Research in Mathematics Education

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Kristen N. Bieda
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Karl W. Kosko
Megan Staples *Editors*

Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof

Springer

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Series Editors

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Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof

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Foreword

The notion of mathematical proof, the ways in which mathematical conjectures or statements are tested formally, has had an important but also controversial role in school mathematics. The importance of proof is obvious as it is related to the nature of mathematics. It is controversial not only because the traditional Euclidean geometry course is about proof, but also there are several emerging notions related to proof, such as justification, argumentation, and reasoning. This volume involves the reader in this controversy in a delightful fashion, directly addressing the different conceptualizations of argumentation, justification, and proof, and how these conceptualizations result in different ways of research, learning, and teaching, but also synthesizing the perspectives in an innovative manner to inform future work.

The editors of this volume in the Research in Mathematics Education Series take a unique approach to unpacking these meanings. A set of scholars in the field of argumentation, justification, and proof were given the same data and asked to analyze it: The meat of the volume consists of these analyses of data at elementary, middle, and high school and tertiary levels. The analyses highlight different ways in which perspectives that focus on argumentation differ from those emphasizing justification, or those emphasizing proof. The editors should be applauded for their attempt to distinguishing proof from argumentation and justification.

The dessert course, to continue the metaphor, engages other scholars in synthesizing the analyses at each grade band, neatly identifying the unique contributions to research afforded by each perspective, as well as their commonalities. Finally, construct syntheses are provided, discussing the theory within each perspective, tying the body of thought together into a meaningful framework to guide future research and development. The result is a comprehensive, projective take on the broader field that has profound implications for practice.

Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared towards highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those in the intersection between researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission of this book series is:

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We are grateful for the support of Melissa James from Springer in developing and publishing this book series, as well as the support in the publication of this volume. And finally, we thank the editors (Kristen Bieda, AnnaMarie Conner, Karl Kosko and Megan Staples) and all of the authors and peer reviewers who have contributed to this volume for their insightful and synthetic work!

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Acknowledgments

This book would not have been possible without the contributions of scholars who served the roles of chapter authors and external reviewers. Authors met with section editors, reviewed each other's work within a part, and served as reviewers for chapters in other parts. Synthesis chapter authors also served a critical role in providing useful insights regarding the contributions of chapter authors. We wish to thank those scholars serving as external reviewers for the book who lent their time and expertise to improve the quality and rigor of this text. All four coeditors contributed equally to the creation of the book.

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Introduction: Conceptualizing Argumentation, Justification, and Proof in Mathematics Education

Megan Staples and AnnaMarie Conner

Proof, argumentation, justification, reasoning, reasoning-and-proof, and many other terms collectively comprise a heavily debated space in mathematics education. Some of the contestation can be seen as a tug-of-war about the purposes of mathematics education. Some of it can be seen as a result of an applied field (math education) drawing on myriad frameworks, perspectives, and disciplines—viewed by some as a blessing and others a curse. Regardless of the forces shaping this disorderly landscape, there is plenty of current activity that aims to draw boundaries in this space to facilitate important tasks in the field of mathematics education such as synthesizing results, offering clear and useful guidance for policy, and supporting teachers in their work.

This book steps into this space from a unique perspective. Its origins began in conversations among small groups of colleagues and moved into a multi-year Psychology of Mathematics Education-North American (PME-NA) Chapter working group, *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof*. The working group took up questions about definitions, conceptualizations, and relationships between and among the constructs of argumentation, justification, and proof (e.g., Cirillo et al., 2015; Staples et al., 2016; Conner et al., 2017). We note that we did not intend to standardize definitions for these terms, but rather to explore how terms were used and the consequences of the specific uses. We felt the competing and overlapping uses of these terms had potential to hinder accumulation of research in these areas. That is, if I call argumentation what you call justification, will we be able to build on each other's work in

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meaningful ways? Or if I understand proof as a subset of argumentation, and you do not, how might we manage synthesizing our results and offer guidance to teachers? Thus, our goal for the working groups was to understand how researchers defined these constructs, how researchers interpreted the constructs as related to each other, and what the consequences of different definitions might be for research and teaching. The working groups led to many productive conversations and of course new questions. After investigating definitions, relationships, and consequences during the working groups, our conceptualization of this book began as a thought experiment: Could we see the consequences of using different constructs if we examined the same data with specific definitions of our constructs? Did the constructs play out differently depending on grade level?

Our hope, as we conceptualized this book, was to draw attention to the importance of how we define constructs related to argumentation, justification, and proof and to explore with our readers the potential consequences of using particular definitions when examining classroom data. Simultaneously, we intended to explore whether the same definition of a construct could be used across multiple grade bands in a way that was meaningful in each band. Finally, we wondered whether examining the same data using the three constructs would reveal different aspects of the mathematical activity within classrooms.

One goal of this introduction is to highlight the unique nature of this book. This book was not intended to be a compilation of viewpoints from key authors in the field, as edited volumes often are, though we did tap into an international group of respected scholars. Rather, the book was intended to be a knowledge-generation exercise. The chapter authors were charged to take a given definition of a particular construct (argumentation, justification, proof), in a particular grade band (elementary, middle grades, high school, tertiary), and draw upon the definition to analyze new-to-them data specific to their grade band. Their work produced 12 analyses of the data—grounded in the construct—accompanied by their reflective commentary and insights. Our synthesis authors—four focusing within a grade band (across constructs) and three focusing on a construct (across-grade bands)—were then charged with using those chapters as their food for thought, discerning themes, ideas, questions, lessons learned, shortcomings, and, as discussants often do, insights and new questions. In this way, our goal was not a review of the current state of the field but an exercise to play out and provide a window into the consequences of these different terms as situated within classroom data from students of different ages. In that sense, the book is a community thought experiment that has been initiated by the editors and authors and now continues with the reader.

In putting together this book, we are aware of the shoulders we are standing on as we try to look further out at the landscape. The reader will find reference to many of these researchers and their seminal works throughout the chapters, and we do not attempt to delineate them here. In the remainder of this introductory chapter, we share the organization and structure of this volume, the definitions we chose for this book, the rationale for our choices, and the guidance we gave to the authors. We also note the data used, though we leave a more formal introduction of the data to the beginning of each grade band section (e.g., Elementary, Middle Grades, High School, Tertiary).

Conceptions of Argumentation, Justification, and Proof in the Literature

Argumentation, justification, and proof each have their own histories and range of conceptions in the field of mathematics education. To provide some background and context for this book, we discuss briefly key conceptions of these terms. The discussion will not be comprehensive but rather informative, positioning our choice of definition of each construct for this book as "a case of" argumentation, justification, or proof. We conclude each of these sections with the definition chosen for this volume and a brief rationale for the choice. In general, we selected our definitions carefully, choosing a process-oriented definition of each construct. Our process orientation aligned with our selection of classroom data; we were interested in the consequences of using these definitions in analyzing what teachers and students are doing in classrooms. Our choice of definitions was based on our own experiences as researchers and teachers in the field of mathematics education. We entertained several possibilities for each construct definition; our final choices, along with specific considerations for each, are in the following paragraphs.

Argumentation

The term *argumentation* is widely used across disciplines and appears in policy documents for the teaching of mathematics, science, social studies, and English language arts and literacy in the United States (National Council of the Social Studies, 2013; National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA & CCSS), 2010a, 2010b; NGSS Lead States, 2013). While in common usage it connotes disagreement, in academic settings it is used to describe how one communicates ideas and support for those ideas, either in written or spoken language.

In mathematics education, argumentation has been defined in several related ways. These definitions often contain aspects of persuading or convincing, of drawing conclusions, and of defending or supporting conclusions with evidence or reasoning. In the *Encyclopedia of Mathematics Education*, Umland and Sriraman (2014) defined argumentation in mathematics as "the process of making an argument, that is, drawing conclusions based on a chain of reasoning" (p. 44), while they defined argumentation in mathematics education as "the mathematical arguments that students and teachers produce in mathematics classrooms" where a "mathematical argument" is "a line of reasoning that intends to show or explain why a mathematical result is true" (Sriraman & Umland, 2014, p. 46). The distinction here seems to be a focus on students' *intentions* in the mathematics education definition; the second definition can be seen as a subset of the first. That is, in mathematics, argumentation is only about the argument that is made; in mathematics education, there is consideration of both audience and intention. Wood (1999) defined argumentation as "discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (p. 172). Wood's definition contains two aspects absent in Sriraman and Umland's definitions: argumentation as part of the discourse of the classroom and argumentation for the purpose of convincing others. Fukawa-Connelly and Silverman (2015) define argumentation as "using definitions and previously established results to develop conjectures and to explore and verify the truth of conjectures" (p. 449). Fukawa-Connelly and Silverman maintain a focus on intention (to explore and verify conjectures) and introduce an element of formality into a definition of argumentation (using definitions and previously established results); Pedemonte and Balacheff (2016) combine this formality with intent to communicate in their definition of argumentation as "a dynamic and reflective tool to communicate content, ideas, [and] epistemic values" (p. 105). The definition of argumentation chosen for this volume contains aspects of several of these definitions without the formality that is captured in our definition of proof.

Our definition for argumentation was chosen to capture a wide range of potential arguments in a classroom. It is written simply, without specific reference to norms or discourse. It is also close to, but not identical with, definitions that have been used by two of the editors of this volume (e.g., Conner et al., 2014; Kosko, 2016). We defined mathematical argumentation as *the process of making mathematical claims and providing evidence to support them*. The actions and processes involved in argumentation are evident in this definition: making claims and providing evidence.

Justification

Justification, and the related term justify, is also widely used, particularly in K–12 settings. Like argumentation, justification applies to academic settings broadly, as well as to everyday life. Its usage seems to be more prevalent in K–12 classrooms and less prevalent as a focus of research studies in mathematics education in comparison with argumentation and proof. Unlike proof and proving, where the objects of inquiry when proving tend to be conjectures, theorems, or other well-formed mathematical claims, in K–12 classrooms, and even at the undergraduate level, teachers ask students to justify their answer, justify their results, justify their method, justify why something is true, justify their reasoning, and justify their thinking. The term is not used as a synonym for proof (e.g., we generally don't say justify the conjecture or theorem) but as a call to explicate one's thinking in order to compel a position (whether that be a result, idea, or choice).

As noted, justification does not have the same research tradition as the other terms but can be found in the literature in mathematics education. In the 1990s, Cobb, Wood, Yackel, and others (Cobb et al., 1992; Wood, 1999; Wood et al., 2006; Yackel & Cobb, 1996) discussed a tight relationship between explanation and justification, with the core difference being the speaker's perception of whether her activity was explicating or defending. Indeed, their focus was on *situations for justification,* and they were interested in the nature of the responses offered in these situations, which then revealed sociomathematical norms guiding the mathematical activity in that classroom. Simon and Blume (1996) similarly thought about justifications not as a particular type of logical chain defined by specific features but rather as the set of responses students offered when called upon to provide mathematical evidence in support of a result. They were curious about the nature of preservice teacher justifications and what criteria emerged in that community to govern justifications that were acceptable to the community. As used in these studies, justifications are the set of responses that are offered when students are in a situation for justification.

These early usages are also reflected in Dreyfus (1999). Citing Margolinas (1992), Dreyfus discusses a difference between a descriptive mode and a justificative mode of thinking. A key difference seems to be the role the activity is playing in the classroom community. The intention of the actor is salient in whether an utterance is interpreted as a justification or an explanation (more descriptive).

Other researchers and documents have more formally defined justification or developed frameworks to categorize the nature of justification offered by students. The National Research Council (Kilpatrick et al., 2001) used the short definition, "to provide sufficient reason for" and elaborated as follows:

> We use justify in the sense of "provide sufficient reason for." Proof is a form of justification, but not all justifications are proofs. Proofs (both formal and informal) must be logically complete, but a justification may be more telegraphic, merely suggesting the source of the reasoning. (p. 130)

This usage of justifying and justification positions justifying as a practice tightly connected to proof and proving; it can describe on-the-way-to-proof reasoning practices that have value in the classroom, but that would feel inaccurate to call proof activity. This is reflected in the use of the phrase "informal justification." Such descriptors, however, along with others, such as "incomplete justification," seem to reveal the field's potential lack of clarity around the term.

Turning to a current guiding curricular document in the United States, we note the use of justify and justification in the Common Core State Standards (NGA & CCSS, 2010b). (It is of interest to note that CCSSM uses versions of all three terms—argument, proof/prove, and justify—but not the specific terms argumentation or justification.) CCSSM's usage of justify seems to indicate activities in which students are called on to warrant a mathematical claim, conclusion, or choice. For example, in CCSSM, the standards indicate that students can justify a result (e.g., justify formulas, Grade 6), an interpretation of data (e.g., justify conclusions from surveys, High School–statistics), or an approach or method (e.g., justify using multiplication to determine the area of a rectangle, Grade 3; justify a solution method, High School-reasoning with equations and inequalities).

The selected definition for justification for the book is consistent with CCSSM's use, the National Research Council's (2001) description, as well as other instances (e.g., Staples & Lesseig, 2020) and was based on the experiences of two members of our editor team. Bieda and Staples (2020) defined mathematical justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (p. 103). The actions or processes referenced in this definition include supporting claims and choices and explaining why your claim or answer makes sense. It is worth noting that the definition also focuses on student activity rather than disciplinary activity.

Proof

The construct of proof is perhaps the most widely used of the three constructs within mathematics and mathematics education. Researchers in mathematics education have investigated students' work with mathematical proof since the 1970s (e.g., Bell, 1976), influenced by, among other sources, Lakatos' *Proofs and Refutations* (1976). Bell provided one of the earliest definitions of proof in mathematics education: He defined a proof as "a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles" (p. 26). However, Bell also noted that proof is "an essentially public activity," that students may have difficulty with a definition of proof that focuses on conviction, and that in mathematics, proof has at least three roles: verification or justification, illumination, and systematization (p. 24). Multiple authors have remarked upon the difficulty of defining proof (e.g., Dreyfus, 1999) and have suggested that mathematicians may disagree about whether a particular argument is or is not a proof (e.g., Weber, 2008).

It seems that most mathematics educators would agree that the definition of proof to be used must consider the context in which it is used as well as the strictures of the discipline of mathematics. However, as Weber (2014) reported, there is no consensus about a definition of proof shared by the mathematics education community. Weber, in the definition chosen for this volume, collated criteria previously proposed by mathematicians and proposed that a proof is a clustered concept that can be described by six criteria. He further explained what he meant by proof is a clustered concept as follows:

> (a) proofs that satisfied all of these criteria should be uncontroversial, but some proofs that satisfy only a subset of these criteria might be regarded as contentious; (b) compound words exist that qualify proofs that satisfy some of these criteria but not others; (c) it would be desirable for proofs to satisfy all six criteria. (Weber, 2014, p. 358)

We chose a definition of proof that we related to the process of proving, given our commitment to exploring classroom data. Arguably, this definition for proof is the most product-oriented definition we put forward, drawing on Weber's (2014) clustered concept of proof. To frame our interest in proving as a process, we define mathematical proving as a process by which the prover generates a product that has either all or a significant subset of the following characteristics:

> (1) A proof is a *convincing argument* that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a *deductive argument* that does not admit possible rebuttals. (3) A proof is a *transparent argument where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation. (4) A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*. (5) A proof is an *argument within a representation system satisfying communal norms.* (6) A proof is an *argument that has been sanctioned by the mathematical community.* (Weber, 2014, p. 357)

One member of the editorial team had seen significant utility of this definition when she used it in a mathematics education course, in that it helped graduate students think flexibly about the conditions under which an argument is a proof as well as the purposes proof can serve from a disciplinary standpoint. We saw this definition as having wide applicability, given that it does not require all characteristics to be satisfied, and as having potential for identifying multiple actions related to proving in classrooms.

Charges Given to Authors/Synthesizers

In conceptualizing the book, we wanted to generate specific and useful-to-look-across examples of using a construct on a data set and to generate reflective commentary about the uses of constructs and the consequences of these uses. As such, we invited authors for two types of chapters. The first group of authors—our chapter authors—were given a construct (with definition) and data from a classroom at a particular grade band and charged with using the construct to make sense of the mathematical activity in the classroom. For example, an author was given the construct *justification*, with definition, and asked to use that to analyze data from a *middle grades classroom* to provide insight and/or understanding into the justifying activity in that classroom episode. These authors, across grade levels and constructs, each brought additional frameworks to bear on the data, depending on their inquiry. They were given leeway to augment or modify the definitions for the purposes of conducting their analyses, as needed, with the request that they note constraints and affordances of the provided definition of their construct. These chapters were peer-reviewed by the editorial team, another chapter author, and an outside reviewer. Many of our authors and reviewers were participants in our PME-NA working groups.

The second group of authors—our synthesis authors—were provided with the analysis chapters. They read the chapters and were asked to provide perspectives and illustrate the consequences of applying different lenses to the same data set (grade band synthesis) or applying the same lens across data sets from different grade bands (construct synthesis). They were also asked to discuss implications for research and teacher education. There were two different foci for the synthesis chapters. One focus was *different constructs on the same data set* (grade band). These construct synthesis authors read the chapters *within a* grade band (i.e., elementary, middle grades, high school, tertiary) to consider what we could glean from the use of the three different constructs on the same data set. (These chapters are found at the end of each grade band section.) The second focus was *the same construct across data sets at different grade bands.* The authors of this second set of synthesis chapters looked *within a construct* (i.e., argumentation, justification, proof) to consider what we could glean from the use of the same construct as it played out at the different grade levels. This latter set of syntheses chapters is located in the last section of the book. The synthesis chapter authors were provided with an overview of the data and the analytic chapters as the basis of their work.

Organization of This Volume

As the structure of this book is both unconventional and meaningful, we pause to reiterate the different sections and their purposes. The book comprises an introductory chapter (which you are reading now) and five parts. The first four of these five sections have parallel structures, organized by grade band—Elementary, Middle Grades, High School, and Tertiary. Each section begins with an introduction to the data set used for that section's analyses and is followed by three chapters—one each focusing on one of justification, argumentation, and proof—and concludes with the across-construct synthesis for that grade band. The fifth and final section comprises the across-grade band synthesis for each of our constructs, and it is followed by a concluding chapter by two of the editors.

The reader is encouraged to read all chapters but need not go linearly through the book. One's interest might take her first to all chapters on a given construct, or all chapters within a grade band. Alternatively, a reader might be interested in first reading the synthesized ideas related to a construct and then looking at the four data-based chapters using that construct. Each chapter can stand alone, though engaging the ideas fully requires entering into multiple chapters with multiple perspectives. We hope that readers will continue this knowledge-generating activity by engaging with authors in exploring the consequences of the definitions of these constructs.

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Part I
Argumentation, Justification and Proof in an Elementary Classroom

Overview of the Elementary Level Data

Karl W. Kosko

This section focuses on exploring the constructs of argumentation, justification, and proof in the context of one elementary grade's lesson. These constructs can often be interpreted differently for the elementary grades, as well as within the elementary grades. Specifically, first-grade teachers consider justification and proof differently than third-grade teachers, and both groups may consider these constructs differently for their students than college mathematics instructors (Rogers & Kosko, 2019). For example, generalization may be fundamental for argumentation and proof in the teaching and learning of advanced mathematics, but it is not always observed as part of these processes in certain elementary grades (Krummheuer, 2007). This might signal a call for more argumentation, justification, and proof in elementary grades, but it may also be contingent on a developmental trend for how and when children develop aspects of argumentation, justification, and proof (Kosko & Zimmerman, 2019). Such issues are also related to how one defines and enacts these constructs: the core emphasis of this book. To this end, the following three chapters include analysis of the same transcript and classroom artifacts from Ms. Kirk's second-grade classroom.

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Ms. Kirk's Lesson

The class includes students from an economically diverse, suburban school district in a Midwestern US state.[1] Ms. Kirk and her students were participants in a quasi-experimental study comparing students' understanding of equivalence when students participate in one of three conditions. Over a 20-week period, one teacher was provided weekly tasks related to the concept of equivalence. Additionally, the teacher had their students explain and justify their reasoning in writing with each task. Ms. Kirk was provided the same tasks over the same period of time but was only asked to engage her students in discussions to explain and justify their reasoning. A third teacher was not provided the tasks but was provided with the same manipulatives and tools as the other two classrooms in the study (number balances, Cuisenaire rods, and various number line resources). In addition to classroom observations, each teacher was also interviewed at three points in the 20-week period regarding their mathematics pedagogy in relation to concepts being observed in their classroom.

The data described in this chapter come from a math lesson at the end of the first week following Winter Break. Prior to the break, Ms. Kirk and her students used number balances for 5 weeks to develop understanding that both sides of an equation should be equal. In the weeks leading up to Winter Break, the class was introduced to Cuisenaire rods so that they might begin to examine why both sides of an equation are equal (see Fig. 1). As a measurement model, Cuisenaire rods can be used to represent various mathematical relationships and provide a context for discussion of such relationships. Figure 1 provides an illustrative example used in Ms. Kirk's lesson in which light green (3 cm) and purple (4 cm) rods are juxtaposed with red (2 cm) and yellow (5 cm) rods to convey equivalent lengths (i.e., 3 + 4 = 2 + 5).

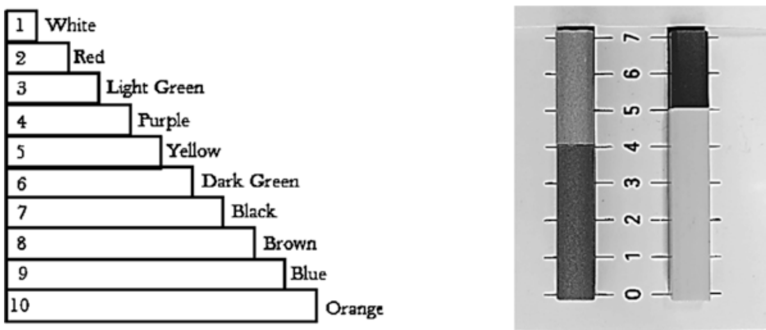

Fig. 1 Cuisenaire rod lengths by color with example illustrating 4 + 3 = 5 + 2

[1] The school district had adopted Investigations in Mathematics (2nd Edition) as their textbook, provided frequent professional development opportunities for elementary mathematics, and advocated reform-oriented pedagogy in each of their schools.

Description of the Lesson

The lesson begins with Ms. Kirk pointing to an equation written on the blackboard: 14 + 3 = 15 + 2. At their tables, students have Cuisenaire rods, centimeter-based number lines, and number balances for reference. Wanting to refresh students' memories after the long break, she points to the equals sign asking students what it means to them. Landon says "the same as." Katie says "equal." Jacob says "combined." Some students recall manipulatives they used to solve equivalence tasks. Hayley notes Unifix cubes, Madeline points to the Cuisenaire rods at their group tables, and Kaylyn describes the number balance. After several minutes of discussing students' meanings of the equals sign and how they used different tools to understand it, the teacher returns to the equation on the board and tells the students to work with their peers, using the Cuisenaire rods, to show whether or not the equation is true.

After several minutes of students working together, Ms. Kirk stops the class to discuss one group's representation of the problem:

> **Ms. Kirk**: So I was talking with Carrie and Amanda, and they represented the problem really well. They showed me the purple, and said it was four. And the green and said it was three. And then they showed me the yellow and said it was five. And the red and said it was two. And I took away from them their number line and I said "just using this…can you prove to me that four plus three is the same as five plus two?"

As Ms. Kirk talks, she notably takes the 4 + 3 from 14 + 3 and 5 + 2 from 15 + 2 that is represented on Carrie and Amanda's table. Following her question about showing they are equal without the number line, which identifies each individual unit, various students loudly shout yes. Surveying the class, Ms. Kirk calls on Sammy to explain how they are equivalent. Moving to the document camera where Ms. Kirk had placed the purple (4) and light green (3) rods next to the yellow (5) and red (2) rods, Sammy states that "if you go make the four a five and the three a two and that would make the seven and it would be equal." The teacher asks how she can be sure that the two are equal, to which Sammy replies that "you can tell if you put them together." Sammy carefully pushes the 4 + 3 length to juxtapose the 5 + 2 length (see Fig. 1 for reference). After confirming that both pairs of rods are the same length, Ms. Kirk and her students move to the next task.

Ms. Kirk passes out a recording sheet with ____ + ____ = ____ + ____ written at the top. "With your partner, I'm asking you to discuss and plan how you can use the Cuisenaire rods to help you design a math equation that is true. Explain how you can prove that it is true." After a brief discussion clarifying the task at hand, students return to their seats and work for several minutes creating their own equations and using the Cuisenaire rods and number lines to model various equations:

> **Ms. Kirk**: Sammy and her partner [Jacob] also worked on something interesting. They tried to go after a larger number that they could still prove with their Cuisenaire rods. Here, I'll move this up here and you guys can explain.

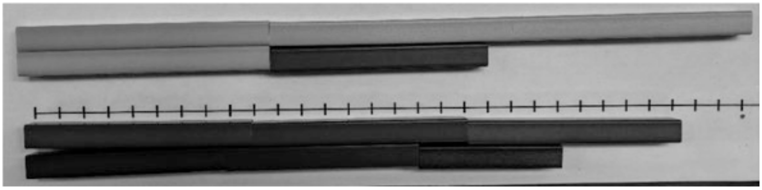

Fig. 2 Recreation of Jacob and Sammy's rod arrangement for 40 + 9 = 27 + 22

Fig. 3 Recreation of Jacob and Sammy's rod arrangement on the floor

Ms. Kirk moves the students' work and places their number line and rods on the document camera. Jacob begins pointing at the two expressions modeled, 40 + 9 and 27 + 22 (see Fig. 2):[2]

Jacob: So this one is 10, this one is 10, 10 and 10. So this one is nine. So that's 49. 40 plus nine equals the same as 27 plus 22.
Ms. Kirk: Woah. So it's a little bit hard to tell that these are the same because they broke their rods in half to fit on their number line.
Katie: Why did they break it in half?
Ms. Kirk: Because they were trying to get it to fit on the number line.
Katie: Oh.
Ms. Kirk: If we just take these, can you guys line those up all in one line for me?

Jacob and Sammy begin to arrange the rods side by side on the floor where the rest of the class can see it from their tables (see Fig. 3). As they are arranging the rods, Ms. Kirk continues the discussion with the rest of the class:

Ms. Kirk: If what they're saying is true [40+9=27+22], if we lined each side up, what should it look like? Tony.
Tony: A big line that is the same on both sides and is equal.
Ms. Kirk: It's the same length, right? The lengths are equal. I'm not sure if you can see from your seats, but when we line them up next to each other….
Katie: They are equal.
Ms. Kirk: They are equal.
Katie: Yeah.

At this point in the lesson, the teacher looks at the clock and asks students to place their Cuisenaire rods back in their containers and to leave everything else at their tables. The students then line up to leave the classroom to go to a different classroom.

[2] Partners worked with individual Cuisenaire rod sets (72 rods), including only four each of orange rods (10 cm), blue rods (9 cm), and brown rods (8 cm). Smaller rod lengths were included at higher frequencies in each set.

Conclusion

The authors in this section conducted secondary analysis of the transcribed data, along with samples of children's written work and images of interactions (student-to-student and students-to-teacher) from the lesson. It represents only a snapshot of classroom practice in Ms. Kirk's classroom. As such, the authors were limited in certain aspects of their analysis (i.e., sociomathematical norms, typical tasks used in the class). However, a shared analysis of a snapshot can be useful in understanding the contributions of different theoretical perspectives and analytic approaches. Independently, each of the next three chapters offers a useful conceptualization of argumentation, justification, and proof. In the final chapter of this section, Stylianides & Stylianides (this volume) consider both the affordances and consequences for how these constructs are considered in relation to this specific dataset but also more generally.

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Argumentation in the Context of Elementary Grades: The Role of Participants, Tasks, and Tools

Chepina Rumsey, Ian Whitacre, Şebnem Atabaş, and Jessica L. Smith

Introduction

We consider the episode from Ms. Kirk's second-grade classroom from the perspective of argumentation. We view mathematical argumentation in a classroom setting as a human activity that is characterized by a participation structure and is profoundly influenced by the teacher's instructional decisions and the sociomathematical norms established in the classroom (Yackel & Cobb, 1996). The mathematical argumentation occurring in a classroom is also influenced by the teacher's and students' beliefs about what an argument is, what is important to attend to, and what counts as evidence. We analyze episodes involving whole-class, co-constructed argumentation in which second graders are reasoning about arithmetic equations and equality. In particular, our analysis focuses on each of the following themes as observed in the episode: the role of definitions and common language, the role of the classroom participants, the role of the task, and the role of tools.

Our findings center on the relationship between argumentation and tool use. We illustrate how the choice of tool and the nature tasks involving that tool may influence opportunities for argumentation and shape the kinds of arguments that students make. Thus, in order to engage students productively in mathematical

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argumentation, it is important to consider the kinds of arguments that students could potentially make and how the resources available might afford or constrain opportunities.

Arguments and Argumentation

Mathematical argumentation is the process of making mathematical claims and providing evidence to support them. The three components—claims, evidence, and reasoning—are recognized by researchers and scholars as constituting the core structure of an argument, although they may be called by different names (Toulmin, 1958; Krummheuer, 1995). In this chapter, we use the term claim to mean the statement under consideration. Evidence provides support for the claim, and reasoning explains the connection between the claim and the evidence. Our analysis focuses less on arguments themselves and more on participation in the activity of argumentation. We see individual arguments functioning as building blocks in episodes of argumentation.

Argumentation in the Classroom

Productive mathematical argumentation requires a participation structure, or set of classroom norms, that lends itself to such interactions. Accountable argumentation describes "a participation structure, embedded in whole-class discussion, that organizes the public disagreements among students and provides interactional resources for clear mathematical reasoning and the production of mathematical generalizations" (Horn, 2008, p. 104). This sort of argumentation is called accountable because students are accountable for making sense of each other's thinking, remembering and using previous arguments and established ideas, and constructing viable arguments. In these ways, students are responsible for participating in mathematical discussions that are productive and respectful. Accountable argumentation addresses the challenges involved in engaging students in argumentation. In particular, this participation structure is intellectually productive and, at the same time, minimizes social discomfort (Horn, 2008).

Social norms that are conducive to mathematical argumentation involve a degree of shared authority between teacher and students (Cobb et al., 2009). If the teacher functions as the sole authority in discussions, then students do not reasonably have the opportunity to engage authentically in attempting to constructing their own, viable arguments and to critique the reasoning of others (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). Cobb and Yackel (1996) describe a process of negotiation (or renegotiation) of social norms that includes explicit discussion of expectations and responsibilities, including "explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives…" (p. 178).

Mathematical Content for Argumentation

Rich mathematical argumentation goes beyond stating solutions to tasks; it is a dynamic process of making mathematical claims, providing evidence in support of those claims, and explaining the reasoning that connects the two. Interesting mathematical arguments can be made at all grade levels. According to Stylianou and Blanton (2018), "While the formality and form of these arguments will vary across grades, all students need to be able to develop, understand, and interpret arguments appropriate at their level of experience in mathematics" (p. 4). Thus, we are interested in opportunities for worthwhile mathematical argumentation around important topics in the elementary mathematics standards. In this case, the tasks and discussions concern topics in Operations and Algebraic Thinking and Number and Operations in Base Ten at second grade.

The specific mathematical content in the episode from Ms. Kirk's classroom involves arithmetic equations that have operations on both sides. Such equations can provide opportunities for discussion of strategies for performing the operations, as well as discussion of number relationships or of the meaning of the equal sign itself. Popular tasks involve asking students to decide whether an equation such as $8 + 4 = 7 + 5$ is true or false or asking them to determine the unknown number needed to make an equation such as $8 + 4 = \_ + 5$ true.

As students share their work and discuss their thinking about these kinds of tasks, they provide important clues regarding their conceptions of the equal sign: a *relational* view involves viewing the expressions on both sides of the equal sign as equivalent. An *operational* view, by contrast, involves viewing the equal sign as one-directional or as an instruction to perform the operation indicated on the left-hand side and to write one's answer on the right-hand side (Baroody & Ginsburg, 1983; Jacobs et al., 2007; Kieran, 1981; McLean, 1964). A student who is interpreting the equal sign operationally might solve the equation $8 + 4 = \_ + 5$ by adding 8 and 4, obtaining a sum of 12, and ignoring the 5 on the right-hand side (or interpreting it as the next operation to be performed). A student who is interpreting the equal sign relationally might solve the same equation by finding the sum of 12 on the left and then asking what number would result in a sum of 12 on the right. A strategy that exemplifies relational thinking would be to notice that 5 is one more than 4 and to reason that the number in the blank must be one less than 8. Interpreting the equal sign relationally opens many possibilities for students to notice and take advantage of number relationships. It is important for students to work on tasks structured to provide opportunities for students to articulate what the equal sign means and engage in discussions to uncover and challenge their existing conceptions.

Brief Summary of the Lesson

The topic of the second-grade lesson is equality, which is a rich area for mathematical argumentation. The teacher, Ms. Kirk, begins with a discussion of the definition of the equal sign and the tools that can be used to picture what the symbol means. Students discuss tools such as Unifix cubes, Cuisenaire rods, and balance scales. Ms. Kirk also reminds the students that they have been working on "how we can prove and explain our thinking to other people." She presents the first task, which is to show with the Cuisenaire rods whether or not 14 + 3 = 15 + 2 is true. Students work with their partners and then are brought back together for a class discussion where Ms. Kirk describes one pair of students' work and then asks another student to explain. As a challenge activity, Ms. Kirk presents a task where students "use the Cuisenaire Rods to help [you] design a math equation that is true." The worksheet shows ▢ + ▢ = ▢ + ▢. Ms. Kirk reminds the students to use Cuisenaire rods, work with their partner, and not to come up with numbers that are too large. After time to work together, the class comes back together for a whole-class discussion. Ms. Kirk highlights several examples that use the Cuisenaire rods before directing the students to clean up the materials.

Analyzing the Lesson Transcript

We started our analysis by reading through the classroom transcript and taking notes about the observations we made. In our analysis, we used methods of grounded theory (Corbin & Strauss, 2015), leading to the emergence of four themes. The themes relate to the roles of definitions, participants, tasks, and tools. The presentation of each theme includes a discussion of what the specific transcript excerpts highlight about these roles, as well as how the theme connects to the research literature. It is important to unpack these four themes as they are essential components for teachers to consider as they integrate argumentation into their lessons.

The Role of Definitions, Established Facts, and Common Language

The first theme we share explores the role of definitions, established facts, and common language. The initial example of introducing definitions or common language occurs before the main task, when Ms. Kirk asks, "What does this mean to you?," while pointing to the equal sign on the board:

Ms. **Kirk**: Lucas, what does it mean to you?
Lucas: The same as.
Ms. **Kirk**: Same as. Good. Does anyone have any other words that they use to help understand it? Kara?

Kara: I forget.
Ms. **Kirk**: You forget? Kylie?
Kylie: Equal.
Ms. **Kirk**: Equal. Equal to. Good. Anything else? Jacob.
Joey: Combined?
Ms. **Kirk**: This means combined?
Joey: No, uh… um… Equals to.
Ms. Kirk: Yeah. Good. Okay.

Ms. Kirk never mentions the name of the symbol but invites students to think about its meaning instead. Students offer ideas such as "the same as" or "equal to," which she accepts as "good." By contrast, Ms. Kirk questions Joey when he responds that the symbol means "combine." It is possible that Joey is suggesting an operational definition of the equal sign or that he would like to combine everything on both sides of the equal sign. In response to Ms. Kirk's question, Joey revises his answer and says that the symbol means "equals to," which she accepts. This questioning of the operational view of the equal sign may inadvertently close the door on discussion of a productive way of reasoning that is relevant to the task. Having this discussion around relational versus operational views of the equal sign could allow space for different ways of reasoning with equivalent expressions and determining whether an equation is true or false. At the heart of evaluating an equation is the idea of comparing two expressions to see if they represent the same quantity, which possibly involves calculations.

Later, there is a connection made between the meaning of the equal sign and Cuisenaire rods. Ms. Kirk prompts students to think about the tools they have used in class and how these might help in understanding the equal sign:

Meredith: Uhmm, those rods.
Meredith points behind her towards her group's table.
Ms. **Kirk**: Those, oh the Cuisenaire Rods? How do those help you?
Meredith: To figure out what both of them equal.
Ms. **Kirk**: Oh, so you use those to help you figure out what each side equals?
Meredith: And it's the same.
Ms. **Kirk**: And it's the same, which makes sense because that can mean the same as. Cool.

Ms. Kirk makes a connection between the rods and the equal sign as meaning "the same as." Meredith says the rods can be used to see what each expression equals, which should be the same for both expressions. This idea connects Joey's operational view of the equal sign with the equal sign as meaning "the same as." Meredith suggests using Cuisenaire rods to see what each expression equals and then comparing those values. The expressions are equivalent if the total lengths of the rods are the same. Meredith offers a method emphasizing computation before comparing magnitudes. This need for computation aligns with aspects of an operational view of the equal sign, which Ms. Kirk previously deemphasized. Ms. Kirk is not always consistent in how the definition of "equals" is conveyed across the discourse.

Stylianides (2007) claims setting a *foundation* is the first element of argumentation in the mathematics classroom. Foundations can include definitions, axioms, or established facts. There needs to be a basis of what can be considered as true in order for argumentation to occur effectively in the classroom. Ms. Kirk is laying the foundation for interpreting the equal sign in order for argumentation around the task to occur. She reinforces the idea of the equal sign as meaning “the same as.” Because of likely institutional constraints (i.e., curriculum, time), she chose to limit the discussion of the operational view, which is a component of students’ reasoning about equations later in the task.

Ms. Kirk also connects the foundational “same as” definition to the tool that students will use during the task. She redirects away from the operational view of the equals sign but accepts student reasoning stressing computation when comparing expressions. While these methods did not directly align with operational thinking, the emphasis on computation is common between the two ways of reasoning about the equals sign. The foundation set by Ms. Kirk centers around operational views of the equal sign and connects to more computational ways of reasoning. Further, she focuses on the relational way of thinking about the equal sign early in the lesson, but then does not accept this way of reasoning with verifying equivalent expressions. This creates a disconnect between the foundation and accepted ways of reasoning with the task. Relating to the theme of definitions and establishing a common language, this episode is rich with opportunities and missed opportunities to establish a foundation where the classroom community could develop argumentation.

The Role of Participants (Teacher and Students)

When students engage meaningfully in mathematical argumentation, “[t]he teacher is not the sole authority in the class. Rather… she supports, facilitates, and coordinates discussions…” (Stylianou & Blanton, 2018, pp. 32–33). The role of participants is an important theme to unpack, and the second-grade lesson offers opportunities to better understand what this looks like in classroom practice.

The students are asked to use Cuisenaire rods with their partners to show whether or not the following equation is true: 14 + 3 = 15 + 2. When the class comes back together, Ms. Kirk describes what she saw two students doing with the Cuisenaire rods to solve the problem. Carrie and Amanda made an arrangement of rods shown in Fig. 1, with the numbers added for the reader’s reference.

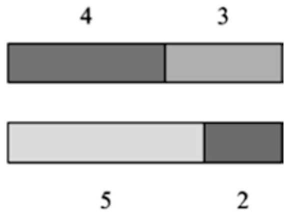
Ms. Kirk asks, “and using this, can you prove to me that four plus three is the same as five plus two?” The excerpt begins with Ms. Kirk looking for someone who she has not yet heard from to explain to the class:

Ms. Kirk: I’m trying to think who I haven’t heard from yet. Sammy do you wanna explain it?

Sammy: Yeah.

Ms. Kirk: All right. Nice and loud voice.

Fig. 1 Depiction of Carrie and Amanda's evidence for equivalence with Cuisenaire rods



Sammy: Because if you did, if you went, and switched the uh… That one if you switch the three and then put it—

Sammy points to the screen at the rods, but does not go very close so others can see exactly where she is pointing.

Sammy: And if you go make the four a five and the three a two and that would make the seven and it would be equal.

Ms. Kirk: Oh wow. So you're saying… So I just wanna use these. Just looking at these, is there any way to tell that these equal each other?

Ms. Kirk is pointing to the rods on the document camera.

Sammy did not point at the rods but explained everything orally while standing in front of the class.

Sammy: [quietly to Ms. Kirk as Sammy puts the rods together]. Um, you can tell if you put them together.

Ms. Kirk: If you turn around and say that to them louder.

Sammy: You can tell because they're the same size if you put them together. [Fig. 1]

Ms. Kirk: They're the same size. Right?

Sammy: Yeah.

Ms. Kirk: Can't really argue with that. That is some hard concrete evidence. What I like about using this tool boys and girls, is it takes a number that sometimes can feel hard to picture, and it gives you something to picture, something you can look at. So not only do you know that four plus three is the same as five plus two because you know that four plus three equals seven, and five plus two equals seven. But now you can also see when they're lined up next to each other, they're the same size.

Using the three elements of argumentation to reconstruct Sammy's argument, we have the new claim of $4 + 3 = 5 + 2$. Even though the initial question was $14 + 3 = 15 + 2$, the example that Ms. Kirk selects to highlight and the argument that Sammy presents are about the new claim. The evidence that is presented by Sammy is an explanation related to compensation that "if you go make the four a five and the three a two and that would make the seven and it would be equal." While the student is beginning to use an explanation that is not reliant on the Cuisenaire rods, Ms. Kirk redirects her to look at the tools, "Just looking at these, is there any way to tell that these equal each other?" Ms. Kirk focuses the reasoning on an argument that uses the tools. To follow with Ms. Kirk's question, Sammy continues, saying, "You can tell because they're the same size if you put them together." Ms. Kirk describes the argument as "hard concrete evidence" that you can't argue with and emphasizes the utility of the tool:

Claim: $4 + 3 = 5 + 2$

Evidence: "And if you go make the four a five and the three a two and that would make the seven and it would be equal." (Sammy)

Reasoning: "Just looking at these" (Ms. Kirk) "You can tell because they're the same size if you put them together." (Sammy)

Integrating argumentation into lessons requires consideration of the roles of both Ms. Kirk and students. While Sammy has an interesting argument to share, perhaps based on compensation, ultimately Ms. Kirk's vision for the argument and the direction of the lesson helped shape what Sammy shared. It would have been interesting to follow Sammy's line of thinking further to see how she might generalize the idea of compensation and, perhaps, how she could use the tool as a generalized representation of the idea. Ms. Kirk also does not go back to Carrie and Amanda to see how they react to Sammy's evidence related to compensation or the reasoning that you can line up the rods. The ways in which participants are invited into the community play a vital role in the way that argumentation is taken up in the classroom.

The Role of the Task

As previously presented, tasks including equations that have operations on both sides provide opportunities to discuss strategies to perform these operations, number relationships, and different conceptions of what equal sign means (operational versus relational view of equal sign). Consistent with this view, the first task, "14 + 3 = 15 + 2", included operations on both sides of the equation and Ms. Kirk asks students to show whether or not the equation was true by using the Cuisenaire rods. During the whole-class discussion, Carrie and Amanda make a new claim of "4 + 3 = 5 + 2" based on the original equation. While there is not an explicit discussion of how this new claim related to the original task, it is an opportunity for the class to discuss whether they could ignore the tens on both sides of the equation and focus on the relationship between numbers on the ones place. Building on this new claim, Sammy argues that "make[ing] the four a five and the three a two and that would make the seven and it would be equal." The numbers in the task afford Sammy to see the "one more, one less" relationship between numbers, which could potentially lead a productive discussion on whether other students in the classroom agree with the claim Sammy made.

The second task requires students to design a math equation fitting the form of "▢ + ▢ = ▢ + ▢" and explain how they could prove the equation they wrote to be true by using the Cuisenaire Rods. The open-ended nature of this task affords students to bring different number combinations. First, discussion taking place in the whole-class discussion is about the combinations of 10. Students seem to recall combinations of 10 from a previous activity and fill in the blanks with different combinations of 10. Importantly, the discussion of different combinations of 10 could potentially lead a discussion on the relationship between number pairs (e.g., one more, one less) as seen below:

Ms. Kirk: All right. I had a few friends discover something really interesting and neat. They were able to find more than just two combinations that equal on here. So, it kind of reminded me of when we play games like combinations to 10.

Kaylyn: Yeah.
Ms. **Kirk**: So we know like five plus five equals 10. What's another one? Milly?
Milly: Seven plus three.
Ms. Kirk: Seven plus three also equals 10. Sam?
Sam: Six plus four.
Ms. **Kirk**: Six plus four also equals 10. Kaylyn.
Kaylyn: One plus nine.
Ms. **Kirk**: One plus nine also equals 10. All right. So that was really neat because Sam and Riley over hear found a bunch of different combinations that equal how much?
Riley: We don't know.

While the combination of 10 could potentially lead a productive discussion on the two addends being "one more, one less" compared to the other two addends in the equation, Ms. Kirk does not act upon the connection between combinations of 10 and the evidence Sammy provided previously. Rather than making the relation between numbers adding up to 10 forefront in the discussion and making explicit connections to Sammy's previous claim, Ms. Kirk asks students to share different combinations of 10 and moves on to another student's equation.

Although Ms. Kirk asks students not to use too big numbers, the open-ended nature of this task allows students to use numbers which leads to some issues in fitting the Cuisenaire rods on the document camera screen or on their number lines. For example, Jacob and Sammy designed the math equation, "40 plus 9 equals the same as 27 plus 22," and they have trouble showing the rod combinations on the document camera because it is not fitting on the screen. This was an important opportunity for students to explain how the math equation they wrote was true without necessarily relying on the use of Cuisenaire rods, which is afforded by the open-ended nature of the task. Thus, the two tasks students worked on create opportunities for students to discuss number relationships, develop some relational thinking strategies, and uncover their conception of the equal sign. Although students use some relational thinking strategies, Ms. Kirk does not make these strategies forefront in the discussion. Rather, she includes some constraints during the launch of the task and leads their attention to the use of Cuisenaire rods to show that the number sentences are correct.

