided two strategies

8.4.5 Analysis Across the Four Number Talks

Across the NTs, we found that in each NT, the most prevalent strategy was our target strategy except for the NT4 (in this one it was the second most prevalent). In NT 1, 16 of 23 PTs used the *Breaking Each Number Into Its Place Value* strategy. In NT 2, 14 PTs drew on the target strategies, *Compensation* (9) and *Making Landmark or Friendly Numbers* (5). In NT 3, 15 PTs used *Compensation* (3) or *Making Tens* (12) strategy as we expected. In NT 4, 10 PTs used *Doubles/Near Doubles* strategy. As such, even in an asynchronous online environment, it seems reasonable to expect to see the anticipated strategies.

We collected a total of 92 responses across the four NTs. The goal of the four NTs was to draw out different strategies for mentally adding numbers. While all NTs could be solved with *Breaking Each Number Into Its Place Value*, and about half of the responses (43 of 92) used that strategy, the rest of the strategies included all of the non-counting strategies from Table 8.1, *Compensation, Adding Up In Chunks, Making Landmark or Friendly Numbers, Doubles/Near Doubles, and Making Tens* (see Table 8.8). The other strategies were aligned with the goals of the NTs. For example, *Adding Up In Chunks, Making Tens* and *Doubles/ Near Doubles,* only appeared in NT 1, NT 3, and NT 4, respectively. *Making Landmark or Friendly*

Numbers was only used in NT 2 and NT 3. In summary, PTs were able to utilize different strategies for the different types of computation problems.

Looking across the strategies we found that only 3 PTs used the same strategy across all NTs. They used *Breaking Each Number Into Its Place Value* to solve all four problems. All but three students used the *Breaking Each Number into its Place Value* strategy at least once, with 7 PTs using it only once, 8 PTs using it twice, 2 PTs three times, and 3 PTs four times. Understanding this strategy deeply will lay the foundation for understanding why the standard algorithm works. The distribution of the strategies across PTs and NTs can be seen in Tables 8.9 and 8.10. We color-coded Table 8.10 to represent.

Throughout the four NTs, PTs relied less and less on a *Concatenated* conception. The number of the PTs with *Reference Units or Groups of One* conception of digits were constantly increased from 15 (NT 1) to 20 (NT 4), and the number of PTs with *Concatenated* conceptions of digits were decreased from 8 (NT 1) to 3 (NT 4) (see Table 8.11).

8.4.6 Prospective Teachers' Reflection on Number Talks

After the third NT, we asked PTs to reflect on NTs and share what they are learning from engaging in NTs. Most of the PTs (15 of 23) responded that they learned that *there are multiple ways to solve a problem.* For example, 1 PT responded, *"It has been really interesting to see the different ways the students are approaching these problems. I often will assume that there is only one way to approach this level of math problem because that is the way that I do them."* Three PTs explicitly stated that they were learning from their peers, for example, *"I am also learning more about how my fellow peers think, and their perspectives on math have given me a new look on how I solve my equations."*

Six PTs' responses focused more on mathematics, such as (a) that strategies depend on the numbers *"I learned that the way I add the numbers in my head depends on what numbers are used,"* (b) NTs strengthen justification skills, *"These exercises have also helped me in forming my justification methods,"* and (c) that NTs help with procedural fluency *"I've noticed lately that it doesn't take me as long because I've learned to group things differently so the numbers don't overwhelm me."*

At the end of the course, we collected a second reflection with 21 PTs responding. The main themes in that reflection were that PTs (1) learned that there are multiple valid ways to solve problems (15 PTs, 71.4%), for example, *"It helped me*

Table 8.9 Summary of prospective elementary teachers' (PTs') use of different strategies (Parrish 2014) across the four number talks (NTs)

Number of PTs who used different strategies across the four number talk			
1 strategy	2 strategies	3 strategies	4 strategies
3 PTs	5 PTs	7 PTs	8 PTs

Table 8.10 Prospective elementary teachers' (PTs') use of strategies (Parrish 2014) across the four number talks (NTs)

	NT1	NT2	NT3	NT4	Number of strategies
PT 1	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	1
PT 2	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	1
PT 3	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	Breaking each number into its place value	1
PT 4	Breaking each number into its place value	Making landmark/friendly numbers	Breaking each number into its place value	Breaking each number into its place value	2
PT 5	Breaking each number into its place value	Breaking each number into its place value	Making tens	Breaking each number into its place value	2
PT 6	Standard algorithm	Breaking each number into its place value	N/A	Breaking each number into its place value	2
PT 7	Standard algorithm	Standard algorithm	Making tens	Standard algorithm	2
PT 8	N/A	Compensation	Compensation	Doubles/near doubles	2
PT 9	Breaking each number into its place value Breaking each number into its place value	Breaking each number into its place value	Compensation	Doubles/near doubles	3
PT 10	Breaking each number into its place value	Breaking each number into its place value	Compensation	Doubles/near doubles	3
PT 11	Breaking each number into its place value	Breaking each number into its place value	Making landmark/friendly numbers	Doubles/near doubles	3
PT 12	Breaking each number into its place value	Making landmark/friendly numbers	Making tens	Breaking each number into its place value	3
PT 13	Breaking each number into its place value	Making landmark/friendly numbers	Making tens	Breaking each number into its place value	3
PT 14	Breaking each number into its place value	Breaking each number into its place value Making landmark/friendly numbers	Making tens	Breaking each number into its place value	3

(continued)

Table 8.10 (continued)

	NT1	NT2	NT3	NT4	Number of strategies
PT 15	Adding up in chunks	Making landmark/ friendly numbers	Breaking each number into its place value	Breaking each number into its place value	3
PT 16	Breaking each number into its place value	Making landmark/ friendly numbers	Making tens	Doubles/near doubles	4
PT 17	Breaking each number into its place value	Making landmark/ friendly numbers	Making tens	Doubles/near doubles	4
PT 18	Breaking each number into its place value	Making landmark/ friendly numbers	Making tens	Doubles/near doubles	4
PT 19	Breaking each number into its place value	Compensation	Making tens	Doubles/near doubles	4
PT 20	Adding up in chunks	Compensation	Breaking each number into its place value	Doubles/near doubles	4
PT 21	Adding up in chunks	Compensation	Making tens	Breaking each number into its place value	4
PT 22	Compensation	Making landmark/ friendly numbers	Making tens	Breaking each number into its place value	4
PT 23	Standard algorithm	Compensation	Making tens	Doubles/near doubles	4

Table 8.11 Prospective elementary teachers' (PTs') conceptions of digits across the four number talks (NTs)

Conceptions of digits	NT1	NT2	NT3	NT4
Reference units or groups of one	15 (65.2%)	19 (79.2%)	18 (81.8%)	20 (87.0%)
Concatenated	8	5	4	3
Total	23	24	22	23

realize that there are so many ways that people can view and solve a problem and all of them are valid," (2) learned the various strategies (12 PTs, 57.1%), *"It is important to see how others solved their problems in order to have a deeper understanding of how different strategies work to solve the same problem,"* and (3) learned that explaining one's own thinking is nontrivial (7 PTs, 33.3%), *"My difficulties were finding a way to explain what I did in a way that would make sense to others."*

8.5 Discussion

In response to RQ1: "Can NTs be successfully executed in an online asynchronous classroom?", we found that NT can indeed be successfully executed in online asynchronous settings. While online asynchronous settings (as we have designed them) have limitations, for example, the fact that no strategy is centered for discussion by all students, they also offer affordances such that every PT gets to share their strategy and get to comment on their peers' strategies. Every PT also gets to hear comments on their own strategy. As such, the online asynchronous setting allows all PTs to participate in both sharing their thinking and responding to other students' thinking. Thus, it may be more equitable with respect to PTs' participation.

The NT also served as a formative assessment tool for the instructor of the course. The instructor was not only able to see the strategies each PT shared but also see which strategies they thought were similar to their own and which were different. This provided more insight into each PTs' strategy. In an in-person classroom, there is typically not enough time for each student to share their solutions and be able to compare across all solutions.

In response to RQ2: "What strategies emerge when students engage in online asynchronous NTs?", we found that the online environment allowed for the same strategies to emerge that emerge in in-person classrooms. We were also able to observe how various strategies emerged and how many PTs used each strategy. All the PTs' responses fit into Parrish's (2014) categorization of strategies for addition except for the vertical algorithm. While *"breaking each number into its place value"* appeared most, PTs were able to utilize various strategies for different NT. Almost all PTs used more than one strategy across the four NT, and more than one-third of the PTs used four different strategies (see Tables 8.8 and 8.9). As in regular classrooms, PTs moved from focusing on digits (first NT) to focusing on units (see Table 8.10). Moreover, since we had written responses from all the PTs, it was helpful to apply other lenses to analyze their strategies.

In response to RQ3: "What do students say they learn when engaging in online asynchronous NTs?", we found that the PTs said NTs helped them to learn multiple ways to solve the problems (15 out of 23 in the first reflection and 15 out of 21 in the second reflection). Also, comparing their strategies to others, which may not happen all the time in NTs in the regular classroom setting, helped them to learn different strategies since they had to think carefully about how others solve the problem in different ways.

8.6 Conclusion and Implications

We found that NTs can be implemented in asynchronous online settings, and we can anticipate similar strategies to emerge as we can in in-person settings. PTs stated that they learned (a) that there are many ways to solve problems, (b) that explaining their thinking is not easy, and (c) that engaging in NTs helped their mental computation. As such, NTs show the same affordances in an asynchronous online setting as they do in standard classrooms. However, we found some differences between the settings. For one, in an asynchronous online setting, all PTs can participate in sharing their initial strategy to solve the problem. As such this format offers participatory equity (Reinholz and Shah 2018). This not only allowed for all PTs to participate in each NT but also for the instructor to track all PTs strategies across all NTs.

In addition, every PT was asked to compare their own strategy to others, both finding PTs who used the same strategy and PTs who used different strategies. This provides additional insight to the instructor of the PTs' understanding of their own strategy. For example, in the first NT, we noticed that several of the PTs who used the *Breaking Each Number Into Its Place Value* but different sub-strategies would claim they are the same. And while those are the same overall strategy, it is important for PTs to learn to distinguish between the sub-strategies. Based on what we learned from this study, we now ask PTs to find other strategies that started the same as theirs, and then some that were completely the same as theirs.

The sudden move to remote learning was a challenge to us, as instructors, and to the PTs. However, we were able to find a way to successfully implement NTs in the online asynchronous course. Moreover, PTs' more equitable participation in online asynchronous NTs than in in-person NTs showed a possibility of NTs as a formative assessment of student thinking.

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Chapter 9
A Three-Part Synchronous Online Model for Middle Grade Mathematics Teachers' Professional Development

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We describe efforts to research an online model of professional development aimed at rural middle school mathematics teachers. To reach these teachers, we created a set of three fully online professional development experiences, the goal of which was to support these teachers to engage in ambitious mathematics instruction. In this chapter, we describe the model, the theory of action on which it is based, the methodology we used to research the model, and results that highlight the utility of the methods. We show how we adapted our research methods to take into account the affordances of the online environment and the tools we incorporated into the environment. We frame the study in terms of the challenges related to supporting teachers to implement ambitious forms of instruction.

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9.1 Supporting Ambitious Mathematics Instruction

Ambitious teaching entails supporting learners to solve complex problems and reason about mathematics (Bieda et al. 2020), listening for and responding to students' intellectual contributions (Lampert et al. 2010; Jacobs et al. 2010; Van Es and Sherin 2008), broadening participation in mathematical discourse (Boaler and Staples 2008), and positioning students as being mathematically competent (Kelley-Peterson 2010). Teacher educators and instructional leaders face the challenge of supporting teachers to enact ambitious forms of instruction, which takes a great deal of support and effort (Munter and Correnti 2017). Researchers have identified or conjectured about a number of factors that support teachers to take up ambitious new instructional practices, including providing teachers with feedback on their practices (Boston and Candela 2018) and creating a professional community that incorporates high-depth collegial interactions (Coburn et al. 2012; Stein and Coburn 2008) and that has appropriate expertise (Cobb and Jackson 2011; Sun et al. 2014). High-depth interactions address "underlying pedagogical principles of the approach, the nature of the mathematics, and how students learn" (Coburn and Russell 2008, p. 212). A particularly notable form of support has been the involvement of mentors or coaches (Hunter et al. 2018). Coaching has the potential to support teachers to engage in critical analysis of their own practices (Kraft and Hill 2020), to focus teachers on students' conceptual understanding (Russell et al. 2020), and to enhance teachers' instructional expertise (Sun et al. 2014). Despite these well-documented efforts, there is still no consensus on what constitutes an adequate support structure, and very little of this research has been conducted in purely online environments. In this chapter, we aim to articulate how a purely online environment can be designed to support teachers to enact ambitious forms of instruction, and how the online environment provides affordances for conducting research on such efforts.

9.2 Theory of Action

The goal of this project was to provide rural teachers access to professional development experiences that would support them to enact ambitious instructional practices. We created a series of professional development experiences based on the following theory of action with respect to teacher change. First, teachers need to be given opportunities to read about and discuss forms of mathematics instruction that may not have been what they experienced as learners (cf. Bekdemir 2010). Second, teachers need to observe and reflect on instantiations of ambitious forms of mathematics instruction; these instantiations serve as images to ground their thinking and discourse on these forms of instruction (Lampert et al. 2013; Strayer et al. 2017). Third, teachers need to engage in ambitious forms of mathematics instruction in a supported way; this involves observation, feedback, and collegial discussion around the teachers' attempts to engage in these practices (Kraft and Hill 2020).

We then conjectured about the support necessary to implement this theory of action. Based on the literature summarized above, we aimed to develop a professional community that engaged in high-depth interactions in the presence of those with expertise in the content and in norm-setting protocols. The professional community would support the first two sets of actions in our theory of change model. We also determined that we needed to provide intensive individualized support in the form of coaching that would provide the teacher with meaningful feedback to help them reflect on their enactment of the ambitious practices. See Fig. 9.1 for how the supports were connected to our theory of action.

Our theory of action did not originally stipulate modality or context. However, we offered this to rural mathematics teachers, which required that we create a fully online experience. Consequently, we focus both on the implementation of this model and how to research it in a fully online setting.

9.3 Intervention and Online Setting

We designed a formal three-part model in which we engaged mathematics teachers in coordinated experiences over 6 months in a single academic year. The three-part model included an online course (six sessions spread out over 5 months), two demonstration lessons that we called Teaching Labs, and two video-assisted content-focused coaching cycles. The three components of the model followed our theory of action presented in Fig. 9.1. The course provided an intensive introduction to the instructional practices emphasized in the project (e.g., the practices articulated by Smith and Stein 2018); these practices were in turn modeled in the two Teaching Labs, in which the teachers reflected on instantiations of the practices discussed in the course; then, teachers engaged in intensive one-on-one work with coaches to implement the practices. Each of these professional development experiences involved synchronous and asynchronous components. See Fig. 9.2 for a visual of the model.

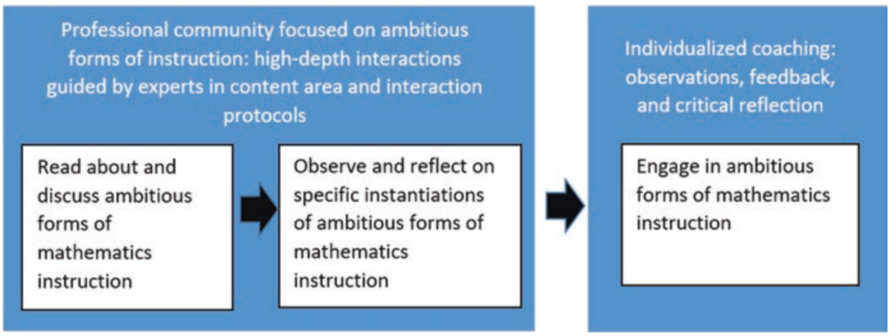

Fig. 9.1 Theory of action and related support

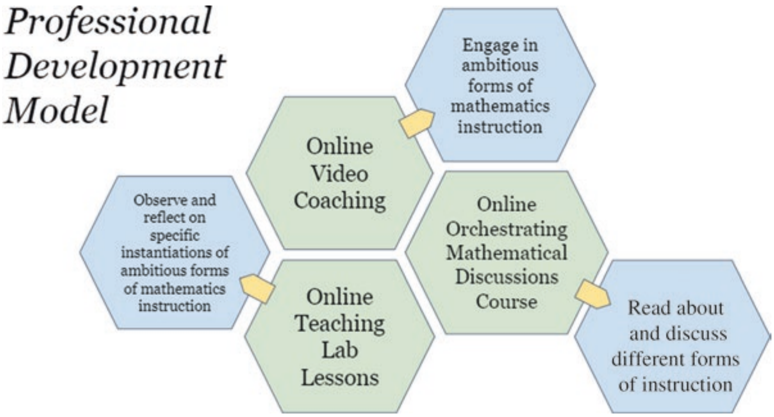

Fig. 9.2 Depiction of three-part model (center, green) with purpose for each part in blue

The course (Orchestrating Mathematical Discussions [OMD]) was interspersed with the Teaching Labs and coaching cycles, with the underlying logic that the course would introduce key instructional practices to the teacher participants. The first Teaching Lab occurred after two class sessions and was intended to model the first set of practices covered in the OMD course (i.e., goals, task selection, anticipating). This Teaching Lab was followed by the first coaching cycle, which allowed teachers to implement and receive feedback on the practices covered in the first two OMD sessions. The second Teaching Lab occurred after three additional OMD course sessions and was followed by a coaching cycle and one additional OMD session.

9.3.1 The Orchestrating Mathematics Discussion Course

The six OMD sessions emphasized elements of ambitious instruction tied to fostering productive classroom discussions in mathematics (Smith and Stein 2018). These discourse practices are catalyzed by five instructional practices emphasized in the course; anticipating, monitoring, selecting, sequencing, and connecting. The sessions also emphasized key aspects of lesson planning, such as goal-setting and selecting high cognitive demand tasks. The specific goals of the sessions were to develop an awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions; to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions; and to further develop teachers' mathematical knowledge, particularly the rich connections around big mathematical ideas (Ball 1991; Ma 1999).

We conducted the OMD course in Zoom, which allowed for synchronous whole class and small group interactions. In addition, we simultaneously used Google Docs and Google Drawings, which allowed the teachers to collectively develop and share artifacts, including approaches to mathematical problems, in real-time. The teachers engaged in *experiences as learners* with high cognitive demand tasks, working in virtual breakout rooms to create a record of their strategies in a shared document, which they displayed to the other groups in the summary portion of the session. This document could then be shared with the other groups. The teachers talked to each other, worked simultaneously on the shared document, and used the chat window to communicate in the virtual space. The course instructors listened to and participated in these group discussions to facilitate the group work.

9.3.2 The Teaching Labs

The Teaching Labs were designed to provide images of the practices emphasized in the OMD course, particularly how those practices provide opportunities to examine student thinking. Our goal was to have teachers notice how the qualities of the tasks implemented in the lessons, in conjunction with teachers' actions, fostered student thinking and to understand how they could productively leverage that thinking to make connections related to big mathematical ideas. To do this, we prompted the teachers to engage in detailed and complex analyses of student thinking in order to make connections between tasks, teachers' discourse moves, and student thinking, a process similar to lesson analysis (Santagata and Yeh 2014). As such, our Teaching Labs include features of the studio model (e.g., Teachers Development Group 2010), lesson study, (e.g., Fernandez and Yoshida 2004), or demonstration lessons (e.g., Barlow and Holbert 2013; Strayer et al. 2017).

Prior to the Teaching Lab, a project team member taught a lesson situated in the classroom of one of the teachers. The tasks enacted in the Teaching Labs were drawn from the OMD course; this helped the teachers see a direct connection between the course content and the instructional practices modeled in the Teaching Labs. The lesson was recorded by project team members using two cameras.

The Teaching Lab facilitators then intentionally selected video clips in order to support the teachers to conjecture how students would engage with mathematical tasks and how teacher moves influenced student thinking. Within each Teaching Lab, we engaged teachers in multiple cycles of video viewing followed by individual or small group reflection time, culminated by a whole group discussion. In the first cycle, the facilitator showed a quick clip and then asked the teachers what the instructor in the video might do. Then they watched the rest of the clip. In the other two cycles, the teachers watched a 10-minute clip and then responded to a prompt in which they were asked to anticipate the instructor's subsequent moves and the potential impact of those moves on what students would do. During these cycles, teachers responded to prompts in Google documents that we termed *capture sheets*. These prompts asked the teachers to identify key moments in the focal lessons and to reflect on them in the light of the discourse practices discussed in the OMD course. The instructors were able to read the capture sheets in real-time, and they provided the research team insights into the teachers' insights and learning. The entire Teaching Lab took place in a 2-hour synchronous online session.

9.3.3 Video-Assisted Coaching Cycles

The video-assisted coaching cycles were grounded in a content-focused coaching model, albeit online. Content-focused coaching focuses on the mathematical goals of the lesson and student engagement with those goals (West and Staub 2003); this model stands in contrast to instructional coaching, in which the focus is typically on an instructional technique. The in-person version of content-focused coaching uses coaching cycles that include pre- and post-lesson conferences situated around a collaboratively taught lesson; in our case, however, the coach and teacher never met in person. Instead, after the planning session, conducted via Zoom, the teacher would teach the lesson, which they video-recorded using Swivl technology. The Swivl platform features the ability to annotate videos; so, both the teacher and coach would annotate the video by identifying moments when the teachers' or students' actions were related to the goal for that coaching cycle. The coach and participant would write a comment or question for each of these key moments. As soon after annotating the video as schedules permitted, the coach and teacher would meet again to debrief the video-recorded lesson.

The Swivl technology enabled the teacher to video-record a lesson without having another person operate the camera. The teacher simply placed an iPad on a Swivl robot and wore a marker on a lanyard. The marker signaled the Swivl robot to rotate the iPad in the direction of the teacher so that the iPad tracked the teacher's movements. Simultaneously, the marker acted as an audio recorder, capturing sounds within the range of the marker (such as the teacher and student voices) to the iPad. The Swivl technology permits the use of multiple markers, and thus multiple audio channels are synced to the same video recording. This made it possible for teachers to record multiple groups by dropping additional markers on groups' tables. Once the lesson was over, the teacher pressed a button on the iPad to stop recording, which triggered an automatic upload of the video to the Swivl library. If the teacher had poor Internet access, not unusual in a rural context, they would wait until they had better connectivity to initiate the upload (Fig. 9.3).

The annotation process, using Swivl software, was similarly simple and highly productive. The teacher or coach logged into their account and selected the appropriate video. As they watched the video, when they noticed a key moment, they clicked in the comment box and began writing, which automatically paused the video. Once the comment was posted, the video would automatically begin to play again. The coach and teacher then viewed each other's annotations before the debriefing session. The annotations typically structured the debriefing session; the coach or teacher referred to an annotation to spur a line of discussion. When the line of discussion concluded, the coach (usually) moved to the next annotation. Figure 9.4 shows the process for one coaching cycle.

Fig. 9.3 Swivl with a marker

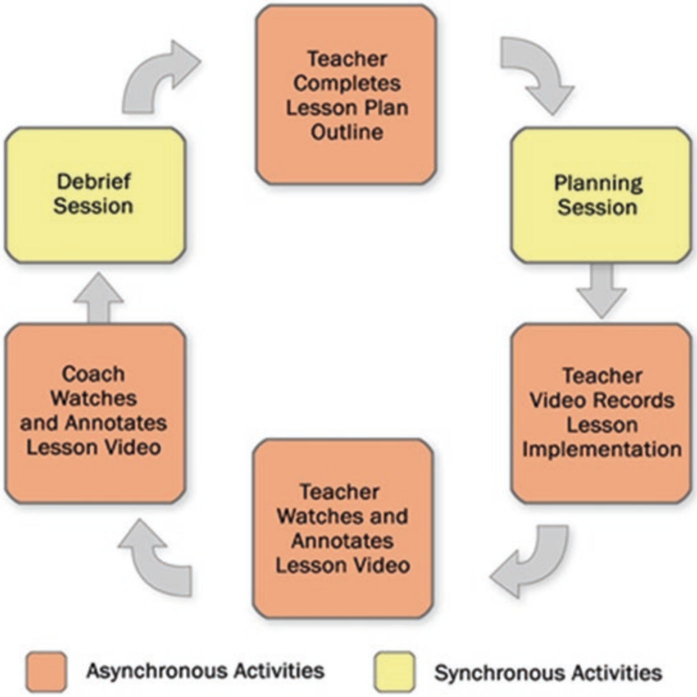

Fig. 9.4 Process for coaching cycle, including annotations

9.4 Researching an Online Professional Development Project

Here we describe how we researched the intervention in an online space. We based our analytic framing on the theory of action previously described. Specifically, for the OMD course and Teaching Labs, we characterized participation in terms of high-depth interactions, using prior work that we had done on a conjecture mapping (Sandoval 2014) exercise (Choppin et al. 2018). For coaching, we focused on elements identified in our theory of action, namely, nonevaluative observation of lessons, critical reflections that connected observations to the elements of ambitious instruction emphasized in the course, and the feedback provided by the teachers' coaches.

To highlight the aspects of the research that were explicitly formulated to take advantage of the online nature of the intervention, we focus on a subset of our data. For the OMD course, we utilized transcripts of the breakout rooms and some whole group interaction. For the Teaching Labs, we utilized data from our capture sheets. For the coaching cycles, we focused on the annotations, a unique feature of the online nature of the coaching. The annotations provided us insights into the elements of the lessons the teachers noticed, the teachers' critical reflections of those moments, and the nature of the coaches' feedback.

9.4.1 Analyzing Transcripts from Breakout Rooms of OMD Course

As part of the course design process, we engaged in a conjecture mapping (Sandoval 2014) exercise in which we identified four key processes in which we wanted our teachers to engage. These four processes map to Coburn and Russell's (2008) definition of high-depth interactions in that they addressed pedagogical principles, focused on mathematics in a meaningful way, and explored the impact of instruction and tasks on student learning. Our four processes were: explaining how task features provide opportunities to engage in mathematical thinking; explaining mathematics in ways that make connections; explaining anticipated or observed strategies or misconceptions; and explaining how teaching moves impact student authority and access to mathematical thinking.

The breakout rooms were recorded using a combination of Zoom and Panopto. Panopto allowed us to record the Zoom screen in addition to the shared Google document the teachers collectively worked on during the course sessions. The transcriber thus had access to the audio of the breakout rooms, the Zoom screen, and the Google documents in order to connect the conversation with the corresponding teacher work. We analyzed the transcripts of the breakout rooms and summary discussions of the OMD course turn-by-turn. For each transcript, two members of the research team coded the transcript and then met to reconcile the codes. We alternated the coder pairings to minimize rater drift (Harik et al. 2009). The coders would identify in a binary manner (present/not present) each of the four processes. If the coder felt that more than one process applied, they chose the one that was dominant and only coded that one process.

9.4.2 Analyzing the Capture Sheets from the Teaching Labs

The capture sheets were automatically saved and the responses pasted into Google sheets for coding. To analyze the capture sheets, we used more parsimonious coding categories for each reflection. We coded the reflections if they (1) identified when teachers connected student strategies to the mathematical goals of the lesson or (2) identified teacher discourse moves and their impact on student engagement or access to mathematical thinking. These categories constituted high-depth interactions and were focused more explicitly on the impact of teacher actions than the codes for the course transcripts, reflecting the purpose of the Teaching Labs.

9.4.3 Analyzing the Annotations

To analyze the annotations from the coaching cycles, we pasted each annotation into a Google sheet and used columns to label who made the annotation and the time stamp to show which part of the lesson video matched the annotation. We then analyzed the teachers' annotations by considering the elements of the lessons they noticed and their critical reflections of those moments. To consider the elements of the lesson they noticed, we coded the annotation for content (what the annotation was about). To code for critical reflection, we coded for stance (how the teacher reflected on the episode), using the codes report, describe, evaluate, and interpret. These themes were adapted from the literature on *noticing* (cf. Van Es and Sherin 2008). Report, describe, and evaluate represent lower-level noticing, where the teacher is primarily marking a moment; by contrast, interpret involves higher-level noticing, because it makes a connection between the moment and a pedagogical principle.

In terms of the coaches' feedback, we analyzed the stance according to two broad categories. One category included the themes describe, evaluate, and interpret, similar to the themes used for teachers. The second category characterized whether the coach's suggestion was in the form of direct assistance (suggest or explain) or invitational (elicit) (see Gillespie et al. 2019 and Ippolito 2010, for a fuller description of this distinction).

9.5 Findings

We describe findings from each of the analyses. The findings demonstrate how the data sources provided insights into the processes in which we engaged teachers, and how the online context provided affordances for understanding these processes.

9.5.1 High-Depth Interactions in the OMD Course

We found that the teachers engaged in all four processes we characterized as high-depth interactions. Table 9.1 shows how the instances of the processes were distributed across the course sessions and reflect the themes explored in each of the sessions. Notably, we found a substantial number of teacher turns in which they explained how teacher moves impacted student authority and access to mathematical thinking, which we felt was the most challenging type of high-depth interaction. Below we provide some examples from the data of each of these types of interactions to provide context and detail.

Explaining How Task Features Provide Opportunities to Engage in Mathematical Thinking

In the second session of the OMD course, the facilitators introduced the task analysis guide (Stein et al. 2009) and asked the teachers to characterize the cognitive demand of four tasks. This was revisited in the third session after the teachers had an opportunity to reflect on their characterizations of the tasks.

One of the tasks involved the following linear relationship: *After 30 people had bought tickets, the register had $205 in it, and after 80 people had bought tickets, the register contained $480*. Students were asked to *write a sentence or a rule that*

Table 9.1 Frequencies of high-depth interactions in course sessions

	Explaining how task features provide opportunities to engage in mathematical thinking	Explaining mathematics in ways that make connections	Explaining anticipated or observed strategies or misconceptions	Explaining how teaching moves impact student authority and access to mathematical thinking
Session 1	0	41	6	4
Session 2	15	8	3	4
Session 3	1	11	6	29
Session 4	0	5	24	13
Session 5	0	0	19	10
Session 6	0	5	3	30

describes how to find the amount of money in the cash register after any number of people have bought tickets. Branson (all names are pseudonyms) stated that the demands of the task were limiting if it were just about coming up with a rule, stating that "if the learning goal becomes how to write rules and sentences and the rest of it is left open and there's many different types of rules," then it would be less limiting. Brown added a similar thought, stating:

> It's not limiting them to all writing the same sentence or writing the same rule or using the same procedure…you could get kids to think about it in more than one way or purposefully choose certain students to share their work and to show the different sentences and rules that they did.

These teachers did not specify what they meant by different kinds of sentences or rules, but at this stage of the course, they recognized that tasks that had multiple approaches were more desirable than tasks that had prescribed methods.

The teachers also examined the following task: *An above-ground swimming pool in the shape of a cylinder has a diameter of 18 feet and a height of 4.5 ft. If the pool is filled with water to 6 in. from the top of the pool, what is the volume of the water in the pool?* Brown reported that this task was procedural, stating "they know 4.5, and then I just do up to 4 feet and, you know, then use that as the height and then they are just using a procedure of plug-and-chug." Harris described how this task did not afford opportunities for sense-making, stating "they wouldn't have an understanding of what volume was and being able to explain volume; the task is just being able to memorize a function and put it in each piece."

Getting teachers to reflect on the connections between tasks and students' opportunity to explore mathematics was critical to their subsequent goal-setting and task selection; while these quotes show that the teachers were beginning to engage with the ideas, it was also clear to us that they could have been more explicit with respect to how task design afforded different approaches that could lead to meaningful classroom discussions.

Explaining Mathematics in Ways that Make Connections

Because we wanted teachers to engage in the kinds of mathematical discourse that they would then facilitate with their students, we engaged them in experiences as learners. We provided the teachers with opportunities to solve problems and then discuss their solutions in ways that demonstrated connections to the underlying

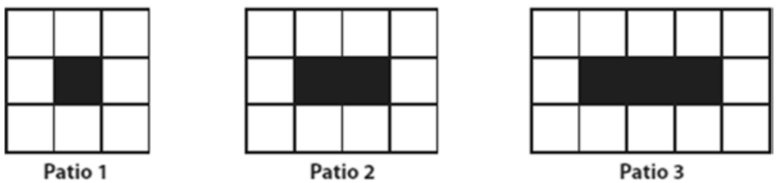

Fig. 9.5 Tiling problem

rationale for their solutions. There were some rich mathematical discussions in the OMD course as a result. Below, we provide some examples of the mathematical explanations that resulted from the tiling task (Smith and Stein 2018), in which the teachers were asked to represent the pattern for the number of tiles around a garden. See Fig. 9.5 below for a diagram of the problem.

Brown described how she used a table to find the pattern:

> So I created a T chart that was just like black and white so patio 1 had one black and 8 white, and patio 2 had 2 black and 10 white, um, and patio 3 had 3 and 12, and so then I made the guess that patio 4 would have 4…4 and 14, and then 5 and 16 for the next two. And then I actually drew the pictures to match the pattern and like, proved that I was correct in my table.

Wilson noted she got the same outcome as Brown, but connected her solution to the geometry of the diagram:

> As, Brown did, that is actually, I also got this 2N, 2 times the N, that is the N is for the number of black tiles then, 2N plus 6, I made it, an equation like that, then I checked whether it works with other uh, number patios by drawing, by putting a diagram for uh…four patios and five patios and it, it worked.

Brown then described how she developed the equation from her table

> Um, I actually went back to the table. (laughs) And I worked backwards from the table to figure out basically the Y-intercept, so if there were zero, then there would…if there were zero black then there would be six. So that gave me the Y-intercept…and then the Y column was increasing by 2 each time so that gave me the slope.

Other teachers provided variations of these explanations; this discussion exemplified how the teachers generated and then considered multiple ways to approach the problem. Furthermore, the facilitator subsequently highlighted how attending to the connections between the approaches was a productive way to engage students in rich mathematical discussions. The ability to easily record and transcribe these breakout rooms greatly facilitated our understanding of the teachers' thinking.

Explaining Anticipated or Observed Strategies or Misconceptions

A key focus of our project was teacher noticing of student thinking (e.g., Jacobs et al. 2010; Van Es and Sherin 2008). Thus, we had the teachers analyze teaching scenarios and artifacts showing student strategies, in addition to having them attend to each other's mathematical explanations. We wanted them to notice the details of other's mathematical thinking and then connect those observations to how others were making sense of the mathematics. Below, we present some examples of the teachers engaging in this process. The examples derive from the analysis of student strategies related to the orange juice problem, in which students would compare four mixtures of concentrate and water to decide which one had the strongest orange flavor and which one the weakest.

Owens noted that his students would employ an additive strategy, reporting they would say "the higher the difference, the least orangey it would be." Jones described a strategy employed in one of the student artifacts, stating "So it looks like group three is all about comparing the fraction of part to part; so they're comparing the cups of concentrate two cups of water and they just turned each into a fraction." Garrett noted that one group converted some of the fractions to a common denominator, noting that the students got "all the concentrates to 15."

These examples demonstrate how the teachers reflected on the mathematical thinking of others, which we hope was a precursor for them doing so in their own classrooms. The examples above were complemented by other occasions in the course when the teachers described others' mathematical strategies in detail.

Explaining the Impact of Teaching Moves

Perhaps the most important process was having the teachers analyze the impact of teaching moves in nonevaluative ways. Specifically, we wanted them to notice how teachers' moves impacted student authority and access to mathematical thinking. To do so, we asked the teachers to analyze brief teaching scenarios (http://mathpractices.edc.org/; Stein et al. 2009) to guide them to more productive ways to analyze and discuss instances of teaching. Below we provide examples of teachers analyzing the impact of teacher moves in the scenarios.

Two teachers commented on the impact of the teacher's questioning on encouraging students to look to themselves as mathematical authorities. Garrett stated:

> The way he was asking questions once that one group got to a 4 by 4 then he was saying 'okay, let's—what if—do all squares work then?' you know what I mean? And kind of that questioning kind of really made them think like, 'let's start to look at patterns.'

Jackson similarly noted the impact of the teacher's questioning on student authority, noting "So his questioning impacted the implementation because…he made sure all of them had the same definition for perimeter and area, which helped them to then at least try to be successful with it." In another case, Jackson noted "she is questioning kids to explain the task in their own words, so each student knows what's expected of them, what their goal is, so they at least all have a starting point."

One participant commented on how teacher moves provided access to mathematical thinking and connected it to her own emerging instructional practices. Harris stated that:

> I have been working on my wait time…I just kind of sat there and waited awhile and I saw a student like he was thinking about answering and finally because I waited so long, he shot up a question and then his questions kind of started to give the snowball effect and more kids asking some questions…it's only been maybe a week or so of me doing more wait time and I am starting to see kids asking questions quicker or leading conversations into stuff more now.

Garrett noted how a teacher in the scenario gave students time to ask each other what the problem was, which impacted both student authority and gave the teacher access to students' thinking:

> I just feel like that the move that the teachers made basically on just having them in the groups and, after reading the problem and just having that discussion, kind of understanding the problem, you know, 'what is it asking?' where they're just discussing to give them that success before they started working on private think time, you know where they actually had some ideas of how organize.

These examples provide insight into the ways the teachers productively analyzed teachers' instructional practices; the teachers considered how those practices impacted what students did and how they provided insights into student thinking.

9.5.2 Teaching Labs

We focused on two categories of high-depth processes in the Teaching Labs when the teachers: connected student strategies to the mathematical goals of the lesson; and identified teacher discourse moves and the impact on student engagement or access to mathematical thinking. These categories constituted high-depth interactions and were focused more explicitly on the impact of teacher actions than the codes for the course transcripts, reflecting the purpose of the Teaching Labs. Below we provide our analysis of quotes from the capture sheets the teachers used to record their observations of the lessons.

Connecting Strategies to the Mathematical Goals of the Lesson

A key intention of our project was to focus the teachers on the mathematical goals of lessons and to attend to the ways students engaged with the mathematics around those goals. We observed a number of the teachers who did this. Jones described how she would sequence the presentation of student strategies to highlight the goals of the lesson in which students were using proportional reasoning to compare juice concentrations:

> I would start with Group Five and then show how they could progress to Group One's work of getting that common amount of water, instead of choosing 18, choosing 90 to make it work for all of the mixtures. It went Five, then One, and then jumping over to Three to show the unit rate because there's the—oh, no, I'm sorry. Group Three was still with the common amount, but now they're looking at concentrate instead of water, so you can make the connection there. Then lastly would be Group Two, looking at concentrate again, but now they're looking at unit rate.

Garrett speculated on the role of goals on the teacher's practice and how those goals guided the teacher:

> Just knowing the goals I think really helped establish the questions that he was going to have when he was interacting with these groups. Now, obviously, he wasn't sure what was going to come about, but the way that they started he was able to see that they were going a common route, something about comparing, and so then he could push them by asking those questions.

In these examples, the teachers shared their observations about how elements from the Teaching Lab video connected to lesson goals. In the first example, Jones described each group's solution strategy to discuss how she would sequence student work for a discussion to accomplish the intended learning goals. In the second example, Garrett highlighted the questioning moves the teacher in the video used to elicit student thinking to better illuminate how students understood the learning goals. The use of the capture sheets allowed us to automatically save and subsequently analyze teachers' reflections on the focal lessons.

Connecting Teachers' Actions to Student Access to Mathematical Goals

Another goal of our professional learning model was for the teachers to understand how teachers' instructional practices can successfully engage students with the mathematical goals of the lesson. The Teaching Labs provided a common referent to reflect on this. This theme was the most productive aspect of the Teaching Lab, with nearly 20 instances from the last Teaching Lab. Harris, for example, described how the teacher provided an opportunity for students to generate their own ideas, stating "I noticed that students had the opportunity to come up with some of their own ideas before given the task at hand. I think the students were more engaged because they didn't know what the teacher was expecting." Another participant, Jones, commented on how the teacher pressed other students to explain a strategy in the group, stating:

> So that one girl that was explaining how she got the unit rate, and then he asked, 'Can someone else in this group share why that is helpful? Why might it be helpful?' He called on them to explain her thought process or the importance of that, which I thought was neat.

Similarly, Wilson noted how the teacher encouraged a student to explain a key concept to other members of the group, stating, "When the students came up with the unit rate, actually he was asking them why the unit rate is helpful, and he was asking them to explain to others why they came up with that unit rate." Brown noted the use of an open-ended question to advance a group's thinking, reporting:

> Even the way that he left both groups was with an open-ended question, because when he left group two, he said, 'Keep running with it, and I'm going to see how you've progressed when I come back.' Both of them still had that, okay, you're doing a good job. You're on the right track, so you know that you're not totally wrong. Keep going. I thought that was a good way to leave them confident in what they're doing but still thinking about where to keep it going.

In addition, teachers noted strategies that kept the students focused, with Branson noting that:

> When he asked students to read, he just didn't ask them to passively read and say, 'Hey, read this.' He seemed to always give some type of direction or prompt or aid to help those kids through the reading process, like when he said, 'As you read this, I want you to think about what this problem's about.'

