Introduction: Conceptualizing Argumentation, Justification, and Proof in Mathematics Education

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Proof, argumentation, justification, reasoning, reasoning-and-proof, and many other terms collectively comprise a heavily debated space in mathematics education. Some of the contestation can be seen as a tug-of-war about the purposes of mathematics education. Some of it can be seen as a result of an applied field (math education) drawing on myriad frameworks, perspectives, and disciplines—viewed by some as a blessing and others a curse. Regardless of the forces shaping this disorderly landscape, there is plenty of current activity that aims to draw boundaries in this space to facilitate important tasks in the field of mathematics education such as synthesizing results, offering clear and useful guidance for policy, and supporting teachers in their work.

This book steps into this space from a unique perspective. Its origins began in conversations among small groups of colleagues and moved into a multi-year Psychology of Mathematics Education-North American (PME-NA) Chapter working group, *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof*. The working group took up questions about definitions, conceptualizations, and relationships between and among the constructs of argumentation, justification, and proof (e.g., Cirillo et al., 2015; Staples et al., 2016; Conner et al., 2017). We note that we did not intend to standardize definitions for these terms, but rather to explore how terms were used and the consequences of the specific uses. We felt the competing and overlapping uses of these terms had potential to hinder accumulation of research in these areas. That is, if I call argumentation what you call justification, will we be able to build on each other's work in

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meaningful ways? Or if I understand proof as a subset of argumentation, and you do not, how might we manage synthesizing our results and offer guidance to teachers? Thus, our goal for the working groups was to understand how researchers defined these constructs, how researchers interpreted the constructs as related to each other, and what the consequences of different definitions might be for research and teaching. The working groups led to many productive conversations and of course new questions. After investigating definitions, relationships, and consequences during the working groups, our conceptualization of this book began as a thought experiment: Could we see the consequences of using different constructs if we examined the same data with specific definitions of our constructs? Did the constructs play out differently depending on grade level?

Our hope, as we conceptualized this book, was to draw attention to the importance of how we define constructs related to argumentation, justification, and proof and to explore with our readers the potential consequences of using particular definitions when examining classroom data. Simultaneously, we intended to explore whether the same definition of a construct could be used across multiple grade bands in a way that was meaningful in each band. Finally, we wondered whether examining the same data using the three constructs would reveal different aspects of the mathematical activity within classrooms.

One goal of this introduction is to highlight the unique nature of this book. This book was not intended to be a compilation of viewpoints from key authors in the field, as edited volumes often are, though we did tap into an international group of respected scholars. Rather, the book was intended to be a knowledge-generation exercise. The chapter authors were charged to take a given definition of a particular construct (argumentation, justification, proof), in a particular grade band (elementary, middle grades, high school, tertiary), and draw upon the definition to analyze new-to-them data specific to their grade band. Their work produced 12 analyses of the data—grounded in the construct—accompanied by their reflective commentary and insights. Our synthesis authors—four focusing within a grade band (across constructs) and three focusing on a construct (across-grade bands)—were then charged with using those chapters as their food for thought, discerning themes, ideas, questions, lessons learned, shortcomings, and, as discussants often do, insights and new questions. In this way, our goal was not a review of the current state of the field but an exercise to play out and provide a window into the consequences of these different terms as situated within classroom data from students of different ages. In that sense, the book is a community thought experiment that has been initiated by the editors and authors and now continues with the reader.

In putting together this book, we are aware of the shoulders we are standing on as we try to look further out at the landscape. The reader will find reference to many of these researchers and their seminal works throughout the chapters, and we do not attempt to delineate them here. In the remainder of this introductory chapter, we share the organization and structure of this volume, the definitions we chose for this book, the rationale for our choices, and the guidance we gave to the authors. We also note the data used, though we leave a more formal introduction of the data to the beginning of each grade band section (e.g., Elementary, Middle Grades, High School, Tertiary).