Here it is important to note the difference between tasks as planned and as implemented in the class (Henningsen & Stein, 1997). While the two tasks afford important opportunities for relational thinking and argumentation, the way Ms. Kirk launches the two tasks appears to lead students to rely on the use of Cuisenaire rods in a straightforward way and limit the variability of student strategies and reasoning to prove the number sentences. The constraints Ms. Kirk includes lead students to rely on the use of Cuisenaire rods to solve the tasks. For example, Ms. Kirk launches the first task by saying, "I want you to show me with your Cuisenaire Rods whether or not that's true." After such an instruction, the teacher could potentially ask students to write symbolically what they were depicting concretely with the use of Cuisenaire rods. This would encourage students to exercise relational thinking and anticipate whether or not the equation will be true, going beyond then finding the answer by laying out the rods. Thus, the instruction - not going beyond the use of rods to find the answer - place a constraint on the task and limits the variability of student strategies and reasoning to prove that 14 + 3 = 15 + 2. When selecting a task to encourage argumentation, many factors influence the extent to which argumentation will flourish. Teachers make several decisions with regard to tasks that can promote or hinder the argumentation that the students have the opportunity to engage in.

The Role of the Tool in Terms of Evidence and Potential Connections

The availability of the Cuisenaire rods (together with the task instructions, as discussed above) lends itself to arguments in which the rods provide the evidence. For example, Jacob claims that "40 plus 9 equals the same as 27 plus 22." With numbers of that size, the rods do not fit on the doc cam. So, Ms. Kirk asks Sammy and Jacob to lay the rods out on the floor for everyone to see (Fig. 2).

Ms. Kirk invites students to make an argument that is directly concerned with this tool:

Ms. **Kirk**: If what they're saying is true, if we lined each side up, what should it look like? Tony.
Tony: A big line that's is the same on both sides and is equal.
Ms. **Kirk**: It's the same length. Right? The lengths are equal. I'm not sure if you can see from your seats. But when we line them up next to each other.
Katie: They are equal.
Ms. Kirk: They're equal.

In the above except, Tony and Ms. Kirk state that if the amounts (on both sides of the equation) are equal, their corresponding lengths should be the same. Indeed, the two lengths of rods are the same, so evidently Jacob is right in asserting that 40 + 9 was equal to 27 + 22.

The tool used in this way provides a compelling form of empirical evidence in support of Jacob's claim, and we believe that there may be important opportunities here for students to make connections between the equation and its representation in terms of rods. On the other hand, from the perspective of opportunities for productive argumentation, the evidence provided by the tool might actually be too strong. When it is used as the primary source of evidence as in the above episode, there is little if any room for disagreement. Being that the sums are represented as

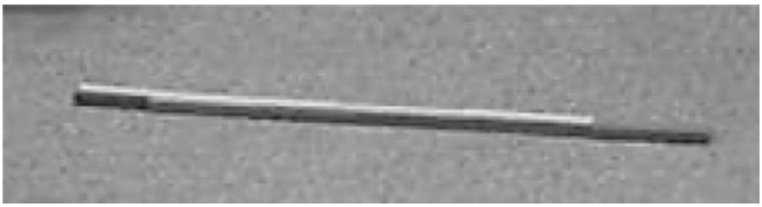


Fig. 2 Sammy and Jacob use rods to show that 40 + 9 = 27 + 22

the lengths that have been laid out side by side, to disagree with the claim that 40 + 9 = 27 + 22 is to disagree with visible evidence that the lengths are the same. In fact, absent that evidence, many second graders would disagree with that claim that 40 + 9 = 27 + 22, due to differing views of the equal sign. The point here is that authentic uncertainty fosters argumentation. Without uncertainty, there is less need for it.

In sum, the use of the tool shapes the nature of argumentation in the lesson such that there is little room for uncertainty and limited opportunities for relational thinking. In the Discussion, we share suggestions regarding how the class could build upon their work with the rods to engage in accountable argumentation and to invite strategies involving relational thinking.

Discussion

Classes like Ms. Kirk's involve students in problem-solving and discussion and thus provide opportunities for the class to engage in mathematical argumentation. We found that argumentation was primarily used as a way to see what students know, how they think about given problems, and how students use Cuisenaire rods as evidence of their thinking and the solution. We conclude by identifying further opportunities for argumentation that relate to each of the themes presented above.

The Role of Definitions and Common Language

Ms. Kirk attempted to create a foundation for argumentation (Stylianides, 2007) by setting a definition of the equal sign. She accepted ideas such as "equal" or "the same as" but questioned Joey's computational description of the equal sign. Joey abandoned this idea and changes his response to "equals to." While Ms. Kirk was laying a foundation for later argumentation, this could have been a moment to argue the different meanings of the equal sign, especially since the task could involve thinking of the equal sign relationally and/or operationally. Ms. Kirk wanted a foundation in the form of a common definition of the equal sign, but the fact is that students think about this symbol in different ways. It could be that a common foundation is not always necessary and that allowing multiple foundations supports different ways of thinking about the same task. Ms. Kirk did not emphasize the operational view of the equal sign when setting the foundation, but she accepted—in fact, she acted as if she preferred—operational methods when working through the task. Preferring operational/computational methods during the task could stem from using Cuisenaire rods when checking the equivalence of two expressions. These specific tools lend more explicitly to computational ways of reasoning, which could explain why Ms. Kirk gravitated toward student thinking involving computation.

A discussion around the different views of the equal sign allows a teacher to give names to those different views, which could then be connected to students' specific ways of reasoning about the task. Such an explicit and flexible foundation might lend itself to accountable argumentation in the classroom because students would then have to reason with and make sense of each other's thinking (Horn, 2008). This approach requires teachers to relinquish some degree of authority and embrace some level of ambiguity. Siegel and Borasi (1996) wrote.

> First of all, inquiry classrooms emphasize the full complexity of knowledge production and expect students to see the doubt arising from ambiguity, anomalies, and contradiction as a motivating force leading to the formation of questions, hunches, and further exploration. Teachers in inquiry classrooms, therefore, are less inclined to take the role of the expert and clear things up for the students and more interested in helping students use this confusion as a starting point for problem posing and data analysis (p. 228).

It is clear Mrs. Kirk is the final authority of discussions (based on her willingness to explicitly evaluate students' responses and solutions), but engaging students in argumentation requires shared authority so that students have the freedom to express their ideas and to critique the reasoning of their peers (Yackel & Cobb, 1996). In this case, argumentation could have started with students' understanding of the equal sign and extended into the task of working with equivalent expressions.

The Role of Participants

While Ms. Kirk is taking steps to introduce argumentation at the second-grade level, there are ways that she could deepen her inclusion of argumentation to involve ownership of the classroom community. Stylianou and Blanton (2018) suggest allowing the students' ideas to take center stage to increase "their agency and their sense of themselves as mathematicians" (p. 33). Notably, Ms. Kirk selected Sammy to share as a student that she had not heard from yet, amplifying the voices of different students in the classroom. Yet, she did not continue to follow the argument that Sammy was making. She steered the argument toward the use of the Cuisenaire rods, whereas Sammy's argument indicated relational thinking and made no reference to the rods. Thus, a next step here would be for the participation structure to move in the direction of greater accountability in the sense of accountable argumentation (Horn, 2008). In order for students to become accountable for making sense of each other's thinking and remembering, Ms. Kirk could set an example by explicitly concerning herself with these responsibilities. In order for the class to engage productively in argumentation, individual students' ideas need to be taken seriously. In this case, that means working to understand Sammy's idea, whether or not it was the argument that Ms. Kirk was expecting.

Likewise, to foster accountable argumentation, it would be important for the class to return to and critique Amanda and Carrie's argument. Instead, Ms. Kirk concluded the discussion without revisiting their arguments, saying, "Can't really argue with that. That is some hard concrete evidence." This final conclusion made by Ms. Kirk could set students up to appeal to authority in future lessons, rather than to look to the community to justify claims and critique arguments. Carpenter et al. (2003) outline three levels of justification that students use to justify that a mathematical claim is true: appeal to authority, justification by example, and generalizable arguments (p. 87). While the first level, appeal to authority, is common, instruction needs to "help students understand that they need to question ideas and use mathematical arguments to justify them. Students need to decide for themselves whether something makes sense and not accept something as true just because someone says it is true" (p. 87). After Sammy shares her ideas, opening up the discussion to the whole class would help the community realize their role in critiquing arguments and could encourage them away from an appeal to authority justification in the future.

The Role of the Task

Ms. Kirk's use of tasks such as true/false equations created an opportunity to uncover her students' conceptions of the equal sign and different perspectives on number relations. However, the constraints that she chose to place on the tasks limited the variability of students' solutions and discouraged argumentation. More specifically, Ms. Kirk asked students to use the Cuisenaire rods in the first task. While Sammy's argument went beyond reliance on the Cuisenaire rods and introduced interesting relational thinking, Ms. Kirk led the attention back to the Cuisenaire rods. Similarly, in the second task, Ms. Kirk asked students to use smaller numbers so that they can use the Cuisenaire rods as they present their work. Thus, the constraints Ms. Kirk provided during the launch of the tasks seem to lead student attention to the use of Cuisenaire rods and limit student work on the tasks as showing two sets to be equal by using the Cuisenaire rods. Ms. Kirk's emphasis on the use of Cuisenaire rods also uncovers what goals she had in mind related to the lesson. Rather than uncovering students' conception of what equal sign means and making claims related to the relations among numbers (relational thinking), she had a goal in mind related to the use of Cuisenaire rods. This goal seems to impact the way she launched the tasks. Possibly, Ms. Kirk believes her role as a teacher is to remove uncertainty for her students and through the additional constraints on the tasks she was trying to make sure that her students know what to do as they work on the tasks. However, she seems to be missing the value of uncertainty in argumentation and how the additional constraints limited the opportunities her students had to engage in argumentation.

Using the same tasks with fewer constraints could open many opportunities for making claims related to numerical relations and different conceptions of the equal sign. We see these opportunities the tasks provided in the evidence Sammy provided as "if you go make the four a five and the three a two and that would make the seven and it would be equal" going beyond a reliance on Cuisenaire rods and attempting to reason about the compensation of numbers within the equation of "$4 + 3 = 5 + 2$."

The combinations of 10 also could lead to a discussion around a similar "one more, one less" relationship among numbers, which is a similar idea that Sammy expressed in the evidence he provided. The equation that Jacob and Sammy presented (40 + 9 = 27 + 22) also could have been an opportunity for students to engage in relational thinking, rather than relying on the tool. For example, students might notice that the total numbers of tens and ones are the same in the expressions on both sides of the equation and conclude that the equation must be true for that reason. Thus, the planned tasks and students' thinking afforded potential opportunities for students to make and justify claims related to number relations—engaging in both the mathematical content and the practice of argumentation at higher levels. Placing fewer constraints on the task instructions could go a long way to encourage such activity.

The Role of Tools

As noted above in our analysis of the role of tools, the Cuisenaire rods provided empirical evidence in support of students' claims that specific sums were equivalent. We wonder about opportunities to move beyond this direct use of the tool in order to encourage students to engage in relational thinking and to make conjectures and generalizations. Note that there are affordances and constraints to any tool. An obvious constraint in this lesson was that only so many Cuisenaire rods were available for students to use and fewer still could be visibly aligned under a document camera. Some designers of tools aim for productive constraints in order to deliberately create opportunities for uncertainty. In this case, the constraint inherent in the rods and the doc cam might have been leveraged productively to invite students to reason about the equation without direct reference to the rods or to use the rods to move away from empirical arguments toward generalizations.

Sums of small numbers can be represented in lengths or rods and discussed, as earlier in the lesson. Then equations involving larger numbers open the door to uncertainty, thanks to the limitations of the tools. Further, going beyond justifying a specific equation, students are also capable of justifying general claims using visual representations (Schifter, 2009) in a process that is "accessible, powerful and generative for students" (p. 76). Beginning with representations for an arithmetic problem can be a good place to begin before moving toward challenging students to "consider how to argue for the truth of a claim about an infinite class" (p. 84).

In Ms. Kirk's classroom, students may be invited to anticipate whether or not the rod lengths should be equal, rather than to merely observe that they are. This might sound like a teacher saying, "Uh-oh! We can't fit so many rods under the doc cam. I'll tell you what: Instead of using the rods for this one, let's just use our brains. Decide whether you think this equation is true or false, and be prepared to share your reasoning." Or, a teacher could encourage students to generalize the situation using the rods at unspecified lengths.

Conclusion

A common thread running through the above themes is that argumentation is messy. In our analysis, we identified ways in which Ms. Kirk avoided that messiness: establishing a common definition of the equal sign, discouraging the use of large numbers, and funneling the discussion away from relational thinking and toward arguments in which the tool served as a source of empirical evidence. The literature points to ways of advancing the sophistication of argumentation in this and other classrooms: greater emphasis on students' ideas and accountability for making sense of these ideas, shared authority among the teacher and students, less restriction on the task instructions and tool use, and encouragement to move away from empirical certainty and toward uncertainty and conjecture. Each of these recommendations introduces messiness and will not help to ensure that lessons go smoothly. Instead, these recommendations prioritize the goal of students engaging in intellectually productive, accountable argumentation in which their mathematical ideas are valued, authority is shared, tasks are open, and tool use inspires progress toward generalizations.

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Justification in the Context of Elementary Grades: Justification to Develop and Provide Access to Mathematical Reasoning

Eva Thanheiser and Amanda Sugimoto

Role of Justification in the Mathematics Classroom

Mathematical justification is a component of mathematical reasoning. The National Research Council (2001) defines justification as the act of providing sufficient reason for mathematical ideas or strategies. An important distinction between mathematical proof and mathematical justification is that justifications do not have to be logically complete (Jaffe, 1997). This is particularly salient in elementary classrooms as students are likely learning both foundational mathematical content and mathematical reasoning skills (National Research Council, 2001). Meaning, if justifications do not have to be complete, then even young students can engage in justification as a way of building their content knowledge and their reasoning skills.

In this chapter, we focus specifically on how mathematical justification has been used and conceptualized in classroom practice, which can be distinct from how it is conceptualized in the larger mathematics community (De Villiers, 1990). In the classroom, justification is often a tool used to engage students in mathematical arguments to support their sense-making of mathematical concepts and procedures (De Villiers, 1990; Staples et al., 2012; Stylianides, 2007). Further, teachers can gain an understanding of how their students are thinking while at the same time providing all students access to each other's thinking. This way multiple arguments can be shared and connected in the classroom, which allows more students access to mathematics and the potential to advance their knowledge. Therefore, justification can serve multiple purposes in the classroom. Firstly, justification can be a means for students to explain their reasoning to their peers and teacher, thus making their reasoning public for classroom analysis and discussion. Secondly, the act of justifying

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repeatedly can build a student's mathematical reasoning skills over time. Thirdly, justification can be a means for increasing students' conceptual and procedural understandings (National Research Council, 2001).

Justifications come from the knowledge and discourse practices that are shared by the community (Simon & Blume, 1996). A justification can be conceptualized as an argument for or against a mathematical claim that utilizes previously established mathematical knowledge and satisfies community-specific requirements for accepted forms of justification (De Villiers, 1990; Staples et al., 2012). If a justification is not within the "conceptual reach" of the individuals within a specific classroom community, then the justification may not advance students' thinking or reasoning (Stylianides, 2007). Based on this view of justification, the editors of this volume defined mathematical justification in the classroom as *the process of supporting mathematical claims and choices when solving problems or explaining why a claim or answer makes sense* (See "Justification" in the chapter by Staples & Conner, this volume).

Justification can be thought of as a social and cognitive process (Cobb et al., 1992; De Villiers, 1990; Ellis, 2007). Cognitively, justifications can support students in developing deep and interrelated conceptual understandings where the why of procedures is explored and eventually known (Simon & Blume, 1996; Stylianides, 2007). Socially, justification involves the establishment of a mathematical community where deep mathematical understanding and validation are a valued classroom activity (Cobb et al., 1992; Simon & Blume, 1996; Yackel & Cobb, 1996) and is available to all students in the classroom. Expecting students to justify (support their mathematical claims and choices when solving problems or explain why a claim or answer makes sense) in a classroom opens up the students' reasoning (rather than simply their answer) to the whole class and allows all students access to various kinds of reasoning. Expecting all students to justify sets community-wide expectations for students.

Establishing classrooms where students engage in justification to explore and explain mathematics is complex and often an under-utilized practice (Bieda, 2010; Jacobs et al., 2006; Schoenfeld, 1988). In the following, we outline types of student justifications and tasks that teachers may pose to engage students in justifications and then turn to how justification could be used as a tool to promote more equitable student gains in the classroom.

Student Justifications and Justification Tasks

The types of justifications that students make in the classroom can be influenced by the student's prior learning as well as the types of tasks that are posed for the classroom to consider (Chua, 2017; Stylianides, 2007). Validation justifications occur when students are asked to provide evidence for why a given claim is or is not true and allow insight into how students are reasoning about a specific claim (Chua, 2017; Harel & Sowder, 2007; Hoyles & Healy, 1999). For example, a student could state that "dividing a fractional piece in half will always create a smaller piece but a larger denominator because the size of one piece will get smaller but the number of total pieces doubles."

Experiential justifications are often used to explain a mathematical claim using specific examples or visuals (Becker & Rivera, 2009; Lannin, 2005). For example, a student could physically demonstrate the previous justification about dividing fractional pieces by drawing a rectangle, dividing the rectangle in half with a vertical line drawn through the middle of the rectangle, and then dividing each half in half again with a horizontal line through the middle of the rectangle as a means of supporting their argument that "the size of one piece will get smaller but the number of total pieces doubles."

An elaboration justification involves students clearly explaining the strategy or method they used when constructing a claim (Becker & Rivera, 2009; Stylianides, 2015). To extend the fraction example, a student could explain their strategy by saying something like, "I drew a rectangle first because they are easy to split up. Then, I drew a line down the middle to make two halves, so the fraction would be ½. Then, I drew a line across the other way to split each half into half again. This made each piece smaller and actually made four total pieces so even though my pieces are smaller, my denominator when I write the fraction is now ¼."

Yet, students do not often provide justifications spontaneously in the classroom, and teachers must intentionally identify and create "situations for justification" (Cobb et al., 1992). One way of creating opportunities for students to justify is by selecting tasks that prompt students to use justification in order to complete the task. Chua (2017) distinguishes *making decisions tasks* and *elaboration tasks* based on the purpose of the task and the expected elements of a justification that would come from the task (Chua, 2017). For example, a making decisions task could involve the teacher presenting a mathematical claim to students and asking them to decide whether or not the claim is true or not (e.g., The first four numbers in a pattern are 1, 3, 5, 7. Explain whether or not 23 is a number in the pattern). Students would be required to provide supporting evidence when they are explaining the validity of or refuting the claim, which aligns with a validation justification (e.g., "One, three, five, and seven are all odd numbers and so is 23, so 23 would be in the pattern if we kept going") (Hoyles & Healy, 1999). In contrast, an elaboration task focuses on how students obtained a mathematical result (e.g., your class is having a pizza party. Your teacher bought two large pizzas and each pizza has 36 slices. If only four students came to the party, how many slices would each student get? If 30 students came to the party, how many slices would each student get? Explain how you determined these) (Lannin, 2005). Students could provide an elaboration justification in response to an elaboration task (e.g., "To figure out how many slices the four students would get, I first drew 72 triangles because there were 72 pieces of pizza in the two large pizzas. Then, I numbered the slices one, two, three, four and kept doing this until all the slices had a number. All the slices with a number one went to the first student and counted up how many they had and it was 18, so each student gets 18 slices") (Becker & Rivera, 2009; Stylianides, 2015).

Potential Relationships Between Justification and Equity

Justification can be positioned as a tool to address inequitable outcomes in classrooms (Boaler, 2006; Boaler & Staples, 2008; Staples et al., 2012). Justification has the potential to increase learning across diverse student populations (Boaler, 2006; Boaler & Staples, 2008), which means it has the potential to go beyond what some call *just good teaching* (Celedón-Pattichis & Ramirez, 2012). By asking all students to share their justifications with the whole class, teachers can use justification to assess student learning, support students' developing conceptual understandings, and build upon the diversity of their students' mathematical understandings (Boaler & Staples, 2008; Staples et al., 2012).

In alignment with this equity focus, Staples' (2007) work focused on supporting students' whole-class collaborative inquiry that "fundamentally relies upon students making their thinking public—their conjectures, their proposed next steps to problems, their ideas and justifications" (p. 172). In order to support this type of whole-class collaboration, Staples (2007) identified three purposes that the teacher plays: (1) supporting students in making contributions, (2) establishing and monitoring common ground, and (3) guiding the mathematics. All three of these purposes can be framed as practices that support students' equitable participation in a mathematics learning community in order to support their mathematical development. For example, teachers can support all students in getting their ideas onto the table through eliciting and scaffolding moves as well as amplifying moves that demonstrate the logic of the student's contribution. Additionally, teachers can work to create common ground for mathematical discussions by building a shared context and understanding of the mathematics and previously established justifications so that students approach new content with a shared understanding to build upon. Finally, teachers can flexibly guide the mathematical work of the classroom by attending to both the mathematical content as well as students' in-the-moment mathematical thinking through the implementation of high cognitive demand tasks. This flexibility teaches both required mathematical content while also positioning students as "thinkers and decision-makers" (Staples, 2007).

Summary

To review, the authors were originally given the definition of justification as the process of supporting mathematical claims and choices when solving problems or explaining why a claim or answer makes sense. In the classroom, this process can support students in developing their mathematical understandings and their reasoning skills. In addition, justification can also be conceptualized as an equity tool. By asking students to justify, the teacher asks the students to build on their own understanding, reason from those understandings, and make their reasoning public for the class. Asking several students to justify and connecting across those justification allows students with varying levels of prior knowledge and understanding to access and connect to each other's reasoning.

Methods

In this chapter we analyzed the transcript of the second-grade classroom described in the introduction to this section. The authors began by reading the transcript to identify the tasks posed as well as the types of justifications that students gave. The authors framed this analysis with the definition of justification that the editors provided and the aforementioned classifications of student justifications and justification tasks.

Distinct tasks were identified based on shifts in student activity and teacher directions. For example, when students were directed to transition from an introductory whole-class discussion that elicited prior knowledge to partner work on a specific teacher-given task, the authors coded these two instances as two distinct tasks. Once the tasks were identified, the authors examined each task with respect to its potential for justification. Tasks were classified as a (1) making decisions task, (2) elaboration task, or (3) not a justification task.

The authors then identified instances of student justification within each task from the transcripts of the whole-class discussions. The unit of transcript analysis, or stanza (Saldaña, 2015), was defined as both the student's given justification and the question(s) that the teacher or other students asked in relation to the justification, as well as any additional text needed for context. In order to identify stanzas of student justification, the authors first used the editor-provided definition (the process of supporting mathematical claims and choices when solving problems or explaining why a claim or answer makes sense). The authors then classified the justifications that the students provided as (1) validation justification, (2) experiential justification, or (3) elaboration justification.

The Nature of the Tasks

This second-grade mathematics lesson was structured around three main tasks focused on equality. In the following, we describe each task in turn and analyze the potential for justification as either a making decisions task or an elaboration task (Chua, 2017).

> **Task 1:** For the first task, the teacher asks the students to focus on an equation written on the board (14 + 3 = 15 + 2) by saying, "Alright, so just to refresh our brains, can someone tell me what this means to you? What does that mean to you?" While she says this, she points explicitly to the equal sign. The class then holds an extended discussion on what the equal sign means to them based on their current knowledge and prior experiences.

The first task focuses on defining the mathematical concept of equality through the mathematical symbol of the equal sign. Students use both mathematical terminology and their prior knowledge and experiences to explain their current understandings of the equal sign. Since the task does not explicitly focus on a mathematical result, it cannot be classified as either making decisions task or an elaboration task (Chua, 2017).

Task 2: In the second task, the teacher prompted students to "show me with your Cuisenaire Rods whether or not that's true. Okay?" The teacher directs the students' attention to the equation written on the board [14 + 3 = 15 + 2] when explaining the task. Students then work with a partner to decide whether or not the equation is true before coming into a whole-class discussion to discuss their findings.

The second task can be classified as a *making decisions* task in that students are asked to decide about whether or not a given mathematical claim is true. With their partners and during the whole-group discussion, students are prompted to provide specific evidence to support their decision.

Task 3: The third task (see Fig. 1) extends work done in the second task. The task is more open-ended in that students are given a blank equation and are asked to fill in expressions that would make the equation true. Additionally, students are explicitly prompted that they will be required to explain, and potentially justify, their strategies to the teacher and eventually during the final whole-class discussion.

The third task can be classified as an *elaboration* task in that students must make a mathematical claim by creating two expressions that are equal and then explain their claim to the teacher and/or class. During this explanation, students are required to explain how they created their two equal expressions in order to support the validity of their mathematical claim.

This task analysis shows that these teacher-selected and teacher-implemented tasks by design (1) were based on establishing common ground (meaning of the equal sign), (2) were high cognitive demand (asked for justification not just solutions), and (3) engaged second-grade students in justification. Task 3 in particular is low floor (allows all students to engage) and high ceiling (allows students to push their limits) (Boaler, 2015). In sum, the three main tasks of this lesson offered opportunities for students to engage in different types of justifications, i.e., explaining whether or which and explaining how.

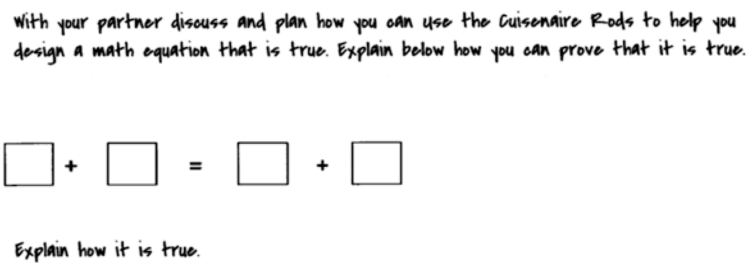

Fig. 1 Task 3 create a valid equation

Students' Justifications

Instances of student justifications occurred in Tasks 2 and 3. In Task 2 the students were asked whether 14 + 3 = 15 + 2 was true. While the students were working in small groups, the teacher worked with a pair (Carrie and Amanda) and then brought their thoughts to the class (see Fig. 2):

Teacher: They showed me the purple, and they said it was four. And the green and said it was three. And then they showed me the yellow and said it was five. And the red and said it was two.

The teacher at this point worked off the students' argument and changed the task from 14 + 3 = 15 + 2 to 4 + 3 = 5 + 2. Neither the teacher nor the students mentioned the drop of the 10 on both sides of the equal sign. This could be due to the fact that this move is an accepted community practice (De Villiers, 1990; Staples et al., 2012) not needing further discussion or due to the fact that the teacher wanted to focus on a visual at this point and the two 10 sticks would have been the same visually.

Teacher: And in using this, can you prove to me that four plus three is the same as five plus two?

The teacher called on Sammy, who provides a decision and a justification for that decision.

Sammy: And if you go make the four a five and the three a two and that would make it seven and it would be equal.

Sammy argued that if you changed the 4 (purple) + 3 (green) to a (4 + 1) (yellow) + (3 – 1) (red), you would get 5 + 2, which is 7. Sammy defended her answer by using a compensation strategy, moving a unit from one of the addends (3) to the other addend (4) and a calculation strategy, calculating the result as being 7. Sammy did not refer to the Cuisenaire rods in her explanation and did not point to them. While Sammy provided a valid justification, the teacher did not comment on it. Instead, the teacher asked Sammy to use the Cuisenaire rods to show that the two sums are equal (see Fig. 2):

Teacher: Oh wow. So you're saying… So I just wanna use these [Cuisenaire rods]. Just looking at these, is there any way to tell that these equal each other?

Sammy then uses Cuisenaire rods blocks to justify.

Sammy: [quietly to the teacher as Sammy puts the rods together]. Um, you can tell if you put them together.

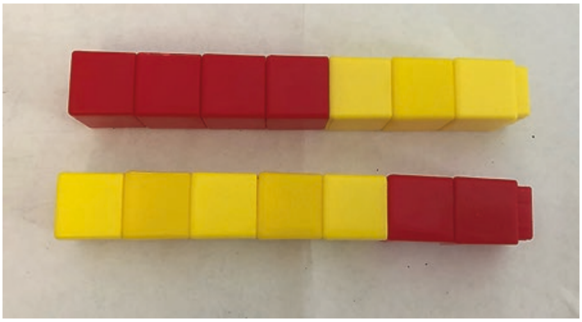

Fig. 2 The teacher modeled each rod on the document camera as she explained Carrie and Amanda's strategy

Teacher: If you turn around and say that to them louder.
Sammy: You can tell because they're the same size if you put them together.

There are different reasons the teacher might have asked Sammy to use the Cuisenaire rods. On the one hand, the teacher might have felt that the compensation strategy as provided was not accessible to all students and thus wanted Sammy to use the Cuisenaire rods to make it more accessible through an experiential justification with visuals. On the other hand, the teacher may have wanted to focus on the length argument. Finally, the teacher may not have fully attended to Sammy's compensation strategy.

Sammy's initial justification is an example of a validation justification (Chua, 2017; Hoyles & Healy, 1999) as she was providing evidence for why the equation is true, i.e., moving a unit from one addend to the next. This justification could be explained by using the Cuisenaire rods and comparing the two sums (see Fig. 2) and looking at what the overlap is, in this case, one unit (see Fig. 3). This would make it both a validation justification and an experiential justification. Combining both or using visuals to explain a validation argumentation would provide more students access to Sammy's reasoning and thus to the compensation strategy.

However, when the teacher prompted Sammy to use Cuisenaire rods, Sammy switched her strategy from a compensation strategy to a measurement strategy (seeing that the length of both sums are the same). This is a case of using the Cuisenaire rods to explain a solution/justification rather than using the Cuisenaire rods to come up with the solution/explanation. It is unclear whether Sammy switched strategies because she thought that is what the teacher wanted or whether Sammy switched strategies due to the use of Cuisenaire rods. In either case, her switch in strategy was not mentioned by the teacher. The teacher summarized Sammy's strategy:

Teacher: So not only do you know that four plus three is the same as five plus two because you know that four plus three equals seven, and five plus two equals seven. But now you can also see when they're lined up next to each other, they're the same size.

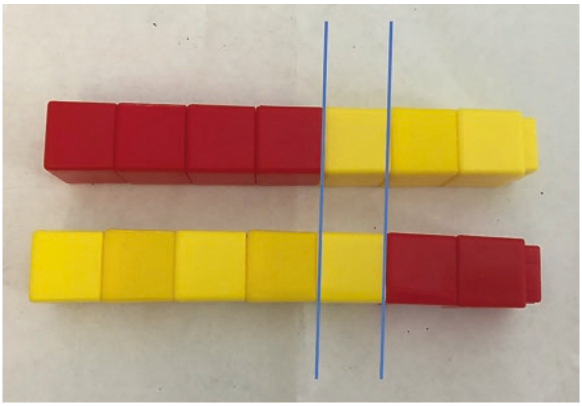

Fig. 3 Using Cuisenaire rods to show that 4 + 3 = 5 + 2 highlighting the overlap

The teacher related the computation strategy and the measurement strategy thus connecting a validation justification and an experiential justification but did not mention the compensation strategy. Bringing out the compensation strategy and relating it to the length argument could have been a powerful move both mathematically in establishing connections and with respect to access as it might have allowed students with different understandings and background knowledge access to the compensation strategy.

Moving to Task 3 (see Fig. 1), the teacher stated:

> **Teacher:** All right. So with your partner, I'm asking you to discuss and plan how you can use the Cuisenaire Rods to help you design a math equation that is true. Explain how you can prove that it is true. So, I just set up some boxes…

During partner work, three different kinds of strategies emerged among the collected work of 27 students. Of the 27 students, 4 used a compensation strategy, similar to Sammy's argument, i.e., moving one or several units from one addend to the next. Of the 27 students, 15 used a calculational strategy, calculating the sums on both sides and comparing those sums to argue that both sides of the equation are equal. Of the 15 students who used a calculational strategy, 5 referred to or mentioned the Cuisenaire rods, while the other 10 did not. However, merely mentioning the Cuisenaire rods did not always lead to a measurement strategy (see Fig. 4 for examples of both).

Of the 27 students, 8 used the teacher's intended strategy of comparing the length of both sums with the Cuisenaire rods (some of them incomplete). Thus, in total 13 students at least mentioned the Cuisenaire rods in their justification, and 14 students justified without them.

During the class discussion, Sammy and Jacob shared that four orange 10 rods (40) and a blue nine rod (9) is the same as three blue 9 rods (27), two brown 8 rods (16), and a dark green 6 rod (6). Thus, that 40 + 9 is the same as 27 + 22. However, the students presented the rods in groupings rather than in a line so it was not visually clear from the rods whether 49 and 27 + 22 are equivalent (see Fig. 5). It is unclear how the students came up with this problem and strategy as the rods were

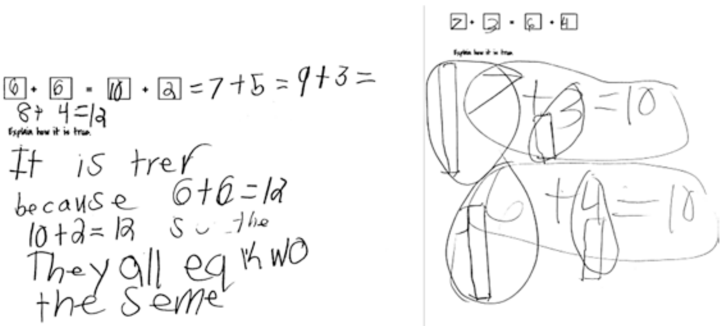

Fig. 4 Madelyn's solution to 6 + 6 = 10 + 2 by calculating both sides and Gino's solution to 7 + 3 = 6 + 4 by calculating and drawing some blocks

Fig. 5 Sammy and Jacob's collection of Cuisenaire rods

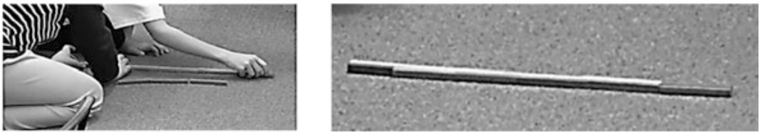

Fig. 6 Lining up the Cuisenaire rods to show equal length

not lined up. Thus, it is likely the students either calculated the answer and then pulled out the corresponding Cuisenaire rods to explain their answer or started with a combination of Cuisenaire rods and calculation until they made it work. In this case again, Cuisenaire rods were not (or not exclusively) used to solve the problem but rather to illustrate the answer, especially considering neither Sammy nor Jacob used the Cuisenaire rods when working on Task 3 individually. Sammy used a compensation strategy on Task 3, and Jacob used a calculation strategy.

The teacher then prompted the class to consider the equality of the sums:

Teacher: If what they're saying is true, if we lined each side up, what should it look like? Tony.

Tony: A big line that's the same on both sides and is equal.

The students then proceeded to create the two equal lengths (see Fig. 6) which used a measurement strategy to ascertain equivalence and thus led to an experimental justification. We describe this discussion in more detail in the next section.

All 27 students provided at least an initial mathematical justification in their written work and, as such, all 27 students had access to the task. However, as in Task 2, the shared problem initially did not (or not exclusively) rely on the length argument (measurement strategy) with the Cuisenaire rods, but the length argument was provided upon teacher prompting.

Providing Access to Participation Through Justification

In this lesson, the teacher implemented several teaching practices that appeared to support more equitable participation from students while engaging in the shared work of justification. These practices can support students' access so that the justification is within the conceptual reach of the community. In the following, we explicate the ways in which this teacher did and did not support students in making contributions, establishing common ground, and guiding the mathematics in alignment with Staples' (2007) conceptualization of the role(s) that teachers can play in supporting equitable participation in classrooms. Since justification is both a cognitive and social process, our aim here is to explore ways that teachers can involve more students in justification through discussion so that more students have the opportunity to develop their content understanding and reasoning skills.

In the first task, where the teacher asked students to explain "what this means to you" referring specifically to the equal sign, the teacher appeared to be establishing common ground. The teacher then further guided the mathematics by explicitly asking students to "think about the tools that we've used in class to help us understand this [equal sign]." The discussion then shifted to manipulatives that students could use to support their work with equations. One student highlighted cubes to "tell people how you solved it." While another student highlighted Cuisenaire rods in the following exchange:

Student 1: Uhmm, those rods. (Points behind her towards her group's table.)
Teacher: Those, oh the Cuisenaire Rods? How do those help you?
Student 1: To figure out what both of them equal.
Teacher: Oh, so you use those to help you figure out what each side equals?
Student 1: And it's the same.
Teacher: And it's the same, which makes sense because that can mean the same as. Cool.

Here the student highlighted a specific manipulative, and the teacher pushed for an explanation for how the manipulative could be used when working with equations. This led to a definition that connected to the previously shared definition from another student, i.e., "the same as." The teacher could have ended the discussion then, but she continued to push students for more ideas by asking "anything else," which led to the next exchange:

Student 2: The balance scales.
Teacher: Ooh. How does the balance scale, what part of the balance scale does that make you think of?
Student 2: The bottom like where you put this stuff on.
Teacher: So like the long yellow part? Or do you mean the red? (*As teacher talks, she motions her hands horizontally for the "yellow part" indicating the balance beam, and vertically for the "red part" indicating the fulcrum).*
Student 2: Yeah. (*The student motions her hands vertically, indicating the red part is her reference.)*
Teacher: …stand part? The red part? Yeah. Very good. All right.

The student's contribution of the balance scales appeared to be a visual metaphor of how an equation works, with the equal sign being the fulcrum, or balancing point, and the expressions on each end of the beam. Again, this connects to previously shared definitions about the equal sign meaning "the same as" where the individual expressions are equal. In total, this opening discussion took approximately 4.5 min, but it laid important foundational work for students by establishing a common understanding of what the equal sign represents through definitions, connections to previous experiences, connections to specific mathematical tools, and mathematical metaphors. The tool conversation provided insight into how the students were conceptualizing the equal sign, e.g., as a balance point or fulcrum between two expressions, and how the class used shared tools to justify their reasoning when solving equations in the past. This emphasis on shared tools also gives a glimpse into what potentially "counts" as a justification within this community's specific norms (Cobb et al., 1992; Simon & Blume, 1996; Yackel & Cobb, 1996). Finally, the time spent in this discussion allowed all students some time to focus on the meaning of the equal sign so that they were ready to start the next two tasks.

During the second and third tasks, the Cuisenaire rods became an important tool for supporting students in sharing their justifications. The teacher guided the mathematics by asking specific pairs of students to share their equations. In the second task, an opportunity to connect the measurement and the calculation strategies by using the Cuisenaire rods is realized, thus providing access to those strategies and building a more connected understanding. This use of multiple modes, e.g., verbal explanations, mathematical representations, gestures, and manipulatives, can be seen as a move to support students in making contributions to discussions (Staples, 2007), particularly when students are learning to explain their justification to their peers. Additionally, using multiple modalities to explain a justification can increase the listeners' access to the mathematical ideas as compared to situations when a justification is only presented verbally (Moschkovich, 2007; Shein, 2012). In other words, the use of multiple modalities to support student sharing of justifications during class discussions may actually extend the shared conceptual reach of the community by presenting the mathematical idea in different modes. However, an opportunity to connect those strategies to the compensation strategy was missed, thus limiting access to that strategy and a deeper understanding of the mathematics. It is unclear from this transcript whether this missed opportunity was due to the teacher not noticing the compensation strategy or due to the teacher's focus on the other strategies.

If justification is to become a practice for supporting equitable outcomes and for learning mathematics, then teachers must learn to use several interrelated skills. These skills include (1) identify high cognitive demand tasks that elicit student justifications, (2) notice students' mathematical thinking and specifically their justifications, (3) thoughtfully engage students in discussions that develop and support their ability to justify, and (4) support students' ability to access their peers' justifications.

Discussion

Justification to develop mathematical reasoning as well as to provide access to mathematical reasoning is a developing line of research in mathematics education but one that has potential to increase student learning, particularly for students who have been underserved in mathematics classrooms (Boaler, 2006; Boaler & Staples, 2008). Yet, mathematical justification is often underutilized in the classroom (Bieda, 2010; Schoenfeld, 1988).

In this lesson, the teacher successfully leveraged justification to develop mathematical reasoning and to provide access to mathematical reasoning but also missed some opportunities. The teacher's goal appeared to center around building students' understanding of equality of sums. During the lesson tasks, students produced several kinds of justifications including validation justifications (Chua, 2017; Hoyles & Healy, 1999) when they argued whether or not a given equation was true, elaboration justifications (Becker & Rivera, 2009; Stylianides, 2015) when they explained how they created their own true equations, and experiential justifications (Becker & Rivera, 2009; Lannin, 2005) when they used Cuisenaire rods as visuals to justify why they believed a claim was true or false. Yet, the teacher guided the mathematics to focus on the experiential justification through the Cuisenaire rods and the length argument.

The level of attention that this teacher paid to eliciting student justifications shows that this type of pedagogical strategy takes careful planning and selecting of tasks that intentionally engage students in justifying, as well as in-the-moment moves to elicit, support, and possibly even extend students' justifications. Yet, she seemed to emphasize one type of reasoning (with the Cuisenaire rods) to the exclusion of other types of reasoning and did not engage students in a conversation about the similarities and differences among the various justifications provided. This limited both the mathematical reasoning from students and the access that students had to varying forms of mathematical reasoning in the class.

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Proof in the Context of Elementary Grades: A Multimodal Approach to Generalization and Proof in Elementary Grades

Candace Walkington and Dawn M. Woods

Introduction

Young children have opportunities to understand mathematics when they are actively engaged in learning (Confrey, 1994), solve problems that build upon existing knowledge (Carpenter et al., 1989), and participate in "genuine conversations" (Wood et al., 1991, p. 601). During these learning activities, children are accountable for their own learning as they participate in sensemaking activities characterized by argument, justification (Kazemi & Stipek, 2001), and proof (Ball & Bass, 2003; Carpenter et al., 2003). Yet, consensus has not been reached on what proof means in the context of elementary mathematics (Reid, 2005), particularly since the concept of proof may be interpreted subjectively depending on the person or community (Harel & Sowder, 2007).

Although proof may be interpreted subjectively, the act of proving plays a central role in mathematics. Through the act of proving, mathematicians (including young children) have the opportunity to consider relationships, make conjectures, and cultivate new ideas as they deepen their mathematical understanding (Schoenfeld, 2009; Tall et al., 2012). Active engagement with proving, rather than knowledge transmitted by a teacher, is necessary for K–12 students to deeply learn mathematics (Ball & Bass, 2003; Carpenter et al., 2003; Lampert, 1990; Stylianides, 2016).

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Consequently, in order characterize the act of proving, three important factors must be considered: (a) construction of new knowledge, (b) integrity of the concept of proof, and (c) the social nature of the proving process (Harel & Sowder, 2007). First, new knowledge is built upon prior knowledge, and young children generate their own understanding by making logical connections between pieces of information (National Academies of Sciences, Engineering, and Medicine, 2018). Hence, the act of proving provides young children with rich opportunities to generalize, categorize, and solve problems. Second, the integrity of the concept of proof must be maintained. That is, instruction should not treat young children as little mathematicians (Stylianides, 2016), but the act of proving becomes indispensable to engagement with authentic mathematics (Stylianides, 2007). Third, the act of proving is social – others (i.e., teachers, peers) must accept what a young child offers as a convincing argument within his/her community.

In this chapter, we characterize the act of proving in elementary school through the analysis of a second-grade mathematics classroom excerpt. We argue that there are two particularly important elements when considering the proof activities of children – a move toward the generalization of mathematical operations and ideas and the multimodality of the act of proving. These two elements lay the foundation for proving activities that allow students to build new knowledge, create authentic proofs, and participate in a community of math learners.

Theoretical Framework

Since the act of proving is social, a situated perspective on learning is the overarching theoretical framework utilized here. A situated perspective on learning argues that people learn as they deepen their participation within a community of practice (Greeno, 2006; Lave & Wenger, 1991). Within this learning community, young children engage in authentic mathematical activities and discussions as they build relationships that enable them to learn from each other as they generate meaning through the development of shared repertoire (Wenger, 1998). This approach to learning is arguably more effective than simply learning by doing because the community positions learning as a process that moves from decontextualized talk of knowledge to a place where developing shared knowledge increases understanding (Franke & Kazemi, 2001; National Academies of Sciences, Engineering, and Medicine, 2018). It is in this space where new knowledge is built as the integrity of proof is maintained. Next, we will define our conceptualization of proof.

Definition of Proof

Weber (2014) frames proof as a discursive concept that can be defined using a clustered approach. In this approach, a number of different models of proof are combined to formulate an overarching definition, with uncontroversial proofs satisfying all models, and contentious proofs satisfying some. According to Weber (2014), this clustered definition of proof makes it such that the essence of what proof is would be difficult to teach through direct instruction and instead may be best understood through participating in practices in a community of mathematics learners. The specific clusters identified by Weber (2014) are:

> (1) A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a deductive argument that does not admit possible rebuttals. The lack of potential rebuttals provides the proof with the psychological perception of being timeless. Proven theorems remain proven. (3) A proof is a transparent argument where a mathematician can fill in every gap (given sufficient time and motivation), perhaps to the level of being a formal derivation. In essence, the proof is a blueprint for the mathematician to develop an argument that he or she feels is complete. This gives a proof the psychological perception of being impersonal. Theorems are objectively true. (4) A proof is a perspicuous argument that provides the reader with an understanding of why a theorem is true. (5) A proof is an argument within a representation system satisfying communal norms. That is, there are certain ways of transforming mathematical propositions to deduce statements that are accepted as unproblematic by a community and all other steps need to be justified. (6) A proof is an argument that has been sanctioned by the mathematical community. (p. 357)

There are two aspects of this cluster concept we would like to discuss in more detail that will have relevance to the excerpt under consideration – the first is the role of generalization and reasoning with examples, and the second is the multimodal nature of proof.