Jones added that the teacher used prompts to focus the students as they read the problem:

> We noticed that he praised both groups before leaving them, which is a really smart move to build up confidence. You said something along the lines of, ‘I really love your strategy. I like what you’re doing. Keep going forward with it,’ and I think that gets the kids to—it helps validate what they’re doing and encourages them to keep moving on with what they’re doing

The teachers reported a range of ways that the teachers’ moves influenced student activity, pushing students to focus on the task, advance their thinking, and promote participation structures. The design of the Teaching Lab facilitated opportunities for the teachers to engage in noticing student thinking and connecting that to mathematical goals and teacher moves.

9.5.3 Coaching Cycles

We analyzed the video annotations as data that emerged from the online nature of the coaching. We focus our analysis on two coach-teacher pairs because they served as representative cases and illustrate the use of annotations as a form of professional development and as an analysis tool. The two pairings differed according to the patterns in both the coaches’ and teachers’ annotations, described in more detail and summarized in Table 9.2.

Patterns in Coaches’ Annotations

In terms of differences between the coaches, Bishop asked questions and made suggestions in addition to describing and evaluating practice, while Lenore primarily evaluated and interpreted episodes in the video. Bishop offered detailed descriptions of practice that were the basis of questions posed to the teacher, such as the following:

> At this point, you are re-introducing the task to the students on the second day of work on the task and then giving instructions to students about how you want them to proceed. So,

Table 9.2 Characterizations of annotations of two coach-teacher pairings

Bishop-Brown pairing				Lenore-Owen pairing			
Coach move		Teacher move		Coach move		Teacher move	
Elicit	3	Report	0	Elicit	0	Report	0
Suggest	2	Describe	0	Suggest	0	Describe	4
Explain	0	Evaluate	2	Explain	0	Evaluate	8
Describe	3	Interpret	0	Describe	1	Interpret	1
Evaluate	2	Ask question	1	Evaluate	12	Ask question	1
Interpret	0			Interpret	7		

> what I think you are thinking about in your comment is 'How can I accomplish these two goals in a manner that engages students in the conversation with both you and with their peers?'

In other cases, Bishop offered a suggestion to the teacher:

> I wonder what would have happened if you had asked the group to work together to recall how they answered the question 'How many tickets were sold if there was $755 in the cash box?' And that you would return in 5–10 minutes to find out how they answered that question?

Most of Bishop's annotations contained a similar level of detail, indicating a close viewing of the video and setting expectations for subsequent reflections of the lesson. We found that Bishop used these annotations to structure the debriefing discussions with the teacher, often going sequentially through them to guide the discussion.

Lenore's annotations were much shorter and more evaluative in nature. Many of the annotations we coded as evaluative were positive and encouraging, such as "love the connection between Dameon and Molly's work here" and:

> A few derailments but that comes with the job of working with kids. There is a ton of evidence regarding student engagement and student-to-student dialogue which is a huge accomplishment. Well done. That was fun to watch.

Some of Lenore's annotations responded directly to one of the teacher's annotations in response to the teacher asking "Was I leading her too much?" Lenore responded "I don't think so. You are repeating back to her everything she just told you and you are asking great questions."

The analysis shows how the annotations provided insights into the feedback the coaches provided to the teachers. In our prior analysis, we found that the patterns in the coaches' annotations were consistent with patterns we found in transcripts of the coaching sessions, suggesting that the analysis of the annotations provides a quick and accessible insight into the coach-teacher dynamic. However, we note the limitations in understanding teachers' critical reflections of their lessons, which were much more evident in the transcripts of the coaching sessions.

Patterns in Teachers' Annotations

In terms of teachers, Brown made only three annotations, two of which were evaluations (e.g., "I feel like I'm talking a lot. It's a lot of directions without a lot of interaction"), while Owen was much more active and had a wider range of annotations, including one coded as interpretation ("Molly was adding 194 repeatedly…forever, going way past of where she had to. 194 was referring to 97 and 97 added together"). In general, these teachers offered shorter and more evaluative annotations, which was consistent with teachers' annotations throughout the project.

9.6 Discussion

The goal of the project was to support rural mathematics teachers to develop ambitious instructional practices. In this chapter, we provided details of the professional development model we designed to accomplish this goal and our efforts to research the model. Specifically, we chose aspects of the analytical methods that took advantage of the online nature of the project. The findings, which reflect a subset of the data corpus generated in the project, offer evidence that the teachers had opportunities to engage in high-depth interactions around images of ambitious instruction and their attempts to enact ambitious instruction.

Of relevance to the focus of this book, this project demonstrates that online environments do not constrain the possibility of providing teachers with high-quality professional development. Teachers had opportunities to read and discuss discourse practices consistent with ambitious forms of mathematics instruction, observe and reflect on images of practice, critically reflect on their own attempts to implement ambitious instruction, and receive timely and detailed feedback on their practice.

The main contribution this chapter makes is the consideration of data sources that were made possible or more accessible by the online context. The ease of recording breakout rooms in Zoom and the generally high quality of the audio gave us insight into teachers' reflections on the tasks and images of practice discussed in the course. The use of Google documents to record in real-time teachers' reflection on the focal lesson in the Teaching Labs provided us insight into the detailed ways teachers reflected on key aspects of practice. The annotations of the video recordings of teachers' lessons were easily generated and immediately accessible; these annotations provided insights into the specific ways coaches and teachers reflected on the lessons in addition to broader dynamics that were evident in other sources of data in the project.

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Chapter 10
The Impact of an Online Teacher Education Program on the Development of Prospective Secondary Mathematics Teachers' Noticing

Ceneida Fernández, Salvador Llinares, and Yoilyn Rojas

10.1 Introduction

The development of digital technologies has generated new questions regarding prospective teachers' learning and knowledge-building practices in formal distance mathematics teacher training programs (Borba et al. 2013, 2018; Borba and Llinares 2012). Technology-mediated interactions between individuals in social interaction spaces influence prospective teachers' learning, creating new opportunities to build knowledge (Bragg et al. 2021; Prilop et al. 2021; Weber et al. 2018). Interactive technologies in distance learning contexts allow prospective teachers to collaborate with their peers and university tutors in new ways. Interaction spaces and the way in which prospective teachers take part in them create new forms of knowledge construction (Cendros-Araujo and Gadanidis 2020) and lead to the development of competencies that are not yet well understood (Silverman and Hoyos 2018).

Although there has been a growth in the design of online professional development opportunities for teachers in recent years, there are still questions about the most effective practices to facilitate teachers' professional development online, and about which design elements of online professional development programs support teachers' learning (Bragg et al. 2021). In fact, Engelbrecht et al. (2020) identified the analysis of knowledge construction processes and competence development in

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online contexts as well as the influence of design elements as domains for further research.

This chapter focuses on the potential of online environments to enhance prospective teachers' noticing skills in teaching-learning situations (Mason 2002). Mason (2011) claimed that "noticing is as a movement or shift of attention" (p. 45), so it implies an increase in sensitivity to the details of learning situations, avoiding generalities, emotional content, or judgments. This shift of attention is articulated by different ways of paying attention: (1) discerning details; (2) recognizing relationships, that is, becoming aware of sameness and differences; (3) perceiving properties, understood as becoming aware of particular relationships as instances of properties that could hold in other situations; and (4) reasoning on the basis of agreed properties going beyond the assembling of things one thinks one knows. Noticing can be understood as a teaching competence that allows teachers to recognize what is relevant in a teaching-learning situation, attending to some details and ignoring others to act accordingly and support students' learning.

Recent research has focused on different tools to develop teachers' noticing competencies online (Fernández et al. 2012, 2020; Fernández and Choy 2020; Ivars and Fernández 2018; Prilop et al. 2021; Weber et al. 2018). This previous research supports the idea that sharing spaces of interaction generates opportunities for participants to reorganize their knowledge and develop aspects related to the noticing competence. Technology-mediated interaction is assumed to have an impact on participants' learning, allowing them to determine focal points of attention and the possibility of using theoretical knowledge to reason about their teaching. However, although previous research has shown some tools for the development of noticing, such as online forums or the expert feedback (Prilop et al. 2021; Weber et al. 2018), little is known about its development during the internship period at schools (Fernández et al. 2020; Stockero 2021).

In this chapter, we contribute to the field of noticing development by characterizing prospective teachers' noticing development in a formal online environment during the internship period at schools. The environment was based on two theoretical perspectives: an enactivism perspective (knowing is doing and doing is knowing) and Mason's conceptualization of noticing. It consists of cycles in which prospective teachers write a narrative about their own practices and share them over online forums, receiving feedback from their fellow classmates and the university tutor (instructor) and then writing a new narrative. These cycles were intended to help prospective teachers to focus their attention on significant events as they teach and to determine new teaching moves. By analyzing prospective teachers' participation in these cycles, we can characterize how prospective teachers develop their noticing skill. We formulated the following research questions:

- How do prospective teachers develop their noticing skills in a formal online environment where they have to write narratives about their own practices and share them over online forums?
- How do the design elements of the formal online environment influence the prospective teachers' learning?

10.2 Becoming Aware: Knowing Is Doing and Doing Is Knowing

This research was at the intersection of two theoretical perspectives: the enactive stance "knowing is doing and doing is knowing" used for deliberative analysis (Brown and Coles 2011) and the process of becoming aware (Mason 2002).

The enactivism perspective makes the link between knowing and doing explicit and can reveal certain behavior patterns which support levels of awareness (Mason 1998). From this viewpoint, the development of prospective mathematics teachers' noticing can be understood as an active process of categorizing their interpretations of situations and deciding what to do (Brown and Coles 2012). Brown and Coles (2011) underline that "over time, if novices are able to analyze their experiences, they literally come to see more linked to their actions" (p. 862). Therefore, the deliberative analysis allows to unpack the reasons that underlie one's actions (Brown and Coles 2012). Brown and Coles (2012) from Varela indicate that "deliberative analysis involves 'after acting spontaneously', being able to 'reconstruct the intelligent awareness' that justifies the action" (p. 220).

In order to engage prospective teachers in this process, we designed a learning environment consisting of the following cycles: writing a narrative (a description of a mathematics situation relevant for the students' learning)—sharing this narrative with others over an online forum in which prospective teachers can provide the reasons underlying their actions; opening themselves up to alternative possibilities from other participants (fellows and university tutor); and then writing a new narrative. Asking for a deliberative analysis obliges one to become aware of a range of actions (Brown and Coles 2012), and, therefore, we assume that it could encourage the development of noticing.

10.3 A Formal Online Environment in a Costa Rican Distance State University (UNED)

The formal online environment was designed for the Degree in Mathematics Teaching at the Universidad Estatal a Distancia de Costa Rica (UNED). This degree qualifies a student to be a mathematics teacher in secondary education (students at that stage range from 13 to 16 years old). It is a 4-year degree (12 4-month periods) that provides training in mathematics, pedagogy, psychology, legislation, and teaching resources for the teaching of mathematics. UNED implements a distance educational model using synchronous (interactive webinars) and asynchronous (online forums) tools in virtual environments supported by the university's institutional Moodle virtual platform. The learning environment was implemented during a traineeship period in secondary education institutions.

The traineeship period consisted of eight 2-week periods (a total of 16 weeks) divided into two phases (Fig. 10.1). During the first phase (three periods making up

Phases during the traineeship period.

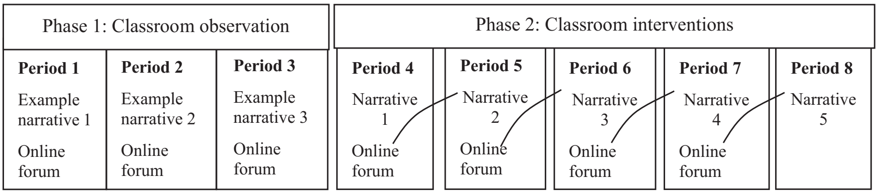

Fig. 10.1 Phases during the traineeship period

a total of 6 weeks), prospective teachers (PTs) observe classrooms. During the second phase (five periods making up a total of 10 weeks), PTs plan and implement a teaching unit. Furthermore, they write five narratives about mathematics classroom situations that they consider relevant for students' mathematical learning, one for each period in the second phase. They share these narratives on asynchronous online forums integrated into the Moodle platform where PTs receive feedback from the university tutor and their fellows and give feedback to their fellow classmates. To avoid difficulties with the writing of narratives, PTs were provided with some examples during phase 1 and were trained on how to participate in online forums to give feedback to others.

The design is characterized by a cycle in which a narrative is written and shared on an online forum, and then a new narrative is written taking into consideration the feedback received in the forum. This cycle provides PTs with the opportunity to link their knowledge to their actions (practice).

10.3.1 Narratives: Guided Questions

PTs were required to write five narratives in which they described classroom situations during their own lessons that they believed contributed to the students' mathematical learning. A narrative is a story in which the author tells about a sequence of events that are significant to him/her and presents an internal logic that makes sense to him/her (Chapman 2008). Previous research has shown that writing narratives can help PTs to structure their attention, particularly when noticing students' mathematical thinking (Cavanagh and McMaster 2015; Ivars and Fernández 2018). The narratives can also become shared objects allowing for better relationships between theory and practice (Pulvermacher and Lefstein 2016).

PTs were provided with guided questions to write the narratives. Guided questions were based on the skills of identifying, interpreting, and making decisions related to the noticing of students' mathematical thinking competence (Jacobs et al. 2010) and taking into account the findings of previous studies (Ivars and Fernández 2018) (Appendix 1). These guiding questions were designed to help PTs to focus their attention on specific aspects of the mathematics teaching-learning situations, thus supporting the development of awareness.

10.3.2 The MOST Analytical Framework

Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOSTs) are "instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas" (Leatham et al. 2015, p. 90). Student instances are understood as observable student actions or a small collection of connected actions. MOSTs are at the intersection of three characteristics: student's mathematical thinking, mathematically significant, and pedagogical opportunity (Leatham et al. 2015):

- Student mathematical thinking. The instance must meet two criteria: (1) student mathematics—if an observer can infer what the student is expressing mathematically—and (2) mathematical point, if there is a mathematical idea that is closely related to the instance's student mathematics.
- Mathematically significant. An instance is characterized as mathematically significant when it meets two criteria: (1) appropriate mathematics—the mathematical point must be accessible to the students based on their prior mathematical experiences and should not be one that most students at this mathematical level would already understand—and (2) central mathematics, if the instance's mathematical point is closely related to a learning goal in the corresponding lesson.
- Pedagogical opportunity. An instance embodies a pedagogical opportunity when it meets two criteria: (1) opening (an instance in which the expression of a student's mathematical thinking seems to create an intellectual need for students to make sense of the student mathematics) and (2) timing (the timing must be right to catch a pedagogical move).

The MOST analytical framework provides a means for identifying how PTs (1) discern instances of student thinking that can be mathematically important in a given lesson; (2) link the particular instance of student mathematical thinking with broader education principles (recognizing and perceiving); and (3) take into account the classroom context when determining whether the instance might provide leverage for moving the student forward in their mathematical understanding (providing next teaching moves).

The MOST analytical framework was used by researchers (one of them was the university tutor) as an analytical tool to analyze PTs' narratives. This framework helped to identify mathematically productive instances of student thinking (MOSTs) in the PTs' narratives (knowing), and whether PTs took advantage of them during the lesson (doing), allowing us to describe changes in PTs' noticing.

10.3.3 Online Forums: Feedback

Once PTs had written a narrative, they shared it with the university tutor and their classmates on an online forum of a Moodle platform (Appendix 2). PTs could provide feedback on the narratives of one or all classmates, and they could see the feedback provided in other narratives. Feedback in online learning environments consists of prompts to promote PTs' learning, providing information that guides them toward the improvement of their narratives (Wang et al. 2019). For instance, the university tutor could suggest a reading during the online discussion in order to support PTs' reflections about what had been noticed, or participants (university tutor and classmates) could question the meanings underlying something that has been said. The tutor provided feedback to all their narratives with the focus on guiding PTs toward the identification of MOSTs and toward the way to take advantage of them.

Feedback from the tutor and other classmates in online forums can be seen as a dialogic process of knowledge construction (Andriessen et al. 2003; Mitchell 2003; Prilop et al. 2021; Weber et al. 2018) that takes place when different perspectives are examined and an agreement is sought on acceptable courses of actions (Fernández et al. 2012; Ivars and Fernández 2018; Llinares and Valls 2009, 2010). Therefore, the written feedback might help PTs focus their attention on the identification of MOSTs during the lesson and on how to take advantage of them. PTs then wrote the next narrative taking into account the feedback provided on the online forum.

10.4 Analyzing the Development of Noticing in an Online Context

Five prospective secondary school mathematics teachers (pseudonyms: JB, AG, GV, AL, NT) participated in the formal online environment. These PTs were in different secondary school institutions in their traineeship period. We analyzed a total of 25 narratives written by these 5 PTs during phase 2 of the traineeship period as well as the feedback provided by fellows and the university tutor on the 4 online forums (Fig. 10.1).

The analysis was performed in two stages. During the first stage, each narrative was analyzed, identifying students' instances that could present the characteristics of a MOST (Leatham et al. 2015) (knowing): student mathematical thinking, mathematically significant, and pedagogical opportunity. We also identified the moment that had generated each MOST using the categories of Stockero and Van Zoest's (2013) study: extending, incorrect mathematics, sense-making, mathematical contradiction, and mathematical confusion.

Table 10.1 shows an example of when a student instance was considered a MOST. The prospective teacher (JB), in the third narrative, describes an interaction

Table 10.1 Analysis example: identifying MOST in PTs' narratives

Excerpt of JB's third narrative	Student instance	MOST identified by the researcher in the analysis of the narrative
JB: We are going to solve the exercise on the blackboard: $4x^2–5x–6$ **Student 1**: This is a trinomial, so I can use the perfect square trinomial **JB**: Well, try to do it. When you finish, let me know	**Student 1**: This is a trinomial, so I can use the perfect square trinomial	The MOST starts when Student 1 makes public incorrect mathematical thinking (incorrect mathematics; with this method the student cannot solve the exercise) Firstly, students' mathematical thinking can be inferred since the student applies an incorrect factoring criterion. Furthermore, recognizing when the factoring method can or cannot be used is key to fulfilling the lesson's learning objectives (mathematically significant) Finally, the student's mathematical thinking creates a need to continue building on his/her mathematical thinking. In this case, the introduction of a new factoring method and its timing in the lesson were in accordance with the objectives and the lesson plan (pedagogical opportunity)

with two students who are trying to factor the polynomial $4x^2–5x–6$ using the methods taught in previous lessons (the full narrative can be found in the following section). However, the polynomial does not meet the criteria to be solvable by the methods previously reviewed. JB uses this fact to introduce a new method, factoring by inspection.

Furthermore, we identified whether PTs took advantage of the MOST (doing) during the lesson, looking at PTs' teaching actions. We used the categories obtained by Stockero and Van Zoest (2013) to classify PTs' teaching actions: ignores or dismisses; acknowledges, but continues as planned; emphasizes the mathematical meanings underlying the questions by highlighting a definition or the mathematics that support a procedure; pursues students' thinking by asking the students to provide more information about their thinking; and extends/makes connections by going beyond the topic in the lesson to revisit and make connections to past learning or to foresee or lay a foundation for future learning.

In the example provided in Table 10.1, JB took advantage of the MOST since he pursued the student's thinking saying "Well, try to do it. When you finish, let me know." JB invited the student to apply his chosen factorization method, although it was not appropriate to solve the exercise. This led the student to think about why he could not use this method.

In the second stage, we compared the five narratives written by a PT to identify changes in the MOSTs identified and how PTs took advantage of them during their traineeships. This comparison enabled us to characterize how noticing develops in a formal online environment.

10.5 Noticing Development and Changes in Prospective Teachers' Practices

From the analysis, PTs' noticing development was evidenced by the changes in their narratives. These changes were based on how PTs took advantage of the MOST (focusing/not focusing on students' understanding) and whether PTs took advantage of different MOSTs by relating/not relating them (Table 10.2). Furthermore, the development of noticing is linked to changes in PTs' practices: changes in class management and changes in lesson planning.

At the end of the learning environment, all PTs showed evidence of change 1, and four out of the five PTs showed also evidence of change 2. We exemplify these changes with two cases: the case of AG who was in a class with 22 ninth graders (15 years old) and the case of JB who was in a class with 26 ninth graders.

10.5.1 The Case of AG

This case exemplifies the change from taking advantage of MOSTs, but not focusing on students' understanding, to taking advantage of them and focusing on students' understanding, illustrating also changes in classroom management.

In the first narrative, AG identified two MOSTs, and took advantage of one during the lesson, but did not continue to build on student thinking. AG described an interaction with two students who had to interpret a frequency distribution table (with intervals) and answer the question: *How many students achieved a mark higher than 80?*

AG: Now, let's look at your table, and tell me how many students achieved a mark higher than 80?
Student 1: ummmm, the table shows that there are only four students. Because 80 is in this interval.
AG: But this interval goes from 80 to 85.
Student 1: That's true, but then how do I know how many students did achieve 80? I have to check the list, right? Let's see…there are two.

Table 10.2 Links between the development of noticing and changes in practices

	Characteristics of PTs' noticing development		Changes in practices
Change 1	**From** identifying MOSTs and not taking advantage of them/or taking advantage of them but not focusing on students' understanding	**To** identifying MOSTs and taking advantage of them focusing on students' understanding	Change in class management
Change 2	**From** taking advantage of different MOSTs without relating them	**To** taking advantage of different MOSTs relating them	Change in lesson planning

AG: I'll repeat the question: how many students achieved a mark higher than 80?
Student 1: ahhhhh, I have to see the frequencies where the marks are higher than 80, so there are 4 in this interval and 6 in the other intervals (pointing out the intervals located above).
AG: Right, have you seen the importance of the frequency distribution table? This allows us to summarize the data so that it is easier to interpret the information.
Student 2: Why do you count the marks of these intervals (the student pointed out the intervals situated above (80–85) that included data such as 90, 91, and 95) if you are asking about marks higher than 80?
AG: Because, for example, 90 and 91 are higher than 80.

The student instances that generated each MOST are shown in Table 10.3. MOST1 started when Student 1 made public incorrect mathematical thinking and MOST2 when Student 2 articulated confusion. AG took advantage of one of the MOSTs by pursuing the students' thinking and emphasizing the mathematical procedures. However, AG's actions did not continue to build on the student's thinking since he made a general comment about the importance of the frequency distribution table.

The feedback provided in the online forum by the tutor and fellow classmates push AG to be more aware of students' thinking, analyzing their understanding and giving students the opportunity to express their thinking and to reason about their confusion (without advancing the correct answer). For instance, the tutor wrote:

> …apart from looking at students' mathematical thinking when interacting with them, you must use it to help the students continue building their thinking, making decisions that help them progress in their learning. You can notice that students seem not have difficulties in

Table 10.3 MOST identified and how AG took advantage of them in the first narrative

Student instance	MOST identified	Evidence of taking advantage of the MOST
Student 1: Ummmm, the table shows that there are only four students. Because 80 is in this interval	MOST1 starts when Student 1 makes public incorrect mathematical thinking and an incorrect solution (incorrect mathematics)	AG takes advantage of MOST1 by emphasizing the mathematical procedures "but this interval goes from 80 to 85" and pursuing Student 1's thinking when he says "I'll repeat the question; how many students have a mark higher than 80?" AG invites Student 1 to investigate and to realize that the result is not correct
Student 2: Why do you count the marks of these intervals (the student pointed out the intervals situated above (80–85) that included data such as 90, 91, and 95) if you are asking about marks higher than 80?	MOST2 starts when Student 2 articulates mathematically what she is confused about (mathematical confusion)	AG does not take advantage of MOST2. He acknowledges Student 2's confusion but continues as planned, since he provides Student 2 with the answer "because, for example, 90 and 91 are higher than 80" but does not continue to build on the student's thinking

reading frequency tables, so we should avoid giving only a correct answer and try to give them the opportunity to think about it.

And JB said: "it seems that not only Student 1 had difficulties in reading the frequency table, so it would have been necessary to continue examining students' thinking." In this sense, the online feedback focused on building awareness around the discerned details of students' mathematical thinking.

In the second narrative, AG was able to identify three MOSTs and took advantage of them, focusing on student's understanding during the lesson. For example, regarding MOST1, AG described an interaction with three students on the subject of differentiating quantitative and qualitative variables.

Student 1: I don't know what a quantitative variable is.
AG: Let's see, what comes to mind when saying QUANtitative (marking the emphasis).
Student 2: It refers to quantity, which can be counted.
AG: Correct, and QUALItative (marking the emphasis)?
Student 1: It refers to qualities or some characteristic.
AG: Correct, so if you have to conduct a survey on some families, give me three questions that refer to quantitative variables and three that refer to qualitative variables.
Student 1: The salary, the number of people living in the house, and the number of rooms would be quantitative variables.
Student 2: Qualitative variables are their job, the color of the house, and…
Student 3: The level of studies.

The student instance that generated MOST1 is shown in Table 10.4. MOST1 started when Student 1 articulated confusion. AG took advantage of this MOST, pursuing students' thinking by asking questions that invited the three students to differentiate between qualitative and quantitative variables. AG's actions helped students to continue building on their thinking.

Table 10.4 MOST1 identified and how AG took advantage of it in the second narrative

Student instance	MOST identified	Evidence of taking advantage of the MOST
Student 1: I don't know what a quantitative variable is	MOST1 starts when Student 1 articulates what he is confused about (mathematical confusion)	AG takes advantage of MOST1 pursuing students' thinking, through questions such as "Let's see, what comes to mind when saying QUANtitative?" or "Correct, so if you have to conduct a survey on some families, give me three questions that refer to quantitative variables and three that refer to qualitative variables"

Changes in Practices: A Change in Class Management

We have shown that, in the first narrative, AG identified students' thinking but without continuing to build on their understanding. AG emphasized some procedures and made a general comment without checking whether students had understood the frequency distribution table with intervals (the lesson's learning objective).

In the second narrative, AG showed a class management change, pursuing students' thinking and trying to build on students' thinking asking them for some examples to ensure they had understood the difference between qualitative and quantitative variables.

10.5.2 The Case of JB

This case exemplifies the change from taking advantage of MOSTs without relating them to taking advantage of them and relating them and illustrates changes in the lesson planning. In the second narrative, JB identified two MOSTs and took advantage of them focusing on students' understanding during the lesson, but he did not relate these MOSTs. JB described an interaction with two students. Both students had to identify the appropriate factoring method and factor the expressions. Student 1 was working on the expression $5x(3x-2)-3x + 2$ and Student 2 on $w^2-z^2 + 4 + 4w-1-2z$.

Student 1: The exercise does not specify which method can be used to obtain the solution.

JB: Yes. The idea is that you identify the factoring method that can be used.

Student 1: In $5x(3x-2)-3x + 2$, you must first solve the multiplication of monomials and then group the expression.

JB: Tell me why you think that is the best option.

Student 1: Because when you have a monomial in front of a parenthesis you have to multiply, in this way, you get $15x^2-10x-3x + 2$, that is, $15x^2-13x + 2$.

JB: That's right, but once you have done it, how would you solve $15x^2-13x + 2$?

Student 1: We can identify that it is a trinomial, so the factorization method of the perfect square trinomial is used.

JB: I'm going to see how [Student 2] is solving the exercise, and then, I will come back to see how you have done it.

**

Student 2: I have a question about this exercise $w^2-z^2 + 4 + 4w-1-2z$. We can group it in two trinomials, but I don't know to which one I have to put the correct constant.

JB: Let's analyze this for a moment. What would happen if we used −1 as the term of c in $w^2 + 4w$, that is, $w^2 + 4w-1$.

Student 2: We cannot do it with the perfect square trinomial factorization method since the square root of −1 is not a real number.

JB: That's right. Now, if we had $w^2 + 4w + 4$, what would happen then?

Student 2: It would be no problem to factorize it. If ax^2 is negative, it is grouped with the negative constant and afterwards we can get a −1 as a common factor. If it is positive, it is grouped with the positive constant.

JB ends with Student 2 and returns to Student 1.

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Student 1: I couldn't solve it with a perfect square trinomial because 2 and 15 do not have an exact root.

JB: So how can you solve this algebraic expression?

Student 1: Looking at this expression 5x(3x-2)-3x + 2, we can take out (3x-2) as a common factor.

The students' instances that generated each MOST are shown in Table 10.5. MOST1 started when Student 1 made public incorrect mathematical thinking and MOST2 when Student 2 articulated a confusion. JB took advantage of the two MOSTs pursuing students' thinking by asking questions that invited both students to investigate why the chosen factorization methods could not be applied. JB's action helped both to continue building on their thinking (without reminding them or giving them the correct procedure). However, JB described these two situations in his narrative without establishing any relationship between them. The two MOSTs showed isolated situations that were only linked by the lesson's mathematical topic.

Table 10.5 MOST identified by JB and how JB took advantage of them in the second narrative

Student instance	MOST identified	Evidence of taking advantage of the MOST
Student 1: In 5x(3x-2)-3x + 2, you must first solve the monomial multiplications and then group the expression	MOST1 started when Student 1 makes public incorrect mathematical thinking and an incorrect solution (incorrect mathematics)	JB takes advantage of MOST1 pursuing Student 1's thinking when he says "Tell me why you think that it is the best option." or "That's right, but once you have done it, how would you solve $15x^2$-13x + 2?" JB invites Student 1 to investigate and to realize that the chosen method is not appropriate
Student 2: I have a question about this exercise w^2-z^2 + 4 + 4w-1-2z. We can group it in two trinomials, but I don't know to which one I have to put the correct constant	MOST2 starts when Student 2 articulates mathematically what he is confused about (mathematical confusion)	JB takes advantage of MOST2 pursuing Student 2's thinking through questions such as "Let's analyze it for a moment. What would happen if we used −1 as the term of c in w^2 + 4w, that is, w^2 + 4w-1" or "That's right. Now, if we had w^2 + 4w + 4, what would happen then?" Again, JB invites Student 2 to inquire about his confusion and error

On the online forum, JB's fellows highlighted the actions that encouraged the students' thinking (inviting students to think about their own difficulties). For example, one of the fellows (AL) wrote:

> JB has created an opportunity during the lesson for building on students' reasoning. Your class exemplifies your students as critics where each procedure is questioned with the help of the teacher clarifying doubts.

On the other hand, the tutor's feedback directs JB toward providing more details of the students' mathematical thinking and explanations about his own teaching actions:

> In your narrative, you described what you did in class. It would be good **if you could interpret, for instance, Student 1's mathematical thinking:** What did you notice in her answer? What did you decide to do during the lesson and why? (Emphasis added)

The tutor provided a comment that pushed JB to explain the reasons for his actions. This type of feedback provides PTs with the opportunity to think more deeply about in-moment actions.

In the third narrative, JB was able to identify four MOSTs. He took advantage of them focusing on student's thinking and established relationships between them (Fig. 10.2). JB described an interaction with two students during a lesson aiming to "Factor and simplify algebraic expressions by means of Inspection." In this situation, the students tried to factor the polynomial $4x^2$-5x-6 using the methods taught in previous lessons. However, the polynomial did not meet the criteria that would have allowed applying the methods seen previously. JB used this fact to introduce a new method, factoring by inspection.

JB: We are going to solve the exercise on the blackboard: $4x^2$-5x-6.
Student 1: This is a trinomial, so I can use the perfect square trinomial.
JB: Well, try to do it. Let me know when you've finished.
Student 2: In this exercise, we should take out a −1 as a common factor to solve it.
JB: Why do you think that would be the best option?
Student 2: Because during the exam practice, I had a similar case where the extremes were negative so with this change of sign, it works.
JB: Ok. Try to do it. When you finish, I will come back.
Student 1: I cannot solve it with this method because, to start with, 6 does not have an exact root and second, it is negative.
JB: Right!
Student 2: If I make a sign change, the 6 is positive with the 5x but the $4x^2$ is negative, so the perfect square trinomial does not work.
JB: That is correct. I have a question for you (for both students), with the methods we know, can this trinomial be factored?
Both: No.
JB: So we need to introduce a new factoring method called Inspection.

The students' instances that generated the four MOSTs are shown in Table 10.6. MOST1 and MOST2 started when Student 1 and Student 2 made public incorrect mathematical thinking, respectively. JB took advantage of MOST1 and MOST2

Narrative 3: MOSTs and JB actions.

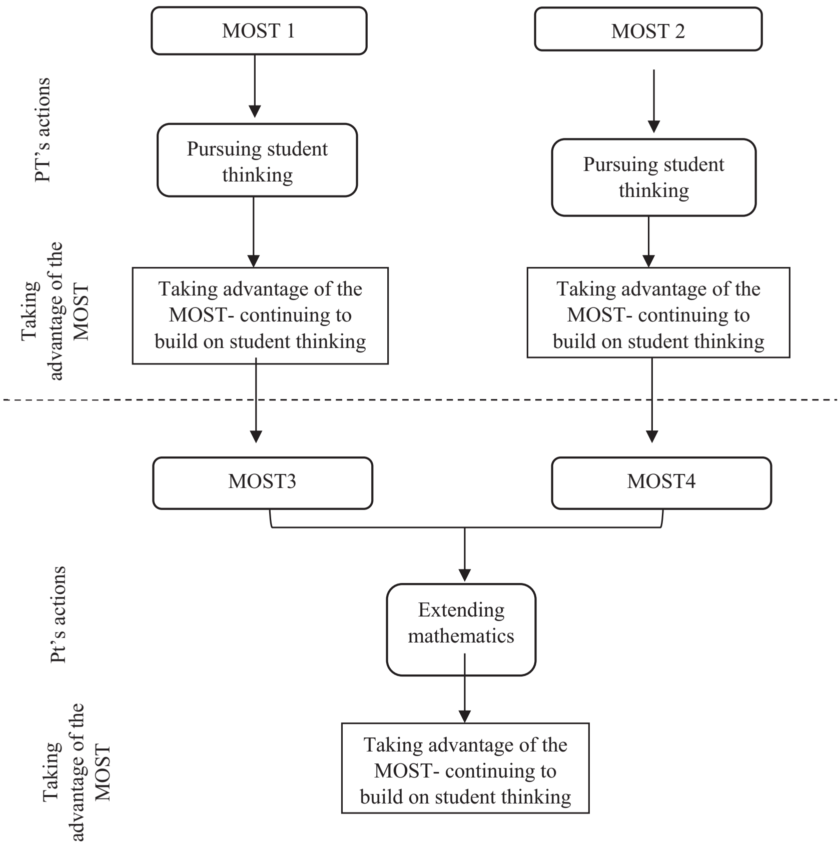

Fig. 10.2 Narrative 3: MOSTs and JB actions

during the lesson, pursuing each of the student's thinking by asking questions that invited them question the viability of the selected factoring method. MOST3 was generated after taking advantage of MOST1, and MOST4 was generated after taking advantage of MOST2. Both MOSTs started when Student 1 and Student 2 tried to justify why he/she could not use the factoring method. JB took advantage of MOST3 and MOST4, relating them and using the same action "expanding the mathematics students know" that helped both students to continue to build on their thinking about factoring polynomials. This behavior may be considered as evidence of categorizing the interpretation of a situation and deciding what to do.

Table 10.6 MOSTs identified and how JB took advantage of them in the third narrative

Student instance	MOST identified	Evidence of taking advantage of the MOST
Student 1: This is a trinomial, so I can use the perfect square trinomial	MOST1 starts when Student 1 uses an incorrect method. With this method he cannot solve the exercise (finding a perfect square trinomial)	JB takes advantage of MOST1 pursuing Student 1's thinking when he says "Well, try to do it. Let me know when you've finished." JB invites Student 1 to apply the factorization method he chose, although this is not the appropriate way to solve the exercise. This led Student 1 to think about why he could not use this method
Student 2: In this exercise, we should take out a -1 as a common factor to solve it	MOST2 starts when Student 2 uses an incorrect method. With this method she cannot solve the exercise (taking out a -1 as a common factor)	JB takes advantage of MOST2, also pursuing Student 2's thinking with questions such as "Why do you think that would be the best option?" Again, JB invites Student 2 to reflect on whether the chosen factoring method is appropriate
Student 1: I cannot solve it with this method because, to start with, 6 does not have an exact root and second, it is negative	MOST3 starts when Student 1 justifies why he cannot use his method to solve the exercise (finding a perfect square trinomial)	JB takes advantage of both MOST3 and MOST4, extending the mathematics students know. Once the reason why they could not use the factoring methods they knew was discerned, JB introduced the need to introduce a new factorization method
Student 2: If I make a sign change, the 6 is positive with the 5x but the $4x^2$ is negative, so the perfect square trinomial does not work	MOST4 starts when Student 2 justifies why she cannot take out a -1 as a common factor to use the perfect square trinomial	

Changes in Practices: A Change in the Lesson Planning

In the second narrative, JB described a situation where students were working on identifying the appropriate factoring method (these factoring methods had been introduced previously). In the third narrative, JB presented a different lesson plan with respect to the situations described in previous narratives. He thus showed that he was starting to categorize the situations that he noticed. In this lesson, JB tried to generate a need in students (the impossibility of using a known factorization method to solve a new type of activity) and to build on "their mathematical thinking introducing a new method of factoring." JB included the following excerpt in his narrative, explaining this change:

> The dynamic of the class in which the activity took place favored the development of the specific skills that were planned (the specific skills of factoring and simplifying algebraic expressions), since it encouraged students to solve an exercise using their knowledge. In other words, if students analyze the use of known methods and conclude that it is not possible to use any of them, they will feel the need to learn about a different method to solve it. Therefore, it awakens their interest in a different way of factoring.

10.6 Discussion and Conclusions

Our study contributes to our knowledge of how PTs learn to notice teaching situations in a formal distance education program. In the online environment designed during the traineeship period, PTs wrote narratives about their own teaching and shared them over an online forum with their fellow classmates and the university tutor. In this context, the forum was understood as an online interaction space allowing PTs and the university tutor to interact without being physically in the same place. The conclusions fall into two sections. First, we discuss how PTs enhanced their noticing competence by integrating knowing and doing. Second, we discuss how the characteristics of the online environment (i.e., the cycles of written narratives and sharing them on online forums, with guided questions) helped to strengthen PTs' noticing competence.

10.6.1 PTs' Noticing Development: Integration of Knowing and Doing

Evidence of prospective teachers' noticing development was found, on the one hand, in the changes in the way they took advantage of the MOSTs (focusing/not focusing on students' understanding) and, on the other, in whether they took advantage of different MOSTs by relating/not relating them. These changes seem to be linked to a more structured ability to notice. This more structured noticing is manifest when PTs take advantage of what they have noticed about the students' thinking, and when PTs relate different situations (MOSTs) to act. In this way, we can thus assume that more articulated ways of noticing (categorization of situations, Brown and Coles 2012) are the precursors of actions taken in a specific direction. This finding supports the claim that the purpose of noticing is increasing the range of actions available to enact (Mason 2002).

Furthermore, results show that the enhancement of PTs' noticing competence is linked to changes of practices, specifically to changes in class management and lesson planning. In fact, the shift from identifying MOSTs and not taking advantage of them/or taking advantage of them but not focusing on students' understanding to identifying MOSTs and taking advantage of them and focusing on student's understanding was shown when PTs changed the way they managed the class, thus allowing students to continue to build on their thinking. The change from taking advantage of different MOSTs, without relating them, to taking advantage of different MOSTs and relating them was demonstrated when PTs changed the way they planned their lessons.

These changes seem to suggest that the cycles of writing and sharing narratives on an online forum may support PTs' noticing competence because the PTs progressively focused more on students' mathematical thinking and provided actions to continue building on students' thinking. In other words, these cycles seem to raise the awareness of links between knowing and doing that configure the prospective teachers' learning.

10.6.2 How the Characteristics of the Online Environment Support the Enhancement of Noticing

One feature of any distance teacher training program is that PTs have little contact with their classmates. However, the characteristics of this formal online environment seem to help to overcome this isolation and to support the enhancement of noticing. First, the cycles of writing a narrative on relevant mathematics teaching-learning situations in their own teaching, sharing and discussing the subject over an online forum, and writing a new narrative led to an online collaborative group between the five PTs and the university tutor. Second, the context of the online interactions—sharing their experiences and discussing the meanings underlying their actions—appears to support PTs' ability to describe the events in their own teaching and to reason about their actions (Fernández & Ivars, 2018; Fernández et al. 2020). The online forum in this intervention opens the door to multiple interpretations and opportunities to justify different ways of approaching the link between the teachers' decisions and students' behaviors. In this case, the possibility to write a new narrative taking into account the feedback provided in the online forum allowed PTs to improve the way they articulate reasons for a specific teaching action or why they chose a specific teaching move in the lesson plan.

Third, writing narratives, sharing them over online forums, and receiving feedback can be understood as exploration spaces in which PTs were involved in actively making sense of their world. Therefore, PTs were more aware of how their actions could influence students' thinking, and this can be interpreted as evidence of the link between the development of noticing and changes of actions.

