Expansion by Regions: An Overview

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Abstract A short review of expansion by regions is presented. It is a well-known strategy to obtain an expansion of a given multiloop Feynman integral in a given limit where some kinematic invariants and/or masses have certain scaling measured in powers of a given small parameter. Prescriptions of this strategy are formulated in a simple geometrical language and are illustrated through simple examples.

1 Historiographical Notes

Expansion by regions is a universal strategy to obtain an expansion of a given Feynman integral in a given limit, where kinematic invariants and/or masses essentially differ in scale. For simplicity, let us consider a Feynman integral $G_\Gamma(q^2, m^2)$ depending on two scales, for example, q^2 and m^2, and let the limit be $t = -m^2/q^2 \to 0$. Experience tells us that the expansion at $t \to 0$ has the form

$$G_\Gamma(t, \varepsilon) \sim (-q^2)^\omega \sum_{n=n_0}^{\infty} \sum_{k=0}^{2h} c_{n,k}(\varepsilon)\, t^n \, \log^k t \, , \tag{1}$$

where $\omega = 4h - 2\sum a_i$ is the degree of divergence, with a_l powers of the propagators, h is the number of loops and $\varepsilon = (4-d)/2$ is the parameter of dimensional regularization. The expansion is often called asymptotic, in the sense that the remainder of expansion has the order $o(t^N)$ after keeping terms up to t^N. However, every power series at a power of logarithm in expansions in various limits of momenta and masses has a non-zero radius of convergence which is determined usually by the nearest threshold.

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There can be different reasons to consider some limit and the corresponding expansion. Typically, different scaling of kinematic invariants and/or masses involved is dictated by a phenomenological situation. Moreover, experience obtained when expanding Feynman integrals in some limit can show a way to construct the corresponding effective theory. At the level of individual Feynman integrals, expanding a complicated Feynman integral in some limit can approximately substitute the analytic evaluation of the integral.

One can use various techniques in order to obtain an expansion of a given Feynman integral in some limit: one can start with a parametric representation, or apply the method of Mellin–Barnes representation, or obtain an expansion within the method of differential equations. However, the *general* strategy of expansion by regions provides the possibility to write down a result for the expansion immediately once relevant regions are known. Such a result looks similar to (1) but now exponents of the expansion parameter depending linearly on ε are not yet expanded in ε,

$$G_\Gamma(t, \varepsilon) \sim (-q^2)^\omega \sum_{n=n_0}^{\infty} \sum_{k=0}^{h} \sum_{j=0}^{h} c'_{n,j,k}(\varepsilon) t^{n-j\varepsilon} \log^k t \tag{2}$$

and the coefficients in the expansion can be represented in terms of integrals over loop momenta or over Feynman parameters. These integrals on the right-hand side of the expansion are constructed according to certain rules starting from the Feynman integral or a parametric integral for the initial Feynman integrals G_Γ. This means that expansion by regions reduces the problem to the evaluation of integrals present in (2).

Logarithms in (2) within dimensional regularization do not appear in limits typical of Euclidean space such as the off-shell large momentum limit and the large mass limit. Rather, they are typical for limits typical of Minkowski space such as the Regge limit and various versions of the Sudakov limit. In fact, one can avoid such logarithms by introducing an auxiliary analytic regularization which can be introduced as additional complex numbers in the exponents of the propagators. One can say that, after this, the various scales in the problem become separated so that the expansion becomes only in powers of the expansion parameter. After turning off this regularization, spurious poles in the auxiliary analytic parameters cancel giving rise to the logarithms, and this happens to be an important consistency check. A lot of examples illustrating this phenomenon can be found, e.g., in [1]. We will come back to this point in Section 2 when discussing the geometrical formulation of expansion by regions.