Generalization and Proof

One element not made explicit in the above clusters is the idea of a proof as being "general" – as holding for all objects in the class under consideration (e.g., Harel & Sowder, 2007). While deductive reasoning for mathematicians might necessarily imply that they are reasoning about the appropriate class of objects, this is not always the case for children. Children often reason about mathematical conjectures using examples (Healy & Hoyles, 2000; Knuth et al., 2019) and can formulate deductive arguments that only take into account subsets of the class under consideration (e.g., prove the triangle inequality deductively but only for isosceles triangles). Moving K–12 students toward making more general arguments (often by leveraging their use of examples; see Ellis et al., 2012) is important pedagogically and also can serve to facilitate an understanding of what proof is and why the mathematical community needs proofs (Stylianides & Stylianides, 2009). Thus, in addition to the clusters given above, we consider generality of arguments as important to not only the act of proving but to a cluster definition of proof.

The Multimodal Nature of Proof

In the fourth cluster given in Weber's (2014) definition, proof is an argument received by a "reader." This highlights an important issue that has particular pedagogical significance when considering K–12 children – the modality in which a proof can or should be communicated as new knowledge is being constructed in a community of learners. A modality is defined here as any channel of communication – writing or drawing is a way humans communicate, as is speech, video, gesture, and actions on objects (like manipulatives). When considering an embodied account of the nature of cognition, human knowledge is conceptualized as being perceptual, action-based, and spatial in nature, rather than being governed by abstract rules and mental structures (Wilson, 2002). Therefore, the act of proving may encompass more than an argument that may be read by a "reader." Proof can take many forms through many communication channels. We further use the term *multimodal* to refer to instances where different modalities are used in conjunction with each other or to the more general idea that human activity involves many different channels of communication being utilized.

Accordingly, research has shown that critical elements of students' proofs can be shown entirely through actions and modalities other than speech and writing. Research and theories on embodied learning suggest that even very abstract mathematical ideas (like algebraic equations) have embodied and perceptual qualities (Landy et al., 2014; Lakoff & Núñez, 2000). Williams-Pierce et al. (2017) describe an instance where a learners' speech implies that he is giving a proof based on one specific example of a triangle, while his gestures reveal that his mathematical argument is general and would apply to all triangles. Marghetis et al. (2014) describe mathematicians' proving processes as "a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (p. 243).

Conceptualizing the act of proving as a multimodal endeavor can remove barriers between students and the formal discursive practices associated with proofs that may obscure mathematical meaning. Students can come to understand that mathematical formalisms and properties result from generations of human activity interacting with the world through perception and action, rather than through orderly revelations of universal truths. Because of this disconnect and lack of a historical trace, formalisms in mathematics can be difficult for novices to grasp (Nathan, 2012), and concrete representations can provide an entry point to understanding mathematical abstractions (Fyfe et al., 2015; Goldstone & Son, 2005).

Modal-Specific Epistemological Commitments and Generalization

This multimodal conceptualization of proof forefronts the important issue that people can have preferences for certain modalities over others that can impact their reasoning – this is referred to as *modal-specific epistemological commitments* by Walkington et al. (2014). An epistemology is how someone interprets or conceptualizes what it means to know a concept. Thus, an epistemological commitment would be a demonstrated belief that one particular way of knowing a concept is superior or preferable to another. A modality is a channel that a concept can be communicated through – like a diagram or a physical model. So, when epistemological commitments are modal-specific, people value one modality as a way to "know" a concept over other modalities.

For example, students and teachers may be so focused on concrete, salient representations like physical objects that they no longer consider mathematical abstractions that underlie the construction of these objects like equations. Walkington et al. (2014) describe a bridge building project where students first learn about mechanical principles of bridges and then later construct bridges out of balsa wood and test them. As the project progresses, students and teachers focus on the physically present model bridges, rather than on the mathematical equations and diagrams covered earlier in the project sequence. These modal-specific epistemological commitments cause missed opportunities for explicitly connecting the latter parts of the project to the core disciplinary ideas from the beginning of the project.

These epistemological commitments are also important to students' opportunity to learn mathematical concepts because a modality is *not* simply a channel that communicates an objective string of reasoning. Instead, the modality influences and co-constitutes the very nature of argument the prover is giving (see Hall & Nemirovsky, 2012). Every modality has its affordances and constraints for representing and communicating mathematical information. For example, concrete objects like manipulatives and familiar everyday representations like pictures of objects can allow learners to leverage prior knowledge and use new strategies based on their informal understandings (Goldstone & Son, 2005; Koedinger et al., 2008). However, students may not understand the abstractions that the manipulatives represent (Laski et al., 2015) or the dual role of the manipulative as both an object and a symbol of a mathematical concept (McNeil & Jarvin, 2007). In addition, concrete representations like manipulatives may be salient or distracting (e.g., McNeil et al., 2009) and inhibit transfer to contexts that are more generalized or more abstract to the learner (Sloutsky et al., 2005).

Other modalities of communication have their own affordances and constraints – speech alone lacks the ability to physically or visually represent spatial, referential, and dynamic information, thus in mathematics classrooms it is often naturally accompanied by gestures (Alibali & Nathan, 2012). It is critical for teachers and students to monitor their modal-specific epistemological commitments and to consider how these commitments will modify the very nature of the mathematical proofs that happen in classrooms, including opportunities for generalization.

Method

The excerpt used for this chapter was a transcript of a second-grade classroom exploring equivalence relationships (See *Overview of the Elementary Level Data*, Kosko, this volume). We conducted multimodal analysis, which involves qualitative analysis of multiple episodes and an examination of the modalities used in those episodes. The nature of this book (comparing analyses of the same transcript of a single lesson) limits the degree to which certain multimodal aspects (like gesture) could be examined. We examine three mathematical tasks descriptively using the two themes from our theoretical framework – the importance of generalization during proof activities and multimodal resources for mathematical proving – in order to understand the act of proving in this second-grade classroom.

Data Analysis and Findings

In the following sections, we deconstruct the act of proving found in three mathematical tasks. First, we examine how Ms. Kirk supported the construction of a more generalized conception of the equals sign using multimodal resources. Then, in the second task, we investigate how the complex layering of two different modalities for proving equality relations (number line and Cuisenaire rods) can create new affordances but may be at odds with the teachers' mathematical goals for the lesson. Finally, in the third task, we explore how prioritizing one modality may hinder generalization.

Excerpt 1: Defining the Equals Sign

Summary

At the beginning of the transcript, Ms. Kirk launches a whole discussion of the meaning of an abstract symbol (the equals sign) in order to practice the act of proving and "explaining our thinking to other people." Landon immediately contributes his well-developed conceptualization of the equals sign, "The same as," as a relational definition of what the sign means. Building upon Landon's sophisticated idea, Ms. Kirk continues to press students to share "any other words that they use to help understand it," and she discovers that her students may not all have this same level of understanding as Landon. For example, Amaya offers "what it equals" signaling that she may understand the equal sign as an operator that means to get an answer. Realizing her students may not have the words to describe the meaning, Ms. Kirk presses "What do you picture in your head when you're trying to understand what this symbol means?," and eventually students consider tools or a concrete, embodied referent (a balance scale).

Pressing for further understanding, Ms. Kirk then explicitly directs students toward using the manipulatives to formulate their definition of the equals sign stating, "Think about the tools that we've used in class to help us understand this," and eventually a student named Kaylyn mentions a balance scale. However, Kaylyn describes the part of the balance scale that corresponds to the equals sign as being "The bottom like where you put this stuff on" rather than the fulcrum, which is the point at the center of the scale where the beam rests. Ms. Kirk uses horizontal and vertical hand gestures to reinforce the distinction between the balance beam and the fulcrum while stating, "So like the long yellow part? Or do you mean the red…." Kaylyn then communicates her updated understanding only via gesture, by making a vertical gesture indicating the fulcrum and simply saying "Yeah." When Kaylyn communicates her understanding, she echoes or repeats the vertical fulcrum gesture Ms. Kirk had just made, showing that they have established a kind of embodied common ground for understanding the equals sign (see Walkington et al., 2019).

Multimodality

This excerpt showcases how Ms. Kirk leveraged her students' multimodal definitions of the equals sign. Modalities like the Cuisenaire rods, gesture, and the balance scale were used in order to create a more generalized definition to support students' developing conception of equality. The balance beams are not explicitly used for the remainder of the lesson given in the transcripts. Indeed, the modality of the balance beam fades into the background as learners focus on a new modality – the Cuisenaire rods. There are important connections to be made between the balance beams and the Cuisenaire rods – as both show how the equals sign can mean "both sides are the same value." But these connections between modalities are left implicit and not explicitly brought to the forefront in the lesson that follows.

Excerpt 2: Discussion of 14 + 3 = 15 + 2

Summary

In the next excerpt of the transcript, Ms. Kirk has asked students to consider the equation 14 + 3 = 15 + 2 and to "show me with your Cuisenaire Rods whether or not that's true." During the subsequent whole-class discussion, she highlights the work of two students Carrie and Amanda, who used Cuisenaire rods to show that a 4 rod and a 3 rod were the same length as a 5 rod and a 2 rod. Ms. Kirk has placed the proper rods on the overhead projector for this discussion (Fig. 1).

Ms. Kirk first points out that "I took away from them their number line" and instead gave them a blank piece of paper. The students' number line page contained five horizontal lines of equal length arranged one on top of another with space in between. One of the number lines had no subdivisions, one had subdivisions that

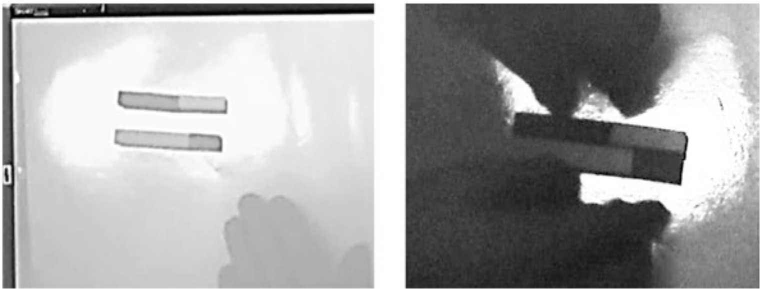

Fig. 1 Left: 4 rod and 3 rod (top) and 5 rod and 2 rod (bottom) on overhead. Right: Sammy pushes the rods together to show equality

had a length of 1 unit on the Cuisenaire rods, and the remaining three had subdivisions of lengths 2, 3, and 4.

Ms. Kirk then says that she had asked Carrie and Amanda "in using this, can you prove to me that four plus three is the same as five plus two?" However, figuring out what kind of proof to construct based on the omittance of the number line is difficult for some of the other students. Ms. Kirk calls on a student named Sammy to describe the visual proof of Carrie and Amanda using the rods. Sammy instead describes an arithmetic compensation strategy where she takes the expression 4 + 3 = 5 + 2 and transforms it into 5 + 2 = 5 + 2 by adding 1 to the 4 and taking 1 away from the 3.

Ms. Kirk responds to Sammy's reasoning by saying "I just wanna use these," referring to the Cuisenaire rods. Ms. Kirk had initially manipulated the rods on the overhead herself, lining the two rows up to show equality. Sammy had given her arithmetic proof verbally, but when Ms. Kirk tells her to use the rods, Sammy physically manipulates the rods and pushes them together in the manner Ms. Kirk asked for (see Fig. 1). Sammy contributes the idea that "they're the same size," integrating a measurement model of equivalency. Ms. Kirk responds by saying "You can't really argue with that. That is some hard concrete evidence." Ms. Kirk further accentuates the importance of the visual, saying that "this tool… it takes a number that sometimes can feel hard to picture, and it gives you something to picture, something you can look at."

Generalization

Ms. Kirk explicitly uses the word "prove" in this excerpt, and her comment about taking away the number line from the students gives an indication as to what she might mean by "prove." The presence of the number line may encourage students to show equality simply by counting cube spaces, whereas the blank paper may be thought to better elicit a strategy where the lengths or the rods are lined up and shown to be equal visually. While counting cube spaces might be viewed as only applying to the particular numerical values in the equation at hand, lining up the rods to show equality might be viewed as more general and transcendent of particular numerical values. Thus Ms. Kirk seems to be pushing for a kind of generalization on the part of her students.

However, it is interesting that Ms. Kirk does not explore Sammy's strategy, when it arises. Sammy's strategy implicitly demonstrates understanding of the reflexive property and of addition and subtraction as inverse operations and would generalize to any two sets of addends in a balanced equation. Furthermore, Sammy's strategy reveals to the class *why* the number sentence is true (i.e., because of the aforementioned properties of arithmetic), in a way that visually showing two rows of blocks being the same length does not. Thus, Sammy's proof could actually be considered stronger in terms of the cluster definition of proof given above. Indeed, comparatively, Ms. Kirk's desired proof using the rods is closer to simply being a visual definition of the equals sign rather than "a perspicuous argument that provides the reader with an understanding of why a theorem is true" (Weber, 2014, p. 357). Ms. Kirk encouraged some generalization but the kind of generalization that was encouraged was tied to a specific modality that did not afford understanding why. Generalization of the concept of equality could have been strengthened by connecting Sammy's arithmetic strategy to the modality of the rods.

Multimodality

In this excerpt, Ms. Kirk seems to privilege the visual representation of the Cuisenaire rods as the appropriate pathway to proof. At one point, she seems to identify one potential shortcoming of Sammy's proof – its lack of a visual perceptual channel of communication. However, the rods have constraints as well. As the rods are a solid nondetachable unit, using them to show adding and subtracting of single units (i.e., Sammy's compensation strategy) is not an affordance of the particular modality. In addition, some students seem to initially struggle to give a proof that utilizes the rods. Thus, this excerpt shows how the complex layering of two different modalities for proving equality relations (number line and Cuisenaire rods) can, when put together, create new affordances which may or may not be consistent with the teacher's mathematical goals.

The prioritization of the visual manipulatives may have caused a missed opportunity for Ms. Kirk to connect mathematical properties – reflexive property and addition and subtraction as inverse operations – that would generalize to any two sets of addends in a balanced equation.

Excerpt 3: Jacob's Proof of 40 + 9 = 27 + 22

Summary

In the third excerpt, Ms. Kirk is discussing Jacob and his partner Sammy's proof of 40 + 9 = 27 + 22. When setting up the task, Ms. Kirk allows the students to choose their own numbers to balance the equation but cautions students: "So I know that some of you like to make problems with like the largest numbers you can come up with. However, right now is not the time for that, because I asked you to use a certain tool."

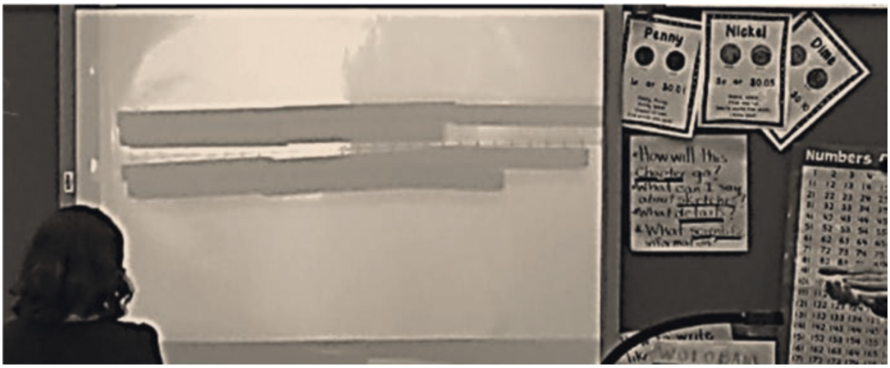

Fig. 2 Jacob and Sammy's proof on the overhead projector

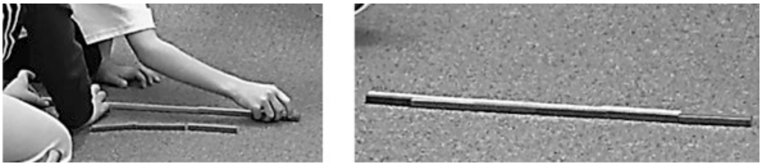

Fig. 3 Students build two rows of rods (left) and push them together (right)

Despite these instructions, Jacob and Sammy still chose a large number (49) that was not ideal to show with the rods. When displayed on the number line (which had approximately 22 spaces), the number 49 was big enough that one side of the equation was represented by two rows instead of one row, which made the "line up to visually observe equality" strategy problematic (see Fig. 2).

Jacob and Sammy's strategy involved representing 40 as four 10 rods, 9 as one 9 rod, 27 as three 9 rods, and 22 as two 8 rods and one 6 rod. Ms. Kirk states that "It's a little bit hard to tell that these are the same because they broke their rods in half to fit on their number line." Ms. Kirk tells the students how to line the rods up on the floor such that they can make two equal rows. As the students line up their rows (Fig. 3), Ms. Kirk asks "If what they're saying is true, if we lined each side up, what should it look like?" Tony responds with "A big line that's the same on both sides and is equal." Kate further adds "They are equal," and Ms. Kirk repeats this phrase.

Generalization

Similar to when Sammy's strategy came up in the second excerpt, Ms. Kirk also does not explore the arithmetic properties that underlie Jacob and Sammy's combinations of rods (i.e., how they made the numbers of 27 and 22). There is overall little explicit focus on generalization in this transcript – the problem-solving is reconciled when an essentially identical observation is made as was made in Excerpt 2 about the lines being visually the same. There is again no discussion ensuing about *why* they are equal or how the rods being visually the same connect to generalizable properties of number. Given that the focus is on just one modality (the rods), there are missed opportunities for generalizing mathematical principles across modalities, as there was in Excerpt 2.

Multimodality

The events in this transcript reveal yet another limitation of the modality of the Cuisenaire rods – it is difficult to represent larger numbers with them. In addition, the modality of the overhead projector, and its limitations in the size of the area it projects, also constrained the students' ability to visually communicate their argument. The complexity of the representation Jacob and Sammy came up which shows a related issue with the modality of the Cuisenaire rods – there were only 4 10 rods available in the Cuisenaire rods kit, so Jacob and Sammy had to improvise and use other rod sizes to represent their numbers. This improvisation could have in and of itself been capitalized on as a learning opportunity, but was not discussed.

Indeed, despite Jacob and Sammy's sophisticated use of the rods, since the rods did not line up into two equal-sized rows, Ms. Kirk does not move forward with their proof. Instead, she has the students line up the rows on the floor to create a visual proof consistent with the proof in Excerpt 2. This prioritization of the visual manipulatives may have caused a missed opportunity for Ms. Kirk to connect mathematical properties to the visual proof thereby limiting the generalizability of the mathematical proof that the students were constructing together. It may have also resulted in the third mathematical task not functioning to continue to push their understanding further, past what they learned in the second excerpt.

Discussion

Although Ms. Kirk frames the activities the class is engaging in as proving, this lesson simply explores the relational meaning of the equals sign (e.g., Matthews et al., 2012). By conflating a focus on definition with that of proof and by privileging a single modality (the Cuisenaire rods), Ms. Kirk sets communal norms for what a proof is that are not consistent with Weber's (2014) cluster concept of proof. It is not surprising, then, that the norms Ms. Kirk is trying to set for how to formulate mathematical arguments are generally not taken up by the students and that explicit prompts from the teacher on what to do are regularly needed.

In terms of multimodality in the present transcript, however, beyond speech and gesture, there are four different representational systems used for understanding the equals sign in these excerpts – the balance scale, the number line, the Cuisenaire rods, and numerical symbols and operations. Ms. Kirk shows preference for the balance in Excerpt 1, and for the rods in Excerpts 2 and 3. The degree to which she holds to her modal-specific epistemological commitment for the rods is striking. Ms. Kirk is so focused on this representation she does not take up any student contributions that are not within this modality, despite the mathematical sophistication of other student contributions. She redirects student reasoning that does not utilize the rods, discusses explicitly with the class the affordances of the rods, and uses the rods in her instructions for what constitutes proving during this lesson.

One reason Ms. Kirk's modal-specific epistemological commitments could be problematic is the lack of flexibility of the rods as a representational system – they are limited when showing large numbers and cannot easily be broken apart to show strategies like compensation. These limitations made it difficult to use the rods to create general mathematical arguments that reveal *why* the statement is true. This shows how modalities in part constitute the argument – they dictate how mathematical ideas can both be formulated and communicated. Combining the rods with the number line might further obscure the kind of generalized definition of equality, as the number line allows for counting.

These excerpts also show the importance of multimodal resources like actions and gestures during proof activities. Performing actions on objects like rods or a balance scale can reveal key mathematical principles, and gestures like pointing gestures (e.g., when Jacob pointed to the rods to communicate his proof) and representational gestures (e.g., when Kaylyn gave her answer of fulcrum only using gestures) are a critical component of students' mathematical reasoning. However, these excerpts tell a cautionary tale about the importance of connecting concrete and abstract representations. Concrete materials like manipulatives can be easier for students to access and allow for new strategies and easier detection of conceptual errors. However, representations that may be more abstract to children like symbols can be more flexible and powerful and easier to generalize from (see Goldstone & Son, 2005; Koedinger et al., 2008). An excessive focus on concrete representations of math concepts can limit students' potential to engage in proof activities and can leave sophisticated mathematical thinkers like Sammy behind.

We argued in the theoretical framework that the generality of students' mathematical arguments should be an important component of the cluster concept of proof. In addition to proofs being generalizable to all mathematical objects under consideration, proofs are strongest when they *generalize across modalities.* In the excerpts, connections could have been forged across the four representational systems – balance, number line, rods, and symbols. This could have created for students an integrated understanding of equality and a mathematically stronger argument for proving each individual number equation. While Ms. Kirk certainly understood the connections between these representational forms, the implicit relations between different modalities can be much more difficult for novices to grasp unless the teacher makes very explicit connections between them (Nathan et al., 2013, 2017).

Nathan et al. (2013) give recommendations for how teachers can connect different modalities in STEM classrooms, by (1) reminding students of their activities involving the same math concept with prior modalities, (2) explicitly coordinating two modalities side by side to show their relation, and (3) explicitly labeling key math concepts as they occur in different modalities. The first move on this list, referred to as "projection," could have been used by Ms. Kirk here to make references back to the balance scale modality in Excerpts 2 and 3. She also could have made projections to students' past or future experiences with compensation strategies when considering arithmetic operations. The second move on this list, referred to as "coordination," could have been used when Sammy offered her numeric strategy. Ms. Kirk could have used gestures over the Cuisenaire rods on the overhead to demonstrate how Sammy was "cutting off" a piece of one rod and "adding" to another to maintain equality. In this way, she would have coordinated the modality of the rods with the mathematical symbols and operations. And finally, the third move on the list, referred to as "identification," was used by Ms. Kirk – she identified the fulcrum as being an embodiment of the equality concept, as she did with the rows of congruent rods. Although there were missed opportunities to identify the role of the equality concept in Sammy's arithmetic strategy, the equality concept and the idea of equality was consistently labeled across different contexts in these excerpts.

It is important to note that even though Ms. Kirk uses the word "prove," the fact that the tasks she presents do not specifically ask for generalization beyond specific cases does limit the degree to which the activities can be understood with proof as the framework. Although generalization was not explicitly mentioned in Weber's (2014) cluster definition of proof, it is a key component of other definitions of proof. Ms. Kirk may guide her class in a very different way when they are confronting activities that involve more explicit generalization beyond specific cases, and analyzing a transcript with this kind of activity would also be useful for understanding the nature of proof and multimodality.

Conclusion

While there is no consensus on the meaning of proof in elementary mathematics, research suggests that the act of proving at this grade level can be collaborative, inquiry-oriented, and embodied. We close this chapter by providing some specific recommendations for mathematics educators when considering the teaching of proof in the elementary grades. Through the descriptive theory-driven analysis of the given transcript and the literature reviewed in this chapter, we highlighted the importance of:

1. Teachers and students conceptualizing proof in a multimodal manner and in valuing the different modalities used to communicate proofs
2. Teachers and students explicitly considering the affordances and constraints of different modalities for proving conjectures
3. Teachers monitoring theirs and their students' modal-specific epistemological commitments

4. Instruction bridging abstract and concrete modalities, and making connections between modalities when discussing mathematical proofs

The analyses in this chapter showed that proof is a richly embodied process that involves the complex coordination and interplay of many different modalities of mathematical representation and communication. Paying attention to the multimodality of proof may enhance students' opportunities to learn, while missing these opportunities may limit potential for mathematical growth. Teaching proof in an integrated, multimodal manner may allow students to come to more fully and deeply understand the underlying mathematical concepts and relations that are central to elementary grades mathematics.

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On the Meanings of Argumentation, Justification, and Proof: General Insights from Analyses of Elementary Classroom Episodes

Andreas J. Stylianides and Gabriel J. Stylianides

The terms (mathematical) *argumentation*, *justification*, and *proof* are or can be related, and, as the editors of this book rightly observed, the terms have been used in various ways in the mathematics education literature and often without an explicit definition. It would also be fair to say that, sometimes, these terms have been used interchangeably in the literature. Accordingly, it is a worthwhile endeavor to reflect on the possible meanings of these terms and on the affordances, constraints, or implications of choosing certain terms and their respective definitions in particular research situations.

The authors of each of the three chapters in this section of the book used as their starting point when posing and exploring their own research questions the definitions of (mathematical) argumentation, justification, and proof that were selected by the editors, summarized in Table 1. Also, all three chapters were based on analysis of the same corpus of data comprising a sequence of episodes from Ms. Kirk's second-grade classroom that were selected again by the editors. The common corpus of data provided an interesting context for the authors to offer various insights into Ms. Kirk's instructional practice. Those insights derived from the different theoretical perspectives and analytical approaches on argumentation, justification, or proof that each team of authors brought into play based on the specific notion they explored in their chapter.

The issues discussed in each chapter are valuable in their own ways, though there is not a concrete basis for comparisons across chapters, partly because of the

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Table 1 The definitions of (mathematical) argumentation, justification, and proof that were selected by the editors for consideration in this book

Term	Definition
Argumentation	Mathematical argumentation is the process of making mathematical claims and providing evidence to support them
Justification	Mathematical justification is the process of supporting mathematical claims and choices when solving problems or explaining why your claim or answer makes sense (Bieda & Staples, 2020)
Proof	Mathematical proof is considered as a clustered concept on the basis of the following criteria: "(1) A proof is a *convincing argument* that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a *deductive argument* that does not admit possible rebuttals. […] (3) A proof is a *transparent argument where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation. […] (4) A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*. (5) A proof is an *argument within a representation system satisfying communal norms*. […] (6) A proof is an *argument that has been sanctioned by the mathematical community*" (Weber, 2014, p. 357; italics in original)

differences among the chapters' research questions and methodological approaches. For example, given the apparent overlap between the definitions of argumentation and justification in Table 1, if the authors of the Argumentation chapter (see the chapter by Rumsey, Whitacre, Atabaş, and Smith, this volume) and Justification chapter (see the chapter by Thanheiser and Sugimoto, this volume) had pursued similar research questions and similar methodological approaches to address them, one would have expected two relatively similar chapters to emerge from that process. Of course, the two chapters are completely different: Rumsey et al. analyzed Ms. Kirk's episodes from the perspective of how the choice of tools (e.g., Cuisenaire rods) and the nature of tasks involving those tools can shape students' opportunities for argumentation, while Thanheiser and Sugimoto analyzed the same episodes from the perspective of how justification can allow students to access more equitably various forms of mathematical reasoning in the classroom. While less variation among research questions and methodological approaches across chapters could have offered crisper insights into affordances, constraints, or implications of using each of the terms in Table 1 and their respective definitions, it would have restricted the authors in pursuing their research interests, and thus it might have compromised the important and varied contributions made in their chapters. Accordingly, the overall setup and structure of this section of the book is both defensible and interesting.

Having said that, we believe that some further reflection on issues of terminology, particularly with regard to the meanings of argumentation, justification, and proof, would still be relevant given the overall focus of the book on the interplay among these terms, and thus we have decided to focus this commentary on such issues. What are the implications of using the particular definitions in Table 1 as opposed to alternative definitions in the literature when analyzing the *same* episodes?

As a context for our discussion, we will consider Ms. Kirk's episodes alongside an episode from Deborah Ball's third-grade classroom that presents work in the same general mathematical domain (arithmetic) and that raises some contrasting issues related to argumentation, justification, and proof. The episode shows Ball's third graders engaging in a lively discussion over the answer to the following problem:

$$6 + \hat{6} = ?$$

An in-depth discussion of the episode from the perspective primarily of proof, but also argumentation and justification, can be found in Stylianides (2016, pp. 81–89) where we draw on for our discussion herein. Substantial background information would need to be offered about the work of the class prior to this episode for the reader to fully appreciate Ball's instructional thinking in posing this problem to her students. For the purposes of this commentary, however, it should suffice for us to note the following as a background for the episode: Ball used the circumflex (ˆ) above the numerals in the place of the minus sign of negative integers (e.g., $\hat{6}$ means "minus 6"), because she considered that this notation could help the students focus "on the idea of a negative number as a number, not as an *operation* (i.e., subtraction) on a positive number" (Ball, 1993, p. 380; italics in original). Also, for their exploration of aspects of integer arithmetic, Ball made available to the students a "building model" that consisted of all floors ranging from floor $\widehat{12}$ (i.e., floor 12 below the ground level) all the way up to floor 12 (above the ground level). The students had used this model in previous lessons.

As you read the following dialogue in the episode, we invite you to consider the following question: In what way, if at all, do you see issues related to *argumentation*, *justification*, and *proof* arising in the work of the class?

1. **Sean**: I got 6 [as the answer to $6 + \hat{6} = ?$].
2. **Betsy**: Instead of Sean's [answer], I got 0.
3. **Ball**: You'd like to put 0 here for 6 plus minus 6?
4. **Betsy**: Do you want to see how I do it?
5. **Ball**: Okay.
6. **Other students:** Yeah!
7. **Betsy:** Here. You're here [she shows the sixth floor on the building model], but you can't go up to 12, because that's 6 plus *6*. So I say it's just the opposite. It's just 6 minus 6.
8. **Sean:** But it says plus, not minus!
9. **Betsy:** But you're minusing.
10. **Riba:** Where'd you get the minus?
11. **Sean:** You should just leave it alone. You can't *add* 6 below 0, so you just leave it. Just say "goodbye" and leave it alone and it is still just 6.
12. **Mei:** But this 6 below 0 would just disappear into thin air?!
13. **Sean:** I know. It would just disappear because it *wouldn't* be able to do anything. It just stays the same, it stays on the same number. Nothing is happening.

14. **Betsy:** But, Sean, what would you do with this 6 below 0 then?
15. **Sean:** You just say "goodbye" and leave it alone.
16. **Riba:** You *can't* do that. It's a *number.*
17. **Sean:** I know, but it's not going down. It's going up because it says plus.
18. **Mei:** I think I disagree with Betsy and Sean because I came up with the answer 9.

The discussion continued a bit longer until the end of the lesson. We return now to the question that we posed earlier about whether there are issues related to argumentation, justification, and proof arising in the work of the class in the episode. The answer depends, of course, on one's definitions of these terms. We will consider the question in some detail with regard to the notion of argumentation for the purposes of illustration, first in the context of Ball's episode and then in the context of Ms. Kirk's episodes. We will conclude with some brief remarks with regard to the notions of justification and proof.

We note from the outset that the meaning that we attribute to "argumentation" is different from the definition of this term in Table 1. Specially, we use the term *argumentation* to describe the discourse or rhetorical means (not necessarily mathematical) that are employed by an individual or a group to convince others that a statement is true or false (cf. Boero et al., 1996; Mariotti, 2006; Stylianides et al., 2016). The term "convince" in this description is important as it indicates that, for us as well as for others, argumentation focuses on the epistemic value of a given statement and it can embody a link with Harel and Sowder's (2007) notions of *ascertaining* (i.e., the process employed by an individual to remove his or her own doubts about the truth or falsity of a statement) and *persuading* (i.e., the process employed by an individual or a group to remove the doubts of others about the truth or falsity of a statement).

The episode from Ball's class presented a lively discussion, characterized by Ball at the end of the lesson as a "scientific debate," over the truth or falsity of students' various claims about the answer to the problem that was posed by the teacher. Was the correct answer 6, as proposed by Sean (line 1)? Zero, as proposed by Betsy (line 2)? Or 9, as declared toward the end of the episode by Mei (line 18)? The episode presented also a variety of rhetorical means used by the students to convince others about the correctness of their own answer or about the falsity of others' answers. For example, the rhetorical means that Sean used to convince the rest of the class that the term + $\hat{6}$ in the sentence $6+\hat{6}=?$ "would just disappear" (line 13) are not mathematical, but they nevertheless fit in with the particular meaning of argumentation that we presented earlier. The rhetorical means that Betsy used to convince Sean about why it would not make sense to interpret the term "$+\hat{6}$" as "plus six" (line 7) is more mathematical in nature, resembling an argument by reductio ad absurdum.

Thus, based on our adopted perspective on argumentation, the episode from Ball's class raises prominently issues of argumentation. At the center of the class's engagement in argumentation was the *uncertainty* that prevailed among the students over the answer to $6+\hat{6}=?$. This uncertainty appeared to have underpinned students' use of *rhetorical means* to *convince* themselves (cf. ascertaining) and others (cf. persuading) about the truth or falsity of their various claims about the answer to the problem so as to remove everyone's doubts and resolve the emerging disagreements.

Turning now to the episodes from Ms. Kirk's class, from our perspective on argumentation, we see little (if any) evidence of students' engagement in argumentation. In these episodes we do see lots of (mathematical) *arguments* – a term that we use to describe connected sequences of assertions for or against (mathematical) claims (Stylianides, 2007) – but little if any *argumentation*. This is by no means a criticism of Ms. Kirk's instructional practice. The lack of evidence for the presence of argumentation (as we perceive it) in the episodes might reflect different instructional priorities for the particular episode.

Let us explain in more detail why we do not see much argumentation in Ms. Kirk's episodes. As the authors of the Argumentation chapter (see the chapter by Rumsey et al., this volume) rightly observed, the nature of the problems that were posed to the class by Ms. Kirk, coupled with Ms. Kirk's emphasis on the use of the Cuisenaire rods to produce evidence for the equivalence of different number sentences, left little room for uncertainty or disagreement over the answers to the problems or over the reasoning through which those answers could be derived. Accordingly, there was little motivation and attempt on behalf of the students to use rhetorical means to *convince* themselves or others that a statement was true or false. In short, the epistemic value of the statements under consideration in the episodes was not a focus of discussion while the act of convincing was virtually absent.

The definition of argumentation that was used by the authors of the Argumentation chapter in this section of the book (see Table 1) is less constraining than ours, and it does not place a similar emphasis on *conviction* linked to uncertainty or disagreement (doubts) over the truth or falsity of the statements. Instead of making these features part of the meaning of argumentation, as in our adopted perspective, Rumsey et al. kept them separated from their definition of argumentation, viewing them rather as features of *productive* (or *rich*) argumentation. Thus, while Rumsey et al. and we made *similar* observations about the problems and their implementation in Ms. Kirk's episodes that allowed little if any room for uncertainty or disagreement, Rumsey et al. and we reached *different* conclusions with regard to the presence of argumentation in these episodes. Rumsey et al. did devote a good part of their chapter discussing missed opportunities on behalf of Ms. Kirk to utilize the value of uncertainty and other factors in argumentation, but this does not change the fact that we reached different conclusions about the presence of argumentation in Ms. Kirk's episodes. We hypothesize that, if Rumsey et al. had analyzed Ball's episode from their adopted perspective of argumentation, we not only would have made again *similar* observations about the teacher's instructional practice related to argumentation, but, most importantly, in this case all of us would have reached also the *same* conclusion about the presence of argumentation in the episode. We already explained how Ball's episode illustrates our definition of argumentation; the episode appears to illustrate also many of the features of Rumsey et al.'s view of *productive* argumentation, which set the bar higher than their definition of argumentation.

The previous discussion about using two different definitions of *argumentation* in the analysis of the same classroom episodes illustrates (1) the complex ways in which the choice of a definition might impact one's research conclusions and (2) how it is possible to reach the same conclusion by using different definitions of argumentation. Similar points could be made if we used different definitions of *justification* and *proof* – say those in Table 1 versus, for example, the definitions discussed in Stylianides (2016, pp. 12–13) – to analyze the same episodes from the classes of Ball and Ms. Kirk. Below we make some brief remarks with regard to justification and proof so as to further illustrate the two points.

Like in the case of argumentation, the meaning that we attribute to the term "justification" differs from the definition of this term presented in Table 1. Specifically, in our prior work, we used *justification* (Stylianides, 2016, p. 12) but also *verification* (Stylianides, 2009, p. 269; see also Bell, 1976; de Villiers, 1999) to describe a key function that proofs (or mathematical arguments more generally) can serve in mathematical work, namely, to establish the truth of a claim. Other key functions that proofs can serve include *refutation* (Stylianides, 2016) or *falsification* (Stylianides, 2009), when they establish the falsity of a claim, and *explanation*, when they offer insight into why a mathematical claim is true or false (e.g., Bell, 1976; de Villiers, 1999; Hanna, 1990). The definition of justification in Table 1 appears to encompass all three of these functions that a mathematical argument might serve. Accordingly, using this definition might make it difficult to detect differences between episodes like those from Ms. Kirk's class that included situations calling for proofs to establish and explain the *truth* of claims (cf. justification/verification and explanation) and other episodes like that from Ball's class that included a situation calling for proofs to establish and explain both the *truth* and the *falsity* of claims (cf. justification/verification, refutation/falsification, and explanation). Of course, detecting such differences might not always be a research concern, like it was not in the chapter by Thanheiser and Sugimoto (this volume), but nevertheless we see again how the choice of a definition might impact one's research conclusions.

With regard to proof, the authors of the respective chapter (see the chapter by Walkington and Woods, this volume) adapted Weber's (2014) definition of proof, selected by the editors (see Table 1), to include also *generality of argument* as one of its key components. While the authors made a good case for the adaptation, it is nevertheless interesting to observe that the adaptation resulted in them observing a limited presence of proof in Ms. Kirk's episodes: "It is important to note that even though Ms. Kirk uses the word 'prove,' the fact that the tasks she presents do not specifically ask for generalization beyond specific cases does limit the degree to which the activities can be understood with proof as the framework" (see "Discussion" in the chapter by Walkington and Woods, this volume). Weber's (2014) original definition and other definitions of proof, including our own (e.g., Stylianides, 2007, 2009), do not restrict proving tasks to those that ask for generalization beyond specific cases. In fact, the episode from Ball's class that we presented earlier is discussed by Stylianides (2016) as an example of a task that could involve proof of a *single* case. On this basis, and without us having to get into the details of our own definition of proof, we see again how the choice of a definition might impact one's research conclusions.

To conclude, our discussion in this commentary, drawing on the three chapters, highlighted the importance of researchers making their definitions of argumentation, justification, and proof explicit so as to enable proper interpretation of their conclusions and facilitate comparisons across studies (see also Balacheff, 2002; Stylianides, 2007; Stylianides et al., 2017). Beyond that, the question arises as to whether any definition would be acceptable provided it was clearly stated and properly followed in the analysis. While it might be neither possible nor desirable for all researchers to adopt the same definitions for different terms, not least because different research goals might be served better by certain definitions than by others (similarly to the function served by different equivalent definitions in the discipline of mathematics), it is less clear to us what the acceptable range of definitions might be and how the range of acceptable definitions might depend on the research context. Reflecting on the implications of using alternative terms and definitions to analyze the same classroom episodes, as in the three chapters and like we did in this commentary, can help deepen understanding about and sensitivity to issues of terminology and meaning in mathematics education research.

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Part II
Argumentation, Justification and Proof in a Middle Grades Classroom

Overview of Middle Grades Data

Megan Staples

In this section, we explore particular definitions of the constructs of argumentation, justification, and proof in the context of one middle school classroom—a 7th-grade classroom in a small, somewhat rural community in New England. The school included students in grades 4–8. The teacher, at that time, had 4 years of experience. There were approximately 19 students in the class. The school did not have any grouping or tracking, and the teacher taught all sections of 7th-grade math. There were no English learners in the class.

These data were collected as part of an NSF-funded grant project. This teacher was one of 12 middle school teachers collaborating with the research team to better understand the nature of justification in middle grades classrooms. These data are from year 1 of the project. We videotaped four "lesson cycles" per teacher per year. The core of each lesson cycle was the implementation of a task designed to engage students in support-for-claims mathematical activity.

The Number Trick task was designed by the research team (cf. Porteous, 1990; Bieda, Holden & Knuth, 2006) and vetted by the project teachers. This task was the second of four such activities to be implemented during that school year as part of the research activities. We videoed the lesson and collected student work from the lesson. The data set provided to the chapter authors comprised of: (a) a transcript of the task implementation along with (b) student written work on the task (9 samples), and (c) a lesson table, to provide context for the lesson, which summarized the activity and major themes of the work students and teachers did for each of the different phases of the entire lesson. Note that the student work samples had pseudonyms, but the transcript tended to say primarily "Student 1," "Student 2," etc., because the original transcription process did not require that an individual's

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contributions be tracked. Thus, the chapter authors could not necessarily link the student work to a student's contribution(s) in class, although often there was a clear connection between the written work and what was said by the student. The transcribed lesson comprised the full task implementation for that class period that included the launch, individual work, small group work, and then whole-class discussion (approx. 35 min).

The task featured in the data set, the Number Trick task, was designed to engage students in ideas related to the distributive property. Students in this class had not yet been introduced to the distributive property, but they had been working to understand order of operations in previous lessons. It presented a "trick" done by Jessie, where she takes a number, does two different processes on that input, and finds that the result for both processes is the same. The two core questions focus on (1) establishing the trick for values from 1 to 10 and offering an explanation and (2) considering whether the explanation for values 1–10 demonstrates that the trick works for all values. The Number Trick task is shown in Fig. 1.

Note the unique phrasing of the second question. Students are not asked directly to demonstrate that the trick does or does not hold for *any* number (i.e., does the trick always work). Rather, students are asked to evaluate whether the reasoning they provided in the first part would also show that the two answers will *always* be the same for *any* number. This unconventional wording potentially opens space for a discussion about approaches to argumentation and when/what type of arguments can be used to establish a general claim versus claims about a finite set of instances.

After launching the task, students worked individually and then in small groups, with the teacher moving from group to group to ask students about their reasoning and to support them in articulating and recording their results. The teacher then selected groups, in a particular order, to share their results with the whole class. This

Jessie discovers a cool number trick. She thinks of a number between 1 and 10, she adds 4 to the number, doubles the result, and then she writes this answer down. She goes back to the number she first thought of, she doubles it, she adds 8 to the result, and then she writes this answer down.

Here is an example:

Jessie thinks of the number.	5
She adds 4 to her number	5 + 4 = 9
She doubles the result	9 x 2 = 18
She writes down her answer.	18

Jessie goes back to the number she thought of.	5
She doubles her number.	5 x 2 = 10
She adds 8 to the result.	10 + 8 = 18
She writes down her answer.	18

Will Jessie's two answers always be equal to each other for any number between 1 and 10? Explain your reasoning.

Does your explanation show that the two answers will always be equal to each other for *any* number (not just numbers between 1 and 10)? Explain your answer.

Fig. 1 The Number Trick task

move resonates with the idea of selecting and sequencing (Smith & Stein, 2011) although these data were collected prior to the publication of the *5 Practices for Orchestrating Productive Mathematics Discussions* book. The teacher ultimately had five students share the work of their group (one student for question 1 and four students for question 2).

As the teacher transitioned from the small group work to whole-class discussion, he framed their conversation in the following way:

> T: I think we all kinda came up with the same answer. That, yes, the trick will work for numbers 1 through 10. Okay. I was talking to all the groups. We all felt that it was a different… Or we all felt that it was true for numbers 1 through 10. A couple of different ways of showing it. So, I want Derek to share his method first.

The first student had checked, with his group, the numbers 1–10. The teacher highlighted this as a valid approach and then turned to the second question in the form of—does the trick always work, which is a slight, but notable, rephrasing of the question on the task. Students started offering a range of different numbers they tested (e.g., 72, 26, 451, 777, a googol).

The teacher then pressed back against this empirical approach, commenting:

> T: So, we have a bunch of bigger numbers that it definitely worked for. So, we don't know, is it going to work for every bigger number? Does just trying a smathering of random numbers that are bigger, does that show me that it is going to work for every bigger number? Does that show me that it's going to work for every bigger number?

Students proceeded to offer their ideas for the second question, in an order designated by the teacher. One group shared a theory that, because they knew that the trick "worked" for 1–10, and since all other numbers had one of those values in the ones place, the trick would work for all numbers. This proposal was refined slightly, when another student noted they would need to test 0 (which was quickly confirmed).

The teacher chose not to pursue this line or reasoning, commenting that he had not thought about that approach. He then redirected the group to share the other approach they would come up with, as he was interested in having that discussed.

