Dirac Structures in Thermodynamics of Non-simple Systems

Hiroaki Yoshimura[1](✉) and François Gay-Balmaz[2]

[1] Waseda University, Okubo, Shinjuku, Tokyo 169-8555, Japan
yoshimura@waseda.jp
[2] Ecole Normale Supérieure, Paris, France
francois.gay-balmaz@lmd.ens.fr

Abstract. We present the Dirac structures and the associated Dirac system formulations for *non-simple* thermodynamic systems by focusing upon the cases that include irreversible processes due to friction and heat conduction. These systems are called non-simple since they involve several entropy variables. We review the variational formulation of the evolution equations of such non-simple systems. Then, based on this, we clarify that there exists a Dirac structure on the Pontryagin bundle over a thermodynamic configuration space and we develop the Dirac dynamical formulation of such non-simple systems. The approach is illustrated with the example of an adiabatic piston.

Keywords: Dirac structures · Non-simple systems · Thermodynamics · Adiabatic piston

1 Variational Formulation of Non-simple Systems

Before exploring Dirac structures underlying the thermodynamics of non-simple systems, we review the variational setting of such non-simple systems by focusing on the internal irreversible processes associated with friction and heat conduction.

1.1 Setting for Thermodynamics of Non-simple Systems

Non-simple Systems with Friction and Heat Conduction. Consider an adiabatically closed system $\mathbf{\Sigma} = \cup_{A=1}^{P} \mathbf{\Sigma}_A$ which consists of P simple thermodynamic systems $\mathbf{\Sigma}_A$, in which we include the irreversible processes due to friction and heat conduction between subsystems. Here a *simple* thermodynamic system

H.Yoshimura is partially supported by JSPS Grant-in-Aid for Scientific Research (17H01097), JST CREST Grant Number JPMJCR1914, the MEXT Top Global University Project, Waseda University (SR 2021C-134) and the Organization for University Research Initiatives (Evolution and application of energy conversion theory in collaboration with modern mathematics).

F. Nielsen and F. Barbaresco (Eds.): GSI 2021, LNCS 12829, pp. 918–925, 2021.
https://doi.org/10.1007/978-3-030-80209-7_98

denotes a system that has only one variable to represent the thermodynamic state, usually denoted by entropy. Since $\mathbf{\Sigma}$ is an interconnected system of simple subsystems $\mathbf{\Sigma}_1, ..., \mathbf{\Sigma}_P$, it becomes a "non-simple" system that has several entropy (or temperature) variables (see [7]) and we note that all the irreversible processes are *internal*. For each simple subsystem $\mathbf{\Sigma}_A$, $A = 1, ..., P$, $S_A \in \mathbb{R}$ indicates its entropy variable. Here, we assume that the mechanical configuration of $\mathbf{\Sigma}$ is given by independent mechanical variables $q = (q^1, ..., q^n) \in Q$, where Q is the mechanical configuration manifold of $\mathbf{\Sigma}$.

Friction, Heat Conduction and External Forces. Let $F^{\mathrm{ext}\to A} : T^*Q \times \mathbb{R}^P \to T^*Q$ be an external force that acts on $\mathbf{\Sigma}_A$ and hence the total exterior force is $F^{\mathrm{ext}} = \sum_{A=1}^{P} F^{\mathrm{ext}\to A}$. Let $F^{\mathrm{fr}(A)} : T^*Q \times \mathbb{R}^P \to T^*Q$ be the friction forces associated with the irreversible processes of each subsystem $\mathbf{\Sigma}_A$, which yield an entropy production for subsystem $\mathbf{\Sigma}_A$. Associated with the heat exchange between $\mathbf{\Sigma}_A$ and $\mathbf{\Sigma}_B$, let J_{AB} be the fluxes such that for $A \neq B$, $J_{AB} = J_{BA}$ and for $A = B$, $J_{AA} := -\sum_{B\neq A} J_{AB}$, where $\sum_{A=1}^{P} J_{AB} = 0$ for all B.