Conceptions of Argumentation, Justification, and Proof in the Literature

Argumentation, justification, and proof each have their own histories and range of conceptions in the field of mathematics education. To provide some background and context for this book, we discuss briefly key conceptions of these terms. The discussion will not be comprehensive but rather informative, positioning our choice of definition of each construct for this book as "a case of" argumentation, justification, or proof. We conclude each of these sections with the definition chosen for this volume and a brief rationale for the choice. In general, we selected our definitions carefully, choosing a process-oriented definition of each construct. Our process orientation aligned with our selection of classroom data; we were interested in the consequences of using these definitions in analyzing what teachers and students are doing in classrooms. Our choice of definitions was based on our own experiences as researchers and teachers in the field of mathematics education. We entertained several possibilities for each construct definition; our final choices, along with specific considerations for each, are in the following paragraphs.

Argumentation

The term *argumentation* is widely used across disciplines and appears in policy documents for the teaching of mathematics, science, social studies, and English language arts and literacy in the United States (National Council of the Social Studies, 2013; National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA & CCSS), 2010a, 2010b; NGSS Lead States, 2013). While in common usage it connotes disagreement, in academic settings it is used to describe how one communicates ideas and support for those ideas, either in written or spoken language.

In mathematics education, argumentation has been defined in several related ways. These definitions often contain aspects of persuading or convincing, of drawing conclusions, and of defending or supporting conclusions with evidence or reasoning. In the *Encyclopedia of Mathematics Education*, Umland and Sriraman (2014) defined argumentation in mathematics as "the process of making an argument, that is, drawing conclusions based on a chain of reasoning" (p. 44), while they defined argumentation in mathematics education as "the mathematical arguments that students and teachers produce in mathematics classrooms" where a "mathematical argument" is "a line of reasoning that intends to show or explain why a mathematical result is true" (Sriraman & Umland, 2014, p. 46). The distinction here seems to be a focus on students' *intentions* in the mathematics education definition; the second definition can be seen as a subset of the first. That is, in mathematics, argumentation is only about the argument that is made; in mathematics education, there is consideration of both audience and intention. Wood (1999) defined argumentation as "discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (p. 172). Wood's definition contains two aspects absent in Sriraman and Umland's definitions: argumentation as part of the discourse of the classroom and argumentation for the purpose of convincing others. Fukawa-Connelly and Silverman (2015) define argumentation as "using definitions and previously established results to develop conjectures and to explore and verify the truth of conjectures" (p. 449). Fukawa-Connelly and Silverman maintain a focus on intention (to explore and verify conjectures) and introduce an element of formality into a definition of argumentation (using definitions and previously established results); Pedemonte and Balacheff (2016) combine this formality with intent to communicate in their definition of argumentation as "a dynamic and reflective tool to communicate content, ideas, [and] epistemic values" (p. 105). The definition of argumentation chosen for this volume contains aspects of several of these definitions without the formality that is captured in our definition of proof.

Our definition for argumentation was chosen to capture a wide range of potential arguments in a classroom. It is written simply, without specific reference to norms or discourse. It is also close to, but not identical with, definitions that have been used by two of the editors of this volume (e.g., Conner et al., 2014; Kosko, 2016). We defined mathematical argumentation as *the process of making mathematical claims and providing evidence to support them*. The actions and processes involved in argumentation are evident in this definition: making claims and providing evidence.

Justification

Justification, and the related term justify, is also widely used, particularly in K–12 settings. Like argumentation, justification applies to academic settings broadly, as well as to everyday life. Its usage seems to be more prevalent in K–12 classrooms and less prevalent as a focus of research studies in mathematics education in comparison with argumentation and proof. Unlike proof and proving, where the objects of inquiry when proving tend to be conjectures, theorems, or other well-formed mathematical claims, in K–12 classrooms, and even at the undergraduate level, teachers ask students to justify their answer, justify their results, justify their method, justify why something is true, justify their reasoning, and justify their thinking. The term is not used as a synonym for proof (e.g., we generally don't say justify the conjecture or theorem) but as a call to explicate one's thinking in order to compel a position (whether that be a result, idea, or choice).