According to the first formulation of expansion by regions [2] one analyzes various regions in a given integral over loop momenta and, in every region, expands the integrand in parameters which are there small. Then the integration in the integral with so expanded propagators is extended to the whole domain of the loop momenta and, finally, one obtains an expansion of the given integral as the corresponding sum over the regions. Although these recipes were formulated in a

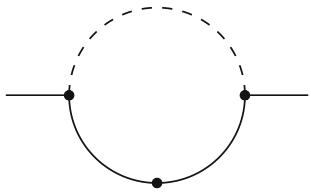


Fig. 1 A one-loop graph

suspicious mathematical language, expansion by regions was successfully applied in numerous calculations.

A very simple example is given by the Feynman integral corresponding to the graph depicted in Fig. 1,

$$G(q^2, m^2; d) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2 (q-k)^2} \tag{3}$$

in the limit $m^2/q^2 \to 0$.

The relevant regions are the region of small loop momenta, $k \sim m$, and the region of large loop momenta, $k \sim q$. According to the above prescriptions, in the first region, the first propagator is unexpanded and the second propagator is expanded in a Taylor series in k. In the second region, the first propagator is expanded in a Taylor series in m the second propagator is unexpanded. The leading terms of expansion give

$$G(q^2, m^2; d) \sim \int \frac{\mathrm{d}^d k}{(k^2)^2 (q-k)^2} + \frac{1}{q^2} \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^2} + \dots \tag{4}$$

The integrals involved can be evaluated by Feynman parameters, with the following result

$$G(q^2, m^2; d) \sim i\pi^{d/2} \left(\frac{\Gamma(1-\varepsilon)^2 \Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)(-q^2)^{1+\varepsilon}} + \frac{\Gamma(\varepsilon)}{q^2 (m^2)^{\varepsilon}} + \dots \right) \tag{5}$$

Although the initial Feynman integral is finite at $d = 4$, there are simple poles above: an infrared pole in the first term and an ultraviolet term in the second term. They are successfully canceled, with the following result

$$i\pi^{d/2} \left(\log \left(\frac{-q^2}{m^2} \right) + \dots \right) . \tag{6}$$

Such an interplay of various divergences is a typical feature of expansions in momenta and masses. Only in rare situations, such as an expansion in the small momentum limit of a Feynman integral without massless threshold in the corresponding channel, there is no such phenomenon. Let me also point out that the first term in (4) is convergent at $\mathrm{Re}(\varepsilon) < 0$ while the second term in (4) is convergent at $\mathrm{Re}(\varepsilon) > 0$. This can be seen from an analysis of convergence of the corresponding integrals over Feynman parameters. Thus, there is no domain in the complex plane of ε where both terms are given by convergent integrals. In fact, using auxiliary subtraction operators, it is possible to write down the result of expansion in such a way that both terms on the right-hand side will be convergent in some domain of ε. However. I prefer to follow the prescription which is implied in practice: to evaluate every term in the result for expansion in a domain of ε where it is convergent and then analytically continue the corresponding result to some desirable domain.

Expansion by regions has the status of experimental mathematics. Usually, when studying a given limit, one starts from one-loop examples, checks results by independent methods and, finally, one understands which regions are relevant to the limit and that one obtains reliable expansion within this strategy. Beneke provided a one-parametric example showing explicitly how expansion by regions works. The example was used in Chapter 3 of [1]. Guided by this example, Jantzen [3] provided detailed explanations of how this strategy works in several two-loop examples by starting from regions determined by some inequalities and covering the whole integration space of the loop momenta, then expanding the integrand and then extending integration and analyzing all the pieces which are obtained, with the hope that 'readers would be convinced that the expansion by regions is a well-founded method'.

However, there is an important class of limits for which there is a mathematical proof. These are limits typical of Euclidean space: for example, the off-shell large momentum limit and the large mass limit. In [4] (see also Appendix B of [1]) that the remainder of such expansion constructed with the help of an operator which has the structure of the R-operation (i.e. renormalization at the diagrammatical level) has the desirable order with respect to the parameter of expansion. This proof was for a general h-loop graph. It was similar to proofs of results on the R-operation and was based on sector decompositions and a resolution of singularities in parametric integrals, with power counting of sector variables.