Thermodynamic Displacements. In our formulation, we introduce the concept of thermodynamic displacements, see [3,4]. For the case of heat exchange, we define the *thermal displacements* Γ^A, $A = 1, ..., P$ such that its time rate $\dot{\Gamma}^A$ becomes the temperature of $\mathbf{\Sigma}_A$. We also introduce a new variable Σ_A associated with the internal entropy production.

1.2 Variational Formulation of Non-simple Systems

The Lagrange-d'Alembert Principle for Non-simple Systems. Now we consider a variational formulation of Lagrange-d'Alembert type for non-simple systems with friction and heat conduction, which is a natural extension of Hamilton's principle in mechanics (see [4]).

Given a Lagrangian $L : TQ \times \mathbb{R}^P \to \mathbb{R}$ and an external force $F^{\mathrm{ext}} : TQ \times \mathbb{R}^P \to T^*Q$, find the curves $q(t)$, $S_A(t)$, $\Gamma^A(t)$, $\Sigma_A(t)$ which are critical for the *variational condition*

$$\delta \int_{t_1}^{t_2} \Big[L(q, \dot{q}, S_A) + \dot{\Gamma}^A (S_A - \Sigma_A) \Big] \mathrm{d}t + \int_{t_1}^{t_2} \left\langle F^{\mathrm{ext}}, \delta q \right\rangle \mathrm{dt} = 0,$$

subject to the *phenomenological constraint*

$$\frac{\partial L}{\partial S_A} \dot{\Sigma}_A = \left\langle F^{\mathrm{fr}(A)}, \dot{q} \right\rangle + J_{AB} \dot{\Gamma}^B, \quad \text{for } A = 1, ..., P, \tag{1}$$

and for variations subject to the *variational constraint*

$$\frac{\partial L}{\partial S_A} \delta \Sigma_A = \left\langle F^{\mathrm{fr}(A)}, \delta q \right\rangle + J_{AB} \delta \Gamma^B, \quad \text{for } A = 1, ..., P, \tag{2}$$

with $\delta q(t_1) = \delta q(t_2) = 0$ and $\delta \Gamma^A(t_1) = \delta \Gamma^A(t_2) = 0$, $A = 1, ..., P$.

By direct computations, we obtain the following evolution equations:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} = \dfrac{\partial L}{\partial q} - \displaystyle\sum_{A=1}^{P} \dfrac{\dot{\Gamma}^A}{\frac{\partial L}{\partial S_A}} F^{\mathrm{fr}(A)} + F^{\mathrm{ext}}, \\ \dfrac{\partial L}{\partial S_A} + \dot{\Gamma}^A = 0, \quad A = 1, ..., P, \\ \dot{S}_A - \dot{\Sigma}_A + \displaystyle\sum_{B=1}^{P} \dfrac{\dot{\Gamma}^A}{\frac{\partial L}{\partial S_A}} J_{BA} = 0, \quad A = 1, ..., P. \end{cases} \tag{3}$$

From the second equation in (3), the temperature of the subsystem $\mathbf{\Sigma}_A$, i.e., T^A can be obtained as $\dot{\Gamma}^A = -\frac{\partial L}{\partial S_A} =: T^A$. Because $\sum_{A=1}^{P} J_{AB} = 0$ for all B, the last equation in (3) yields $\dot{S}_A = \dot{\Sigma}_A$. Hence, together with (1), we obtain the following Lagrange-d'Alembert equations for the curves $q(t)$ and $S_A(t)$:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = \displaystyle\sum_{A=1}^{P} F^{\mathrm{fr}(A)} + F^{\mathrm{ext}}, \\ \dfrac{\partial L}{\partial S_A}\dot{S}_A = \left\langle F^{\mathrm{fr}(A)}, \dot{q} \right\rangle - \displaystyle\sum_{B=1}^{P} J_{AB}\left(\dfrac{\partial L}{\partial S_B} - \dfrac{\partial L}{\partial S_A} \right), \quad A = 1, ..., P. \end{cases} \tag{4}$$