This started the sharing of four different arguments addressing why the number trick would work for all values. Here is a brief summary of the arguments, numbered in the order presented:

1. Grace: In the first set, we add before we multiply. And when we multiply by two, it's going to be a bigger number. But in the second set, you multiply first. So we have to add a bigger number.
2. Emma: Okay. I don't know if it makes sense, but okay. [Reading her paper] Jessie basically in the second equation broke down the first equation. When she added 8, she might have imagined first the 8 equals 4 times 2, which she did in the first equation when she added 5 plus 4 and then doubled it. But to understand that, you must realize that 4 is still part of the equation even though it was smushed in with 5. You did double 4, but it was part of the 5 then.
3. Tracie: Alright. So, if n represents any number, you're doing n plus 4 [writing $n + 4$] times 2 [adding parentheses around the $n + 4$ and writing *2 to the right].

But that also equals *n* times [S: She's taking our equation!] two plus 8 [writing (*n* * 2) + 8].

4. Jenna comes to the SmartBoard. She writes *e* = # between 1 and 10 and then (*e* + 4)*2 = *e**2 + 4*2 = *e**2 + 8.

The teacher was out of instructional time at that point. He highlighted the differences in Jenna's and Tracie's responses (i.e., the inclusion of the intermediary step on Jenna's), and then connected Tracie's, Jenna's, and Emma's, asking what they all did that helped show the trick worked for any number. A student responded: "They gave us a letter instead of a specific number, saying that it could be any number." The teacher reiterated the point, praised the class for its good work that day, and class ended.

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Argumentation in the Middle Grades: Exploring a Teacher's Support of Collective Argumentation

Carlos Nicolas Gomez Marchant, Stacy R. Jones, and Hilary Tanck

An essential part of doing mathematics is learning how to develop an argument (Koestler et al., 2013). Moreover, national policies and documents have emphasized argumentation as fundamental to learning mathematics (see Council of Chief State School Officers and the National Governors Association Center for Best Practices, 2010; National Council of Teachers of Mathematics, 2014). For students to be able to develop mathematical proofs—a specific kind of mathematical argument—they need to conceptualize what makes an argument convincing (Bieda & Lepak, 2014). The act of argumentation is a social phenomenon (Krummheuer, 1995) requiring the participation of the teacher and students in the development of arguments. Teachers and students need to work together to co-construct sociomathematical norms about argumentation including what makes an appropriate mathematical argument (Yackel & Cobb, 1996). The sociomathematical norms of the mathematics classroom provide a structure for students' opportunities to participate in mathematical discourse (Jansen, 2008).

A teacher's role, therefore, is significant in supporting students in the construction of arguments (Yackel, 2002). Teachers need to decide on the use of mathematical tasks promoting argumentation (Ayalon & Hershkowitz, 2018), and determine how to promote participation from nondominant students (Civil & Hunter, 2015) and the purposes for argumentation in their mathematics lessons (Staples & Newton, 2016). Moreover, teachers need to highlight and provide opportunities for students to share their mathematical thinking (Philipp et al., 2007). The facilitation of

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collective argumentation—when a group attempts to validate a claim (Krummheuer, 1995)—is a necessary skill for teachers to make explicit students' mathematical thinking. Exploring and understanding how teachers support the construction of student arguments can provide insight into how to better prepare teachers for facilitating these arguments. In this chapter, we focus on how a middle grade mathematics teacher, Mr. MC, supported his students during whole-group conversations in constructing various arguments. The question leading this exploration was how does Mr. MC support and facilitate the collective mathematical argumentation in his classroom? Our goal with this exploration was to focus on how Mr. MC used direct contributions, questioning, and other supportive moves to promote student contributions to the construction of arguments in his mathematics classroom.

Exploring Collective Argumentation in the Mathematics Classroom

Toulmin (1958/2003) explored how arguments are used by individuals and developed a model to deconstruct arguments. "Toulmin's model offers both a language to describe argumentation and a means to structure the components of an argument" (Wagner et al., 2014, p. 10). There are three components at the core of any argument structure (Toulmin, 1958/2003). The *claim* is the statement of truth an individual or collective is attempting to establish. *Data* is the information used by individuals or a collective as a basis for the claim. Finally, the *warrant* is the justification used by the individual or a collection of individuals linking the data to the claim (see Fig. 1).

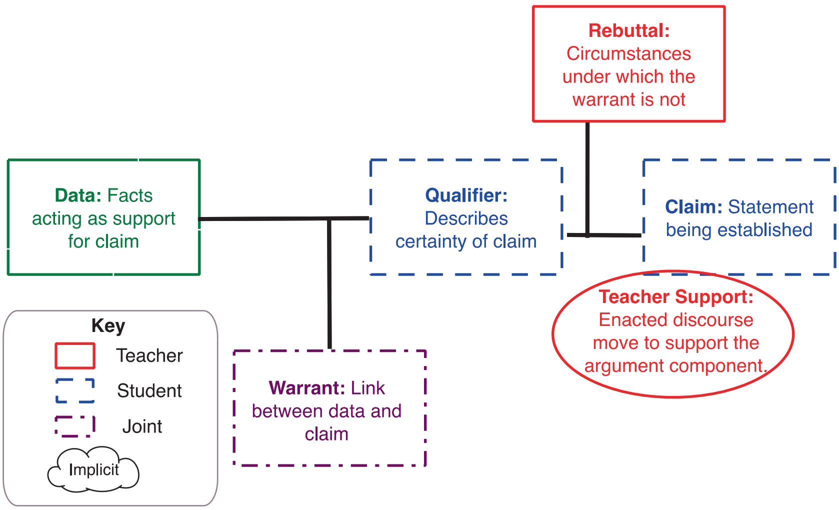

Fig. 1 Core factors of Toulmin diagrams (Conner, 2008; Conner et al., 2014)

Toulmin's model has been used in mathematics education research as a tool to analyze mathematical argumentation (e.g., Giannakoulias et al., 2010). Like other chapters in this volume, our work here is informed by a conceptualization of argumentation as the process of making mathematical claims and providing evidence to support them (Krummheuer, 1995).

Krummheuer's (1995) use of Toulmin's model in his ethnography of argumentation provided a theoretical background of argumentation and conceptualized how it can be considered specifically in the learning of mathematics. "The concept of argumentation will be bound to interactions in the observed classroom that have to do with the intentional explication of the reasoning of a solution during its development or after it" (Krummheuer, 1995, p. 231). Krummheuer explicitly linked argumentation in the mathematics classrooms to students' sharing their reasoning for a solution to a problem or task. Researchers have also incorporated Krummheuer's argumentation conceptualization in highlighting specific kinds of argumentation components (e.g., Giannakoulias et al., 2010).

Conner (2008) described an expanded version of Toulmin's model to aid in deeper explorations of the collective argumentation occurring in mathematics classrooms. The modifications recommended were the use of colors and line styles designating teacher, student, or co-constructed contributions and extending diagrams to demonstrate the intricacies of the mathematical arguments (Conner, 2008). Additionally, chains of arguments can be constructed. For example, the claim of one argument will be the data for the next claim: written as data/claim. A data/warrant would be a warrant for one argument that is also the data for another argument. In further expansions of Toulmin's model, Conner et al. (2014) included how the teacher supported specific components of the argument. Each of these modifications helps in attaining an understanding of the form and structure of the arguments of the classroom, as well as the role the teacher and students are playing in the construction of arguments. With these expanded models, we were able to explore how collective argumentation in Mr. MC's mathematics classroom was structured. We used the Teacher Support for Collective Argumentation (TSCA) framework (Conner et al., 2014) to understand how Mr. MC supported students' development of arguments.

Teacher Support for Collective Argumentation Framework

Conner et al.'s (2014) TSCA framework highlights teachers' actions used to promote students' participation in collective argumentation. Conner et al. suggested "the power of the framework can be seen when the three types of support are analyzed together in the context of collective argumentation" (Conner et al., 2014,

p. 417). In conjunction with expanded Toulmin diagrams, the TSCA framework supplements the analysis of the form and structure of the argumentation by providing insight into the roles teachers and students play in the construction of mathematical arguments.

The TSCA framework documents three types of support for collective argumentation: (1) directly contributing an argumentation component, (2) asking questions prompting an argumentation component, and (3) other supportive actions. These three types of support make up the initial shell of the TSCA framework. Conner et al. (2014) compartmentalized how teachers support collective argumentation through questioning and other supportive actions into five subcategories. Table 1 provides a description of subcategories for each of the teacher support moves (direct contributions, questions, and other supportive actions). The TSCA framework afforded a way to understand how Mr. MC supported students' construction of arguments.

Table 1 Teacher Support for Collective Argumentation framework (Conner et al., 2014, p. 418)

Direct contributions		Questions		Other supportive actions	
Claims	Statements whose validity is being established	Requesting a factual answer	Asks students to provide a mathematical fact	Directing	Actions that serve to direct the students' attention and/or the argument
Data	Statements provided as support for claims	Requesting a method	Asks students to demonstrate or describe how they did or would do something	Promoting	Actions that serve to promote mathematical exploration
Warrants	Statements that connect data with claims	Requesting an idea	Asks students to compare, coordinate, or generate mathematical ideas	Evaluating	Actions that center on the correctness of the mathematics
Rebuttals	Statements describing circumstances under which the warrants would not be valid	Requesting elaboration	Asks students to elaborate on some idea, statement, or diagram	Informing	Actions that provide information for the argument
Qualifiers	Statements describing the certainty with which a claim is made	Requesting evaluation	Asks students to evaluate a mathematical idea	Repeating	Actions that repeat what has been or is being stated

Methodology

In this section, we describe the data analysis procedures followed to explore how Mr. MC supported the construction of arguments while facilitating a conversation about his students' thinking regarding Jessie's Number Trick. The data set and the definitions of mathematical argumentation have been discussed (see the chapter by Staples, this volume and the chapter by Staples and Conner, this volume). So our attention will be mainly on our analysis of the given data. We begin with a brief exchange on the limitations of using transcripts of classroom discourse created by an outside party to investigate collective argumentation.

Limitations

Communication in the mathematics classroom is multimodal. Teachers and students use gestures, facial expressions, representations, manipulatives, and other forms of discourse to share conjectures, ideas, feelings, warrants, and other pertinent information with each other. It is difficult to capture all of this within a transcript. Because we were provided with only a transcript and did not transcribe from the video recording ourselves, we are limited by the information the transcriber decided as relevant to include in the transcript. Lankshear and Knobel (2004) argue the construction of a transcript is a political process:

> Transcripts are not neutral representations of what was said and how it was said. A transcript is a representation of the researcher's views…. The researcher makes decisions about what will be written down from the audio recording, how it will be written down and what will be edited out. (Lankshear & Knobel, 2004, p. 268)

Consequently, for our analysis, we used the TSCA framework and the expanded Toulmin diagrams on transcripts that were not constructed with our research objectives in mind. The TSCA framework is designed to be used with a combination of video data and transcript. This provides the opportunity to explore the nonlinguistic forms of discourse used to support the students in their development of mathematical arguments. We fully recognize the incompleteness our analysis, but we still feel there is enough with the transcript to begin a conversation about the form and structure of the arguments being facilitated and supported by Mr. MC. Without the video, we felt the transcript did not provide enough information to analyze the teacher's interactions with students while monitoring their work. We, therefore, focused our attention to the last part of the lesson where Mr. MC is facilitating the students' responses to the task as a collective.

Data Analysis Procedures

Our analysis began by looking over the transcript and identifying final claims and the associated data, warrants, and data/claims. Final claims were identified by determining the conclusion of a discussion and the shift in the conversation to a new topic. Based on the identified final claim, the other components are demarcated using Toulmin's conceptualization of arguments. As data and warrants are determined, it is possible the final claim shifts. Then, we constructed Toulmin diagrams to be able to see the relationship across components. We refer to these occurrences as episodes of argumentation and identified seven episodes within Mr. MC's facilitation of whole-class discourse (Lines 325–553). Thereafter, the lead author constructed expanded Toulmin diagrams (Conner, 2008; Conner et al., 2014) for each episode of argumentation. The completed diagrams were sent to the other members of the research group for validation. The group met to collectively analyze each of the arguments and reach consensus that the constructed Toulmin models accurately represented the episode of argumentation.

Once agreement was reached for all the expanded Toulmin diagrams, a spreadsheet was constructed to organize the information from the Toulmin models and as a space to code Mr. MC's support moves (see Appendix A). Initial coding for the teacher support moves using the TSCA framework was conducted by the first author and then sent to the other members of the research team for confirmation. The group met again to discuss the codes and come to consensus on any disagreements. Pivot tables were constructed to provide insight into the frequency of specific kinds of questions that were used, other support moves enacted, and who contributed particular components to the arguments. Together, the expanded Toulmin models and the TSCA framework supplement each other by providing a rich way to consider the argumentation occurring in Mr. MC's classroom.

Results

In this section, we discuss the results of our analysis of the support for collective argumentation Mr. MC used while facilitating the classroom conversation on the results of Jessie's Number Trick task (see the chapter by Staples, this volume). We organize this section based on the three types of support described by the TSCA framework. Each type of support provides insight into the form and structure of the mathematical argumentation and the supports provided by Mr. MC for collective argumentation. We conclude this section by looking across each of the types of support and how Mr. MC used them together to provide a space for students to construct arguments and share their mathematical thinking. To facilitate conversation in this section, we begin by providing an example of an episode of argumentation. Thereafter, we briefly describe each of the ways Mr. MC supported collective argumentation within the episode to illustrate how he used specific teacher supports to aid collective argumentation. Finally, we share how Mr. MC supported collective argumentation across the whole-group conversation.

Episode of Argumentation and Teacher Support

The following is the third episode of argumentation identified. The final claim was provided by a student, but Mr. MC also contributed directly to the argument. Students' computational work (e.g., tried googol in Jessie's two algorithms) provided the bridge (i.e., warrant) between the data the collective had access to and the claim that the number meets the criteria of Jessie's Number Trick. Figure 2 shows the Toulmin diagram constructed based on episode three:

MC: When you said, "Well, this is too easy. I know it's going to work for one through 10." Which was, will is will this *always* work for any number?
Bryant: I tried googol.
MC: You tried one followed by 100 zeros.
B: Yes.
MC: Did it work?
B: Yes.
MC: Okay. Good.
MC: Grace, what's up?
Grace: It works for 72.
MC: It works for 72. I know Emma's group tried 26. And that worked.
MC: *turning to a particular student* You did 400 what?
S2: 51.
MC: 451. Did that work?
S2: Yes.
S3: I did 777.
MC: So, we have a bunch of bigger numbers that it definitely worked for. So, we don't know, is it going to work for very bigger number? Does just trying a smattering of random numbers that are bigger, does that show me that it is going to work for every bigger number? Does that show me that it's going to work for every bigger number? Tracie?
Tracie: No, but I came up with a theory.

In the episode, Mr. MC makes three direct contributions—two claims and one warrant. He begins by directly contributing a claim that collectively will be developed through his questioning ("So we have a bunch of bigger numbers that it definitely worked for"). Mr. MC then proceeds to provide two more direct contributions to the argument that were of students he had seen while monitoring ("I know Emma's group tried 26" and "that worked"). Even though Mr. MC directly contributed these aspects to the collective argument, he attributed the work to Emma and her group. Moreover, Mr. MC supported the students in their collective argumentation by putting together the mathematical claims they made and then supporting them through his questioning in making the final claim. He used the students' previous claims as warrants to push students to a larger, more powerful mathematical claim and idea. Although he did not stress making the warrant explicit for the final claim, his direct contribution supported the students in reaching the final claim. We hypothesize the warrant was not explicit because of previous experiences Mr. MC provided in the classroom. Mr. MC attempted to make the discussion more student-centered by having students contribute most of the direct contributions.

Mr. MC asked a variety of questions to elicit direct contributions from students in the episode. His questions were used to support students by *requesting an idea*

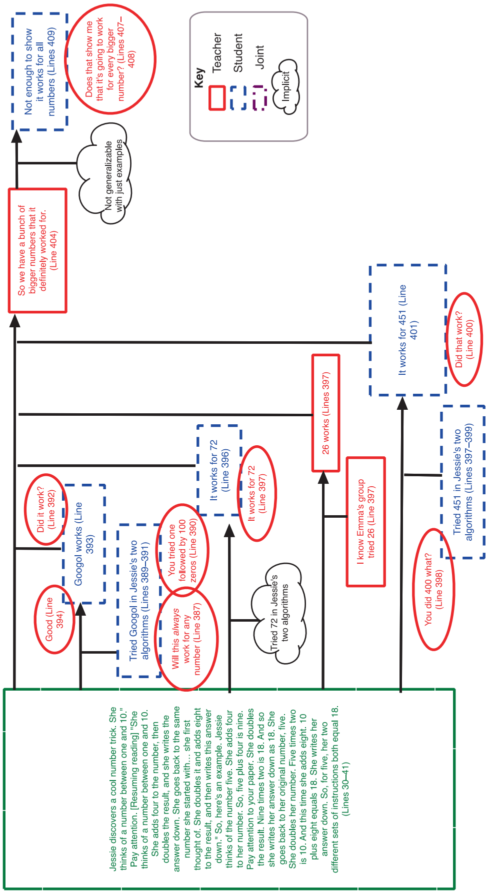

Fig. 2 Toulmin diagram for 3rd episode of argumentation

(e.g., "Did it work?" "Will this always work for any number?"). Mr. MC's direct contribution of the claim, along with his question, "Does that show me that it's going to work for every bigger number?" emphasized how all the previous claims had helped build to the final claim. Mr. MC modeled how arguments can be collectively constructed through the amalgamation of warrants. This support also helped in students making the final claim about against only using examples to generalize.

Mr. MC used other supportive moves, specifically evaluating and repeating, to support the students' collective argumentation. Displaying student work (in the case of directing) or revoicing students' contributions (i.e., repeating) can help in emphasizing students' mathematical thinking and position students as contributors to the mathematical activity (Herbel-Eisenmann et al., 2009). We conjecture Mr. MC's goal was to motivate students to participate and contribute to the collective argumentation by providing direct contributions.

Directly Contributing an Argumentation Component

There were 47 direct contributions made across the 7 episodes of argumentation (see Table 2) with the majority of direct contributions (n = 26) made by students. Mr. MC, in contrast, made eight direct contributions, and one co-constructed contribution was made by the students and Mr. MC. The given task, Jessie's Number Trick, provided the data for the collective argumentation seven times. Finally, five warrants were left implicit and were not spontaneously given by the students nor did Mr. MC follow up with a supportive move to make the warrant explicit.

This demonstrates the students' thinking was central to the discussion in the classroom because students provided the majority of warrants, warrant/claims, and claims. Although Mr. MC did directly contribute to the arguments, the students' thinking was valued more so than the teachers' ways of thinking. This is evidenced by the lack of direct contributions, particularly warrants, Mr. MC contributed to the collective argumentation. Furthermore, there were times when Mr. MC directly contributed but only because it was the work of a student he observed while monitoring that he wanted other students to build on.

Table 2 Direct contributions by contributors

Contributor	Claim	Data	Data/ claim	Rebuttal	Rebuttal/ data	Warrant	Warrant/ claim	Total
Students	7	0	2	0	1	9	7	26
Teacher	2	0	2	1	0	2	1	8
Co-constructed	0	0	0	0	0	1	0	1
Task	0	7	0	0	0	0	0	7
Implicit	0	0	0	0	0	5	0	5
Total	9	7	4	1	1	17	8	47

Asking Questions Prompting an Argumentation Component

Mr. MC asked 23 questions throughout the discussion on students' findings after completing Jessie's Number Trick. The majority of questions requested an idea (n = 8)—questions requesting students to "compare, coordinate, or generate mathematical ideas" (Conner et al., 2014, p. 419). The next category of questions used most were those requesting a method—questions asking students to show how they did something or describe how they did it—and questions requesting an evaluation, asking students to assess a mathematical idea (Conner et al., 2014). In relation to what component Mr. MC's questions elicited, more than half his questions provoked a warrant or warrant/claim (n = 12). The next highest elicitations were for claims. Table 3 demonstrates the frequency for each category of questions in relation to the component the question elicited.

Mr. MC's questioning demonstrates, similar to his direct contributions, a valuing of students' mathematical thinking and reasoning. His questions were focused on eliciting students' ways of knowing mathematical claims were true. In episode three, Mr. MC used his requests to elicit a claim from students by asking, "Did it work?" Additionally, he used questioning as a way of focusing students on the claims of multiple students and their warrants. This was achieved by both direct contributions and his use of questions. We conjecture this is done to help students listen to each other and value other students' mathematical thinking. It is possible Mr. MC is using argumentation to develop particular sociomathematical norms (Yackel & Cobb, 1996) about justifications and sharing mathematical thinking.

Mr. MC supported students in determining the final claims in five of the seven episodes of argumentation. Positioning students to provide the final claim gave them the opportunity to make connections between their claims and the other components of the argument (e.g., data, warrant, data/claim). Mr. MC providing students with opportunities to make connections and supported them in seeing how argument components work together to reach a final conclusion. In episode three, he directly contributed a claim to then support–through his questioning–students in

Table 3 Question category by elicited argument component

Component	Requesting a method	Requesting evaluation	Requesting an idea	Requesting a factual answer	Requesting elaboration	Total
Claim	1	1	4	0	1	6
Data	0	0	0	0	0	0
Data/claim	1	1	0	0	1	3
Rebuttal	0	0	1	0	0	1
Warrant	2	3	2	1	0	8
Warrant/claim	1	1	2	0	0	4
Rebuttal/data	0	0	0	0	1	1
Total	5	5	9	1	3	23

making the final claim of the argument. Mr. MC's direct contribution of the claim, along with his question, "Does that show me that it's going to work for every bigger number?" emphasized how all the previous claims had helped build to the final claim.

Other Supportive Actions

Mr. MC used a variety of other supportive moves to encourage students' participation in collective argumentation (n = 18). Directing, evaluating, informing, and repeating supportive moves were all used by Mr. MC; however, in the seven episodes, there was no evidence of *promoting* components of the argument. While he used a variety of other supportive moves, the frequency of their use was fairly even. Warrants were the most common component supported by Mr. MC's other support moves (n = 15). The other components (claims and data/claims) were also supported by other teacher moves (n = 2 and n = 1, respectively). Table 4 shows the frequency for other supports in relation to the argument components supported.

Mr. MC's other support moves continue the trend of his focus on the warrants of arguments. Previously in Table 1, we demonstrated how students provided the majority of warrants throughout the episodes. Mr. MC then continued to support their warrants by directing, evaluating, informing, and repeating components of the arguments. Mr. MC valued making students' mathematical reasoning explicit.

What We Learned from Mr. MC's Facilitation of Argumentation

In this chapter, we sought to understand and explore how Mr. MC facilitated collective argumentation and supported students' participation in the construction of mathematical arguments. Mr. MC's support for collective argumentation through direct contributions, questioning, and other support moves (Conner et al., 2014) focused on students' reasoning as evidenced by his actions making students'

Table 4 Other support moves category by elicited argument component

Component	Directing	Promoting	Evaluating	Informing	Repeating	Total
Claim	0	0	1	0	1	2
Data	0	0	0	0	0	0
Data/claim	1	0	0	0	0	1
Rebuttal	0	0	0	0	0	0
Warrant	5	0	1	5	2	13
Warrant/claim	0	0	1	0	1	2
Rebuttal/data	0	0	0	0	0	0
Total	6	0	3	5	4	18

warrants explicit. Furthermore, Mr. MC supported students to use the claims they were making collectively to reach a final claim. This kind of scaffolding was common across the episodes and can be seen in episode three above as well. Mr. MC positioned students' mathematical thinking as being central in this mathematics lesson.

We conceptualized mathematical argumentation as the processes followed to make mathematical claims and provide warrants or evidence of support. Typically, students were guided to work backward. Mr. MC elicited their final claim, and then he supported them to provide explicit warrants and warrant/claims to construct their argument. It was, however, always Mr. MC who supported the mathematical arguments. There did not seem to be a norm established where students could question or support other students in the construction of mathematical arguments. Mr. MC empowered students to construct arguments but, with the exception of one rebuttal, did not seem to give space to critique the reasoning of others. This is a possible space for growth for Mr. MC and shows a needed line of inquiry.

How Mr. MC supported students' collective argumentation aligns with suggestions from Jansen (2008) to promote students' participation in classroom discussions. Student participation was valued by Mr. MC. His questioning and other support moves pushed students to participate in the collective argumentation. These moves cannot work alone but supplement the actions the teacher needed to take toward creating a classroom environment where students felt they could share their thinking without negative evaluation. Therefore, the moves Mr. MC took in promoting and validating particular discursive acts constructed the sociomathematical norms (Yackel & Cobb, 1996) about what appropriate mathematical reasoning and participation looks like. What we see from Mr. MC's facilitation of argumentation is how empowering supporting argumentation can be to students in developing their identities as doers-of-mathematics.

Herbel-Eisenmann et al. (2017) found, however, issues of power and identity are not typically discussed in regard to the use of Toulmin diagrams and studies on mathematical arguments. We stress the call for more studies using a critical lens on collective argumentation in the mathematics classroom. For example, one problematic aspect of Toulmin models is how the method decontextualizes the argument discourse. Toulmin models focus strongly on the discourse of the classroom, but they do not help in considering how context impacts the argumentation. Toulmin models can be more powerful for researchers if additional theories work alongside Toulmin's framework. We conclude this paper by discussing how we have attempted to include a more critical lens on the larger spheres of influence to mathematical arguments.

Conclusion: Larger Spheres of Argumentation

Although argumentation is a powerful perspective for considering mathematical discourse, we contend argumentation is a productive lens for understanding phenomena beyond the mathematical reasoning of students. Argumentation has been used as a lens to explore other phenomena of learning and doing mathematics. For example, Metaxas et al. (2016) used argumentation as a way to frame the pedagogical arguments of John, an in-service teacher in a course for his master's degree. Gomez (2018) used Toulmin's (1958/2003) conceptualization of argumentation to capture how a prospective elementary teacher argued her identity as a teacher of mathematics.

These studies demonstrate how Toulmin's conceptualization of argumentation can be used to explore social phenomena. There have been, however, few previous attempts to conceptualize argumentation as a broader lens. Researchers have considered how individuals make arguments about how they want to be seen by others (Gee, 2001). As Goffman (1959) wrote:

> When an individual plays a part he implicitly requests his observers to take seriously the impression that is fostered before them. They are asked to believe that the character they see actually possesses the attributes he appears to possess, that the task he performs will have the consequences that are implicitly claimed for it, and that, in general, matters are what they appear to be. (p. 17)

We argue it is limiting to only consider mathematical arguments because it decontextualizes them from the social forces at work in the classroom. Argumentation is based on the ideas that individuals learn through social interactions (Krummheuer, 1995). We claim mathematical arguments are embedded within at least two other types of arguments: (1) social and (2) sociomathematical arguments (see Fig. 3). Social arguments are about the kind of person one wants to be seen as by others. This includes the individual's use of particular discourses (Gee, 2001) and their presentation of self in everyday life (Goffman, 1959). Sociomathematical arguments, however, are attempts to convince an audience of the ways individuals do mathematics within the social order. These are related to the construction of

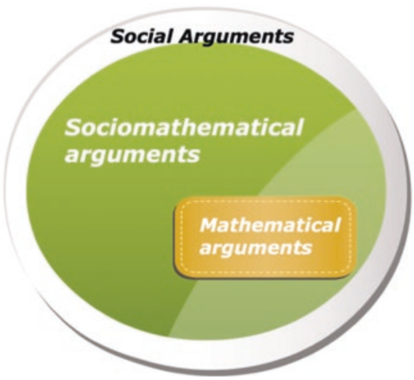

Fig. 3 Three types of argumentation

sociomathematical norms (Yackel & Cobb, 1996), or "normative aspects of mathematics discussions specific to students' mathematical activity" (p. 361). In the episode above, Mr. MC could be seen as making an argument about what makes a strong mathematical argument. In particular, he argues not only how the use of examples is not enough to make generalizations but also the importance of checking the boundary cases of claims stating "for any" or "for all."

We believe a framework is needed to bring to the forefront, not just the mathematical arguments made in the classroom but also the social arguments occurring in school settings. We believe the use of an argumentation lens can provide new understandings of how dominant narratives are perpetuated in mathematics education. Researchers can dismantle how teachers and students perpetuate deficit narratives and aid in the reconstruction of more productive narratives. We hope this brief discussion begins a larger conversation needed in our field.

Appendix A: Sample from Spreadsheet Deconstructing Argument

Arg. #	Class	Part	Implicit	Question	Kind question	Other support	Kind other support	Content of component	Who contributed
MC_3	WC	W		Will this always work for any number?	Requesting an idea			Tried googol in Jessie's two algorithms	Blue
MC_3	WC	W*				You tried 1 followed by 100 zeros	Informing	Tried googol in Jessie's two algorithms	Blue
MC_3	WC	W/C		Did it work?	Requesting an idea	Good	Evaluating	Googol works	Blue
MC_3	WC	W	I					Tried 72 in Jessie's two algorithms	
MC_3	WC	W/C				It works for 72	Repeating	It works for 72	Blue
MC_3	WC	W						I know Emma's group tried 26	Red
MC_3	WC	W/C						26 works	Red

Arg. #	Class	Part	Implicit	Question	Kind question	Other support	Kind other support	Content of component	Who contributed
MC_3	WC	W		You did 400 what?	Requesting a factual answer			Tried 451 in Jessie's two algorithms	Blue
MC_3	WC	W/C		Did that work?	Requesting an idea			It works for 451	Blue

Note: The asterisks are added as to not double count the part of the Toulmin diagram. This provides the opportunity to keep track and code multiple supports for one component.
WC whole-class conversation, *W* warrant, *W/C* warrant/claim

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Justification in the Context of Middle Grades: A Process of Verification and Sense-Making

Kristin Lesseig and Jerilynn Lepak

Mathematical justification as defined by the editors of this volume is "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020). Key to this definition is the idea that justification is a *process* that is grounded in sense-making. Engaging students in the process of justification is recognized as a means through which to foster (or an indicator of) other reform-based practices (e.g., promoting discourse and the use of multiple representations) central to student-centered instruction (Staples, 2007; Yackel & Cobb, 1996). As such, justification is a valued disciplinary practice that has been embraced by mathematics educators as a means to elicit, develop, and assess students' understanding of mathematical practices and concepts.

In this chapter, we sought to describe the justification process and identify elements that encouraged students to engage in justification in one middle-grade classroom. Our theoretical framework guided our analysis to show that, for this particular episode, the justification process was somewhat proceduralized. This process was supported by the task design and specific teacher moves designed to provide access and set expectations for students' written justification. We claim that while the deliberate scaffolding may have provided access, the step-by-step nature of the justification process that resulted may have inadvertently limited opportunities for student sense-making.

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Theoretical Framework

We view mathematical justification as a cognitive and social process in which individuals work to collectively determine what is mathematically valid (Simon & Blume, 1996; Yackel, 2004) and provide reasons for why they believe a claim to be true. This process can be supported through questioning, probing, and pushing students to think deeply about the choices they make in validating a claim. This is important as it has the potential to solidify understanding and reveal inherent mathematical structures that make a particular statement universally true. But the processes of sense-making and making choices can also be done internally and are not always visible to others. This conceptualization of justification aligns with features of the provided definition while placing greater emphasis on the explanatory role of justification.

The justification process is not necessarily linear, as students can take various paths in making sense of the claim, revising their stance on the truth value of the claim, or settling on a statement that is acceptable to their peers. Yackel and Cobb (1996) use the term sociomathematical norms to describe the normative understandings of what counts as an acceptable mathematical explanation or justification within a community. What is deemed to be "acceptable" cannot be outlined in advance but rather is interactively constituted through the classroom interactions among students or between the students and teacher (Lannin et al., 2006; Yackel & Cobb, 1996). As representatives of the mathematical community, teachers must be aware of and communicate expectations that are appropriate to their classroom community. What is appropriate depends largely on students' age, mathematical sophistication, and experience with mathematical practices such as justification.

It is also worth noting that mere exposure to a logical argument, even one that adheres to the established classroom norms, does not guarantee common understanding within the community (Simon & Blume, 1996). In other words, students can accept a justification or claim without necessarily developing an intuitive understanding of the mathematical concepts and processes involved. This complex interaction between the social and cognitive elements of the justification process makes it difficult to determine how and what individual students learn—especially when analysis is limited to, as it is in this chapter, a transcript of one classroom episode.

Literature Review

In light of the data analyzed for this chapter, two aspects of the justification process highlighted in the literature are particularly salient. First is recognition that the teacher plays a critical role in establishing what and how the practice of justification is enacted in the classroom. Second is the importance of justification as a process and product of learning mathematics.

The Teacher's Role in Promoting Justification

As representatives of the mathematical community, teachers play a central role in determining the mathematical quality of classroom discussions and in establishing norms for students' mathematical contributions (Yackel, 2002; Yackel & Cobb, 1996). Teachers' use of talk moves, such as revoicing or repeating student ideas (Chapin et al., 2013) during discussions, can initiate shifts in the discourse and highlight salient aspects of student contributions. These decisions about which student ideas to pursue, and in what manner, can communicate to students what it means to justify and, more generally, what it means to do mathematics in the given classroom community.

Teachers are responsible not only for supporting the development of productive sociomathematical norms but also for creating and capitalizing on justification opportunities within the classroom (Bieda, 2010; Simon & Blume, 1996; Staples, 2007). Deliberate attention to the selection and setting up of tasks is critical. Tasks inform students of what it means to engage in mathematical activity (Stein et al., 1996), shape the content students learn and how they learn to process it (Doyle, 1983), and set expectations for mathematical practices like justification. During implementation, teaching strategies to promote justification include making student thinking public, creating formal and informal opportunities for students to exchange ideas, encouraging the use of multiple representations, and pressing for conceptual explanations that are consistent with disciplinary norms (James et al., 2016; Staples, 2007).

Moving students toward conceptual explanations or implementing other moves to encourage or extend student reasoning is not easy. For example, teachers in Bieda's (2010) study had difficulty promoting substantive discussions centered on student justifications. Teacher follow-up, when present, often consisted of quick approval and provided limited opportunities for students to understand elements of appropriate justifications, critique their peers' justifications, or reflect upon the mathematical concepts underlying the justification. Further, teachers' ways of answering student questions, responding to student ideas, or lecturing may inadvertently position the teacher as the sole arbitrator of mathematical correctness (Harel & Rabin, 2010). Such actions do little to promote student autonomy and often curtail the learning potential of the justification process. This last theme is closely tied to a second aspect of justification implied by the provided definition—that the process of justification entails sense-making and as such can contribute to student learning.

Justification as a Learning Practice

The justification process, as envisioned here, can evoke multiple student actions (e.g., making decisions about whether a mathematical claim is true or false, providing supporting evidence, or elaborating how one obtained a result) that naturally lend themselves to deeper learning. Staples et al. (2012) characterize justification as a *learning practice* in the sense that it "promotes understanding among those engaged in justification—both the individual offering a justification and the audience of that justification" (p. 449). Through the process of justification, students not only build understanding but also reveal their current understandings, giving teachers insight into what students have learned.

Lannin et al. (2006) demonstrated how the process of justification can encourage students to observe how a general rule applies across specific cases and develop understandings that can help them construct generalizations in related situations. The justification process can also push students to reflect on their own reasoning about the viability of rules and procedures they construct. While students' reliance on empirical justification has sometimes been framed as a deficit to be overcome (e.g., Harel & Sowder, 2007; Stylianides & Stylianides, 2009), there is also a considerable body of research demonstrating the important role examples play during the justification process (e.g., Pedemonte & Buchbinder, 2011; Zazkis et al., 2008). By testing specific examples, students begin to formulate conjectures and test the boundaries of generalizations. More importantly, students' empirical explorations have the potential to reveal mathematical relationships and structures that can lead to a deeper understanding of mathematical concepts underlying the claim (Knuth et al., 2019).

Method

We focused our analysis on the transcript of the lesson in Mr. MC's class, as that allowed us to see how interactions among the task, teacher, and students contributed to the justification process enacted in the classroom. As needed, we reviewed the remaining data (i.e., the lesson summary table and samples of student work) to gain additional contextual information and check our interpretations.

We first chunked the transcript into three lesson activities marked by clear transitions in the transcript: introducing and becoming familiar with the task, small group work, and sharing results—segments that align with the Mathematics Task Framework ([MTF], Stein et al., 1996). This framework (Fig. 1) captures a task from its original form through different phases of setup and implementation, culminating in student learning. Central to their work, Stein and Lane (1996) documented how task features (e.g., the use of multiple representations and demands for justification) can change during the setup and implementation phases in ways that may lower the cognitive demand of the task.

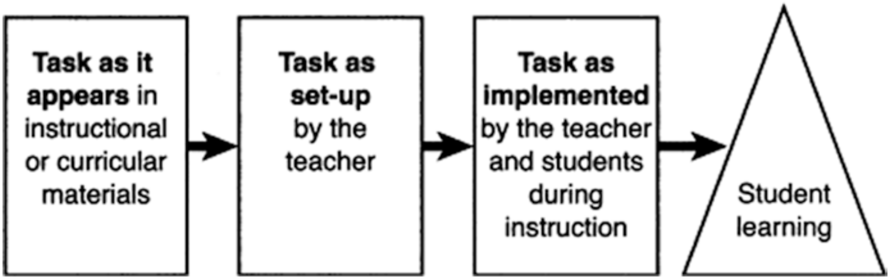

Fig. 1 The Mathematics Task Framework (Stein & Lane, 1996)

The first phase of the MTF signifies the task in its "raw" form as teachers may encounter it in a textbook or other curricular documents. To identify how the task supported students to make justifications, we considered various attributes of the task, including language, mathematical content, and the justification prompt. With regard to language, we looked for specific mathematical terms or symbols students would need to decipher and considered how easily a seventh grader would understand the goal of the task. The mathematical content was analyzed in relation to grade-appropriate expectations.

The second phase, task setup, refers to the extent to which the task as announced by the teacher promotes task features that engage students in mathematical reasoning. Here, we noted how the teacher might have influenced how students took up the task by elaborating, highlighting, or detracting from the task as written. This part of the classroom lesson included the whole group and individual settings.

The third phase, task implementation, refers to the ways in which the teacher supports students as they work on the task. Here we concentrated on how the interactions among the teacher and students during the small group work and final whole group discussion supported student sense-making and justification. We identified eight distinct exchanges in which the teacher interacted with a particular student or group of students during the small group work (the transcript did not include any group interactions without the teacher). These exchanges were clearly marked in the transcripts with "stage cues" such as "teacher moves to another group."

Within each of these interactions and subsequent whole-class discussion, we categorized teacher moves that have potential to support student reasoning and sense-making as operationalized by the Teacher Moves for Supporting Student Reasoning (TMSSR) framework (Ellis et al., 2019). Following this framework, we looked for instances in which the teacher *elicited, responded to, facilitated,* or *extended* student reasoning. For eliciting, we identified what the teacher did to establish the initial conversation and gain access to student thinking. Responding to involved the teacher follow-up to students' initial thinking, including moves such as revoicing or asking for clarification. Facilitating student reasoning described teacher moves that pushed students toward formulating their justification and extending referred to moves the teacher did to help students generalize—in this case, helping students understand whether the Number Trick would work for values beyond 10.

Results

In this chapter, we describe the justification process that emerged during Mr. MC's classroom lesson and present an analysis of the factors that contributed to that process. Our analysis revealed that the justification process followed a somewhat linear path with students directed to concentrate on the finite case before moving toward generalization. The written task, teacher moves, and teacher-student interactions more generally all played a role in shaping the nature of the justification process and the extent to which justification was promoted as a sense-making activity.

Justification Process

For the Number Trick task, the justification process that unfolded in this classroom consisted of two distinct but related parts: (1) verify the trick worked for the numbers 1–10 and (2) reason about whether the Number Trick works in general. For part one of the process, arriving at a claim was intricately tied to verification. Students performed the required calculations for every number from 1 to 10. Their calculations served a dual purpose: they led students to an answer that the Number Trick worked (the claim) and provided the explanation to support their claim. All that the students had left to do was write a complete sentence summarizing their calculations (i.e., we know it works [claim] because we tried all the numbers [explanation].) During this initial activity, we saw limited evidence that students were engaged in making sense of *why* the Number Trick worked. Students were often encouraged to limit their exploration to verifying that the Number Trick worked for 1–10 and provide a written justification for 1–10 before moving on to the general case. Thus, the first part of the justification process can be summarized as a somewhat straightforward verification of finite cases.

Toward the end of the small group work, two groups offered ideas related to the second part of the process: reasoning about whether the trick worked for all numbers. In particular, these groups recognized that adding 4 and then doubling was the same as doubling and adding 8. "When she added 8, she might have imagined first that 8 equals 4 x 2 she did in the first equation…. It must be that the 4 is still part of the equation even though it was smushed in with 5. You did double 4, but when it was part of the 5." This reasoning process allowed the group to go beyond the finite case to make a general statement for all numbers. For the students who proceeded to the second part of the task, we observed more sense-making along with the uncertainty and "messiness" that is often a part of the reasoning process.

The deliberate selecting and sequencing of student work during the final whole-group discussion reinforced the progression of first justifying the claim for the numbers 1–10 before considering the general case. The first group was asked to share what they did to show that the Number Trick was true for the numbers 1–10, with a specific request to recreate one of the calculations on the board. After this group read their complete justification aloud (i.e., "The answer to the problems will always be the same. I did the problems for 1 through 8 and my groupmates did 9 through 10."), the teacher moved on to "that separate question underneath… will this *always* work for any number." After acknowledging that several groups had started checking larger numbers (i.e., 26, 777, a googol), the teacher selected three groups to share their "theories" about whether the Number Trick would work for every bigger number.

Factors That Shaped the Justification Process

Different stages of task implementation yielded different classroom moves but with the same intent: to make sure that students were supplying both a claim and an explanation for why the claim was valid. To highlight how the progression of the justification activity was supported, we have organized these findings according to the stages of the MTF (Stein & Lane, 1996).

Task as Written and Setup

Features of the written task that supported student access included language use, mathematics content, and the use of a specific example. The task is introduced in colloquial language as a "cool number trick," with nontechnical step-by-step directions. A specific numeric example is presented alongside the verbal directions, providing a possible scaffold to ensure students understood the sequence of computational steps in the trick. The mathematics underlying the task (whole number computations and properties) is content that should be familiar to seventh-grade students. Immediately prior to introducing the Number Trick task, the teacher reviewed the order of operations and led students through a series of examples demonstrating how to use "PEMDAS." Thus the use of properties to simplify expressions was content that was immediately accessible.

Students were first asked to explain whether the Number Trick works for a subset of numbers (1–10) before considering if their explanation would work for *all* numbers. The task as written has the potential for students to recognize the structure of the computations through their examination of the finite case and then extend to all numbers. In this way, the task prompts could scaffold the justification process if students are pushed to make sense of their computations and explore why they continue to get the same results for each set of computations.

During the task setup phase, the teacher did not deviate from the task as written, though he stopped short of introducing the second prompt (to extend to the general case). He read through the first part of the task verbatim, including the example that was provided and the initial questions students were to work on: will Jesse's two answers be equal to each other for any number between 1 and 10? He emphasized that students should work through the calculations to check each number, thus providing both the claim and the explanation for why the claim worked (proof by exhaustion). While introducing the task, the teacher also reminded students of the RACE (reword, answer, cite, explain) acronym and their classroom definition of justification. These reminders reinforced the expectation that students were to create a written justification and also offered a potential resource for students. Students had worked with the RACE acronym before, and as this was the second formal justification task of the year, they were also familiar with the classroom criteria for justification in a mathematics context.

Following this brief verbal introduction, students were given a few minutes to process the task individually before working in small groups. The limited student-teacher interactions during the individual work time focused on clarifying the task, making sure students understood what it meant to say the trick "worked," thus providing an entry into the verification process.

Task as Implemented

After working to make sense of the task individually, students began working in small groups. With each group encounter, the teacher consistently asked two questions (paraphrased): Did you find an answer? and Why does it work? During this time, the classroom teacher followed a familiar routine of eliciting student thinking, responding, and facilitating student reasoning (Ellis et al., 2019). Initial questions to reveal how students were approaching the problem ranged from relatively direct questions (e.g., did you check/try them all?) to more open probing (e.g., what are you thinking?). These eliciting questions allowed the teacher to check group progress and determine if students had established a claim—most of the groups recognized the Number Trick worked for 1–10. Once students shared their initial thinking regarding the validity of the Number Trick, the teacher responded to the groups by affirming their claim and reminding them that they also had to include a written explanation.

Although this teacher demonstrated moves to facilitate student reasoning, there were relatively few questions that prompted students to extend their reasoning or make their sense-making visible. Questions like "Did you check them all?" occurred frequently at the beginning of the group work and may have limited students' exploration of why the Number Trick made sense. In addition, there were instances when the teacher seemed to discourage a general line of reasoning in favor of a justification for 1–10, perhaps to ensure that students would record their thinking. For example, he told a group to "Save that thought, but what I want you to do now is get an answer down for that first question. We know that it always works for the numbers one through 10. Okay. Get an answer down and explain why you know it will always work." Again, in a different group, he pushed students to consider the concrete examples, 1–10: "I'm not saying that it's wrong, but what I want you to do is think about the specific cases one through ten. Can your reasoning help you answer that question?" In short, we see the teacher strongly encouraging students to write down their explanation for why the Number Trick works for 1–10 and pausing their thinking about the general case for the time being. This could be an intentional move on the part of the teacher who wanted to shift students' conceptions of doing math as just writing down an answer to the mathematical process of justification.

For the two groups that were further along in their thinking, the teacher listened to the student justifications and followed with various questions or revoicing to elicit a full expression of their thinking, such as "So you're saying that adding four to a number is the same thing as…?" As the teacher gained insight into students' thinking and progress, he was able to press students to acknowledge and meet the expectations of the second part of the problem—explaining their reasoning for why the Number Trick worked. To help students establish their justification, he asked more pressing questions that allowed students to articulate how and why they arrived at their claim. These included "Why do we know it's true?" and "Show me what you're talking about." In the segment of the transcript analyzed, the teacher reminded students that they needed an explanation of why the Number Trick worked, but there is no evidence that Mr. MC facilitated students' written justifications for the general case.

