Argumentation in the Context of Elementary Grades: The Role of Participants, Tasks, and Tools

Chepina Rumsey, Ian Whitacre, Şebnem Atabaş, and Jessica L. Smith

Introduction

We consider the episode from Ms. Kirk's second-grade classroom from the perspective of argumentation. We view mathematical argumentation in a classroom setting as a human activity that is characterized by a participation structure and is profoundly influenced by the teacher's instructional decisions and the sociomathematical norms established in the classroom (Yackel & Cobb, 1996). The mathematical argumentation occurring in a classroom is also influenced by the teacher's and students' beliefs about what an argument is, what is important to attend to, and what counts as evidence. We analyze episodes involving whole-class, co-constructed argumentation in which second graders are reasoning about arithmetic equations and equality. In particular, our analysis focuses on each of the following themes as observed in the episode: the role of definitions and common language, the role of the classroom participants, the role of the task, and the role of tools.

Our findings center on the relationship between argumentation and tool use. We illustrate how the choice of tool and the nature tasks involving that tool may influence opportunities for argumentation and shape the kinds of arguments that students make. Thus, in order to engage students productively in mathematical

C. Rumsey (✉)
University of Northern Iowa, Iowa, USA
e-mail: chepina.rumsey@uni.edu

I. Whitacre · J. L. Smith
Florida State University, Tallahassee, FL, USA
e-mail: iwhitacre@fsu.edu; jls17r@my.fsu.edu

Ş. Atabaş
College of Education, Florida State University, Tallahassee, FL, USA
e-mail: satabas@fsu.edu

K. N. Bieda et al. (eds.), *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-80008-6_3

argumentation, it is important to consider the kinds of arguments that students could potentially make and how the resources available might afford or constrain opportunities.

Arguments and Argumentation

Mathematical argumentation is the process of making mathematical claims and providing evidence to support them. The three components—claims, evidence, and reasoning—are recognized by researchers and scholars as constituting the core structure of an argument, although they may be called by different names (Toulmin, 1958; Krummheuer, 1995). In this chapter, we use the term claim to mean the statement under consideration. Evidence provides support for the claim, and reasoning explains the connection between the claim and the evidence. Our analysis focuses less on arguments themselves and more on participation in the activity of argumentation. We see individual arguments functioning as building blocks in episodes of argumentation.

Argumentation in the Classroom

Productive mathematical argumentation requires a participation structure, or set of classroom norms, that lends itself to such interactions. Accountable argumentation describes "a participation structure, embedded in whole-class discussion, that organizes the public disagreements among students and provides interactional resources for clear mathematical reasoning and the production of mathematical generalizations" (Horn, 2008, p. 104). This sort of argumentation is called accountable because students are accountable for making sense of each other's thinking, remembering and using previous arguments and established ideas, and constructing viable arguments. In these ways, students are responsible for participating in mathematical discussions that are productive and respectful. Accountable argumentation addresses the challenges involved in engaging students in argumentation. In particular, this participation structure is intellectually productive and, at the same time, minimizes social discomfort (Horn, 2008).

Social norms that are conducive to mathematical argumentation involve a degree of shared authority between teacher and students (Cobb et al., 2009). If the teacher functions as the sole authority in discussions, then students do not reasonably have the opportunity to engage authentically in attempting to constructing their own, viable arguments and to critique the reasoning of others (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Cobb and Yackel (1996) describe a process of negotiation (or renegotiation) of social norms that includes explicit discussion of expectations and responsibilities, including "explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives…" (p. 178).

Mathematical Content for Argumentation

Rich mathematical argumentation goes beyond stating solutions to tasks; it is a dynamic process of making mathematical claims, providing evidence in support of those claims, and explaining the reasoning that connects the two. Interesting mathematical arguments can be made at all grade levels. According to Stylianou and Blanton (2018), "While the formality and form of these arguments will vary across grades, all students need to be able to develop, understand, and interpret arguments appropriate at their level of experience in mathematics" (p. 4). Thus, we are interested in opportunities for worthwhile mathematical argumentation around important topics in the elementary mathematics standards. In this case, the tasks and discussions concern topics in Operations and Algebraic Thinking and Number and Operations in Base Ten at second grade.

The specific mathematical content in the episode from Ms. Kirk's classroom involves arithmetic equations that have operations on both sides. Such equations can provide opportunities for discussion of strategies for performing the operations, as well as discussion of number relationships or of the meaning of the equal sign itself. Popular tasks involve asking students to decide whether an equation such as $8 + 4 = 7 + 5$ is true or false or asking them to determine the unknown number needed to make an equation such as $8 + 4 = \_ + 5$ true.