The First Law of Energy Balance. For the total energy $E : TQ \times \mathbb{R}^P \to \mathbb{R}$ given by $E(q, v_q, S_A) = \left\langle \frac{\partial L}{\partial v_q}(q, v_q, S_A), v_q \right\rangle - L(q, v_q, S_A)$, we have $\frac{d}{dt}E = \langle F^{\mathrm{ext}}, \dot{q} \rangle = P_W^{\mathrm{ext}}$ along the solution curve of (4). If the Lagrangian is given by $L(q, v, S_1, ..., S_P) = \sum_{A=1}^{P} L_A(q, v, S_A)$, the evolution equations for $\mathbf{\Sigma}_A$ are

$$\frac{d}{dt}\frac{\partial L_A}{\partial \dot{q}} - \frac{\partial L_A}{\partial q} = F^{\mathrm{fr}(A)} + F^{\mathrm{ext}\to A} + \sum_{B=1}^{P} F^{B\to A}, \quad A = 1, ..., P,$$

where $F^{B\to A}$ is the internal force exerted by $\mathbf{\Sigma}_B$ on $\mathbf{\Sigma}_A$. From Newton's third law, we have $F^{B\to A} = -F^{A\to B}$. Denoting by E_A the total energy of $\mathbf{\Sigma}_A$, we have

$$\frac{d}{dt}E_A = P_W^{\mathrm{ext}\to A} + \sum_{B=1}^{P} P_W^{B\to A} + \sum_{B=1}^{P} P_H^{B\to A}, \tag{5}$$

where $P_W^{\mathrm{ext}\to A} = \left\langle F^{\mathrm{ext}\to A}, \dot{q} \right\rangle$ is the mechanical power that flows from the exterior into $\mathbf{\Sigma}_A$, $P_W^{B\to A} = \sum_{B=1}^{P} \left\langle F^{B\to A}, \dot{q} \right\rangle$ is the internal mechanical power that flows from $\mathbf{\Sigma}_B$ into $\mathbf{\Sigma}_A$, and $P_H^{B\to A} = \sum_{B=1}^{P} J_{AB}\left(\frac{\partial L}{\partial S_B} - \frac{\partial L}{\partial S_A} \right)$ is the internal heat power from $\mathbf{\Sigma}_B$ to $\mathbf{\Sigma}_A$. It follows that the power exchange can be written as $P_H^{B\to A} = J_{AB}(T^A - T^B)$.

The Second Law and Internal Entropy Production. The total entropy of the system is $S = \sum_{A=1}^{P} S_A$. Therefore, it follows from (4) that the rate of total entropy production of the system is given by

$$\dot{S} = -\sum_{A=1}^{P} \frac{1}{T^A} \left\langle F^{\mathrm{fr}(A)}, \dot{q} \right\rangle + \sum_{A<B}^{P} J_{AB} \left(\frac{1}{T^B} - \frac{1}{T^A} \right) (T^B - T^A),$$

which becomes always positive because of the second law. This is consistent with the phenomenological relations of the form

$$F_i^{\mathrm{fr}(A)} = -\lambda_{ij}^A \dot{q}^j \qquad \text{and} \qquad J_{AB} \frac{T^A - T^B}{T^A T^B} = \mathscr{L}_{AB}(T^B - T^A). \tag{6}$$

In the above, λ_{ij}^A and $\mathscr{L}_{AB}$ are functions of the state variables, where the symmetric part of λ_{ij}^A are positive semi-definite and with $\mathscr{L}_{AB} \geq 0$ for all A, B. From the second relation, we get $J_{AB} = -\mathscr{L}_{AB} T^A T^B = -\kappa_{AB}$, with $\kappa_{AB} = \kappa_{AB}(q, S_A, S_B)$ the heat conduction coefficients between $\mathbf{\Sigma}_A$ and $\mathbf{\Sigma}_B$.