Discussion

For this chapter, we sought to describe the justification process and identify classroom elements that encouraged students to engage in justification. We claim that justification was portrayed as a step-by-step, linear procedure and that the initial focus on verification may be at odds with the notion of justification as a learning practice (Staples et al., 2012). The step-by-step process began with verifying that the Number Trick works for the numbers 1–10. To establish their claim, the teacher led students toward a singular mode of argumentation—proof by exhaustion—by confirming with multiple groups that they had checked all of the numbers and encouraging students to record this fact as their justification for the claim that the trick works for this finite set of cases. Once that initial claim was established, students began considering whether the Number Trick will always work.

In this discussion, we consider the teacher's moves that promoted this process of justification in light of student learning—the final phase of the MTF. Given the limited data, we make no claims about what individual students did or did not understand as a result of this lesson, nor do we mean to diminish the work that was accomplished in this classroom. Instead our aim is to describe ideas about justification made available to students as evidenced in the lesson transcript. Our considerations of what students may have learned about the justification process highlight the complex work of teaching about and through the process of justification, leaving us with a number of lingering questions as a field.

What Might Students Be Learning About the Justification Process?

In this classroom lesson, students had the opportunity to learn that justification builds from specific examples to reasoning about the more general case. Given the mathematical content (numeric operations) and the fact that the claim is made about a finite set of cases, using examples is a natural entré into this task. As noted, examples can play a critical role in helping students make sense of conjectures and can support further generalization and justification (Knuth et al., 2019). In this case, the teacher's questions and prompts to maintain a focus on the numbers 1–10 provided access to the task and engaged all students in mathematical work that could serve as the basis for a written justification. Students were able to convince themselves that the trick worked for 1–10 and create a written explanation or proof by exhaustion to communicate their thinking.

Through this work, students had an opportunity to learn that you can justify a claim about a finite number of cases by verifying each case. For students who only addressed this part of the task, however, we question if they made sense of the problem. The majority of their justifications were akin to "I tried it and it worked." Generally, their rationales did not reference the underlying mathematical operations. For these students, we wonder whether their use of examples gave them a false understanding of the process of justification—especially as defined here to require sense-making. Though the two questions in the task are clearly related, the focus on first cases 1–10, and second, the general case, may have felt disjoint to students. We would hope that students' reflections on their repeated calculations for 1–10 would reveal the mathematical structures and concepts underlying the trick and support student sense-making about why it always works (Knuth et al., 2019). A focus on the results of calculations, without understanding the connections between the examples and the general case, does little to promote the sense-making aspect of justification and may actually contribute to future difficulties in learning to prove (Pedemonte & Buchbinder, 2011).

Put simply, the intentional scaffolding of the task turned the process of justification (at least for some students) into a linear procedure that may have delayed students' opportunities to grapple with and make sense of the structure of the Number Trick. A possible alternative approach that would have allowed for more sense-making in the first half of the task would be for the teacher to ask questions about the repeated calculations to focus students' attention on the underlying concepts (e.g., order of operations, meanings of multiplication). Further, the teacher might encourage more strategic use of the examples by highlighting what students noticed during the exploration phases and connecting those noticings to structural aspects of the Number Trick (Knuth et al., 2019). For example, the teacher could have used the student's explanation of "seeing" the 4 as being "smushed in" with the 5 to make explicit connections to the distributive property. Although a more formal version of these ideas was presented by a group at the end of class, it is uncertain if students were able to make the connections between the informal reasoning ("smushed in") with the formal reasoning (a symbolic representation of the distributive property) because they were packing up to go to their next class.

Further, though the teacher referenced the RACE rubric and justification definition during task setup, there was no evidence that he returned to it. Rather than using the rubric and definition to evaluate their own work, students instead appeared to rely on the teacher for feedback. This positioned the teacher as the sole authority of mathematical correctness (Harel & Rabin, 2010), potentially limiting what students may have been able to do in the justification process.

In short, we wonder if this portrayal of justification as a step-by-step process might restrict students' view of justification to that of verification or summarizing a procedure rather than seeing justification as a sense-making practice. Moreover, this proceduralization might inadvertently limit students' ability to identify the underlying structure of the trick in support of further generalization. In other words, rather than using their empirical exploration to make sense of the mathematical operations at play, students may work as though the two activities (i.e., verifying the specific cases and reasoning about the general case) are independent. From this perspective, teacher moves to redirect student groups back to checking 1–10 despite the fact that they seemed to be reasoning more generally seem particularly salient.

Conclusion

Our reflection on what the lesson afforded and constrained for student learning highlights the complexities involved in promoting justification as a sense-making process in classrooms. The teacher must balance the need to make the justification process accessible for students with the need to establish expectations that are consistent with the discipline of mathematics. These sometimes competing obligations place significant demands on teachers. Although we raised a number of questions about Mr. MC's decisions to focus on the first part of the task, we also saw where students in this classroom did engage in sense-making activity. We also identified a number of conceptually oriented questions the teacher asked, (e.g., "what do you mean you're making up for what you lost?") that could further student reasoning about the mathematics operations at play.

In conclusion, we set out to capture the process of justification in one classroom episode. Our working definition of justification, as a process grounded in sense-making, aligns with a more organic practice where students have the freedom to move back and forth between specific examples and the general case as they explore properties and relationships—a practice that necessarily requires more than one classroom episode to develop. When students are allowed that type of freedom, we can get closer to realizing justification as a learning practice. This resonates with Lannin et al.'s (2011) description of mathematical reasoning as an evolving, nonlinear process in which "Students may move back and forth among conjecturing and generalizing, investigating *why*, and justifying or refuting at any time in the reasoning process" (p 11). Although we only had the opportunity to analyze a relatively brief episode of this class, it is possible that students in this class did or will experience more freedom in other situations involving justification. Nor is this meant to diminish the work the teacher did in this lesson to initiate students into the process of mathematical justification. Rather, this lesson provided introductory experiences with justification through proof by exhaustion that could lead to later discussions about the limitations of example use when providing a justification of the general case. Indeed, for some students, the structure of the Number Trick was used as grounds to develop a more robust, general justification—an opportunity the teacher could capitalize on in future lessons to move all students further along in their understanding and use of mathematical justification.

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Proof in the Context of Middle Grades: Can We Label Middle School Arguments as Proof with a Capital P?

David A. Yopp, Rob Ely, Anne E. Adams, and Annelise W. Nielsen

Introduction

The concept of *proof* is central to the field of mathematics, but defining the term has challenged educators and researchers (see Czocher & Weber, 2020; Weber, 2014, for a discussion). In theory, the term seems easy to define. A *proof* is a sequence of statements involving logic and prior results. These statements are arranged in a logical order to support or refute a quantified claim. *Prior results* include axioms, definitions, and previously established results. The logic is mathematical logic, which is a metatheory about what types of logical inferences can be made and what methods and modes of reasoning (e.g., modus ponens and modus tollens) are acceptable. This metatheory also establishes what types of statements can be made (e.g., quantified statements that are either true or false), which statements are logically equivalent, and the norms for representing generality and inferences. This metatheory provides the "rules of the game" for mathematical proof and proving.

The challenge in using this definition is that the metatheory is tough to unpack in school mathematics and it does not account for all the types of arguments some mathematicians have accepted as proofs (Czocher & Weber, 2020). Moreover, it

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seems impractical and, arguably, imprudent to teach middle school students all the rules that mathematicians have learned to follow when producing a proof. Teachers pursue many instructional goals in the middle school mathematics—content, skills, and other mathematical practices—and have limited amounts of time to pursue these goals. Yet, many educators, researchers, and policy makers ask that middle-grade mathematics students produce arguments that can be taken as proofs, have elements of proof, or at least provide pathways toward proofs. In the USA, Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) recommend that students use "stated assumptions, definitions, and previously established results in constructing arguments" (p. 3) and "build logical progression[s] of statements to explore the truth of their conjectures" (p. 3). Similar recommendations have motivated researchers to search students' mathematical arguments for elements of proof amidst the various types of reasoning students use when they are asked to prove. Researchers have noted proof-like behaviors among students who responded to proving tasks, such as attending to the general case and using definitions and prior results, and have sometimes likened these behaviors to the practices of trained mathematicians. Stylianides (2007), for example, found proof-like practices among elementary students' responses and likened them to the proof practices of mathematicians. Yet, noting "proof- and proving-like" behaviors among students does not necessarily mean that students have an understanding of a metatheory for mathematical proof and proving. In other words, the students who produce arguments akin to proofs may not understand why their reasoning is valid nor understand how their reasoning fits into a valid proof and proving scheme.

Students' lack of a proof metatheory is perhaps one reason the editors of this book asked us to use a broader definition of proof that more readily adapts to the reasoning of students in the middle grades. In this definition, which we will call "our working definition," a mathematical proof has all or most of the following characteristics, and proving is an activity that leads to such a product:

1. A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true.
2. A proof is a deductive argument that does not admit possible rebuttals.
3. A proof is a transparent argument *where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation.
4. A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*.
5. A proof is an *argument within a representation system satisfying communal norms.*
6. A proof is an *argument that has been sanctioned by the mathematical community* (Weber, 2014, p. 537).

In this book chapter, we use a version of this definition to analyze a collection of middle-grade student work presented to us by the editors to determine the degree to which the students demonstrated a level of proficiency with proof and proving as described by this definition. We also consider the teachers' contributions during activities related to proof, proving, and justification from the classroom episode from which this student work was collected. Through this lens we aim to discuss whether or not students in the class exhibited characteristics of the above definition in their arguments and justification practices as well as whether or not these students were exposed these "rules of the game" for proof and proving in mathematics. Consequently, we discuss possible mismatches between this definition and the guidelines students were given for developing acceptable justification.

Ultimately, we will consider the possibilities for proof and proving in the middle grades and the implications of exposing students to the same definition of proof and proving that is used to analyze their work. In other words, we ask, "What can happen when students are presented with the same rules of the game for proof and proving in mathematics as those used to analyze their data?" But, first, we consider this above definition in the context of middle school mathematics and discuss factors to consider when adapting this definition into a rubric used to examine middle-grade students' proving activities.

Reframing the Definition of Proof for the Current Context

As described in the chapter "Overview of Middle Grades Data" (this volume), the current context involved a small seventh-grade class of 19 students in New England (USA) who engaged in the Number Trick task. This class was taught by a teacher who actively worked to engage students in argument and justification as part of the Justification and Argumentation: Growing Understanding of Algebraic Reasoning Project (the JAGUAR Project, funded by NSF, DRL 0814829). More will be said in the methods section about the materials the students received for understanding characteristics of justifications and argumentation, and later we will discuss how well these resources aligned with our working definition of mathematical proof. For now, we will reflect on the definition of mathematical proof we were provided and aspects we considered when adapting it to be an analytic tool.

In our opinion, the six characteristics in our working definition of mathematical proof attend to purposes and norms for proving in the community of mathematicians, and consequently, the definition can be difficult to apply to the work of school-age children, particularly when we do not know whether the children received explicit training in the norms and practices expressed in the definition or frequent experiences with these norms and purposes. Applying the definition also relies on the judgment and imagination of the reader, which is a source of subjectivity. Nonetheless, we believe it to be valuable to adapt the criteria in the definition to our current context for the sake of understanding the types of reasoning students used in their arguments and justifications and how their reasoning can be developed to better fit with the norms and practices of the broader mathematical community.

One characteristic that was particularly challenging to apply to our context—middle-grade students engaged in the Number Trick task—was the *convincingness* characteristic. A knowledgeable mathematician or mathematics educator would already be convinced that Jessie's two computational approaches equate prior to reading a student's argument because Jessie's approaches can be equated using the distributive property. Thus, to ascertain convincingness, the analyst must imagine, "Were I unfamiliar with this claim, and the distributive property, would this argument convince me of its truth?" We wonder how reliable an analyst's image of a convincing argument can be in such a hypothetical situation.

Another source of subjectivity when applying the convincingness characteristic is "who gets to decide?" Who exactly are these "knowledgeable mathematicians," and how can we assume they would all agree? Can mathematics teachers of various mathematical backgrounds count as knowledgeable mathematicians? This question is particularly troubling if we view proof and proving as a means for students and teachers to take ownership of and authority over their mathematical knowledge and learning. Relying on the authority of more knowledgeable others to assess the teachers' and students' arguments seems to defeat this purpose of proof and proving.

The *representation system* and *community-sanctioned* characteristics (5 and 6) presented similar challenges. The *community-sanctioned* characteristic applies readily to classic proofs like those of the Pythagorean theorem and those showing $\sqrt{2}$ is irrational, but no such prototypical proofs exist for claims associated with the Number Trick task, such as "For all real numbers, the two computational approaches in the Number Trick task produce the same result." Therefore, it was unclear how to compare students' arguments in this context to arguments sanctioned by the mathematical community.

The *representation system* characteristic (5) presented similar challenges. This characteristic refers to norms for representing arguments, but whose norms are to be met? The mathematical norm for representing algebraic proofs is to use variables to express the general case. However, Yopp and Ely (2016) argued that noncanonical representations such as generic example referents can be leveraged in proofs provided that certain criteria were met. One critical criterion is that the example (a.k.a. referent) is not appealed to in any way that is specific to the example chosen. Here, specific means a trait not shared by all cases in the domain of the claim. To us, it is more important to assess how representations are leveraged in the argument than it is to assess whether or not the representations are canonical. But this view can be in tension with the social role of proof and proving. In other words, our view acknowledges that a student might produce a proof that her/his peers would not sanction as proof but a reader with an open mind about the power of alterative representation systems would sanction as proof.

Indeed, all six of the characteristics in the definition of proof depend on socially accepted norms and meanings. Even the characteristics that seem most objective such as deductive inference, *proof* characteristic (2), are subjective when applied to middle school students' work. Deduction is a well-defined method of reasoning where the arguer identifies a rule $p \rightarrow q$ and a case of p and concludes that because of the rule, the case of p also has the property (or properties) q. But in practice, we rarely find middle school reasoning expressed in this form. Instead, we, the authors, have found ourselves searching student arguments for deductive-like reasoning, as we envision it, without any knowledge of whether or not the students who produced the arguments were conscious of the deductive inferences being made and the logical necessities such inferences produced. Are we sometimes using wishful thinking when we attribute deductive reasoning?

In fact, a student who appears to be reasoning deductively may be appealing to modes of reasoning that arise outside of mathematics courses. Various psychology frameworks describe "naturally occurring" and spontaneous modes of reasoning that can mimic deduction but are distinct from deduction (e.g., mental models, Johnson-Laird, 1983, and pragmatic reasoning schemes, Cheng & Holyoak, 1985). Even Harel and Sowder (1998) ground their analytic and deductive reasoning schemes not just in deductive inferences but in schema-based transformations of mathematical objects. We assert that it would be quite difficult to know whether or not deductive reasoning is truly present in any piece of student work unless the student is explicit about this mode of thought.

Another subjective aspect of applying the *deductive* characteristic is judging whether the general rules being leveraged by students are taken as general prior knowledge in the classroom community. Stylianides (2007) and Stylianides and Stylianides (2008) proposed, as part of a definition of proof for school mathematics, that "[proof in school mathematics] uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification" (Stylianides & Stylianides, 2008, p. 107). Although this criterion specifies the community, it also requires that the reader assessing the argument is knowledgeable in the background of that classroom community.

A final point we wish to make about the *deductive* characteristic is that sometimes the characteristic does not apply at all to a proof. For instance, a proof by exhaustion might contain no deductive reasoning. In some of the proofs we analyzed in this chapter, the students simply checked that the claim was true for all ten possible cases. This is a completely valid method of proving that uses no deductive reasoning.

Moreover, a proof by exhaustion also might have no explanatory power, which would also prevent us from applying the *explains why* characteristic (4) when evaluating it. Another factor when applying this proof characteristic (to any argument, not just to an exhaustion argument) is that a proof's explanatory power depends upon the reader. For instance, Hanna (1990) asserts that mathematical induction may not have this explanatory trait, but Stylianides, Sandefur, and Watson (2016) point out that many mathematical induction arguments do have this trait and can be distinguished from those that do not. Perhaps what we mean is that explanatory proofs! demonstrate why the defining properties of the mathematical objects specified in the conditions of the claim *must* also have the properties specified in the conclusion of the claim. If so, the *explains why* characteristic may be better described in terms of conceptual insights (Sandefur et al., 2013), which provide structural links between the conditions and conclusions.

In the context of the Number Trick task, an example of a conceptual insight that provides such a structural link is the distributive property, which transforms one expression into another equivalent expression. Yet students might see other structures in the expressions that equate the approaches' outputs. The need for "balance" by adding twice as much to the doubled value in the case where doubling comes first is one such structural link that does not explicitly employ the distributive property. Moreover, what if it is clear that a student is searching for a conceptual insight but does not find one in the time allocated? This could be evidence that the student is aware that a proof of a general claim needs more than just an empirical check but that the student got "stuck" and was unable to find an insight that shows why the conditions imply the conclusion. We consider any search for a conceptual insight as evidence of deeper understanding of proof than expressed in an empirical argument.

Although our discussion so far has pointed out ambiguities and potential problems with assessing these six characteristics of proof, we wrap up with a more optimistic look at one of these, the *fillable gaps* characteristic (3). This characteristic turned out to be one of the most useful of the six and perhaps truest to the analytic methods we typically use. Inevitably, when analyzing middle-grade students' arguments, including arguments we collected in other projects, our group's discussions of these arguments turn to our own knowledge and practices. To make sense of the students' work, we often spend a great deal of time constructing our own proofs from the sparks of insights found in students' work. We find merit in novel approaches, and we discover proof paths that we overlooked. Of course, this type of analysis also presents challenges. Reconstructions, gap-fillings, and extensions may project reasoning onto student responses that the students did not intend. Perhaps we might also fail to value certain insights because we cannot see the proof path, even when one is present.

Methods

Our critical discussion in the introduction was not meant to dismiss our task as impossible. Instead, our critical discussion was an attempt to make sense of what we could assess and how we could assess it and to determine what we could not assess. From our ponderings, we developed a rubric (Fig. 1) for assessing the student data that maintains, in slightly altered form, the six characteristics provided to us and elaborates on them in the context of the Number Trick task.

The rubric applies only to arguments that attempt to address a general case. It does not apply to exhaustive proofs. We found that we needed to create two scores for each student because the Number Trick task had two prompts: whether the Number Trick worked for natural numbers from 1 to 10 and whether the Number Trick worked for all real numbers. For the first prompt, the second and fourth characteristics in Levels 1 and 2 had to be modified to accommodate proof by exhausting all cases, which may contain little or no explicit deduction and may provide no explanatory power.

We also wish to note that while our rubric articulates criteria for every characteristic at every level, the notion of cluster category does not mean that every argument

2	If we were to set our conviction aside, then the argument would convince us that the general claim, "for all real numbers, the two approaches produce the same result," is true. The argument illustrates deductive-like reasoning, leveraging what are assumed to be prior results to make inferences that we assume are logical necessities. The argument needs little or no gap filling, meaning all the important inferences needed are provided. The argument provides a conceptual insight showing why for every real number, the two approaches produce the same result. We were able to understand the representations used, see how these representations illustrate the two approaches, and see how the transformations on the representations illustrate how the structure of the approaches necessitate the conclusion that the approaches produce the same values for any number in the domain of the claim. We were able to sanction the argument as proof, based on our experiences as mathematicians.
1	If we were to set our conviction aside, then the argument would not persuade us that the general claim, "for all real numbers, the two approaches produce the same result", is true. The argument leaves room for doubt. The argument illustrates a search for a conceptual insight that could lead to deductive-like reasoning, yet the prior results and inferences made are vague. The argument needs gap filling, meaning some important inferences must be filled in by the reader. The argument indicates a search for a conceptual insight that could show a structural reason why for every real number, the two approaches produce the same result, even if the structural reason was not found or is unclear. We made significant assumptions about how the representations were used to illustrate the two approaches and/or how the transformations on the representations illustrate how the structure of the approaches necessitate the conclusion that the approaches produce the same values for any number in the domain of the claim. We were not able to sanction the argument as proof based on our experiences as mathematicians. Overall: The argument provides evidence that the arguer was aware that either all cases in the domain of the claim must be tested or awareness that a general argument based in mathematical structures of the conditions and the conclusion and awareness that inferential links between these structures must be illustrated.
0	Perhaps the argument provides empirical support for the general claim, perhaps illustrating only a proper subset of the cases in domain of the claim. Nevertheless, the argument fails to provide evidence that the arguer was aware that either all cases in the domain of the claim must be tested or that a general argument based in mathematical structures of the conditions and the conclusion and that inferential links between these structures must be illustrated.

Fig. 1 Proof rubric

that can be labeled as "proof" must satisfy all six characteristics. Weber (2014) explains that such an argument that satisfies all six characteristics is likely to generate wide agreement regarding its status as a proof, while arguments that only satisfy a subset may generate more debate among the community of mathematicians as to whether or not it suffices as a proof. However, because one of our goals is to help translate the characteristics presented in Weber's (2014) cluster concept of proof for a middle-grade context, we included revised descriptions for each of the six characteristics at each level of proof.

Our first pass through the student work included a search for (1) empiricism, which tests only a subset of cases to which the claim applies, (2) conceptual insights, and (3) searches for conceptual insights. This pass helped us understand the nature of students' proving processes and prepared us to apply the rubric. We asked ourselves questions such as:

1. What type of claim is being made, and what is the claim's domain?
2. What prior results and knowledge does the student leverage to support their claim?
3. Are these results and knowledge leveraged in a manner consistent with what was assumed to be taught or with what is canonical to mathematics?
4. Are the inferences, implicit or explicit, logical necessities?

Ultimately, we used the rubric more as a guide than a checklist, as it was difficult to assess every characteristic in any piece of student work because the students were often vague and perhaps unfamiliar with ways of expressing their mathematical thinking. Although a score of 2 required that most or all of the characteristics were met and the four questions above were answered in the affirmative, a score of 1 was awarded if there was compelling evidence that the student searched for a conceptual insight, as noted in the overall summary of a score of 1 in the rubric. To us, all six characteristics are implicit in the expression of a conceptual insight explaining why the two approaches must produce the same outcome.

Each author reviewed the student work separately and scored it using the rubric. The scores were compared and discussed until the authors reached agreement on the scores.

We also reviewed the materials associated with the lesson from which the student work was generated. This analysis also involved our rubric and included transcripts of the teaching episode and other materials provided to the students, including a description of "what makes a good justification." Our review of the transcript was performed for two purposes. One purpose was to triangulate our analysis of the student work with any discussion of the student work, particularly by the student. Here, we looked for any comments about the students' work that might shed light on meaning in the student work that might have been overlooked. The other purpose was to better understand the context in which the student work was constructed and presented, including attending to Mr. MC's prompts and contributions that related to the six elements of proof as articulated in our rubric. This later analysis provided us with an opportunity to consider consistencies and possible mismatches between the teacher's justification and argument goals and our analysis scheme based on the definition of mathematical proof we were provided.

Findings

Classroom Episode

Based on Mr. MC's activities during the lesson and the materials provided to the students, we found that the teacher valued and at times sanctioned student activities that showed promise for developing a proof, as defined by our working definition of mathematical proof and as described in our rubric. We found numerous occasions throughout the transcript where Mr. MC asked students to consider whether or not the students' examples or reasoning demonstrated that Jessie's Number Trick works for all cases or every case. Such teacher moves are likely to encourage *convincing arguments* and *arguments that do not admit rebuttals*. These requests were given in both the finite and infinite-domain contexts, when students were addressing Jessie's Number Trick for natural numbers 1 through 10, and again when students were addressing Jessie's Number Trick for any number, which we assumed to be any natural number.

We also found numerous occasions throughout the transcript where Mr. MC asked students to explain their thinking, explain why their claim about Jessie's Number Trick is true, and explain how they knew it worked for all cases. These requests were also made in both contexts, when students were addressing the finite-domain prompt and when students were addressing the infinite-domain prompt. We found it interesting that the terms "explain" and "explanations" were used to encourage students to be more explicit about their reasoning in both these contexts. In lines 134–135, when students are addressing the finite-domain prompt, Mr. MC tells students, "You're going to have to explain how you think… why you think it true," and in lines 239–240, Mr. MC refocuses students with the prompt, "Explain your reasoning." Both of these statements were made during exchanges in which Mr. MC was also encouraging students to test every case from 1 to 10. This observation is not to criticize Mr. MC's uses of these phrases but to point out that the use of the expressions "show why" and "explain why" in the classroom could lead to a mismatch between students' notions of showing why and notions of "a perspicuous argument…[offering] an understanding of why a theorem is true," (Weber, 2014, p. 537), as well as the notion of conceptual insights as described in our rubric. Exhaustive arguments may be included in the classroom community's concept image of arguments that explain or show why a theorem/claim is true, while our use of the term refers to arguments that express conceptual insights, which typically do not include exhaustive arguments. We wonder if "showing [or explaining] why" was a catch-all phrase for being explicit about your thinking and argument approaches or why you believe your argument is viable as opposed to its intended meaning in the working definition of mathematical proof and in our rubric. This lack of clarity in the terms "explain why" and "show why" could lead to confusion among students as to what constitutes viable arguments and proofs in mathematics class.

Having acknowledged this possible mismatch between our use of these phrases and the classroom community's use of these phrases, we did find evidence that Mr.

MC valued and encouraged arguments that expressed conceptual insights often by revoicing students' arguments. The exchange below was representative of these teacher activities when encouraging searches for conceptual insights:

S4: When you add these two numbers together, it's going to be higher than when you… first and then you add it: since eight is more than four, you need to have a higher number.
S6: They come in doubles… The second number that you add needs to be higher, so they need to be the same [inaudible].
T: So, you're saying that here you're multiplying the number by something before you add something to it. Up here, you're adding something and then multiplying. So what you are multiplying is going to be bigger here… So, that's some great thinking in terms of the general case….

In exchanges like this, we see the teacher focus students on structural aspects of the situation (here, the order in which multiplication and addition occur and how that affects the size of the outcome) that could lead students to understand why 8 must be added instead of 4 when the input number is doubled first. We interpreted this type of teacher move as encouraging and valuing conceptual insights, as well as encouraging perspicuous arguments that show why.

Yet, we found little evidence that Mr. MC encouraged or explicitly described the other properties of a proof as described in our rubric and working definition. Most of the episode involved discussion of student approaches both in small-group and whole-class settings. There was little emphasis on how students represented their ideas in their written work and no explicit instruction on what it meant to provide a deductive argument. Mr. MC sanctioned several students' approaches with phrases such as "perfectly valid explanation" and "good logic," but there was no explicit standard for validity and good logic. Finally, the convincingness of an argument was implicit in the teacher's emphasis on why the two approaches would equate for any real number; yet, the standard for convincing others, such as knowledgeable mathematicians, was not mentioned.

The lack of evidence of Mr. MC explicitly discussing these properties of proof raises the question of whether the definition of proof we applied was a match for his goals for this lesson. Our analysis of the transcribed episode suggests that Mr. MC's primary goals were to introduce students to the distributive property and encourage good justifications for a general claim. Further, this speculation aligns with the two descriptions of a good justification that Mr. MC had provided to the students and referenced during the classroom episode we analyzed. One of these was a sheet of paper titled "R.A.C.E. What makes a good mathematical justification?" This document suggested: (1) reword[ing], restate the question; (2) answering, include your answer and make it reasonable; (3) cite[ing], use information from the problem and what you previously learned; and (4) explain[ing], draw pictures, show work, explain thinking in words, and give specific details. The second document was titled, "What makes a good justification," which restated the above suggestions without referencing the R.A.C.E acronym. Early during the teaching episode, the teacher reminded students of the R.A.C.E. sheet and directed students to "follow your steps in R.A.C.E." Because these documents do not completely align with the properties of proof on our definition, it is not surprising that students' arguments would not attend to all of these features. However, it is reasonable to anticipate that the students' work would score at Level 1 in our rubric, because our overarching indicator for this score is that the student at least searched for a conceptual insight.

Students' Written Work

Findings from our analysis of the students' written work were similar to our findings from our analysis of the teaching episode and the associated materials. When addressing generalizations with infinite domains, more than half of the students (six of nine) described structural reasons why the two approaches must produce the same result. Yet, these reasons were generally too vague to be viewed as proof. These students appeared to be searching for conceptual insights, which suggested that the students were at least aware of what they needed to do to prove their general claims. While the Number Trick task asked students to argue for a claim with the finite domain of natural numbers between 1 and 10 as well as for a claim with the infinite domain of all real numbers, most students did not specify which of these domains they were addressing in their written work. Thus we include in our analysis below our own inferences about which claim or claims each student was attempting to prove.

In total, we analyzed the nine student work samples provided to us by the editors. Due to space limitations, we discuss several representative cases and then summarize our findings from the sample.

Shawn, Finite-Domain Argument Score, N/A; Infinite-Domain Argument Score, 0

Shawn explicitly asserted that the two approaches will equate for any input numbers, including large natural numbers outside the originally proposed domain. He offered several examples that conformed to his claim. No conceptual insight was found in his response, and we find no evidence that he searched for a structural link between the two approaches and the results. His argument was purely empirical.

Hope, Finite-Domain Argument Score, 2; Infinite-Domain Argument Score, N/A

Hope tested every natural number case from 1 to 10 but made no claim about all real numbers. We judged her argument to be a proof of a claim with a finite domain.

Jared, Finite-Domain Argument Score, 2; Infinite-Domain Argument Score, 0

Jared claimed the equivalence "works with all" numbers. Jared compared the two approaches for all natural numbers 1 to 9 and placed check marks beside the work, as if noting that every case had been tested. If Jared's claim was restricted to this domain, then his support was an exhaustive proof. (The case of $n = 10$ was not included; perhaps he interpreted "between 1 and 10" as not including 10.) Because Jared included no structural link equating the approaches and his work does not suggest any search for structure, his work is not proof of the more general claim.

Emma, Finite-Domain Argument Score, 1; Infinite-Domain Argument Score, 1

We coded Emma's argument in regard to both the finite-domain and the infinite-domain claims because at first she claimed that the Number Trick works for numbers 1–10 (assumed to be natural numbers) but scratched that out. In both cases, Emma's argument, and our analysis of it, focused on structure that she leveraged to equate the two approaches. Emma noted that the 4 was to be doubled regardless of whether it was added to the 5 prior to doubling or after doubling.

In particular, Emma wrote, "In the first equation when she added the 5 + 4 and doubled it, but you must realize that 4 is still part of the equation even though it was smushed [sic] in with the 5, you did double the 4 but when it was part of the 5 [sic]." Emma clearly searched for a conceptual insight and found one, but we had to make *significant assumptions* about Emma's approach. We assumed that Emma noted that both the 5 and the 4 were ultimately doubled in both approaches (perhaps she implicitly invoked the commutative and associative properties), but our assumptions required considerable *gap-filling* given that Emma's response was vague and clumsily worded.

Jenna, Finite-Domain Argument Score, 2 or 1, Infinite-Domain Argument Score, 2 or 1

Jenna wrote, "Jessie's trick will work for any number between 1-10." We assumed Jenna referred to natural numbers in this range, a finite domain, but her argument could also be interpreted as addressing an infinite domain: all real numbers between 1 and 10. Jenna wrote, "$(e + 4) \bullet 2 = e \bullet 2 + 4 \bullet 2$, or $e \bullet 2 + 8$." as her key representation of the structure she observed. These equations illustrate Jenna's thinking about how to transform one of the general expressions into the other using the distributive property. Jenna's argument could be viewed as proof, if we assume that the distributive property was a prior result to her. If this were so, the distributive property served as a tool for linking the two structures, a conceptual insight explaining why the two approaches produce the same outcome for numbers in the specified domain.

However, this interpretation assumes that the distributive property was a prior result for Jenna, which brings out a dilemma. As noted earlier in chapter "Overview of Middle Grades Data" (this volume), the teacher used the Number Trick task as a way of introducing the distributive property. Consequently, if Jenna's argument was viewed as communication between her and her classroom community, then the distributive property could not be taken as shared knowledge. From this perspective, Jenna's response did not have backing in a prior result. Instead, Jenna's key equations could be viewed as merely stating a generalized version of what Jessie found in her specific instance, $e = 5$, as given in the task. In that case, Jenna's score would be a 1 because she expressed awareness that *a general argument is based in mathematical structures of the conditions and the conclusion* and awareness that *inferential links between these structures must be illustrated.* However, Jenna failed to provide reasoning based in knowledge assumed to be prior knowledge. Thus, Jenna's score depended on whether the argument was a communication with herself, with her teacher, or with her peers, because students in a single classroom may have very different mathematics backgrounds.

Summary

Although only one student in the data set presented an argument that could be taken as proof of the general claim with infinite domain—under certain assumptions—six of nine students presented evidence that they searched for conceptual insights. We found few characteristics of "proof" as described in the definition provided to us among the students' work, but we found evidence that students engaged in a practice critical to proof construction: finding a structural link between the two approaches that allows one to equate the two approaches. This finding was also supported by exchanges found in the transcripts, where students discussed with peers and teachers the reasons why the two approaches would equate for any number. Practices such as searching for structure linking two approaches/expressions can be groundwork for learning about the other characteristics of proof and proving such as being explicit about the prior results leveraged in the reasoning and explicit about the deductive inferences made in a progression toward writing proofs in the canonical sense.

Is a Proof Accessible to This Classroom Community?

After our analysis, we wondered, "Can we envision an argument that does not rely on the distributive property and could be taken as proof?" We also wondered, "Would such an argument be accessible?" We cannot answer these questions definitively, as we lack knowledge of these students' mathematical backgrounds and past experiences. Yet, based in our knowledge of many US states' mathematical content standards, we developed an argument that we believe could be accepted as proof and is within the conceptual reach of these students.

The argument we developed relies upon the interpretation of multiplication as repeated addition. This would serve a prior result, an informal definition. In notation, $2(n + 4) = (n + 4) + (n + 4)$. Applying the associate and commutative properties, $(n + 4) + (n + 4) = 2n + 2 \cdot 4 = 2n + 8$. We could also envision prose or a generic example argument that accomplishes a similar proof. To label the argument as proof, we would not require the students to explicitly name the prior results used.

The argument above could be leveraged as a generic example proof of the distributive property if the domain was restricted to natural numbers. Extending the argument to other multipliers (e.g., non-integer rational numbers and irrational multipliers) would be difficult or impossible for this classroom community. After all, the distributive property is generally taken as an axiom in advanced mathematical classes where the real number system is developed.

Data from Another Project

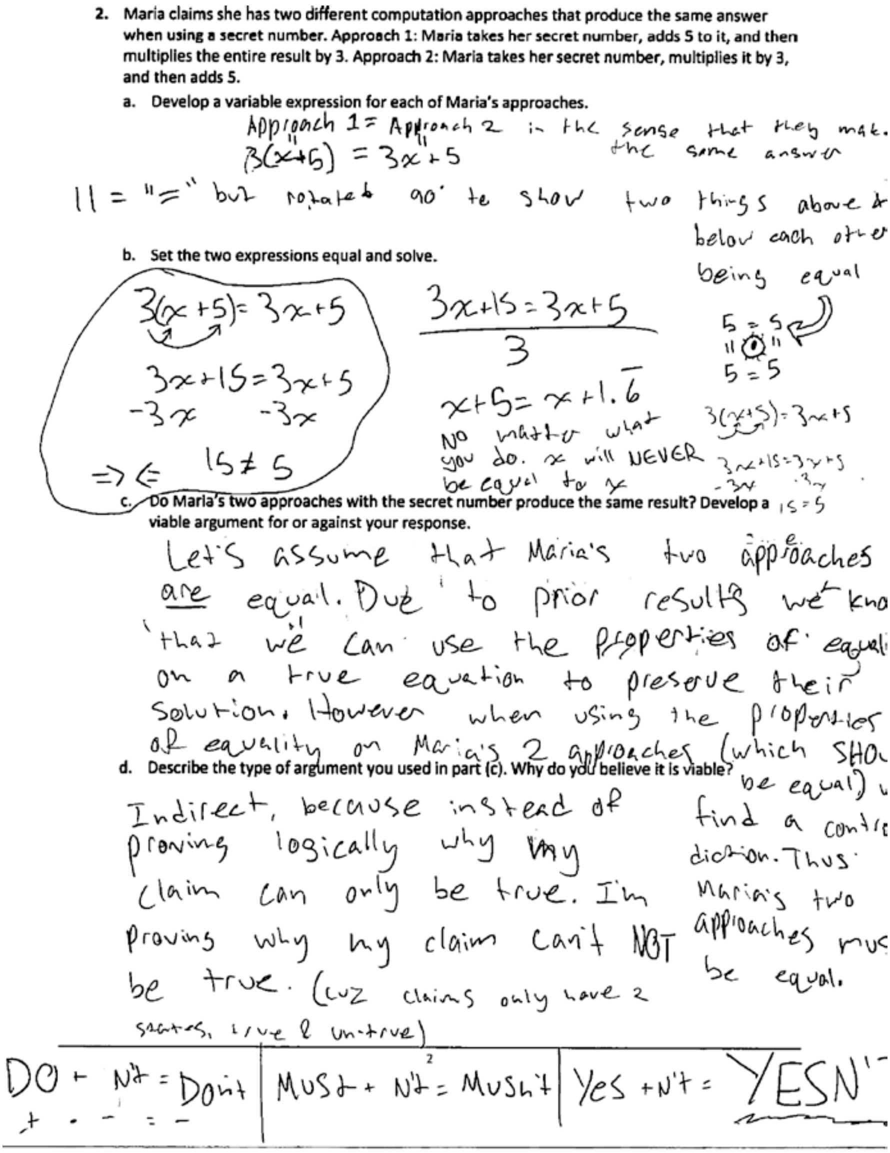
We were concerned that the data provided to us came from students who had not received explicit instruction on “the rules of the game” for proof and proving as described in the working definition provided to us and in the rubric we developed from this definition. Yopp (2015) pointed out that students benefit from explicit instruction on types of claims in mathematics and how these claims are written. Yopp also noted that students benefit from explicit instruction on how to present their arguments and what modes of argumentation are acceptable in mathematics.

Below, we include data on a different but related task from our own project, Longitudinal Learning of Argument Methods for Adolescents (LLAMA) (see acknowledgements). We do this only to illustrate the possibilities for proof and proving in classrooms where the “rules of the game” for proof and proving are explicitly taught. Students in the LLAMA project were taught by teachers who learned about our models and methods for viable argumentation and proving, and the teachers and students had access to our project-developed lessons. These lessons developed Common Core Grade 8 content (NGA & CCSSO, 2010) through viable argument activities. A complete description of LLAMA is beyond the scope of this chapter, but for our current purposes, we summarize key features of LLAMA in terms of what students were taught:

1. Viable arguments/proofs include well-worded general claims or existence claims using language such as “for all,” “if-then,” or “there exist.”
2. Viable arguments/proofs for general claims with large or infinite domains use representations (referents) that illustrate a general case(s) and the logical steps/transformations pertinent to showing the claim is true. Explicitly, viable arguments/proofs *eliminate the possibility of counterexamples* (Yopp, 2015) to the claim.
3. Viable arguments/proofs include a narrative that links the representations/referents in the argument to the claim, notes the prior results used, the method/mode of argumentation used, and how the argument’s steps are logical and demonstrate the truth of the claim.
4. Viable arguments/proofs of general claims use established methods/modes of argumentation such as exhaustion, direct (e.g., a sequence of modus ponens transformations/steps), contrapositive, and contradiction.

We used the term viable argumentation in place of proof and proving to acknowledge that axiomatic systems are not necessarily in place in the middle grades. We also wished to emphasize the phrasing in Common Core Mathematical Practice 3, *Construct viable arguments and critique the reasoning of others* (NGA & CCSSO, 2010).

Figures 2 and 3 illustrate sample work from two LLAMA students, Students A and B. This work is not representative of all students who received the intervention but illustrates the possibilities. Our point is that if we make explicit to students the



2. Maria claims she has two different computation approaches that produce the same answer when using a secret number. Approach 1: Maria takes her secret number, adds 5 to it, and then multiplies the entire result by 3. Approach 2: Maria takes her secret number, multiplies it by 3, and then adds 5.

a. Develop a variable expression for each of Maria's approaches.

Approach 1 = Approach 2 in the sense that they make the same answer
3(x+5) = 3x+5
|| = "=" but rotated 90° to show two things above & below each other being equal

b. Set the two expressions equal and solve.

3(x+5) = 3x+5
3x+15 = 3x+5
−3x −3x
15 ≠ 5
⇒⇐

(3x+15 = 3x+5) / 3
x+5 = x+1.6̄
No matter what you do. x will NEVER be equal to x

5 = 5
3(x+5) = 3x+5
3x+15 = 3x+5
−3x −3x
15 ≠ 5

c. Do Maria's two approaches with the secret number produce the same result? Develop a viable argument for or against your response.

Let's assume that Maria's two approaches are equal. Due to prior results we know that we can use the properties of equality on a true equation to preserve their solution. However when using the properties of equality on Maria's 2 approaches (which should be equal) we find a contradiction. Thus Maria's two approaches must not be equal.

d. Describe the type of argument you used in part (c). Why do you believe it is viable?

Indirect, because instead of proving logically why my claim can only be true. I'm proving why my claim can't NOT be true. (cuz claims only have 2 states, true & untrue)

DO + N'T = Don't | Must + N'T = Mustn't | Yes + N'T = YESN'T

Fig. 2 Response from Student A, a US eighth-grade student who participated in the LLAMA project

"rules of the game" for proof and proving, we are more able to assign the label of proof to their arguments. For example, we call Student A's argument "proof" of the claim "For all real numbers, none solve $3(x + 5) = 3x + 5$" because we can identify:

1. A general claim that Maria's two approaches are unequal no matter the choice of x—although the wording and labeling of the claim could be improved

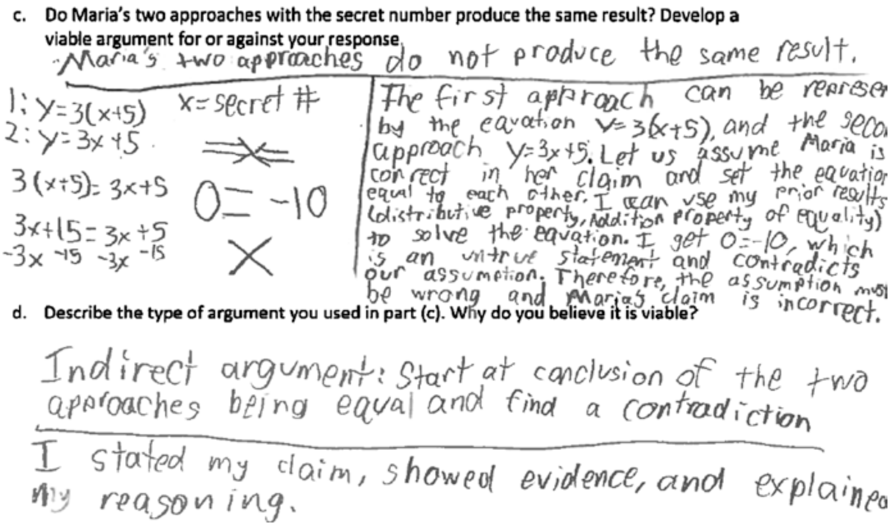


Fig. 3 Response from Student B, a US eighth-grade student who participated in the LLAMA project

2. A referent, the equation-solving steps, with the appropriate generality and a narrative discussing the logic expressed in the referent: that the use of "solution-preserving steps" (the key prior result) leads to a contradiction, rendering the assumption of a solution false
3. A clear, unambiguous expression of the method/mode of argumentation, proof by contradiction, and an explicit discussion of this method/mode as proof

Moreover, when we applied the rubric developed for this chapter, we still arrived at a Level 2 rating. The argument convinces us that the general claim is true (Characteristic 1), uses deductive-like inferences from prior results (Characteristic 2), and needs little gap-filling (Characteristic 3). The referent provides the structural links (Characteristic 4), the referent (equation-solving approaches) is canonical (Characteristic 5), and we all sanctioned the argument as proof (Characteristic 6). We also labeled Student B's argument as proof for similar reasons. We included Student B's argument as a more succinct proof that explicitly names the prior results used in the logical inference: the distributive property.

Conclusion

Applying a definition of proof to middle-grade students' arguments is challenging when the definition is presented as a list of characteristics derived from social-cultural norms in mathematics and formal logical constructions like deduction. Middle-grade students may have access to very different guidelines on what characteristics should present in proofs, arguments, and justification. Our focus was on the "rules of the game" for proof and proving, but the students in Mr. MC's classroom were focused on rules for justifications as expressed in the R.A.C.E sheet. As we noted in the beginning, our theoretical framework did not align well with the goals of instruction in this particular episode, and our focus may have misconstrued, and even overlooked, learning opportunities presented to students in this class. Consequently, our chapter can serve as a cautionary tale for researchers about applying a definition of proof or proving to a classroom where the students have different goals and notions about what it means to argue that a claim is true and justify their thinking.

Having acknowledged this possible mismatch, we did find that the definition provided to us and the rubric we generated, which valued conceptual insights, proved useful in finding proof-like features among students' work. Most students in Mr. MC's class at least searched for conceptual insights that explained why the two approaches must produce the same result. Perhaps searches for conceptual insights can be encouraged in classrooms that use very different guidelines for developing acceptable justifications, arguments, and proofs.