For this class of limit, the expansion of a given Feynman integral corresponding to a graph Γ is given [4–6] (see also [7] and Chapter 9 of [8]) by the following simple formula:

$$G_\Gamma \sim \sum_\gamma G_{\Gamma/\gamma} \circ \mathcal{T}_{q_\gamma, m_\gamma} G_\gamma \ . \tag{7}$$

which is written for the off-shell large-momentum limit, i.e. where a momentum Q is considered large and momenta q_i as well as the masses m_j are small. The sum runs over subgraphs γ of Γ which can be called asymptotically irreducible (AI): they are one-particle irreducible after identifying the two external vertices associated with the large external momentum Q. Moreover, $\mathcal{T}$ is the operator of Taylor expansion in internal masses and external momenta of a subgraph γ, the symbol $\circ$ means the insertion of the polynomial obtained after this Taylor expansion into the vertex of the reduced graph Γ/γ to which γ is reduced.

In the case of limits typical of Euclidean space, there is a natural one-to-one correspondence between AI subgraphs and regions in the description of the expansion within expansion by regions, so that we obtain an indirect justification of expansion by regions for such limits. The set of relevant regions exactly corresponds to the set of AI subgraphs. There are two kind of regions for each loop momentum: small and large. For a given AI subgraph γ, the corresponding region is defined by considering each loop momentum of γ as large and the rest of the loop momenta of Γ (i.e. loop momenta of Γ/γ) as small. For example, two subgraphs are AI for Fig. 1: the graph Γ and the subgraph consisting of the massless line. As a result, we obtain the same contributions as above.

For limits typical of Minkowski space, to reveal the set of relevant regions is not so simple. For example, for the threshold limit in the case where the threshold in the q channel is at $q^2 = 4m^2$ and the small expansion parameter is introduced by $y = m^2 - q^2/4 \to 0$, the following four kind of regions for a loop momentum are relevant [2]:

$$\text{(hard)},\ \ k_0 \sim \sqrt{q^2}\,,\ \ \mathbf{k} \sim \sqrt{q^2}\,,$$

$$\text{(soft)},\ \ k_0 \sim \sqrt{y}\,,\ \ \mathbf{k} \sim \sqrt{y}\,,$$

$$\text{(potential)},\ \ k_0 \sim y/\sqrt{q^2}\,,\ \ \mathbf{k} \sim \sqrt{y}\,,$$

$$\text{(ultrasoft)},\ \ k_0 \sim y/\sqrt{q^2}\,,\ \ \mathbf{k} \sim y/\sqrt{q^2}\,.$$

where $q = (q_0, \mathbf{0})$.

An alternative version of expansion by regions was formulated and illustrated via examples in [9] within the well-known Feynman parametric representation. This representation in the case of propagators with $-k^2$ propagators with general indices a_i (powers of the propagators) is

$$G(q_1, \ldots, q_n; d) = \left(i\pi^{d/2}\right)^h \frac{\Gamma(\sum a - hd/2)}{\prod_i \Gamma(a_i)}$$
$$\times \int_0^\infty \ldots \int_0^\infty \delta\left(\sum x_i - 1\right) \prod x_i^{a_i-1}\, U^{a-(h+1)d/2} F^{hd/2-a} \mathrm{d}x_1 \ldots \mathrm{d}x_n \quad (8)$$

where n is the number of lines (edges), $a = \sum a_i$, h is the number of loops of the graph,

$$F = -V + U \sum m_l^2 x_l\ , \quad (9)$$

and U and V are two basic functions (Symanzik polynomials, or graph polynomials) for the given graph,

$$U = \sum_{T \in T^1} \prod_{l \notin T} x_l\ , \quad (10)$$

$$V = \sum_{T \in T^2} \prod_{l \notin T} x_l \left(q^T\right)^2 . \tag{11}$$

In (10), the sum runs over trees of the given graph, and, in (11), over *2-trees*, i.e. subgraphs that do not involve loops and consist of two connectivity components; $\pm q^T$ is the sum of the external momenta that flow into one of the connectivity components of the 2-tree T. The products of the Feynman parameters involved are taken over the lines that do not belong to a given tree or a 2-tree T. As is well known, one can choose the sum in the argument of the delta-function over any subset of lines. In particular, one can choose just one Feynman parameter, x_l, and then the integration will be over the other parameters at $x_l = 1$. The functions U and V are homogeneous with respect to Feynman parameters, with the homogeneity degrees h and $h+1$, respectively.