2 Dirac Formulation of Non-simple Systems

In this section, we develop the Dirac formulation for the dynamics of non-simple systems by means of an induced Dirac structure the Pontryagin bundle; for the details, see [1,5,6].

2.1 Dirac Structures in Thermodynamics

Thermodynamic Configuration Space. For our class of non-simple systems, let $\mathscr{Q} = Q \times V$ be a *thermodynamic configuration space*, where Q denotes the mechanical configuration space with mechanical variables $q \in Q$ as before and $V = \mathbb{R}^P \times \mathbb{R}^P \times \mathbb{R}^P$ is the thermodynamic space with thermodynamic variables $(S_A, \Gamma^A, \Sigma_A) \in V$. We denote by $x = (q, S_A, \Gamma^A, \Sigma_A)$ an element of $\mathscr{Q}$, by (x, v) an element in the tangent bundle $T\mathscr{Q}$ where $v = (v_q, v_{S_A}, v_{\Gamma^A}, v_{\Sigma_A}) \in T_x\mathscr{Q}$, and by (x, p) an element of the cotangent bundle $T^*\mathscr{Q}$, where $p = (p_q, p_{S_A}, p_{\Gamma^A}, p_{\Sigma_A}) \in T_x^*\mathscr{Q}$.

Nonlinear Constraints of Thermodynamic Type. Let $C_V \subset T\mathscr{Q} \times_{\mathscr{Q}} T\mathscr{Q}$ be the *variational constraint* locally given as

$$C_V = \left\{ (x, v, \delta x) \in T\mathscr{Q} \times_{\mathscr{Q}} T\mathscr{Q} \;\middle|\; \frac{\partial L}{\partial S_A} \delta \Sigma_A = \left\langle F^{\mathrm{fr}(A)}, \delta q \right\rangle + \sum_{B=1}^{P} J_{AB} \delta \Gamma^B, A = 1, ..., P \right\}. \tag{7}$$

For every $(x, v) \in T\mathscr{Q}$, we consider the subspace of $T_x\mathscr{D}$ given by

$$C_V(x, v) := C_V \cap \left(\{(x, v)\} \times T_x\mathscr{Q} \right) \subset T_x\mathscr{Q}.$$

The *kinematic constraint* associated to C_V is defined by

$$C_K = \{(x,v) \in T\mathscr{Q} \mid (x,v) \in C_V(x,v)\}, \tag{8}$$

which is given locally as

$$C_K = \left\{ (x,v) \in T\mathscr{Q} \;\middle|\; \frac{\partial L}{\partial S_A} v_{\Sigma_A} = \left\langle F^{\mathrm{fr}(A)}, v_q \right\rangle + \sum_{B=1}^{P} J_{AB} v_{\Gamma^B},\, A = 1, ..., P \right\}. \tag{9}$$

Variational and kinematic constraints C_V and C_K are called *nonlinear constraints of thermodynamic type* if they are related as in (8).

Note also that the annihilator of $C_V(x,v) \subset T_x\mathscr{Q}$, defined by

$$C_V(x,v)^\circ = \left\{ (x,\zeta) \in T_x^*\mathscr{Q} \;\middle|\; \langle \zeta, \delta x \rangle = 0, \;\; \forall \delta x \in C_V(x,v) \right\} \subset T_x^*\mathscr{Q},$$

is given, in coordinates $\zeta = (\zeta_q, \zeta_{S_A}, \zeta_{\Gamma^A}, \zeta_{\Sigma_A}) \in T_x^*\mathscr{Q}$, by

$$C_V(x,v)^\circ = \left\{ (x,\zeta) \in T_x^*\mathscr{Q} \;\middle|\; \zeta_q + \frac{\zeta_{\Sigma_A}}{\frac{\partial L}{\partial S_A}} F^{\mathrm{fr}(A)} = 0, \quad \zeta_{S_A} = 0, \right.$$
$$\left. \zeta_{\Gamma_A} = \frac{\zeta_{\Sigma_A}}{\frac{\partial L}{\partial S_A}} \sum_{B=1}^{P} J_{AB}, \;\; A = 1, ..., P \right\}.$$