As students share their work and discuss their thinking about these kinds of tasks, they provide important clues regarding their conceptions of the equal sign: a *relational* view involves viewing the expressions on both sides of the equal sign as equivalent. An *operational* view, by contrast, involves viewing the equal sign as one-directional or as an instruction to perform the operation indicated on the left-hand side and to write one's answer on the right-hand side (Baroody & Ginsburg, 1983; Jacobs et al., 2007; Kieran, 1981; McLean, 1964). A student who is interpreting the equal sign operationally might solve the equation $8 + 4 = \_ + 5$ by adding 8 and 4, obtaining a sum of 12, and ignoring the 5 on the right-hand side (or interpreting it as the next operation to be performed). A student who is interpreting the equal sign relationally might solve the same equation by finding the sum of 12 on the left and then asking what number would result in a sum of 12 on the right. A strategy that exemplifies relational thinking would be to notice that 5 is one more than 4 and to reason that the number in the blank must be one less than 8. Interpreting the equal sign relationally opens many possibilities for students to notice and take advantage of number relationships. It is important for students to work on tasks structured to provide opportunities for students to articulate what the equal sign means and engage in discussions to uncover and challenge their existing conceptions.

Brief Summary of the Lesson

The topic of the second-grade lesson is equality, which is a rich area for mathematical argumentation. The teacher, Ms. Kirk, begins with a discussion of the definition of the equal sign and the tools that can be used to picture what the symbol means. Students discuss tools such as Unifix cubes, Cuisenaire rods, and balance scales. Ms. Kirk also reminds the students that they have been working on "how we can prove and explain our thinking to other people." She presents the first task, which is to show with the Cuisenaire rods whether or not 14 + 3 = 15 + 2 is true. Students work with their partners and then are brought back together for a class discussion where Ms. Kirk describes one pair of students' work and then asks another student to explain. As a challenge activity, Ms. Kirk presents a task where students "use the Cuisenaire Rods to help [you] design a math equation that is true." The worksheet shows ▢ + ▢ = ▢ + ▢. Ms. Kirk reminds the students to use Cuisenaire rods, work with their partner, and not to come up with numbers that are too large. After time to work together, the class comes back together for a whole-class discussion. Ms. Kirk highlights several examples that use the Cuisenaire rods before directing the students to clean up the materials.

Analyzing the Lesson Transcript

We started our analysis by reading through the classroom transcript and taking notes about the observations we made. In our analysis, we used methods of grounded theory (Corbin & Strauss, 2015), leading to the emergence of four themes. The themes relate to the roles of definitions, participants, tasks, and tools. The presentation of each theme includes a discussion of what the specific transcript excerpts highlight about these roles, as well as how the theme connects to the research literature. It is important to unpack these four themes as they are essential components for teachers to consider as they integrate argumentation into their lessons.

The Role of Definitions, Established Facts, and Common Language

The first theme we share explores the role of definitions, established facts, and common language. The initial example of introducing definitions or common language occurs before the main task, when Ms. Kirk asks, "What does this mean to you?," while pointing to the equal sign on the board:

Ms. **Kirk**: Lucas, what does it mean to you?
Lucas: The same as.
Ms. **Kirk**: Same as. Good. Does anyone have any other words that they use to help understand it? Kara?

Kara: I forget.
Ms. **Kirk**: You forget? Kylie?
Kylie: Equal.
Ms. **Kirk**: Equal. Equal to. Good. Anything else? Jacob.
Joey: Combined?
Ms. **Kirk**: This means combined?
Joey: No, uh… um… Equals to.
Ms. Kirk: Yeah. Good. Okay.

Ms. Kirk never mentions the name of the symbol but invites students to think about its meaning instead. Students offer ideas such as "the same as" or "equal to," which she accepts as "good." By contrast, Ms. Kirk questions Joey when he responds that the symbol means "combine." It is possible that Joey is suggesting an operational definition of the equal sign or that he would like to combine everything on both sides of the equal sign. In response to Ms. Kirk's question, Joey revises his answer and says that the symbol means "equals to," which she accepts. This questioning of the operational view of the equal sign may inadvertently close the door on discussion of a productive way of reasoning that is relevant to the task. Having this discussion around relational versus operational views of the equal sign could allow space for different ways of reasoning with equivalent expressions and determining whether an equation is true or false. At the heart of evaluating an equation is the idea of comparing two expressions to see if they represent the same quantity, which possibly involves calculations.