One can consider quite general limits for a Feynman integral which depends on external momenta q_i and masses and is a scalar function of kinematic invariants and squares of masses, s_i, and assume that each s_i has certain scaling ρ^{κ_i} where ρ is a small parameter.

An algorithmic way to reveal regions relevant to a given limit was found in [10]. It is based on the geometry of polytopes connected with the basic functions U and F in (8). This was a real breakthrough, both in theoretical and practical sense because, on the one hand, it became possible to formulate expansion by regions in an unambiguous mathematical language and, on the other hand, the authors of [10] presented also a public code `asy.m` which was later successfully applied in various problems with Feynman integrals.

Ironically, this algorithm and the code didn't find, in this first version, the potential region for the threshold expansion. Later, this algorithm was updated and, in its current version, it can reveal potential region as well as Glauber region. This was done by introducing an additional decomposition of the integration domain and introducing new variables. Consider, for example, one-loop diagram with two massive lines in the threshold limit $y = m^2 - q^2/4 \to 0$

$$G(q^2, y) = i\pi^{d/2}\,\Gamma(\varepsilon) \int_0^\infty \int_0^\infty \frac{(x_1+x_2)^{2\varepsilon-2}\,\delta(x_1+x_2-1)\;\mathrm{d}x_1\mathrm{d}x_2}{\left[\frac{q^2}{4}(x_1-x_2)^2 + y(x_1+x_2)^2 - i0\right]^\varepsilon} . \tag{12}$$

The code `asy.m` in its first version revealed only the contribution of the hard region, i.e. $x_i \sim y^0$. To make the potential region visible, let us decompose integration over $x_1 \le x_2$ and $x_2 \le x_1$, with equal contributions. In the first domain, let us turn to new variables by $x_1 = x_1'/2,\ x_2 = x_2' + x_1'/2$ and arrive at

$$i\pi^{d/2}\,\frac{\Gamma(\varepsilon)}{2} \int_0^\infty \int_0^\infty \frac{(x_1+x_2)^{2\varepsilon-2}\,\delta(x_1+x_2-1)\;\mathrm{d}x_1\mathrm{d}x_2}{\left[\frac{q^2}{4}x_2^2 + y(x_1+x_2)^2 - i0\right]^\varepsilon} .$$

Now we observe two regions with the scalings (0, 0) and (0, 1/2). The second one, with $x_1 \sim y^0$, $x_2 \sim \sqrt{y}$, gives

$$i\pi^{d/2}\frac{\Gamma(\varepsilon)}{2}\int_0^\infty \frac{\mathrm{d}x_2}{\left(\frac{q^2}{4}x_2^2+y\right)^\varepsilon} = i\pi^{d/2}\frac{1}{2}\Gamma(\varepsilon-1/2)\sqrt{\frac{\pi y}{q^2}}y^{-\varepsilon} \ .$$

Taking into account that we have two identical contributions after the above decomposition, we obtain a result for the potential contribution equal to the previous expression with omitted 1/2.

Observe that the expression for the function F in the Feynman parametric representation is non-negatively defined and only some individual terms are negative but this brings problems when looking for potential contributions. In the current version [11] of the code `asy.m`, one can get rid of the negative terms due to additional decompositions and introduction of new variable. Let me emphasize that this code can work successfully also in situations with a function F not positively defined even without additional decompositions—see, e.g. [12, 13].

For completeness, let me refer to [14, 15] where two specific ways of dealing with expansion by region were applied.

Let us realize that the very word 'region' is used within the strategy under discussion in a physical rather mathematical way. By region, we mean some scaling behaviour of parameters involved. I will present expansion by regions in a mathematical language in the next section using another form of parametric representation, rather than (8) and illustrate it through simple examples.