Dirac Structures on the Pontryagin Bundle. $\mathscr{P} = T\mathscr{Q} \oplus T^*\mathscr{Q}$. The Pontryagin bundle $\mathscr{P}$ is defined as the Whitney sum bundle of $\mathscr{P} = T\mathscr{Q} \oplus T^*\mathscr{Q}$, with vector bundle projection, $\pi_{(\mathscr{P},\mathscr{Q})} : \mathscr{P} = T\mathscr{Q} \oplus T^*\mathscr{Q} \to \mathscr{Q}$, $\mathrm{x} = (x,v,p) \mapsto x$. Given a variational constraint $C_V \subset T\mathscr{Q} \times_{\mathscr{Q}} T\mathscr{Q}$ as in (7), we define the *induced distribution* $\Delta_{\mathscr{P}}$ on $\mathscr{P}$ by

$$\Delta_{\mathscr{P}}(x,v,p) := \left(T_{(x,v,p)} \pi_{(\mathscr{P},\mathscr{Q})} \right)^{-1} (C_V(x,v)) \subset T_{(x,v,p)}\mathscr{P},$$

for each $(x,v,p) \in \mathscr{P}$. Locally, this distribution reads

$$\Delta_{\mathscr{P}}(x,v,p) = \left\{ (x,v,p,\delta x,\delta v,\delta p) \in T_{(x,v,p)}\mathscr{P} \mid (x,\delta x) \in C_V(x,v) \right\}.$$

Further, the presymplectic form on $\mathscr{P}$ is defined from the canonical symplectic form $\Omega_{T^*\mathscr{Q}}$ on $T^*\mathscr{Q}$ as $\Omega_{\mathscr{P}} := \pi^*_{(\mathscr{P},T^*\mathscr{Q})} \Omega_{T^*\mathscr{Q}}$, which is locally given by using local coordinates $(x,v,p) = (q, S_A, \Gamma^A, \Sigma_A, v_q, v_{S_A}, v_{\Gamma^A}, v_{\Sigma_A}, p_q, p_{S_A}, p_{\Gamma^A}, p_{\Sigma_A})$ for each $\mathrm{x} = (x,v,p) \in \mathscr{P}$ as

$$\Omega_{\mathscr{P}} = dq \wedge dp_q + dS_A \wedge dp_{S_A} + d\Gamma_A \wedge dp_\Gamma + d\Sigma_A \wedge dp_{\Sigma_A}.$$

Definition 1. *The Dirac structure $D_{\Delta_{\mathscr{P}}}$ induced on $\mathscr{P}$ from $\Delta_{\mathscr{P}}$ and $\omega_{\mathscr{P}}$ is defined by, for each $\mathrm{x} \in \mathscr{P}$,*

$$D_{\Delta_{\mathscr{P}}}(\mathrm{x}) := \left\{ (u_\mathrm{x}, \alpha_\mathrm{x}) \in T_\mathrm{x}\mathscr{P} \times T_\mathrm{x}^*\mathscr{P} \;\mid\; u_\mathrm{x} \in \Delta_{\mathscr{P}}(\mathrm{x}) \text{ and } \right.$$
$$\left. \langle \alpha_\mathrm{x}, w_\mathrm{x} \rangle = \Omega_{\mathscr{P}}(\mathrm{x})(u_\mathrm{x}, w_\mathrm{x}) \text{ for all } w_\mathrm{x} \in \Delta_{\mathscr{P}}(\mathrm{x}) \right\}.$$