Later, there is a connection made between the meaning of the equal sign and Cuisenaire rods. Ms. Kirk prompts students to think about the tools they have used in class and how these might help in understanding the equal sign:

Meredith: Uhmm, those rods.
Meredith points behind her towards her group's table.
Ms. **Kirk**: Those, oh the Cuisenaire Rods? How do those help you?
Meredith: To figure out what both of them equal.
Ms. **Kirk**: Oh, so you use those to help you figure out what each side equals?
Meredith: And it's the same.
Ms. **Kirk**: And it's the same, which makes sense because that can mean the same as. Cool.

Ms. Kirk makes a connection between the rods and the equal sign as meaning "the same as." Meredith says the rods can be used to see what each expression equals, which should be the same for both expressions. This idea connects Joey's operational view of the equal sign with the equal sign as meaning "the same as." Meredith suggests using Cuisenaire rods to see what each expression equals and then comparing those values. The expressions are equivalent if the total lengths of the rods are the same. Meredith offers a method emphasizing computation before comparing magnitudes. This need for computation aligns with aspects of an operational view of the equal sign, which Ms. Kirk previously deemphasized. Ms. Kirk is not always consistent in how the definition of "equals" is conveyed across the discourse.

Stylianides (2007) claims setting a *foundation* is the first element of argumentation in the mathematics classroom. Foundations can include definitions, axioms, or established facts. There needs to be a basis of what can be considered as true in order for argumentation to occur effectively in the classroom. Ms. Kirk is laying the foundation for interpreting the equal sign in order for argumentation around the task to occur. She reinforces the idea of the equal sign as meaning "the same as." Because of likely institutional constraints (i.e., curriculum, time), she chose to limit the discussion of the operational view, which is a component of students' reasoning about equations later in the task.

Ms. Kirk also connects the foundational "same as" definition to the tool that students will use during the task. She redirects away from the operational view of the equals sign but accepts student reasoning stressing computation when comparing expressions. While these methods did not directly align with operational thinking, the emphasis on computation is common between the two ways of reasoning about the equals sign. The foundation set by Ms. Kirk centers around operational views of the equal sign and connects to more computational ways of reasoning. Further, she focuses on the relational way of thinking about the equal sign early in the lesson, but then does not accept this way of reasoning with verifying equivalent expressions. This creates a disconnect between the foundation and accepted ways of reasoning with the task. Relating to the theme of definitions and establishing a common language, this episode is rich with opportunities and missed opportunities to establish a foundation where the classroom community could develop argumentation.

The Role of Participants (Teacher and Students)

When students engage meaningfully in mathematical argumentation, "[t]he teacher is not the sole authority in the class. Rather… she supports, facilitates, and coordinates discussions…" (Stylianou & Blanton, 2018, pp. 32–33). The role of participants is an important theme to unpack, and the second-grade lesson offers opportunities to better understand what this looks like in classroom practice.

The students are asked to use Cuisenaire rods with their partners to show whether or not the following equation is true: 14 + 3 = 15 + 2. When the class comes back together, Ms. Kirk describes what she saw two students doing with the Cuisenaire rods to solve the problem. Carrie and Amanda made an arrangement of rods shown in Fig. 1, with the numbers added for the reader's reference.

Ms. Kirk asks, "and using this, can you prove to me that four plus three is the same as five plus two?" The excerpt begins with Ms. Kirk looking for someone who she has not yet heard from to explain to the class:

Ms. Kirk: I'm trying to think who I haven't heard from yet. Sammy do you wanna explain it?

Sammy: Yeah.

Ms. Kirk: All right. Nice and loud voice.

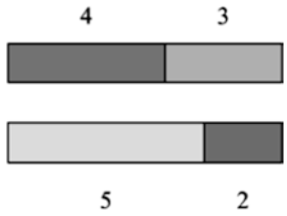

Fig. 1 Depiction of Carrie and Amanda's evidence for equivalence with Cuisenaire rods

Sammy: Because if you did, if you went, and switched the uh… That one if you switch the three and then put it—

Sammy points to the screen at the rods, but does not go very close so others can see exactly where she is pointing.

Sammy: And if you go make the four a five and the three a two and that would make the seven and it would be equal.

Ms. Kirk: Oh wow. So you're saying… So I just wanna use these. Just looking at these, is there any way to tell that these equal each other?