Our rubric was also useful in determining what was missing from an argument that prevented it from being sanctioned as proof. Scores of 0 and 1 were most easily determined by assessing whether or not a student searched for a conceptual insight. Category 1 ended up being a broad category containing responses that expressed a conceptual insight that could be leveraged toward proofs and responses that made clear the students at least searched for conceptual insights. Category 0 contained responses that included empirical support or no evidence that the student searched for a conceptual insight. Perhaps, for practitioners, the rubric can serve as a formative tool for giving students feedback as they learn the "rules of the game" for mathematical proof and proving.

Our assessment did however rely heavily on our abilities to recognize insights among the "noise" found in middle-grade students' writing. This type of assessment poses risks. Researchers might overlook potentially fruitful student reasoning that is novel and expressed ambiguously, or researchers might inadvertently impose sophisticated reasoning onto an argument that the student who wrote it did not possess.

In closing, we argue that giving students access to the "rules of the game" for proof and proving in mathematics, including the language of mathematics and the accepted methods/modes of proving, can help students understand what they are required to do when proving as well as help them to clearly communicate their reasoning to teachers, researchers, and peers. The rules of the game can also empower students with knowledge of the mathematical obligations for proving a general claim, such as representing the general case and demonstrating through logical applications of prior results that every case satisfying the conditions also satisfies the conclusion. And yet, we acknowledge that the "rules of the game" as we describe them may not be appropriate for all classrooms and may be inconsistent with some teachers' and researchers' goals.

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Argumentation, Justification, and Proof in Middle Grades: A Rose by Any Other Name

Eric Knuth, Orit Zaslavsky, and Hangil Kim

Introduction

We were tasked by the editors to provide a synthesis of the three chapters, highlighting, in particular, the similarities and differences among the chapters as well as the consequences associated with analyzing the classroom data from the three different perspectives of argumentation, justification, and proof. In our discussion, we also highlight the consequences associated with the analytic framework that each set of authors brought to bear on the data and suggest that these latter consequences may contribute more to the differences among the chapters than the differences in perspective. In addition, we also offer an alternative definition for the activities associated with argumentation, justification, and proof as we think it may serve as a more comprehensive way for describing and studying such activities.

The chapter authors were provided definitions of the three focal constructs, definitions that the editors acknowledge are one of many ways that each particular construct might be defined. Indeed, as Hanna (2020) claimed: "Argumentation, reasoning, and proof are concepts with ill-defined boundaries" (p. 561). Likewise, we would add to that list of concepts, *justification*, as it too falls into the category of concepts with ill-defined boundaries. Nevertheless, in our discussion we used the following definitions of each construct as stated by each of the chapter authors:

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Argumentation as "the process of making mathematical claims and providing evidence to support them" (see "Exploring Collective Argumentation in the Mathematics Classroom" in the chapter by Gomez Marchant, Jones, and Tanck, this volume).

Justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020)" (see the introductory paragraph in the chapter by Lesseig and Lepak, this volume).

Proof as an argument that consists of six characteristics, with arguably the most critical characteristic being that the argument "does not admit possible rebuttals" (see "Introduction" in the chapter by Yopp, Ely, Adams, and Nielsen, this volume).

Based on these working definitions, we view proof as a special case of argumentation and view justification as the second "half" of argumentation (i.e., providing evidence to support a claim). The relationship among these three constructs serves to inform our discussion of the three chapters. In our discussion of the chapters, we also consider aspects of practice not necessarily captured with argumentation, justification, or proof, namely, the practices of developing and exploring conjectures. We refer to the collective set of practices as *proving-related activities* (Knuth et al., 2019) and use this expression throughout our discussion in this chapter.

The Task and Its Possibilities

In discussing the three chapters, we first consider the potential of the task to engage students in proving-related activities (cf. Arbaugh et al., 2019) and the instructional goals associated with the Number Trick task as each of the chapter authors based aspects of their analyses on assumptions about the task and on its implementation.[1] In their chapter, Lesseig and Lepak introduced the Mathematics Task Framework (Stein et al., 1996) as a means of capturing "a task from its original form [as written] through different phases of set up and implementation, culminating in student learning" (see "Method" in the chapter by Lesseig and Lepak, this volume). We adopt this particular lens as well to frame our discussion as it offers a way to portray the possibilities afforded by the written task, the potential instructional goals associated with the task, and the opportunities, both taken and missed, by the teacher to meaningfully engage students as the task was set up and enacted.

[1] We are uniquely positioned to discuss the potential outcomes and instructional goals of the Number Trick task as the first author used this same task in a project which examined middle school students' justifying and proving competencies (e.g., Knuth & Sutherland, 2004; Knuth et al., 2009), and much of our discussion is based on this prior work.

Potential Proving-Related Activities

In the first part of the task, students are asked whether Jessie's two answers will always be equal to each other for any number between 1 and 10 and then asked to explain their reasoning. Responses that might be anticipated from students include (i) testing several, but not all, numbers between 1 and 10 and concluding that the two answers *will* always be equal based on the results of their computations; (ii) testing every number between 1 and 10 and concluding that the two answers *are* always equal; or (iii) testing several numbers between 1 and 10, gaining a critical insight from the computational process (i.e., an insight related to the distributive property), and concluding that the two answers *will* always be equal based on that insight. In all three cases, students' explanations involve reasoning with examples: in the first case, an explanation based on a limited set of empirical evidence; in the second case, an explanation based on exhausting the domain of the claim; and in the third case, an explanation based on observing generality in the computational process. In the last two cases, one might consider the explanations to constitute proof: a proof-by-exhaustion and a generic proof (Zaslavsky, 2018) or an algebraic proof (i.e., $2(x + 4) = 2x + 8$).

In the second part of the task, students are asked, with respect to their explanation to the first part, whether their explanation shows that the two answers will always be equal to each other for any number (not just between 1 and 10) and to explain their response. Anticipated student responses might include (i) concluding that the answers will always be equal for any number based on their work from the first part of the task (either their computations with several numbers or their computations with every number); (ii) testing several additional numbers (beyond the domain from the first part of the task) and concluding that the two answers *will* always be equal for any number based on these additional computations; (iii) concluding that they cannot be sure because they only tested a limited set of numbers (in the first part) and cannot test every possible number; or (iv) concluding that the two answers will always be equal based on the critical insight related to the distributive property. Similar to the first part of the task, students' explanations again involve reasoning with examples: the first two cases both rely on a limited set of empirical evidence, and the fourth case relies on noticing the generality of the computational process. With respect to whether the explanations constitute proof, as above, only the fourth case might be considered as such.

In considering the three chapters in light of the preceding discussion, two of the three chapters do focus, to varying degrees, on aspects of the task and possible student responses. Yopp et al. noted that the task offers the possibility for students to recognize "that a proof of a general claim needs more than just an empirical check" (see "Reframing the Definition of Proof for the Current Context" in the chapter by Yopp et al., this volume), to engage in proof-by-exhaustion (for the first part of the task) and to gain a conceptual insight regarding regularities in the computational process of testing several numbers. Although Lesseig and Lepak also noted this last possibility, it was only in the context of the second part of the task as they claimed that the computational process students used for the first part of the task only served the dual purpose of leading "students to an answer that the Number Trick worked (the claim) and provided the explanation to support their claim" (see "Justification Process" in the chapter by Lesseig and Lepak, this volume). Gomez, Marchant Jones, and Tanck, however, did not discuss the nature of the task and its possibilities as their analytic framework focuses solely on the implementation of the task.

Potential Instructional Goals

In considering the task setup and its implementation, the three anticipated student responses for the first part of the task provide various instructional opportunities for a teacher to engage students in proving-related activities. And, of course, these instructional opportunities are also enabled (and constrained) by a teacher's particular instructional goals. A teacher might initially ask students who tested several examples how they know the two answers will be the same for the numbers they did not test, and in the classroom conversation that might follow, other students might offer critiques of the responses. A teacher's instructional goal in this case might be to highlight the limitation of examples as a means of establishing a general result. Alternatively, a teacher might facilitate a conversation between students who tested several examples and those who tested the entire domain, asking the students to compare (and critique) the respective strategies. In this case, such a conversation might serve an instructional goal of contrasting the limitation of examples as a means of establishing a general result with their power to prove a general result (as well as to foreshadow what is to come in the second part of the task). Moreover, a teacher might also have a goal of gaining insight about the rationale underlying students' decisions to test every number in the domain: on the one hand, it might be that students did so because they assumed that the task (or teacher) required it, and on the other hand, it might be that students viewed the strategy as a means of gaining absolute certainty about the truth of the claim (a strategy that, as Yopp and his colleagues might say, does "not admit rebuttals"). Finally, for those students who gained a critical insight from noticing regularities in the computational process, having these students share their thinking might serve as a natural transition into the second part of the task (or a teacher might wait until the class has thought about the second part of the task before asking these students to share their thinking).

As the class moves on to the second part of the task, a teacher might first ask for responses from those students who concluded the answers will always be the same based on their computational results from the first part of the task (or based on their testing of additional examples). In the class discussion that follows, other students might suggest that it is impossible to test every number, so one cannot be absolutely sure that the two answers will always be the same for every possible number—an opportunity to highlight the limitations of examples as a means to establish a general result, beyond any doubt (or without admitting rebuttals). Finally, in asking students if there is any way to know for certain whether the two answers will always be the same for any number, students who gained the critical insight (or at least searched for one) could offer their explanations—an opportunity to discuss the idea of a general argument (versus an argument based on empirical evidence).

In examining how the task was set up and implemented in Mr. MC's classroom, each set of chapter authors highlighted different aspects of the classroom interactions as students engaged with the task. Gomez Marchant and his colleagues do not discuss the way in which the task was set up but rather used the Teacher's Support for Collective Argumentation [TSCA] framework (Conner et al., 2014) to document teacher and student contributions in the argumentation process and, importantly, highlighted the teacher moves in that process. The TSCA framework provided a means for examining the nature of the teacher's contributions to the process, whether adding contributions himself or eliciting contributions from students, and documents the critical role the teacher played in supporting the classroom's collective argumentation. Interestingly, and somewhat in contrast to the Lesseig and Lepak chapter, Gomez Marchant et al. do not comment about any of the opportunities Mr. MC may have missed to engage students in thinking more deeply about the task. For example, Lesseig and Lepak noted that Mr. MC pushed students to compute every number for the first part of the task ("Did you check them all?") and "seemed to discourage a general line of reasoning in favor of a justification for 1–10 early in the process" (see "Task as Implemented" in the chapter by Lesseig and Lepak, this volume). In this case, by pushing students to compute every number between 1 and 10, the teacher removed the possibility, at that moment, for discussing different student responses (e.g., testing only a few numbers, noticing the underlying structure). It is also not clear whether Mr. MC intended for students to test every number in the domain (part 1) with proof-by-exhaustion in mind. If a goal was to use the task as context for introducing the distributive property through a discussion of the computational regularities students may have noticed, and to then use a generic example to illustrate the structure, this opportunity was missed.

Framing Matters

Across the three chapters, the analytic framework applied to the data, in large part, seemed to make the biggest difference in what the chapter authors reported. Gomez Marchant et al.'s use of the TSCA framework (Conner et al., 2014) resulted in a primary focus on documenting both student and teacher moves in creating an argument, whereas Lesseig and Lepak's use of the Mathematics Task Framework (Stein et al., 1996) resulted in a primary focus on the teacher's instructional practices and the resulting opportunities for students to learn from their engagement in the justification process. In their chapter, Yopp and his colleagues focused less on the actual classroom implementation of the task and more on the "product" of students' engagement in the proving-related activities. In particular, they analyzed the student work relative to six criteria for what constitutes a proof and characterized the responses students produced in light of these criteria. An important distinction that

Yopp et al. noted relates to the critical insight about the distributive property: they distinguish between students who relied solely on empirical evidence and those students who either searched for a "conceptual" insight (but did not find) or determined the conceptual insight. The search for a conceptual insight is important to note as that suggests that these students do recognize the limitation of empirical evidence.

In the end, it is not surprising that each set of chapter authors, given their respective constructs and definitions (i.e., argumentation, justification, or proof) and particular analytic framework, detailed different aspects of the classroom participants' interactions (and products). We also think it is worth noting that none of the chapter authors discussed in any detail the various possible outcomes and instructional goals associated with the Number Trick task as written nor the potential opportunities for deepening students' understanding of argumentation, justification, and proof. From our perspective, the Number Trick task offered several opportunities for students to learn important ideas underlying proving-related activities including the limitation of examples as a means of proof as well as their power as a means of proof (in the case of proof-by-exhaustion or in generic proving), the role of examples can play in noticing regularity or gaining conceptual insights (from computations with examples), and what constitutes a general argument (proof). Yet, in their analyses of the classroom episode, the chapter authors did not discuss in any depth such opportunities (whether taken or missed by the teacher). For example, the authors all noted to varying degrees the fact that Mr. MC pushed his students to test every case between 1 and 10, yet they did not comment that, as a result, Mr. MC was taking away the opportunity to engage the class in a conversation about the role of empirical evidence (its limitations as well as its power).

Capturing the Breadth of Proving-Related Activity

In considering the perspectives of the three chapters—argumentation, justification, and proof—we are reminded of a quote from William Shakespeare: "A rose by any other name would smell as sweet." In this case, whether one uses argumentation, justification, or proof, it seems that the descriptive name of the activity is of less importance than the actual nature of the activity in which students engage. Stylianides (2009), for example, defined reasoning-and-proving:

> to describe the overarching activity that encompasses the following major activities that are frequently involved in the process of making sense of and establishing mathematical knowledge: identifying patterns, making conjectures, providing non-proof arguments, and providing proofs. The choice of a hyphenated term to encompass these four activities reflects my intention to view the activities in an integral way. (pp. 258–259)

We agree with Stylianides in the need to describe the overarching activity but feel his definition is not adequately comprehensive in that it excludes some related activities. As we mention in the introduction, we prefer the expression *proving-related* *activities* as we view it as encompassing aspects of practices related to argumentation, justification, and proof as well as aspects of practices not necessarily captured by these three practices (Ellis et al., 2019). In particular, we characterize proving-related activities as including the development of conjectures, exploration of conjectures, justification of conjectures (including proof of conjectures), and refutation of conjectures. Inherent in our definition is also explicit attention to the role examples play in these activities as we view the time spent thinking about and analyzing examples as playing a foundational and essential role in the development, exploration, and understanding of conjectures, as well as in subsequent attempts to develop proofs of those conjectures.

A focus on argumentation, justification, or proof (as defined and applied by the authors), from our perspective, does not adequately capture all of the critical aspects of proving-related activity, including the development of conjectures as well as the role of examples-based reasoning. As the Number Trick task highlights, examples played a major role in the students' activities, from providing initial conviction about the claim's truth to serving as a means of justification for the claim's truth to providing insight based on regularities observed in the computations with examples.

One final consideration for a more comprehensive way to describe the instructional episode that was the focus of the three chapters relates to our collective work with teachers. If we want the activities associated with argumentation, justification, and proof to play a more central role in middle school classrooms, it may be more powerful and instructive to focus on all the activities involved in such practices. It is easy to get lost in trying to categorize teachers' practices or students' activities as related to argumentation, justification, or proof, when in the end what perhaps really matters is teachers' efforts to meaningfully engage students in the proving-related activities of developing, exploring, justifying (including proving), and refuting conjectures.

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Part III
Argumentation, Justification and Proof in High School Mathematics

Overview of High School-Level Data

AnnaMarie Conner

In the following section, three chapters include analyses of the same transcript and classroom artifacts from Ms. Bell's (all names are pseudonyms) ninth-grade classroom, with the fourth chapter providing a synthesis and comparison of results, focusing on consequences of using the constructs argumentation, justification, and proof with these data. The class included approximately 20 students from an economically diverse, rural school district in a southeastern US state.[1] Ms. Bell was a student teacher who was participating in a longitudinal study investigating prospective teachers' beliefs and their support for collective argumentation. These data were collected during the second month of the school year. Students in the class were identified as higher-achieving students; they investigated topics in analytic geometry and coordinate algebra during the semester-long class. Based on her comfort with the material and the timing of teaching it, Ms. Bell selected the unit from which these data were taken; the entire unit was recorded for the original research purposes.[2] The topic of the shared lessons was finding a formula for the sum of the interior angles of any polygon.

The data set provided to the authors of this chapter included transcripts of parts of two lessons (referred to as part 1 and part 2), an overview of the data (included here as well), and a copy of the worksheet around which the lesson was framed (included here as Fig. 1).

[1] The school district was using state-created frameworks as teacher resources for a new set of standards adopted this year. The task used in these class periods was adapted, by the teacher, from these frameworks.

[2] Participants in this study were asked for a unit that they felt comfortable teaching and in which they would support collective argumentation.

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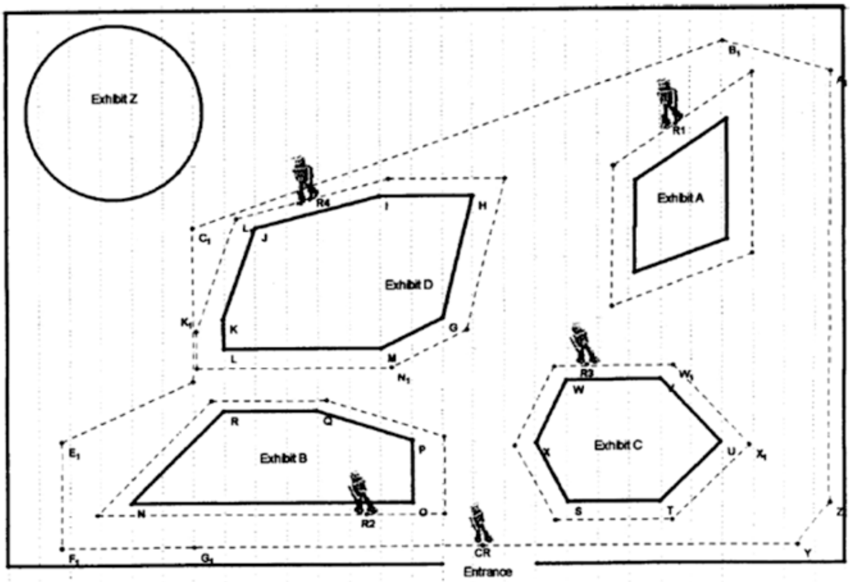
Robotic Gallery Guards Learning Task

The Asimov Museum has contracted with a company that provides Robotic Security Squads to guard the exhibits during the hours the museum is closed. The robots are designed to patrol the hallways around the exhibits and are equipped with cameras and sensors that detect motion.

Each robot is assigned to patrol the area around a specific exhibit. They are designed to maintain a consistent distance from the wall of the exhibits. Since the shape of the exhibits change over time, the museum staff must program the robots to turn the corners of the exhibit.

Below, you will find a map of the museum's current exhibits and the path to be followed by the robot. One robot is assigned to patrol each exhibit. There is one robot, Captain Robot, CR, who will patrol the entire area.



1. When a robot reaches a corner, it will stop, turn through a programmed angle, and then continue its patrol. Your job is to determine the sum of the angles that R1, R2, R3, and R4 will need to turn as they patrol their area. Keep in mind the direction in which the robot is traveling and make sure it always faces forward as it moves around the exhibits.

2. What do you notice about the sum of the angles (for R1, R2, R3, and R4)? Do you think this will always be true? Why?

3. Determine the sum of the measures of the interior angles of the paths the robots travel (try doing this without a protractor). Use this information to help you write a function that gives the sum of the interior angles of any *n*-sided polygon (*n*-gon). Justify your answer.

Fig. 1 Worksheet for small group work, given to students by Ms. Bell at start of class

Overview of the Data

Ms. Bell was a student teacher in a ninth-grade accelerated classroom in a rural district. She was teaching in an integrated mathematics class, and this unit involved various geometric concepts. The lesson began with a task involving the exterior angles of polygons and proceeded into interior angles and sums of both interior and exterior angles. The parts of the lesson included in the transcript are the very end of class in which a student shared his group's solution for a formula for the sum of the interior angles of an n-sided polygon and a whole-class discussion the next day in which Ms. Bell introduced another way to think about the formula.

Ms. Bell began class by discussing the problem presented on the worksheet (see Fig. 1). Students were divided into small groups, and each group was assigned a different shape and asked to measure the angles of the figure and record their results on the board. Small groups worked on questions 1–3 of the worksheet. The whole-class discussion in part 1 of the transcript (approx. 40 lines) began about 75 min into the class period. Two students shared their groups' work to end the class.

Part 1

In part 1, Ms. Bell selected Travis to share his group's work. To give a flavor of the class and the nature of the teacher's questioning, we share the following exchange:

Teacher: So, Travis, tell us what you did.
Travis: All right. [Writing: 360 540 720 900.]
Teacher: So what is this you're writing on the board?
Travis: That was the numbers of all the—these A, B, C, and D
Teacher: What numbers? The numbers of what?
Travis: What do you mean? That was the
Teacher: Sum of the?
Travis: Sum of the interior angles.
Teacher: Interior angles, okay.
Travis: So to get these, it would be—you're just—
Teacher: What are the side lengths that correspond with that, Travis?
Travis: n minus 2—*n* would be the number of sides minus two, times 180. So this 360, was 4-sided, so 4 minus 2 is 2, times 180 is 360. This was 5-sided, so you would do 5 minus 3—or 5 minus 2 is 3, times 180 is 540.
Teacher: Okay.
Travis: So it would just be the number of sides minus two—
Teacher: So did you find that by trial and error? You were just trying different—?
Travis: Yeah
Teacher: Okay, cool. Go ahead and write your formula you got up there for it, a function in terms of *n*.
Travis: All right. I'm going to write both of them. [Travis writes $(n-2)180$ and $F(n) = 180n - 360$]
Teacher: So that's what Travis originally had up top, and then Kevin said, 'hey, what about the distributive property? We can just multiply that through.' Good deal. Thanks, Travis.

Ms. Bell then asked another student to share his approach, and class ended.

Part 2

At the beginning of the next class period, Ms. Bell asked students to find the sum of the interior angles of a 12-sided polygon and the sum of the exterior angles of an 8-sided polygon. The students' answers to these questions were collected and discussed. About 15 min into the lesson, Ms. Bell engaged students in a whole-class discussion related to finding the sum of the interior angles of any polygon, captured in part 2 of the transcript (approx. 204 lines) given to authors.

During this discussion, several mathematical ideas were proposed and discussed. Ms. Bell began:

> Actually, before that, I want to show you guys. I forgot about this. I don't want to say—It's kind of an easier way to think about finding the interior angles of any polygon.

Subsequently, she asked students to come up and divide the polygon (a quadrilateral) into triangles. As it was not clear if the dividing should be into the minimum number of triangles, any number of triangles, or the maximum number of triangles, students proposed different ideas and asked questions. They established the interior angle sum to be 360 degrees (as the sum of two triangles). After this, they looked at a pentagon. Ms. Bell prompted:

> Thank you so much. Does someone want to come and draw the smallest amount of triangles we can make? Okay, come on Karin.

The student who attempted to draw the triangles crossed the diagonals, which resulted in the figure being divided into five triangles (and not all angles of the triangles contribute to the angles of the pentagon). The students readily made suggestions for how to adjust the diagram (essentially deleting one of the diagonals) to produce the minimum number of triangles. They established the sum is 540 degrees.

A student then proposed that they do a hexagon "Let's do six," to which another quickly responded that, "Six would just be 720." The teacher drew a hexagon and the class proceeded to draw diagonals to make triangles. A student came to the board and drew the minimum number of diagonals as seen in Fig. 2.

The teacher prompted, "Is that the only way I could have drawn those triangles?" Multiple students offered suggestions, and the teacher again had to clarify that they were looking for the minimum number of triangles. After establishing the interior angle sum is 720 degrees, a student offered "There's a pattern," and another student said "We already have an equation for this." Ms. Bell took up these ideas:

> We already have the equation for this, right. Okay. So let's see if we can find a different way of finding that equation. So we have an *n*-sided polygon. Let's do number of triangles and what, total degrees, I guess. Sum of interior angles. [Ms. Bell makes a table on the board with the headings '*n*-sided', '# of triangles' and 'sum of int angles' – see Fig. 3] Okay. So I started with—what could I start with? What would be the smallest one you could start with to have a triangle?

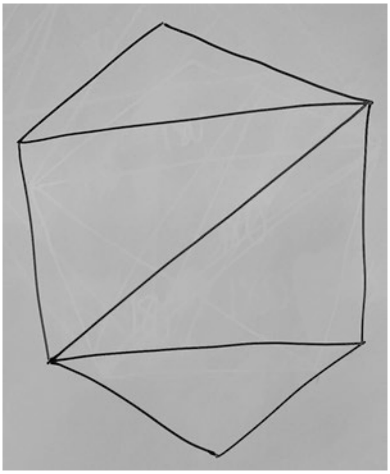
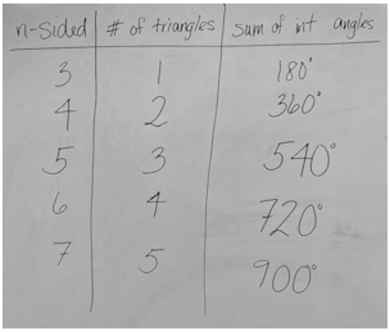


Fig. 2 Student's triangulation of a hexagon



n-Sided	# of triangles	sum of int angles
3	1	180°
4	2	360°
5	3	540°
6	4	720°
7	5	900°

Fig. 3 Table filled in by teacher with class's contributions

The students readily offered the values she requested, and Ms. Bell completed the table (see Fig. 3).

After seven sides, Ms. Bell requested, "All right, so somebody tell me, what was our function again that we got for finding the sum of interior angles?" The students offered both formulas from the day before (i.e., f(n) = (n-2)*180 and "180 x minus 360").

Ms. Bell concluded by asking whether the equations were the same or different, to which a student identified that they used the distributive property on the first to obtain the second, and she asked if they like one of the methods better than the other.

That concluded the transcript shared with the authors. For context, after this discussion, the class applied the results to a 12-sided figure. Then they continued as a whole class to define "regular polygon" and extended the sum of interior angles formula to a formula for the measure of any interior angle of a regular n-gon. Finally, in small groups, they explored, and then as a whole class discussed, the sum of the exterior angles of a convex polygon and finding the measure of an exterior angle of a regular polygon along with applications of these formulae.

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Argumentation in the Context of High School Mathematics: Examining Dialogic Aspects of Argumentation

Markus Hähkiöniemi

Introduction

According to the definition used in this book, "mathematical argumentation is the process of making mathematical claims and providing evidence to support them" (see "Argumentation" in the introductory chapter by Staples and Conner, this volume). Argumentation can be considered from several perspectives that emphasize different aspects. We can examine different types of arguments such as abductive, inductive, generic, or deductive arguments (e.g., Reid & Knipping, 2010), students' conceptions of proofs (e.g., Harel & Sowder, 2007), the structure of argumentation by identifying components of argumentation (e.g., Ayalon & Even, 2016; Conner et al., 2014), or how students interact with their peers, for example, by challenging ideas (e.g., Asterhan & Schwarz, 2009; Chen et al., 2016).

In this study, I examine argumentation from the perspective of dialogic argumentation that is defined as "a specialized way of arguing in which the participants not just defend their own claims, but also engage constructively with the argumentation of their peers" (Nielsen, 2013, p. 373). Thus, dialogic argumentation draws on the more general definition of argumentation but emphasizes collaboration in the process of constructing claims and supporting evidence as well as critical examination of the claims and evidence provided by others. These aspects are not highlighted in the more general definition of argumentation, but they are not excluded either.

By focusing on dialogic argumentation, I study argumentation following the definition provided in this book with special attention to *dialogicity*. I use the term dialogicity in a more strict meaning than some authors who consider argumentation always to be a dialogic process (Ford & Foreman, 2015). I use dialogicity in the

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same sense as Mortimer and Scott (2003) who differentiate between interactivity and dialogicity where the former means that different people participate in the discussion and the latter means that different points of view or ideas are openly explored and worked on. This kind of dialogicity is also included in Alexander's (2004) features of dialogic teaching: collectivity, reciprocity, supportiveness, cumulativeness, and purposefulness. Dialogic argumentation has some connection to collective argumentation that is often used in mathematics education (Conner et al., 2014; Krummheuer, 1995). Collective argumentation emphasizes students and the teacher working together, whereas dialogic argumentation emphasizes that working together is a dialogic process that includes students engaging with the argumentation of their peers.

Hähkiöniemi et al. (2019) proposed that dialogicity can be seen in three important ways in whole-class argumentation. First, dialogicity is present in students' actual arguments in the form of student moves, such as elaborating, which indicate engagement with other students' ideas. Webb et al. (2014) provided evidence that students' engagement with their peers' ideas predicted student achievement more than explaining one's own ideas. Thus, this kind of engagement with others' ideas is a feature of productive discussion. In addition, the concept of exploratory talk by Mercer et al. (2004) emphasizes this kind of dialogicity. Second, dialogicity may be present in the communicative approach of the teacher in the sense that the teacher values, explores, and works on students' ideas without evaluating them based on whether they correspond with the teacher's view (Mortimer & Scott, 2003; see also Lehesvuori et al., 2017). Mortimer and Scott (2003) argue that appropriately sequencing dialogic and authoritative communicative approaches benefits learning. Third, dialogicity may show up in more general organizing for dialogic teaching, such as designing appropriate learning tasks, structuring the lesson in appropriate phases, and making decisions during the lesson to create opportunities for dialogic interaction.

As stated in the definition, argumentation, and thus dialogic argumentation, must include providing evidence. Posing only claims is not argumentation. Generally in mathematical discussion, there is an essential difference between explaining methods and explaining reasons (Kazemi & Stipek, 2001). Similarly, in argumentation, providing evidence may consist of describing facts that support a claim or articulating reasoning that leads to the claim (see Hiltunen et al., 2017). Furthermore, as we are interested in dialogic argumentation involving students, it is particularly interesting how the students provide evidence to support claims.

As argued above, dialogic argumentation includes students providing support and engaging with each other's ideas. The teacher uses communicative approaches and structures the lesson to create opportunities for dialogic argumentation. The aim of this study is to examine how dialogic argumentation exists and how the teacher supports it in the provided data set. Therefore, the following research questions were set:

1. How do the students engage with other students' ideas and describe support for claims or articulate reasoning?

2. How does the teacher support dialogic argumentation by structuring argumentation and using communicative approaches?

Conceptual Framework

Students' Dialogic and Justifying Moves

Several studies have analyzed students' moves or speech turns based on what they contribute to argumentation (e.g., Asterhan & Schwarz, 2009; Chen et al., 2016). Some of the turns indicate that students are engaging with other students' ideas and some of the turns contain components of argument that have some similarities with the elements in Toulmin's model (1958). Hähkiöniemi et al. (under review) created a coding scheme for dialogic argumentation that includes two dimensions that are coded independently: students' dialogic and justifying moves (see Table 1 for descriptions and examples). Students' dialogic moves consist of questioning, challenging, elaborating, commenting, and responding. The first three moves are considered indicating higher engagement with others' ideas than the latter two because commenting and responding to a question do not necessarily mean that a student has thought thoroughly about the preceding idea or question (Hähkiöniemi et al., under review). Justifying moves are either describing support or articulating reasoning. Because the dimensions are coded independently, it is possible to locate afterward the student moves that are both dialogic and justifying (e.g., articulated reasoning which is given as a response to a student question). The coding scheme by

Table 1 Dialogic argumentation coding scheme (Hähkiöniemi et al., under review)

Student move	Description
Dialogic moves	
Questioning	Student asks a question about an idea presented by someone else
Challenging	Student points out a deficiency in another student's idea
Elaborating	Student analyses, develops, or clarifies another student's idea
Commenting	Student comments or takes a stand on another student's idea without questioning, challenging, or elaborating
Responding	Student responds to another student's question without questioning, challenging, elaborating, or commenting
Justifying moves	
Articulating reasoning (AR)	Student explicitly explains why a claim can be concluded from what is known. In other words, a student explains the line of reasoning leading to a claim, making the reasoning visible
Describing support (DS)	Student presents facts, calculations, observations, figures, etc. to support the claim without articulating reasoning. The support has to be related to the content of the lesson

Hähkiöniemi et al. (under review) is used in this study as it allows examining how students engage with their peers' ideas and how they produce components of arguments.

Communicative Approaches and Structuring Argumentation

Communicative approaches are introduced in Mortimer and Scott's (2003) framework. The framework was created for science teaching but can be applied to mathematics teaching as well (Essien, 2017; Lehesvuori et al., 2017). Mortimer and Scott (2003) describe four communicative approaches that a teacher can use:

- Interactive/dialogic: the teacher and students explore ideas, generating new meanings, posing genuine questions and offering, listening to and working on different points of view.
- Non-interactive/dialogic: the teacher considers various points of view, setting out, exploring and working on the different perspectives.
- Interactive/authoritative: the teacher leads students through a sequence of questions and answers with the aim of reaching one specific point of view.
- Non-interactive/authoritative: the teacher presents one specific point of view. (p. 39)

The framework offers a simple way to differentiate the form and function of communicative approach. For example, a teacher can introduce a scientific point of view without considering alternative views (authoritative) either through questioning (interactive) or lecturing (non-interactive). This distinction brings to mind the two interaction patterns by Wood (1998): funneling and focusing. Funneling and focusing both appear in the form of questioning, but funneling leads the students through the path laid out by the teacher, whereas focusing helps all the participants to understand a student's idea. Thus, funneling is one example of interactive/authoritative approach, and focusing is one example of interactive/dialogic approach. As Wood (1998) states, funneling is "univocal" despite the question-answer sequence, and in focusing, students and teachers "participate more equally in the dialogue" (p. 172). Although the focus is on the interaction patterns, individual questions play a role in the patterns. In interactive/authoritative approach, closed questions are often used to have the student respond what the teacher intends. On the other hand, interactive/dialogic approach may include open or genuine questions for which there is not only one expected answer.

Argumentation may also contain several subarguments that are connected to build a larger argument (Conner et al., 2014). Similarly, a teacher can change between communicative approaches in orchestrating a lesson (Mortimer & Scott, 2003). Thus, argumentation can be composed of steps containing smaller arguments, and the teacher can change communicative approach between the steps.

Methods

First, I read the transcript several times to become familiar with the data. To answer the first research question, I coded the data for students' dialogic and justifying moves (Table 1). The unit of analysis is usually a student turn, but I consider a single turn to include several utterances in the following two cases: (1) a student is interrupted by another student, but the interruption does not cause changes in student's turn, and (2) a student continues talking about the same topic after a teacher utterance. Quite often, a student begins, for example, to describe support and continues because the teacher asks a follow-up question. The student moves had to be interpreted related to the context because a statement (e.g., 3 + 7 = 10) can be given, for example, to answer a question (e.g., what is the sum of 3 and 7) or to support another statement (e.g., the sum of two odd numbers is even).

It should be noted that only the transcript was used in the analysis although using a transcript together with video would help to recognize whether a certain student move was posed as a reaction to a preceding student turn or just happened to be said next.

To address the second research question, I first examined how the teacher structured the argumentation into steps. The steps were identified by recognizing where the teacher transferred to achieve a new piece of information with the students. The teacher set up each step by asking a question such as "what is the smallest number of triangles I can cut this into" (line 124–125) or by stating what they are going to do next, e.g., "I love to make tables to find patterns" (line 223–224). After identifying the steps, I examined how the teacher supported dialogic argumentation within the steps and, in particular, coded the communicative approach within each step as dialogic/interactive (D/I), authoritative/interactive (A/I), dialogic/non-interactive (D/N-I), authoritative/non-interactive (A/N-I) according to Mortimer and Scott (2003).

Results

Students' Dialogic and Justifying Moves

Table 2 shows the frequencies of students' dialogic and justifying moves. Altogether, students made ten dialogic moves. Six of the dialogic moves were elaborations that indicated high-level engagement with another student's idea. For example, in lines 120–123, Martin elaborated Angela's drawing by introducing the idea that that there are "infinitely many" triangles and pointing out that the lines are drawn "across the center." Four of the dialogic moves were commenting that indicated some engagement with others' ideas. For example, in line 155, Martin comments on Karin's drawing by saying "It looks funny." Martin does not explicate what kind of deficiency the drawing has and, thus, is not challenging but only commenting.

Table 2 Instances of students' dialogic and justifying moves

Student move	Number of instances	Lines
Dialogic moves		
Questioning	0	
Challenging	0	
Elaborating	6	120–123, 121–122, 160, 161–163, 260–262, 268
Commenting	4	155, 198, 200, 259
Responding	0	
Justifying moves		
Articulating reasoning (AR)	1	14–33
Describing support (DS)	10	121–122, 127, 147–152, 160, 161–163, 169, 171, 189, 195–196, 205–209
Justifying and dialogic moves (DS and elaborating)	3	121–122, 160, 161–163

Students' justifying moves contained one AR and ten DS. The only AR was constructed in the first lesson when Travis explained that he had constructed the formula $(n - 2)180$ by measuring the angles in four polygons. This is an inductive argument in which Travis concludes that, because the formula works in four cases, it works in all cases. Travis articulated his reasoning by explaining why the cases supported the formula:

> *n* minus 2—*n* would be the number of sides minus two, times 180. So this 360, was 4-sided, so 4 minus 2 is 2, times 180 is 360. This was 5-sided, so you would do 5 minus 3—or 5 minus 2 is 3, times 180 is 540. (lines 24–26)

In the other justifying moves, students described support for a claim without articulating reasoning. For example, in lines 147–152, Karin was implicitly claiming that the smallest number of triangles into which a pentagon can be divided is five and supported this by drawing the triangles. Here Karin, or anyone else, did not explain how the claim could be concluded from the drawing. Another example of DS occurs later when the number of triangles is reduced from Karin's first suggestion. Martin proposed that the interior angle sum of pentagon could be found if "You add them" (line 171), where "them" was the 180, 180, and 180 mentioned by the teacher. Here no one articulated why, in the case of the triangles drawn this way (unlike in the case of triangles drawn as Karin first proposed), one can sum the angles of the triangles instead of summing the angles in the pentagon.

Three of the students' justifying moves existed jointly with dialogic moves, which indicates that these justifying moves were posed in reaction to another student's idea. In all these instances, a student turn included DS (justifying move) and elaborating (dialogic move). For example, when Karin had divided a pentagon into five triangles, Angela suggested improving the drawing by removing one line from the figure (line 160). Micah continued by saying "Take that going across. … Now there are three" (lines 161–163). Thus, Angela's and Micah's moves were elaborating Karin's idea and describing support for the claim that the smallest number of triangles is three.

Structuring Argumentation and Using Communicative Approaches

The argumentation in the provided data consists of two strands that were connected in the end. In the first strand (day 1), a formula $(n - 2)180$ was justified inductively by measuring angles in several polygons. In the second strand (day 2), polygons were divided into triangles, and this was used to conclude the sum of interior angles in each of the examined polygons. Finally, the strands were connected by noticing that the number of triangles examined in the second strand corresponds to the expression $n - 2$ in the formula presented in the first strand. The teacher structured the second strand by dividing it into several steps where the class cumulatively achieved more information about the triangle method. The steps, students' dialogic and justifying moves within the steps, as well as the communicative approach of the teacher within the steps, are presented in Table 3.

Table 3 The steps in argumentation, students' dialogic and justifying moves within the steps, and the communicative approaches within the steps

#	Step	n-gon	lines	Students' moves	Approach
0	Formulas for the sum of interior angles	4–7, n	1–37	AR	D/I
1	Number of triangles [quadrilateral]	4	100–124	Elaborating DS and elaborating	D/I
2	Smallest number of triangles [quadrilateral]	4	124–132	DS	A/I
3	Sum of interior angles [quadrilateral]	4	132–142	–	A/I
4	Smallest number of triangles [pentagon]	5	143–165	DS Commenting Two DS and elaborating	D/I
5	Sum of interior angles [pentagon]	5	166–175	Two DS	A/I
6	Smallest number of triangles [hexagon]	6	176–219	Three DS Two commenting	D/I
7	Sum of interior angles [hexagon]	6	219–221	–	A/N-I
8	Collecting results in a table and noticing patterns	3–7	222–248	–	A/I
9	Connecting the table to the formulas in step 0		248–278	Commenting Two elaborating	D/I

In step 0 (day 1), one student, Travis, presented an argument including AR. There were no students' dialogic moves. The communicative approach was dialogic/interactive as the teacher elicited Travis's ideas and allowed Travis to present his own idea. In addition, after Travis had presented his idea, the teacher gave a small talk in which she included Travis's and Kevin's ideas (lines 34–37).

In steps 1–9 (day 2), the teacher was most likely composing a coherent line of argument aiming to arrive to the same formula that was presented in step 0. However, the dialogue unfolded in such a way that students could be thinking about what was done in individual steps detached from the larger argument under construction.

In step 1, the teacher posed the question "How many triangles can I divide this [quadrilateral] up into?" (line 117). Angela responded to the open question by beginning to think about the number of triangles and drawing diagonals to the quadrilateral. Martin elaborated Angela's idea by adding that there are infinitely many triangles, which Angela elaborated further by drawing more lines intersecting in a same point. Thus, the step included dialogicity in the sense of students' dialogic moves. In addition, the teacher used dialogic/interactive approach as she elicited students' ideas by open questions.

In step 2, the teacher asked about the smallest number of triangles that the quadrilateral could be divided into. This question can be considered as a closed question because in the case of one particular quadrilateral, the answer is so obvious that the teacher is most likely expecting a certain answer. Thus, the teacher used authoritative/interactive approach as she introduced the idea of having two triangles through closed questions. A student drew the two triangles to describe support for the claim. No reasoning was provided. In this case, the claim was so evident and followed so directly from the drawing that there was no genuine need for articulating reasoning.

In step 3, the teacher asked about the sum of the angles in the quadrilateral. However, the teacher did this by first asking the interior angle sum of any triangle. Then, she pointed out that the angle sum in one of the triangles within the quadrilateral is 180 and asked the angle sum of the other triangle. After the students responded 180, the teacher asked the interior angle sum of the entire polygon. The students only needed to complete the teacher's idea by adding silently 180 and 180 and responding 360. Thus, the teacher used authoritative/interactive approach within this step. This may have also affected that there were no dialogic nor justifying student moves as students only needed to answer questions concerning facts.

In step 4, the teacher started to examine a pentagon and asked for the smallest number of triangles into which the pentagon can be divided. Because the question concerns pentagons, it is not as simple as in the case of quadrilateral. Thus, the question can be considered as an open question. Indeed, it happened that the first idea presented by Karin was not complete and other students elaborated it. The teacher used dialogic/interactive approach as she elicited students' ideas openly without evaluating Karin's idea by herself. Thus, the ideas originated from students, and furthermore, were posed in a dialogic manner.

In step 5, the teacher led the students to add the sum of the angles of the triangles in the pentagon much in the same way as in step 3. Thus, the teacher used authoritative/interactive approach within this step as the ideas originated from the teacher.

The difference between steps 3 and 5 is that in step 5, the teacher did not only ask for the answer but how the answer could be found. This resulted in two students describing shortly support for the answer: "three sixty plus" (line 169) and "you add them" (line 171). However, the teacher did not focus students to think about the reasons why the sum of the angle sums of the triangles gives the sum of the angles in the pentagon.

In step 6, the teacher asked about the number of triangles in a hexagon. Now the teacher asked the question before drawing the hexagon. The question was open as the students could be thinking in different ways. Indeed, a student answered the question, not based on a figure but based on the pattern she noticed in the already examined polygons: "because there's two there, three there" (line 189). The teacher used dialogic/interactive approach within this step as the main ideas originated from students.

In step 7, the teacher presented how the sum of the angles of the hexagon can be calculated using the triangles. The teacher completed this step alone, and, thus, the teacher used authoritative/non-interactive approach.

In step 8, the teacher collected the information into a table by requesting students to fill in the facts. Thus, the teacher used authoritative/interactive approach. There were no student dialogic nor justifying moves in this episode. Instead of argumentation, the teacher focused on filling the table. However, there were two important moments as regards argumentation. In the beginning of this step, a student said that there is a pattern (line 222). This idea was put aside while building the table. Then, toward the end of step 8, the teacher seemed to promote observing a pattern from the table by saying "Okay, let's look at this for a second" (line 245). At that point, Micah proposed to add a row in the table: "Seven (-gon), five (triangles), nine hundred (sum of angles)" (line 246). Because this case was not yet drawn, it seems that Micah generalized based on a pattern that he had noticed. However, although the teacher recorded this as an additional row in the table, this pattern was not discussed more, and therefore the class did not come to know about Micah's pattern. For example, the pattern could have been recursive one in which the number of triangles is increasing by one and the sum of the angles is increasing by 180°. Alternatively, the pattern could have been more like a function rule in which number of triangles is two less than the number of sides in a polygon and the sum of the angles is 180° times the number of triangles. Nevertheless, it seems that because of the teacher's authoritative approach following the agenda of filling the table, Micah's pattern did not come into the discussion.

In step 9, the teacher decided to connect the table to the formula constructed in the first lesson (as opposed to constructing a new rule based on using the triangles and/or the table). The teacher first asked what functions the students constructed in the first lesson and then where the expression $n - 2$ in one of the students' functions appeared in the table. Kyle proposed that it is the number of triangles (line 268). From the students' perspective, Kyle was just proposing an answer to where the expression $n - 2$ appeared in the table. Thus, Kyle posed a claim. In the next turn, the teacher supported Kyle's claim by showing that $n - 2$ is actually calculated in two rows of the table (e.g., "I have three sides; three minus two is 1." (line 269–270)).

Because the teacher was the one who supported the claim, this step did not include students' justifying moves. The teacher used mainly a dialogic/interactive approach in step 9 because the teacher first brought two students' functions to the discussion and then let the students propose a connection between $n – 2$ and the table.

Discussion

Dialogic Argumentation and Teacher Support

Based on the results, students made some dialogic moves and justifying moves. The dialogic moves were of two types, elaborating and commenting on others' ideas. Challenging others' ideas, asking questions about others' ideas and responding to other students' questions did not occur. One of the justifying moves was articulating reasoning (AR), and others were describing support (DS).