Proposition 1. *The local expression of the Dirac condition, for each* $\mathrm{x} = (x, v, p)$,

$$\big((x, v, p, \dot{x}, \dot{v}, \dot{p}), (x, v, p, \alpha, \beta, \gamma)\big) \in D_{\Delta_{\mathscr{P}}}(x, v, p)$$

is equivalent to

$$(x, \dot{x}) \in C_V(x, v), \quad \beta = 0, \quad \gamma = \dot{x}, \quad \dot{p} + \alpha \in C_V(x, v)^{\circ}.$$

In coordinates $(\alpha, \beta, \gamma) = (\alpha_q, \alpha_{S_A}, \alpha_{\Gamma^A}, \alpha_{\Sigma_A}, \beta_q, \beta_{S_A}, \beta_{\Gamma^A}, \beta_{\Sigma_A},\ \gamma_q, \gamma_{S_A}, \gamma_{\Gamma^A}, \gamma_{\Sigma_A})$, *this condition reads as*

$$\begin{cases} \dot{p}_q + \alpha_q + (\dot{p}_{\Sigma_A} + \alpha_{\Sigma_A}) \dfrac{1}{\frac{\partial L}{\partial S_A}} F^{\mathrm{fr}(A)} = 0, \ \ \dot{p}_{S_A} + \alpha_{S_A} = 0, \\ \dot{p}_{\Gamma_A} + \alpha_{\Gamma_A} = \dot{p}_{\Sigma_A} + \alpha_{\Sigma_A} \dfrac{1}{\frac{\partial L}{\partial S_A}} \displaystyle\sum_{B=1}^{P} J_{AB}, \\ \beta_q = 0, \ \ \beta_{S_A} = 0, \ \ \beta_{\Gamma^A} = 0, \ \ \beta_{\Sigma_A} = 0, \\ \dot{q} = \gamma_q, \ \dot{S}_A = \gamma_{S_A}, \ \dot{\Gamma}_A = \gamma_{\Gamma^A}, \ \dot{\Sigma}_A = \gamma_{\Sigma_A}, \\ \dfrac{\partial L}{\partial S_A} \gamma_{\Sigma_A} = \left\langle F^{\mathrm{fr}(A)}, \gamma_q \right\rangle + \displaystyle\sum_{B=1}^{P} J_{AB} \gamma_{\Sigma_B}. \end{cases}$$

2.2 Dirac Formulation for Thermodynamics of Non-simple Systems

Dirac Dynamical Systems on $\mathscr{P} = T\mathscr{Q} \oplus T^*\mathscr{Q}$. For a given Lagrangian $L(q, v_q, S_A)$ on $TQ \times \mathbb{R}^P$, we introduce an *augmented Lagrangian* by $\mathscr{L}(q, S_A, \Gamma^A, \Sigma_A, v_q, v_{\Gamma^A}) := L(q, v_q, S_A) + v_{\Gamma^A}(S_A - \Sigma_A)$.

In the above, note that the augmented Lagrangian may be regarded as a (degenerate) Lagrangian function $\mathscr{L}(x, v)$ on $T\mathscr{Q}$. Further, define the *generalized energy* $\mathscr{E}$ on $\mathscr{P} = T\mathscr{Q} \oplus T^*\mathscr{Q}$ as

$$\mathscr{E}(x, v, p) := \langle p, v \rangle - \mathscr{L}(x, v).$$

Given an external force $F^{\mathrm{ext}}(q, v_q, S_A)$, which may be regarded as a map $F^{\mathrm{ext}} : T\mathscr{Q} \to T^*\mathscr{Q}$, a horizontal one-form $\widetilde{F}^{\mathrm{ext}} : \mathscr{P} \to T^*\mathscr{P}$ is induced by

$$\left\langle \widetilde{F}^{\mathrm{ext}}(x, v, p), u \right\rangle = \left\langle F^{\mathrm{ext}}(x, v), T\pi_{(\mathscr{P}, \mathscr{Q})}(u) \right\rangle, \quad \text{for all } \ u \in T_{(x,v,p)}\mathscr{P}.$$