Ms. Kirk is pointing to the rods on the document camera.

Sammy did not point at the rods but explained everything orally while standing in front of the class.

Sammy: [quietly to Ms. Kirk as Sammy puts the rods together]. Um, you can tell if you put them together.

Ms. Kirk: If you turn around and say that to them louder.

Sammy: You can tell because they're the same size if you put them together. [Fig. 1]

Ms. Kirk: They're the same size. Right?

Sammy: Yeah.

Ms. Kirk: Can't really argue with that. That is some hard concrete evidence. What I like about using this tool boys and girls, is it takes a number that sometimes can feel hard to picture, and it gives you something to picture, something you can look at. So not only do you know that four plus three is the same as five plus two because you know that four plus three equals seven, and five plus two equals seven. But now you can also see when they're lined up next to each other, they're the same size.

Using the three elements of argumentation to reconstruct Sammy's argument, we have the new claim of 4 + 3 = 5 + 2. Even though the initial question was 14 + 3 = 15 + 2, the example that Ms. Kirk selects to highlight and the argument that Sammy presents are about the new claim. The evidence that is presented by Sammy is an explanation related to compensation that "if you go make the four a five and the three a two and that would make the seven and it would be equal." While the student is beginning to use an explanation that is not reliant on the Cuisenaire rods, Ms. Kirk redirects her to look at the tools, "Just looking at these, is there any way to tell that these equal each other?" Ms. Kirk focuses the reasoning on an argument that uses the tools. To follow with Ms. Kirk's question, Sammy continues, saying, "You can tell because they're the same size if you put them together." Ms. Kirk describes the argument as "hard concrete evidence" that you can't argue with and emphasizes the utility of the tool:

Claim: 4 + 3 = 5 + 2

Evidence: "And if you go make the four a five and the three a two and that would make the seven and it would be equal." (Sammy)

Reasoning: "Just looking at these" (Ms. Kirk) "You can tell because they're the same size if you put them together." (Sammy)

Integrating argumentation into lessons requires consideration of the roles of both Ms. Kirk and students. While Sammy has an interesting argument to share, perhaps based on compensation, ultimately Ms. Kirk's vision for the argument and the direction of the lesson helped shape what Sammy shared. It would have been interesting to follow Sammy's line of thinking further to see how she might generalize the idea of compensation and, perhaps, how she could use the tool as a generalized representation of the idea. Ms. Kirk also does not go back to Carrie and Amanda to see how they react to Sammy's evidence related to compensation or the reasoning that you can line up the rods. The ways in which participants are invited into the community play a vital role in the way that argumentation is taken up in the classroom.

The Role of the Task

As previously presented, tasks including equations that have operations on both sides provide opportunities to discuss strategies to perform these operations, number relationships, and different conceptions of what equal sign means (operational versus relational view of equal sign). Consistent with this view, the first task, "14 + 3 = 15 + 2", included operations on both sides of the equation and Ms. Kirk asks students to show whether or not the equation was true by using the Cuisenaire rods. During the whole-class discussion, Carrie and Amanda make a new claim of "4 + 3 = 5 + 2" based on the original equation. While there is not an explicit discussion of how this new claim related to the original task, it is an opportunity for the class to discuss whether they could ignore the tens on both sides of the equation and focus on the relationship between numbers on the ones place. Building on this new claim, Sammy argues that "make[ing] the four a five and the three a two and that would make the seven and it would be equal." The numbers in the task afford Sammy to see the "one more, one less" relationship between numbers, which could potentially lead a productive discussion on whether other students in the classroom agree with the claim Sammy made.

The second task requires students to design a math equation fitting the form of "▢ + ▢ = ▢ + ▢" and explain how they could prove the equation they wrote to be true by using the Cuisenaire Rods. The open-ended nature of this task affords students to bring different number combinations. First, discussion taking place in the whole-class discussion is about the combinations of 10. Students seem to recall combinations of 10 from a previous activity and fill in the blanks with different combinations of 10. Importantly, the discussion of different combinations of 10 could potentially lead a discussion on the relationship between number pairs (e.g., one more, one less) as seen below:

Ms. Kirk: All right. I had a few friends discover something really interesting and neat. They were able to find more than just two combinations that equal on here. So, it kind of reminded me of when we play games like combinations to 10.