Finally, the guided questions that helped PTs to write their narratives focused their attention on students' thinking and on what the teacher's role should be to promote students' learning. This aspect plays an important part because the guided questions centered PTs' attention on how their actions could influence students' learning outcomes (Cavanagh and McMaster 2015).

These findings suggest that specific design elements may have the potential to support the enhancement of PTs' noticing and provide knowledge about the successful delivery of online professional development for teachers (Bragg et al. 2021). We are concerned that other factors of the online program may influence PTs' changes, such as the mathematical content chosen for writing their narratives or their beliefs. Furthermore, PTs and the tutor had to adapt to a new way of interacting through online forums, not being the writing of narratives and providing feedback easy tasks. Nevertheless, we think that this type of learning environment seems to be useful to support PTs' learning in formal distance mathematics teacher education programs.

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Appendix 1: Summary of the instruction to write the narratives and the guided questions provided

From the lessons given during the week, choose a situation in which students are developing some aspects of the mathematical competence that you considered relevant, and describe this situation (you can use the guided questions provided). During the next week, you have to share the narrative with your fellows and give feedback to other narratives. To write the new narrative, you have to consider the suggestions given by your fellows and the tutor.

Guided questions:

- Describe the situation (identifying): Provide a detailed description of the activity (curricula contents, materials, resources, etc.), what students do (students' answers to the activity, difficulties, etc.), and what the teacher does (methodology, interactions, etc.).
- Interpret the situation (interpreting): Indicate the activity's mathematical objectives and provide evidence from students' answers that they have achieved the objectives (students' understanding of the mathematical content and difficulties).
- Complete the situation (taking decisions): Complete the situation indicating how you will proceed in order to help students progress in their learning of the mathematical content.

Appendix 2: Instruction provided in each online forum

In this forum, you can upload your narrative and provide and see feedback to/from other fellows' narratives.

To provide feedback to your fellows, you can consider the guided questions, for example, you can observe whether in the narrative:

- The situation is described in detail providing a description of the activity (with materials, resources, etc.), what students did (writing some interactions) and what the teacher did.
- The situation is interpreted with regard to whether students' have achieved the learning objectives, providing evidence from students' thinking.
- The situation is completed indicating how the teacher would proceed in order to help students progress in their mathematical understanding.

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Chapter 11
Theory-Based Intervention Framework to Improve Mathematics Teachers' Motivation to Engage in Online Professional Development

Nathan A. Hawk, Margaret A. Bowman, and Kui Xie

Teacher professional development (TPD) is an often used, well-resourced, and indispensable activity that allows primary and secondary mathematics teachers to continue developing and improving both their content and pedagogical knowledge. Recently, mathematics TPD is moving toward online or blended formats (Wasserman and Migdal 2019) in part because online TPD is generally considered more cost-effective, less intensive, can be completed asynchronously, and can occur over a longer period of time (Dede et al. 2016; Goos et al. 2018; Heck et al. 2019). The COVID-19 pandemic enhanced the need for flexibility, more sustainable and scalable programs, and new professional learning approaches, making the relevance and usage of online TPD even more evident.

Teachers, however, often have difficulties engaging in an online TPD (or the online portion of a blended TPD) experience. As a result, this disengagement has at times negatively impacted the effectiveness of TPD, including possible dropout (Parsons et al. 2019; Russell et al. 2009b; Xie et al. 2017). One factor found to be critical to successful online and blended learning professional development (PD) is sustaining teachers' motivation and engagement (Kowalski et al. 2017). In fact, teachers' motivational perceptions and beliefs about TPD impact their engagement in professional development courses (Russell et al. 2009a, b). To address these concerns, researchers have been looking for different motivational frameworks while examining teachers' TPD experiences. One empirically tested and well-validated motivational framework is the Expectancy-Value Theory (EVT; Eccles 1983). EVT

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posits that one's expectancies for success combined with four distinct task values help determine one's likelihood to engage in a learning task (Wigfield 1994; Wigfield and Eccles 1992). While EVT has been widely used in examining students' motivation in academic settings, empirical research exploring teachers' motivational and personal value beliefs on engagement in TPD is still lacking. In particular, few well-structured analyses and syntheses of these findings exist in the extant literature.

As a result, the goal of this chapter is to collect and synthesize the available, relevant literature concerning mathematics TPD and teachers' values and expectancy for success toward TPD. Synthesizing these research findings will allow instructional designers to discern important design principles to utilize when developing online PD experiences. Contextualizing the EVT in mathematics TPD settings, we develop and promote several key design principles *with a focus on increasing mathematics teachers' positive value toward, and their engagement in, online TPD experiences*.

In the first section, we briefly outline the existing research on the importance of teacher motivation. In particular, we discuss its impact on teachers' engagement in PD and argue that we should view motivational values using a more concise and multidimensional framework. Second, we present the Expectancy-Value Theory, discussing why EVT is significant for online engagement in mathematics education. We note that teacher motivation in online TPD is understudied within mathematics education; however, the topic is well-established within other areas. In our literature review, we briefly discuss the breadth of research on teacher motivation, TPD, and online learning in other areas to help illuminate these topics within mathematics education. Third, we propose five design principles, based on the EVT framework, noting how these design principles are intended to support and improve positive value perceptions for mathematics teachers engaged in online learning. We conclude with some final remarks on the implications for mathematics teacher education and practice.

11.1 Teacher Motivation and Engagement Toward Mathematics Professional Development

Karabenick and Conley (2011) stated, "Motivational concerns…remain a critical yet understudied component of teacher PD interventions" (p. 9). Motivation is broadly described as the process through which goal-directed behaviors are both initiated and sustained (Cook and Artino Jr 2016). Often, academic motivation theorists attempt to describe the reasons why learners initiate, sustain, and engage in specific learning tasks. Empirical research has also supported the strong relationships between learners' motivation and engagement in classrooms (Greene et al. 2004; Liem et al. 2008; Xie et al. 2020) as well as in online learning settings (Xie et al. 2006, 2011; Xie and Ke 2011). Engagement and motivation are intertwined, with one often influencing the other. When a person is motivated to learn, they often engage more in the learning process (Martin 2007; Skinner and Belmont 1993). Mathematics teachers' motivation for learning engagement, for instance, may help explain whether they engage in online TPD (Parsons et al. 2019; Wentzel and Miele 2009; Wigfield 1994).

According to de Araujo et al. (2018,), "PD should focus not only on teachers' content and pedagogical knowledge but also on their attitudes and dispositions about learning and teaching mathematics" (p. 324). Engagement in TPD faces multiple challenges, as some mathematics teachers may resist difficult or demanding professional development (Pasley 2011). Masuda et al. (2013), for example, found that the professional development opportunities that teachers are provided are often in-school and district-wide, involving quick lectures with little to no opportunity to observe, practice, or implement what was learned in a meaningful way. Teachers are often given little choice in the timing or topic of TPD, feeling that it has been imposed upon them (Francis-Poscente and Jacobsen 2013; Masuda et al. 2013). Dede et al. (2009) stated that many TPD programs in the past have been of poor quality, ineffective, and often do not provide ongoing support, leaving teachers frustrated and unable to implement what they learned. This, coupled with the amount of time it takes for teachers to participate in TPD, leaves many teachers feeling that TPD isn't valuable or worth engaging in. This leads to poor engagement and thus, little classroom use of what was conveyed in the TPD program. In addition, mathematics teachers often have difficulties engaging and persisting in online TPD (Parsons et al. 2019; Xie et al. 2017).

To address these challenges within online mathematics TPD, we believe that an exploration of how a focus on a specific motivational framework could impact TPD is useful. Within this chapter, we target the various reasons why teachers engage in professional development, from a motivational values perspective. Along these lines, four primary motivational values are discussed.

First, interest (domain-specific) is a salient personal variable that impacts teachers' sustained engagement. In one study, mathematics teachers involved in unmoderated (non-facilitated), self-paced workshops showed that their increased interest positively related to engagement and motivation toward the work experience (Renninger et al. 2011). When TPD experiences are meaningful, well-designed, and provide intrinsically motivating outcomes, teachers are more inclined to participate (Lebec and Luft 2007).

Second, other frameworks have examined perceived usefulness. In one study, Smith and Sivo (2012) found that teachers who perceived an e-learning course experience as useful were more likely to engage in future e-leaning experiences. One model, the technology acceptance model (TAM; Davis 1989), historically has been used to describe teacher-technology integration (Teo et al. 2008; Xie et al. 2019). Extending its use, Smith and Sivo (2012) utilized the TAM framework to predict teachers' intentions to continue in online TPD.

Third, perceived importance is another influential determinant of teacher engagement in PD (Battle and Looney 2014). Generally, the perceived importance of PD refers to the extent a teacher thinks a learning experience is important to their work.

In some past research, importance and interest were statistically related and predicted the continuation in teaching and teaching activities (e.g., Battle and Looney 2014). Thus, when teachers perceive an experience as both important to their teaching and intrinsically interesting, they are more likely to remain engaged.

Finally, several studies promote the importance of ability beliefs, ease of use, or self-efficacy beliefs. Self-efficacy or ability beliefs generally refer to how confident one feels about accomplishing a task (Bandura 1977; Wigfield and Eccles 2000). Conceptually, there is often a significant relationship between motivational values and one's self-efficacy or ability beliefs (Wigfield and Eccles 2000). When teachers are personally confident in their capabilities, they are more likely to learn for intrinsic reasons (Kao et al. 2011). For example, according to Appova and Arbaugh (2018), mathematics teachers may feel motivated to become better teachers—they may feel they are a good teacher, but there is always room to learn more. As a result, these positive motivational perceptions often lead mathematics teachers to change their instructional practices (Kao et al. 2011; Palermo and Thomson 2019).

While past research has separately discussed different motivational values, we believe that a more comprehensive and concise exploration of how these values are interconnected and can impact TPD is needed. One under-examined area in adult motivational research describes one's expectancy for success and reasons for engaging in a task. In searching for a widely used and validated theoretical framework with which to examine teachers' values toward TPD, the Expectancy-Value Theory has emerged as one that is well-defined and well-suited to this chapter's objectives. Past research and measurement instruments help demonstrate its strength in empirical research. This empirically tested framework is explored further below.

11.2 Expectancy-Value Theory and Its Implications for Online Mathematics TPD

Expectancy-Value Theory (EVT) focuses on achievement-related choices and performance outcomes (Eccles and Wigfield 2002). Earlier research with this framework focused on students, but more recent research has studied teachers' attitudes and values through the lens of EVT (Bowman et al. 2020; Cheng et al. 2020; Vongkulluksn et al. 2020). For instance, empirical research has shown the importance of the teacher's value beliefs, such as usefulness, along with personal characteristics, and internal and external barriers to technology integration, suggesting the importance of perceived values on whether or how technology is integrated (Vongkulluksn et al. 2018; Cheng and Xie 2018). EVT encompasses four motivational values as well as one's expectancies for success to describe one's reasons for engaging in learning tasks (Wigfield 1994; Wigfield and Eccles 1992). According to Wigfield and Eccles (2000), a distinction is made on what one perceives he or she is good at (expectancy) and what one values (task values). Expectancy for success is conceptually similar to the construct of self-efficacy (Bandura 1977), or perceived capability to complete a task successfully. Expectations of success involve an individual's belief about how well they expect to do on a task. Subjective task value is further broken down into four factors: interest or enjoyment value, utility value, attainment value, and cost. Interest describes a learner's perception of whether a task is enjoyable or personally of interest. Those who espouse interest might personally enjoy learning about, or practicing with, new mathematics techniques or new classroom technologies. Utility value describes the perceived usefulness of the task. Mathematics teachers exhibiting utility value may find a particular instructional method useful to their overall development. In turn, these methods may ultimately benefit students. Attainment value describes whether the task is personally important to the learner. Mathematics teachers who espouse attainment value might decide to engage in a task because it is important to their overall professional development goals. For instance, the professional development experience might help them to develop instructional practices that better support students. Cost describes what one believes they would need to sacrifice to engage in another task. Mathematics teachers exhibiting time cost value might have to give up other activities or personal family time to attend new learning opportunities.

The expectancy for success and task values play separate roles. "In other words, in choosing whether to learn something the task value matters most; once that choice has been made, expectancy of success is most strongly associated with actual success" (Cook and Artino Jr 2016, p. 1003). Research shows that lower levels of one construct may be compensated for by higher levels of another construct, suggesting that both expectancy and values may be required to drive learning behavior (Putwain et al. 2019).

Prior research has shown the importance of EVT in explaining learning engagement for a wide variety of learners and contexts, including in mathematics (Wentzel and Miele 2009). First, both expectancy and value predict engagement, and high expectancy can compensate for perceived low value (and vice versa), according to Putwain et al. (2019). Next, positive professional development experiences are related to improved motivation (Brinkerhoff 2006; Kim et al. 2017; Kleickmann et al. 2016). Further, teachers who value a TPD experience will engage more in the learning, leading to increased perceived value for both the TPD in general and for the specific learning goals and experience (Rutherford et al. 2017). Finally, when these experiences are perceived as valuable, this can outweigh the perceived time cost for their participation. This suggests the need to significantly add opportunities for teachers to perceive added value in exchange for their time (de Araujo et al. 2018; McCourt et al. 2017). Indeed, teachers may be able to overcome challenges to implement instructional changes if they value these changes in their pedagogy highly enough. Grove et al. (2009) stated, "If the participants believed they had the ability to make changes in their classroom and valued a particular element, some changes were seen in their teaching practices" (p. 258).

EVT is an important framework to examine across online mathematics TPD. Mathematics teachers express that one reason for the lack of online learning engagement is a perceived lack of time, particularly if there is not an extrinsic reason to participate (Lebec and Luft 2007). On the other hand, when considering specific values, teachers are more likely to seek additional learning opportunities. Further, these differences were observed between online and traditional learners (Renninger et al. 2011). According to other research, adult online learners are more likely to learn when their perceived task values are positive; however, the expectancy-value model was not able to detect differences between learners who are normally face-to-face versus online (Zimmerman 2017). In sum, evidence suggests that in online learning, experiences should be of high quality and target specific values in order to emphasize impact.

11.3 Guidelines for Enhancing Motivation in Online TPD Experiences

In recent years, the ways in which TPD is conferred has changed, and online TPD has become more prevalent. While it may be more convenient and often less costly, there remain concerns about TPD quality and the extent of participant engagement. Online TPD participants often do not feel a strong connection to the instructor or other participants. As a result, this leaves them feeling dissatisfied with the experience (Holmes et al. 2011) and sometimes not completing the program (Reeves and Pedulla 2011). This may also contribute to poor attitudes and low engagement in TPD.

There are, however, teachers that are not unhappy with or disengaged from TPD. It may be beneficial to closely examine these teachers' motivations and incorporate findings into future TPD to address the lack of motivation and engagement by others. Occasionally, teachers are given provisions to seek out their own TPD or to participate in research-based TPD led by institutions of higher education. Often the participants in these types of TPD programs are highly motivated, see the value in the TPD, or acknowledge the need for a change in their own teaching practices. Thus, they choose to register and participate in the TPD program. While research has shown that these teachers do engage more and perceive the TPD as valuable, much of the research is limited to self-selected teachers (Barrett et al. 2013). Perhaps if more teachers are provided high-quality, engaging TPD and the value of the TPD is conveyed to teachers in a meaningful way, teachers' attitudes toward TPD would improve. As a result, they may also engage more in the TPD, and their teaching practices would be more positively improved (Holmes et al. 2011; Reeves and Pedulla 2011).

Masuda et al. (2013) found that "Teachers' attitudes and willingness to engage in PD were closely tied to the perceived value or importance that the PD experience held for them. In turn, the value of the PD was closely tied to its perceived quality based on their experiences" (p. 10). For teachers to gain the most benefit from professional development, they need to appreciate what they are learning. Hargreaves and Preece (2014) emphasized the need to focus teacher development on "philosophically important values rather than just the practical details" (p. 131). According to the literature, it has been shown that TPD instructors have to emphasize, explain, and model the value affordances associated with the TPD content.

Using the Expectancy-Value Theory as a framework, we propose five principles with which to develop mathematics teachers' motivation to engage in online professional development. These principles are designed to support and improve each of the three positively associated value beliefs, decrease beliefs about the perceived cost of participation, and enhance teachers' expectancies for success.

11.3.1 Principle 1: Promote Intrinsic Value

Teachers are more likely to participate in TPD if they know it will be interesting, enjoyable, and fun (Karabenick and Conley 2011) and that there is a meaningful purpose (Thomas 2009). The purpose of the TPD should be conveyed in a way that helps the teachers see its value and is linked to their current and future work. When teachers have a sense that they will accomplish something meaningful by engaging, they may perceive greater intrinsic value toward the TPD. Furthermore, they may be increasingly motivated when they know at the outset that they will have enjoyable and fun interactions throughout the TPD. Therefore, TPD may benefit from the addition of games, awards, and badges to increase interest, accountability, and record of accomplishment (Diamond and Gonzalez 2016).

Another way to increase teacher-perceived intrinsic values in the TPD is to provide active participation opportunities (Bayar 2014; Kanaya et al. 2005). These opportunities may include engaging with the instructor, engaging with other teachers in the TPD, and engaging with the content. The instructor should model his or her own intrinsic value for engaging in the TPD. Research shows that "intrinsic motivation can be facilitated through the mere perception that the teacher is intrinsically motivated" (Patrick et al. 2000, p. 219). Furthermore, when the instructor presents the content energetically and enthusiastically, the participants' intrinsic value is more likely to increase.

Creating opportunities for teachers to engage with each other is another way of promoting intrinsic value. Open discussions with other teachers involved in the TPD can create a sense of belonging and trust (Thomas 2009). These discussions could be during live, synchronous sessions, or through written discussion boards, which are designed to provide social interaction (Hoskins 2012). The more complex these interactions are, the more likely the participants are to be engaged. For example, if TPD participants are assigned to groups to debate for or against a particular mathematical method, the surrounding discussions can be meaningful, valuable, and motivating.

Engagement with the content is also important and can increase intrinsic value (Brophy 2008; Hoskins 2012). Online professional development opens up opportunities for more interactive learning. Providing teachers with engaging, interactive experience may increase teachers' value for the content and ultimately increase their integration of the learned content into their classrooms.

Implications for Online Mathematics TPD

To make TPD meaningful for mathematics teachers, it should be linked to the teachers' sense of purpose. For example, some teachers may choose to teach mathematics because it may be viewed as difficult and less valued by many students (Eccles et al. 1989). They may also be motivated to engage in their own learning opportunities if it is closely tied to their students' learning. Appova and Arbaugh (2018) found that "teachers are motivated to learn from observing their students' struggles with understanding mathematics and, as a result, from developing a feeling of dissatisfaction with their own teaching. This sense of responsibility for students' learning encourages teachers to want to engage in PD to become 'better' teachers" (p. 15). Developers and instructors of TPD for mathematics teachers may benefit by emphasizing the meaningful nature of the TPD, tapping into teachers' intrinsic value for both teaching and learning.

Mathematics teachers may also enjoy engaging in and be intrinsically motivated by mathematical puzzles and games or researching historical backgrounds of specific mathematical concepts. For example, teachers may enjoy learning about the discovery of π by various mathematicians, or how the Pythagorean theorem was discovered. This may both engage them in the TPD and provide activities and learning opportunities that can be incorporated into their classrooms.

Finally, implementing active participation into the TPD is important. Research on TPD has shown repeatedly that collaboration is a strong motivating factor. Participants should engage in meaningful discussions with other participants and with the instructor, whether through video conferencing or online discussion forums. Additionally, including games in which the TPD participants work together in small groups may increase intrinsic motivation further.

11.3.2 Principle 2: Highlight Utility Value

Utility value is the belief that something (e.g., a workshop or conference) is useful or relevant. For a teacher to value the PD, they need to see and believe that the content is useful to them and relevant to their work. For example, Kanaya et al. (2005) found that teachers who perceived the content of the TPD as both useful and relevant were more likely to engage and have successful outcomes from the TPD. Two recent specific examples focused on TPD for technology integration. Bowman et al. (2020) determined that teachers who believe that technology is useful and valuable are more likely to perceive technology-related TPD as useful, and thus, integrate technology into their classrooms in more meaningful ways. Similarly, when examining profiles of teacher value beliefs toward technology using EVT, Vongkulluksn et al. (2020) found that more adaptive value profiles were more likely to integrate classroom technology in different ways. While this was specific to teachers participating in technology-focused TPD, it may be generalized to other types of TPD. Perhaps, if teachers perceive the TPD as relevant to them, to their students, or to their school, they may exhibit more positive utility value (Bayar 2014) for the TPD.

One way to encourage utility value in the beginning or to encourage participation is to create dissonance. Teachers need to believe that change needs to occur, and therefore the TPD is necessary and useful (Timperley et al. 2007). Additionally, within the TPD program, instructors should not only present content relevant to the teachers but connect the content to classroom practices. Masuda et al. (2013) found that teachers in any stage of their careers wanted an application component and that the content had to be relevant to their own contexts. This improves the likelihood that teachers will engage and use what they learn in the TPD (Saderholm et al. 2017).

Once teachers see the need for change or growth, certain TPD experiences can be successful in changing or developing teachers' utility value. In other words, if teachers see the content of the TPD as being useful, they are more likely to have positive views of the TPD. For example, a study of the professional development program for one school district's iPad initiative revealed that teachers who had a positive view of technology and felt capable of using iPads had more positive views of the iPad TPD (Liu et al. 2018). In another instance, a 1-year-long PD for digital content evaluation, for 171 teachers from 5 Central Ohio school districts, incorporated a focus on utility value because the content of the PD was new and the facilitators felt that teachers would need to see the relevance and utility of the new skills being learned in order to motivate them to engage and go on to use what they learned in their classrooms (Kim et al. 2017; Xie et al. 2017).

Implications for Online Mathematics TPD

Research has shown that it is important for mathematical experiences to be meaningful (Di Martino and Zan 2001). For this to occur, teachers need to see and believe its relevance to their own life and work. The utility value should be linked to the teachers' specific needs. For example, for teachers who struggle to understand and explain a specific mathematical concept to their own students, the TPD program could provide resources and activities to increase teachers' deep conceptual understanding, which is promoted as part of the value of the program. The utility value of the TPD should be explicitly stated and demonstrated early on in the TPD program. To help teachers see the connection between the TPD and their own teaching practices, it may be useful to have teachers present examples of student work or their own teaching practices that they would like to see improved. In this way, the program can create dissonance for teachers who may be less prone to see the need for the TPD. Then, once teachers understand the need for improvement, the program can introduce interventions regarding the relevance of the TPD.

Two relevance-related interventions have been shown to increase utility value for mathematics content. In the first, students reflected on quotations that explicitly stated the usefulness of course materials (Durik and Harackiewicz 2007). In the second, students generated written arguments for why the course material was useful (Hulleman et al. 2010; Hulleman and Harackiewicz 2009). Perhaps these same interventions could have the same effect on teachers' utility value for mathematics TPD. The TPD program could open with a statement explicitly conveying the usefulness of the program in relation to teachers' classroom practices as well as student learning. The TPD could close with a time of discussion and reflection on the usefulness of the program, allowing teachers to express what they found most relevant and how they plan to incorporate what they learned into their classrooms.

11.3.3 Principle 3: Foster Attainment Value

Attainment value is the importance one places on an activity as it relates to their identity or self-concept. If a teacher's identity is tied to being a mathematics teacher, they are more likely to set goals or engage in tasks and activities that will improve their teaching knowledge and performance. To support attainment value, the TPD should be closely tied to teachers' identities. Teachers have reported they are most willing to participate in TPD when they are discipline-specific (Garet et al. 2001) and the objectives are tied to mathematics course-specific needs, such as improved subject-matter knowledge (Karabenick and Conley 2011; Krille 2020; Qian et al. 2018; Xie et al. 2017). Additionally, teachers are more likely to engage and participate when they play a role in the selection and development of the TPD itself, that way the TPD topics are directly tied to their needs and are relevant (Bayar 2014).

Teachers' identities may also be tied to their educational community. Teachers should feel supported by other teachers, support staff, and administration. Additionally, collective participation, collaboration, social presence, and social opportunity to exhibit support are shown to impact learning engagement (Kowalski et al. 2017; Palermo and Thomson 2019). Grove et al. (2009) stated,

> Teachers may attend a professional development program, and gain insights and knowledge, to return to their classroom and implement new ideas along with their renewed excitement for their content area. However, if the teacher feels that the changes are not appreciated and/or not supported by peers or administration, there may be little motivation to put the changes to thinking and planning into practice. (p. 258)

It may be beneficial for the TPD instructors to provide information to the school, explaining the purpose and value of the TPD. In doing so, teachers can feel more supported by their community and their identity as a teacher among a community of educators can be further encouraged.

Implications for Online Mathematics TPD

While some teachers may be interested in or need more general development in mathematical content knowledge or pedagogical development, for example, it may be most beneficial to provide professional development that targets specific mathematical domains. At the high school level, a teacher seeking improvement in teaching Geometry may have different needs from a teacher of Algebra I. At the middle school level, a teacher may be confident in teaching the number sense domain but desires to improve his or her pedagogical knowledge in the expressions and equations domain. More general TPD can be planned for a larger group of mathematical teachers, but it may be useful to group the participants by their needs. In this way, the teachers can have some control and choice over the topics with which they engage.

By forming small groups, teachers can also form communities of practice. According to one review of multiple online TPD intervention programs, among the design principles were collective participation, collaboration, and communities of practice (Kowalski et al. 2017). Increasing the support mathematics teachers feel in their work, and tying that feeling of support to improved communities of practice, might help enhance their identity and attainment value. Furthermore, teachers may be more engaged in online TPD if they take ownership of some of the TPD components. For example, having a teacher or group be responsible for teaching a certain topic may help them to see the value of the topic as it relates to their own identity as both a teacher and as a member of the online TPD community.

11.3.4 Principle 4: Reduce Perceived Cost

Cost is negatively associated with the other three task values. If teachers feel the cost is too high and that they are giving up too much in order to participate in TPD, they are less likely to engage or value the TPD. For example, if teachers feel they must take additional time to find ways to integrate what they learned into their own classrooms, they may become overwhelmed and perceive the additional costs outweigh the value of the experience (Masuda et al. 2013). On the other hand, more positively adaptive intrinsic, utility, and attainment value may help to compensate for the lower perceived cost. Teachers report that a determining factor that reduces the perceived cost of attending and engaging in PD is its specific application. Masuda et al. (2013) stated, "Teachers must have something tangible to show for their investment of time" (p. 12).

Teachers are more willing to engage if TPD is short, concise, efficient, well-designed, and organized, reducing the time and effort it takes to complete the TPD (Karabenick and Conley 2011). In contrast, long-term TPD is shown to be more effective in terms of improved teaching practices and student success (Bayar 2014; Clary et al. 2017). The amount of time the TPD takes may not be as important as how effectively the time is used. Making every minute of the TPD count will reduce what might be seen as time lost (Loucks-Horsley et al. 2009).

Implications for Online Mathematics TPD

Online mathematics TPD should be well-designed, easy to use and navigate, and easily accessible to reduce the time a teacher might spend on simply learning how to use and engage in the online learning platform. In addition, there is a strong relationship between the ease of use or technology self-efficacy and other motivational values such as perceived usefulness (Teo et al. 2008; Davis 1989; Zimmerman 2017). As mathematics teachers become more comfortable and confident in the learning platforms, other values increase. In turn, this may reduce perceived cost.

Another way to reduce perceived cost specifically for online TPD is to spend time focusing on the technology as it relates to the participants' students. Most recently, many TPD efforts are spent on instructing teachers how to teach online (Lay et al. 2020). It may be valuable for the TPD online learning platform to be the same as that of the students. This allows teachers to learn and use the platform in meaningful ways while also engaging in the content of the TPD. This may reduce the cost that teachers feel in participating in an online TPD.

11.3.5 Principle 5: Increase Expectancy for Success

Improving teachers' value for TPD has been shown to increase their expectations for success (Rutherford et al. 2017). Implementing mastery experiences into the TPD may allow for the greatest increase in teachers' expectations for success (Tschannen-Moran and McMaster 2009). Mastery experiences, which are past experiences of success, are generally considered one of the four sources of self-efficacy (Bandura 1977; Usher and Pajares 2008). Self-efficacy is a closely related concept to expectancies for success. For example, the previously described PD program designed to train teachers to evaluate digital content also focused on expectancy for success (Kim et al. 2017; Xie et al. 2017). Teachers were given opportunities to evaluate digital content in small groups with the help of the PD instructors. Later, they were asked to find and evaluate digital content on their own and were provided feedback. In this way, teachers were afforded the opportunity to learn and be supported, increasing their expectations for their own success.

Implications for Online Mathematics TPD

First, regarding expectations of success for engaging in the online TPD itself, considering that it is online and requires the use of technology,

> online PD may fail to motivate or engage teachers who do not feel comfortable or skillful using technology. Another challenge to online PD, therefore, is to find technical support for those in need, and to differentiate resources and tasks for teachers with different levels of comfort and expertise with technology. (CADRE 2017, p. 6)

In terms of expectations for success for learning and eventually implementing the content of the TPD, teachers should be offered mastery experiences. Again, collaboration with other participants and with the instructors is key. Another source of self-efficacy includes the social context in which teachers are involved, and vicarious experiences and verbal and social persuasions are key sources to one's adaptable self-efficacy or positive perceptions of expectancy for success (Usher and Pajares 2008). Teachers should be given opportunities to practice new concepts and skills with other teachers while being supported, eventually practicing, and perhaps presenting their learning on their own. Mathematics teachers especially may need opportunities to practice mathematical concepts to build confidence before designing and implementing teaching practices (see a summary of these principles in Table 11.1).

11.4 Implications and Significance to the Field of Mathematics Education

Overall, our synthesis of the available research provides some overarching implications for researchers, instructional designers, and practitioners. First, research shows that when mathematics teachers' expectancies and values increase and their perceived costs decrease, they are likely to engage more, persist longer, and increasingly find ways to integrate their gained knowledge and skills into their teaching practices. Thus, instructional designers and online learning developers should consistently find ways to build value-based instructional practices into online mathematics TPD. For example, building opportunities to increase interest or perceived values of usefulness may help to mitigate other concerns, such as the diminished social presence or lack of traditional forms of teacher engagement or collaboration. Second, research has shown that increased engagement also leads to greater expectations of success. Consequently, more positive expectancies of success in mathematics teachers' own abilities to transfer their TPD knowledge and skills often lead to increased student achievement (Rutherford et al. 2017). Finally, empirical interventions and mathematics TPD experiences should focus on multiple values simultaneously. Our research (e.g., Vongkulluksn et al. 2020) has shown a strong interrelation between task values. Similarly, other research has found that a high task value can compensate for another low task value or expectancy for success. For instance, when the perceived attainment value is low, overall engagement was managed with more adaptive expectancies of success (Putwain et al. 2019).

Using a theory-based approach to recommend several design principles, our research synthesis also promotes opportunities for future empirical research. First, our research synthesis should encourage future quality intervention-based research to empirically examine our design principles. Using our guidelines, future researchers may consider how to both incorporate and evaluate teacher-related outcomes across new online TPD experiences. Evaluating both qualitative teacher data and

Table 11.1 Summary of design principles

Value-focused design principle	Description	Examples
Promote intrinsic value	Provide enjoyable and meaningful activities with opportunities for active engagement with the instructor, other teachers, and the content	• The instructor should model intrinsic motivation • Mathematical puzzles and games • Provide or have teachers research the historical background of how certain mathematical concepts were discovered or developed (e.g., the discovery of π or the Pythagorean theorem) • Discussions with other teachers about the content • Teachers share students' work
Highlight utility value	Provide content that is useful and relevant to the teachers' needs	• Teachers write about and/or verbally explain areas where personal improvement is needed, such as specific mathematical content knowledge or pedagogical knowledge • Explain and support deep conceptual understanding about specific mathematical ideas (e.g., why the Pythagorean theorem works)
Foster attainment value	Tie course content and objectives to teachers' identities	• Teachers select specific mathematical content to be closely examined and have them present their learning to others • Group teachers by specific mathematical domains, forming communities of practice
Reduce perceived cost	Emphasize the benefits of time spent in PD to reduce any drawbacks to participation	• Ensure the learning platform is easy to use. Use the same platform that teachers and students use if possible • Use technology and tools that teachers will use in their own classes, such as virtual manipulatives, *Geometer's Sketchpad*, *Geogebra*, etc.
Increase expectancy for success	Support teachers' expectations that they can successfully implement what they learn into their own classrooms	• Provide opportunities for teachers to practice new mathematical concepts and skills with other teachers while being supported • Have teachers select specific mathematical concepts or skills that they then teach other PD participants

self-report quantitative data could lend further support to our proposed guidelines while adding to the research base of adult-focused EVT research. Second, across much of the empirical literature, EVT-based research has typically focused on children in K-12 settings, according to Wigfield and Eccles (2000). Although we have conducted recent research utilizing EVT among K-12 teachers (e.g., Bowman et al. 2020; Vongkulluksn et al. 2020; Xie et al. 2017), additional research in online learning contexts could inform more effective TPD practices. Specifically, building TPD programs that include our recommendations and then examining whether they lead to increased engagement, motivation, and application or transfer of learned material is suggested. Finally, research has shown that collective participation, or the ability to participate with those in similar grades or content areas, is an important component in TPD (de Araujo et al. 2018; Kowalski et al. 2017; Tirosh et al. 2015). Future research should consider a specific focus on either similar grades or content areas. That is, if teachers are engaged in TPD within their subject area, they may be more likely to view the experience as useful and important to their professional development.

11.5 Conclusion

In summary, we briefly overviewed and described a multifaceted motivational values' framework, the Expectancy-Value Theory. Subsequently, we described this framework within the general educational literature as well as how it is situated within the mathematics education literature. From the available research, we developed five design principles that we recommended to implement in online mathematics TPD to promote increased engagement and transfer of new knowledge and skills. Our hope is that future designers of online mathematics TPD will consider including value-based principles in their instructional decisions to better maintain engagement and more positive teacher affect.

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Chapter 12
Mathematics for the Citizen, m@t.abel, and MOOCs: From Paper to Online Environments for Mathematics Teachers' Professional Development

Ferdinando Arzarello, Ornella Robutti, and Eugenia Taranto

With the goal of addressing the current standards and expectations for math learning, math teachers face challenges in changing their teaching to incorporate effective pedagogical practices, technological tools, and new curriculum resources (Hollebrands and Lee 2020). Often the resources available to support professional development (PD) are limited, and in recent years many teachers look for such opportunities for online PD such as in massive open online courses (MOOCs). In fact these online environments are spreading more and more and allow their users to increase their chances to engage in a variety of learning opportunities (Avineri et al. 2018; Borba et al. 2016; Pebayle and Rossini 2017). Research shows that online PD that is accessible, meaningful, collaborative, and responds to the different needs and abilities of participants can lead to changes in teachers' educational practices (e.g., Renninger et al. 2011; Vrasidas and Zembylas 2004). It is therefore important to devote attention and care to the design phase of an online PD. Qian et al. (2018) have led to three recommendations for designing online PDs: use activities that match teachers' knowledge and experience; align activities with curricula; and use motivational design to improve teachers' engagement. Moreover Kleiman et al. (2015) consider four principles of effective online PD that include self-directed learning, learning from multiple voices, job-connected learning, and peer-supported learning. Considering these design features in the context of teaching and learning mathematics and statistics, Hollebrands and Lee (2020) designed three MOOCs for educators (MOOC-Eds) for mathematics and statistics teachers and examined how

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these design principles were enacted in the development of the MOOC-Eds and how they influenced the engagement of a massive number of teachers in the development of their knowledge ofbeliefs about and attitudes toward teaching mathematics and statistics. This study stresses the importance of examining not only the design of a MOOC for teachers but also the way in which such MOOCs are implemented and experienced by participants. In this chapter we will deal with both of these aspects—the importance of examining the design of MOOCs for mathematics teachers and the way in which they are implemented and experienced by participants—showing the "Math MOOC UniTo" project. We will use as a theoretical framework the meta-didactical transposition (Arzarello et al. 2014) that allows us to go deeper into aspects related to the design practices of our MOOCs. The meta-didactical transposition does not contradict the approaches of Qian et al. (2018) and Kleiman et al. (2015). In fact we will highlight in the analysis when necessary the links with them.

Math MOOC UniTo is a project of online PD for mathematics teachers, by the University of Turin, aimed at increasing their professional competencies and improving their classroom practices. It lasted for 5 years, and its mathematics MOOCs (delivered one per year) are contained in the Moodle platform "DI. FI.MA. in rete" (https://difima.i-learn.unito.it/), the institutional platform for professional development for the university. Every MOOC deals with the didactic of a different topic and has a duration of 10 weeks; is divided into modules of 1 or 2 weeks; and is made of online resources, weekly tasks, and a final project work, everything in line with the Italian National Curriculum. Resources and tasks have been designed by a team made of researchers and teacher researchers (Aldon et al. 2019; Taranto et al. 2020), in tune with national teachers' professional programs.

In this chapter, we would like to focus on some examples of activities in the MOOC to discuss the following issues: (a) how the authors adapted the content of activities from the previous programs to a specific online learning module for mathematics teacher education according to the National Curriculum and (b) how mathematics teachers, in a totally online environment, learnt and made use of new teaching practices, new resources, and new technologies for teaching mathematics, while collaborating at a distance (Robutti et al. 2016).

We will base the discussion on showing examples of teachers' online interactions and sharing the guidelines we have followed to set the module design of the MOOCs. We will share with the research community the design and monitoring practices that in our experience work.

12.1 Theoretical Framework

We use the meta-didactical transposition (MDT) framework (Arzarello et al. 2014; Aldon et al. 2013) to properly describe the role of researchers and teachers when working together in an educational program according to their specific roles. Roughly speaking, researchers have to design an educational program according to some research issues, and to consequently coach it, while teachers actively participate in it, basing on their professional experience. In both roles there is a deep intertwining between practical and theoretical issues. These two aspects have been described by Chevallard's Anthropological Theory of Didactics (Chevallard 1999) as praxeologies: precisely he described what he calls teachers' praxeologies in teaching mathematics. The word *praxeology* is a Chevallard's neologism made of *praxis* as the "know how" and *logos* as the "knowledge," namely, the theoretical discourses on praxis. According to Chevallard, a praxeology is made of the following components: a task; a technique, that is a way of performing the task (praxis); and two kinds of theoretical justification (logos). While Chevallard uses this construct to analyze teachers' didactical praxeologies in their school activity, the MDT (Arzarello et al. 2014; Robutti 2020) transposes this idea to the analysis of PD programs for teachers, introducing the meta-didactical praxeologies: those of the teachers, because of their professional experience in the school, and those of researchers as designers and coachers of the teachers' educational program, because of their competence in mathematics education. During the development of the program, the meta-didactical praxeologies of researchers and teachers (different at the beginning) may evolve toward *shared praxeologies* (Arzarello et al. 2014), through the passage of some of their components from external to being internal (viz., used by individuals or by a community). The evolution of the praxeologies does not mean that all the teachers (or researchers) involved in the educational program evolve in the same way with the same transformation of components: in fact, different teachers may evolve in different ways, with respect to their histories and experiences.

In what follows we will use the MDT to describe our MOOCs and the previous PD programs in which they have their roots.

12.1.1 The First Part of Our Story

The experience of our MOOCs, upon which we base the main part of this chapter, has its roots in a series of emblematic events, which feature the main changes in the official and implemented Italian curriculum of mathematics. It is a 20-year-long story, in which for a series of circumstances, two of the authors of the paper (F. Arzarello and O. Robutti) had a relevant role (the third author, E. Taranto, had a relevant role only in the MOOCs). The story marks not only the changes in the curriculum but also the evolution in the structure and rationale of programs for teachers' PD and shows a link between national issues, due to the long tradition of Italian mathematics teaching, and international instances, due to the radical changes that math teaching was having at the turn of the century (see, e.g., the PISA 2021 framework: Carr 2018).

This evolution is presented using the MDT framework, illustrated above. The presentation follows the change in programs for teachers' PD, from being based on paper materials and face-to-face meetings to online resources shared among the participants through synchronous and asynchronous virtual meetings in a technological platform that fosters different types of interactions with its different tools (e.g., forum, padlet (https://it.padlet.com/), tricider (https://www.tricider.com/)).

The presentation is divided into three parts, of which the discussion about our MOOCs will be the last. Hence in this section, only the first two will be sketched:

(a) The experience of the *Matematica per il Cittadino* (*Mathematics for the Citizen*): 2000–2005
(b) The m@t.abel project: 2006–2012
(c) Math MOOC UniTo: 2015–2020

We will point out two structural indicators in order to underline their evolution as programs for teachers' PD: (a) the format of the used materials and (b) the format of the interactions between the teachers and the researchers who designed the programs.