As noted, justification does not have the same research tradition as the other terms but can be found in the literature in mathematics education. In the 1990s, Cobb, Wood, Yackel, and others (Cobb et al., 1992; Wood, 1999; Wood et al., 2006; Yackel & Cobb, 1996) discussed a tight relationship between explanation and justification, with the core difference being the speaker's perception of whether her activity was explicating or defending. Indeed, their focus was on *situations for justification,* and they were interested in the nature of the responses offered in these situations, which then revealed sociomathematical norms guiding the mathematical activity in that classroom. Simon and Blume (1996) similarly thought about justifications not as a particular type of logical chain defined by specific features but rather as the set of responses students offered when called upon to provide mathematical evidence in support of a result. They were curious about the nature of preservice teacher justifications and what criteria emerged in that community to govern justifications that were acceptable to the community. As used in these studies, justifications are the set of responses that are offered when students are in a situation for justification.

These early usages are also reflected in Dreyfus (1999). Citing Margolinas (1992), Dreyfus discusses a difference between a descriptive mode and a justificative mode of thinking. A key difference seems to be the role the activity is playing in the classroom community. The intention of the actor is salient in whether an utterance is interpreted as a justification or an explanation (more descriptive).

Other researchers and documents have more formally defined justification or developed frameworks to categorize the nature of justification offered by students. The National Research Council (Kilpatrick et al., 2001) used the short definition, "to provide sufficient reason for" and elaborated as follows:

> We use justify in the sense of "provide sufficient reason for." Proof is a form of justification, but not all justifications are proofs. Proofs (both formal and informal) must be logically complete, but a justification may be more telegraphic, merely suggesting the source of the reasoning. (p. 130)

This usage of justifying and justification positions justifying as a practice tightly connected to proof and proving; it can describe on-the-way-to-proof reasoning practices that have value in the classroom, but that would feel inaccurate to call proof activity. This is reflected in the use of the phrase "informal justification." Such descriptors, however, along with others, such as "incomplete justification," seem to reveal the field's potential lack of clarity around the term.

Turning to a current guiding curricular document in the United States, we note the use of justify and justification in the Common Core State Standards (NGA & CCSS, 2010b). (It is of interest to note that CCSSM uses versions of all three terms—argument, proof/prove, and justify—but not the specific terms argumentation or justification.) CCSSM's usage of justify seems to indicate activities in which students are called on to warrant a mathematical claim, conclusion, or choice. For example, in CCSSM, the standards indicate that students can justify a result (e.g., justify formulas, Grade 6), an interpretation of data (e.g., justify conclusions from surveys, High School–statistics), or an approach or method (e.g., justify using multiplication to determine the area of a rectangle, Grade 3; justify a solution method, High School-reasoning with equations and inequalities).

The selected definition for justification for the book is consistent with CCSSM's use, the National Research Council's (2001) description, as well as other instances (e.g., Staples & Lesseig, 2020) and was based on the experiences of two members of our editor team. Bieda and Staples (2020) defined mathematical justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (p. 103). The actions or processes referenced in this definition include supporting claims and choices and explaining why your claim or answer makes sense. It is worth noting that the definition also focuses on student activity rather than disciplinary activity.

Proof

The construct of proof is perhaps the most widely used of the three constructs within mathematics and mathematics education. Researchers in mathematics education have investigated students' work with mathematical proof since the 1970s (e.g., Bell, 1976), influenced by, among other sources, Lakatos' *Proofs and Refutations* (1976). Bell provided one of the earliest definitions of proof in mathematics education: He defined a proof as "a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles" (p. 26). However, Bell also noted that proof is "an essentially public activity," that students may have difficulty with a definition of proof that focuses on conviction, and that in mathematics, proof has at least three roles: verification or justification, illumination, and systematization (p. 24). Multiple authors have remarked upon the difficulty of defining proof (e.g., Dreyfus, 1999) and have suggested that mathematicians may disagree about whether a particular argument is or is not a proof (e.g., Weber, 2008).