Kaylyn: Yeah.
Ms. **Kirk**: So we know like five plus five equals 10. What's another one? Milly?
Milly: Seven plus three.
Ms. Kirk: Seven plus three also equals 10. Sam?
Sam: Six plus four.
Ms. **Kirk**: Six plus four also equals 10. Kaylyn.
Kaylyn: One plus nine.
Ms. **Kirk**: One plus nine also equals 10. All right. So that was really neat because Sam and Riley over hear found a bunch of different combinations that equal how much?
Riley: We don't know.

While the combination of 10 could potentially lead a productive discussion on the two addends being "one more, one less" compared to the other two addends in the equation, Ms. Kirk does not act upon the connection between combinations of 10 and the evidence Sammy provided previously. Rather than making the relation between numbers adding up to 10 forefront in the discussion and making explicit connections to Sammy's previous claim, Ms. Kirk asks students to share different combinations of 10 and moves on to another student's equation.

Although Ms. Kirk asks students not to use too big numbers, the open-ended nature of this task allows students to use numbers which leads to some issues in fitting the Cuisenaire rods on the document camera screen or on their number lines. For example, Jacob and Sammy designed the math equation, "40 plus 9 equals the same as 27 plus 22," and they have trouble showing the rod combinations on the document camera because it is not fitting on the screen. This was an important opportunity for students to explain how the math equation they wrote was true without necessarily relying on the use of Cuisenaire rods, which is afforded by the open-ended nature of the task. Thus, the two tasks students worked on create opportunities for students to discuss number relationships, develop some relational thinking strategies, and uncover their conception of the equal sign. Although students use some relational thinking strategies, Ms. Kirk does not make these strategies forefront in the discussion. Rather, she includes some constraints during the launch of the task and leads their attention to the use of Cuisenaire rods to show that the number sentences are correct.

Here it is important to note the difference between tasks as planned and as implemented in the class (Henningsen & Stein, 1997). While the two tasks afford important opportunities for relational thinking and argumentation, the way Ms. Kirk launches the two tasks appears to lead students to rely on the use of Cuisenaire rods in a straightforward way and limit the variability of student strategies and reasoning to prove the number sentences. The constraints Ms. Kirk includes lead students to rely on the use of Cuisenaire rods to solve the tasks. For example, Ms. Kirk launches the first task by saying, "I want you to show me with your Cuisenaire Rods whether or not that's true." After such an instruction, the teacher could potentially ask students to write symbolically what they were depicting concretely with the use of Cuisenaire rods. This would encourage students to exercise relational thinking and anticipate whether or not the equation will be true, going beyond then finding the answer by laying out the rods. Thus, the instruction - not going beyond the use of rods to find the answer - place a constraint on the task and limits the variability of student strategies and reasoning to prove that 14 + 3 = 15 + 2. When selecting a task to encourage argumentation, many factors influence the extent to which argumentation will flourish. Teachers make several decisions with regard to tasks that can promote or hinder the argumentation that the students have the opportunity to engage in.

The Role of the Tool in Terms of Evidence and Potential Connections

The availability of the Cuisenaire rods (together with the task instructions, as discussed above) lends itself to arguments in which the rods provide the evidence. For example, Jacob claims that "40 plus 9 equals the same as 27 plus 22." With numbers of that size, the rods do not fit on the doc cam. So, Ms. Kirk asks Sammy and Jacob to lay the rods out on the floor for everyone to see (Fig. 2).

Ms. Kirk invites students to make an argument that is directly concerned with this tool:

Ms. **Kirk**: If what they're saying is true, if we lined each side up, what should it look like? Tony.
Tony: A big line that's is the same on both sides and is equal.
Ms. **Kirk**: It's the same length. Right? The lengths are equal. I'm not sure if you can see from your seats. But when we line them up next to each other.
Katie: They are equal.
Ms. Kirk: They're equal.

In the above except, Tony and Ms. Kirk state that if the amounts (on both sides of the equation) are equal, their corresponding lengths should be the same. Indeed, the two lengths of rods are the same, so evidently Jacob is right in asserting that 40 + 9 was equal to 27 + 22.