12.1.2 Matematica per il Cittadino *(*Mathematics for the Citizen*)*

At the beginning of the new century, one of the authors (F. Arzarello) on behalf of the Italian Ministry of Education and of UMI (Italian Mathematical Union) chaired a working group made up of about 30 experts (one was O. Robutti), academics in mathematics and mathematics education, policy makers, and mathematics teachers in order to elaborate an updated and compact curriculum of mathematics for primary and secondary schools: the *Matematica per il Cittadino* (MpC: *Mathematics for the Citizen*). All the people involved in the group had a strong expertise in mathematics education, and most of them had been part of research groups promoted since the 1970s by the Italian National Council of Researches (CNR) in order to improve the teaching of mathematics in the schools (Arzarello and Bartolini Bussi 1998).

MpC was inspired by the NCTM *Principles and Standards for School Mathematics* (https://www.nctm.org/Standards-and-Positions/Principles-and--Standards/) and by other curricular projects appeared in those years (e.g., Belgium project), by the results of the international mathematics education research and, of course, took also into consideration the Italian tradition and practices in mathematics teaching. The proposed curriculum was based on the idea of *mathematics laboratory*, not so much a place but a methodology based on activities in which the students can learn by doing, seeing, imitating, and communicating with each other, under the guidance of the teacher — that is "practicing," as in an Italian Renaissance workshop (AAVV 2001; Chapman and Robutti 2008). The proposed teaching practices fostered a close interaction between novices (students) and expert (teacher), in the frame of *cognitive apprenticeship*, namely, it stressed "the learning-through-guided-experiences based on cognitive and meta-cognitive, rather than on physical, skills and processes" (Collins et al. 1989). Another important issue of the project was the particular attention to *processes* (and not only to *contents*), such as the use of different communication strategies, conjectures, argumentation and proof, problem-posing and problem-solving, measuring, and modelling. In consonance with the NCTM documents, MpC considered four main areas of mathematics contents (numbers and algorithms; space and figures; relations and functions; data and forecasts) and three main transversal areas (arguing, conjecturing, and proving; measuring; problem-posing and problem-solving), essentially the same for all the grades of pre-university schools. The curriculum, which distinguished between skills and knowledge, was accompanied by about 150 examples of teaching activities, which illustrated the meaning of the curriculum itself: for each of the areas, the examples made explicit the skills and the knowledge, at which the activity aimed. Each activity was also accompanied by the relative assessment tests. All this huge work was published in three volumes, (AAVV 2001, 2003, 2004: *La Matematica per il cittadino*, vol. 1, 2, 3 [*Mathematics for the Citizen*: https://umi.dm.unibo.it/materiali-umi-ciim/]), which were sent as a book to all the Italian schools and later made freely available on the web as a PDF book. The new proposal was spread through the country with many conferences, where its main points were showed to the schools and its content was used in many programs for teachers' PD; moreover, the Ministry of Education used it to design—in the school reform—the new mathematics curriculum (in 2003, 2006, 2010, and 2012, respectively).

The MpC project can be classified according to the two indicators as follows:

1. The materials were books, available on the website.
2. The interactions happened through the traditional format programs for teachers' PD: meetings promoted by the Ministry of Education and by the schools, where the proposal was illustrated by experts, who had shared experiences in mathematical education through the CNR research groups mentioned above.

12.1.3 m@t.abel

Although the new Italian National Curriculum inspired by the MpC project mirrored the influence of this work, school reality was however quite far from being broadly influenced by the new perspectives: innovation was bounded to isolated cases and to primary or middle schools, more than to secondary schools. Hence, in order to improve school mathematics education at the secondary level, the Ministry of Education and the Agency of School (INDIRE) promoted in 2006 a new project, m@t.abel (it is not a web address but an acronym, **Mat**ematica: **a**pprendimenti di **b**ase con **e-l**earning—Basic mathematics with e-learning—http://www.scuolavalore.indire.it/superguida/matabel/). The aim of m@t.abel was to disseminate the activities and teaching practices of MpC in a mathematics teacher PD program, carried out over Italy from 2006 until 2012 in blended modality with an online platform, through the recruitment of teacher trainers, trained on their side by researchers. For this dissemination, most of the activities designed for m@t.abel came from MpC, and adapted to a double modality—multimodal and static, as a downloadable file—presentation in the platform online. In the blended course, researchers typically entered into contact with teachers through a *two-step process*, which can be framed through the lens of the MDT (Table 12.1).

The entire process happened in two steps. In the *first step*, tutors' education (made by university researchers): the meta-didactical praxeologies of the researchers were shared with those of tutors, giving rise to shared praxeologies. In the *second step*, trainers: the shared praxeologies of the first step became the base of the second step as researcher/trainer praxeologies and were shared with teachers praxeologies.

In the second step, the teachers were organized in communities of inquiry (Jaworski and Goodchild 2006), composed of 15–20 teachers and supervised by tutors. The communities of teachers worked first in some face-to-face meetings with tutors: the tutors presented the activities and the spirit of the project in some meetings, asking the teachers to analyze them from a didactical point of view. Then, the teachers experimented with some chosen activities in their own classrooms and observed their students' processes, writing their notes in a logbook (see Fig. 12.1) to be uploaded on the platform: in this way, teachers' praxeologies started to change. Finally, the tutor coordinated the group of teachers from remote, through synchronous meetings and asynchronous discussions.

During the educational program, the tutors' and teachers' meta-didactical praxeologies evolved and changed toward the convergence of shared praxeologies as a result of the MDT. m@t.abel shared praxeologies included, for instance:

- At task-technical level, the use of exploration-conjecture-argumentation tasks, the mathematical laboratory methodology, and the introduction of new tools (like DGS, spreadsheets, etc.)
- At the theoretical level, a new vision of mathematics learning and teaching that was shared by the communities of teachers and researchers/tutors

Figure 12.2 shows an example of the way the documents uploaded by the teachers can concretely show this evolution.

Table 12.1 Researchers' praxeology to move from MpC to m@t.abel

Task	Designing tasks for teachers' PD, which includes activities to be used with their students and teaching practices
Techniques	Modifying activities of MpC according to the m@t.abel frame and writing down teaching practices to be used (group works, mathematical discussion, use of tools, etc.); providing resources (texts, videos, diagrams); and referring to the institutional frame of the National Curriculum
Theoretical justification	Current international research in mathematics education, in particular: theoretical frameworks of meta-didactical and didactical transposition (Arzarello et al. 2014), communities of practice (Wenger 1998), and the mathematics laboratory (Anichini et al. 2004)

General aspects/Context of the activity	• title of the activity • name of the teacher • name of the school • involved class • period of time within which the activity was developed • number of hours devoted to the activity (in the class and before or after the lessons)
Aspects related to the organization and the development of the activity	• methodology of work with students (working group activities and composition of groups, class discussion, interconnections with other subjects); • reactions of the students to the proposed activity; • collaboration between students and students and between teacher and students; • difficulties met by the teachers in developing the activity and strategies adopted to overcome them; • planning of the final test.
Aspects related to students' motivation and to students' learning	• difficulties met by students during the proposed activity (from both the metacognitive and cognitive point of view) and choices made by the teacher in order to make them overcome these difficulties; • positive metacognitive results (changes in students' attitude toward the discipline, growth in their interest…); • positive cognitive results; • students' results in the final test;
General evaluation of the activity	• effectiveness of the activity in terms of recovery of students' difficulties; • role of the activity as a stimulus for gifted students; • effects of the activity on the teacher in relation to his/her didactical planning and to this/her attitude toward the discipline; • suggestions for the improvement of the activity.

Fig. 12.1 The structure of the logbook. *Note:* The logbook served to keep track of the teaching experiments that the teachers conducted in the classroom, both from the point of view of the carrying out of the activity and the learning processes that emerged from the students

The two indicators for the m@t.abel can be summarized:

1. The material was constituted at the beginning of the m@t.abel examples, elaborated from the MpC ones; then they increased through the contributions of the teachers, who uploaded their comments and logbooks in the platform. Hence it was a dynamic material, made of different components, which increased and could be shared in time.
2. The interactions changed from step 1 (traditional face-to-face courses) to step 2 (synchronous and asynchronous interactions between teachers and tutors). It must be said that, generally, the online interactions happened between a tutor and a group of teachers who worked together and physically met in some school to connect with the tutor (in the 2010s, Internet connectivity at schools in Italy was limited). In the asynchronous interactions, generally, the teachers uploaded their comments and questions.

Figure 12.3 pictures the evolution of researchers' and tutors' meta-didactical praxeologies during m@t.abel.

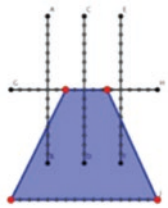
	WHAT I THOUGHT BEFORE THE TEACHING EXPERIMENT ON QUADRILATERALS	WHAT I THINK NOW
Working modalities with students	Work in pairs with concrete materials	• Work in pairs in laboratory with software GeoGebra
Teacher's role	Teaching, explaining, exemplifying	• Guiding students in discovering properties of quadrilaterals • Discussing with them about description, definition, properties, • Coordinating discussions giving stimuli, ordering conjectures • Institutionalizing knowledge
Tools and their functions	Concrete materials (paper and pencil) as a model where constructing quadrilaterals according to their symmetry properties	• Software GeoGebra for constructing the same model of concrete materials (Fig 4)  • Paper sheets of every activity • Instruction for constructing quadrilateral in GeoGebra

Fig. 12.2 The evolution of a teachers' praxeology

12.2 Methodology

Drawing on the MDT, in a MOOC for teacher's PD, we can consider the two communities of researchers and teachers. Considering the researchers, we will refer to those who made up the MOOC team, namely, the people who collaborate in the project. This team is composed of university researchers (the three authors of the chapter) and a group of teacher researchers who have been collaborating for several years with the research group in the didactics of mathematics at the University of Turin. We will refer to the members of the MOOC team from here on as instructors. We pointed out some essential meta-didactical types of tasks that, according to our experiences, any instructors of a MOOC for mathematics teacher education should address. Precisely, we consider four topics related to the design principles: team for designing and monitoring; teaching activities to be proposed; interaction with and among the participants; and assessment. For each topic, we describe the researchers' meta-didactical praxeologies. In fact, we identify the related meta-didactical types of tasks, the techniques adopted to solve such tasks, and the related theoretical

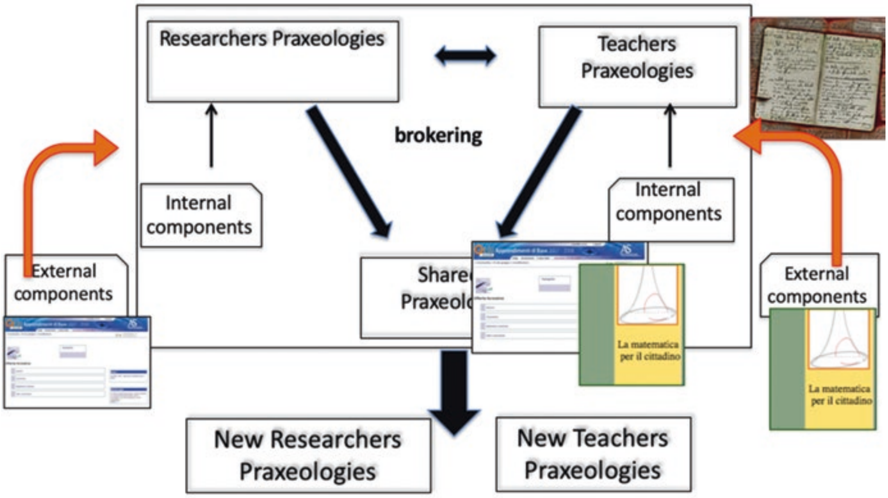


Fig. 12.3 The evolution of a teachers' and tutors' praxeologies

justification. For the latter, we particularly wondered how the chosen techniques were justified and supported by theories in mathematics education or more generally in the educational field. The identification of these meta-didactical praxeologies has been possible by reflecting on the design phases in which we were involved during the five seasons of our MOOCs. Some of them evolved during the MOOC seasons. The reasons for this evolution (intended as an improvement of the PD program) came from the researchers' self-analysis of the respective experiences but also from some MOOC-teachers' comments (e.g., posts in communication message boards).

Considering the community of MOOC-teachers, i.e., the recipients of the researchers' meta-didactical praxeologies, we focus in particular on one of the MOOCs we have provided, MOOC numeri, the second one. We chose this one because the first MOOC was our first experience, and from the second one onward, meta-didactical praxeologies had already evolved. We will take one of its modules into consideration and illustrate its structure, i.e., the implementation of meta-didactical praxeologies. We will focus in particular on some interactions between MOOC-teachers to show how they learned and made use of new teaching practices, new resources, and new technologies for teaching mathematics. The communication message boards we will consider are the forum and the padlet. The former allows having nested discussions, keeping track of those who left the post (we will only quote the teachers' initials), and the date and time the post was written. The second does not allow replies to the uploaded posts but keeps track of those who wrote the post (here too, we will only report their initials) and the typology of the school (lower or higher secondary school) to which he/she belongs.

12.3 Findings

12.3.1 Participants and MOOC Completion Rates

Table 12.2 shows data referring to the Math MOOC UniTo project (Taranto and Arzarello 2020). The participants were Italian mathematics teachers of all school levels (from primary to higher secondary school). The target audience we requested should be in-service teachers, so that they would be able to implement in their classrooms the activities presented in the MOOC. However, as one of the characteristics of MOOCs is their "openness," among the participants, there were also pre-service teachers, although these constituted a clear minority. Usually, we started to advertise each MOOC about a month before it started, spreading the information through social networks, mailing lists, and mostly on the platform hosting the MOOC itself, the DI.FI.MA platform, already mentioned in the introduction. This platform is managed by the Turin Department of Mathematics and organizes initiatives (seminars, blended courses, projects, etc.) addressed to teachers of scientific subjects. To date, it has reached about 2000 teachers. All those who have participated in our MOOCs have voluntarily chosen to enroll in them.

From Table 12.2, it can be seen that the periods of delivery have changed during the course of the editions. This happened because we understood that, for Italian teachers, the period from September to December is a very busy period of schooling. While, from January onward, we observed more participation and willingness to follow and complete the online training.

In addition, we stress that the completion rates of these MOOCs[1] were on an average of 39%. The literature states that, for MOOCs in general, "[…]the completion rate[…]is below 13%" (Onah et al. 2014, p. 5825). For MOOCs aimed at

Table 12.2 MOOCs of the math MOOC UniTo project

Math MOOC	Timeline	Enrolled	Completion rate (%)
MOOC Geometria	October 2015–January 2016	424	36
MOOC numeri	November 2016–February 2017	278	42
MOOC Relazioni e Funzioni	February 2018–May 2018	358	39
MOOC Dati e Previsioni	January 2019–April 2019	450	40
MOOC Modelli	January 2020–April 2020	262	38

[1] In order to calculate the completion rate, we proceed as follows: we consider all mathematics teachers who have enrolled in the MOOC and that a teacher can be said to have completed the MOOC if two criteria are met. The first one is related to the tasks that are contained in the various MOOC modules. Each time all the tasks of each module are fulfilled, the platform issues a badge (in line with the third recommendation of Qian et al. (2018)). So the first criterion is to have collected all the badges of the MOOC. The second criterion is related to the final module activities, which consist of designing a teaching activity and revising the activity designed by another MOOC teacher. If both criteria are met, we consider the MOOC completed and issue a certificate of participation, on behalf of the Turin Department of Mathematics, which is equivalent to 30 h of PD. This is certainly an incentive for Italian teachers, for whom PD is a right and a duty.

training mathematics teachers, the completion rate is 12% (Panero et al. 2017). Our completion rates differ from those reported in the literature, and we believe that this is also related to the design choices and monitoring that have been made in the various editions (Taranto 2020). We will illustrate these aspects below.

12.3.2 Design and Monitoring of MOOCs in Math MOOC UniTo Project

In this section we will focus on the instructors' community and its meta-didactical praxeologies.

MOOC Team

The members of the MOOC team (Table 12.3) worked for at least 1 year on a single MOOC. The design and digitization of platform content take 6–8 months. The monitoring phase of the MOOC coincides with the period of its delivery (about 3 months) and then follows a final check/assessment for the issuance of certificates of participation which takes 1–2 months. During the design, there are collective and collaborative moments, where everyone puts together ideas and shares the first results. In developing the activities, we are a community of practice (Wenger 1998): we all pursue the common goal of creating a MOOC for mathematics teachers, which is carried out in a collaborative way. Once the MOOC is ready on the platform, all team members commit themselves to a beta-test in order to test that everything works technically and didactically. Constructive criticism arises, an important input to improve the content and/or its exposure or how it is shown. Both when we monitor the MOOC and when we reflect on its conclusion, we are a community of inquiry (Jaworski and Goodchild 2006).

In the first two MOOCs, the team was large, i.e., in addition to the three researchers, there were 15 teacher researchers. Over the years the number of researchers has remained unchanged, while the number of teacher researchers has decreased. Since the third edition, in the design phase, there were at most six teacher researchers, and in the monitoring and conclusion phase always three teacher researchers. We have experienced that the small group is more fruitful. With larger groups it is difficult to

Table 12.3 MOOC team

Task	Designing, digitizing, and monitoring a MOOC for mathematics teachers
Techniques	Create a group composed of university researchers and teacher researchers
Theoretical justification	Community of practices (Wenger (1998)) and community of inquiry (Jaworski and Goodchild (2006))
Evolution	The group's size has decreased

Table 12.4 Teaching activities

Task	Showing teaching activities to MOOC-teachers that they could replicate in their classes
Techniques	Choice of a topic Identify a m@t.abel activity focused on that topic
Theoretical justification	Italian curriculum Theoretical frameworks as meta-didactical and didactical transposition First and second recommendations of Qian et al. (2018) Principles of "self-directed learning" and "job-connected learning" by Kleiman et al. (2015)
Evolution	Instead of working on topics, we worked on macro-themes Taking a cue from the setup of m@t.abel, we designed our own new activities

satisfy everyone, while smaller groups are advantageous because you can agree more quickly to meetings and on the division of tasks.

Teaching Activities

The teaching activities (Table 12.4) of the modules of the first two MOOCs were organized as follows. Each module intended to focus on a specific topic (e.g., triangle height, angle, sense of number, arithmetic language vs. algebraic language). For each topic, a specific m@t.abel activity was identified, which served as the opening activity of the module, and to this about 2–3 activities for both lower and higher secondary schools were added, in order to propose variations of activities on the same topic for different school levels. The variants of the activities were designed by the teacher researchers in collaboration with the researchers. Subsequently, they were digitized by Sway,[2] always including the description of the activity and pedagogical suggestions to replicate it in class with the students. In each module, about 5–6 activities were then proposed on a specific topic. The MOOC-teachers could freely consult the various activity proposals, regardless of their school level, and could experiment with them with their students.

From the third MOOC onward, we did not work on topics, but on macro-themes (e.g., functional thinking, statistics, probability). We continued to take inspiration from m@t.abel, but some of the activities that opened the module were also the result of the design by the MOOC team. In particular, from the fourth MOOC, we chose to consider for each module a unique activity for all the school levels and present its simplifications or insights for the various school levels (note that for each edition, 6% of teachers taught at the primary school).

In all MOOCs, in each module, the teaching activities were preceded by a short video (5–7 min). In the videos instructors shared reflections on the topic or

[2] Sway (https://sway.offic e.com/): Microsoft tool that allows users to combine text and media to sustain the showing of online content.

Table 12.5 Interaction with and among MOOC-teachers

Task	Making the interaction with and among the trainees possible
Techniques	Setting different communication message boards and reducing instructors' speeches, monitoring behind the scene Sending periodic e-mails Organizing webinars for creating occasions of synchronous contact
Theoretical justification	"Communicate"[a] from the 7Cs by Conole (2014); principle of "learning from multiple voices" by Kleiman et al. (2015); community of practices by Wenger (1998)
Evolution	None

[a]Mechanisms to foster communication: How are the learners interacting with each other and their tutors? (Conole 2014, pp. 3–4)

macro-theme of the module and gave didactic-pedagogical suggestions. Suggestions included anticipating how the various activities would enable teachers to propose similar topics to their students in a different way or how they could use these activities to overcome misconceptions or their occurrence.

A constant in all five editions, in line also with the second recommendation of Qian et al. (2018), has been the fact that all the activities proposed are based on the National Curriculum, because we want to propose activities that can be spent in class, on topics that the teacher is required to deal with, perhaps in an innovative way, different from the standard ones. In addition, with the proposed activities, we have promoted the methodology of the mathematics laboratory, the use of simple materials (e.g., paper, pencil, string) and also technologies (e.g., dynamic geometry system, spreadsheets, MathCityMap (Gurjanow et al. 2019)). The activities have always been accompanied by suggestions on teaching practices: we suggest to start a mathematical discussion in a certain way with specific stimulus questions and to make working groups in a collaborative/cooperative way. From the third edition onward, when we started working on macro-themes, we also presented general pedagogical suggestions, related to the macro-themes of the MOOC. They consisted of paying attention to using more student-centered than teacher-centered practices and to students' processes and their own teaching practices and possibly contextualizing the suggested practices in their own classroom environment.

Interaction with and Among the MOOC-Teachers

We have observed, in the course of our five experiences, that in a totally asynchronous environment, fostering interaction proves to be a strength (Table 12.5). On the platform, we have always proposed three types of communication message boards: the forum, the padlet, and the tricider.[3] The presence of the communication message boards was on the one hand due to the fact that we wanted to enable teachers to

[3]For more information about these communication message boards in our MOOCs, see Taranto et al. (2017).

communicate with each other in an online environment. On the other hand, we wanted to have feedback on the activities we proposed (e.g., how teachers evaluate them, if and how they used them in class). Therefore, it was important for us that teachers share their thoughts and comments on the activities proposed by the MOOC and how they implemented them in class with their students. For this reason, each board was preceded by one or more specific questions that were asked to be answered or a title that served as a talking point. Encouraging conversations and making them a habit in all modules helped to create a community of practice (Wenger 1998) among the participants.

In monitoring the communication message boards, we have always followed this methodological choice: we stayed behind the scenes, intervening only if strictly necessary (e.g., for precise clarifications on activities), because we did not want to inhibit the teachers, nor to direct their discussion. In each edition, it happened that, at a certain point, the teachers seemed to forget that on the other side there were us, instructors, reading them and they started to put in place practices that we did not expect. They started to confide in each other, telling about their perceived frustration because of the students who don't seem to be particularly interested in the discipline; others are enthusiastic about events that happened in the classroom during the experimentation and also share personal materials (that they had already had before the MOOC), although nobody asked; and others said they perceive the environment of the MOOC as that of a family (for more details see Taranto 2020; Taranto and Arzarello 2020). These externalities convinced us that not intervening on the communication message boards but being vigilant was a good practice to continue to maintain.

We also sent weekly e-mails to keep the community up to date on the progress of the distance learning experience, anticipating what would await them in the modules that gradually opened. The only moments of synchronous interaction between the MOOC-teachers and us instructors were webinars, i.e., online meetings, in which one of us spent 30–40 min explaining a topic relevant to those dealt with in the MOOC and chosen also on the basis of the interactions that were emerging on the communication message boards between the teachers themselves. The instructor then devoted another 30 min to answer questions from the MOOC-teachers, who asked them via chat. We organized about three webinars for each edition. These webinars gave the teachers the idea of the vigilant and attentive presence of the instructors and helped to encourage a constant commitment to bringing the MOOC to a conclusion among the teachers.

Assessment

As mentioned earlier, a badge has been associated with each module. Once earned, these badges are visible in each user's profile, and this has a twofold advantage: on the one hand, inserting gamification elements encourages the user to engage more (Surendeleg et al. 2014) and thus to complete the MOOC; on the other hand, it is useful for us instructors to have an overview of the progress of the MOOC in terms of involvement and commitment of the participants. The badge of each module was automatically issued by the platform if certain tasks were fulfilled (e.g., filling in a questionnaire, reading the teaching activities, interaction on the communication message board setup). In the first MOOC, the issuing of the badge was also associated with a score obtained in a multiple-choice test aimed at ascertaining that the MOOC-teachers had actually viewed the module resources. However, this instructors' choice did not meet with the MOOC-teachers' approval (for more information see Aldon et al. 2019). From the second edition onward, we have decided to no longer include the test.

In addition to the badges, to assess the teachers' engagement (Table 12.6), we set two final activities. The MOOC-teachers had to demonstrate acquired teaching skills and expertise by designing an individual project work (PW), using a web-based tool, Learning Designer (Laurillard 2016). They were free to choose the theme of their PW, in line with the theme dealt with in the MOOC (e.g., in the MOOC Geometria, the PW had to be based on a geometry topic). To make the teachers familiar with the Learning Designer software, a video and a PDF tutorial were created. In the first MOOC, they were made available 2 weeks before the opening of the last module. However, this time was not enough, so in subsequent editions, the choice of the software and the task that the teachers would take at the end of the MOOC were announced from the first module, making the tutorials immediately available. The deadlines for accomplishing the PW were announced

Table 12.6 Assessment

Task	Assessing the degree of participation of the MOOC-teachers
Techniques	(a) Multiple choice test related to the video content and module activities (b) Release of the badge (the test was a necessary and sufficient condition for its release) (c) MOOC-teachers are asked to design a PW (in 1 week), using the learning designer software and freely choosing the content to address in their project according to the theme of the MOOC (d) MOOC-teachers are asked to do a peer review (1–1) of a project they choose at the same school level (in 1 week), using a review grid (e) Tutorials to accomplish the PW were given 2 weeks before this activity; review grid was given in the same week devoted to PR
Theoretical justification	(a) Assess massive number of users (b) Qian et al. (2018) third recommendation: "use motivational design to improve teachers' engagement" (c, d) Kleiman et al.'s (2015) four principles (e) Time for appropriation of new praxeologies: a tool as learning designer and the criteria of the review grid
Evolution	(a) Test was present in the first season, but removed from the second one (b) The deadline to carry out the PW was extended by 2 weeks (c) The deadline to accomplish the PR was extended. The project to be reviewed was assigned by the instructors to each teacher taking into account the school level (d) The tutorials for LD and the review grid were given at the beginning of the MOOC

immediately after the creation and sharing of the PW. The PW also had to be peer reviewed (PR). The PR was performed via a review grid, which was delivered to MOOC-teachers in the week of the PR. In the first MOOC, we allocated 1 week for the PW and 1 week for PR. However, some teachers expressed the need to have more time to accomplish their PW. Thus, in the subsequent seasons of the MOOCs, the deadline was extended by 2 weeks for both requests (PW and PR). The teachers also asked us to look at the review grid first, as it is useful to take its criteria into account during the design phase. From the second MOOC onward, the review grid was delivered together with the PW tutorials. Finally, in the MOOC Geometria, the choice of which PW to review had been left to the teachers. However, this meant that some PWs were not chosen, while others reviewed the same PW several times. From the second edition onward, we assigned the PWs to be reviewed to each teacher.

We emphasize that the PW could be based either on activities already seen in the MOOC, but to be adapted to their own school context, or it could be based on activities from scratch, but relevant to the programing core in question. The final aim was to see also their design skills, not only from the point of view of mathematical content but also of the teaching practice they chose to adopt. This would have allowed us to understand whether the proposed activities and the pedagogical suggestions attached to them (laboratory activities for the most part) had become part of the teachers' practices. The instructors did not want to evaluate the PW due to the possibility of teachers feeling judged, but rather the feedback on what they produced comes from a peer: the feedback was thought of in a constructive way, in order to possibly improve any aspects of the project (e.g., timing, pedagogical suggestions).

12.3.3 An Example from MOOC Numeri

In this section we will focus on the MOOC-teachers' community, and we take into consideration a module of MOOC numeri, the first thematic module entitled "Meteorites, bacteria, rice grains…numbers and their meaning." The mathematical topic of this module focuses on the orders of magnitude and the number sense, as explained in a short video, placed at the opening of the module, in which one of the instructors speaks. The first activity proposal offered is the activity m@t.abel "Il livello del mare"—the sea level—(the activity is available in Italian here[4]: http://www.scuolavalore.indire.it/nuove_risorse/il-livello-del-mare/). It addresses problems related to estimates of orders of magnitude. In fact, being able to correctly estimate an order of magnitude, or to approximate values, is a widespread difficulty (AAVV 2001, 2003). By proposing problems also linked to real situations, it is clear

[4]A short English version of the activity is available in the Appendix to give the reader an idea of the activity most relevant aspects.

that in certain cases, the order of magnitude is important, while the number of significant figures becomes secondary. At the beginning of the activity, the scientific notation is introduced to represent numbers and discuss the use of this writing to evaluate the order of magnitude. These are fundamental tools for a citizen because information often arrives from the mass media which is received with little capacity for analysis and without an adequate sense of numbers. Analyzing and reasoning on an important topical issue (How much would the sea level rise if all the glaciers would melt?), in a concrete case, the different problems concerning the order of magnitude, precision, and approximation are dealt with.

A forum follows, in which teachers are invited to share any teaching experiences related to this topic. Here are some examples of discussions, from which you can see examples of teachers' didactical praxeologies on the topic of the module:

> P.M.—07/11/2016, 11:50: […] in science I started chemistry and we are dealing with molecules and atoms. I thought I would make a connection with the magnitudes of these structures and thus analyze the microscopic world with the negative powers of 10. What do you think?
>
> E.L.I.—08/11/2016, 11:07: I share your idea. This year I have a grade 6 class and, in chemistry, I was thinking of applying the activity topic using scientific notation with negative exponent powers.
>
> E.Z.—15/11/2016, 19:07: […] I agree with you that for a middle school it's hard! However, I will try […] to propose it in science in a grade 8 class: I'm showing a long documentary on the history of the earth and many questions about the future of the earth and in particular about Venice have already emerged from the students; instead of answering, I will propose the activity […]: mathematics is the tool through which they can answer their own questions. In middle school I believe that intriguing is still the most important aspect; […]

This discussion excerpt involves lower secondary school teachers. All three agree that the activity is feasible at upper secondary school. However, they try to find a didactical praxeology that will allow them to meet the task they are setting themselves: to adjust the activity for lower secondary school. P.M. and E.L.I. call for an interdisciplinary link with chemistry. E.Z. instead takes advantage of an experience that has already happened in class: the vision of a science documentary that has aroused the interest of the students. This will be the stimulus from which to lead the students to answer their curiosity about Venice and whether or not it will be submerged.

Let us look at this other example of teachers' discussion:

> S.F.—07/11/2016, 21:38: […] In high school the activity could be proposed as an 'open problem', without too much data, to force students to search for the data needed to solve it. Then I would force them to use the calculator as little as possible forcing them to exploit the advantage of applying the power properties! Very often they make trivial calculations because they are lazy enough to use the calculator!!!! Does this happen to your students too?
>
> E.L.I.—08/11/2016, 11:16: Yes yes…away from the calculator!!!! making 'mechanical' use of it they do not learn and do not use the powers and relative properties when things would be simpler this way. […]
>
> T.A.—08/11/2016, 22:32: Yes, I also fully agree. I teach in [a higher secondary school]: the use of the calculator takes students away a lot from acquiring a sense of number. It is much better to make students see reason by using the power properties.

In this second excerpt of discussion, we observe how the three teachers agree that the use of the calculator is a practice to be discouraged, because it limits the reasoning that students can do by exploiting the properties of powers and distances them from understanding and assimilating a correct sense of number. They all seem to agree on a didactical praxeology that invites students to reason on open issues.

Subsequently, four activities are proposed (two for lower secondary school and two for higher secondary school), such as variations of activities on the same topic, and a fifth activity valid for all school levels, in which exercises and examples are proposed on the use of some technologies (the scientific calculator and excel) for the development of the sense of number. The module ends with a request to comment on the padlet by answering at least one of the following questions left by the instructors as a stimulus to the teachers, in order to allow them to interpret the content and pedagogical notion of the activity themselves:

- How could you develop one of the activities in the classroom? Are you planning to do it?
- At what point in your programming do you address the orders of magnitude?
- What are the strengths of the activity to work on orders of magnitude?
- How can problem-posing/problem-solving activities help to address the conceptual nodes of the proposed unit?

We report some examples of conversations, both by lower and higher secondary school teachers, from which emerge the didactical praxeologies adopted by them in dealing with the concept of order of magnitude and sense of number, together with their considerations on the activities presented:

> M.L. (higher secondary school): The various activities proposed have interesting ideas: working with 'open problems', unconventional, often taken from situations that have actually occurred […] Mathematics, therefore, as a tool to deal with the study of reality. They also pose important questions about the approach to our teaching: the teacher becomes a director, lets the pupils organize themselves and be independent in their work, the process is important and not the result. […] my difficulty is to find problems […] and the proposals herein reported have really opened up a world for me. I will be able to use them to introduce the concept of order of magnitude in a more significant way than I have done so far.
>
> F.G. (lower secondary school): […] Usually, I introduce the scientific notation and the orders of magnitude at grade 6, immediately after having treated the powers in arithmetic […] I take the power with negative exponent back at grade 8, but only with those pupils who are particularly motivated. At grade 8, I also have greater possibilities to use exponential notation in science, both when we talk about the birth and evolution of the Earth and when we deal with astronomical distances. […] Problem-solving activities [proposed here] can be stimulating for the pupils, who put into practice in real contexts the mathematical knowledge and skills acquired.
>
> D.R. (higher secondary school): I teach in a professional institute […] I am also trying […] to modify not only the contents but also the practices. What I am trying to do now can be summed up in three points: fewer calculations, more problems, and collaborative work. It is not always easy to work like this and in some cases, it is the students who find it hard to approach the problems and would prefer 'some nice expressions'. […] I could propose the sea level activity […] The strengths are related to the adherence of the problem to reality, I think that […] is really important for a real understanding of concepts. […]

> M.S. (lower secondary school): I have always introduced the exponential notation in grade 6, in successive stages. Already when dealing with natural numbers, kids start asking how to call very large numbers and I always answer that scientists use a different way of calling numbers. At the end of the activities on powers, I present the order of magnitude and the scientific notation. Talking about the cell and the measurement of its elements, I explain that negative powers are used. At grade 8 the pupils are already accustomed and the interplanetary and interstellar measurements are already presented in this way. I will take my cue from the activities of this module to make innovative examples.

We observe how the teachers not only appreciated the activities offered but also showed a willingness to use them in the classroom with their students. The teachers tell us about the didactical praxeologies they used before the MOOC to present the topics dealt with in this module (e.g., F.G. talks about interdisciplinary between mathematics and astronomy and M.S. with chemistry; D.R. explains that she aims a lot at collaborative work). Some of them seem to have learned new teaching practices and proposed them to their students, possibly adapting them. For example, M.L. states that MOOC activities invite the teacher to be a director and leave the role of actor to the students. The MOOC, in fact, invites a lot of laboratory activities, where the student learns by doing, starting from real problems. This last aspect has been captured by all these four teachers and seems to be appreciated: a world has opened up to M.L. and M.S. who talks about new innovative examples.

We would like to remind that the final module of MOOC engages the teachers in the design of a teaching activity on the theme of numbers. Out of 278 teachers, 42% of them have worked on the PW. Among them, 11% designed an activity based on the sense of numbers.

12.4 Conclusion

In this paper, we have described and investigated our experience in designing and coaching some MOOCs aimed at teachers' PD. In this work, we based on the experience we had acquired since the beginning of the century in promoting and managing some previous important national programs.

We pointed out how the structure of these projects evolved in the course of time considering two indicators: (a) the format of the used materials and (b) the format of the interactions between teachers and researchers who designed the programs. We could show how our current MOOCs' materials and interactions are a concrete result of this evolution. For example, the forum and the padlet, massively used in the MOOCs, make possible a type of interaction that partially emerged in the blended experience of the m@t.abel project and that was noticed as positive by the tutors. As well, the m@t.abel experience of logbook compilation by the teachers in the project had revealed a promising way for having detailed information about what happens in the classrooms, but it showed difficulties for its compilation (too much time required to the teachers) and for effectively sharing it among the participants to the program (too many pages to read). The way of sharing experiences and comparing different praxeologies among the participating teachers was partial and limited in the m@t.abel communities: in our MOOCs the asynchronous affordances allowed by such devices like the forum and the padlet fostered the sharing of experiences among the participants in a simpler and more direct way. We can observe that there has been a change in the type of community, induced by the tools used. Indeed, the examples discussed above illustrate how the higher autonomy of the teachers in managing the discussion through the affordances of the platform allowed them to share their praxeologies in a deeper and less guided modality. This has multiplied the amount of teachers' conversations compared with those in m@t.abel. Indeed, we observed that, in the m@t.abel experience, interactions were either orchestrated by the instructors and the teachers took part in them or the interactions took place in a one-to-one mode between the teacher authoring the logbook and the instructor. In contrast, in MOOCs, all interactions take place on communication message boards, freely and spontaneously, and in countless numbers (forum postings are in the hundreds for each topic). Instructors limit their posts while remaining vigilant behind the scenes. In Taranto et al. (2020), we use the metaphor of explosion to describe this massive increase:

> The explosion concerns both space and time: the latter because the sequence of conversations increases in an 'unprejudiced' and exponential manner and everyone has freedom of speech, there are no time constraints that limit the sharing of one's thoughts (as instead happens in face-to-face courses); the space, insofar as the networks of knowledge both of the MOOC and of the single individuals are filled with nodes and connections, thanks to all the inputs that the MOOC offers and to the other participants' support (who share their own experience, reflections, etc.). (p. 14)

At the same time, the availability of such spontaneous data has made possible for instructors to enter more directly into teachers' meta-didactical praxeologies. Of course, this has been possible since the evolution of technology and particularly of ICT (information and communication technology) in the society and has made people fonder of their use, and consequently, teachers could use the MOOC platform in a "natural" way: this marks a big difference with the way ICT had been used in the m@t.abel project. Indeed, as noted above, in m@t.abel experience, the community of teachers occasionally met to discuss with the instructor who was directing the interactions. The habit of communicating and sharing that society has learned through social networks allows for freer and more spontaneous interactions on MOOC communication message boards (Taranto and Arzarello 2020).

While these observations point out the elements of innovation realized in the MOOCs with respect to the previous project, another aspect must be underlined as an element of continuity in this evolution: in fact, the influence of the way m@t.abel elaborated the activities of the *Mathematics for the Citizen* project into its examples of didactical situations for the classroom continues in the teaching situations, from which teachers are asked to start in the MOOC standard plans. The MDT has allowed to properly frame these aspects of continuity: this is shown in Table 12.4, where the evolution from the choice of a topic within the m@t.abel examples to its elaboration according to the classroom necessities within a chosen macro-theme is clearly hinted.

As to the acquired didactical capability of teachers who took part in the MOOCs, described above, we involve teachers in two final activities: the design of a project work and the consequent peer reviews of another project work made by another MOOC-teacher. Also, in this case, the MDT frame allowed us to get some global results about this issue, as illustrated in Table 12.6.

The four meta-didactical praxeologies of the instructors' community (Tables 12.3, 12.4, 12.5, and 12.6) are examples of how to extend the preexisting PD programs into the mode of remote delivery, and this could provide insight for other researchers interested in developing curriculum and activity design for mathematics teachers.

From the instructors' point of view, from one MOOC edition to another, one can see an evolution in their meta-didactic praxeologies. In fact, there has been changes in design, allowing teachers to meet their assignments in a longer time frame; assigning less complex tasks; and interweaving more mathematical task aspects with pedagogical aspects. In particular, we, the instructors, took into account the different school levels of the teachers participating in the MOOCs with proposals of activities useful for them all. In fact, if in the first editions activities were proposed only specific to a certain school level, in the following editions, activities were proposed on the same mathematical topic, but with different degrees of depth depending on the teachers' school level.

From the teachers' point of view, we stress again that MOOCs generate online communities that are very different from those of face-to-face courses, due to the different way in which interactions take place (Taranto 2020). However, the following limitations should be noted. Where there are face-to-face or blended courses, it is true that numbers are lower, but face-to-face interactions can lead to higher results in terms of praxeologies. We mean that the instructor has the possibility to interact synchronously with the teachers and the so-called shared praxeology between the instructors' community and teachers' community is achieved. On the other hand, in the MOOCs, where interactions take place totally asynchronously, it is possible to observe the meta-didactic praxeologies of the teachers, but a shared praxeology is not achieved. In fact, detailed reports from the logbooks are missing; therefore, the instructors do not know the teaching praxeologies of the whole community of teachers. Moreover, the teachers do not all follow the same pedagogical indications, because the community is large, it is formed by teachers of different school levels: they all start to follow the basic pedagogical indications indicated by the activity (written by the instructors), but these are then personalized not only according to their own classroom context but also to the other interactions that take place on the platform, which are not orchestrated by the instructors. This, therefore, does not allow for the generation of a shared praxeology between instructors and teachers' communities, but it certainly generates more richness in teachers' professional practices.