2 Geometrical Formulation

Lee and Pomeransky [16] have recently derived another form of parametric representation which turns out to be preferable in certain situations

$$G(q_1,\ldots,q_n;d) = \left(i\pi^{d/2}\right)^h \frac{\Gamma(d/2)}{\Gamma((h+1)d/2-a)\prod_i\Gamma(a_i)} \times\int_0^\infty\ldots\int_0^\infty\prod_i x_i^{a_i-1}P^{-\delta}\mathrm{d}x_1\ldots\mathrm{d}x_n \ , \qquad (13)$$

where $\delta = 2-\varepsilon$ and $P = U + F$. One can obtain (8) from (13) by [16] inserting $1 = \int\delta(\sum_i x_i - \eta)\mathrm{d}\eta$, scaling $x \to \eta x$ and integrating over η.

The parametric representation takes now a very simple form: up to general powers of the integration variables, there is only one polynomial raised to a general complex power. I believe that the fact that this function is the sum of the two basic functions in Feynman parametric representation is not crucial and expansion by regions holds for any polynomial.

Let us formulate, following [17], expansion by regions for integral (13) with a polynomial with positive coefficients in the case of limits with two kinematic invariants and/or masses of essentially different scale, where one introduces one parameter, t, which is the ratio of two scales and is considered small. These can be such limits typical of Minkowski space as the Regge limit, with $t \ll s$ and various versions of the Sudakov limit. Then the polynomial in Eq. (13) is a function of Feynman parameters and t,

$$P(x_1, \ldots, x_n, t) = \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}} , \tag{14}$$

where S is a finite set of points $w = (w_1, \ldots, w_{n+1})$ and $c_w > 0$.

By definition, the Newton polytope $\mathcal{N}_P$ of P is the convex hull of the points w in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ equipped with the scalar product $v \cdot w = \sum_{i=1}^{n+1} v_i w_i$. A facet of P is a face of maximal dimension, i.e. n.

The Main Conjecture (Expansion by Regions) The expansion of (13) in the limit $t \to +0$ is given by

$$G(t, \varepsilon) \sim \sum_{\gamma} \int_0^\infty \ldots \int_0^\infty \left[M_\gamma \left(P(x_1, \ldots, x_n, t) \right)^{-\delta} \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n , \tag{15}$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of P, for which the normal vectors $r^\gamma = (r_1^\gamma, \ldots, r_n^\gamma, r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$. Let us normalize these vectors by $r_{n+1}^\gamma = 1$. Let us call these facets *essential*.

The contribution of a given essential facet is defined by the change of variables $x_i \to t^{r_i^\gamma} x_i$ in the integral (13) and expanding the resulting integrand in powers of t. Let us write this procedure explicitly. For a given essential facet γ, the polynomial P is transformed into

$$P^\gamma(x_1, \ldots, x_n, t) = P(t^{r_1^\gamma} x_1, \ldots, t^{r_n^\gamma} x_n, t) \equiv \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w \cdot r^\gamma} . \tag{16}$$

The scalar product $w \cdot r^\gamma$ is proportional to the projection of the point w on the vector r^γ. For $w \in S$, it takes a minimal value for all the points belonging to the considered facet $w \in S \cap \gamma$. Let us denote it by $L(\gamma)$.

The polynomial (16) can be represented as

$$t^{L(\gamma)} \left(P_0^\gamma(x_1, \ldots, x_n) + P_1^\gamma(x_1, \ldots, x_n, t) \right) , \tag{17}$$

where

$$P_0^\gamma(x_1, \ldots, x_n) = \sum_{w \in S \cap \gamma} c_w x_1^{w_1} \ldots x_n^{w_n} , \tag{18}$$

$$P_1^\gamma(x_1,\ldots,x_n,t) = \sum_{w\in S\setminus\gamma} c_w x_1^{w_1}\ldots x_n^{w_n} t^{w\cdot r^\gamma - L(\gamma)} \,. \tag{19}$$

The polynomial P_0^γ is independent of t while P_1^γ can be represented as a linear combination of positive rational powers of t with coefficients which are polynomials of x.