The tool used in this way provides a compelling form of empirical evidence in support of Jacob's claim, and we believe that there may be important opportunities here for students to make connections between the equation and its representation in terms of rods. On the other hand, from the perspective of opportunities for productive argumentation, the evidence provided by the tool might actually be too strong. When it is used as the primary source of evidence as in the above episode, there is little if any room for disagreement. Being that the sums are represented as

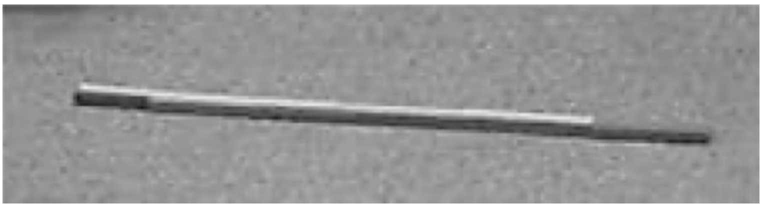

Fig. 2 Sammy and Jacob use rods to show that 40 + 9 = 27 + 22

the lengths that have been laid out side by side, to disagree with the claim that 40 + 9 = 27 + 22 is to disagree with visible evidence that the lengths are the same. In fact, absent that evidence, many second graders would disagree with that claim that 40 + 9 = 27 + 22, due to differing views of the equal sign. The point here is that authentic uncertainty fosters argumentation. Without uncertainty, there is less need for it.

In sum, the use of the tool shapes the nature of argumentation in the lesson such that there is little room for uncertainty and limited opportunities for relational thinking. In the Discussion, we share suggestions regarding how the class could build upon their work with the rods to engage in accountable argumentation and to invite strategies involving relational thinking.

Discussion

Classes like Ms. Kirk's involve students in problem-solving and discussion and thus provide opportunities for the class to engage in mathematical argumentation. We found that argumentation was primarily used as a way to see what students know, how they think about given problems, and how students use Cuisenaire rods as evidence of their thinking and the solution. We conclude by identifying further opportunities for argumentation that relate to each of the themes presented above.

The Role of Definitions and Common Language

Ms. Kirk attempted to create a foundation for argumentation (Stylianides, 2007) by setting a definition of the equal sign. She accepted ideas such as "equal" or "the same as" but questioned Joey's computational description of the equal sign. Joey abandoned this idea and changes his response to "equals to." While Ms. Kirk was laying a foundation for later argumentation, this could have been a moment to argue the different meanings of the equal sign, especially since the task could involve thinking of the equal sign relationally and/or operationally. Ms. Kirk wanted a foundation in the form of a common definition of the equal sign, but the fact is that students think about this symbol in different ways. It could be that a common foundation is not always necessary and that allowing multiple foundations supports different ways of thinking about the same task. Ms. Kirk did not emphasize the operational view of the equal sign when setting the foundation, but she accepted—in fact, she acted as if she preferred—operational methods when working through the task. Preferring operational/computational methods during the task could stem from using Cuisenaire rods when checking the equivalence of two expressions. These specific tools lend more explicitly to computational ways of reasoning, which could explain why Ms. Kirk gravitated toward student thinking involving computation.

A discussion around the different views of the equal sign allows a teacher to give names to those different views, which could then be connected to students' specific ways of reasoning about the task. Such an explicit and flexible foundation might lend itself to accountable argumentation in the classroom because students would then have to reason with and make sense of each other's thinking (Horn, 2008). This approach requires teachers to relinquish some degree of authority and embrace some level of ambiguity. Siegel and Borasi (1996) wrote.

> First of all, inquiry classrooms emphasize the full complexity of knowledge production and expect students to see the doubt arising from ambiguity, anomalies, and contradiction as a motivating force leading to the formation of questions, hunches, and further exploration. Teachers in inquiry classrooms, therefore, are less inclined to take the role of the expert and clear things up for the students and more interested in helping students use this confusion as a starting point for problem posing and data analysis (p. 228).

It is clear Mrs. Kirk is the final authority of discussions (based on her willingness to explicitly evaluate students' responses and solutions), but engaging students in argumentation requires shared authority so that students have the freedom to express their ideas and to critique the reasoning of their peers (Yackel & Cobb, 1996). In this case, argumentation could have started with students' understanding of the equal sign and extended into the task of working with equivalent expressions.

The Role of Participants

While Ms. Kirk is taking steps to introduce argumentation at the second-grade level, there are ways that she could deepen her inclusion of argumentation to involve ownership of the classroom community. Stylianou and Blanton (2018) suggest allowing the students' ideas to take center stage to increase "their agency and their sense of themselves as mathematicians" (p. 33). Notably, Ms. Kirk selected Sammy to share as a student that she had not heard from yet, amplifying the voices of different students in the classroom. Yet, she did not continue to follow the argument that Sammy was making. She steered the argument toward the use of the Cuisenaire rods, whereas Sammy's argument indicated relational thinking and made no reference to the rods. Thus, a next step here would be for the participation structure to move in the direction of greater accountability in the sense of accountable argumentation (Horn, 2008). In order for students to become accountable for making sense of each other's thinking and remembering, Ms. Kirk could set an example by explicitly concerning herself with these responsibilities. In order for the class to engage productively in argumentation, individual students' ideas need to be taken seriously. In this case, that means working to understand Sammy's idea, whether or not it was the argument that Ms. Kirk was expecting.