The relevance of our experience with MOOCs is twofold: from the one side, it has its roots in the previous Italian programs for teachers learning development, while from the other side it has been a rich and fruitful practice to be exploited in these times of online courses because of the COVID-19 pandemic. In fact, it is suggesting to us promising fresh ways to follow in such forms of teaching, and we are now investigating these new aspects according to which learning and teaching are changing, using the MDT and also other theoretical frames: so in this sense, our research story in mathematics education is continuing.

Appendix

Briefly, we report here the most salient parts of the activity “the sea level.” First of all, school students are asked to find some data, asking them to reflect on the following question: “What ice can affect the rise in seawater?” The ice at the North Pole floats on the sea: its melting will not change the sea level (school students verify this statement by means of an experiment with water and ice placed in a glass). We can also disregard the glaciers on the high peaks, above 6000 m altitude: the rise in temperature may have the effect of increasing snowfall rather than melting glaciers. We look for data on continental glaciers, i.e., resting on the land above sea level. The data search will produce the following results:

Glacier	Surface area in km^2	Volume of ice in km^3
Antarctic continental glacier	14 million	30 million
Greenland continental glacier	1.8 million	2.7 million
Remaining continental glaciers (e.g., Perito Moreno in Patagonia) in total	500,000	200,000

School students will be made aware of the inconvenience of ordinary number writing in dealing with these quantities (it is easy to make mistakes in reading or writing), and the scientific notation will be presented as the most suitable for this situation. Exercises will be proposed for its application.

We then return to the starting question, linking it to a real problem: Will Venice be flooded in the (near) future due to melting glaciers? We have to estimate by how much the sea level would rise if all the continental glaciers on earth melted. School students have already found data on the total volume of ice; what is needed now is the extent of the area over which the melted water will expand (they can look up the data again or reason, and derive this empirically—a digression is proposed for interested teachers). The surface area of the Earth (SE) is equal to

$$\text{SE} = 4\left(6.37\ \text{E}\ 3\right)2\ \ 3.14\ \ \text{km}^2 = 509.645864\ \ \text{E}\ \ 6\ \ \text{km}^2 = 5.1\ \ \text{E}\ \ 8\ \ \text{km}^2\ \ \text{circa.}$$

Ice has a greater volume than the corresponding water, which is why it floats like icebergs in the oceans. As the water freezes, it increases in volume by about 10%:

$$\text{V\_ice} = 1.1\ \ \text{V\_water.}$$

In order to find the volume of water produced by the melting of all continental glaciers, we need to divide their volume by 1.1:

$$3.3\ \mathrm{E}\ 7\ \mathrm{km}^3 : 1.1 = 3\ \mathrm{E}\ 7\ \mathrm{km}^3.$$

Water covers the earth for about 70% of its surface. The extent of the earth's surface over which water will be distributed will therefore be

$$5.1\ \mathrm{E}\ 8\ \mathrm{km}^2 \cdot 0.7 = 3.57\ \mathrm{E}\ 8\ \mathrm{km}^2.$$

The expected height is then

$$\mathrm{h} = \mathrm{V} / \mathrm{S} = \left(3\ \mathrm{E}\ 7\ \mathrm{km}^3\right) / \left(3.57\ \mathrm{E}\ 8\ \mathrm{km}^2\right) = 0.084\ \mathrm{km} = 84\ \mathrm{m}.$$

It is easy to see from the Venice municipality website that, for the time being, Venice is subject to recurring episodes of "high water" around 80 cm and therefore two orders of magnitude lower than forecast due to the melting of the glaciers.

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Chapter 13
Tweeting or Listening to Learn: Professional Networks of Mathematics Teachers on Twitter

Anne Garrison Wilhelm and Jaymie Ruddock

Research has demonstrated that the average professional development program for teachers is largely ineffective (Hanushek 2005; Kennedy 2016; Sykes 1996). These challenges have led to the notion that we might reconsider designs for professional learning in light of what we know about how teachers work and effective supports for teacher learning (Opfer and Pedder 2011). Conceptualizing informal teacher collaboration as professional development that values teachers' agency and problems of practice is an alternative to more formal professional development.

Social networks provide teachers with access to resources and expertise (Daly 2010), and there is a growing body of evidence supporting the notion that teachers learn through their social network interactions (e.g., Frank et al. 2004; Sun, Wilhelm et al. 2014, N. Sun et al. 2014). However, most research on teachers' social network interactions has focused on school-based networks. There is some evidence that networks that extend beyond the walls of the school, do, in fact, support teachers' learning (e.g., Spillane et al. 2015). Expertise and resources are not equitably distributed across schools; therefore, networks that extend beyond school walls have the potential to expand access to expertise and resources. Similar to other studies of teachers' professional networks, in this study, we consider "personal networks of the individuals that the teachers selected to collaborate and interact with to solve professional problems" (Baker-Doyle 2012, p.78).

A large number of mathematics teachers have extended their networks beyond the walls of their schools by collaborating on Twitter (among other social media platforms). Several studies have described how teachers use Twitter (e.g., Carpenter and Krutka 2015), and even how one or more mathematics teachers have used Twitter for professional reasons (e.g., Risser 2013). One dominant group on Twitter

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started as a grassroots network of mathematics teacher bloggers (Risser and Bottoms 2014), which developed into a group self-named the MathTwitterBlogoSphere. On one website, created by the members of the group, they describe their community:

> We communicate via Twitter and blogs so we use the nickname Math Twitter Blogosphere (MTBoS). First and foremost it's a support group for teachers in year 0 to 50 (pre-service to retired!)…This community is what you make of it: it is professional development if you ask professional questions, it is personal development if you ask personal questions. It can also be a mentoring opportunity if you choose to answer others' questions, give support or feedback. (https://exploremtbos.wordpress.com/overview/)

Twitter plays a central role in connecting mathematics educators in this network. In this study, we explore the scope and nature of the MTBoS network along with an overlapping network of mathematics educators on Twitter. In July of 2017, Dan Meyer, a prominent mathematics educator and member of the MTBoS, wrote a blog post entitled "Let's Retire #MTBoS" and tweeted it out to the MTBoS community. That blog post was not well-received by many in the MTBoS community and Meyer added to the original post to clarify his assertion. He added "I hope that the thousands of people who find community around '#MTBoS' will continue to enjoy it! But I'm hopeful that '#iteachmath' will be a better invitation for the hundreds of thousands of math teachers who don't yet know how great we have it" (Meyer 2017). His intent was to open up the MTBoS community to more people by making the hashtag more inclusive, but the MTBoS community felt it as an attack on their inclusiveness and perceived their identity being challenged (another indication of the robustness of the MTBoS community). Following that blog post and ensuing conversation, a number of mathematics educators are using #MTBoS and #iteachmath as hashtags to reach out to other mathematics educators. In this study, we investigate the use of both of those hashtags, separately and simultaneously, when considering how mathematics educators are collaborating on Twitter.

13.1 Literature Review

13.1.1 Teachers' Social Networks Support Learning

In considering networks as providing teachers with access to resources and opportunities for learning, we are taking a social capital perspective (Lin 2001) by assuming that access to material resources and expertise through social relations provides a foundation for individual, and collective, change (Daly 2010). In particular, a teacher professional network, or social network more generally, is a set of individuals or organizations connected to one another by relations or interactions. A large number of studies have investigated teachers' networks in schools in the USA and in other countries. Within-school networks are bounded by the number of individuals working in the school, and often more practically bounded by grade level or content team membership (e.g., Spillane et al. 2012). For example, Spillane et al.

(2010) studied 28 elementary schools in one school district in the southeastern USA. Schools in this sample averaged 41 faculty members. This means that teachers in these schools had access to approximately 40 other individuals on their campus. In secondary schools, it is even more common to just consider the subset of content teachers and instructional leaders as the bounded network. For example, in a study of middle school mathematics teacher networks in one school district in the USA, the number of mathematics teachers ranged from 6 to 20 teachers (Gibbons et al. 2019).

As described above, studies have found that social network interactions are related to change in teachers' knowledge, conceptions, and practice (e.g., Daly et al. 2010; Penuel et al. 2009; Spillane et al. 2015; Sun et al. 2014b). Further, studies have found that the expertise of individuals in their networks has an impact on the extent to which teachers learn through interactions (e.g., Penuel et al. 2010; Sun et al. 2014b). For example, Sun et al. (2014b) found that the expertise of instructional coaches positively moderated the extent to which teachers developed deeper mathematical knowledge for teaching through interactions with colleagues. In other words, access to expert colleagues was critical for teachers' development. Therefore, teachers who do not have access to more expert colleagues are not as likely to improve. This lack of access to more expert colleagues is exacerbated for historically underserved students due to the fact that schools that serve non-White students and students from families with lower incomes tend to have the least qualified and least experienced teachers (Darling-Hammond 2010; Goldhaber et al. 2018; Lankford et al. 2002). Until we, as a society, find ways to redistribute teaching expertise across schools, we need other ways to provide instructional expertise to teachers who may not have access. One way to do this is to look to social media for expertise outside of one's school.

13.1.2 Twitter as a Site for Teacher Professional Learning

Twitter has the potential to provide teachers with access to an extended professional network. As of January 2021, Twitter had 353 million monthly active users (Hootsuite 2021). Twitter is a micro-blogging service, meaning that users describe events in their lives in 280-character descriptions augmented by other media (e.g., pictures, link to websites, etc.).

There are several different ways to interact with other users and content on Twitter. First, the majority of tweets are public, so you can search for people, words, or phrases and see what is returned. Alternatively, users can populate their newsfeed by "following" other Twitter uses. Unlike some other social media platforms, permission is not required to follow other users. Therefore, users often follow large numbers of other users on Twitter (Kwak et al. 2010). Another way that people find things of interest to them is to use hashtags. A hashtag, a word or phrase preceded by a # (in the case of this study, #MTBoS and #iteachmath), allows people to tag content with an identifier so that it can systematically be filtered or directed to appropriate audiences (Huang et al. 2010). While hashtags are not assigned or owned, groups of individuals adopt hashtags to connect to one another (Rosenberg et al. 2016). Some hashtags are used for long periods of time (e.g., #edchat or #MTBoS), while other hashtags are used briefly in response to conferences or current events. For example, 1 group of 3598 French teachers used a Twitter hashtag, #educattentats, to support one another as they prepared to discuss recent terrorist attacks with students (Greenhalgh and Koehler 2017). In addition to tweeting new content, users can also like tweets, retweet other users' tweets, or reply to tweets. Differing from live conversation between two people, conversations on Twitter can happen synchronously or asynchronously. For example, conversations can be ongoing for days or weeks and involve large numbers of people in numerous locations.

There is a growing body of recent research describing how teachers use Twitter. The bulk of this research has used surveys to understand for what professional purposes teachers use Twitter (e.g., Biddolph and Curwood 2016; Visser et al. 2014). For example, in studies published in 2014 and 2015, Carpenter and Krutka explored how and why educators use Twitter by surveying 755 K-16 educators. They found that their sample of teachers used Twitter quite frequently, with 84% of the participants indicating that they used Twitter daily or multiple times per day. Further, Twitter was more often used for professional learning than for communication with students and families. Respondents reported multiple (mean = 4.7) professional purposes for Twitter engagement. The most popular uses, indicated by more than 50% of respondents, were resource sharing/acquiring (96%), collaboration with other educators (86%), networking (79%), and participating in Twitter chats (73%) (Carpenter and Krutka 2014). Sixty-nine percent of survey respondents provided open-ended responses to the prompt "Explain what aspects of Twitter you find most valuable, and why." (Carpenter and Krutka 2015, p. 713). These responses underscored some of the reasons why teachers choose to use Twitter including ease of use, timeliness, and broad reach. Sharing and/or acquiring of information and/or knowledge was alluded to in 51% of responses. Eight percent of respondents directly commented on how Twitter combats isolation they experience (Carpenter and Krutka 2015).

One recent study used machine coding in combination with the hashtag #edchat to identify and analyze a pool of more than 1.2 million education-related tweets over 8 months, by more than 200,000 Twitter users (Staudt Willett 2019). The intent was to compare the purposes observed within the sample of tweets with the purposes identified by Carpenter and Krutka (2014). First, far less information about the purposes of using Twitter was discernable using machine coding compared with Carpenter and Krutka's (2014) survey. However, this level of analysis did allow the author to identify particular trends in intent associated with different participation patterns. One finding related to collaboration was that when teachers replied to one another, those tweets were classified based on content as focused on "working together" (Staudt Willett, p. 9).

Overall, based on both survey data and observational tweet data, there is evidence that teachers use Twitter for professional collaborative purposes. Therefore, it makes sense to consider Twitter as a site for professional learning for teachers.

13.1.3 Mathematics Teachers' Learning through the MathTwitterBlogoSphere

Most previous research on the MTBoS has focused on interactions in the context of the blogs, rather than the interactions the context of Twitter (e.g., Parrish 2017; Risser and Bottoms 2015, 2018; Risser et al. 2019). However, several recent studies have explored MTBoS interactions on Twitter. For example, Staudt Willet and Reimer (2018) analyzed 6985 unique tweets from 2828 different contributors over 1 month in 2017. They found that 36.5% of the tweets were original posts, 3.7% of them were replies, and 58.2% were retweets.

One additional study of the MTBoS sheds light on the learning potential of this community through their close analysis of one conversation that stemmed from a single tweet asking people to respond to a student's mathematical thinking (see Fig. 13.1). In their study, Larsen and Liljedahl (2017) describe how this one tweet "elicited 254 replies from a total of 87 users, 52 of whom identify as mathematics teachers" (p. 131). They provide examples of different types of responses (e.g., suggesting students should check their work, describing subsequent activities to encourage student reasoning) and related conversations. Through this study, Larsen and Liljedahl document how different ways of thinking and interacting that are shared among Twitter conversation participants, along with a diversity of ideas thanks to a wide range of participants, have the potential to support learning.

Given these studies and other findings about the utility of teacher networks on Twitter in supporting professional learning (e.g., Carpenter and Krutka 2014), we sought to broadly describe the networks that have emerged through mathematics teachers' collaboration on Twitter. Given that we know access to expertise is critical for mathematics teacher learning, we wanted to describe how it was that individuals participated within these networks and understand just how broadly reaching these networks were in terms of participation. In particular our research questions were:

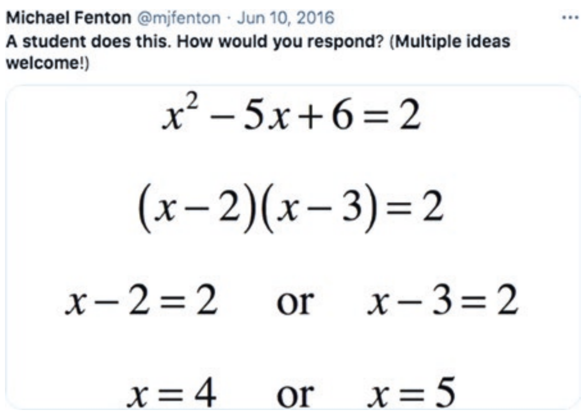

Fig. 13.1 Example of mathematics educator tweet

1. How do mathematics educators use #MTBoS and #iteachmath on Twitter?
2. How many mathematics educators are participating in the #MTBoS and #iteachmath networks?

13.2 Method

We utilized digital methods (Snee et al. 2016) to aggregate, observe, and describe educators' participation by examining their digital traces (Welser et al. 2008) on Twitter. We considered mathematics teachers as representing a subgroup of the teachers who collaborate on Twitter and utilized social network analysis to present a descriptive case study of mathematics teachers' collaboration on Twitter.

13.2.1 Sampling

To understand the scope of the networks as well as the structure of network interactions, we used large-scale data techniques (e.g., pulled tweets through the Twitter application programming interface, API) to pull tweets marked with #MTBOS or #iteachmath for 4 weeks (November 11 to December 8, 2019), as they were tweeted. We chose this 4-week period because it falls toward the end of the first semester of schooling in most schools in the USA, so we assumed that this might allow us to capture an image of typical school year Twitter use for math teachers in these networks. In addition to pulling tweets, we also harvested information about the relations between the contributors of those tweets (i.e., followers). Because the original data set was a live stream, pulling tweets with #MTBoS or #iteachmath for only 4 weeks, it may not have included all replies to the collected tweets. Therefore, we took all of the tweets that we originally collected and returned to Twitter to collect any tweets in reply to those tweets. Then, we used the tweets and replies to create a data set of conversations. We define a conversation to include at least 1 reply tweet. The next step was to categorize each conversation as an #MTBoS conversation, an #iteachmath conversation, or a conversation marked with both hashtags and divide them into separate data sets. This was done by examining whether either hashtag appeared within any tweet within the conversations. In what follows, we describe how we drew on these different data sets of tweets, followers, and conversations to answer our research questions.

13.2.2 Analysis

To answer our first research question pertaining to how mathematics educators use #MTBoS and #iteachmath, we engaged in a descriptive analysis of the tweets and users to examine: (1) the number of original tweets compared with retweets, (2) the number of unique participants, (3) the different hashtags used, and (4) and the tweeting frequency of different users. We also sought to describe the frequency of question posing, by hashtag. We hypothesized that given the robustness of the MTBoS community, there might be more questions posed when users used #MTBoS.

The conversation data sets were initially analyzed to describe the number of conversations and the mean and maximum length of conversation within the data set. Then, we reformatted the data to perform social network analysis: each conversation data set was used to create an edge list, a data set describing conversation-based relations between Twitter users. Each reply to a tweet constituted a directed tie (i.e., an arrow or edge on the sociogram) from the author of the reply to the previous tweet. We used social network analysis software Gephi (Bastian et al. 2009) to analyze the conversation networks by creating a sociogram and describing the number of nodes (i.e., participants), edges (i.e., following relationships), and average in-degree, a measure of the average number of edges coming into a node (Scott 2013), for example, the average number of followers per participant in a follow network. We used this descriptive information to compare the different conversation networks.

To answer the second research question pertaining to the size of the networks themselves, we utilized additional analyses to extend beyond the basic number of participants in our 4-week sample. To do this, we used lists of followers for each of the original participants to estimate the broader sphere of influence of the network. Following Crawford (2011), we call people who followed participants but did not tweet using the hashtag itself during the month of data collection a "listener," rather than a "lurker." We took the approach of defining listener empirically: To determine who might be considered a #MTBoS or #iteachmath listener, we aggregated the follower lists of all of the participants who had used each hashtag and then looked to see how many listeners would be included if we define a listener as someone who follows $n = 1, 2, 3, \ldots$ participants. We used that data to give estimates of the scope of the networks that extend beyond the number of active participants.

Finally, to analyze the #MTBoS follow network from a social network perspective, we created an edge list for the original Twitter participants and the listening participants for the #MTBoS tweets. We created the follow networks (Myers et al. 2014) by considering following as a directed relation from a participant to a follower, constituting a directed tie on the network graph. We created an edge list that described all such directed ties for #MTBoS participants. Again, we used social network analysis software Gephi (Bastian et al. 2009) to analyze the list of participants and their individual followers by describing the number of nodes (i.e., participants), edges (i.e., following relationships), and average in-degree (Scott 2013). We use this descriptive information to compare the follow network with the conversation networks.

13.3 Results

To answer our first research question pertaining to how mathematics educators are using Twitter, we randomly selected a conversation that involved a question from the sample to exemplify the types of conversations taking place on Twitter as well as explain several analytical tools. The initial tweet was by @tracyjoproffitt on November 10, 2019, at 7:08 pm and stated: "I'm compiling a list of specifically Number Talk resources for teachers who are working on building multiplication fact fluency. See below for what I have so far. What else am I missing? #mtbos." In Fig. 13.2, you can see a simplified treeverse representation of the conversation (for the original treeverse representation, see https://treeverse.app/view/AxtdwOdi). Treeverse is a tool for visualizing and navigating Twitter conversations, offered as an Internet browser extension (Butler 2017). This conversation began with a tweet that included the question "what else am I missing?" and was followed by additional tweets from the author with specific resources they had complied. In response to one such tweet, describing some models for mathematics teaching, one teacher @ nancy_estapa replied on November 11, 2019, at 7:39 am, "…I'm going to use these visuals instead of the virtual reckenreck I was gonna use!" This full branch of the conversation is visible in Fig. 13.2. In our analysis, we use sociograms to describe

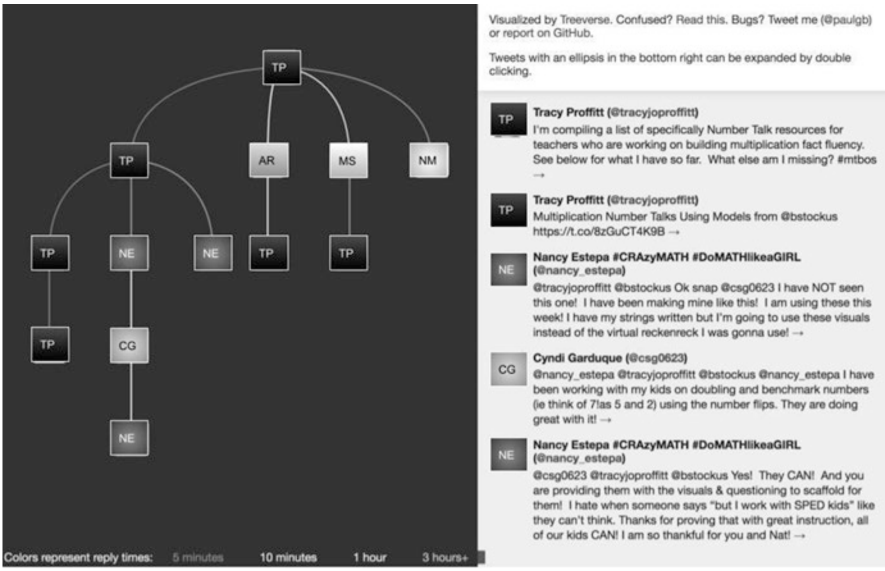

Fig. 13.2 Simplified treeverse representation of a #MTBoS conversation from our sample

relations between participants. The example sociogram in Fig. 13.3 shows the interactions between the different participants in the example conversation. In this sociogram, node 1 represents the author of the original tweet, @tracyjoproffitt, and node 2 represents @nancy_estapa. The sociogram shows that the conversation involved a total of six Twitter users and the author replied to some of the other conversation participants' tweets (denoted by a bidirectional arrow, as with node 5) and did not reply to others (denoted a unidirectional arrow, as with node 6).

For the remainder of this analysis, we zoom out to describe the nature of mathematics educators' participation in the networks more broadly, keeping in mind that the example tweet and related conversation represent the type of data we are reporting on in the aggregate. First, we continue to address the first research question about how mathematics educators are using #MTBoS and #iteachmath on Twitter. We began our broad analysis by examining all of the tweets we extracted from Twitter over the 4 weeks in the fall of 2019. We found that, overall, there were 14,754 tweets included in the #iteachmath or #MTBoS data set, 9384 of which were retweets, 1478 were unable to be classified, and the rest we considered original tweets (which includes replies). Across the full set of tweets, there were 6589 unique participants, 1074 (16.3%) of whom authored original tweets. This means that the remaining 83.7% participated in other ways, the majority of which was retweeting. Of the 1074 participants who authored original tweets, 30.7% of them only used #MTBoS, 49.4% of them only used #iteachmath, and the remainder used both of the hashtags in some way. Based on our comparison of tweeting activity by week, we found that there was considerable variation in tweeting activity (weekly tweets ranging from 2669 to 5031) and who participated by week, with approximately 75% of the participants tweeting in only one of the 4 weeks, 16% tweeting in two of the 4 weeks, 6% tweeting in three of the 4 weeks, and only 3% tweeting in all 4 weeks. Given this distribution, it could be that stopping at 4 weeks of data collection underestimates the true size of the networks.

We also examined the tweets by hashtag to explore variation in the types of original tweets, with specific attention to questioning to broadly understand how

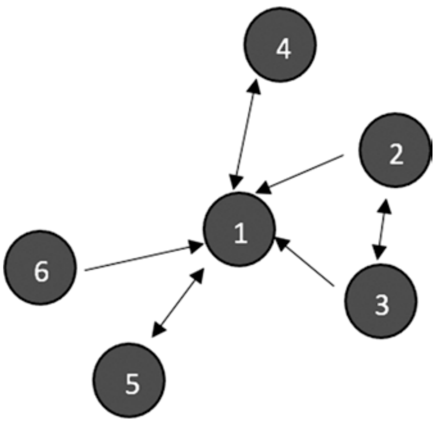

Fig. 13.3 Sociogram representing the example #MTBoS conversation

mathematics educators use the different hashtags to collaborate (and potentially learn from one another) on Twitter. We first considered the entire set of 14,754 tweets and removed the 63.6% of the tweets that were retweets and the 10.0% of the tweets that were unclassified. Then, we examined the original tweets, which are broken down into three categories, #MTBoS tweets, #iteachmath tweets, and #MTBoS and #iteachmath tweets, to understand the extent to which they involved questions. There were more #iteachmath tweets (1429) than #MTBoS tweets (1103) or tweets that included both hashtags (1360). Further, the percentages of original tweets that were questions was higher for #MTBoS (33.3%) than for #MTBoS and #iteachmath or #iteachmath (26.9% and 16.3%, respectively; see Fig. 13.4). This suggests that the #MTBoS network might be a place where more advice-seeking happens than in the #iteachmath network more broadly.

Another way to understand whether the #iteachmath and #MTBoS networks were utilized in different ways is to investigate the number of tweets involved in conversations (i.e., "the length of the conversation") within the networks. Overall, we analyzed 6496 conversations which stemmed from the 14,754 tweets collected for 4 weeks. It was most common for both hashtags to appear within a conversation (3771/6496 = 58%), and conversations tagged with both hashtags tended to be longer than other conversation (see Table 13.1). The average conversation length was two tweets for conversations including #MTBoS or #iteachmath hashtags and nearly three for conversations including both hashtags. By our definition of conversations, it is not possible for conversations to include fewer than two tweets. Therefore, generally, the conversations in this data set were quite short. Overall, conversations ranged from 2 tweets in length to 95 tweets in length.

Sociograms of the conversation networks help to characterize the connections between different members of the networks. The conversation network employing

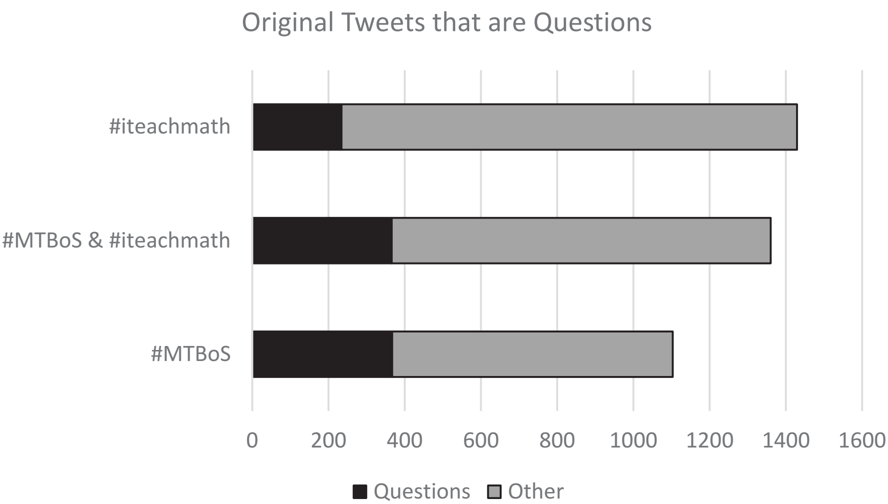

Fig. 13.4 Original tweets categorized as questions or other, by hashtag

Table 13.1 Descriptive statistics of conversations with different hashtags

	Number	Mean length	Max length
#MTBoS	1533	2.26	16
#iteachmath	1192	2.15	17
#MTBoS and #iteachmath	3771	2.98	95

Table 13.2 Conversation and follow network descriptive information

	Conversation network			Follow network
	#MTBoS	#iteachmath	#MTBoS and #iteachmath	#MTBoS
Number of nodes	701	673	1141	26,405
Number of edges	704	580	1583	1,092,481
Average in-degree	1.00	0.86	1.39	41.37

both hashtags was considerably larger than the other two (see Table 13.2 and Fig. 13.5). In addition, the average in-degree of the conversation network was 0.862 for the #iteachmath network, 1.004 for the #MTBoS network, and 1.387 for the #MTBoS and #iteachmath network. An average in-degree of about 1 can be interpreted as meaning that for every participant in the conversation network, they participated in a conversation with, on average, one other person in the month. Based on the clusters of nodes and edges in all three sociograms, it is clear that there were a number of participants who interacted with much larger groups on Twitter, especially when conversations were marked with #MTBoS and #iteachmath (see Fig. 13.5). As an example of the variation in conversation networks, in the enlarged image of a piece of the #MTBoS and #iteachmath network (see Fig. 13.5, box D), at the top, you can see a pair of individuals who conversed, whereas toward the bottom of the square, there are larger subgroups that are connected to other subgroups, likely suggesting that there were key individuals who participated in multiple conversations, bridging different subgroups.

While a network of 6589 unique participants is quite large, especially compared to a school-based network, we were interested in describing the scope of the network by including information about people who might listen to the conversations on Twitter, but might not actively participate by tweeting. We chose to address this question empirically. We found that the set of Twitter users who follow at least 1 #iteachmath participant was 1,978,140 users and the set of Twitter users who follow at least 1 #MTBoS participant was 493,289 users (see Fig. 13.6). This disparity in initial size is likely explained by the fact that #iteachmath was used by more people, and the network is likely less dense because of its relative newness. By requiring that they follow more than 1 participant, in fact, up to 30 participants, we can increase the likelihood that they are closely listening to what might be taking place in the #MTBoS or #iteachmath networks. The magnified version of the graph reveals the end behavior (see Fig. 13.7). As the follower restriction increases, the change in network size decreases. When restricting the minimum number of in-network followers to 20, both listener networks exceed 22,000 Twitter users. In

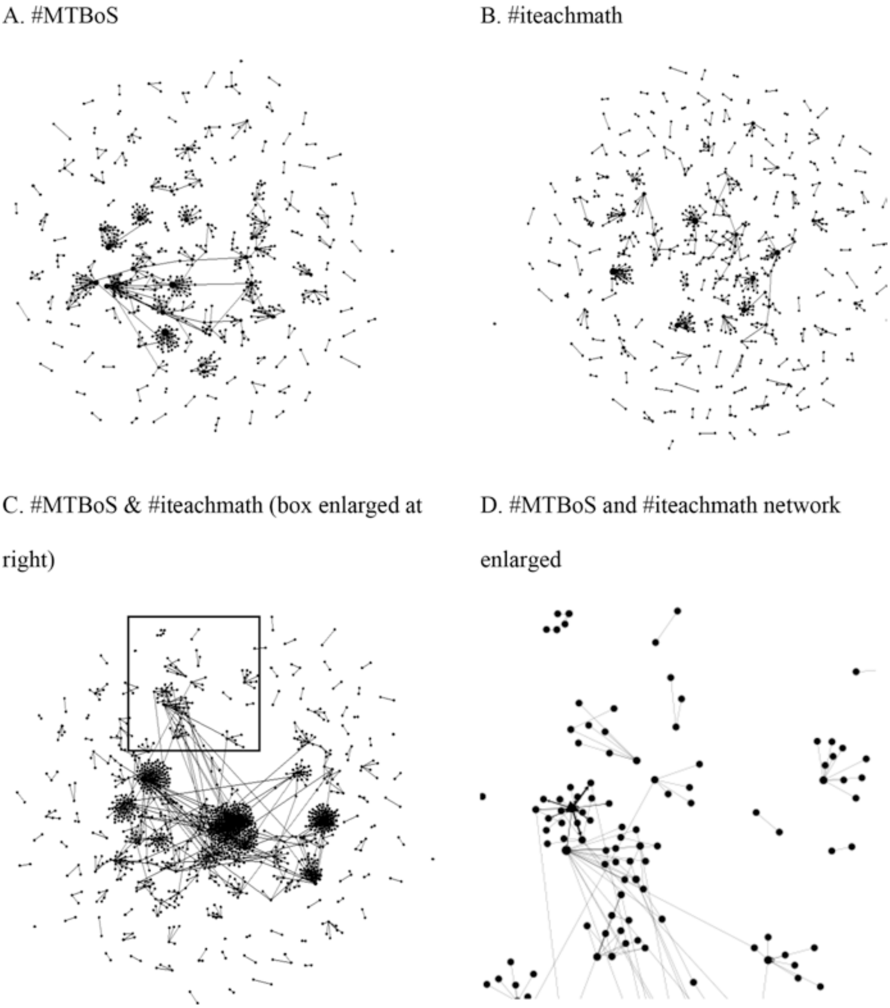

Fig. 13.5 Sociograms of #MTBoS, #iteachmath, and #MTBoS and #iteachmath conversation networks

other words, for those networks, those 22,000 people each follow at least 20 people who tweeted and used #MTBoS or #iteachmath over the 4 weeks of data collection.

Our final analysis sheds light on both research questions as it describes both the following behavior of the #MTBoS members and the scope of the network. We produced a sociogram of the follow network and included descriptive information from the sociogram in Table 13.2. We found that there were 26,405 nodes (i.e., Twitter users) and 1,092,481 edges (follow relationships) and that the average in-degree was 41.37. This in-degree can be interpreted as meaning that for every user in the follow network, they were followed by, on average, 41 other users. In

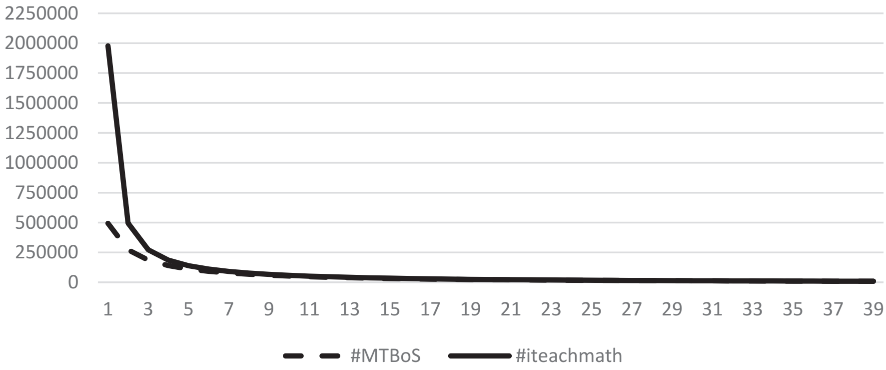

Fig. 13.6 Listener network size by follow requirement

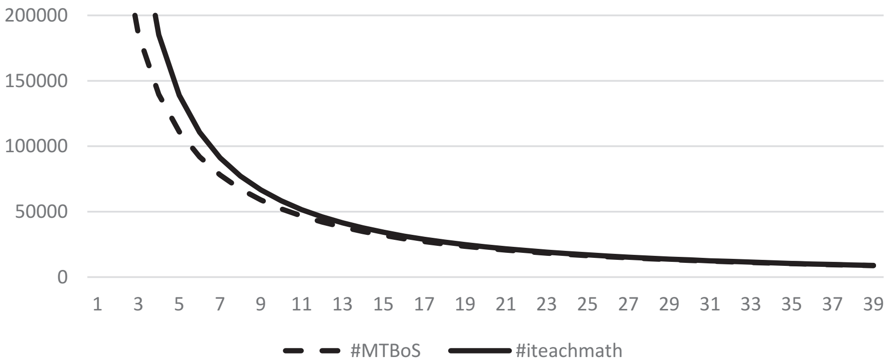

Fig. 13.7 Listener network size by follow requirement, limiting the range to between 0 and 200,000

contrast, the conversation networks were much smaller (see Table 13.2) and much less connected, as was expected.

13.4 Discussion and Conclusion

In this study, we extend prior work demonstrating how teachers learned through their participation on Twitter generally and specifically within the MTBoS community to describe how Twitter has the potential to support mathematics teacher learning at a large scale. We sought to describe how teachers make use of #MTBoS and #iteachmath on Twitter by analyzing 4 weeks of tweets and characterizing how mathematics educators participated. First, we found that the #MTBoS and #iteachmath networks are quite large and overlap. In particular, over 6500 Twitter users actively participated in the #MTBoS or #iteachmath networks, and that number grows to over 20,000 Twitter users if we consider the listeners who follow more than 20 people. One key finding from this analysis is the amplified scope of the network when considering listeners (Crawford 2011). Recall that school-based networks typically max out around 40 people, with mathematics teacher networks tending to be closer to 10–15 people in size in a given school (e.g., Gibbons et al. 2019; Spillane et al. 2010). Therefore, the potential of this network to provide teachers with greater access to expertise is significant, just based on scope.

Second, the conversation networks are much more sparse than the follow networks, but they demonstrate that there is quite a bit of interaction taking place within the hashtags #MTBoS and #iteachmath. Sociograms revealed that the #MTBoS and #MTBoS and #iteachmath conversation networks were more connected than the #iteachmath conversation network. In other words, people interacted with greater numbers of other people when conversations included #MTBoS or both hashtags. It could be that the relative newness of the #iteachmath hashtag makes for less uptake of tweets, that the greater size of the network decreases the likelihood of people responding to tweets, or that the sense of community within the MTBoS contributes to the quantity and quality of conversations (Larsen and Parrish 2019; Risser and Bottoms 2018).

Further, we found that conversations in our data set varied in length but were generally quite short, with an average of two or three tweets per conversation, depending on the network. When both the #MTBoS and #iteachmath hashtags were used, those conversations were slightly longer, on average. It could be that this is because the combination of the hashtags allowed for the conversation to reach more people, or tap into the community that is the #MTBoS network. The fact that the #MTBoS and #iteachmath network seems to connect more people to each other and those conversations are longer constitute two distinct findings. For example, it could be that two people could interact back and forth on Twitter to produce a very long conversation, but that would be a conversation that connected only two people in the network. It is likely that when more people participate in a conversation, the potential for a long conversation is greater, but it is not a given. Further research should investigate the relation between the number of people who participate in a Twitter conversation and the length of those conversations.

Finally, by using basic text analysis, we investigated the prevalence of retweets and questions. First, 63.6% of the tweets we scraped were retweets. This indicates that retweets were the predominant way that people participated. This is consistent with Staudt Willet and Reimer's (2018) study of #MTBoS over one month and Staudt Willet's (2019) study of #edchat over the course of 8 months. In both cases, they found that approximately 58% of the tweets they analyzed were retweets and approximately 40% original posts (including replies). In our #MTBoS and #iteachmath sample, questions constituted a significant portion of the original posts. We found that questions were more prevalent in the #MTBoS network (33.3%) than in the #iteachmath network (16.3%). This might be explained by several differences between the #MTBoS and #iteachmath communities. First, recall that #MTBoS has been consistently in use much longer than #iteachmath. Second, the #MTBoS community self-identifies as an advice network. It is likely that not all tweets that contain questions are actually seeking advice, and conversely, not all tweets seeking advice actually contain questions. However, a common way to ask for advice in the context of Twitter is to ask a question as was true for the question Larsen and Liljedahl (2017) studied which states “A student does this. How would you respond? (Multiple ideas welcome)” (see Fig. 13.1).

Overall, this study sought to add to the literature on mathematics teachers’ use of Twitter by scraping and analyzing data from Twitter to describe the basic structure of tweets and conversations in two overlapping mathematics teacher networks and the scope of those networks. The #MTBoS and #iteachmath networks are broad reaching and have the potential to serve as advice networks for mathematics teachers, among other roles, to support mathematics teacher learning. Because mathematics teachers can bring their own problems of practice to the broad network, they can design their own professional learning experiences (Risser et al. 2019).

We offer several implications for practice and future research. First, teacher networks on Twitter have the potential to extend teachers’ advice networks. This is particularly important in countries like the USA where teachers are not equitably distributed in schools (Adamson and Darling-Hammond 2012), and we know that access to expertise is critical to supporting teachers’ development (Bruner 1996). Studies of teacher collaboration suggest that allocated time for collaboration is a necessary but not sufficient condition for learning (e.g., Horn and Little 2010; Peterson et al. 1996). Similarly, access to a broader network does not imply access to expertise or learning. Studies of teachers’ collaboration on Twitter suggest that teachers find value in the collaboration (e.g., Carpenter and Krutka 2014; Larsen and Parrish 2019), but future research should specifically attend to the access to quality expertise and learning opportunities within mathematics teacher collaboration on Twitter. Also, future research should continue to investigate how to introduce novice or more experienced teachers to Twitter communities in ways that make them want to continue to participate. This is likely very nuanced given several studies demonstrating that teachers find Twitter useful when assigned to participate as a course activity, but many of them do not continue to actively participate once the course has completed (e.g., Carpenter 2015).

Second, the data for this study was collected prior to the COVID-19 pandemic, but future research should investigate how mathematics teacher collaboration on Twitter has changed as a result of the COVID-19 pandemic. We hypothesize that this type of support network becomes even more useful and accessible when teachers are conducting much of their work online already and they have fewer opportunities for informal interactions with their school-based colleagues.