The teacher structured argumentation by sequencing it into steps so that each step established a new piece of information. The teacher used different communicative approaches in the steps. In the following, I discuss how structuring and communicative approach affected students' dialogic and justifying moves.

Structuring argumentation in steps divided the argumentation into smaller pieces that, from the students' perspective, were answered separately one at a time. For this reason, the students only needed to provide support for the small pieces (e.g., drawing a diagonal to show that a quadrilateral can be divided in two triangles). Thus, the structuring seemed to simplify the claims so that there was no genuine need to articulate reasoning. The only articulated reasoning existed in step 0. That step differed from the other steps in the sense that Travis was presenting a complete argument that he had worked on. Structuring argumentation in steps may also explain why the students did not ask questions of or challenge others' ideas. If the achieved step is small, the presented ideas may be so clear that there is no need to ask questions or challenge it.

In addition, the teacher's communicative approach within the steps affected the argumentation. As expected, students' dialogic moves existed in the steps in which the approach was dialogic/interactive. Because dialogic teacher talk does not evaluate students' ideas but rather explores them (Mortimer & Scott, 2003), dialogic teacher talk creates a space for students to explore presented ideas. For example, in step 4, the teacher received Karin's imperfect drawing neutrally and when Martin wanted to make a different drawing, the teacher asked, "What's your problem with this one?" (line 154), which supported students to explore and build on Karin's idea. However, dialogic/interactive approach does not ensure that students' engage with others' ideas (e.g., step 0).

Besides enabling students' dialogic moves, dialogic/interactive approach enabled student moves that were both dialogic and justifying. These instances can be seen as the heart of dialogic argumentation in the analyzed data as the students were engaging with other students' ideas (Webb et al., 2014) and at the same time produced evidence. These instances existed in steps 1 and 4 that included situations where a student responded to the teacher's open question by proposing something original and the teacher received this neutrally. In step 1, Martin proposed infinitely many triangles, and Angela elaborated this claim by describing support for it. Similarly, in step 4, Angela and Micah improved Karin's imperfect support for a claim.

On the other hand, the teacher's authoritative approach may reduce students' justifying moves. In steps 3, 7, and 8, the teacher controlled the discussion so much that there was no space for students to contribute more than giving factual answers to the teacher's questions. Here the teacher used authoritative/interactive approach by means of a funneling pattern (Wood, 1998) so that the students did not need to consider the actual argument. In steps 2 and 5, students provided support for a claim, but the supports were straightforward responses to the teacher's initiations (drawing a diagonal and adding three 180s, mentioned by the teacher).

Besides affecting argumentation within the steps, the structuring affected the whole argumentation chain that consisted of the steps 0–9. The sequence of steps seemed to be strictly controlled by the teacher so that the steps funneled (Wood, 1998) students toward the teacher's aim. Thus, besides funneling within some of the steps, the teacher funneled the argumentation by laying out the steps. The students could be only thinking about one particular step without considering where the steps are leading. Things might have been different if, in step 8, the teacher had continued to explore Micah's pattern instead of reminding students of the functions that were constructed previously. Using the terms of Wood (1998), the teacher could have focused on Micah's pattern instead of funneling students to connect one component of the previously constructed rule to the table. This alternate move could have led to inductive or generic argument (Reid & Knipping, 2010) depending on whether a common property from the examined cases was generalized or if one of the examined cases was used as a generic case.

Dialogic Argumentation in Studying Mathematical Argumentation

In this book, mathematical argumentation is defined as "the process of making mathematical claims and providing evidence to support them" (see "Argumentation" in the introductory chapter by Staples and Conner, this volume). According to the definition, making claims without supporting them with evidence is not argumentation. In line with this, focusing on instances of students describing support or articulating reasoning helped to recognize crucial aspects of students' argumentation in this study.

Another important feature of the definition is that it emphasizes argumentation as a process. In line with this, I analyzed argumentation as it evolved, considering students' and the teacher's turns in relation to each other and examining the sequence of several steps into which the teacher structured the argumentation.

The provided definition of argumentation does not explicitly emphasize dialogicity. However, when argumentation is understood as a process, dialogicity is an essential ingredient in the process. Thus, a focus on dialogic argumentation enriches the analysis of argumentation. In this study, identifying students' dialogic moves enabled recognizing those instances where students engaged with each other's ideas. Furthermore, overlapping dialogic and justifying moves enabled recognizing the three instances where a student described support for a claim by building on another student's idea.

Examining communicative approaches (Mortimer & Scott, 2003) brings in another perspective to dialogic argumentation. Differentiating between dialogic and authoritative approaches helps to conceptualize the role of students in argumentation. While a dialogic approach opens opportunities for students to engage in argumentation, an authoritative approach may constrain these opportunities. In an authoritative approach, students may just be providing facts as a response to the teacher's questions. For example, in the analyzed data, a student stated that the sum of the angles in any triangle is 180°. The teacher seemed to be heading toward justifying that the sum of interior angles in quadrilateral is 360°, but the student was just responding to the question about triangles. Thus, the student was not constructing a justification for the interior angle sum being 360° although his statement contained parts of the argument that the teacher had envisioned.

Considering how the teacher structures argumentation in several steps helps to differentiate between argumentation within the steps and argumentation composed of the steps. In this study, some episodes clearly included dialogic argumentation as the students engaged constructively with their peers' argumentation. In these episodes, different viewpoints were present, which is an essential feature of dialogicity (Alexander, 2004; Mortimer & Scott, 2003). However, when considering the whole sequence, it seemed to be dominated by the teacher's view although she included students' ideas when they fit the overall agenda. The approach of analyzing each step and the sequence of steps has some similarity to the argumentation diagrams containing subarguments (Conner et al., 2014). However, this study emphasizes that it is important to consider whether the steps are connected in a funneling manner or through a dialogic approach.

The definition of argumentation provided in this book emphasizes that argumentation is a process. This study has shown that dialogicity is a relevant aspect to be examined in this process. Considering students' dialogic moves helps to understand how, for example, supporting evidence is constructed as an elaboration of another student's idea. In addition, examining how a teacher structures argumentation and uses communicative approaches helps to understand how the teacher controls the process and how students' ideas steer the process forward.

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Justification in the Context of High School: Co-constructing Content and Process

Jill Newton and Erna Yackel

Introduction

In this chapter, we investigate the transcript from Ms. Bell's mathematics class through the lens of "mathematical justification," defined for the purposes of this book as the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense (Bieda & Staples, 2020). In this definition, two closely related verbs, "supporting" and "explaining," are used to describe justifying, the key verb under investigation. The use of these other verbs in the definition, in some ways, underscores the complex relationships among several of the constructs highlighted in this book and foreshadows the challenges of conducting analyses in which the goal is to isolate or foreground one of them, in this case, mathematical justification. Even acknowledging these challenges, it is a worthwhile endeavor to examine these constructs to clarify their connections and distinguishing features.

In addition to the book's definition of mathematical justification, we drew on ideas from Cobb et al. (1992). They investigated, using a micro-level analysis of teacher and student turns, two classroom communities, noting the use of mathematical explanations and justifications during the lessons. They distinguished between the two classrooms in terms of the ways in which mathematical ideas were constituted, concluding that students in one class had opportunities to learn mathematics

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with more understanding than those in the other class. Important for our work from this study was the authors' differentiation between *situation for explanation* and *situation for justification*. They recognized both explaining and justifying as collective activities and characterized the purpose of explaining as clarification in contrast to the purpose of justifying as accountability. For them, the primary difference between the two resulted from whether or not the approach or solution was challenged; the perception of challenge of import was that of the person presenting their work. If the presenter perceived a challenge, the situation shifted from one of explanation to one of justification.

Theoretical Perspectives

Blumer (1969) posited that symbolic interactionism rests on three premises: (1) human beings act toward things on the basis of the meanings that things have for them; (2) the meaning of such things is derived from, or arises out of, the social interaction that one has with one's fellows; and (3) these meanings are handled in, and modified through, an interpretative process used by the person in dealing with the things he encounters (p. 2). Here we talk about the second and third premises in slightly different terms, as *co-construction* of meaning; however, the foundations of the ideas are found here. If justification is, as described earlier, a practice that requires an audience (i.e., someone to challenge), then justifications are necessarily co-constructed. Yackel (2001, 2004) used Blumer's (1969) symbolic interactionism perspective, along with Toulmin's (1969) argumentation scheme, and Krummheuer's (1995) elaboration of these approaches into mathematics education to describe explanation, justification, and argumentation in a variety of mathematics classrooms, including elementary contexts and a university-level differential equations course. Yackel's use of these multiple perspectives lends support to the previously mentioned notion that the interactions among these constructs are complex. Yackel also drew on her earlier work with Cobb (Yackel & Cobb, 1996), which differentiated between social norms (e.g., students are expected to explain and justify their thinking) and sociomathematical norms (e.g., what counts as an acceptable mathematical explanation or justification). That is, sociomathematical norms are "specific to the mathematical aspects of students' activity" (p. 458).

Staples et al. (2012) identified justification as a core mathematical practice, emphasizing its role as both a disciplinary practice and a learning practice. "As a learning practice, justification is a means by which students enhance their understanding of mathematics and their proficiency at doing mathematics; it is a means to learn and do mathematics" (p. 447). They investigated teachers' perceptions of the purposes of justification in the middle school mathematics classroom and identified the following purposes: (a) promoting conceptual understanding, (b) fostering valued math skills and dispositions, (c) assessing – displaying and monitoring, (d) fostering valued lifelong skills and dispositions, (e) managing diversity, and (f) influencing social relationships. They noted "strong overlap with the mathematician community's purposes of explanation and incorporation" and the absence of "the purpose[s] of verification, systematization and discovery" (p. 458). In their reflections on the differences between the mathematician's purposes and the learning purposes, they proposed that the purposes of justification were closely related to the overall purposes of the community.

Like Yackel, Cobb, and colleagues, we acknowledge the collective nature of explanation and justification and seek to investigate the social and sociomathematical norms at play in Ms. Bell's mathematics classroom. In the spirit of Blumer and Krummheuer, we explored the co-construction of explanation and justification as the students make sense of the mathematics at hand, i.e., finding the angle sum of a polygon. We also attended to how explanation and justification were enacted as both disciplinary practices and learning practices as described by Staples and colleagues. We strived to maintain our focus on justification, that is, what was or was not being justified, how was it being justified, and why was it being justified. The goal for our analysis was to understand how both justification and the associated mathematical ideas were co-constructed in this classroom on these two days. We recognize that the norms that we discuss in this chapter were not created on these 2 days in isolation; developing norms is an ongoing and interactional process. However, even in this short transcript, we find evidence of norms that were operative in the classroom. In our analysis, we were guided by three questions: (1) how were norms for justification enacted and co-constructed? (2) which mathematical ideas were justified and which mathematical ideas were left unjustified? and (3) to what extent were the members of the classroom community convinced by the justifications provided?

Methods

In an effort to answer our guiding questions, we searched the transcript for instances in which members of the mathematics classroom community used justification to co-construct meaning and develop mathematical understandings. In our efforts to conceptualize the classroom community of learners, we focused not only on the pedagogical strategies of Ms. Bell but also on the co-construction of meaning by all members of the community (i.e., students and teacher). We reviewed the transcript line by line using a symbolic interactionism perspective, noting when participants explained, justified, or asked questions that led to explanations or justifications. Students' meaning making, highlighted in the social interactionism perspective, relies on both the explanations and justifications shared in the classroom and iteratively contributes to these explanations and justifications through continual co-construction of mathematical ideas. In addition, we highlighted instances that suggested particular classroom norms (social or sociomathematical) related to the practice of justification.

We used the following definitions to guide our identification of particular constructs related to the co-construction of justification in a mathematics classroom:

- Social Norm: A norm which is promoted in the classroom to sustain classroom microcultures (e.g., students are expected to explain and justify their thinking) (Yackel & Cobb, 1996).
- Sociomathematical Norm: A "social norm" that is specific to the mathematical aspects of students' activity (e.g., what counts as an acceptable mathematical explanation or justification) (Yackel & Cobb, 1996).
- Justification: The process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense (Bieda & Staples, 2020).
- Situation for Explanation: A situation in which students "explicate their interpretations, validate what they have made explicit, or discuss the legitimacy of particular mathematical constructions" (Cobb et al., 1992, p. 577).
- Situation for Justification: A "situation for explanation" in which an approach or solution has been challenged from the perspective of the person presenting their work (Cobb et al., 1992).

As a note of clarification, what a student offers as part of a situation for explanation or a situation for justification may or may not be a justification by the given definition (Bieda & Staples, 2020). In looking at the definitions, we note that, as situations for explanation include validating and discussing the legitimacy of ideas, these situations can prompt students to offer a justification by the given definition, though that is not a requirement. Similarly, as a situation for justification requires a perceived challenge, we would expect many student contributions in these situations to be justifications, but they do not have to be. The nature of student contributions in these situations for explanation and situations for justification in any one classroom will vary but likely be patterned and would depend on, and reflect, the norms and sociomathematical norms.

To examine the enactment and co-construction of social and sociomathematical norms (i.e., Question 1), we noted who was talking, who responded to whom, and how these interactions led to the co-construction of situations for explanation and justification. We were interested in patterns across the interactions as we sought to identify norms which had been established in the classroom community. In order to gain insights into the level of justification of particular mathematical claims (i.e., Question 2), we attended to the mathematical ideas that were presented and if and how they were taken up by Ms. Bell and the students, including if some claims were accepted with less justification than others. For Question 3 (i.e., how were the community members convinced by the justifications), we documented students' actions and language which indicated their level of acceptance of claims. We also searched for particular instances of co-construction of justification (how it was being developed as a disciplinary practice) and co-construction of mathematical knowledge (justification as a learning practice), while at the same time taking a more holistic view of the classroom culture and the presence of both social and sociomathematical norms. In the next sections, we discuss our findings related to the guiding questions and provide instances in which community members interacted with one another to co-construct meaning and justify their mathematical claims.

Findings

How Were Norms for Justification Enacted and Co-constructed?

Evidence related to norms for mathematical explanation and justification was found throughout the transcript. Here, we highlight three salient aspects of these norms: (a) students' willingness and eagerness to explain and justify their strategies and solutions, (b) use of multiple mathematical representations, and (c) concerted effort to co-construct mathematical ideas.

First, there was an implicit expectation for students to share, explain, and/or justify their thinking as well as to ask questions about, or challenge, the explanations and justifications of others; evidence was present throughout the transcript. An important norm seen throughout the 2 days was that students were aware that they should contribute to the conversation whenever they had an idea; that is, participation and co-construction were expected and valued. According to Yackel and Cobb (1996), this is a social norm as students explain and justify their answers in classrooms beyond mathematics classrooms. Although there were more teacher turns than student turns, many students were engaged in the co-construction of mathematical meaning. There was also some discussion about what to share, who should share, and why they should be selected to share their work. The process of co-construction became especially salient as we selected excerpts for inclusion and were unable to shorten them by eliminating lines/turns, thus supporting that each interaction was an important part of the overall co-construction of mathematical meaning for the community. On Day 1 (Lines 1–13), students had just finished working with their groups to develop a formula for finding the sum of the measures of the interior angles of a polygon and were preparing to share their work with the whole class:

Teacher:	Does anyone want to come up and share how they got their formula?
Martin:	Me.
Teacher:	I'm going to let Travis do it because he is really proud.
Martin:	My formula is different.
Adam:	Ours is better.
Teacher:	Y'all's is better? Come over here, Travis. All right, everybody needs to pay attention to Travis, especially if you didn't get this yet. So you can go home and have this formula. I wanted you to explain how you got it, Travis.
Travis:	Okay.
Adam:	Hurry up before I explode.
Teacher [to Martin's group]:	And then maybe one of y'all can do it if you got it a different way.
Martin:	She said her way was different.
Teacher:	Okay. All right, so one of y'all will go up and show next.

In this excerpt, we saw Ms. Bell setting up a situation for explanation and both a willingness and eagerness from multiple students to share their ideas. Martin volunteered and stated that his formula was different, suggesting that "different" strategies are good ones to share and that the students in the class (at least Martin) had some notion of what it meant for something to be mathematically different. Adam followed up with "Ours is better" which is interesting both because he credited the work to his group using "ours" instead of "mine" and because he said that it was "better" which suggested that all strategies may not be seen as equally valuable to the community. Martin's and Adam's comments were moving into the realm of sociomathematical norms as they expressed ideas about what it meant to have "different" and "better" mathematical strategies. Adam went so far as to say "Hurry up before I explode," indicating that he was excited at the prospect of sharing his work and seemed to think he would get his chance. Ms. Bell confirmed this, suggesting that groups that did it "a different way" may have such an opportunity. This willingness and eagerness continued throughout the transcript, ultimately resulting in many students involved in the co-construction of meaning in a range of ways. We even saw students use language associated with amazement and wonder as the process unfolded on Day 2. For example, students collectively replied "Ohhh" when Ms. Bell showed another possible process for calculating the interior sum. A bit later, Brenda said, "Wow" as collaborative work continued among Adam, Angela, Martin, and Ms. Bell. Angela exclaimed "Oh, snap" to express her enthusiastic approval near the end of the transcript as the class co-constructed a table which tied together their work from Day 1 and a new strategy proposed by Ms. Bell.

The co-construction of justification and mathematical meaning in this classroom also included the use of multiple representations of the interior angles of polygons and their sums. The use of a range of mathematical representations seemed, from this transcript, to be a norm in the classroom—this would likely qualify as a sociomathematical norm given that there was discussion about how these various representations were both different from each other and supported one another. Prior to the beginning of the transcript on Day 1, the students had been collaborating in groups to determine the sum of the measures of the interior angles of polygons with the goal of writing a symbolic formula for any n-sided polygon (n-gon). Students were given a worksheet containing several polygons and a protractor to measure the angles. Additional polygons were later drawn on the whiteboard during the whole-class discussions on Days 1 and 2. Finally, on Day 2, yet another representation, a table, was co-constructed on the whiteboard in order to organize information (i.e., number of sides, number of triangles, and sum of interior angles). The extensive work related to mathematical representations on these 2 days established justification as both a disciplinary practice as students had opportunities to explain their work and answer questions about it, as well as a learning practice as, through these discussions, they co-constructed ideas about how to establish the formula for the sum of the interior angles of a polygon using multiple representations.

The students engaged in the co-construction of mathematical meaning throughout the transcript, interacting with both Ms. Bell and one another. Ms. Bell worked alongside Travis to publicly establish a formula on Day 1:

Travis:	All right. [Writing: 360 540 720 900. See Fig. 1.]
Teacher:	So what is this you're writing on the board?
Travis:	That was the numbers of all the—these A, B, C, and D
Teacher:	What numbers? The numbers of what?

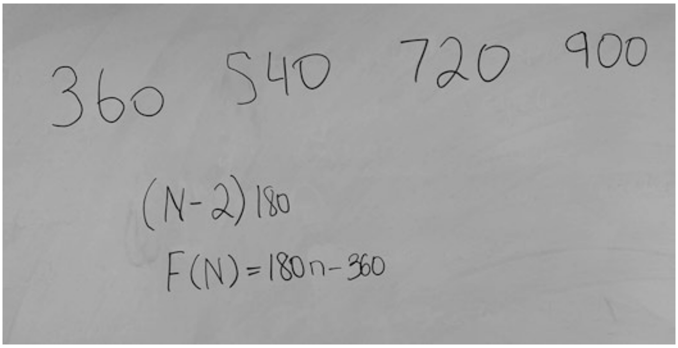

Fig. 1 Travis's formula written on board (transcript Figure 1)

Travis:	What do you mean? That was the
Teacher:	Sum of the?
Travis:	Sum of the interior angles.
Teacher:	Interior angles, okay.
Travis:	So to get these, it would be--you're just
Teacher:	What are the side lengths that correspond with that, Travis?
Travis:	*n* minus 2–*n* would be the number of sides minus two, times 180. So this 360, was 4-sided, so 4 minus 2 is 2, times 180 is 360. This was 5-sided, so you would do 5 minus 3—or 5 minus 2 is 3, times 180 is 540.
Teacher:	Okay.
Travis:	So it would just be the number of sides minus two
Teacher:	So did you find that by trial and error? You were just trying different?
Travis:	Yeah
Teacher:	Okay, cool. Go ahead and write your formula you got up there for it, a function in terms of *n*.
Travis:	All right. I'm going to write both of them. [Travis writes $(n–2)180$ and $F(n) =180n – 360$]
Teacher:	So that's what Travis originally had up top, and then Kevin said, 'hey, what about the distributive property? We can just multiply that through.' Good deal. Thanks, Travis.

Throughout Ms. Bell's efforts to co-construct this formula with the students, she asked many questions for what appeared to be multiple purposes but all with the ultimate goal of helping Travis explicate his group's strategy. Ms. Bell had established a situation for explanation; however, the question of whether this situation for explanation became a situation for justification is not obvious. As described earlier, Cobb et al. (1992) talked about this in terms of whether or not there was a challenge presented, thus moving from explanation to justification. What is interesting here is that even though an explicit challenge was not posed by another member of the community, the students often seemed to work under the impression that a real or perceived challenge existed when they were explaining their work. Therefore, the need to justify seemed to be understood and, in this case, was often facilitated by Ms. Bell's questioning, (e.g., "What numbers? The numbers of what?" "What are the side lengths that correspond with that, Travis?"). Ms. Bell also inquired about the process used to develop the formula, asking, “So did you find that by trial and error?” Travis answered in the affirmative, suggesting an inductive approach used by his group.

Another situation for justification and example of co-construction including Ms. Bell and at least eight students occurred on Day 2 (Lines 108–217) when students began to publicly draw lines in the polygons to explore the proposed strategy of dividing them into triangles. At one point in this interaction (Lines 145–172), they were engaged in a discussion about how to draw the lines:

Teacher: Thank you so much. Does someone want to come and draw the smallest amount of triangles we can make? Okay, come on Karin.

[Karin comes to the board and starts drawing diagonals that intersect in the pentagon.]

Teacher: Oh, okay.

Adam: Wait

Karin: Like, go from

Martin: Can I try?

Teacher: Karin, thanks [See Karin’s drawing in Figure 2]. Yes, Martin?

Martin: Can I try it?

Teacher: What's your problem with this one?

Martin: It looks funny.

Teacher: It just looks funny?

Adam: I know another way

Teacher: The smallest amount of triangles. This one has one, two, three, four, five triangles [points at each triangle in Figure 6, starting with the top of the diagram].

Angela: Take this one, going across. [points to one vertex and motions across the pentagon]

Micah: Take that going across

Teacher: Take one of these away?

Micah: Yeah, now there are three.

Teacher: Okay, so I had five triangles. How many do I have now? [Erases a diagonal. See Figure 7]

Adam: Three

Teacher: So the triangle sum of this [the topmost of the newly formed triangles] is one eighty, one eighty [writes one eighty inside each triangle], one eighty. So how would I find the interior angle sum of all those?

Angela: three sixty plus

Adam: Five forty

Martin: You add them.

Teacher: Add them together.

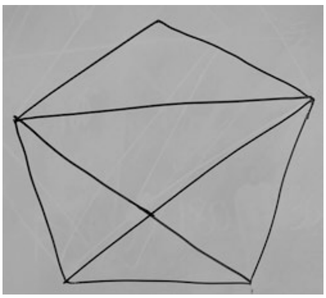

Fig. 2 Karin’s drawing of diagonals for the pentagon (transcript Figure 6)

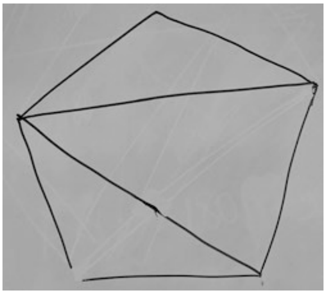

Fig. 3 Revised figure of pentagon with diagonals (transcript Figure 7)

Karin was asked to draw the minimum number of triangles in a pentagon. She began drawing intersecting diagonals. Ms. Bell said, “Oh, okay” followed by Adam saying, “Wait” and Martin asking, “Can I try?” Adam and Martin seemed to be challenging Karin’s work, establishing this as a situation for justification. Ms. Bell asked Martin, “What’s your problem with this one?” and a conversation ensued with students working together to figure out how the lines could and could not be drawn in the pentagon to result in the minimum number of triangles as requested by Ms. Bell. The co-construction began with Karin’s figure that was used by the class throughout the discussion even though changes were made to it along the way. Micah and Angela proposed removing one of the diagonals that left the result in Fig. 3.

So far, we have focused on the co-construction of norms related to justification, highlighting the nature of these norms and recognizing the limitations of this small data set for drawing conclusions about the classroom community. Aspects of these justification norms included students’ willingness and eagerness to explain and justify their strategies and solutions, the use of multiple mathematical representations, and a concerted effort, by Ms. Bell and the students, to co-construct both the process of justification and the mathematical ideas under development.

Which Mathematical Ideas Were Justified, and Which Mathematical Ideas Were Left Unjustified?

In the process of identifying the mathematical ideas that the classroom community justified and those that were left unjustified (using the given definition), we noticed that, although the teacher and many of the students interacted in order to co-construct, explain, and justify ideas related to *how* to draw lines in polygons to end up with the minimum number of triangles, the question of *why* to draw these lines was left largely unexplored. Here, we use the cases of the quadrilateral, pentagon, and hexagon to illustrate how Ms. Bell and the students collaborated to establish the process for constructing the minimum number of triangles in a polygon. It was Ms. Bell who first called for the “smallest number of triangles I can cut this into”; in fact, she reiterated this request three more times on Day 2; this requirement was also mentioned by Travis twice. A method for drawing lines in order to achieve the minimum number of triangles was first discussed in relation to the quadrilateral (Lines 111–130):

Teacher:	So, exhibit A I think was something like that? A four-sided figure, right? [Ms. Bell draws a quadrilateral on the board.] Somebody come up and show me how I could divide this into triangles. Angela, come on up.
Angela:	How many?
Teacher:	Huh?
Angela:	How many?
Teacher:	How many triangles can I divide this up into? Well, that's kind of my question.
Angela:	Wait, that's kind of [Angela draws a diagonal across the quadrilateral.]
Teacher:	So, all right. [Angela draws in the second diagonal.]
Martin:	Infinitely many
Teacher:	Infinitely many. Okay. [Angela continues drawing in lines that go through the center of the quadrilateral (Figure 3).]
Martin:	Across the center
Teacher:	Okay. [laughter] That's really good. Okay, what is the smallest number of triangles I can cut this into? [Ms. Bell draws another quadrilateral like the first one.]
Martin:	Two
Teacher:	Okay, so draw those two. [Angela draws in a single diagonal (see Figure 4).] Okay. All right, so we're cut into two triangles. Could I have done it the other way?
Students:	Yes.
Teacher:	Doesn't matter.

Several issues arose in this interaction related to how to draw the lines. First, Angela drew a series of lines across the figure all going through the center (and therefore all intersecting) (see Fig. 4). Then, Ms. Bell pointed out the need for the “smallest number of triangles” and drew a new quadrilateral for Angela to try again. Angela drew one diagonal across the quadrilateral. Here again, Ms. Bell has provided a situation for explanation and Martin and Angela have engaged in this

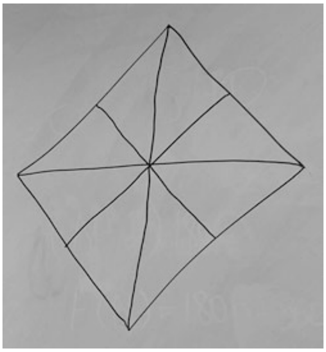

Fig. 4 Angela’s drawing of diagonals for the quadrilateral (transcript Figure 3)

discussion related to *how* to draw the lines. Notable, however, is the lack of any questions about *why* to draw the lines. That is, the students do not question why or provide information about the purpose of dividing the quadrilateral into triangles. Therefore, in the absence of a perceived or real challenge, this situation does not become a situation for justification.

The conversation related to drawing lines continued in the discussion of the pentagon. Ms. Bell again emphasized her desire for the "smallest amount of triangles," and Karin drew intersecting lines (see Fig. 2). As mentioned above, the community suggested removing lines until Fig. 3 was agreed upon as having the minimum number of triangles. In the pentagon case, the class again discussed how to draw the lines in order to end with three triangles (the minimum number using their conventions) and to add up the angle sum of the three triangles (180 + 180 + 180) to justify the angle sum of 540 degrees that they had calculated earlier. The class moved on to investigating a hexagon and additional considerations for drawing lines were discussed (Lines 194–217):

Teacher:	All right, someone want to come up here and draw that [hexagon] hasn't drawn before? Okay, Martin. Here you go. [Martin goes to the board and draws non-intersecting diagonals, see Figure 10.] Okay. Is that the only way I could have drawn those triangles?
Students:	No.
Angela:	You could [inaudible]
Adam:	You could have done them the other way.
Teacher:	There could have a lot of different ways to do it. You could have gone one right here, one right here, one right here. What's the important thing when you're drawing these triangles. What do you notice that we're not doing that maybe we were doing before?
Adam:	Crossing the lines
Travis:	But wouldn't the least amount of triangles, couldn't you draw just one line, that would be one triangle?
Teacher:	Where? If I drew one line here I could make one triangle? Where? Show me.
Travis:	I mean, if we're trying to find the least amount of triangles, if you just drew one line across the top, that would be one triangle.
Teacher:	Okay, but then, what is this shape going to be down here?
Travis:	Oh, I got it.
Teacher:	We're trying to divide the polygon into triangles.
Travis:	I get what you're trying to do.
Teacher:	Okay. I can see. I'm glad you said something; that's important to figure out. So how many triangles did we have here?
Students:	Four
Teacher:	Four [writes 4 triangles on board].

Interestingly, Martin did not draw all of his diagonals (see Fig. 6) from the same vertex (as the class did with the pentagon) which resulted in a figure which looked quite different. Perhaps noting this difference, Ms. Bell asked if that was the only way to draw the triangles and "What do you notice that we're not doing that maybe we were doing before?" Adam answered, "Crossing the lines." Travis then asked about just drawing one line to make one triangle at the top and Ms. Bell explicated, "We're trying to divide the polygon into triangles." During this co-construction of

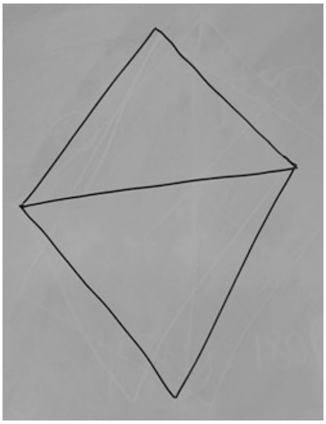

Fig. 5 Angela's revision to drawing diagonals of the quadrilateral (transcript Figure 4)

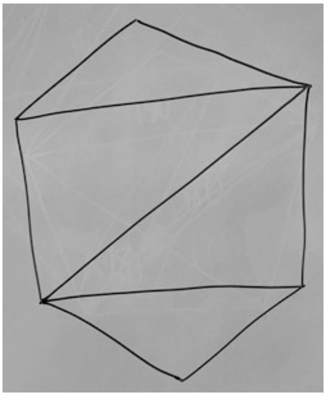

Fig. 6 Martin's drawing of nonintersecting diagonals (transcript Figure 10)

the hexagon case, Travis challenges the process, explicitly asking about the possibility of just drawing one line, establishing this as a situation for justification.

Throughout these whole-class discussions on Day 2, Ms. Bell and the students were co-constructing a process or set of rules for drawing the lines to divide the polygon into the minimum number of triangles. The students interacted with Ms. Bell and one another to establish this process, often making comments and asking questions to support or clarify an action or idea. The norms at play here seemed to be social as students shared their work but extended into sociomathematical as students were challenged and needed to justify their responses. This dynamic calls the question whether the norms in situations for explanation are largely social, but once there is perceived challenge and the situation becomes one requiring justification, then the norms become sociomathematical in nature, as these justifications require students to be familiar with acceptable forms of mathematical justification in their classroom.

While the class worked through how to draw in lines to get the minimum number of triangles, one thing that they did not establish (or justify) during these lessons was the relationship between the triangles and the sum of the interior angles of the polygon, that is, why did this process work? This question seems critical for exploration as all work subsequent to Ms. Bell stating this requirement was based on the fact that dividing the polygon into the minimum number of triangles was the only way to calculate its interior sum (and result in the "correct" solutions). In terms of the definition of justification under consideration in this chapter, this claim (that drawing the minimum number of triangles was required) was never explicitly supported, and a justification was neither requested nor given as to why the claim made sense. We do not know that this assumption was left unjustified indefinitely as we cannot see into future lessons for this classroom community, but at least during the parts of these two lessons that we have access to, this claim was not justified.

From the first time Ms. Bell mentioned that finding the smallest number of triangles was required (which was repeated multiple times), the claim was never explicitly questioned. Students did demonstrate some agency in that Angela drew intersecting lines resulting in many triangles (see Fig. 5) and, even after Ms. Bell had stated the requirement, Karin drew intersecting lines resulting in too many triangles (see Fig. 2). Martin questioned "At most or at least?" when Ms. Bell asked how many triangles she could get out of a six-sided polygon. Finally, Travis seemed not to have made the connection between drawing lines to create triangles and finding the sum of the interior angles of the polygon when he asked, "But wouldn't the least amount of triangles, couldn't you draw just one line, that would be one triangle?" In spite of the fact that the class did not have an explicit conversation about why the smallest number of triangles was required, the students seemed convinced that drawing the smallest number of triangles would lead to the correct calculation for the angle sum of the polygon, leaving us with questions about the factors that contributed to their persuasion.

To What Extent Were the Members of the Classroom Community Convinced by the Justifications Provided?

Even though the need for dividing the polygon into the minimum number of triangles was not fully explored or justified and the general case was never established, the students seemed, at the end of the transcript, to be satisfied with their formula for the sum of the interior angles of an n-gon. We propose several possible explanations for this satisfaction. Researchers (e.g., Chazan, 1993; Healy & Hoyles, 2000) have found, in a range of contexts, that empirical evidence is often more convincing than abstract arguments. This phenomenon, inductive reasoning as preferred over deductive reasoning, is likely at play here. However, at least two additional factors may be helping to build the convincing case for this particular classroom community.

One factor that may influence the students' confidence in their formula and results is that they approached the question in multiple ways that all led to the same result. They began by using a protractor to measure the angles of four types of polygons (quadrilateral, pentagon, hexagon, and heptagon) and used a "trial-and-error" method to write their formulas: $f(n) = 180(n-2)$ and $f(n) = 180n-360$. We know that at least two groups were able to share their results with the class; therefore, we have reason to believe that the community entered Day 2 with a shared formula for calculating the sum of the measures of the interior angles of a polygon. On Day 2, Ms. Bell introduced a different approach, and the community co-constructed the method for drawing lines to separate the polygon into triangles and then informally multiply this minimum number of triangles by 180 degrees. After some negotiation, their results were the same as those from Day 1. Adam asked, “Are we going to make an equation out of this?”; both Ms. Bell (twice) and Angela suggested that they already had one (the one established on Day 1). Ms. Bell then suggested using a table as a “different way of finding that equation,” saying that she “love[s] to make tables to find patterns.” The table was co-constructed by the class, and the formulas were discussed and reaffirmed (Lines 226–272). These various methods (i.e., measuring, using guess and check, writing a formula, separating polygons into triangles, making a table) all led to the same conclusion, thereby seemingly persuading the class of its accuracy and illustrating another type of co-constructed justification.

Finally, in spite of the collaborative nature of this classroom community, the mathematical authority still resided with Ms. Bell. This is not an evaluative statement, as students’ perceptions about who in the room knows the mathematics comes from years of experiences with adults as the knowers and students as the learners in most aspects of their lives; this phenomenon is certainly the norm in most schools. Ms. Bell was the one who initially suggested the strategy carried out on Day 2, saying “Actually, before that, I want to show you guys. I forgot about this. I don’t want to say, it’s kind of an easier way to think about finding the interior angles of any polygon.” In addition, Ms. Bell expressed support for the method throughout Day 2, saying, “Cool, right?”, “Oh snap” and asking “So do you guys like this way better?” It seems likely that the claim that a minimum number of triangles had to be drawn did not require support or justification in this classroom because the claim was authored by Ms. Bell. The empirical evidence, along with Ms. Bell’s support, and the multiple strategies used to confirm the results for the polygons combined to give the students confidence that they had justified the result. Of course, this formula is correct, and Ms. Bell’s method works; however, we are left with the question of whether the students know why the method works.

Discussion

The goal for our analysis was to understand how justification and the associated mathematical ideas were co-constructed in this classroom on these 2 days and to consider how a definition of justification supported and/or shaped that effort and the findings. Four major ideas emerged for me in this work as I co-constructed this analysis with Erna Yackel and wrote this chapter; I will speak only for myself in this section.

First, I return to the situations for explanation and situations for justification as described by Cobb et al. (1992). Several phenomena and dilemmas emerged from our analysis of the “situations” in the transcript. On the one hand, it seemed that situations for explanation precede situations for justification; that is, an explanation is necessary in order for a challenge to be presented or perceived. However, as mentioned in the findings, the line between the two types of situations was often blurred because challenges were rarely explicitly stated, and even when they were, they were subtle. More often, it seemed like the challenge was implicit and that students were working under the impression that a challenge was present; that is, the need to justify was understood. Perhaps the difficulty of identifying these implicit challenges provides support for a more inclusive definition of justification that alludes to both explaining and justifying such as the one proposed in this book: the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense. These implicit challenges (i.e., calls for justification) also seem to support the need for attention to sociomathematical norms as that which is "normal" within the mathematics community may be difficult to identify as the challenges are understood by the community members without being explicitly stated. In addition, I wondered about the actors involved in the construction of the situations. We suggested earlier that Ms. Bell was setting up a situation for explanation and the students were participating in the situation, but in hindsight, it is more likely that situations were co-constructed by the members of the classroom community. In fact discussions about explanations involved interactions among students and between students and the teacher. It is notable, however, that a key claim (i.e., the minimum number of triangles must be drawn in the polygon in the process of finding the sum of the interior angles) that was ultimately accepted by the community did not seem to require justification, perhaps because the claim was made and reiterated multiple times by Ms. Bell.

Second, we saw significant evidence in this transcript that both the co-construction of how, when, and why situations for justification occurred and the mathematical meaning making of, in this case, the sum of the measures of the interior angles of a polygon, are complex endeavors, made more complex because these are happening simultaneously. These complexities are inherent in learning situations in which multiple community members bring their ideas to the task at hand and the goal is some version of shared knowledge. Many times during these 2 days, Ms. Bell collaborated with multiple students, asking questions and commenting on their answers to help them develop mathematical ideas *and* develop justification skills. There is a reflexive relationship between the practice of justification and the learning of mathematics; each contributes to the development of the other, and neither exists without the other. Staples et al. (2012) suggested that introducing justification as a learning practice is likely more compelling to teachers than introducing justification solely as a disciplinary practice, because teachers value, first and foremost, student learning, and justification's role as a powerful pedagogical tool addresses this goal directly. Fortunately, however, at the same time justification is serving student learning, it is also developing students' understanding about the disciplinary practice, including the importance of its role in evaluating and verifying claims. In spite of these often competing and complex goals (Staples & Newton, 2016), navigating this terrain in which justification serves both learning and verification of claims is essential for the development of users and lovers of mathematics.

Third, as we reviewed this transcript, it became apparent that, although mathematical ideas and the practice of justification are co-constructed in classroom communities, the agency to both navigate and control the construction is not necessarily evenly distributed among the members. In mathematics classrooms we co-construct how to find angle sums, how to justify (or not), and ideas around the nature of mathematics as a sensemaking endeavor (or not). In spite of the collaborative nature of many classroom communities, the teacher, in this case Ms. Bell, nevertheless usually plays a key role in controlling the agenda and setting the goals, also determining the balance between justification as a learning and disciplinary practice. In these lessons, Ms. Bell chose to focus the class discussion on drawing in lines to create the minimum number of triangles in the polygon and using multiple mathematical representations to establish the necessary formula. Alternatively, or in addition to this, she might have focused the discussion on why triangles drawn within the polygon can be used to figure out the sum of the interior angles of the polygon. Ms. Bell had to make many decisions in action which ultimately influenced how the class proceeded. Inevitably, a different focus may have resulted in justifications of a different nature or of different mathematical claims. Although such instantaneous decisions are unavoidable, this suggests the need for teachers' purposeful goal setting related to justification, asking questions such as how can I develop classroom norms that support the co-construction of justifications and the associated mathematical ideas?

Finally, the analysis of a transcript of just 2 days in a mathematics classroom from the point of view of justification can give insights into how communities of learners co-construct norms for justification (process) and important mathematical ideas (content). Such analyses are useful for researchers and teachers alike, as we strive to better understand how to facilitate and support these co-constructions. For researchers, the complexity of deciding which constructs to highlight and how to incorporate closely related and tangential constructs presents a challenge as illustrated in this analysis. Ultimately, researchers make decisions based on the goals of the study—it may be that sometimes isolating a construct is essential in order to investigate how study participants conceptualize that construct and its roles. In other cases, for example, in much of Erna Yackel's work, the synergy of multiple constructs to tell the story of a complex classroom environment may be more helpful. In the case of this exploration, the consideration of aspects of Cobb et al.'s (1992) situations for explanation and justification in addition to the book's given definition of justification was useful to investigate boundaries between explanation and justification. The implicit nature of many of the challenges (calls for justification) in the classroom community lends support for the book's inclusive definition. For teachers, considering the roles of justification as described in the work of Staples and colleagues (2012) is helpful as they work to maximize the potential of justification as a practice to both enhance students' mathematical learning of content and develop their skills for evaluating and defending their claims in the mathematics classroom and beyond.

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Proof in the Context of High School: A First Approach Through Discussion, with Occasions and Missed Opportunities

Francesca Morselli

Introduction and Background

Mathematics educators agree on the crucial role of proof and proving in students' learning of mathematics. Accordingly, there is a wide range of research in the teaching and learning of proof (see the work by Stylianides et al. (2016) for a recent overview). In the literature, it is possible to find different conceptualizations of proof. As outlined by Balacheff (2002), it is important that researchers specify their perspective on proof so as to promote mutual understanding and comparison between studies. In this chapter, I refer to the following conceptualization of proof, as proposed by Weber (2014):

Mathematical proving is a process by which the prover generates a product that has either all or a significant subset of the following characteristics:

(1) A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true.
(2) A proof is a deductive argument that does not admit possible rebuttals.
(3) A proof is a transparent argument *where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation.
(4) A proof is a perspicuous argument *that provides the reader with an understanding of why a theorem is true*.
(5) A proof is an *argument within a representation system satisfying communal norms.*

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(6) A proof is an *argument that has been sanctioned by the* mathematical community. (p. 537)

This conceptualization seems promising because it takes into account both the mathematical perspective and the learner's perspective on proof. Moreover, the conceptualization considers both the process of proving and the final product and suggests the importance of considering the process of proving as a process that takes place in a social environment. The conceptualization does not refer to the ways of introducing proof in the classroom via specific activities (such as class discussion) and does not discuss specificities for the different actors that are involved in the class discussion.

To complement the theoretical tools at my disposal, I refer to research that addresses the issue of promoting the approach to proof in the classroom. Lin et al. (2012) present a series of principles for task design aimed at promoting conjecturing, proving, and the transition between conjecture and proof. In relation to conjecturing, the authors point out the importance of providing students with an opportunity to engage in (1) observing specific cases and generalizing; (2) constructing new knowledge based on prior knowledge; (3) transforming prior knowledge into a new statement; and (4) reflecting on the conjecturing process and on the produced conjectures. Concerning the transition from conjecture to proof, the authors point out that the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317), emphasizing that the acceptance/rejection is based on the logical structure of the argument and not on the authority of the instructor. Moreover, the teacher should propose tasks that raise students' need to prove.

In relation to proving, the authors argue that it is important to lead students (1) to express in different modes of argument representation (verbal arguments, symbolic notations, etc.); (2) to understand that "different modes of argumentation are appropriate for different types of statements" (p. 318); (3) to create and share their own proofs and to evaluate proofs produced by the teacher, thus "changing roles"; and (4) to become aware of the problem of sufficient and necessary[1] proof.

The aforementioned principles bring to the fore the crucial role of the teacher in mediating the approach to proof, both in the sense of understanding what a proof is and in learning how to prove (Balacheff, 1987). The mediating role of the teacher is deeply addressed by Cusi and Malara (2016), who, adopting a Vygotskian perspective (Vygotsky, 1978), study the role played by the teacher in effectively guiding students to develop those skills that are crucial for proving, with a special focus on the use of algebraic language as a proving tool. Starting from the analysis of real class episodes, the authors describe in detail the roles the teacher should play, with

[1] By sufficiency I mean that what stands as a (minimal) proof depends on the type of statement: valid universal statements require a general proof, for proving a valid existential statement a supportive numerical example is sufficient, to refute an invalid universal statement a counterexample is sufficient, to refute an invalid existential statement one must provide a general proof (Tabach et al., 2010).

the explicit aim of "making thinking visible" (Collins et al., 1989) and promoting meta-reflection on the actions that are performed. The first group of roles refers to the fact that the teacher acts as a learner who faces the problem and possibly becomes a "model" of effective actions for students. Such roles are (a) investigating subject and constituent part of the class in the research work, with the aim of stimulating students' attitude of research; (b) practical/strategic guide, with the aim of sharing the thinking processes and discussing the possible strategies to be activated; and (c) "activator" of interpretative processes and anticipating thoughts. The second group of roles refers to the fact that the teacher must also help students to reflect at meta-level, evidencing effective strategies adopted during the activity and promoting meta-reflections on the actions that were performed. Such roles are (d) reflective guide in identifying effective strategies and approaches during class activities; (e) maintaining a balance between semantic and syntactic aspects of algebraic language; and (f) acting both as an "activator" of reflective attitudes and as an "activator" of meta-cognitive actions.