Likewise, to foster accountable argumentation, it would be important for the class to return to and critique Amanda and Carrie's argument. Instead, Ms. Kirk concluded the discussion without revisiting their arguments, saying, "Can't really argue with that. That is some hard concrete evidence." This final conclusion made by Ms. Kirk could set students up to appeal to authority in future lessons, rather than to look to the community to justify claims and critique arguments. Carpenter et al. (2003) outline three levels of justification that students use to justify that a mathematical claim is true: appeal to authority, justification by example, and generalizable arguments (p. 87). While the first level, appeal to authority, is common, instruction needs to "help students understand that they need to question ideas and use mathematical arguments to justify them. Students need to decide for themselves whether something makes sense and not accept something as true just because someone says it is true" (p. 87). After Sammy shares her ideas, opening up the discussion to the whole class would help the community realize their role in critiquing arguments and could encourage them away from an appeal to authority justification in the future.

The Role of the Task

Ms. Kirk's use of tasks such as true/false equations created an opportunity to uncover her students' conceptions of the equal sign and different perspectives on number relations. However, the constraints that she chose to place on the tasks limited the variability of students' solutions and discouraged argumentation. More specifically, Ms. Kirk asked students to use the Cuisenaire rods in the first task. While Sammy's argument went beyond reliance on the Cuisenaire rods and introduced interesting relational thinking, Ms. Kirk led the attention back to the Cuisenaire rods. Similarly, in the second task, Ms. Kirk asked students to use smaller numbers so that they can use the Cuisenaire rods as they present their work. Thus, the constraints Ms. Kirk provided during the launch of the tasks seem to lead student attention to the use of Cuisenaire rods and limit student work on the tasks as showing two sets to be equal by using the Cuisenaire rods. Ms. Kirk's emphasis on the use of Cuisenaire rods also uncovers what goals she had in mind related to the lesson. Rather than uncovering students' conception of what equal sign means and making claims related to the relations among numbers (relational thinking), she had a goal in mind related to the use of Cuisenaire rods. This goal seems to impact the way she launched the tasks. Possibly, Ms. Kirk believes her role as a teacher is to remove uncertainty for her students and through the additional constraints on the tasks she was trying to make sure that her students know what to do as they work on the tasks. However, she seems to be missing the value of uncertainty in argumentation and how the additional constraints limited the opportunities her students had to engage in argumentation.

Using the same tasks with fewer constraints could open many opportunities for making claims related to numerical relations and different conceptions of the equal sign. We see these opportunities the tasks provided in the evidence Sammy provided as "if you go make the four a five and the three a two and that would make the seven and it would be equal" going beyond a reliance on Cuisenaire rods and attempting to reason about the compensation of numbers within the equation of "$4 + 3 = 5 + 2$."

The combinations of 10 also could lead to a discussion around a similar "one more, one less" relationship among numbers, which is a similar idea that Sammy expressed in the evidence he provided. The equation that Jacob and Sammy presented (40 + 9 = 27 + 22) also could have been an opportunity for students to engage in relational thinking, rather than relying on the tool. For example, students might notice that the total numbers of tens and ones are the same in the expressions on both sides of the equation and conclude that the equation must be true for that reason. Thus, the planned tasks and students' thinking afforded potential opportunities for students to make and justify claims related to number relations—engaging in both the mathematical content and the practice of argumentation at higher levels. Placing fewer constraints on the task instructions could go a long way to encourage such activity.

The Role of Tools

As noted above in our analysis of the role of tools, the Cuisenaire rods provided empirical evidence in support of students' claims that specific sums were equivalent. We wonder about opportunities to move beyond this direct use of the tool in order to encourage students to engage in relational thinking and to make conjectures and generalizations. Note that there are affordances and constraints to any tool. An obvious constraint in this lesson was that only so many Cuisenaire rods were available for students to use and fewer still could be visibly aligned under a document camera. Some designers of tools aim for productive constraints in order to deliberately create opportunities for uncertainty. In this case, the constraint inherent in the rods and the doc cam might have been leveraged productively to invite students to reason about the equation without direct reference to the rods or to use the rods to move away from empirical arguments toward generalizations.