For a given facet γ, let us define the operator

$$\begin{aligned} M_\gamma\left(P(x_1,\ldots,x_n,t)\right)^{-\delta} &= t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t\left(P_0^\gamma(x_1,\ldots,x_n) + P_1^\gamma(x_1,\ldots,x_n,t)\right)^{-\delta} \\ &= t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta}\left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} + \ldots \end{aligned} \tag{20}$$

where $\mathcal{T}_t$ performs an expansion in powers of t at $t=0$.

Comments

- An operator M_γ can equivalently be defined by introducing a parameter ρ_γ, replacing x_i by $\rho^{r_i^\gamma} x_i$, pulling an overall power of ρ_γ, expanding in ρ_γ and setting $\rho_\gamma = 1$ in the end.
- The leading order term of a given facet γ corresponds to the leading order of the operator M_γ^0:

$$\begin{aligned} &\int_0^\infty\ldots\int_0^\infty \left[M_\gamma^0\left(P(x_1,\ldots,x_n,t)\right)^{-\delta}\right] \mathrm{d}x_1\ldots\mathrm{d}x_n \\ &= t^{-L(\gamma)\delta+\sum_{i=1}^n r_i^\gamma}\int_0^\infty\ldots\int_0^\infty \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta}\mathrm{d}x_1\ldots\mathrm{d}x_n \,. \end{aligned} \tag{21}$$

- In fact, with the above definitions, we can write down the equation of the hyperplane generated by a given facet γ as follows

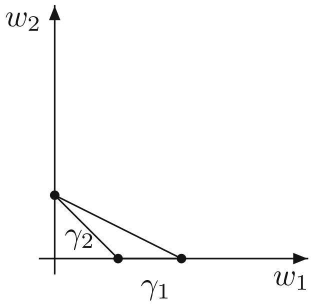
$$w_{n+1} = -\sum_{i=1}^n r_i^\gamma w_i + L(\gamma) \,. \tag{22}$$

- Let us agree that the action of an operator M_γ on an integral reduces to the action of M_γ on the integrand described above. Then we can write down the expansion in a shorter way,

$$G(t,\varepsilon) \sim \sum_\gamma M_\gamma G(t,\varepsilon) \tag{23}$$

- In the usual Feynman parametrization (8), the expansion by regions in terms of operators M_γ is formulated in a similar way, and this is exactly how it is implemented in the code `asy.m` [10]. The expansion can be written in the same form (23) but the operators M_γ act on the product of the two basic polynomials U

Fig. 2 The Newton polytope for (24)



and F raised to certain powers present in (8). Now, each of the two polynomials is decomposed in the form (17) and so on.

- Of course, prescriptions based on representation (13) are algorithmically preferable because the degree of the sum of the two basic polynomials is smaller than the degree of their product UF (previously used in `asy.m`) so that looking for facets of the corresponding Newton polytope becomes a simpler procedure.[1] Therefore, the current version of the code `asy.m` included in `FIESTA` [18] (called with the command `SDExpandAsy` is now based on this more effective procedure.
- It is well known that dimensional regularization might be insufficient to regularize individual contributions to the asymptotic expansion. As it was explained in the discussion after Eq. (2), the natural way to overcome this problem is to introduce an auxiliary analytic regularization, i.e. to introduce additional exponents λ_i to powers of the propagators. This possibility exists in the code `asy.m` [10] included in `FIESTA` [18]. One can choose these additional parameters in some way and obtain a result in terms of an expansion in λ_i followed by an expansion in ε. If an initial integral can be well defined as a function of ε then the cancellation of poles in λ_i in the sum of contributions of different regions serves as a good check of the calculational procedure, so that in the end one obtains a result in terms of a Laurent expansion in ε up to a desired order.

To illustrate the above prescriptions let us consider a very simple example of the integral

$$G(t,\varepsilon) = \int_0^\infty (x^2+x+t)^{\varepsilon-1}\mathrm{d}x \tag{24}$$

in the limit $t \to 0$. The polynomial involved is $P(x,t) = \sum_{(w_1,w_2)\in S} c_{(w_1,w_2)} x^{w_1} t^{w_2}$. The corresponding Newton polytope (triangle) is shown in Fig. 2.