Third, we found that mathematics teachers engage on Twitter in a wide range of ways. We argue that a significant number of mathematics teachers engage by listening. In our case, the number of listeners extends the network by approximately 250%. Studies across social platforms suggest that there’s variation in how people listen with some participants being considered active listeners and others being considered more passive listeners (N. Sun et al. 2014a). Future research should investigate how it is that mathematics teachers listen on Twitter. It is likely that newer Twitter users may start by listening or retweeting and then move to more active posting and engaging in conversations. Further research should study teachers’ engagement with Twitter over time to see how their participation changes as they become more comfortable with the platform and the community as well as considering other personal or contextual factors that influence their participation (cf. Risser and Bottoms 2014).

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Chapter 14
A Distributed Leadership Model for Informal, Online Faculty Professional Development

Erica R. Miller and Emily Braley

Distributed leadership and communities of practice are two theoretical concepts that have been applied to analyze teachers and their professional networks. However, few studies have examined how these two concepts complement each other and can be applied in the online context. Our purpose for writing this chapter is to examine these two theories in detail while simultaneously illustrating how these theories can be used to structure informal, online faculty professional development. In particular, we will illustrate how we knit these two theories together to create an online book study group for college mathematics faculty members.

We will first discuss distributed leadership and communities of practice independently and then highlight the ways in which they are connected. After describing the literature related to these two concepts, we go into more detail about the structure and format of our distributed leadership model for online book study groups. In particular, we focus on the three macro-activity tasks of (a) launching the book study group, (b) supporting the participants, and (c) supporting the facilitators. Finally, we wrap up the chapter by discussing implications for informal, online faculty professional development. Within the theoretical foundation section, we use our book study group as an illustrative example to demonstrate how the theories of distributed leadership and communities of practice can be applied to informal, online professional development. Before focusing on the theories, we first provide a brief background of the book study group to help orient the reader.

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14.1 Background for Our Informal, Online Faculty Professional Development

In Spring 2020, the academic world drastically changed due to the impact of the COVID-19 pandemic. Instructors were forced to rapidly transition to teaching online (some for the first time), with no clear idea of when they would return to teaching in a face-to-face setting. Around the same time, individuals in the United States took a strong stance for anti-racism by protesting police brutality and supporting the #BlackLivesMatter movement. In response, many faculty turned their attention to issues of inclusion, access, equity, and institutionalized racism within academia. Recognizing that faculty were searching for support in relation to these two national crises, we decided to run an online book study group in the summer of 2020 to provide mathematics faculty members with the opportunity to practice facilitating conversations about teaching mathematics equitably and in an online setting. In particular, we chose to read the *MAA Instructional Practices Guide* (*IP Guide*, MAA 2018) and the corresponding *Book Study Guide* (Braley et al. 2020) because they focus on the overarching themes of equity and technology.

From the beginning we knew that we wanted to create a community of practice and use a distributed leadership model in our online book study group, so we specifically recruited mathematics faculty members who were willing to actually help facilitate the book study group sessions with our support. We were not able to find a single weekly meeting time that worked for everyone, so we decided to split into two subgroups. Each subgroup met weekly, and participants had the choice to attend either of the weekly meetings (as they focused on the same topics), but most members chose to attend the same meeting time each week. Having two separate weekly subgroups covering the same material provided facilitators the opportunity to co-plan their sessions.

Before each session, participants were expected to reflect on prompts from the *Book Study Guide* (Braley et al. 2020) and read sections of the *IP Guide* (MAA 2018). As the community coordinators, we supported the facilitators while they co-planned their sessions. During each session, the facilitators selected discussion questions from the *Book Study Guide* (or wrote their own) and asked participants to engage in small and whole group discussions. We encouraged individuals to create a written record of their discussions in order to curate a collection of tools and artifacts that we could reference and share after the book study group ended. Finally, at the end of each session, we solicited feedback from the participants and shared this feedback with future facilitators so that they could adapt and respond to the needs of the community. Now that we have provided a brief overview of the online book study group, we will go into more detail about the theoretical foundations, structure, and format in the following sections.

14.2 Theoretical Foundations

Distributed leadership and communities of practice are two theoretical frameworks that can easily complement each other and provide experienced teachers with the opportunity to both build their leadership skills and expand their professional networks. In particular, we found that these two frameworks work together within the context of informal, online faculty professional development because they foster active participation, allow participants to step into leadership roles, and build community. In this section, we provide an overview of distributed leadership and communities of practice separately and then discuss how they are connected. Along the way we will interweave examples from our own experience implementing these two theories in our online book study group.

14.2.1 Distributed Leadership in Action

Characteristics of Distributed Leadership and the Book Study Group Launch

In our informal model for faculty professional development, distributed leadership provides the context within which a community of practice develops (Jones and Harvey 2017). While there is no single definition or model of distributed leadership that is accepted within the literature, researchers have identified common characteristics that help define this term. In their systematic literature review, Woods et al. (2004) identified three distinctive characteristics of distributed leadership; distributed leadership is an *emergent property* of a group where members *lead according to expertise* and there is an *openness of boundaries* within the leadership structure. We will expand on these characteristics below, discuss leadership tasks, and discuss the related artifacts born from leadership tasks.

An emergent property of a group. First, distributed leadership is an *emergent property* of a group or network of individuals in which "members pool their initiative and expertise" (Woods et al. 2004, p. 441). This sits in contrast to traditional models of leadership that focused on the individual and individual leaders (Gronn 2000). Distributed leadership depends on "multiple leaders working together, each bringing somewhat different resources, skills, knowledge, perspectives" (Spillane et al. 2004, p. 18). While the term "emergent" can be used to describe something unexpected or urgent, the key idea for this characteristic (according to Woods et al. 2004) is that distributive leadership emerges or comes from groups or networks of people, as opposed to individuals.

When we first conceived of hosting the online book study group in the summer of 2020, we specified the target audience and goals for the community we hoped would form. The audience was faculty who might be leading professional development sessions or conversations on teaching (in person or online) in the fall of 2020.

We advertised that the book study group would focus on developing online facilitation skills and building community around teaching equitably within the group. In other words, we intentionally recruited individuals who might be working in leadership roles and formed our community to specifically focus on building leadership skills through the facilitation of sessions.

The initial goal for the book study group was to create an opportunity for participants to practice facilitating conversations about online learning experiences, while attending to the cross-cutting themes of equity and technology. Participants were invited to join the book study group and asked on an initial interest survey to volunteer to facilitate a session. Instead of using the traditional model where a single individual leads the book study group, we wanted each session to be led by different faculty. Over the 11 weeks, participants pooled their initiative and expertise as rotating facilitators brought their unique perspectives and experiences.

Leadership according to expertise. Another characteristic of distributed leadership identified by Woods et al. (2004) is *leadership according to expertise*. Distributed leadership is “a leadership approach in which individuals trust and respect each other’s contributions and collaborate together to achieve identified goals” (Jones and Harvey 2017, p. 316), which happens most effectively when people at all levels accept leadership in their particular areas of expertise. This characteristic highlights that the execution of leadership tasks can be distributed among multiple leaders and followers with the outcome being greater than the sum of the individual contributions (Woods et al. 2004). This is similar to what Wenger-Trayner and Wenger-Trayner (2015) called synergy—when the interaction or cooperation of two or more members produce a combined effect greater than the sum of their separate effects.

In our online book study group, there was enough faculty interested in participating that a schedule was developed with 2 weekly subgroups that would cover the same reading assignment each week, and 17 out of 21 members volunteered to facilitate at least 1 session. Faculty members primarily volunteered to lead sessions that they had experience with, like engaging students, designing practices, and selecting appropriate mathematical tasks. The two faculty participants who were assigned to lead the two sessions during the same week collaboratively planned with each other and us, the community coordinators. Facilitators often thanked their collaborators (even if they were not in attendance at their subgroup meeting), acknowledged that the session plan would not have been as rich without the co-planning work, and emphasized the importance of collaboration.

Openness of boundaries. Woods et al. (2004) identified *openness of boundaries* as the third characteristic of distributed leadership. This is the idea that distributed leadership widens the net of leaders and does not suggest how wide the net of leadership should be set. Openness of boundaries relates to the characteristic of *leadership according to expertise*; when members accept leadership roles within their area of experience and expertise, they contribute to widening the net of leadership and the collective body of knowledge of the community, making way for a community of practice to form. In our book study group, the rotation of facilitators allowed for members to participate at different levels throughout the 11 weeks that the group met. Using a distributed leadership model, faculty participants stepped into leadership roles in co-planning and facilitating sessions and therefore widened the net of leadership.

Leadership Activities

Distributed leadership provides a lens for examining how multiple leaders and followers work together, and separately, to execute leadership functions and tasks (Spillane et al. 2004). Viewing leadership through this lens helps to highlight the "interdependencies among constituting elements - leaders, followers, and the situation - of leadership activity" (p. 18). Spillane et al. argued that leadership activity is defined by the interactions between leaders, followers, and their situation in the execution of particular tasks. Building on Spillane et al., we define leadership tasks as consisting of macro-activity tasks that are intended to create change and the micro-activity tasks that are necessary to make progress toward enacting the change.

For example, one macro-activity task that we engaged in, as the book study group coordinators, was the initial launch and planning. Launching the book study group required a series of micro-activity tasks, such as inviting participants, recruiting facilitators, and creating and organizing tools and artifacts (e.g., a schedule, access links, session folders, virtual whiteboards, polls) and resources within a shared online community space. Initially, as the community coordinators, we needed to envision and design a virtual space for the book study group in order to create a productive space for faculty members to participate and step into the leadership role of facilitating.

While the micro-activity task of creating the shared online community space fell to us, all faculty participants made contributions to the corresponding folders over the course of the summer, and it became a space where facilitators collaborated on the development of facilitation tools and artifacts. Facilitators had the option to use any platform they wanted to host the weekly sessions, but most facilitators asked us to create Zoom links on their behalf, and all of the sessions were hosted via Zoom. Chat transcripts became part of the regular set of artifacts produced in the sessions. We also encouraged members to share resources and record discussions in the shared community folder.

The Role of Artifacts

Related to leadership tasks are the artifacts that help direct leadership activity or are produced in the execution of leadership tasks. We use Wenger's (2000) definition of artifacts as the set of documents, tools, stories, symbols, and websites that are produced and maintained by the community. These artifacts aren't limited to at-hand material items but include abstract artifacts like the schedule of meetings and calendar. In the distributed leadership literature, artifacts are identified as helping to establish the "hidden rhythms" that inform the practice of a community (Spillane et al. 2004). For example, a reading group schedule and faculty member schedules "shape the space and temporal resources available" (p. 25) to address macro-activity tasks. Meanwhile, community agreements and available technologies are key artifacts that direct and define leadership activity. Within a community of practice, artifacts benefit the group when members negotiate how time should be used, or might be better spent, and give members agency. Within a distributed leadership model, members can feel confident that they all share in the interests and planning of the community (Woods et al. 2004). Now that we have summarized the literature and characteristics of distributed leadership, we will discuss the second theoretical foundation—communities of practice—that our book study group model was built upon.

14.2.2 Communities of Practice

The idea of communities of practice stems from situated and social learning theory (Lave 1991; Lave and Wenger 1991; Wenger 1998), which assert that learning is a situated activity requiring sociocultural participation on the part of the learner. Wenger et al. (2002) defined communities of practice as "groups of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (p. 4). Our book study group fits this definition in that it is a group of mathematics faculty members who share a concern about and passion for teaching equitably online. Initially, we interacted together weekly in an online environment in order to deepen our knowledge and expertise in this area and have continued to share tools and resources beyond the end of the scheduled book study group. Wenger (1998) states that as part of learning in a community of practice, "we define…enterprises and engage in their pursuit together, we interact with each other and with the world and we tune our relations with each other and with the world, accordingly" (p. 45). Learning, then, encompasses how a community adapts its work to accommodate new circumstances, situations, or events. This process of adapting and accommodating was illustrated in our book study group as we tuned our focus to meet the needs of our community members and respond to world events such as the COVID-19 pandemic, #BLM, and #ShutDownSTEM.

Situated and Social Learning Theory

The concept of communities of practices was originally founded in the idea of situated learning theory (Lave 1991; Lave and Wenger 1991). Lave and Wenger (1991) viewed learning that occurs naturally within authentic contexts as more meaningful and more likely to transfer. In particular, this theory of learning came out of their study of apprenticeships, which situates learning within the same real-world context where it is expected to be applied. In our community of practice, all of the members were mathematics faculty facing the reality of teaching online during the COVID-19 pandemic, so the work that we did together as a community was situated within the real-life context of teaching equitably online and helping others do the same.

Building on the idea that learning should be situated within authentic contexts, Wenger (1998) extended the concept of communities of practice and focused more on the social aspects of learning. In particular, Wenger's social learning theory focuses on *community* (learning as belonging), *practice* (learning as doing), *identity* (learning as becoming), and *meaning* (learning as experience). Within our book study group, we focused on building a *meaningful* online learning *community* that provided a space for community members to engage in the *practice* of creating tools and artifacts for facilitating conversations about teaching mathematics equitably online while also attending to our ever-evolving *identities*.

The Domain, Community, and Practice

While Lave and Wenger first introduced communities of practice in 1991, Wenger further developed the concept in his 1998 book and has continued to refine it through his ongoing work. According to Wenger et al. (2002), communities of practice are defined by three fundamental elements: the *domain*, the *community*, and the *practice*. The *domain* is the issue, concern, set of problems, or passion that all members of the community share and helps distinguish the group from others. The *community* element is essential as it distinguishes communities of practice from other groups of people who may have a common domain or practice, but do not actively engage together as a community. Notably, "in pursuing their interest in their domain, members engage in joint activities and discussions, help each other, and share information" (Wenger-Trayner and Wenger-Trayner 2015, p. 2). Finally, "members of a community of practice are practitioners. They develop a shared repertoire of resources: experiences, stories, tools, ways of addressing recurring problems—in short a shared *practice* [emphasis added]" (p. 2).

We conceptualized the three fundamental elements of our community of practice in the following way. The *domain* that all members of the community shared and helped distinguish the group from others was our shared concern and passion for teaching mathematics equitably online. The *community* consisted of ourselves, the community coordinators, the rotating weekly pair of facilitators, and the remaining community members who actively participated in the weekly book study group discussions. The *practice* that we developed was our shared repertoire of tools and artifacts for facilitating conversations about teaching mathematics equitably online, which included our experiences, artifacts like notes, stories and tools, and ways of addressing problems (Wenger-Trayner and Wenger-Trayner 2015).

Levels of Participation

Members can participate in communities of practice at varying levels (Wenger 1998; Wenger et al. 2002). These levels are conceptualized as being nested within each other, although members may participate in different levels at different times. The ebb and flow of community members participating at different levels is similar to the ideas of *leadership according to expertise* and *openness of boundaries* in distributed leadership. At the center of the community are the *core members*, who help identify and tackle issues. From within the core group, the *coordinator*(*s*) organize and connect the community. Next, the *active members* are involved in the community, but less so than the core group. Finally, *peripheral members* watch from the sidelines. However, it is important to note that peripheral members are not seen as external to the community. Rather, communities of practice are founded upon the idea of *legitimate peripheral participation* (Lave and Wenger 1991), which refers to the idea that newcomers to a community can learn effectively from longer-standing members by participating peripherally in the practice.

We conceptualized levels of participation in our online book study group in the following way. We took on the role of *coordinators*. In this role, we engaged in the micro-activity tasks of recruiting facilitators and participants, deciding upon and managing the structure of the book study group, working with facilitators as they co-planned the sessions, and actively participating in the weekly sessions. We conceptualize the *core group* as including the coordinators and the two rotating weekly facilitators and the *active group* and *peripheral group* to be the other book study group members (see Fig. 14.1). Defining the *core group* as including the rotating weekly facilitators aligns with Wenger et al.'s (2002) claim that "this core group

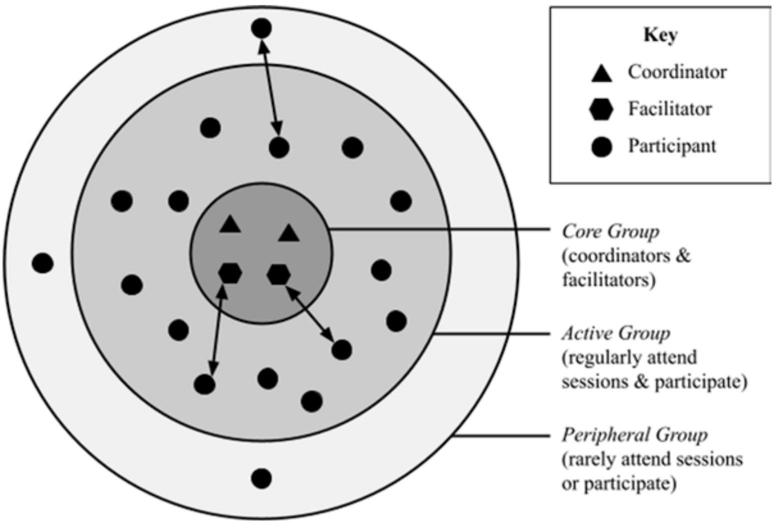

Fig. 14.1 Flexible membership structure for our book study group

takes on much of the community's leadership…becoming auxiliaries to the community coordinator" (p. 27). When taking on a leadership role and facilitating a session, core group members were responsible for identifying the topics to address and leveraging the group meeting to collaboratively work toward our identified goals (Jones and Harvey 2017). Because some members attended regularly at the beginning and then stopped attended or attended rarely, disentangling members of the *active* and *peripheral groups* was more difficult. However, we conceptualized the *active group* as the members that attended sessions regularly and the *peripheral group* as the members that rarely attended sessions. The arrows in Fig. 14.1 are intended to capture the movement between these groups that was a result of the flexible participation structure. Now that we have summarized the literature and characteristics of distributed leadership and communities of practice separately, we will discuss how they are connected.

14.2.3 Connecting Distributed Leadership and Communities of Practice

In this section, we will dive deeper into the ways that distributed leadership and communities of practice complement each other. In exploring the nexus between distributed leadership (DL) and communities of practice (CoP), Jones and Harvey (2017) found that:

> [S]ynergies between DL and CoP can be identified whereby, on the one hand, DL provides the supportive context for, and action by which, CoP can be created and sustained. On the other hand, CoP contribute one of the means by which a DL approach is enabled. (p. 316)

As the community coordinators, we set up the initial distributed leadership structure, which provided the context for the rotating facilitators and participants to engage in the community of practice. Because of the distributed leadership structure and commitment to working together as a community, participants supported and encouraged one another as they stepped into the role of facilitator. On the other hand, the formation of the community of practice provided participants with the opportunity to engage in distributed leadership roles.

Jones and Harvey (2017) also identified synergy between the behavior expectations of distributed leadership (which focuses on relational leadership) and community of practice (which focuses on relational interdependence). This is illustrated in our book study group as participants were asked to both lead and learn together. In particular, participants worked together in the practice of the community to create artifacts (like the facilitation plan and slides) and reflected on the community collaboration (by revisiting and adjusting the community agreements at the start of each session) and leadership (by providing weekly feedback at the end of each session). We see artifacts as another common element to both distributed leadership and communities of practice because artifacts help direct leadership activities and are essential to understanding the practice in the community.

We have also found that these two conceptual frameworks complement each other in other ways. As mentioned above, one distinctive characteristic of distributed leadership is that it is an *emergent property* of a group or network of people where collective work exceeds the work of individuals. In this context, it is difficult to discern the work of leaders from the work of contributing members, as opposed to traditional leadership structures where it is clear that leadership work stems from specific individuals. This emergent property of distributed leadership mirrors the varying levels of participation we see in communities of practice. In our book study group, we observed participants moving in and out of the core group in the weeks they were serving as facilitators and in and out of the active and peripheral groups as the summer went on. The other two characteristics of *leadership according to expertise* and *openness of boundaries* is also reflected in the idea that members of a community of practice may participate in different levels at different times. In the next section, we explain how we used the concept of distributed leadership to create opportunities for members within a community of practice to become active members and engage in leadership tasks normally taken on by the core group.

14.3 Illustrating Theory in Informal, Online Faculty Professional Development

In this section, we will talk about the macro- and micro-activity tasks that we engaged in as part of our book study group. In particular, we identify three macro-activity tasks—(a) launching the book study group, (b) supporting the participants, and (c) supporting the facilitators—and describe the related micro-activity tasks and work required.

14.3.1 Launching the Book Study Group

In the above distributed leadership and communities of practice sections, we used our online book study group as an illustrative example of how to build informal, online faculty professional development using these two theoretical foundations. In particular, we talked about our process for recruiting facilitators and participants and establishing a virtual space. Our distributed leadership model resulted in a flexible membership structure where participants had the opportunity to engage in different levels of the community of practice. Also, as the community coordinators, we built the virtual space where members collaborated and shared artifacts and resources.

One aspect of launching the book study group that is not addressed above is how the leaders and community established a rhythm. In order to establish the "hidden rhythm" (Spillane et al. 2004) of our book study group, we sent reminder emails each week containing the session reading assignment, the facilitators' contact information, the necessary access links for Zoom, and the link to the community folder. If there were particular themes or questions that the facilitators wanted participants to think about ahead of the session, those were also included in the reminder email. Feedback was collected at the end of every session to help inform the planning of future sessions.

While this quickly became the natural rhythm of the community, there were also moments where flexibility was required and the community was polled to decide how to proceed. One example of this was when #ShutDownSTEM called on academics to stop "business as usual" on June 10, 2020, to reflect on what actionable steps could be taken toward addressing institutional and systemic racism. Dr. Douglas Ensley was scheduled to facilitate a session on choosing appropriate mathematical tasks on June 10, but instead volunteered to host a conversation around the themes of #ShutDownSTEM. Based on further feedback, we added a week to the book study schedule to incorporate a second conversation related to #ShutDownSTEM. A common concern that emerged in the feedback from these sessions was that facilitators should prioritize questions of equity each week and not leave them to the end "if there is time."

14.3.2 Supporting the Participants

While the facilitators were asked to take responsibility for encouraging discussion by and among all participants during each session, we still took on an active role in supporting the participants. In particular, we regularly engaged in three micro-tasks, (a) creating community and honoring individual identities, (b) establishing and refining community guidelines and agreements, and (c) allowing participants to choose their level of participation, all while adapting to the needs of the community. We will talk about each of these micro-tasks in more detail in the following subsections.

Creating Community and Recognizing Individual Identities

Our book study group consisted of individuals who may have not met before but shared interest in facilitating conversations about teaching mathematics equitably online. We worried that establishing a sense of community in an online setting may be difficult, so we dedicated most of the first session to getting to know each other. We also wanted to recognize the diversity of individuals in our groups and their individual identities. To support these goals, we included questions in our interest survey about participants' preferred name and gender pronouns (Medina 2011), personal context (type of institution, job responsibilities, etc.), and individual goals in joining the group. By asking these questions, we were able to get a sense of who our participants were and tailor our first session to meet their individual needs. Since we wanted the preferred names and gender pronouns to be communicated to everyone in the group, we started the first session by asking participants to make sure that their name on Zoom reflected the name they wished to be called and included their preferred gender pronouns. We also asked participants to take time to introduce themselves to each other during breakout groups. As the coordinators, we also made an effort to join every session early to engage in informal conversations with the participants before the session started. This provided us with the opportunity to continue to build relationships with members of the community.

Establishing and Refining Community Guidelines and Agreements

We decided to facilitate the first session to model what a session might look like, establish virtual conferencing guidelines, and model the co-creation of community agreements with participants. In planning our first session, we came up with two virtual conferencing guidelines concerning wearing headphones and agreements for whole group discussion (see Fig. 14.2). However, during the first session, participants suggested adding the final three guidelines to the list. We continued to revisit these guidelines at the beginning of each session and made refinements as needed, which we discuss more below.

In addition to virtual conferencing guidelines, we also took time during our first session to establish general conversation agreements (see Fig. 14.3). Because some of the discussion questions in the *Book Study Guide* (Braley et al. 2020) asked participants to share personal opinions and speak from personal experience, we felt that establishing and maintaining agreements for respectful and equitable conversation was important. Again, we came up with an initial list of agreements (taken from the *Book Study Guide*), and one participant suggested that we add the "ouch/oops rule." Establishing conversation agreements when teaching undergraduate classes is also recommended, so we wanted to model a practice that applied to the everyday work of our participants. In particular, the *IP Guide* (MAA 2018) illustrates how establishing agreements for active engagement and taking steps to increase a student's

Original ~~Norms~~ Guidelines

- Wear headphones to prevent feedback
- During whole group discussions (5+ people)
  - Keep your microphone on mute
  - ~~Raise your hand if you have something you want to say~~ Turn on your microphone when you want to speak up
  - Freely contribute to the chat

Added ~~Norms~~ Guidelines

- Minimize distractions (visual or audio) as much as possible
- If you need to step away briefly, you're welcome to stop your video until you return.
- Expectation: Everyone communicates verbally!

Fig. 14.2 Virtual conferencing guidelines (strikethrough indicates refinements that were made later)

Original ~~Norms~~ Agreements

- Be an active listener.
- Listen for understanding.
- ~~Assume positive intentions.~~ Focus on impact over intent.
- Share talk time.
- Be present and aware of your virtual presence.
- Recognize that everyone in the group has expertise.
- Give wait/think time.

Added ~~Norms~~ Agreements

- Ouch/Oops (http://teachpaperless.blogspot.com/2011/06/oops-ouch.html)

Fig. 14.3 General conversation agreements (strikethrough indicates refinements that were made later)

sense of belonging to the mathematics community can positively impact the quality of student engagement in the mathematics classroom (p. 11).

Initially, we referred to both of these lists (in Figs. 14.2 and 14.3) as "norms." However, one of the facilitators, Dr. Moira McDermott, brought to the attention of the group during her session that these "norms" may privilege some voices over others. In contrast, if a group co-constructs agreements, instead of abiding by handed-down norms, the participants have more agency to speak up if an agreement is problematic or no longer relevant for the group. Dr. McDermott was inspired by the following tweet, which she used to start the discussion in the session she facilitated.

> Workshop agreements I no longer use. "Assume best intentions." "Engage with civility." Also, I no longer call them "norms" because we are actually upholding white norms. So. Nah. What are the ones you no longer use? (Talusan 2020)

In preparation for broaching this subject during her session, Dr. McDermott read through the full Twitter thread as well as related papers (Chandler and Wiborg forthcoming). In particular, Dr. McDermott brought to our attention that we often use the term "norms" when we actually mean "aspirations" (Chandler 2020). As a result, we decided to move away from labeling our lists as "norms" and also decided to replace "assume positive intent" with "focus on impact over intent." In doing this, we hoped to recognize that even within a community of practice, statements made can have a harmful impact on others, though that may not have been the speaker's intent.

In another session, we also chose to modify the virtual conferencing guideline of raising your hand and waiting to be called on during whole group discussions. We noticed that some individuals followed this rule religiously, while others chose to speak up whenever they wanted to. Sometimes this resulted in one person raising their hand and then getting passed over by another person who chose to just speak up, so we decided to stop using the "raise hand" feature and allow participants to talk as desired. Our subgroups were small enough that this change seemed to work well and did not cause any major problems. We also discussed how this guideline could be implemented in online mathematics classes for undergraduates and which option might be more appropriate for that setting.

14.3.3 Allowing Participants to Choose Their Level of Participation

Because most of our participants volunteered to facilitate a session, they moved in and out of the *core*, *active*, and *peripheral groups*, depending on the week (see Fig. 14.1). While we encouraged participants to attend each session, we wanted them to participate in ways that best meet their needs, even if that meant limiting their participation due to other responsibilities. There were participants who started out in the *active group* and attended sessions regularly in the early weeks of the book study group, but for various reasons (such as needing to prepare for teaching online in the fall of 2020) rarely attended or stopped attending altogether. We see these members as moving from the *active group* to the *peripheral group*, but want to emphasize that their contributions in the early weeks helped shape the community and establish the rhythm. For example, one participant who facilitated in week 2 was integral in helping us refine our process for recording and creating artifacts even though they stopped attending half way through the summer.

Wenger et al. (2002) used the term “peripheral” to describe members of a community of practice who rarely participate. While this term may have the connotation of being an outlier or less important, Wenger et al. emphasize that peripheral members are essential, often not as passive as they may seem, and “gain their own insights from the discussions and put them to good use” (p. 27). Therefore, we don’t want to downplay the role of our book study group members who were in the *peripheral group*. Lave and Wenger (1991) emphasized the *legitimacy* of peripheral participation and noted that through peripheral participation newcomers can become part of the *active group*. In our book study group, there were two members in particular who, in the first 2 weeks, wanted to simply listen and prefaced their contributions with statements like “I am not an expert…” or “I haven’t really thought deeply about this but…”. This type of peripheral membership aligns with Wenger et al.’s (2002) claim that “some [members] remain peripheral because they feel that their observations are not appropriate for the whole or carry no authority” (p. 27). However, over time we noticed that these participants became more confident contributors and active members.

Other examples of peripheral members were participants who attended sporadically or who changed which of the weekly sessions they attended on a regular basis. This type of peripheral membership aligns with Wenger et al.’s (2002) claim that some members remain peripheral because they “do not have the time to contribute more actively” (p. 27). Participants who frequently switched which weekly subgroup they attended may have appeared as peripheral members to the active members who attended strictly the Wednesday or Thursday subgroup. However, the occasional presence of a “new” active member created a different dynamic that participants seemed to like, since they helped draw connections between the two groups and brought in new ideas. In the feedback form, one participant commented that, “I like the random breakouts…[you] never know what you’re gonna get.”

14.3.4 Supporting the Facilitators

Co-Development of Weekly Facilitation Plans

In order to support the rotating pairs of facilitators, we provided them with suggestions on how to collaboratively prepare for and plan their session (see Fig. 14.4). Collaboration between facilitators helped participants step into the leadership role of facilitator with more confidence and helped participants move from roles as *active* or *peripheral members* to *core members* in the community. Facilitators co-created leadership artifacts like facilitation plans and accompanying slides and occasionally met with each other or with us to troubleshoot technical issues like creating breakout rooms, screen-sharing, and distributing necessary links for participants. Artifacts from previous sessions, like slide decks, collaborative white boards, and spreadsheets designed for collective note taking, were offered as models for new facilitators to consider implementing.

Thank you so much for being willing to facilitate a session for the IPG Book Study Group!

Your main role as facilitator is to focus on using the **After You Read** prompts to discuss the reading with the group and give space for folks to talk about possible **Action Steps** from the Book Study Guide. Here are some prompts/ideas for you to consider while planning.

- Are there questions from the **Before You Read** section that you want to discuss during the session?
- What questions from the **After You Read** section are relevant for our audience? Are there any that you want to omit or modify? Are there any that you want to use for small group discussions?
- Can you integrate time for people to work on their **Mini-Action Plan** or will you need to delegate this to "homework"?
- How will you maintain audience **attention** and **participation**?
- What **talk moves** can you use to encourage discussion by and among all participants?
- Does the **pace** of the session maintain interest but also allow for deep discussions?
- How will you attend to the **established agreements** for respectful and equitable conversation?

If you look in the Session 1 folder on Google Drive, you will find the facilitation plan, slides, and activities that Emily and Erica used. You are welcome to copy any of these and use them as a template for your own session. We suggest that you **create a subfolder** for your session so that others can go back and access the resources you created.

We encourage you to work with the other person who is facilitating the same session on the other day, but also let us know if you would like to work more closely with us to plan the session. We are here to support you in whatever capacity you need! Just let us know what level and type of support you would like.

One technical detail we do need to decide in advance is who will create the **Zoom link** for each session. We are happy to do this for you and give you host privileges during the session, but some people have expressed a preference for creating their own Zoom link. Just let us know what your preference is.

Best regards,

Erica & Emily

Fig. 14.4 Email sent to the new pair of facilitators each week

The introduction of a new pair of facilitators to the *core group* each week allowed for growth and change in the community agreements that were revisited and, some weeks, revised. It also allowed for the introduction of new technology and platforms to enable participant collaboration and contributed to the groups' overall knowledge base of online learning tools.

Sharing Participant Feedback with New Facilitators

Each week we sent curated participant feedback to the facilitators to consider as they co-planned their session. Many comments included in the participant feedback were related to (a) the use of time in the breakout rooms and (b) technology (see Fig. 14.5). As the summer progressed, we read through the new participant feedback and appended responses before sharing the curated comments with the new pair of facilitators. Based on participant feedback, facilitators decided that instead of having three short breakout conversations, planning for only two breakout sessions allowed participants more time to talk, process, and create a record of their conversation and action steps. We also noticed that facilitators began to experiment more with different kinds of tools and artifacts, including PollEverywhere, Google Jamboards, Miro Boards, and Google Forms. The core group members brought their own experience and expertise with these platforms to showcase how they could be used in online teaching and facilitation while serving the goals of the group. The use of the feedback collected in each session is another example of how the adaptive nature of the distributed leadership and community of practice model allowed the core group to react and respond to the needs of the members.

- The hardest part seems always to be not enough time in breakout rooms, but not sure how to "fix" and keep on schedule or fit within a specific time frame. **[Comment 1]**
- The breakout rooms were well timed and I liked the structure of having us record something on the slides; I think it kept us more on track and focused on the prompt since we were supposed to write things down. **[Comment 2]**
- I appreciate gathering info from breakout groups by having groups type on slides, but this has been tricky. First, the technology can be cumbersome at times….For note taking purposes, maybe a Google Doc with 2-3 prompts would be simpler. **[Comment 3]**
- I liked using the poll to draw out areas for discussion. **[Comment 4]**

Fig. 14.5 Examples of curated participant feedback that we shared with the weekly facilitators

Coordinators Supporting Facilitators During the Sessions

During the weekly sessions, we acted as "helpers" for the facilitators. We often would brainstorm ahead of time with the facilitators about exactly what they would need help with during the session and how we could provide support. This included sharing links to documents, creating and managing breakout rooms, and copying pertinent information into and from the chat.

While we primarily used this setup to support the facilitators, we also discussed how to transport this setup to teaching in the fall by utilizing teaching assistants in virtual classrooms. The groups discussed what teachers and facilitators could ask of their students/participants in terms of help during an online learning session. For example, asking for help re-pasting a link into the chat for participants arriving late is a reasonable task to ask a teaching assistant or even a student in the class. In fact, in visiting breakout rooms during the online book study group, we would often hear our participants make statements like "I missed that; could you put it in the chat" as they took on these roles themselves. As a community, we developed a level of comfort with strategies like this for supporting highly active sessions that would prove crucial to our own online teaching of mathematics in the summer and fall of 2020.

14.4 Implications and Conclusion

The distributed leadership model we used in this informal, online faculty professional development setting gave participants a level of agency that allowed them to gain confidence in facilitating conversations about online learning experiences. Many of the participants joined because they would be teaching online in the fall of 2020 and they wanted to practice facilitating online. This distributed leadership model lifted some of the load of planning the sessions off of the coordinators, while also giving participants more agency in driving the direction of the practice, bringing in the issues that were most relevant to them. Wenger (1998) states that as part of learning in a community of practice, "we define…enterprises and engage in their pursuit together, we interact with each other and with the world and we tune our relations with each other and with the world, accordingly" (p. 45). This quote rang true for this informal professional development experience as we adapted the schedule, content, and platforms to meet the needs of the participants and respond to current events happening in the world.

The rotation of facilitators helped maintain participant engagement. As participants got to know one another better through weekly interactions, they began to feel accountable to each other and the community. They wanted to show up for one another in the sessions. This illustrated Gronn's (2000) view of agency, "people taking shared responsibility for the successful outcomes of their joint work" (as cited in Woods et al. 2004, p. 447). Additionally, each facilitator brought their own perspectives, experiences, and personalities to the sessions they ran, which resulted in a diverse and lively community.

14.4.1 What Makes This Informal Faculty Professional Development Work Online?

The structure of the online setting allowed participants to broaden their professional networks outside of their home institutions, provided more flexibility for participation, and was a longer timeline than typical professional development opportunities. The online setting lends itself to participation across many institutions. Participants noted how helpful it was to get ideas and feedback from faculty outside of their departments. In the end of summer feedback, participants reported that they integrated things they had learned as well as shared resources and artifacts with colleagues at their home institutions.

Because the reading group was virtual, members could attend while traveling or switch which day they attended if they had a scheduling conflict. The sessions each had a stand-alone reading assignment, which made it flexible for participants; if members missed a session, they could rejoin the following week without needing to "catch up." If they wanted to review or recap what they missed, notes and artifacts from the session were available in the shared community folder that was part of the space where participants regularly interacted.

This type of long-term professional development is much better suited to the online environment. The majority of faculty attended sessions consistently over the 11 weeks, but there was some attrition. In particular, we were careful to emphasize that while consistent participation was encouraged, there was no expectation to attend every session knowing that commitments would shift over the 11 weeks of sessions and during such uncertain times (during the first summer of the COVID-19 pandemic). This time commitment felt very different than meeting for the same number of contact hours in a 2-day intensive workshop, for example, and also made way for a community to really form. It was made clear from the start that participants were welcome to come and go as needed. As Wenger et al. (2002) found, we believe that this flexibility in participation was a strength of our community as it allowed all participants to feel like full members.

14.4.2 What Makes This Informal, Online Faculty Professional Development Math Specific?

While it is possible for teacher educators in other disciplines to implement a distributed leadership and communities of practice model for informal, online faculty professional development, we feel that these two theoretical frameworks address specific needs within the postsecondary mathematics community. The *IP Guide* (MAA 2018) challenges undergraduate mathematics faculty to "gather the courage to advocate beyond our own classroom for student-centered instructional strategies that promote equitable access to mathematics for all students" (p. viii). The contrast between teacher-centered instruction and student-centered instruction in many ways mirrors the contrast between traditional and distributed leaderships. In teacher-centered instruction and traditional leadership, one figure tends to have the authority and decision-making power. However, in student-centered instruction and distributed leadership, authority and power are decentralized. Our distributed leadership model allows mathematics faculty the opportunity to experience what it feels like to be on equal ground while also being asked to present and facilitate conversations even when they don't yet feel qualified to be called "experts" and teach others. This is similar to the common phenomenon (in student-centered classrooms) where students question why the teacher is asking the students to explain and present their ideas when they are not "experts."

In addition to advocating for student-centered instruction, the *IP Guide* also challenges undergraduate mathematics faculty to "extend the reach of our efforts beyond our own students in our own classrooms" and take responsibility "to help our colleagues improve and to collectively succeed at teaching mathematics to all students" (p. vii). We aimed to respond to this call through our online book study group through the formation of a community of practice. The mathematics community at large still has a lot of work to do when it comes to implementing evidence-based teaching practices, and this work cannot be done by individuals alone. We must take up the challenge of transforming the teaching and learning of mathematics as a community. Learning how to facilitate student success in mathematics is hard work, and we chose to engage in both doing that hard work together, as a community of practice, and supporting one another as we continued to do that work within our own departments and institutions.

For those who are involved in mathematics teacher education and professional development, our distributed leadership model for book study groups can be used as a way to informally engage teachers in conversations about mathematics education literature. Unlike traditional book study groups, who may have a single leader that does the planning and guides the discussion, our model involved participants as facilitators. However, this distributed model of leadership still involved a high level of organization by the coordinators. In particular, the book study group coordinators engaged in the following three macro-activity tasks: (a) launching the book study group, (b) supporting the participants, and (c) supporting the facilitators. Taken together, this model of distributed leadership helped us form a close-knit community of practice that can be applied in other settings.

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Chapter 15
Confronting Teachers with Contingencies to Support Their Learning About Situation-Specific Pedagogical Decisions in an Online Context

Amanda M. Brown, Irma Stevens, Patricio Herbst, and Craig Huhn

This chapter builds on the literature that explores how emerging digital technologies can be leveraged to support the delivery of online, practice-based professional learning experiences for teachers (see also Herbst et al. 2016; Herbst et al. 2019). Specifically, we share an innovation we call *contingency cards*—developed in the context of recent online implementations of Story*Circles* (Herbst and Milewski 2018, 2020)—that can help address concerns that online professional learning too often lacks opportunities for engagement with the subject-specific realities of classroom practice (Dede et al. 2009; McCrory et al. 2008; Wallace 2003). Story*Circles* is a form of professional education that gathers teachers to collectively represent a lesson through iterative phases of scripting, visualizing, and arguing about alternatives—with teachers' visualization of lesson details supported through the production of storyboards. Distinct from face-to-face and school-based forms of practice-based professional development, we have used Story*Circles* to gather teachers across geographically-distant districts to work together on the design and improvement of lessons (Herbst and Milewski 2018, 2020; Milewski et al. 2018). Story*Circles* mediates teachers' online interactions by providing them with a virtual space to engage in collective professional experimentation—with the production of storyboard frames necessitating that teachers move beyond the more typical vague

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narrative accounts that make it challenging for them to engage with one another (Horn and Little 2010).