Theorem 1. *Given* C_V *and* C_K *as in* (7) *and* (9), *the solution curve* $\mathrm{x} = (x(t), v(t), p(t))$ *of the Dirac system*

$$\Big((x, v, p, \dot{x}, \dot{v}, \dot{p}), \mathbf{d}\mathscr{E}(x, v, p) - \widetilde{F}^{\mathrm{ext}}(x, v, p)\Big) \in D_{\Delta_{\mathscr{P}}}(x, v, p), \qquad (10)$$

satisfies the equations of motion

$$(x, \dot{x}) \in C_V(x, v), \quad p - \frac{\partial \mathscr{L}}{\partial v} = 0, \quad \dot{x} = v, \quad \dot{p} - \frac{\partial \mathscr{L}}{\partial x} - F^{\mathrm{ext}}(x, v) \in C_V(x, v)^{\circ}.$$

In coordinates, we obtain the system

$$\begin{cases} \dot{p}_q = \dfrac{\partial L}{\partial q} - \displaystyle\sum_{A=1}^{P} \frac{\dot{\Gamma}^A}{\frac{\partial L}{\partial S_A}} F^{\mathrm{fr}(A)} + F^{\mathrm{ext}} = 0, \;\; \dot{p}_{\Gamma^A} + \displaystyle\sum_{B=1}^{P} \frac{\dot{\Gamma}^A}{\frac{\partial L}{\partial S_A}} J_{BA} = 0, \\ \dot{q} = v_q, \; \dot{\Gamma}_A = v_{\Gamma^A}, \; \dot{\Sigma}_A = v_{\Sigma_A}, \\ p_q = \dfrac{\partial L}{\partial v_q}, \;\; p_{\Gamma^A} = S_A - \Sigma_A, \;\; \dfrac{\partial L}{\partial S_A} + \dot{\Gamma}^A = 0, \\ \dfrac{\partial L}{\partial S_A} v_{\Sigma_A} = \left\langle F^{\mathrm{fr}(A)}, v_q \right\rangle + \displaystyle\sum_{B=1}^{P} J_{AB} v_{\Gamma^B}. \end{cases} \tag{11}$$

This system yields the Lagrange-d'Alembert evolution Eq. (4) for non-simple thermodynamic systems with friction and heat conduction.,

Along the solution curve $(x(t), v(t), p(t)) \in \mathscr{P}$ of the Dirac dynamical system in (10), the energy balance equation holds as

$$\frac{d}{dt} \mathscr{E}(x, v, p) = \left\langle F^{\mathrm{ext}}(x, v), \dot{x} \right\rangle.$$

2.3 Example of the Adiabatic Piston

The Adiabatic Piston. Now we consider a piston-cylinder system that is consisted of two cylinders connected by a rod, each of which contains a fluid (or an ideal gas) and is separated by a movable piston, as in Fig. 1 (see [2]).

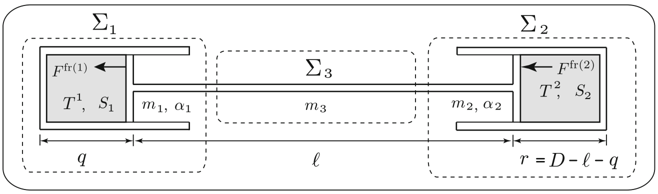

Fig. 1. The adiabatic piston problem

The system $\mathbf{\Sigma}$ is an interconnected system that is composed of three simple systems; the two pistons $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2$ with mass m_1, m_2 and the connecting rod $\mathbf{\Sigma}_3$

with mass m_3. As in Fig. 1, q and $r = D - \ell - q$ denote the distances between the bottom and the top in each piston where $D = \text{const}$. Choose the state variables (q, v_q, S_1, S_2) (the entropy of $\mathbf{\Sigma}_3$ is constant), and the Lagrangian is