[1] In fact, this step is performed within `asy.m` with the help of another code `qhull`. It is most time-consuming and can become problematic in higher-loop calculations.

There are two essential facets γ_1 and γ_2 with the corresponding normal vectors $r_1 = (0, 1)$ and $r_2 = (1, 1)$. For the facet γ_1, we obtain the contribution given by expanding the integrand in t. In the leading order, we have

$$\int_0^\infty (x^2 + x)^{\varepsilon-1} \mathrm{d}x = \frac{\Gamma(1-2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-\varepsilon)} \ . \tag{25}$$

For the facet γ_2, we obtain t times the integral of the integrand with $x \to tx$ expanded in powers of t. In the leading order, we have

$$t^\varepsilon \int_0^\infty (x+1)^{\varepsilon-1} \mathrm{d}x = -\frac{t^\varepsilon}{\varepsilon} \ . \tag{26}$$

The sum of the two contributions in the leading order gives

$$G(t, \varepsilon) \sim -\log t + O(\varepsilon) \ . \tag{27}$$

Let us now consider again the example of Fig. 1. The two basic functions of Feynman parameters are

$$F = x_1(t(x_1 + x_2) + x_2) \ , \quad U = x_1 + x_2 \ . \tag{28}$$

The set S involved in the definition (14) consists of the vertices

$$A(2, 0, 1), B(1, 1, 1), C(1, 1, 0), D(1, 0, 0), E(0, 1, 0)$$

of the Newton polytope for the polynomial $P = U + F$, as it is shown in Fig. 3.

There are two essential facets. The first one is CDE which belongs to the plane $w_3 = 0$ and has the normal vector $(0, 0, 1)$. It gives the contribution obtained by expanding the integrand in t.

The second essential facet is ACD which belongs to the plane $w_1 - w_3 = 1$ and has the normal vector $(-1, 0, 1)$. It gives $t^{-\varepsilon}$ times the integral

$$\frac{\Gamma(2-\varepsilon)}{\Gamma(1-2\varepsilon)} \int_0^\infty \int_0^\infty x_1 \left[x_1 + x_1^2 + x_1 x_2 + t x_2 + t x_1 x_2 \right]^{\varepsilon-2} \mathrm{d}x_1 \mathrm{d}x_2$$

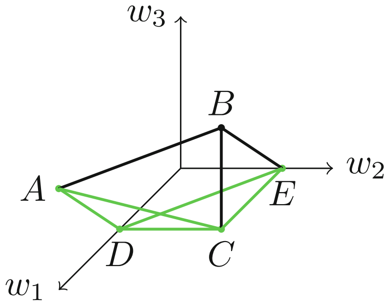


Fig. 3 The Newton polytope for Fig. 1

with the integrand expanded in t. Taking the leading orders in both contributions we reproduce (5).

3 Conclusion

As it was argued in [17], the more general parametric representation (13), with a general polynomial not necessarily related to Feynman integrals, looks mathematically more natural for the proof of expansion by regions. Moreover, first steps of analysis of convergence of integrals (13) were made and expansion by regions was proven in a partial case in the leading order of expansion. Hopefully, expansion by regions will be sooner or later mathematically justified in the case of a general polynomial P.

Practically, expansion by regions is a very important strategy which is successfully applied for several purposes. Let me, finally, point out that one can use expansion by regions in various ways.

- One can apply the code `asy.m` included in `FIESTA` [18] (i.e. the command `SDExpandAsy`) to obtain an expansion in some limit treating all the involved parameters numerically. In particular, one can check analytic results.
- One can use `SDExpandAsy` with the option `OnlyPrepareRegions = True` in order to reveal relevant regions and to construct contributions to the expansion as parametric integrals which can then analytically be evaluated. Here the method of Mellin-Barnes representation can serve as an appropriate additional technique.
- One can study expansion in multiscale limits, applying `asy.m` several times, in various orders.

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