While our prior design and online implementation of Story*Circles* embodied some of the elements we felt crucial for a practice-based approach (Ball and Cohen 1999; Hiebert and Morris 2012; Lampert 2010), it was less clear that our previous design for Story*Circles* contained structures to support a facilitator to intervene on teachers' activity in ways that could promote the production of a rich lesson. In this conceptual paper, we report on the design of materials and their use with experienced teachers toward their eventual use on a broader scale to support teacher learning. We use this chapter to illustrate the ways we have been responding to this design challenge through the creation and implementation of contingency cards for ushering teachers into moments of a lesson (see Fig. 15.1). *Contingency cards* are representations of practice that can help occasion conversations about realistic contingencies teachers may face in practice.

Specifically, we (1) illustrate ways that contingency cards helped elicit teachers' discussion of subject-specific instructional decisions in the context of online professional development and (2) suggest professional learning activities centered on contingency cards that can help support teachers' online interactions with opportunities to gain competence for making such decisions.

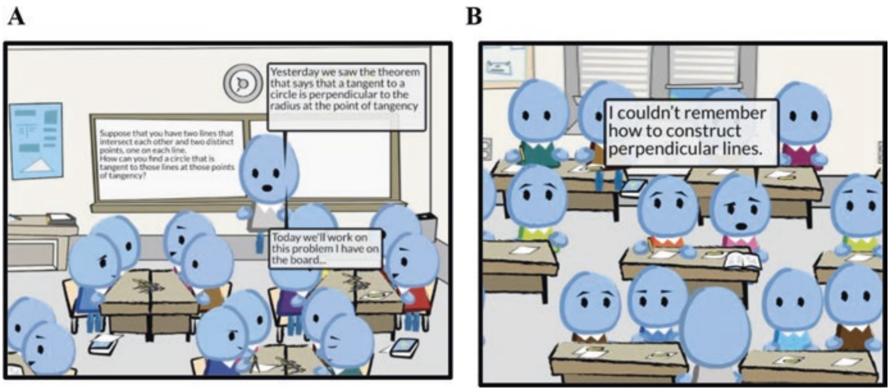


Fig. 15.1 Contingency card examples. © 2021. The Regents of the University of Michigan, all rights reserved, used with permission. *Note:* **Panel A:** Opening frame provided to a group of Story*Circles* participants to support them in envisioning together how a lesson based on the Tangent Circle Problem could unfold. **Panel B:** Contingency card used in the context of that Story*Circle* to support subject-specific interactions about how to handle students' difficulties with construction

15.1 Supporting Teachers to Learn in, from, and for Practice in Online Settings

Historically, the practice of teacher development has struggled with a tension between two orientations to the tasks of promoting instructional change and supporting teacher growth, which Richardson (1990) classified as *teacher change* and *learning to teach*. Driven by well-intended efforts to reform schools, the *teacher change* literature constructed idealistic visions of teaching but rarely paused to question the appropriateness of the changes suggested—even in the face of teacher resistance (see also Ball and Cohen 1996; Chazan and Ball, 1999; Cohen 1990). The *learning to teach* literature, in contrast, demonstrated ways that teachers can and do grow naturally across their careers. However, this natural growth, when taken as a strategy for teacher development, had the major liability of being overly reliant on teachers' proclivities toward reflection (Schön 1982; Shulman 1986)—which have been shown to be substantially influenced by teachers' idiosyncratic beliefs and thus unlikely to support wide-scale educational improvement (Feiman-Nemser and Buchmann 1986).

Furthermore, both approaches failed to account for research documenting the challenges inherent in the attempts to take hold of the egalitarian ideals held by reformers (Romagnano 1994). These challenges emerge quite naturally from the routine structures of schooling—including both the kinds of dispositions held by teachers and students but also the organizational realities of schooling such as how resources are allocated (e.g., Cusick 1983; Wilson, 2008; Webel and Conner 2017). The failures of these approaches suggested that teachers need collaborative support in working toward such ideals and that improvement efforts must proceed in ways that seeks to understand, acknowledge, and work collaboratively to resolve the challenges inherent in pursuing such ideals in actual practice (e.g., Ball and Cohen 1999; Chazan et al. 2009).

Practice-based approaches have developed, in part, out of the growing consensus that to make progress toward such ideals, teachers need opportunities to learn **in**, **from**, and **for** practice (Lampert 2010). Practice-based approaches presume the importance of teachers having opportunities to learn *in* contexts where they are engaged in doing the *work of teaching*—with the work of teaching defined by actual practice rather than by a theory of practice (Ball and Cohen 1999). Such approaches also aim to deliberately focus novices' learning on competencies deemed necessary *for* carrying out ambitious professional practice that aligns with the kinds of ideals outlined by reformers and draws from research on teaching (Ball and Forzani 2009; Lampert et al. 2013). Finally, practice-based approaches seek to draw teachers' learning *from* the kinds of lessons learned within actual teaching practice (Hiebert and Morris 2012).

While the majority of practice-based innovations have been designed for face-to-face professional education settings, there is an emerging recognition regarding the need to develop online practice-based approaches to support teachers' collaboration with colleagues across larger networks (National Research Council 2007). One particularly compelling justification for such work emerges out of the challenge of supporting teachers to develop subject- and even course-specific knowledge in contexts where teachers do not have colleagues that teach the same course, or even the same subject-area, such is the case with many of the upper division mathematics courses and for teachers working in rural locales.

To that end, some recent online innovations include opportunities for teachers to learn by engaging collaboratively *in* the work of teaching. For example, some projects have explored ways to support teachers learning to notice and attend to students' ways of thinking by engaging them in collaboratively reviewing, annotating, and discussing classroom-based artifacts such as student work (e.g., Fernández et al. 2012), narrative cases (e.g., Koc et al. 2009), or classroom video (e.g., Llinares and Valls 2010; Santagata 2009). In other work, researchers have sought to provide teachers with opportunities to learn aspects of practice crucial *for* ambitious practice; augmenting teachers' review of classroom artifacts with research-based frameworks (e.g., Hollebrands and Lee 2020). Teacher educators have also sought to create opportunities for teachers to learn *from* their own classroom practice. In some cases, teachers are given opportunities to share aspects of their practice in the form of artifacts (video, student work) or narrations and receive feedback from others (e.g., Fernández et al. 2020). In other cases, teacher educators use the artifacts to create opportunities for teachers to approximate and receive feedback on aspects of simulated practice (e.g., Campbell et al. 2020; Casey et al. 2018; Herbst et al. 2017; Webel and Conner 2017).

Parallel to these efforts to develop online, practice-based support for teachers, there has been a growing concern among some about practice-based approaches, including those delivered online. In particular, there have been concerns regarding the ways that some practice-based approaches favor a narrow technical focus at the expense of teachers' opportunities to learn *from* practice, as it is situated in local contexts (e.g., Philip et al. 2019; Richmond et al. 2017; Zeichner 2012). For example, Zeichner (2012) suggested that this kind of narrow focus on learning particular skills is not only insufficient to reform teaching, but it also runs the risk of deprofessionalizing teaching—denying teachers the opportunity to develop the more crucial ability of reasoning about decisions that are responsive to context. More recently, Philip et al. (2019) described concerns with the ways that findings emerging from practice-based scholarship (e.g., Grossman 2018) have been taken up in rather shallow ways. Furthermore, they suggest the dangers of representing crucial practices, such as those related to issues of equity, using language alone. These critiques point to an important challenge related to the design of online professional learning for teachers where groups of teachers are frequently gathered across schools to learn together and yet are situated across disparate contexts (Ellis et al. 2016). If online professional development is to play a role in supporting teachers' learning to handle the complexities of actual practice, it must be designed in ways to draw crucially *from* the settings in which teaching takes place and do so in ways that authentically accounts for the problems of practice teachers face in such settings.

15.2 Designing Story*Circles* to Support Teachers' Learning in, from, and for Practice

Story*Circles* is a process for practice-based teacher development that builds crucially on Lampert's (2010) argument about the importance of supporting teachers to learn *in*, *from*, and *for* practice. Story*Circles* can support teacher to learn *in* practice—with engaging teachers in a form of collective professional experimentation where teachers take turns describing how a lesson could play out, representing their ideas visually using the *LessonDepict* storyboarding software, discussing alternatives, and formulating arguments about the relative merits of the various alternatives. Story*Circles* has the potential to support teachers learning *from* practice—with teachers having the agency to steer the conversation toward particular problems of practice they themselves identify as worthwhile. Through their exploration of alternatives, teachers have the opportunity to consider, share, and receive feedback regarding their own and others' suggestions for practice (see Fig. 15.2).

Finally, we have also observed ways that Story*Circles* participants have opportunities to develop knowledge and competencies important *for* supporting particular ends deemed valuable in practice (Herbst et al. 2020; Milewski et al. 2018). The aspirations and experiences that Story*Circles* participants bring and share to justify one alternative over another are a primary source for teacher learning. For example, Story*Circles* often argue for one discursive move over another on the basis of how well it aligns with instructional regimes they aspire to. In some iterations of Story*Circles*, teachers teach the lessons in their own classrooms, bringing back feedback in the form of contingencies that are then used to further inform the group's refinement of the lessons (e.g., Milewski et al. 2019). While participants'

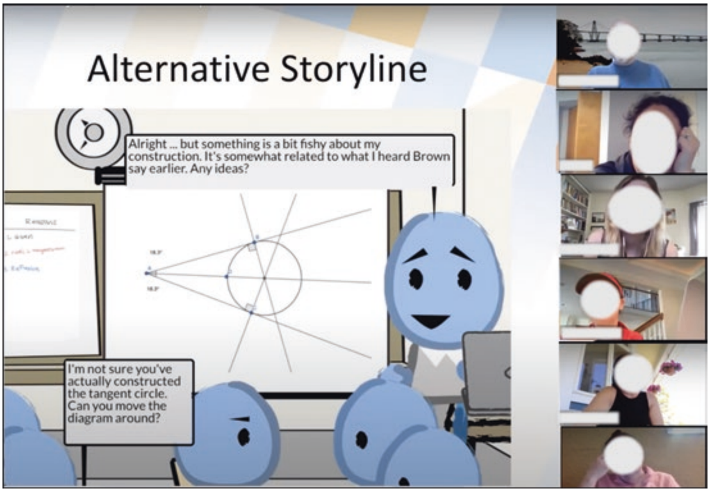


Fig. 15.2 Story*Circles* meeting hosted with video conferencing software. © 2021. The regents of the University of Michigan, all rights reserved, used with permission

aspirations and experiences can serve as a critical source for orienting the groups' learning, we acknowledge the limitations in counting on such sources exclusively. Thus, one of the affordances of Story*Circles* is that it does not rely on teachers having to collect and share data from their own classrooms for collective consideration. The contingency cards, specifically, allow the teachers to discuss hypothetical situations that draw on realistic possibilities informed by research on teaching and learning. We were interested in finding ways to enable a Story*Circles* facilitator to intervene *for* practice—supporting teachers' growth toward particular ends.

*15.2.1 Challenges of Intervening on Story*Circles *Interactions*

As we advance Story*Circles* as a virtual site for professional learning, mathematics teacher educators have challenged us with the question of what to do if the knowledge needed to provoke particular aspects of learning from practice is not present among the teachers who participate (Ed Silver, personal communication, November 24, 2015). Traditional forms of professional development sometimes consider the facilitator as akin to the teacher in the classroom—positioning the professional development participants as students (Carroll and Mumme 2007). If professional development is framed as a kind of instruction, it may feel legitimate for the facilitator to intervene by providing the knowledge that is missing or creating activities in which practitioners construct that knowledge. The facilitator of Story*Circles* is deliberately encouraged not to act like a teacher to participants but rather like a host of a gathering. In cases when facilitators engaged in direct intervention, participants have expressed confusion and even frustration. For example, in a focus group interview drawn from one of our earlier iteration of Story*Circles*, one participant said:

> [The facilitator is] great but sometimes [they] come in with their experience and we don't have that. And we don't get to really hash out what we think so sometimes I feel like we're led in a direction that might not have been where we would have gone. We may have gotten there eventually but we just didn't have enough time to think about it in person. (Story*Circles* participant, April 18, 2016)

Thus, the problem remains. The design of Story*Circles* includes the expectation that teachers will bring their experience to the task of scripting a lesson and that as they visualize the lesson, the diversity in their experiences and contexts will help them improve the script by adding expectations of what students could do and arguing for how those events could be handled. If the facilitator is perceived as intent on a set way to teach a lesson, participants may refrain from sharing any experiences or expectations that differ from that of the facilitator.

Our design and use of contingency cards have enabled facilitators to introduce knowledge which is not (yet) present among participants while also avoiding the impression that the facilitators are intent on pushing the lesson in a particular direction. Rather than appearing like a text to be studied, the cards appear like reasonable events in a journey, and they are incorporated to the work of scripting a lesson in a game-like challenge: Since these contingencies could occur in a lesson, is it possible to include them in the lesson being scripted?

15.2.2 Designing Innovative Forms of Engagement to Support Facilitator Intervention

During the 2019–2020 academic year, we engaged in new rounds of design-based research in which we developed and implemented contingency cards in the context of online Story*Circles*. This work was conducted with a group of secondary algebra teachers and a group of secondary geometry teachers working collectively to design lessons that featured whole class discussions around problem-based tasks. Our design of contingency cards drew on our prior research in which we had observed experienced teachers implement these same tasks—documenting the student conceptions that emerged during an implementation of these lessons (e.g., Stevens et al. 2020) and the kinds of subject-specific decisions that teachers faced regarding how to elicit and respond to students' work in these lessons (Boileau et al. 2020). One of our foci in this round of Story*Circles* was to develop and refine a set of contingency cards for supporting teachers to engage in discussions about subject-specific instructional decisions.

The emergence of COVID-19 heightened our dependence on the contingency cards in the spring of 2020 (Milewski et al. 2020). Specifically, the cards played a critical role in allowing teachers to continue learning from and adapting to realistic classroom contingencies. This suggested to us the importance of this innovation for supporting practice-based teacher education in times when opportunity for participants to implement a particular lesson with actual students was scarce. Furthermore, the contingency cards helped gather evidence that teachers' management of student whole class contributions can benefit from a subject-specific language of description. In the next sections, we elaborate on what we mean by subject-specific.

15.3 Theoretical Perspectives

15.3.1 Toward a Subject-Specific Account of the Teachers' Role in Managing Classroom Discussions

Whole class discussions call for different kinds of specificity for teacher moves, and contingency cards that contain student contributions contextualized in whole class discussions can elicit responses from teachers that reveal this specificity. Likely influenced by NCTM's 1991 Professional Standards for Teaching Mathematics, the last two decades have included a substantial focus on teachers' management of classroom discourse, with many important contributions to educational research (Boaler and Brodie 2004; Ghousseini and Herbst 2016; Hufferd-Ackles et al. 2004; Milewski and Strickland 2020; Truxaw and DeFranco 2008) and practice (Chapin et al. 2009; Herbel-Eisenmann et al. 2013; Milewski and Strickland 2016; Smith and Stein 2011). This literature has largely considered those discursive moves that, while applied in the context of mathematically-specific work, do not depend on this work for an account of their meanings but rather have meanings posited to be generalizable across the mathematical work onto which they are applied. Without discounting the value of having repertoires of discursive moves for use across different mathematical contexts, we argue that problem-based classroom discussions call for subject-specific moves for managing the complexities related to eliciting and responding to students' contributions while attending to instructional goals.

A canonical example of a subject-specific move might be a move that includes lexical choices pointing to particular kinds of mathematical work or ideas (e.g., "can you prove that?" versus "what is the measure of that?" allude to different kinds of mathematical work). Yet, we do not merely refer to lexical choices within clauses. More generally, teacher moves belong in larger chunks of mathematical work that provide context to any one move. By saying that the move could be subject specific, we mean more generally that in the work context in which it is made, it points to specific mathematical meanings, regardless of whether the move itself uses mathematical or generic words. For example, "why would you say that?" sounds like a generic why question and may be generic in many occasions, yet when asked after a student has written a line in the solution of an equation, it calls for the student to say what transformation they made to a prior equality. In contrast, when "why would you say that?" is asked after a student has written a statement in a geometric proof, it calls for the student to spell out a reason to warrant the statement (a reason that can be procured from a limited repertoire of sanctioned statements). Finally, in other contexts, the question might be interpreted generically, either as an appeal to the personal reason the student said something or to suggest that whatever the student said was inconvenient. In sum, by saying that a move is subject-specific, we mean that it elicits specific mathematical meanings and calls for specific mathematical actions or utterances associated to the mathematical work context in which the move is made, and regardless of whether the move itself employs technical lexical choices. Obviously, because moves are or are not subject-specific, depending on the mathematical work context where they are made, these contexts need subject-specific accounting.

Teacher moves are subject-specific in a variety of ways, including specific to (1) the instructional situations that can be used to frame the problem at hand, (2) the student conceptions activated in students' work on the problem at hand, and (3) the students' contributions made at a particular moment in their work on the problem. In all cases, attention to specificity enables the teacher to manage the transaction between work on a problem and the instructional goal on whose behalf the problem was chosen. In what follows we elaborate on these cases.

15.3.2 Instructional Situations

In Herbst and Chazan's (2012) theory of practical rationality, the authors provide a theoretical framework for accounting for mathematics teaching. One of the primary assumptions of that work is that embedded within particular courses of study, there are sets of recurrent situations in which teachers deploy mathematical objects of study in the form of tasks for students to do. These tasks are typically deployed with some regularity—with the tasks taking familiar forms that can provide evidence that allow teachers to make claims about students' understanding of the mathematical objects they are intended to represent. The authors posit that these recurrent situations, called *instructional situations*, carry within them a set of tacit expectations about how the teacher will present the tasks and what work students are expected to do. These expectations are subject-specific inasmuch as they are specific to the mathematical work students are expected to do in response to the type of task or knowledge at stake.

Two such instructional situations in US high school geometry are those of *doing proofs* and *construction*. In the instructional situation of doing proofs, teachers are tacitly expected to provide a clear statement of the proposition to be proven in the form of separate statements about the proposition's assumptions (the givens) and the proposition's conclusion (the prove)—with both the statements of assumptions and conclusion given in terms of a labelled geometric diagram that is also provided by the teacher (Herbst 2006; Herbst and Brach 2006). In the instructional situation of construction, teachers often feel responsible to provide diagrammatically a set of initial geometric objects and verbally a target geometric object, and it often is presumed that students will engage in a construction by describing or demonstrating how they would use geometric tools (e.g., straightedge, compass) with the initial geometric objects to produce the target geometric objects.

One sense to which discourse moves are subject-specific draws from Herbst and Chazan's (2012) notion of instructional situation. In particular, we use the concept of instructional situations to make sense of how teachers contextualize mathematical work, in particular to manage the complex work of facilitating problem-based lessons. We take the perspective that instructional situations contain useful resources that can inform both teachers and students about how they are expected to act in a given mathematical task (e.g., Aaron 2011; Herbst et al. 2011). Furthermore, when operating outside of these more familiar situations, such as when participating in a problem-based lesson centered on a novel task, the uncertainty teachers and students might have about how to act within such lessons can be overcome by leveraging routines from instructional situations with subject-specific cues—we call this phenomena *framing moves* (Boileau et al. 2020; Milewski et al. 2019).

Teachers' use of framing moves can summon expectations from a related instructional situation in ways that help students get started on a novel problem situation. For example, after launching a novel geometry task, a teacher could cue the instructional situation of construction or proof with the infusion of mathematical tools (e.g., compasses and straightedges strewn on the students' tables or a two-column proof frame drawn on the board) or the use of discursive moves that include lexical elements specific to the situation (e.g., "How about you try constructing the circle?" or "Can you prove the assertion?"). These framing moves are subject-specific in the sense that they support particular kinds of mathematical work.

While the choice to *frame* a novel task by evoking a familiar instructional situation may have the effect of discouraging other strategies and encroach on goals the teacher may have for using the novel task (Stein et al. 1996), we argue they may also have the benefit of encouraging students' engagement in the task—creating conditions for students to know what work to do—and thereby enabling the class to make speedier progress toward the lesson's stated content objective. Framing moves also create conditions where teachers' discursive moves can take on particular meanings. For example, we can anticipate a responding move like "How can we be certain?" will have different meanings when spoken in the context of an Algebra 1 lesson about solving linear equations (e.g., used to respond to a student who solves an equation $ax + b = cx + d$ by graphing) than when spoken in a Geometry lesson exploring a new theorem (e.g., used to respond to a student who has just formulated a conjecture based on a constructed diagram). When used in the context of the instructional situation of solving equations, the students often rightly understand the question as a bid for more canonical forms of solving (see Buchbinder et al. 2019); while the same discursive move used in the context of formulating conjectures from a diagram is more likely to be interpreted as a transition to the new instructional situation of doing proof. We account for such interpretations in practice as sensible given how the underlying epistemologies of each situation differ: the way one typically determines certainty in algebra is distinct from the way one determines certainty in geometry. In the context of a Story*Circle* in which a problem has been framed after an instructional situation and a contingency is drawn that shows a student contribution to the problem, we expect that the move the teacher uses to respond to that contribution will be subject-specific inasmuch as it will be chosen to fit the instructional situation that frames the work.

15.3.3 Students' Contributions in Light of the Lesson's Goal

The moves teachers make in response to students' contributions can also elicit teachers' subject-specific reasoning in regard to the role the contribution can play in supporting the class to make progress on the lesson's goals. Much of the literature on responding builds on Mehan's (1979) IRE recitation pattern—describing pervasive patterns in mathematics instruction. In that pattern, the *evaluation* of the student's *reply* is generic in that it only depends on the correctness or incorrectness of the student's *reply*. In a problem-based lesson aimed at achieving a particular instructional goal, however, the sense to which students' contributions are correct is insufficient to justify an evaluative response: A student contribution with errors could still be serviceable to the instructional goal of the lesson, and one without errors might not be particularly serviceable in building toward that goal. We expect that, among other factors, teachers' handling of student contributions to classroom discussions in the context of problem-based lessons will depend on the serviceability of the student contribution toward the instructional goal of the lesson.

To the extent that achieving the instructional goal of the lesson is distinct from merely solving the problem posed, teachers' responsive moves need to make use of how students solve the problem in ways that intentionally build toward claiming the instructional goal. Thus, the moves need to be subject-specific in this second sense—specific to the exchange between students' work on the problem and the instructional goal being targeted. Contingency cards administered in a Story*Circle* to represent a possible student contribution thus support teachers in considering not only the correctness of a contribution but also in what way the contribution is serviceable toward the instructional goal of the lesson.

15.3.4 Students' Conceptions

Finally, teachers' moves responding to students' contributions are also subject-specific in regard to the students' mathematical conceptions. The student's written work and associated utterances and gestures provide the teacher with ways to construe models of students' mathematics—that is the second-order models that teachers make of "students' mathematical concepts and operations to explain what students say and do" (Steffe and Ulrich 2020, pp. 134–135). These models may be refined and adapted throughout the class as the teacher engages with students over various mathematical ideas and attempts to imagine the mathematics from the students' perspectives in the classroom (Confrey 1990; Davis 1997; Steffe and Thompson 2000). Beyond assessing the correctness of a students' solution, the teacher may try to understand how a student reached their solution (Teuscher et al. 2016). The teacher can use these second-order models to support a whole class discussion of a problem chosen to develop students' understanding toward the instructional goal. In this way student contributions are not necessarily replies that could be evaluated on their own. Rather they likely are building blocks that can be used by the teacher to support students' learning—transitioning from what is known to what is not yet known. As the students work on the problem during the lesson, they construct representations from their existing knowledge that enable them to reason about the problem in ways they anticipate will help them make progress toward a solution.

Those practices that a problem may elicit in different students instantiate the different conceptions at play in the lesson. We contend that teachers make use of students' contributions in a problem-based lesson to both anchor the new problem to existing conceptions and use those conceptions to make progress toward a solution of the problem. While a teacher may be looking for particular contributions which are serviceable to the goals of the lesson, the teacher depends on the work the students do, and in that sense, they need to make do with what students actually produce. It becomes essential for the teacher to try and understand where students are coming from so as to know how likely it is that the direction in which they are headed will produce serviceable contributions. Moreover, constructing these second-order models supports teachers to be able to anticipate what students might find confusing, and then the teacher can use those anticipations to ask questions that offer students with time and support to think through those confusing components (see Johnson and Larsen 2012).

While teachers' responses to what students do or say may be generic in their expression, to understand what role those responses play in a classroom discussion, one needs to know what student conceptions it responds to. Teachers can use student contributions (e.g., utterances, work) to make inferences about a student's conceptions. Thus, the conceptions are the "explanatory model[s] used to explain observed abilities and limitations of mathematics learners in terms of their (inferred) ways of knowing" (Simon 2017). In that sense, teacher responding moves are subject-specific. Story*Circles*' contingency cards support exercising teachers' knowledge of responding moves by presenting to participants students' contributions which may not yet be serviceable to the goals of the lesson but may already suggest conceptions that the teacher could use to encourage work toward serviceable responses.

15.4 Description of 2019–2020 Geometry Story*Circles*

The data used in this chapter is drawn from a design-based research project in which we use Story*Circles* to support teachers' professional growth related to the facilitation of whole class discussion of open, novel mathematical tasks to support students' learning of new mathematical concepts. We draw specifically on the online interactions that took place among four secondary in-service geometry teachers located within a 100-mile radius from a large university located in the Midwest. These teachers expressed initial interest in partaking in the study after being contacted by the research team from a database of local teachers. The teachers received an interest form which also provided the research team with information about availability, institution, current teaching schedule, and teaching experience. Those teachers who (1) expressed interest in joining the geometry group, (2) had more than 5 years of experience teaching geometry, and (3) were currently teaching geometry were subsequently invited to participate. The group varied in terms of demographics (gender[1] and age), years of teaching experience (10–27 years), and institutional settings (size; public, private, and charter; and locale). As noted earlier, and due to space constraints, we elected to focus our sharing here on participants' interactions about how a lesson around the *Tangent Circle Problem* (Fig. 15.1a) might unfold in the context of a high school geometry class. The teachers' activities with the lesson were framed in two ways. First, we provided them with a set of introductory frames that outlined how the task was launched and some of the exchanges a teacher might overhear during the small group portion of the lesson. Also, we provided them with a goal for the lesson; namely, through their exploration of the *Tangent Circle Problem*, students will learn the following theorem: *A*

[1] Though we had one male participant (and because we only had one), we elect to use "Ms." for all pseudonyms to ensure we maintain the anonymity of participants' individual contributions.

circle is tangent to two intersecting lines if and only if the points of tangency are equidistant to the point of intersection of the lines.

All teachers participated in the round of the virtual Story*Circles* focused on the *Tangent Circle Problem*, which occurred during a 6-week period in the late spring of 2020. These interactions included 3-hour-long synchronous, video conference meetings alongside weekly asynchronous work that took place in an online learning management system. At the end of the 2019–2020 Geometry Story*Circles*, they also participated in an hour-long individual interview and a focus group in which we revisited the teachers' work on the *Tangent Circle Problem*. The following illustrations draw on these interactions and include contingency cards we provided them. Lastly, we also draw on prior classroom implementations of the *Tangent Circle Problem*.[2]

15.5 Illustrating and Eliciting the Subject-Specific Nature of Teachers' Discursive Choices

Addressing concerns that online professional learning too often lacks opportunities for engagement with subject-specific realities of classroom practice, we start each of the sections below by illustrating how contingency cards helped provide an efficient way to structure teachers' engagement in online Story*Circles* to attend to such realities. We conclude each section by suggesting various professional learning activities centered around contingency cards for use in supporting teachers to gain increased competence with these subject-specific decisions and avoid overgeneralizing about the utility of particular discursive moves by supporting teachers' discussions regarding the use of particular moves across classroom episodes in which they may be more or less appropriate.

15.5.1 Teachers' Rationality Regarding Whether and How to Frame a Novel Task

When introducing the *Tangent Circle Problem*, there are different choices a teacher confronts when launching the lesson. The choice to leave the task unframed by saying something like "How about you take some time to think about this in your group?" leaves it open to the students to decide how to proceed with the problem. In the remainder of this section, we discuss two ways to frame the problem: construction framing and proof framing.

Construction framing. Rather than leaving the problem unframed, the teacher might elect to frame the problem by cueing the instructional situation of

[2] Implementations took place in Ms. Keating's classroom during the 2018–2019 school year.

construction. For example, in her actual implementation of the *Tangent Circle Problem*, Ms. Keating offered a construction framing to her students, stating, “You’re trying to draw a circle that’s tangent to the lines at those points ... You should still have a ruler ... or if you want to [use a] compass, they’re up there [gesturing towards the class construction tools].” While we observed Ms. Keating using these kinds of more normative framing moves (those that make explicit reference to construction or formal construction tools) in her implementation of the *Tangent Circle Problem*, she did not always. Sometimes, she elected to use a less normative version of the construction framing with requests to “draw more diagrams.” She justified those calls with statements like “that’s sort of the big thing in Geometry is a picture really is worth a thousand words”—indicating to her students not only the importance of diagrams in Geometry, but also suggesting it as a viable means for making progress on this problem.

Throughout the Story*Circles* cycle, teachers offered up reasons related to their choices about how to frame the *Tangent Circle Problem* that help us understand the kinds of subject-specific considerations they use when deciding whether and how to frame a problem as one of construction. For example, when asked to deal with a contingency card that features a student who sketched a circle tangent to the two given lines (Fig. 15.3), Ms. Zion indicated that she would “Ask [the student shown in Fig. 15.3] how he constructed his circle ... to help [the student] realize that he was just guessing”—simultaneously indicating her own preference for a construction over a sketch and the unexpected nature of the students’ work for the situation.

Sometimes teachers disagreed on the need for a more normative construction framing—arguing instead on waiting to see whether students needed such framing or for a less normative version of the construction framing. For instance, during her interview, Ms. Kortez expressed doubt about the need for early and more normative framing saying, “I feel like some students might just try and do a rough sketch and see if it gives them any insight into how to do the construction”—naming a value for sketching prior to construction and waiting to see what students do that might naturally lead to more formal construction. She also saw value in just sketching, saying that students working on a sketch in which the given points were not equidistant from the intersection might help them notice, “like, ‘hey, this thing doesn’t seem to be working.’”

Whereas Ms. Zion saw the more normative construction framing as a way to support a student who sketched to make progress on the problem, Ms. Kortez thought

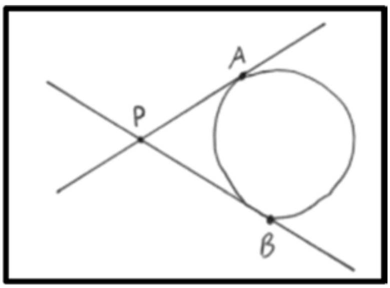


Fig. 15.3 Contingency card containing student response to the Tangent Circle Problem

that a less normative framing might enable students with sketches to contribute something complementary to those who constructed—namely, formulating a conjecture serviceable to the goal of the lesson. These justifications for different framing moves suggest to us the importance of teachers' subject-specific considerations on how to elicit students' work on open, novel tasks, in terms of both the timing for such a move and reasoning afforded to students through the production of more and less formal geometric diagrams.

Proof framing. The teacher may also (or instead) decide to use *proof framing*, cueing a set of expectations for students' work related to the instructional situation of proof. For example, in the context of an asynchronous scripting activity, Ms. Zion suggested that following some exploration of the diagram, a teacher could transition the class's work to proof by suggesting what we see as a more normative proof framing—namely, "So let's take a few minutes and see if you can prove that those points are equidistant from the point of intersection of the two lines." In a different asynchronous forum interaction in which the teachers needed to consider different framing moves we provided, Ms. Bonny argued for a less normative framing move that read, "Before we go further, what are the givens in this situation?" noting it as "a good transition from the construction to the proof," although she would wait to see if a student would ask it first. Ms. Kortez saw the same provided framing move and qualified that she could only imagine using this move if there was a need to clarify either the starting place or the goal of the proof.

In a later asynchronous activity in Story*Circles*, Ms. Kortez suggested an alternate way to transition to proof, again showing preference for a less normative framing and the importance of timing. Specifically, after envisioning a portion of the lesson in which a student formulates the desired conjecture in front of the class, Ms. Kortez scripted the following dialogue for the teacher, "Can anyone say why this might need to be true? Talk to your groups for a minute ... this might need to be an idea we come back to, but let's think about it first." Ms. Bonny's and Ms. Kortez's reactions to our presentation of more normative framing moves illustrate the subject-specific nature of the decision to provide a proof framing—the appropriateness of the decision to offer a more or less normative framing for proof is contingent on students' demonstrating a need for more structure in their exploration of the problem.

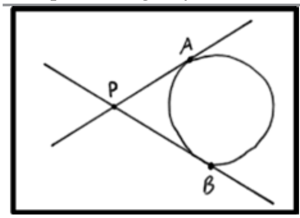
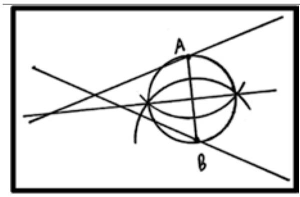
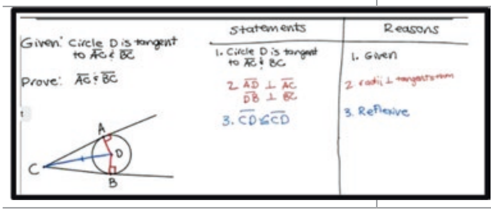
15.5.2 Activity for Eliciting Teachers' Subject-Specific Considerations Related to Framing

This activity starts by providing teachers with various contingency cards featuring the variety of ways a student might work on the problem (see left column of Table 15.1) and asking teachers to reason about how a student that produced such work might be thinking and some of the possible affordances such work might have for helping the class make progress toward the goal of the lesson. After having a chance to consider the affordances of different approaches, teachers are asked, "How could a teacher ensure solutions like this (in which we provided a variety of work including construction and sketches, formal and informal proof) emerge from students' work on the problem?" (see right column of Table 15.1). Finally, after teachers have the chance to script such moves, they could be asked to visualize where in the lesson such moves could be used—engaging them in a discussion about whether, how, when, and why one might elect to use such moves.

15.5.3 Teachers' Rationality Regarding Selecting and Responding to Students' Contributions

After making choices regarding whether and how to frame the *Tangent Circle Problem*, teachers are confronted with a series of decisions regarding how to handle students' contributions. When observing experienced teachers' reason about such

Table 15.1 Tangent Circle Problem contingency cards with possible discursive moves. *Note*: **Row 1**. Contingency card representing a servicable, non-normative piece of student work for a construction framing of the Tangent Circle Problem—sketching a wonky circle rather than constructing the diagram. **Row 2**. Contingency card representing a non-servicable, normative piece of student work for a construction framing of the Tangent Circle Problem—constructing a non-tangent circle that fails to identify the fundamental elements necessary for locating the center of a tangent circle. **Row 3**. Contingency card representing a non-servicable, normative piece of student work for a proof framing of the Tangent Circle Problem—wrongly asserting a corollary of the theorem to be proved

Sample contingency cards	Sample discursive move that could be used to elicit the work shown on the contingency card
	I'd like you to take a few minutes and try drawing the circle that is tangent to two intersecting lines PA and PB
	Please grab a compass and straightedge, and try to construct the circle that is tangent to two intersecting lines PA and PB
	I'd like you to take a few minutes and consider whether you can prove the following theorem [writing "If a circle is tangent to two intersecting lines, the points of tangency are equidistant to the point of intersection of the lines." on the board]

choices, we noticed ways that the instructional situation, as framed by the teacher, became a resource that helped teachers make decisions about how to handle students' mathematical contributions.

For this section, we focus our illustration on how teachers elected to handle the contingency card featured in Fig. 15.4c (we refer to this student as *Green*). To be clear, this card was presented to teachers after they had already interacted with the card shown in Fig. 15.3 (containing the same work). When presented with this later contingency card, teachers were asked to cope with the contingency that Green would at some point in the whole class discussion find himself at the board with his work and requested they share how they would handle that contingency. Finally, we note that the teachers were coping with this contingency card in the context of a storyline in which a less normative construction framing was at play (see Fig. 15.4a). In this way, Green's work can be understood as nonnormative in that it breaches the

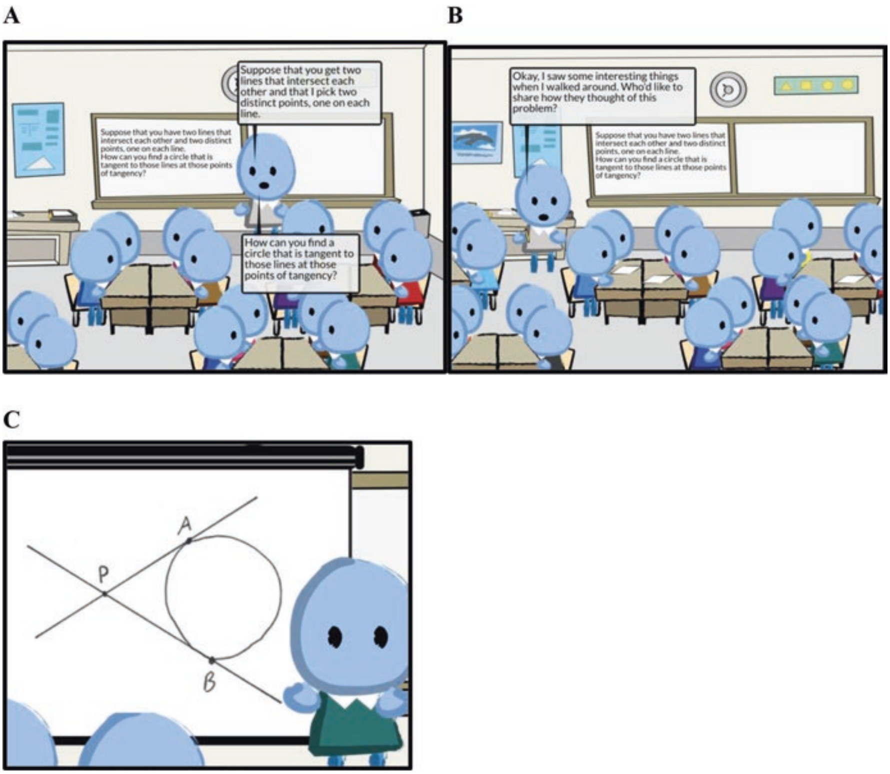


Fig. 15.4 Introductory storyline and contingency card example. © 2021, The regents of the University of Michigan, all rights reserved, used with permission. *Note:* **Panel A:** The second frame of the introductory storyline we provided to teachers. Note the less normative construction framing teachers were operating within. **Panel B:** The eighth frame of the introductory storyline we provided to teachers. Note that the monitoring of group work had already happened. **Panel C:** A contingency card that features a student standing at the board with their work. Teachers were asked to incorporate this into their scripting of the whole class discussion

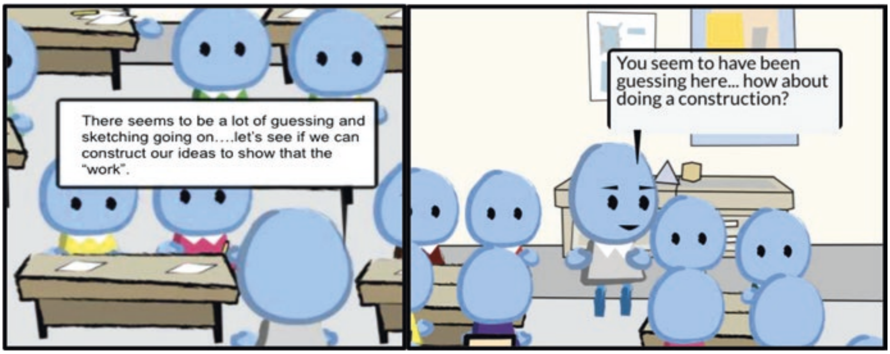


Fig. 15.5 Teacher changes to introductory frames. © 2021, The Regents of the University of Michigan, all rights reserved, used with permission. *Note:* Scripts from Ms. Bonny (left) and Ms. Zion (right) representing how to handle Green's work

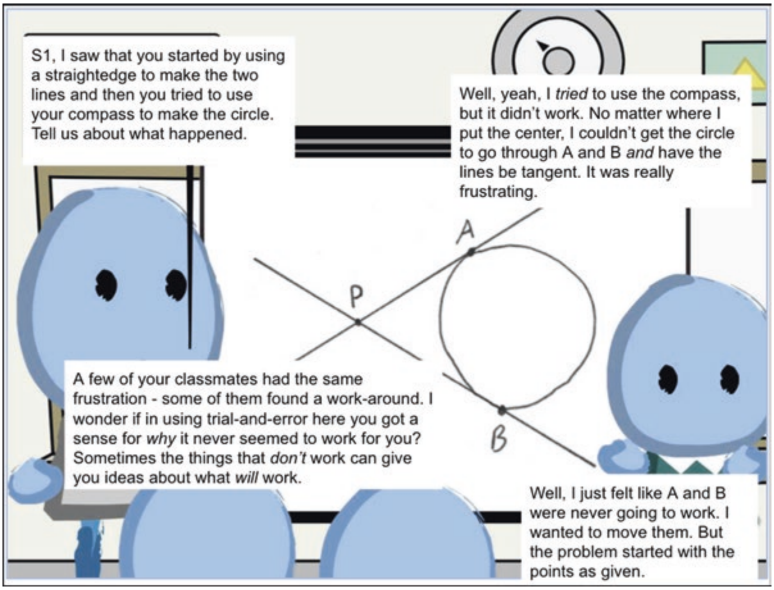
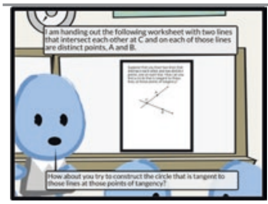


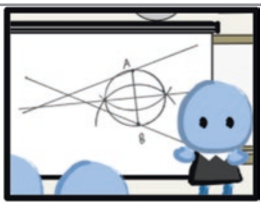
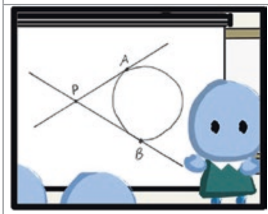
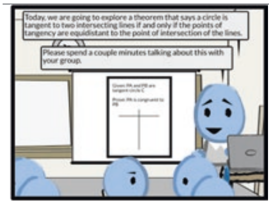
Fig. 15.6 Ms. Kortez's representation of how to handle Green's work. © 2020, The Regents of the University of Michigan, all rights reserved, used with permission

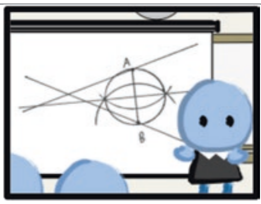
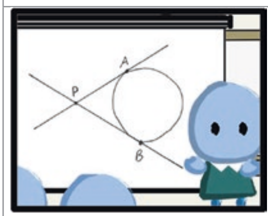
tacit set of expectations for the situation of construction that students will complete such work with a compass and straightedge. But given the less normative framing, it is reasonable to wonder how strongly the norms of the situation of construction sway teachers' decisions.