$$L(q, v_q, S_1, S_2) = \frac{1}{2} M v_q^2 - U_1(q, S_1) - U_2(q, S_2),$$

where $M := m_1 + m_2 + m_3$, $U_1(q, S_1) := \mathsf{U}_1(S_1, V_1 = \alpha_1 q, N_1)$, and $U_2(q, S_2) := \mathsf{U}_2(S_2, V_2 = \alpha_2 r, N_2)$, with $\mathsf{U}_i(S_i, V_i, N_i)$ the internal energies of the fluids, N_i the constant numbers of moles, and α_i the constant areas of the cylinders, $i = 1, 2$. As in (6), we have $F^{\text{fr}(A)}(q, \dot{q}, S_A) = -\lambda^A \dot{q}$, with $\lambda^A = \lambda^A(q, S^A) \geq 0$, $A = 1, 2$ and $J_{AB} = -\kappa_{AB} =: -\kappa$, where $\kappa = \kappa(S_1, S_2, q) \geq 0$ is the heat conductivity of the connecting rod.

From the Dirac system formulation (11), we obtain the evolution equations as

$$\begin{cases} \dot{p}_q = \Pi_1(q, S_1)\alpha_1 - \Pi_2(q, S_2)\alpha_2 - (\lambda^1 + \lambda^2)\dot{q}, \\ \dot{q} = v_q, \quad p_q = M v_q, \\ T^1(q, S_1)\dot{S}_1 = \lambda^1 \dot{q}^2 + \kappa\left(T^2(q, S_2) - T^1(q, S_1)\right), \\ T^2(q, S_2)\dot{S}_2 = \lambda^2 \dot{q}^2 + \kappa\left(T^1(q, S_1) - T^2(q, S_2)\right), \end{cases}$$

where $T^i(q, S_i) = \frac{\partial U_i}{\partial S_i}(q, S_i)$, $\frac{\partial U_1}{\partial q} = -\Pi_1(q, S_1)\alpha_1$, and $\frac{\partial U_2}{\partial q} = \Pi_2(q, S_2)\alpha_2$.

Since the system is isolated, we recover the first law $\frac{d}{dt}E = 0$, where $E = \frac{1}{2}M\dot{q}^2 + U_1(q, S_1) + U(q, S_2)$. The second law is also recovered as

$$\frac{d}{dt}S = \left(\frac{\lambda^1}{T^1} + \frac{\lambda^2}{T^2}\right)\dot{q}^2 + \kappa\frac{(T^2 - T^1)^2}{T^1 T^2} \geq 0.$$

References

1. Courant, T.J.: Dirac manifolds. Trans. Amer. Math. Soc. **319**, 631–661 (1990)
2. Gruber, C.: Thermodynamics of systems with internal adiabatic constraints: time evolution of the adiabatic piston. Eur. J. Phys. **20**, 259–266 (1999)
3. Gay-Balmaz, F., Yoshimura, H.: A Lagrangian variational formulation for nonequilibrium thermodynamics. Part I: discrete systems. J. Geom. Phys. **111**, 169–193 (2016)
4. Gay-Balmaz, F., Yoshimur, H.: From Lagrangian mechanics to nonequilibrium thermodynamics: a variational perspective. Entropy **21**(1), 8 (2019)
5. Gay-Balmaz, F., Yoshimura, H.: Dirac structures in nonequilibrium thermodynamics for simple open systems. J. Math. Phys. **61**(9), 092701 (2020)
6. Gay-Balmaz, F., Yoshimura, H.: Dirac structures and variational formulation of port-Dirac systems in nonequilibrium thermodynamics. IMA J. Math. Control Inf. **37**(4), 1298–1347 (2020)
7. Stueckelberg, E.C.G., Scheurer, P.B.: Thermocinétique Phénoménologique Galiléenne. Birkhäuser (1974)