Sums of small numbers can be represented in lengths or rods and discussed, as earlier in the lesson. Then equations involving larger numbers open the door to uncertainty, thanks to the limitations of the tools. Further, going beyond justifying a specific equation, students are also capable of justifying general claims using visual representations (Schifter, 2009) in a process that is "accessible, powerful and generative for students" (p. 76). Beginning with representations for an arithmetic problem can be a good place to begin before moving toward challenging students to "consider how to argue for the truth of a claim about an infinite class" (p. 84).

In Ms. Kirk's classroom, students may be invited to anticipate whether or not the rod lengths should be equal, rather than to merely observe that they are. This might sound like a teacher saying, "Uh-oh! We can't fit so many rods under the doc cam. I'll tell you what: Instead of using the rods for this one, let's just use our brains. Decide whether you think this equation is true or false, and be prepared to share your reasoning." Or, a teacher could encourage students to generalize the situation using the rods at unspecified lengths.

Conclusion

A common thread running through the above themes is that argumentation is messy. In our analysis, we identified ways in which Ms. Kirk avoided that messiness: establishing a common definition of the equal sign, discouraging the use of large numbers, and funneling the discussion away from relational thinking and toward arguments in which the tool served as a source of empirical evidence. The literature points to ways of advancing the sophistication of argumentation in this and other classrooms: greater emphasis on students' ideas and accountability for making sense of these ideas, shared authority among the teacher and students, less restriction on the task instructions and tool use, and encouragement to move away from empirical certainty and toward uncertainty and conjecture. Each of these recommendations introduces messiness and will not help to ensure that lessons go smoothly. Instead, these recommendations prioritize the goal of students engaging in intellectually productive, accountable argumentation in which their mathematical ideas are valued, authority is shared, tasks are open, and tool use inspires progress toward generalizations.

References

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *The Elementary School Journal, 84*(2), 199–212.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Heinemann.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40–68.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3–4), 175–190.

Corbin, J., & Strauss, A. (2015). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524–549.

Horn, I. S. (2008). Accountable argumentation as a participation structure to support learning through disagreement. In A. H. Schoenfeld (Ed.), *A study of teaching: Multiple lenses, multiple views, Journal for Research in Mathematics Education Monograph* (Vol. 14, pp. 97–126). National Council of Teachers of Mathematics.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258–288.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). LEA.

McLean, R. C. (1964). Third-graders and the equal sign: Report of an experience. *The Arithmetic Teacher, 11*(1), 27.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Schifter, D. (2009). Representation-based proof in elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 71–86). NCTM and Routledge.

Siegel, M., & Borasi, R. (1996). Demystifying mathematics education through inquiry. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematical education* (pp. 218–232). The Falmer Series.

Stylianides, A. J. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics, 65*, 1–20.

Stylianou, D., & Blanton, M. (2018). *Teaching with mathematical argument*. Heinemann.

Toulmin, S. E. (1958/2003). *The uses of argument* (Rev. ed.). Cambridge University Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

Chepina Rumsey is an associate professor of mathematics education at the University of Northern Iowa. She is interested in how students in primary grades playfully explore mathematics, make conjectures, and share their ideas with others.

Dr. Ian Whitacre is an Associate Professor in the School of Teacher Education at Florida State University. His research has focused on the following topics: the mathematical thinking and learning of preservice elementary teachers, especially their development of number sense; the mathematical thinking of K-12 students, especially regarding integers and integer-related tasks; and mathematics teaching with technology, especially PhET Interactive Simulations.

Şebnem Atabaş is a doctoral candidate at Florida State University. She is doing her Ph.D. in Mathematics Education. Her dissertation work focuses on teacher learning, specifically how supplementary curricular materials combined with instructional technologies play a catalyst for change in mathematics classrooms—instructional practices, social norms, sociomathematical norms, and teachers' views of teaching and learning mathematics. Her research interests focus on designing learning experiences for teachers to reflect on their instructional practices and views and position their students to engage in mathematical thinking and reasoning in meaningful ways.

Jessica L. Smith is a doctoral candidate at Florida State University. Her research interests include equity and ways students think about and learn concepts in secondary and undergraduate mathematics courses. Within these interests tend to be ideas around classroom discourse, argumentation, and whose ways of reasoning are being highlighted in front of the class. She currently is a part of a project focused on integrating mathematics and computer science in middle school classes (CSIMMS) and another project involving the development of inquiry-oriented linear algebra units (IOLA-X).