When confronted with the request to depict how they would handle Green's work in the context of a whole class discussion, two teachers went out of their way to change the introductory frames we provided them with (i.e., inserting Green's work in between the comic frames we provided— which already included a

Table 15.2 Contingency card, student contributions, and discursive moves. *Note*: **Left Column for Rows 1.1 & 1.2**. Contingency card representing a normative construction framing for the Tangent Circle Problem. **Middle Column for Row 1.1**. Contingency card representing a servicable, non-normative piece of student work for a construction framing of the Tangent Circle Problem. (Same as that shown in Middle Column for Row 2.1) **Middle Column for Row 1.2**. Contingency card representing a non-servicable, normative piece of student work for a construction framing of the Tangent Circle Problem. (Same as that shown in Middle Column for Row 2.2) **Left Column for Rows 2.1 & 2.2**. Contingency card representing a normative proof framing for the Tangent Circle Problem. **Middle Column for Row 2.1**. Contingency card representing a non-servicable, nonnormative piece of student work for a proof framing of the Tangent Circle Problem. (Same as that shown in Middle Column for Row 1.1) **Middle Column for Row 2.2**. Contingency card representing a servicable, non-normative piece of student work for a proof framing of the Tangent Circle Problem. (Same as that shown in Middle Column for Row 1.2)

<table>
<tr><th>Contingency card representing how a lesson was framed</th><th>Contingency card representing various student contributions</th><th>Sample discursive moves that respond to students' contributions and fit the instructional situation that frames the work</th></tr>
<tr><td rowspan="2">

More normative construction framing</td><td>
</td><td>How do you know that circle is tangent at point A?</td></tr>
<tr><td>
</td><td>Maybe you could try using a compass and straightedge to get a more accurate construction. Remember a picture is worth a thousand words.</td></tr>
<tr><td rowspan="2">

More normative proof framing</td><td>
</td><td>You don't need to be so accurate with your diagram, remember a picture is just a picture.</td></tr>
<tr><td>
</td><td>Based on this diagram, any ideas on what might be true?</td></tr>
</table>

Note: A sampler of contingency card sets (left and middle) representing student work as contextualized within instructional situations and the discursive moves (right) teachers might use to respond to such work representation of small group time, see Fig. 15.4b) rather than finding a place for the contingency card as provided (i.e., with Green standing next to his work at the front board to simulate his public contribution to the whole class discussion)—suggesting a strong preference for not featuring Green's sketch during the whole class discussion following a construction framing. Furthermore, these teachers' scripted responses contained both a negative evaluation of Green's work (e.g., "You seem to have been guessing") and suggestion that the student should try constructing (see Fig. 15.5)—suggesting teachers saw a need for Green to repair his work to comply with the norms of the situation of construction (more discussion on this in the conclusion).

While Ms. Kortez managed to take up our request as intended (see Fig. 15.6), she did so by constructing a storyline in which Green's work had changed quite dramatically after the teacher had last seen it. Crucially, Green is purported to have started with a construction and then abandoned that approach for the sketch, only after realizing the construction was not working. This suggests that Ms. Kortez had some qualms about bringing Green up to the board. We hypothesize this is not because she did not preferred Green's work (drawing on her earlier comments), but rather because the instructional situation, even if framed nonnormatively, shapes teachers' sensibilities about how to handle students' mathematical contributions that do not comply with the norms of the situation.

15.5.4 Activity to Elicit Subject-Specific Considerations for Handling Students' Contributions

From our interactions with experienced teachers, we designed a professional learning activity aimed at supporting teachers to learn about the subject-specific considerations related to the various ways that one can handle students' mathematical contributions. This activity starts by providing Story*Circles* participants with a set[3] of contingency cards (see Table 15.2).

Using those cards, teachers are asked to reason collectively about which pieces of student work they might prefer to feature for the given framing condition. Next, teachers are asked, "Suppose a lesson was launched in the following way [left card] and this particular student [middle card] somehow ended up at the board. What

[3] We note that here we have intentionally represented two sets of cards—representing two very distinct framing conditions (e.g., more normative construction framing vs more normative proof framing)—in anticipation that teachers, in general, may not yet be ready to perceive differences between the more nuanced framing distinctions the experienced teachers in our work were exploring (e.g., more normative construction framing vs less normative construction framing; less normative construction framing vs less normative proof framing). That said, we anticipate that over time, more nuanced differences could be introduced to help teachers realize some of the affordances and constraints offered by those variations as well.

should the teacher do or say next?". After the participants have a chance to walk through one complete set of contingency cards, they would repeat these two activities with a different set of cards where the framing card changes but the student work stays the same. Finally, after completing two or more sets of contingency cards, teachers are asked to reflect on how their response to the same piece of work differed across framing conditions.

By completing the activity with a single set, teachers will have the opportunity to practice what the NCTM (1991, p. 30) says that "teachers must filter and direct students' exploration by picking up on some points and leaving others behind." We also anticipate teachers will have the chance to practice formulating discursive moves that are responsive to different kinds of student contributions—both those they prefer to feature and filter. By completing the activity with more than one set, teachers will have the chance to unpack and question the kinds of tacit expectations they hold for students' work across different instructional situations. Teachers may also begin gaining the kinds of sensibilities we observed among some of the experienced teachers about the affordances of less normative framing moves.

15.6 Conclusion

Before closing, we take a moment to reflect back on a particularly salient illustration in which our notion of subject-specific moves became, we hope, more clear for the reader. But just in the case it did not, we pause to take a closer look at the responses provided by Ms. Bonny and Ms. Zion in Fig. 15.5. We note that this was not the only moment in which we observed experienced teachers, who in many regard would be counted as experts in the craft of facilitating productive mathematical discussions, elected to use a move they might not otherwise explicitly advocate for or use regularly. In fact, perhaps because of our aims (and design of materials) in this project, we report that it was a fairly regular occurrence to see teachers (1) independently suggesting the need for discursive moves that might be broadly thought of as less productive (e.g., evaluation, telling) or (2) collectively rejecting moves that have generally been lauded as productive (e.g., probing, inviting other students to build upon a contribution) on the grounds that they were not considered viable in the given situations.

This phenomena is at the heart of what we are arguing for and we suggest was not the result of any lack of knowledge or fluency our participants had for more productive moves. Rather, we see these decisions as emanating from teachers' practical rationality (Herbst and Chazan 2012) and have provided evidence that experienced teachers have some shared sensibilities about what is appropriate or called for in a given situation. We suggest this kind of expertise is distinct—both in its source and usefulness—from the kind of knowledge about practice encoded in more generic prescriptions that recommend to teachers sets of productive talk moves. It is this practical rationality that we seek to document, as something that emerges from the practice of teaching mathematics, contextualized in specific courses of study and instructional situations.

Moreover, we highlight specifically how the virtual environment in which these observations took place afforded opportunities for our teachers to learn *in*, *from*, and *for* practice. The teachers in this study were located in different schools, but still had the opportunity to discuss their teaching practices with one another in the virtual setting. The opportunity for teachers to engage outside of their local school districts also has the potential to afford teachers with opportunities to engage with teachers with differing backgrounds and school/political climates. Moreover, particularly when classroom data collection was not possible during the start of the COVID-19 pandemic, the teachers could still approximate practice using the Lesson*Depict* software in ways that would not have been possible if video data from classrooms was needed. Using Lesson*Depict* to illustrate hypothetical situations also provides the teachers with opportunities to consider contingencies with their colleagues and engage in argumentation around those ideas without impacting/disrupting their own students in their classrooms.

In this chapter, we have illustrated how we are adapting the Story*Circles* process with the design of contingency cards to elicit teachers' practical rationality about the subject-specific moves teachers find useful for facilitating whole class discussions around open, novel tasks. Specifically, we have shown how contingency cards can enable teachers and researchers alike to explore and become more aware how the givens of a pedagogical situation (e.g., the framing moves used by the teacher) can play a role in shaping teachers' decisions within that situation (e.g., how to handle student work that does or does not manage to comply with the norms of the situation). Furthermore, we have demonstrated how the materials created and knowledge elicited by such work can quickly be recycled and repackaged in the form of professional learning activities for use with larger groups of teachers. The contingency cards provide an opportunity for the facilitator to support teachers in learning to have discussions about how to handle unexpected student contributions—terrain for which teachers new to facilitating discussion typically struggle. Being able to understand where students are coming from (even when what students' offer is nonnormative for a situation) is often the key to perceiving how students' contributions might be made serviceable for the goals of the lesson. Finally, the online administration of such activities can help in tackling one of the enduring challenges in teacher education (Ball and Forzani 2009; Sweeney et al. 2018), namely, ensuring that teachers have regular and routine opportunities to practice and receive feedback on subject-specific pedagogical decisions.

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Chapter 16
Virtual Field Experiences as an Opportunity to Develop Preservice Teachers' Efficacy and Equitable Teaching Practice

Liza Bondurant and Joel Amidon

The leading professional organizations for mathematics teachers and mathematics teacher educators, the National Council of Teachers of Mathematics (NCTM) and the Association of Mathematics Teacher Educators (AMTE), have a significant influence on mathematics education policy and practice. NCTM's strategic framework states "NCTM advances a culture of equity where each and every person has access to high-quality teaching empowered by the opportunities mathematics affords" (NCTM 2017) and AMTE goals include "Equitable practices in mathematics teacher education, including increasing the diversity of mathematics teachers and teacher educators" (AMTE 2019). These statements make it clear that equitable mathematics instruction is a high priority for mathematics educators. Our intent is to make visible the ways in which equitable instructional practices may be supported or undermined in such environments by leveraging a virtual field experience (VFE) with the intent of providing access to the community or practice (Lave and Wenger 1991) centered on the equitable teaching of mathematics.

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16.1 Literature Review

Below is a brief review of the literature on practice in teacher education as extended into VFEs, as early field experiences or "on-ramps to professional practice" (Sweeney et al. 2018, p. 671). In addition, a look at the literature around the call for promoting equitable mathematics teaching and how that overlaps with using technologically mediated approximations of practice, or VFEs, will be discussed.

16.1.1 Practice in Teacher Education

The work below is grounded in the situative perspective of learning (Lave and Wenger 1991). Accordingly, learning occurs best when it takes place in the context in which it is applied and in interactions with others. This implies that preservice teachers (PSTs) should assume the role of an apprentice within an authentic community of practice where learning opportunities arise situationally. Then, as PSTs gain experience and competence within the community of practice, they will move more central within the community of practice (Lave and Wenger 1991).

The influence of situated learning theory can be seen in a shift in the focus of the curriculum and pedagogies of teacher education programs. Practice, defined as "the orchestration of understanding, skill, relationship, and identity to accomplish particular activities with others in specific environments" (Grossman et al. 2009, p. 2059), has been highlighted as a valued characteristic of teacher education programs. Programs that previously focused on theory and knowledge are now focusing on a practice-based curriculum (Darling-Hammond and Sykes 1999; Forzani 2014; McDonald et al. 2013).

In addition to the traditional student teaching field experience at the culmination of the program, programs commonly include early field experience opportunities for PSTs. Some of these field experience opportunities could be labeled approximations of practice (Grossman et al. 2009), which is an environment of limited complexity for a novice to carry out professional practice. These early field experiences provide PSTs the opportunity to develop their teaching skills and professional judgment through pedagogies of enactment.

Early field experiences, or "on-ramps to professional practice" (Sweeney et al. 2018, p. 671), can take different forms. One example is moving portions of teacher training into school settings where students rehearse specific routines in real classroom settings with real students (McDonald et al. 2013). Another example is simulating such experiences where PSTs enact abbreviated lessons to their peers, who in turn simulate realistic interactions with their peers in the mathematics method classroom (McGarvey and Swallow 1986).

Additional options for early field experiences, or "on-ramps to professional practice" (Sweeney et al. 2018, p. 671), are needed. The circumstances surrounding the global pandemic from COVID-19 has limited access to early field experiences in school settings. In addition, early field experiences in the mathematics method classroom have also been limited due to the need to create socially distanced spaces in reduced-capacity classrooms. Technology can be leveraged to create virtual environments for PSTs to engage in teaching, called VFEs.

16.2 Virtual Field Experiences

VFEs are a digital intervention that can be used to provide PSTs opportunities to practice teaching, or interact within the community of practice (Lave and Wenger 1991). Sweeney et al. (2018) defines VFEs as "a mechanism that mediates the practice of teaching and teacher behaviors through interactions with virtual students" (p. 677). Further, VFEs have the following characteristics:

1. It allows one to practice behaviors and skills related to teaching.
2. It allows the preservice teacher to interact with simulated students who can display a variety of characteristics.
3. It involves the use of digital technologies to aid and facilitate the experience.

(Sweeney et al. 2018, p. 677)

VFEs allow PSTs to gain access to early field experiences to practice the dynamic, complex, and interactive work of teaching, when conditions limit access to classrooms, such as those imposed by COVID-19. Additionally, scholars have argued that VFEs present unique affordances, including the ability to stimulate PSTs' self-assessment of their instructional approach, enabling PSTs to examine responses to uncomfortable interactions, removing of any possible negative impact on real children, and targeting PST learning in desired ways (Dotger et al. 2014; Sweeney et al. 2018).

One unique affordance of VFEs central to this chapter is the opportunity to examine and shape practice toward a desired way of teaching that is more equitable for learners of mathematics, especially those traditionally marginalized in the mathematics classroom. To merely use the VFE as a setting to replicate mathematics teaching practices that are currently enacted in field experiences may just perpetuate preexisting inequities. A better vision of mathematics teaching can be the focal point of the community of practice.

16.2.1 Equitable Teaching Practices

Researchers in the United States have described inequitable access to educational opportunities. Not surprisingly, inequities often lead to low academic achievement in mathematics and underrepresentation in math-related professional fields. Such research and enlightenment has led the major professional organizations in mathematics education to create statements calling for the teaching of mathematics to be reimagined and delivered in ways more equitable for those traditionally marginalized (Darling-Hammond 2010; Oakes 2005).

Gutiérrez (2007) states that equity is achieved, in part, by "being unable to predict students' mathematics achievement and participation based solely upon characteristics such as race class, ethnicity, gender, beliefs, and proficiency in the dominant language" (p. 41). One way student participation can be observed is through classroom discourse.

Reinholz and Shah (2018) developed a classroom observation tool, called EQUIP (Equity QUantified In Participation), that focuses on dimensions of classroom discourse, which are cross-tabulated with demographic markers (e.g., gender, race) to identify patterns of participation within and across lessons. Before the development and use of EQUIP, equitable mathematics discourse investigations predominantly use qualitative methods, such as analysis of interviews, observations, and focus groups (Esmonde and Langer-Osuna 2013; Herbel-Eisenmann et al. 2011; Moschkovich 2011). EQUIP provides investigators and/or practitioners with analytics to support the development of equitable teaching practices in the mathematics classroom, thus, allowing a means for considering the degree of equitable discourse that is occurring in a mathematics classroom. For example, Herbel-Eisenmann and Shah (2019) used EQUIP to investigate implicit biases, or the unconscious attitudes and stereotypes that impact our actions in an unconscious manner (Staats et al. 2016), in teacher questioning.

EQUIP was also used to investigate how the quality of talk and opportunities to participate are distributed across individual students based on race and gender (Bondurant 2020). These studies focused on helping educators identify, acknowledge, and address the biases that influenced their teaching practices. The teacher-researchers in these studies learned through EQUIP-generated analytics who needed to participate more, and they were able to incorporate new practices to include students in discussions in high-quality ways to mitigate biases. The use of EQUIP helped reveal the PSTs' implicit biases, and once the implicit biases are made explicit, the PST can reflect on their biases and set goals for their practice (Bondurant 2020, Herbel-Eisenmann and Shah 2019).

16.2.2 Research Questions

The problem that this study was designed to address is the lack of opportunities PSTs have to practice the complex skill of teaching toward more equitable outcomes, which became accentuated during the COVID-19 global pandemic. This study was designed to explore the affordances and constraints of VFEs and their ability to influence PSTs efficacy, skills, and equitable teaching practices of PSTs. The following questions were generated from the review of literature above and guided the design of the study described below.

1. To what degree do PSTs exhibit equitable teaching practices?
2. How can VFEs be leveraged to influence the efficacy and equitable teaching practices of PSTs?

16.3 Methods

This section of the chapter includes a description of the methods used to study this use of a VFE to foster equitable mathematics teaching practices. What follows is a description of the context, participants, and the VFE that was used for this study. Next is a description of the data generated from the experience and the methods of analysis that were used in an attempt to address the questions that guided the design of the study.

This mixed methods study explored the impact of VFEs on the efficacy, skills, and equitable teaching practices of PSTs. Herbel-Eisenmann and Shah (2019) suggest using a mixed methods approach and point out that equity analytics can provide a useful complement to qualitative analyses.

The authors understand that the small number of participants limits the claims that can be made based on this study of practice. Instead, the authors perceive this chapter as an instrumental case (Creswell 2013) that not only highlights interesting findings, but also provides a pathway for future uses of VFEs that aim to promote equitable teaching practices and the use of freely available tools to promote and interrogate such teaching practices.

16.3.1 Context and Participants

The use of a VFE was added to the lead author's Methods of Teaching Secondary Mathematics and Directed Teaching Internship courses during the fall of 2019 semester (before the COVID-19 pandemic) to supplement the in-person field experiences. The researchers were interested in seeing how the VFEs could be used as a setting for PSTs to develop as teachers and if it was worthwhile to permanently add them to the courses in question. Demographic information about the participants can be found in Table 16.1. Analysis will focus primarily on the PSTs enrolled in the Methods course, because a more comprehensive set of data was collected from

Table 16.1 PST demographic information

Name	Race	Gender	Course
Kayla	White	Female	Methods
Ken	Black	Male	Methods
Angel	Black	Female	Student teaching
Polly	White	Female	Student teaching

Note. Pseudonyms were used to protect the identities of the PST participants.

those participants. When available, data from the PSTs enrolled in Directed Teaching Internship will be used to create a better picture of the findings/implications of this work.

16.3.2 Setting

As stated above, a VFE is defined as "a mechanism that mediates the practice of teaching and teacher behaviors through interactions with virtual students" (Sweeney, Milewski, & Amidon, p. 677). In the case of this context, Mursion™ was used as the VFE platform in which PSTs interact with virtual students. Prior to the individual virtual simulation session, with the support of the mathematics teacher educator (MTE), the PSTs planned a short discussion among five student avatars on a topic of their choice. The PST then had 20 minutes in the simulated classroom to guide the five student avatars in discussing their ideas and coming to consensus around specific ideas. Behind the scenes, the five student avatars in the simulated classroom are controlled by a human-in-the-loop called a simulation specialist, who uses voice modulation and other technology to sound and move like secondary students and to respond in real time, creating a realistic experience for the PST.

Each of the five secondary student avatars has a designed character profile that the simulation specialist is trained to consistently enact within each simulation. According to Mursion™, the avatars are designed to be racially ambiguous (Mursion™, personal communication, February 10, 2020). Further, the racial/ethnic background of each character is not explicitly defined in their character profiles. This provides the end users of the platform the opportunity to develop the character's profile to include these characteristics, in service of the simulations they are running. Due to the racial ambiguity of the avatar students, PSTs were asked to share their perceptions of each avatar's race/ethnicity, and the PSTs' perceptions of the avatars' races/ethnicities were used as the basis of the EQUIP coding and analytics.

In each VFE session, the PST facilitated the instruction of a task that they planned which included two problems. They explained the first problem using the think-aloud strategy. For the second problem, they called on students to guide them through how to solve the problem. The enactor was not provided with the problem

Fig. 16.1 Components of Mursion™ simulation in the current study. ©Mursion, 2021

before the sessions and did not have a script. The PSTs in Methods had three VFE sessions, each lasting 10–20 minutes, while the PSTs who were in Directed Teaching Internship had one VFE session. In addition, both groups of PSTs engaged in a simulation where they practiced facilitating a parent-teacher conference. The parent-teacher conference portion of the simulation falls outside the bounds of this chapter.

16.3.3 Data Generation

Analytics of student avatar participation from the EQUIP observation tool were used to determine the degree that PSTs exhibited equitable teaching practices. A survey was created to gather the observed race/ethnicity and gender perceptions of the student avatars by PSTs in order to better interpret the EQUIP analytics. In addition, the scores and sub-scores of the Mathematics Teaching Efficacy Belief Instrument (Enochs et al. 2010), as well as a general impressions survey and interview were used to address how VFEs could be leveraged to influence the efficacy and equitable teaching practices of PSTs.

Perceptions of Student Avatars

PSTs were shown the Mursion™-provided descriptions and images of each of the student avatars and asked to identify the gender and race/ethnicity of each. Again, the race/ethnicity and gender of the avatars are not revealed in the Mursion™-provided descriptions. This survey of perceptions was done so that participation within the VFE by the avatars was marked in EQUIP according to the perceptions of the PST and not the perceptions of the enactor and/or the authors. We understand that the categories of identification are limiting and may not promote the best understanding of the complexity of identity but for the purposes of analysis we decided to include this in the study (Tatum 1992; Reinholz et al. 2019). This is an issue the authors struggle with and address below in the discussion section of the chapter.

EQUIP

Reinholz and Shah (2018) created EQUIP, a customizable observation tool, as a research tool for tracking patterns of participation in mathematics classrooms. The EQUIP observation tool has been developed into a free, customizable web application (see https://www.equip.ninja).

The EQUIP instrument includes seven default discourse dimensions (see Table 16.2), each supported by prior research (Reinholz and Shah 2018). One goal for PSTs was to avoid initiation-response-evaluation (IRE) discourse patterns. Requiring students to justify their answers promotes equity (Gutiérrez 2007). When

Table 16.2 Dimensions of EQUIP

Dimension	Levels
Discourse type	Content Logistics
Student talk length	21 or more words 5–20 words 1–4 words
Student talk type	Why How What Other
Teacher solicitation method*	Random Called on Not called on
Wait time	More than 3 seconds Less than 3 seconds N/A
Teacher solicitation type*	Why How What Other N/A
Explicit evaluation	Yes No

Note. Teacher solicitation type and teacher solicitation method were the two dimensions focused on in this study

students justify their answers, they gain access to the mathematics behind the answers. Moreover, the students who justify their answers are positioned as thinkers and doers of mathematics (Beida and Staples 2020). To investigate PSTs' practices, we focused on the following discourse dimensions: teacher solicitation type and teacher solicitation method. We also looked at the percent of virtual students that participated overall and by gender, perceived race/ethnicity, and the intersectionality of those dimensions.

The forms of the teacher solicitation type dimension are why, how, what, other, and N/A. Henningsen and Stein (1997) found that the cognitive demand of a task is dependent on the teacher's questioning. Lower-level questions, such as "what" questions, can remove the cognitive demand of a task. On the other hand, higher-level questions, such as "why" and "how" questions, raise the cognitive demand of a task because they require deeper student thinking (Boyd and Rubin 2002). Therefore, the goal was to have the PSTs ask more "why" and "how" questions and less "what" questions.

The levels of the teacher solicitation method dimension are random, not called on, and called on. Teachers solicit participation from students in a variety of ways. An important note is that "random" means that a randomization method (e.g., a random name generator using the class roster) was used to call on students. According to Tanner (2013), randomization methods can help ensure equal participation. Randomization methods also show students that contributions from all members in the class are valued. If explicit methods to promote equal participation are not used, teachers can subconsciously let some students dominate the discussion. Teachers may call on certain subgroups more based on implicit biases (Sadker et al. 2009; Staats et al. 2016). Lastly, students may participate without the teacher calling on them. Although this could be interpreted as a classroom where students have agency to take ownership over their own learning (Engle 2012), the lack of an official system for soliciting participation can result in inequitable distributions of participation opportunities.

The EQUIP observation tool is not designed to evaluate educators. Rather it provides a starting place for deeper conversations about race, gender, and other social markers and how they play out in the classroom. Also, the analytics do not prescribe how an educator should teach. There is no "target distribution" for EQUIP analytics. EQUIP does not establish a particular goal, such as equal participation for all students. It is up to the educator to make sense of the data and what they will do with them, based on how they conceptualize "equity."

MTEBI

The Mathematics Teaching Efficacy Belief Instrument (MTEBI) is an instrument used to measure the connection between beliefs and behaviors for preservice mathematics teachers (Enochs et al. 2010). The MTEBI consists of two subscales, personal mathematics teaching efficacy (SE) and mathematics teaching outcome expectancy (OE). OE measures the belief that specific behaviors by mathematics teachers result in desirable outcomes, and SE measures the belief that the mathematics teacher has the ability to execute those specific behaviors. PSTs who were enrolled in Methods (Kayla and Ken) completed the MTEBI before their first VFE session and then after their last VFE session.

General Impressions of VFE

We also surveyed and interviewed the PSTs after each experience. Participants were asked the following questions (items 1–5 were Likert scale survey questions, and item 6 was an open-ended interview question):

1. How authentic (like teaching in the real world) was the simulation you just participated in?
2. How much did the simulation you just participated in help you improve your classroom management skills?
3. How much did the simulation you just participated in help you improve your questioning skills?
4. How much did the simulation you just participated in help you improve your content knowledge?

5. How likely are you to recommend using the simulation to a peer (another preservice teacher)?
6. Please share your impressions of the experience you just had in the simulator. What went well? What did not go well? Were you nervous?

16.3.4 Analysis

The methods of analysis described below are meant to provide a response to the research questions that were used in the design of this study. Also, the methods described below are meant to be approachable by MTEs and PSTs as a way to examine practice within the context of a VFE.

Perception of Student Avatars and Equitable Teaching Practices

To investigate PSTs' equitable teaching practices, we focused on the following discourse dimensions: teacher solicitation type (why, how, what, other, N/A) and teacher solicitation method (random, called on, not called on). We also looked at the percent of student avatars that participated overall and by the PSTs' perception of the student avatar's gender and race. The amount of participation for each of those broad categories was compared to the percentage of the class represented by each of those categories to understand if there was an over- or underrepresentation of participation from any of the groups or the intersection of groups.

MTEBI

The MTEBI sub-scores were calculated for each instance the instrument was administered to each of the PSTs enrolled in the Methods course. Changes in efficacy were calculated by identifying the difference between the sub-scores from the first and second administrations of the instrument. MTEBI subscales, SE and OE, were calculated and compared to the previous administration of the MTEBI for the same PST. MTEBI subscales were not compared across PSTs to avoid confusing differences in interpretation of MTEBI items to differences in beliefs (Kieftenbeld et al. 2011).

General Impressions of VFE

A grounded theory approach was used to code the interview responses (Strauss and Corbin 1990). Initially, each word, line, or segment of the transcribed data was coded and then examined for themes. Subsequently, the initial codes were collapsed into the most significant and frequently occurring initial codes. Finally, the data was sorted and synthesized according to the finalized dominant codes.

16.4 Findings

The findings below are meant to provide responses to the research questions that guided the design of this examination of teaching practice within a VFE. In review, the research questions are:

1. To what degree do PSTs exhibit equitable teaching practices?
2. How can VFEs be leveraged to influence the efficacy and equitable teaching practices of PSTs?

16.4.1 Perceptions of Student Avatars

The PSTs were provided with the Mursion™ descriptions of the student avatars and asked to identify the race/ethnicity and gender perceptions of each of the student avatars (see Table 16.3). Again, this was done to gauge the perceptions of the PSTs in order to categorize participation within EQUIP based on the PST perception and not the perception of the enactor or the authors.

16.4.2 Equitable Teaching Practices

To investigate PSTs' equitable teaching practices, attention was given to the EQUIP dimensions: teacher solicitation type and teacher solicitation method. We also looked at the percent of students (avatars) that participated overall and by gender and perceived race/ethnicity. Across these measures, findings suggest that PSTs with more field experiences exhibited more equitable teaching practices. Those with more field experiences, the students in Directed Teaching Internship versus Methods students, the first session versus the second session, and PSTs who identify as Black versus those who identify as White exhibited more equitable teaching practices. These more equitable practices included greater student participation (overall and by gender and race/ethnicity), greater equitable distribution of participation, and greater percentage of high-level questions. We also found that intersectionality of gender and race/ethnicity played into the participation. We saw an overrepresentation of participation by avatars identified as White. We also observed an underrepresentation of participation by avatars identified as non-White and female.

Table 16.3 PSTs' perceptions of student avatars

Student avatar	PSTs				
	Personality, interests	Kayla	Ken	Angel	Polly
 Jasmine	Empathic, idealistic, very bright, naturally does well in school, passionate about certain subjects (especially science), people pleaser, always seeking acceptance, thinks Ava is her best friend, but Ava may disagree	Asian Female	White Female	Black Female	White Female
 Ethan	Extroverted, sensitive, attention and approval seeking, always angling for a laugh, difficulty paying attention when work is detailed, often volunteers, doesn't mind stepping out of his comfort zone	White Male	White Male	White Male	White Male
 Savannah	Loyal, introverted, potential to take criticism personally, excellent memory and ability to recall data, trouble connecting socially with classmates, works best on her own or with a friend	White Female	White Female	White Female	White Female
 Dev	Enthusiastic, intelligent, enjoys working through a challenge, potential to be condescending, relaxed, friendly, especially with peers with similar interests	Asian Male	White Male	White Male	Hispanic Male
 Ava	Quick thinking, decisive, enjoys leadership roles, dislikes slowing down because someone does not get it, charming to the teacher, but classmates know better, she can be condescending or sarcastic	Black Female	Black Female	Hispanic Female	Hispanic Female

16.4.3 Efficacy

The MTEBI was administered to Kayla and Ken before their first VFE and again after their fourth VFE. Again, the MTEBI consists of two subscales, personal mathematics teaching efficacy (SE) and mathematics teaching outcome expectancy (OE). For the OE subscale, or the belief that specific behaviors by mathematics teachers result in desirable outcomes, Kayla also showed a slight decrease in her OE scale score (from 32 to 30), and Ken also showed a more pronounced increase in his OE scale score (from 28 to 34). For the SE subscale, or the belief that the mathematics teacher being evaluated has the ability to execute those specific behaviors that result in desirable outcomes, Kayla showed a slight decrease (from 57 to 53) in her SE scale score, and Ken showed a slight increase (from 54 to 57) in his SE scale score (Fig. 16.2).

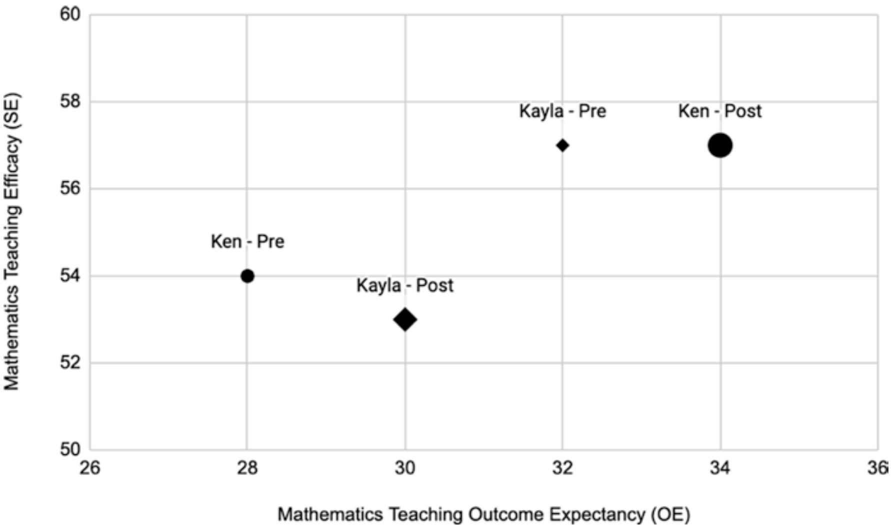

Fig. 16.2 Pre-/post-MTEBI subscale scores for methods PSTs

Table 16.4 PSTs general impressions of VFEs

	Polly		Angel		Kayla			Ken			μ	σ
How authentic (like teaching in the real world) was the simulation you just participated in?	4	5	4	5	4	4	5	5	5	5	4.6	0.52
How much did the simulation you just participated in help you improve your classroom management skills?	3	4	5	5	4	4	4	5	5	5	4.4	0.7
How much did the simulation you just participated in help you improve your questioning skills?	3	5	5	5	3	4	4	5	5	5	4.4	0.84
How much did the simulation you just participated in help you improve your content knowledge?	3	3	4	5	4	4	3	5	5	5	4.1	0.88
How likely are you to recommend using the simulation to a peer (another preservice teacher)?	5	5	5	5	4	4	5	5	5	5	4.8	0.42

Note: A five-point Likert scale was used with 1 representing not at all and 5 representing very much.

These beliefs tie to the PSTs understandings about the ability of a mathematics teacher (including the PSTs) to influence outcomes in a mathematics classroom. The next section shares findings related to whether they see the Mursion™ VFE as representative of a real-life mathematics classroom.

Table 16.5 Interview responses

	Polly	Angel	Kayla	Ken	Only student teacher[a]	Only method students[a]	Total[a]
Pedagogical content knowledge & skills	1	0	1	1	1/4	2/6	3/10
Behavior management	1	1	2	1	2/4	3/6	5/10
Realistic	2	1	2	1	3/4	3/6	6/10
Felt nervous	2	2	1	1	4/4	2/6	6/10
Safe to make mistakes	0	1	0	0	1/4	0/6	1/10
Valued extra practice	1	2	1	0	3/4	1/6	4/10
Watching peers helpful	0	1	0	0	1/4	0/6	1/10
Appreciated MTE feedback afterward	0	1	0	0	1/4	0/6	1/10

[a]The number of times mentioned per interview divided by the number of interviews

16.4.4 General Impressions of VFE

PSTs general impressions of the VFEs were gathered through surveys (Table 16.4) and interviews (Table 16.5). Survey data indicated that PSTs consistently had strong positive impressions of the VFEs. PSTs considered the experiences very authentic ($\mu = 4.6$, $\sigma = 0.52$). They also found the VFEs very helpful for improving their classroom management skills ($\mu = 4.4$, $\sigma = 0.7$), questioning skills ($\mu = 4.4$, $\sigma = 0.84$), and content knowledge ($\mu = 4.1$, $\sigma = 0.88$). Finally, PSTs were highly likely to recommend using the VFEs to a peer ($\mu = 4.8$, $\sigma = 0.42$).

In addition to the survey, PSTs were asked to share their impressions of the VFE after each session. Dominant themes included comments about how realistic the experience was and how nervous they felt (each mentioned in 60% of interviews). Comments about behavior management of the avatars, how much they valued the extra practice, and how the experience helped improve their pedagogical content knowledge and skills were also common (mentioned in 50%, 40%, and 30% of the interviews, respectively). Topics mentioned once included that the PST felt safe making mistakes and benefited from both watching their peers and from receiving feedback from the MTE after the session.

16.5 Discussion

16.5.1 Limitations

In this section we aim to discuss the limitations of this study. The main limitation of our study is that it included only four PSTs; therefore, it is not reasonable for us to assume our findings are generalizable. We acknowledge that more research is needed to confirm our findings and to develop repeated VFE interventions to address the issue of inequitable practices.

16.5.2 Implications

In this section we aim to discuss the implications of this instrumental case (Creswell 2013) for MTEs. We see this instrumental case as representative of how MTEs can use VFEs to supplement culminating field experience placements as a means for developing equitable teaching practices in the mathematics classroom. Survey and interview information from PSTs indicated that they consistently had strong positive impressions of the VFEs and recognized the experience as a place to execute and improve practice. PSTs considered the experiences authentic; helpful for improving their classroom management skills, questioning skills, and content knowledge; and were highly likely to recommend them to a peer. These results suggest that PSTs enjoy and benefit from VFEs. Based on the positive results from this and other related studies, we anticipate an increase in the use of simulations in PST preparation programs during the COVID-19 pandemic, but also continuing afterward.

This study has potential implications on the design of the simulation learning environment. It is important to critically consider the impact of the design features of the environment (see Sweeney et al. 2018). One area to consider is the demographic ambiguity of avatars. This led PSTs to have different impressions of the race/ethnicity of the avatars. MTEs may wish to dial up or down the level of racial ambiguity depending on their goals (see "Authorable Students," Sweeney et al. 2018).

Another important consideration is the design of the enactor training. The researcher may wish to incorporate training for the enactor specific to their area of interest. For example, for this study, we would have liked to provide the enactor with training on equitable teaching practices and racial and gender biases. Most PSTs did not direct their questions at a specific student. In this situation interactors are trained to take turns responding to the teacher. Particular avatars are trained to exhibit particular behaviors (sleep, play with cell phone) when they are not called on. Depending on MTE goals, it would be beneficial to have interactors respond to questions that are not directed at a specific student in a manner that is realistic/authentic. Future research on VFEs could examine the patterns of participation, using such a tool as EQUIP, that are exhibited by the enactor during instances of freedom in the simulation. This examination of patterns may reveal biases, or stereotype reinforcement, that may lead to the perpetuation of prejudice rather than the reinforcement of equitable teaching practice. Finally, in future studies, it would be informative to ask both the PSTs and interactor to complete targeted Harvard Implicit Bias surveys to see if associations between beliefs and actions exist.

We found that most students directed a low percentage of their questions at a specific student (by calling on them) and asked a low percentage of questions that required justification. These findings suggest that these are two areas that MTE may need to place more emphasis on in the preparation of PSTs. Interestingly, the students of color and students with more field experiences directed a greater percentage of their questions at a specific student and asked more questions that required justification. This suggests their peers could benefit from observing them teach and having discussions with them about why and how they made these instructional decisions.

All four PSTs exhibited preferential treatment toward White males. However, PSTs of color and PSTs with more field experiences exhibited more equitable practices. These findings suggest that it is critically important to provide PSTs with ongoing field experiences. Additionally, there is a definite need for MTEs to provide PSTs with explicit instruction in equitable teaching practices. More research is needed to determine best practices for teaching PSTs to use equitable teaching practices. Given that PSTs of color exhibited more equitable teaching practices, it seems their peers may benefit from observing their teaching and having discussions regarding why and how to use equitable teaching practices.

Regarding PSTs' efficacy, findings were inconsistent. The PST that had a higher GPA and consistently scored higher on teaching observations showed a slight decrease in her MTEBI scores. We believe that this PST's MTEBI scores could have decreased, because the realistic simulations exposed the PST to the complexity of teaching. Interview data support this theory, as the PST shared that she found the simulations realistic and challenging. This finding suggests that higher-performing students' self-efficacy may decrease after VFEs. On the other hand, the PST with a lower GPA who consistently scored lower on teaching observations showed an increase in his MTEBI scores. We believe that this PST's MTEBI scores could have increased, because he was less self-critical; therefore, the experiences built up his confidence and provided him with confirmation regarding his beliefs about his abilities.

As it was said previously, the small number of participants does not allow us to generalize the findings in this illustrative case to a broader audience, but the findings are still meaningful. Each participant who engaged in the VFE is a teacher whose practices will influence the relationship with mathematics of a generation of students. Considering how to best leverage the tools available is a challenge MTEs need to take on as we work to develop the best teachers of mathematics we can.

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