It seems that most mathematics educators would agree that the definition of proof to be used must consider the context in which it is used as well as the strictures of the discipline of mathematics. However, as Weber (2014) reported, there is no consensus about a definition of proof shared by the mathematics education community. Weber, in the definition chosen for this volume, collated criteria previously proposed by mathematicians and proposed that a proof is a clustered concept that can be described by six criteria. He further explained what he meant by proof is a clustered concept as follows:

> (a) proofs that satisfied all of these criteria should be uncontroversial, but some proofs that satisfy only a subset of these criteria might be regarded as contentious; (b) compound words exist that qualify proofs that satisfy some of these criteria but not others; (c) it would be desirable for proofs to satisfy all six criteria. (Weber, 2014, p. 358)

We chose a definition of proof that we related to the process of proving, given our commitment to exploring classroom data. Arguably, this definition for proof is the most product-oriented definition we put forward, drawing on Weber's (2014) clustered concept of proof. To frame our interest in proving as a process, we define mathematical proving as a process by which the prover generates a product that has either all or a significant subset of the following characteristics:

> (1) A proof is a *convincing argument* that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a *deductive argument* that does not admit possible rebuttals. (3) A proof is a *transparent argument where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation. (4) A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*. (5) A proof is an *argument within a representation system satisfying communal norms*. (6) A proof is an *argument that has been sanctioned by the mathematical community*. (Weber, 2014, p. 357)

One member of the editorial team had seen significant utility of this definition when she used it in a mathematics education course, in that it helped graduate students think flexibly about the conditions under which an argument is a proof as well as the purposes proof can serve from a disciplinary standpoint. We saw this definition as having wide applicability, given that it does not require all characteristics to be satisfied, and as having potential for identifying multiple actions related to proving in classrooms.

Charges Given to Authors/Synthesizers

In conceptualizing the book, we wanted to generate specific and useful-to-look-across examples of using a construct on a data set and to generate reflective commentary about the uses of constructs and the consequences of these uses. As such, we invited authors for two types of chapters. The first group of authors—our chapter authors—were given a construct (with definition) and data from a classroom at a particular grade band and charged with using the construct to make sense of the mathematical activity in the classroom. For example, an author was given the construct *justification*, with definition, and asked to use that to analyze data from a *middle grades classroom* to provide insight and/or understanding into the justifying activity in that classroom episode. These authors, across grade levels and constructs, each brought additional frameworks to bear on the data, depending on their inquiry. They were given leeway to augment or modify the definitions for the purposes of conducting their analyses, as needed, with the request that they note constraints and affordances of the provided definition of their construct. These chapters were peer-reviewed by the editorial team, another chapter author, and an outside reviewer. Many of our authors and reviewers were participants in our PME-NA working groups.

The second group of authors—our synthesis authors—were provided with the analysis chapters. They read the chapters and were asked to provide perspectives and illustrate the consequences of applying different lenses to the same data set (grade band synthesis) or applying the same lens across data sets from different grade bands (construct synthesis). They were also asked to discuss implications for research and teacher education. There were two different foci for the synthesis chapters. One focus was *different constructs on the same data set* (grade band). These construct synthesis authors read the chapters *within a* grade band (i.e., elementary, middle grades, high school, tertiary) to consider what we could glean from the use of the three different constructs on the same data set. (These chapters are found at the end of each grade band section.) The second focus was *the same construct across data sets at different grade bands*. The authors of this second set of synthesis chapters looked *within a construct* (i.e., argumentation, justification, proof) to consider what we could glean from the use of the same construct as it played out at the different grade levels. This latter set of syntheses chapters is located in the last section of the book. The synthesis chapter authors were provided with an overview of the data and the analytic chapters as the basis of their work.

Organization of This Volume

As the structure of this book is both unconventional and meaningful, we pause to reiterate the different sections and their purposes. The book comprises an introductory chapter (which you are reading now) and five parts. The first four of these five sections have parallel structures, organized by grade band—Elementary, Middle Grades, High School, and Tertiary. Each section begins with an introduction to the data set used for that section's analyses and is followed by three chapters—one each focusing on one of justification, argumentation, and proof—and concludes with the across-construct synthesis for that grade band. The fifth and final section comprises the across-grade band synthesis for each of our constructs, and it is followed by a concluding chapter by two of the editors.

The reader is encouraged to read all chapters but need not go linearly through the book. One's interest might take her first to all chapters on a given construct, or all chapters within a grade band. Alternatively, a reader might be interested in first reading the synthesized ideas related to a construct and then looking at the four data-based chapters using that construct. Each chapter can stand alone, though engaging the ideas fully requires entering into multiple chapters with multiple perspectives. We hope that readers will continue this knowledge-generating activity by engaging with authors in exploring the consequences of the definitions of these constructs.

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