The crucial role of the teacher is recognized also by studies on mathematical discussion, defined as a "polyphony of articulated voices on a mathematical object, that is one of the motives of the learning activities" (Bartolini Bussi, 1996, p. 16). The form of mathematical discussion I refer to is "the scientific debate that is introduced and orchestrated by the teacher on a common mathematical object in order to achieve a shared conclusion about the object that is debated upon (e.g. a solution of a problem)" (Bartolini Bussi, 1996, p. 17). The teacher is the initiator and the orchestrator of the discussion, as well as one of the voices of the polyphony. A crucial feature of the discussion is that students give their own contribution (voices), not merely answering the teacher's prompts, but proposing their own ideas and interacting with their mates, so as to create a real "polyphony." There are many types of discussion, according to the object of the debate: for the aim of this study, I refer to the resolution discussion (where all the students are engaged in the solution of a problem; the debate is aimed at working together towards a class solution) and the balance discussion (where the debate concerns the comparison of different solutions that were previously produced by the students, working individually or in small groups; the aim of the debate is to compare individual/group solutions so as to reach a class solution). Adopting again a Vygotskian perspective, the mathematical discussion is a promising context for the social construction of knowledge. In particular, it is important to study to which extent it can be a good context for the approach to proof.

To sum up, the theoretical framework starts from a conceptualization of proof as a cluster (Weber, 2014) and focuses on ways to promote the approach to proof in the classroom. The principles for task design (Lin et al., 2012) suggest how to plan promising activities, and the work of Cusi and Malara (2016) highlights that the teacher is essential in fostering a meaningful approach to proof. Although Cusi and Malara mainly refer to the use of algebra as a proving tool, I argue that their description of the teacher's roles is efficient also to analyze a task that encompasses algebraic and geometric representation systems.

My analysis of the corpus of data addresses the following overarching questions: Which characteristics of proof can be addressed in lower secondary school by means of a mathematical discussion? What is the role of the teacher in promoting the approach to proof?

More specifically, drawing on the abovementioned theoretical tools, I single out the following research questions:

1. Does the proposed task have some potential in promoting the approach to proof in the sense of Weber (2014)?
2. Does the planned activity align with the principles for task design, as discussed by Lin et al. (2012)?
3. Do some occasions to address the characteristics of proof as a cluster emerge during the teaching sequence?
4. What is the role of the teacher in promoting the approach to proof? Does the teacher play the roles that are outlined by Cusi and Malara (2016)?

Method

First, I performed an a priori analysis of the proposed task, in reference to the principles for task design, as discussed by Lin et al. (2012), and to the characteristics of proof as a cluster (Weber, 2014). The analysis is aimed at evidencing potentialities of the task in relation to the aim of promoting an approach to proof.

Afterwards, I turned to the data and performed a line-by-line qualitative analysis of the transcript, focusing on each discourse turn of the teacher and of the students. The teacher's interventions and students' utterances were analyzed in terms of occasions to address the characteristics of proof that are outlined by Weber (2014). Moreover, the teachers' moves were analyzed in terms of the teacher's roles as described by Cusi and Malara (2016). In the subsequent paragraphs, after the a priori analysis of the task, I present a narrative reconstruction of the whole session, considering the analysis of the interventions.

Results

Analysis of the Task

At first, the students are asked to explore a geometric situation. From that exploration, they have to generalize and write the function that gives the sum of interior angles of any *n*-sided polygon. They are also asked to justify their answer.

In the first part of the task, the work is performed in the geometric representation system (drawings of polygons). Afterwards, the students are asked to reflect on the measures of the interior angles and write a function. This request causes the shift to an arithmetic-algebraic representation system. The final justification is to be made again in the geometric representation system. There are two possible geometric justifications:

- Take a vertex of the polygon and connect it to all the vertices of the polygon (except the two consequent vertices). Then you have the polygon divided into $n-2$ triangles. For each triangle, the sum of the interior angles is 180 degrees. Then, the sum of the interior angles of the polygon is $(n-2)$ times 180 degrees.
- Take a point within the polygon and connect this point to all the vertices of the polygon. You get the polygon divided into as many triangles as are the sides of the polygon. For each triangle, the sum of interior angles is 180 degrees. Thus, the sum of the interior angles of the polygon is n times 180, minus the angles that are not adjacent to the sides of the polygon, which create a circle around the designated point. That is to say, n times 180 degrees minus 360 degrees.

The task is promising in terms of the approach to conjecture and proof. Referring to the principles that are discussed by Lin et al. (2012), the students are asked to work on specific polygons and measure their angles; in this way, they are led to observe specific cases so as to come to a generalization (the algebraic formula).

Afterwards, the task requires a transition from conjecture to proof. Lin et al. (2012) indicate the teacher should find ways to raise students' need for proof. Finally, in the proving process, it is important for the teacher to guide students to coordinate different modes of argument representation (verbal arguments, reference to geometric facts, symbolic notations).

Relevant features of the task, that make it interesting in reference to Weber's (2014) construct, are the following:

- The necessary double generalization (one must find the sum of interior angles for all the polygons with a given number of sides; one must find the sum of interior angles for any number n of sides), that should pave the way to the need for proof, moving beyond empirical verification (one cannot measure all the interior angles!); this could lead to proof as a convincing argument (1).
- The possibility of constructing a deductive proof, drawing on previous knowledge (the sum of the interior angles of a triangle is 180 degrees) (2).
- The necessity of moving in a dialectic way between geometric and arithmetic-algebraic representations; this could also lead to the proof as an explaining argument (4).

The same features make the task complex to deal with: in order to construct a deductive and complete proof, many intermediate passages need to be made explicit. For instance, the fact that the minimum number of triangles a polygon can be divided into is $n-2$ should be explicitly addressed.

Analysis of the Class Discussion

The corpus of data refers to two sessions: the first session consists of a balance discussion (Bartolini Bussi, 1996), where students share their individual solutions to the generalization task; the second session is a resolution discussion (Bartolini Bussi, 1996), where the teacher should lead students towards the justification for the formula. In this section, I reconstruct a narration of the two sessions, focusing on the occasions to promote the approach to proof in the sense of Weber (2014). As an additional theoretical tool, I refer to the roles of the teacher as discussed by Cusi and Malara (2016).

Part 1: Discovering the Formula

In lines 1–37, that is to say, the whole transcript of day 1, the discussion is focused on the *discovery* of the formula for the sum of interior angles of a polygon of n sides. As we will see in detail, the teacher asks one student to describe the trial-and-error reasoning that led to discover the general formula. The work starts in the geometric system of representation but rapidly shifts to the arithmetic-algebraic one.

In lines 7–8, the teacher asks the students to explain how they got the formula ("*I wanted you to explain how you got it, Travis*"). Even if the task requires also a justification, the student just describes the process that led to the discovery of the formula. The difference between procedural description and justification is crucial and should be addressed explicitly.

In line 11, the teacher values the existence of multiple solutions ("*And then maybe one of y'all can do it if you got it a different way*"). This prompt suggests a good opportunity to promote a reflection on the issue of multiple solutions. However, the issue is not properly developed in the subsequent part of the session.

In lines 15 and 17, the teacher intervenes to make clear to the other students what Travis is doing. The focus here is still on the *description* of the process, and the aim is to make the other students understand the generalization.

In line 23, the teacher asks Travis to make explicit the correspondence between the number of sides and the sum of interior angles ("*What are the side lengths that correspond with that, Travis?*"). This is coherent with the aim of identifying a function.

In line 24, Travis enounces the function he discovered and illustrates it by means of the examples. There is no description of how he found it, nor justification for it. This issue is raised by the teacher, who, in line 29, asks for information about the process of discovery of the function ("*So did you find that by trial and error? You were just trying different—?*"). In principle, the request could pave the way to a justification, but this is not the case, since the students found the function by arithmetic reasoning on data, not in reference to the figure. Since the students got the function by trial and error, the teacher could have posed another question (e.g., "Why does the formula of the function contain n−2? Is there any link with the geometric situation?"), to move the discussion towards the justification of the formula on a geometric basis.

In all this discussion from day 1, the focus of the teacher is involving all the students in understanding Travis's solution. The focus is not promoting the justification of the solution.

In line 33, Travis writes the formula for the function he discovered and then transforms it by means of the distributive property. The session ends with the two formulas written on the board. We could say that the justification was not addressed at all during this segment of day 1.

Part 2: Constructing Triangles

One might expect a second session starting with the teacher promoting the transition to proof, thus involving the students into a guided proof for the conjecture (the formula from day 1). However, lines 100 and 101 seem to suggest that the teacher has not the justification as a clear goal in mind, or at least is not efficient in communicating this goal to the students, since she declares she is going to present "an easier way to think about finding the interior angles of any polygon." The teacher does not activate the principle concerning the transition to proof, as we see students are not motivated to prove, and are not even made aware of the goal of proving. Thus, the teacher does not address proof as a convincing or perspicuous argument (Weber, 2014).

The term "proof" appears in lines 102–108, after the intervention of the student Adam:

102 Adam:	What are they talking about when they say conjecture?
103 Teacher:	Conjecture? Does anybody know what the word conjecture means?
104 Micah:	Proof.
105 Teacher:	Proof? What did you say?
106 Travis:	Proving, like, what you say.
107 Martin:	But it says "prove your conjecture," so are you going
108 Teacher:	But prove your conjecture [lots of students providing suggestions]. So you're saying hypothesis, your idea. Right? Okay. All right, we're going to get back to number three. And we're going to see another way to measure the interior angles without a protractor. So, exhibit A I think was something like that? A four-sided figure, right? [Ms. B draws figure on the board.] Somebody come up and show me how I could divide this into triangles. Angela, come on up.

In lines 108–109, the teacher intervenes to clarify the meaning of the term conjecture ("*So you're saying hypothesis, your idea. Right?*"), but she does not clarify what a proof is. This response could be in line with Weber (2014), who asserts that the concept of proof should not be addressed by direct instruction; rather, it should be learned "through practice in a community" (p. 358). On the other hand, practice should also be followed by a reflection on what was done, as stressed also by the second set of roles that are discussed by Cusi and Malara (2016).

Again in lines 110–111, the teacher seems to present the proof just as an alternative way to determine the measure of the sum of interior angles ("another way to measure the interior angles without a protractor"). No emphasis is given to the different function of proof in comparison to direct measuring. Referring to Weber's proof construct, we see here a missed opportunity to address proof as a convincing argument, or as a perspicuous argument.

As we may see in the subsequent excerpt, the teacher gives direct prompts and sometimes she has to refine the request to make the exploration more efficient, without sharing with the students the reason for the refinement. In this way, the teacher does not act as an investigating subject and part of the class, nor as a reflective guide (Cusi & Malara, 2016).

109 Teacher:	[…] Somebody come up and show me how I could divide this into triangles. Angela, come on up.
110 Angela:	How many?
111 Teacher:	Huh?
112 Angela:	How many?
113 Teacher:	How many triangles can I divide this up into? Well, that's kind of my question.
114 Angela:	Wait, that's kind of [Angela draws a diagonal across the quadrilateral.]
115 Teacher:	So, all right. [Angela draws in the second diagonal.]
116 Martin:	Infinitely many.
117 Teacher:	Infinitely many. Okay. [Angela continues drawing in lines that go through the center of the quadrilateral.]
118 Martin:	Across the center.
119 Teacher:	Okay. [laughter] That's really good. Okay, what is the smallest number of triangles I can cut this into? [Ms. Bell draws another quadrilateral like the first one.]
120 Martin:	Two.
121 Teacher:	Okay, so draw those two. [Angela draws in a single diagonal.] Okay. All right, so we're cut into two triangles. Could I have done it the other way?
122 Students:	Yes.
123 Teacher:	Doesn't matter.

Lines 112–113 open the exploration ("*how I could divide this into triangles*"). Asking to divide the quadrilateral into triangles by connecting one vertex to all the other nonconsecutive vertices would have led directly to the division into n–2 triangles.

In line 114, Angela raises a good point, asking how many triangles are to be traced. The impression is that, at first, the teacher had not clear in mind the issue of the minimum number of triangles (*Angela: "How many?"; Teacher: "Huh?"; Angela: "How many?"; Teacher: "How many triangles can I divide this up into? Well, that's kind of my question.*"), or she had not realized that the request had to be clarified to all the students.

In line 117, the teacher asks again to divide the polygon into triangles. In lines 124–125, the teacher asks to divide into the *smallest* number of triangles ("*Okay, what is the smallest number of triangles I can cut this into?*"). Dividing the polygon into the minimum number of triangles is crucial for the proof for the formula. However, students do not have this goal clearly in mind; towards this end, the teacher could have stated it from the beginning so as to guide in an efficient way the exploration.

In line 128, the teacher asks whether there are other ways of dividing the polygon into triangles ("*Could I have done it the other way*?"). Although we do not have direct information on the intention of the teacher, we may hypothesize that the teacher wanted to elicit other ways to divide the polygon into the minimum number of triangles: it is sufficient to choose another vertex and connect it to the other non-consecutive vertices, and the choice of the vertex does not affect the number of triangles one obtains. The question, formulated in a very open way, could also lead to an exploration on different ways to divide the polygon into triangles (for instance, intersecting diagonals). This exploration is not efficient to lead to the proof of the formula. In line 130, the teacher stops this trend of discussion ("*Doesn't matter*"), possibly realizing that opening the discussion on the construction of triangles too much deviates from the focus of the lesson. We may note that the teacher does not add any explanation on her decision to stop the exploration. She does not play the role of strategic guide (Cusi & Malara, 2016), since she does not share with the students her thinking processes. Moreover, we may say that addressing the issue of the minimum number of triangles would have contributed to the characteristic of proof as a transparent argument where a mathematician can fill every gap (Weber, 2014).

The subsequent excerpt shows a missed opportunity for the teacher to mediate the idea of proof as a perspicuous argument (Weber, 2014).

124 Teacher:	All right, so we divided this into two triangles, right? What—how many—what is the interior angle sum of any triangle?
125 Adam or Micah:	One eighty.
126 Teacher:	One eighty. So this triangle, all of its angles sum to one eighty. This triangle, what do all the angles sum to? [Ms. Bell points to the two triangles formed in the quadrilateral.]
127 Students:	One eighty.
128 Teacher:	So what is the interior angle sum of my entire polygon?
129 Students:	Three sixty.
130 Teacher:	Three sixty. [writes the number 360 on the board]
131 Students:	Ohhh.
132 Teacher:	Cool, right? Let's try a five-sided polygon if I can do it. [Ms. Bell draws a pentagon.]
133 Angela:	It's beautiful.

In line 142, students express surprise/satisfaction ("*Ohhh*") because they got the same result they obtained via direct measuring in the previous lesson. For the moment, what is evident is the fact that by drawing triangles into the polygon, one gets the same measure that was previously obtained by means of the protractor. In line 144 again, one student expresses aesthetic appreciation ("*It's beautiful*") for the geometric method. The teacher could have taken the opportunity to clarify that they were working towards a proof, which is a perspicuous argument (Weber, 2014) that provides the learner with an understanding of why the statement is true. This could have been done if she had played the role of strategic guide during the process, or reflective guide at the end of the process (Cusi & Malara, 2016).

In line 158, the teacher recalls the request: to find out the minimum number of triangles. The teacher clarifies the local goal of the activity (dividing the polygon into triangles), without any reference to the bigger goal (proving the conjecture about the sum of interior triangles). Again, she is not acting as a strategic guide (Cusi & Malara, 2016). Moreover, we may note that the approach of dividing into n triangles (with n = number of sides of the polygon) is a way to prove the general formula (see the second justification described in the a priori analysis of the task), but the teacher does not seem to be aware of this or does not want to address this.

In line 173, Brenda expresses surprise and satisfaction for the result. In line 175, Angela says the method is *easier* in comparison to using the protractor. This suggests that, for the students, the perceived aim of the activity is to find *another method*, not to prove it. Students' difficulty in grasping the aim of proving is possibly linked to what was mediated by the teacher with her interventions. Overall in the session, students seem to conceive the geometric method as another way of determining the measure of the sum of the interior angles. From the other side, it is evident that the teacher did not explicitly address the status of proof.

In line 180, the teacher asks the students to make a hypothesis about the minimum number of triangles before dividing a polygon ("*How many triangles—first of all, how many triangles, before I draw it, do you think I can get out of a six-sided polygon?*"). This is the crucial point: by promoting this anticipation, the teacher wants to promote a connection with the function they discovered in the previous session. We may note that this reflection could also promote a connection between algebraic and geometric representations.

134 Teacher:	How many triangles—first of all, how many triangles, before I draw it, do you think I can get out of a six-sided polygon?
135 Adam:	Wait, hold on.
136 Teacher:	Raise your hand. So only two people know how many triangles I could get out of a six-sided polygon?
137 Martin:	At most or at least?
138 Teacher:	At least.
139 Students:	Four.
140 Teacher:	Four. Why do you think four?
141 Angela:	Because there's two there, three there.
142 Teacher:	Let's write this down. [Ms. Bell writes under each picture as she speaks.]

143	We have two triangles, three triangles, and you're saying we're going to have four triangles with a … does anyone know what the name of a six-sided polygon is?
144 Students:	Hexagon.

From one side, it seems the teacher acts as an activator of anticipating thoughts (Cusi & Malara, 2016), but this would be true only if the anticipation had been possible in the same system of representation. Here the risk is to move again to an arithmetic reasoning, losing the reference to the geometric figure. This risk is evident in Angela's answer (line 189) that seems to be grounded on an arithmetic reasoning: she found a numerical pattern in the examples. This is correct, but it does not pave the way to proof.

145 Teacher:	There could have a lot of different ways to do it. You could have gone one right here, one right here, one right here. What's the important thing when you're drawing these triangles? What do you notice that we're not doing that maybe we were doing before?
146 Adam:	Crossing the lines.
147 Travis:	But wouldn't the least amount of triangles, couldn't you draw just one line, that would be one triangle?
148 Teacher:	Where? If I drew one line here I could make one triangle? Where? Show me.
149 Travis:	I mean, if we're trying to find the least amount of triangles, if you just drew one line across the top, that would be one triangle.
150 Teacher:	Okay, but then, what is this shape going to be down here?
151 Travis:	Oh, I got it.
152 Teacher:	We're trying to divide the polygon into triangles.
153 Travis:	I get what you're trying to do.
155 Teacher:	Okay. I can see. I'm glad you said something; that's important to figure out.
156	So how many triangles did we have here?
157 Students:	Four.

In line 203, the teacher wants to draw the attention of the students to the way of creating the smallest number of triangles, i.e., nonintersecting diagonals. Travis's intervention is interesting ("*I mean, if we're trying to find the least amount of triangles, if you just drew one line across the top, that would be one triangle*"): he is focusing on one constraint of the task (smallest number) but losing the control for the other one (all the figures must be triangles). The teacher recalls the second constraint (all the figures must be triangles) (line 212: "*We're trying to divide the polygon into triangles*") and gives Travis positive feedback (line 214: "*I'm glad you said something; that's important to figure out*"). The teacher's intervention is appropriate for the aim of constructing a proof as a perspicuous argument (Weber, 2014), but Travis's intervention suggests that possibly the students give their contributions to the local aim (constructing triangles) but are not aware of the bigger aim (proving).

Part 3: Towards a Formula (Again)

In the subsequent excerpt, the teacher promotes a reflection on the discovered measures so as to come to a generalized formula. The issue of "finding patterns" is introduced.

158 Adam:	Are we going to make an equation out of this?
159 Teacher:	I, maybe we already have one. I don't know. Let's see, so we've got one eighty, one eighty, one eighty, one eighty [Ms. B writes 180 inside each triangle].
160	Seven twenty is what I'm hearing.
161 Student:	There's a pattern.
162 Teacher:	Okay, there's a pattern. So let's come over here, because I love to make tables to find patterns.
163 Angela:	We already have the equation for this.
164 Teacher:	We already have the equation for this, right? Okay. So let's see if we can find a different way of finding that equation. […]

In line 218, Adam shows the understanding that they are working towards a generalization ("*Are we going to make an equation out of this?*"). However, his question seems to confirm that students perceive the activity as another way of finding the measure and not as a way of proving the previous conjecture about the measure. Again, the students give contributions for a local aim (finding a measure) but are not aware of the bigger aim (proving). If the teacher had clarified it, acting as a reflective guide (Cusi & Malara, 2016), she would have been more efficient in approaching proof as a perspicuous argument (Weber, 2014).

A crucial point is line 219. The teacher here seems reluctant: she does not want to say that they are going to prove the validity of the function they already found ("*I, maybe we already have one. I don't know.*"). Rather, she seems to want the students to find again the function and realize this is the same one. This connection is made by Angela (line 225: "*We already have the equation for this*") who points out that the equation is already at their disposal. The subsequent intervention of the teacher (lines 226–227) suggests that the aim is to find the same equation in a different way ("*So let's see if we can find a different way of finding that equation*"). Referring to the characteristics of proof (Weber, 2014), it is not clear what kind of proof (convincing? Transparent? Perspicuous?) she is going to promote in this way.

We may note that it seems the teacher would like to act as constituent part of the class (Cusi & Malara, 2016), without giving instructions and making the students explore and realize connection by themselves, but after some time she closes the exploration and "forces" one direction, without giving any explanation about this. This general approach is also evident in the last excerpt.

165 Teacher:	Okay, we're going to keep going? Seven, five, nine hundred [finishes writing entries in table; see figure 13]. I don't want to keep going any further. Tired. All right, so somebody tell me, what was our function again that we got for finding the sum of interior angles?

166 Angela: *n* minus two times one eighty.
167 Teacher: So with function notation, it was f of *n* equals *n* minus two times one eighty [Ms. Bell writes f(*n*) = *n*−2 (180)].
168 Adam: Parentheses are on the wrong side.
169 Teacher: I don't know what's wrong with me [Ms. Bell corrects board: f(*n*) = (*n*−2) 180]. What's your equation, Martin?
170 Martin: One eighty *x* minus three sixty.
171 Teacher: Is that a different equation?
172 Martin: No.
173 Travis: No, it's just the distributive property.
174 Teacher: What property?
175 Travis: The distributive property.
176 Teacher: Distributive property, right? All right, so, hmm. Where is this *n* minus two 264 [points at *n*−2 in equation written on the board] showing up over here [points at table on board]?
177 Students: The number of sides minus two. The triangles.
178 Teacher: Hmm?
179 Martin: The number of sides minus two.
180 Kyle: The number of sides minus two is the number of triangles.
181 Teacher: The number of sides minus two is the number of triangles. I have three sides 270 [points at top left corner of table], three minus two is one; I have one triangle [points 271 at top middle section of table]. Four sides [points at 4 in table], four minus two is two; 272 I have two triangles [points at 2 in table].

The intervention by the teacher in line 248 ("*I don't want to keep going any further. Tired*") suggests the need for a generalization. As we outlined in the a priori analysis of the task, two generalizations are present: one generalization is evident (number of sides) and the other one is implicit (many polygons with the same number of sides). The second generalization could suggest the need for a proof (one cannot check all the polygons). This direction of reflection is not taken by the teacher, however, who does not play the role of a reflective guide (Cusi & Malara, 2016).

The teacher (lines 249–250) asks students to recall the function they found in the previous session ("*what was our function again that we got for finding the sum of interior angles?*"). Rather than promoting a reflection on the fact that the function is actually efficient in describing the pattern coming from geometric exploration, the teacher asks the students to recall the other function (lines 255–256: "*What's your equation, Martin?*"). This moves the focus of the discussion towards algebraic manipulation rather than on the connection between geometry and algebra. Lines 261–262 confirm that the new focus is on algebraic functions and manipulation (*Teacher: "What property?"; Travis: "The distributive property"*). We may note that also in the subsequent part of the discussion the second equation is not worked on. This could have been interesting because it is connected with the other way of dividing the polygon into triangles (drawing lines to each vertex from a point in the polygon, obtaining as many triangles as the number of sides), that could lead to another geometric justification (see the a priori analysis of the task).

In the last part of the session, the teacher leads the students to connect the algebraic formula to the geometric representation (they realize that *n*-2 is the minimum number of triangles). No reflection is done on the geometric meaning of the second formula, that is obtained by applying the distributive property to the first one.

In line 263, the teacher promotes the connection between the function and the geometric representation of the problem ("*Where is this n minus two showing up over here?*"). One student, Kyle, evidences the connection between function and geometric representation (line 268: "*The number of sides minus two is the number of triangles*"). However, the teacher is not efficient in promoting a reflection on the relationship between algebra and geometry, apart from mirroring Kyle's intervention (line 269: "*The number of sides minus two is the number of triangles. I have three sides, three minus two is one; I have one triangle. Four sides, four minus two is two; I have two triangles*").

The teacher only partially acts as an activator of interpretative processes (Cusi & Malara, 2016), for instance, when she asks to connect the algebraic formula with the geometric representation (see lines 263–272). Moreover, she does not seem to value the geometric way of finding the function in comparison to the previous one (generalization from measures); correspondingly, she does not convey the idea of proof. She does not examine the characteristics that the second method has in comparison to the first one. The teacher could have done this at the end of the process, or elsewhere, in the form of meta-reflection, thus promoting a balance between semantic and syntactic aspects of algebraic language.

Discussion

Referring to the principles for task design concerning the approach to conjecturing and proving (Lin et al., 2012), the task was suitable to promote the approach to conjecture and proof, since it could have given the opportunity to engage in observation of regularities and patterns, construct starting from previous knowledge, and reflection, and to express in different modes of argument representation (verbal, geometric, algebraic, etc.). In reference to Weber's construct, the task offered occasions to address relevant features of proof as a cluster concept.

The qualitative analysis showed that not all the opportunities given by the task were caught, as I summarize here in reference to the features listed by Weber (2014):

1. "A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true." Conviction was already achieved during the first session, by means of direct measure and generalization. The students were satisfied with the generalization carried out in the arithmetic system of representation and did not express any doubt about it. The teacher started the second session by proposing "another way of measuring," not addressing the issue of proving, neither in the sense of convincing nor explaining.

2. "A proof is a deductive argument that does not admit possible rebuttals." As already observed, there is not a final institutionalization of the proof. Thus, there is not a final product in the form of deductive argument. The teacher did not focus on the "strength" of the argument and did not guide the students to organize it into a deductive form.
3. "A proof is a transparent argument where a mathematician can fill in every gap (given sufficient time and motivation), perhaps to the level of being a formal derivation." The argument that emerged during the discussion, even if taken as a finite product, has some missing steps: for instance, the fact that the minimum number of triangles is n-2 was not addressed.
4. "A proof is a perspicuous argument that provides the reader with an understanding of why a theorem is true." In principle, the reference to the decomposition of the polygon into triangles is a promising step towards an understanding. Anyway, the teacher did not convey the idea of understanding, since she presented the process as another way of measuring the sum of interior angles, without the protractor. Students' reactions to the method revealed satisfaction and surprise, not necessarily understanding.
5. "A proof is an argument within a representation system satisfying communal norms." The a priori analysis shows the necessity of a dialectic between arithmetic-algebraic and geometric systems of representation. The analysis of the transcript, on the contrary, evidences the lack of such a dialectic. In particular, a connection between algebraic formulas and geometric proof was not made. The proof here lives in two systems of representation: algebraic and geometric one. The shift from one register to the other one could have been further discussed in the second session.
6. "A proof is an argument that has been sanctioned by the mathematical community." The teacher seemed to aim at presenting one proof that is recognized by the mathematical community, but the impression is that she did not include the details of the proof; she only reproduced the main steps in the lesson (the treatment of the issue of the minimum number of triangles is neglected). Moreover, she did not take advantage of students' interventions, that could lead to interesting developments (for instance, the geometric meaning of the second algebraic formula).

It is necessary to hypothesize reasons why the approach to conjecture and proof was promising, but many occasions were not caught. The teacher has the responsibility of dealing with all the potentialities of the task and also tackling it with the unavoidable complexity. The qualitative analysis shows that the teacher did not seem to have clearly in mind the idea of promoting a transition from verification to proof; as a matter of fact, she presented the proof as *another way of finding the formula*, never, in these discussions, bringing to the fore the issue of explanation. The teacher did not turn her guided procedure into a deductive proof, since she did not institutionalize the process into a final product. Moreover, during the process, it is not clear if the significance of the minimum number of triangles was understood by the students, and the teacher dealt with only part of the meaning of the formula, promoting a connection between arithmetic-algebraic and geometric representations. In fact, she did not treat at all the geometric meaning of the second formula.

In reference to the set of roles that are illustrated by Cusi and Malara (2016), the analysis outlines occasions when the teacher could have played the role of strategic guide, sharing with the students the aim of proving and highlighting specific characteristics of the proving process. Also, in the role of activator of interpretative processes, the teacher could have promoted a deeper connection between the geometric situation and the two algebraic formulas. The second set of roles evidenced by Cusi and Malara (2016) was missing, namely, all roles aimed at promoting awareness of the process (reflective guide, “activator” of reflective attitudes, “activator” of meta-cognitive actions, maintainer of a balance between semantic and syntactic aspects of algebraic language). This absence has clear consequences for the way the first approach to proof and proving was promoted during the activity.

Conclusions

This chapter addresses the first approach to proof in lower secondary school, starting from a geometric problem. The principles for task design (Lin et al., 2012) confirmed the potentialities of the task in promoting the approach to conjecture and proof. The subsequent analysis aimed to understand whether all the potentialities were caught during the class discussion, and draw hypotheses about the reasons why some occasions were missed. Proof as a cluster concept, in the sense of Weber (2014), was used to highlight the characteristics of proof that were treated during the sessions and to point out the parts of the discussion that could have been further elaborated to address the characteristics of proof as a cluster. Additional theoretical tools were essential to perform a fine-grained analysis of the roles that were played (or not played) by the teacher (Cusi & Malara, 2016). Integrating the concept of proof by Weber and roles by Cusi and Malara helped to draw hypotheses about what did not work in the sessions. Importantly, the integration helped to move beyond the acknowledgment of missed occasions to a more specific description of what could have been done in relation to the many valued qualities of proof products. Even if it is not possible to establish direct connections, one can hypothesize that the non-activation of roles such as strategic guide and reflective guide had some influence on the limited attention to the characteristics of proof. For instance, if the teacher had acted more as a reflective guide, she could have made the students aware of the bigger goal (proving, not just finding a pattern in measures).

The analysis suggests that the task had a great potential to foster the approach to proof, but many occasions were not caught by the teacher or not managed to promote the approach to proof, thus turning into missed opportunities. This paves the way to a reflection on teacher education concerning proof and proving, as well as on teaching methods (such as orchestrating mathematical discussions), and on the specific roles the teacher should play to promote a meaningful approach to proof and proving.

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Reasoning Is in the Eye of the Lens-Holder: Observations Made Through the Lenses of Justification, Argumentation, and Proof at the High School Level

Michelle Cirillo and Dana C. Cox

Introduction

Working on this chapter reminded us of the fable of the six blind men and the elephant. Touching only one part of an elephant led each blind man to a different conclusion. In William J. Bennett's version of the fable, as told in *The Moral Compass*, the man who touched the tail concluded that an elephant is "long and round like a snake." While the man who touched a leg claimed, "It is tall and firm, like a tree." Still another man, who had only touched the elephant's tusk, believed them both to be wrong and shouted, "Hard and sharp, like a spear!" Eventually, a sighted man came along and told the men that they were all right, and they were all wrong, adding: "One man may not be able to find the whole truth by himself—just a small part of it. But if we work together, each adding our own piece to the whole, we can find wisdom" (Bennett, 1995, p. 193).

Likewise, in this section of the book, we see that when analyzing the same high school lesson transcript with somewhat similar research questions, three researchers come to different conclusions about the reasoning happening in the classroom. A justification researcher feels the tail and finds, "There are many situations for justification!" An argumentation specialist squeezes the trunk and says, "This is an

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inductive argument." Finally, after touching the leg, the proof analyst concludes, "This is not a proof!" We have much to learn by considering all three perspectives.

One way to think about reasoning, particularly at the high school level, is that it can take on many forms ranging from informal explanations and justifications to formal deduction, as well as inductive observations (National Council of Teachers of Mathematics [NCTM], 2009). Here, we review three analyses of the reasoning that occurred during a 2-day classroom episode conducted by three different researchers, through the lenses of justification, argumentation, or proof. We also consider the important role of the teacher, who, NCTM (2009) suggested, "must consistently support and encourage students' progress towards more sophisticated forms of reasoning" (p. 14).

Defining and Framing Different Forms of Reasoning

Although each researcher began his or her chapter with the definition provided by the book editors, they then layered in additional frameworks for their respective analyses. In the case of justification, Newton drew on ideas from Cobb et al. (1992) who distinguished between a "situation for explanation" and a "situation for justification." A "situation for justification" was described as a "situation for explanation" in which an approach or solution was challenged from the perspective of the person presenting his or her work (Cobb et al., 1992). Within this framing, justification was viewed as a co-constructed classroom practice and as a "learning practice" that serves as a means to learn and do mathematics (Staples et al., 2012). Through Newton's lens, justification only occurs when student presenters are pressed to justify their approach—whether explicitly or perceived—and they respond accordingly.

Hähkiöniemi examined argumentation through the lens of dialogic argumentation—a way of arguing such that the participants not only defend their own claims but also engage constructively with the argumentation of their peers (Nielsen, 2013). The two dimensions considered in the analyses were students' *dialogic* and *justifying* moves, where dialogic moves consist of questioning, challenging, elaborating, commenting, and responding, and justifying moves consist of describing support or articulating reasoning (Hähkiöniemi et al., under review).

After citing Weber's (2014) definition of proof, Morselli highlighted the importance of the task and the role of the teacher in proposing tasks that "raise students' need to prove." She also discussed the "mediating role of the teacher" (Cusi & Malara, 2016) in effectively guiding students to develop skills that are crucial for proving. Morselli acknowledged the important role of the teacher during mathematical discussions whereby *the teacher is the initiator and the orchestrator of the discussion*. However, she also acknowledged the importance of surfacing students' ideas and fostering engagement with one another's ideas.

A common element of these framings is that of community. In each case, the reasoning occurs within a community of learners working together to make sense of the mathematics. Another common thread is the idea that students should present their ideas to their classmates, and the class should interact with these ideas. It is interesting to note that the choice of task and "crucial role of the teacher" is far more prominent in the framework chosen by Morselli than the other authors. Morselli's choice is interesting because she extends beyond documenting the reasoning that is present and places responsibility for what is or is not there on the teacher. These additional frameworks were related to the questions posed by each researcher.

The Research Questions and Their Influence

To get a sense of what was of interest to the authors, we looked across the set of research questions. This review caused us to think about Cohen et al.'s (2003) instructional triangle. In Fig. 1, we classified the research questions as focusing predominantly on the ways in which the task, the student interactions, and the teacher influenced the extent to which reasoning was achieved. (P2 indicates Proof Question #2 posed by Morselli.) The location of the authors' research questions in the modified instructional triangle provided us with important information about each researcher's focus. These details have implications for the conclusions drawn by the researchers.

Morselli's research questions related to proof interrogated the potential of the task and ways in which the teacher influenced the extent to which proof occurred in the classroom. Hähkiöniemi's decision to use a dialogic argumentation framework focused the analysis on how students engaged with the ideas of others. Through this lens, the argumentation is nearly indistinguishable from a discourse analysis of the interactions. Newton's focus was similar to Hähkiöniemi's in that her evidence of

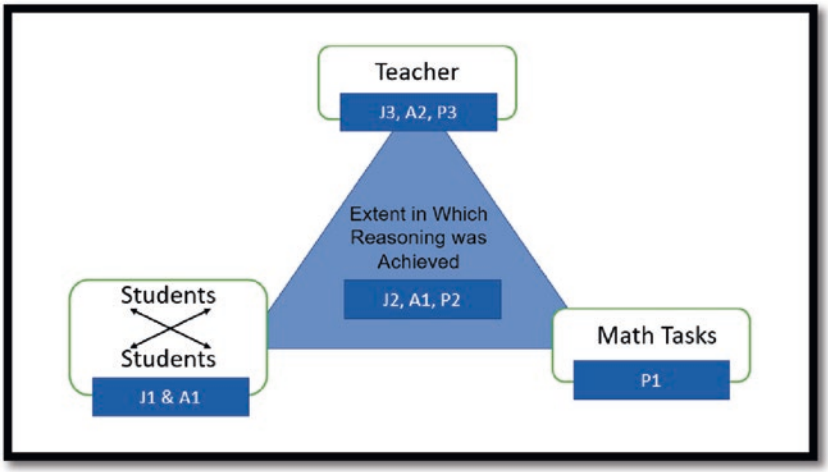

Fig. 1 The research questions set in the instructional triangle

justification was closely linked to an examination of student interactions. However, like Morselli, Newton also attributed the extent to which students engaged in justification to the teacher. She took a broader account in her analysis, however, by taking into consideration what we know about power structures and teachers' implicit status within classrooms.

Framing of the Task and the Classroom Activities on Days 1 and 2

Although the three researchers analyzed the same transcript and 2-day classroom episode, they framed both *the potential for* and *the enactment of* the classroom activity in different ways. These framings impacted the reasoning observed. The student task appears in Table 1. On day 1, students started out by measuring the interior angles of four irregular polygons with 4, 5, 6, and 7 sides.

The framing of the work done on the tasks across the 2 days varied by researcher. Like the characters in the opening fable, the researchers examined the same episode and came to different conclusions about the nature of the reasoning that occurred around the task. For example, Newton concluded that justification occurred through the process of co-constructing a "set of rules" for dividing each polygon into the minimum number of triangles. Hähkiöniemi claimed that through argumentation, the class determined that the sum of interior angles of any polygon could be found by $(n-2)180$, and that the 2 days of work were connected on day 2 when it was noticed that the minimum number of triangles was equivalent to the expression $n-2$ in the formula. In contrast, Morselli concluded that although the task had potential, not only did Ms. Bell not support the class to achieve proof, but she seemed to view the suggestion to "prove your conjecture" as simply "another way" to measure the sum of the angles. Considering these summaries together, we noticed that the researchers had different expectations regarding the level of rigor that the teacher could have or actually did achieve in the lesson. This notion is related to the teacher's role in supporting progress towards sophisticated forms of reasoning, as discussed in the introduction.

Table 1 Ms. Bell's day 1 and day 2 assigned task and solution

Day 1 and 2 task	Determine the sum of the measures of the interior angles of the [polygons]. Use this information to help you write a function that gives the sum of the interior angles of any n–sided polygon (n–gon). Justify your answer
Day 1 and 2 solution	At the end of each day, the class concluded that the solution was "$(n–2)180$" and "$F(n) = 180n–360$" [Boardwork from day 1]

Table 2 Transcript for analysis parsed into common and non-common episodes

Brief Description of Episode from Analyzed Transcript	Line Numbers
DAY 1	
✪ Discussion of Solution to the Day 1 Task	1–35
An Alternative Solution is Presented	36–37
DAY 2	
Ms. Bell Suggests An "Easier Way" to Complete the Task	100–101
A Student Asks About Conjecture & Proof	102–110
✪ Dividing a Quadrilateral into Triangles	111–130
Calculating the Sum of the Angles	131–141
A New Task is Suggested: Pentagon	142–143
Student Attempts the Pentagon	144–157
Ms. Bell Clarifies that they are Looking for the "Smallest Amount of Triangles"	158–159
The Class Finds the Sum of Interior Angles in the Pentagon	160–184*
The Class Begins Working on the Hexagon	185–186
They Divide Up the Hexagon	187–200*
Discussion About Dividing Up the Hexagon	201–219*
A Student Notices a Pattern	220–222
Ms. Bell Verifies a Pattern and Suggests a Table	223–226
Ms. Bell and the Class Co-Construct a Table	227–246
✪ Returning to the Day 1 Function	247–272
Wrapping Up	273–279

✪ indicates a "key episode"; * indicates a single line was not included; white rows are common episodes

A Series of Commonly Analyzed Episodes

Looking across the three chapters, we identified nine common episodes that all three researchers considered to be important, regardless of lens. Common episodes—episodes which were explicitly discussed by all three authors—appear in the white rows in Table 2. Here, we explore the nature of particular "key episodes" (indicated by ✪) selected from the common episodes and discuss how they were interpreted. Key episodes are excerpts that the researchers identified as being particularly important to the reasoning happening in the classroom. While one picture emerges from looking at commonly addressed episodes, we also wondered about non-common episodes, which are indicated by gray shading.

Key Episode 1: Discussion of Solution to the Day 1 Task (Lines 1–35)

All three researchers started from the premise that the mathematical work of this episode was to identify a formula. Several students, including Travis, responded to Ms. Bell's invitation to share their work. At the conclusion, Travis's answers included the sum of the measures of the angles of all four polygons and two different, but equivalent, versions of the formula (see Table 1).

Through the lens of justification, in this episode, Newton viewed Ms. Bell as setting up a situation for explanation. She also described multiple students' "willingness" and "eagerness" to share their ideas. Newton described Ms. Bell as working "alongside Travis to publicly establish a formula" (see "How Were Norms for Justification Enacted and Co-Constructed?" in the chapter by Newton, this volume). She characterized this part of the episode as being about student engagement in the co-construction of mathematical meaning as students interacted with Ms. Bell and one another. Having stated that a challenge heralds the arrival of justification, it was important to Newton's analysis to identify one. In the absence of an explicit challenge, Newton stated: "the students often seemed to work under the impression that a real or perceived challenge existed whenever they were explaining their work" (see "How Were Norms for Justification Enacted and Co-Constructed?" in the chapter by Newton, this volume). She claimed that the need to justify seemed to be understood by the students.

Through the lens of argumentation, Hähkiöniemi, who divided the entire transcript into ten discrete steps, considered this episode Step 0: formulas for the sum of interior angles. Hähkiöniemi noted the inductive nature of Travis's argument and characterized the argument as including articulating reasoning but lacking students' dialogic moves. Step 0 was, in fact, the only one of the ten steps that included *articulating reasoning* in the sense that Travis presented a complete argument that he had worked on. He also made his thinking visible, rather than making smaller contributions to a larger argument. Hähkiöniemi viewed the episode as dialogic/interactive, where the interaction was primarily between Travis and Ms. Bell, who elicited Travis's ideas and allowed him to present these ideas.

Through the lens of proof, Morselli viewed the day 1 classwork as being focused on *discovering* the formula, with the work beginning in the geometric system of representation and then "rapidly" shifting to the arithmetic-algebraic one. Even though the task asked students to *justify* the formula, Morselli noted that Ms. Bell merely asked Travis to explain the process that led to the formula. Although Travis shared the function he discovered and illustrated it by examples, according to Morselli, there was no explanation of how he found it, nor justification for it. The teacher's focus, she said, was on "involving all the students in understanding Travis's solution," rather than "promoting the justification of the solution" (see "Part 1: Discovering the Formula" in the chapter by Morselli, this volume). Morselli concluded that "justification was not addressed at all" in this episode (see "Part 1: Discovering the Formula" in the chapter by Morselli, this volume).

Key Episode 2: Dividing a Quadrilateral into Triangles (Lines 111–130)

In this episode, after suggesting that she would like to consider "another way" to measure the interior angles without a protractor, Ms. Bell asked "somebody" to come up and divide a quadrilateral into triangles. After Angela presented a solution that was not what Ms. Bell was looking for, Ms. Bell asked what would be the smallest number of triangles that would divide the polygon. Angela drew one diagonal in the quadrilateral, and Martin concluded "Two." Ms. Bell asked about and then verified that they also could have done this by drawing the other diagonal.

Newton noted that out of the four total times that she did so on day 2, in this episode, Ms. Bell called for the minimum number of triangles for the first time. This issue became important for Newton, who claimed that the classroom community did not establish (or justify) any kind of relationship between the triangles and the sum of the interior angles of the polygon. So, although Ms. Bell repeatedly called for drawing the minimum number of triangles as the class divided up the four polygons, no attention was given to *why* this process worked. This was important to Newton because even though Ms. Bell repeatedly stated this requirement, the implicit claim that it was required in order to get the "correct" answer was never explicitly supported, and no justification was given "as to why the claim made sense" (see "Which Mathematical Ideas Were Justified and Which Mathematical Ideas Were Left Unjustified?" in the chapter by Newton, this volume). Newton concluded that, at least during the interactions that the researchers examined, this claim was never justified.

Because Martin interacted directly with Angela's drawings and ideas, Hähkiöniemi classified this episode as including dialogic moves, such as elaborating. For Hähkiöniemi, the transition in argumentation from Step 1 (number of triangles) to Step 2 (smallest number of triangles) occurred when Ms. Bell asked about the smallest number of triangles. Because this question is closed, there was no genuine need for *articulating reason*. In this way, Hähkiöniemi viewed many of the interactions as being "controlled by the teacher" (see "Dialogic Argumentation and Teacher Support" in the chapter by Hähkiöniemi, this volume). More specifically, the students could only be thinking about one particular step at a time, rather than considering the big picture of where the steps were leading.

Like the other researchers, Morselli also honed in on Ms. Bell's request to divide the polygon into the smallest number of triangles. Morselli, however, saw this requirement as "crucial for the proof for the formula" (see "Part 2: Constructing Triangles" in the chapter by Morselli, this volume). Because it was so crucial, she argued that Ms. Bell should have stated this requirement from the beginning. Instead, Morselli claimed that Ms. Bell was not actually "clear in mind" about this issue (see "Part 2: Constructing Triangles" in the chapter by Morselli, this volume). She then hypothesized that Ms. Bell ended the discussion about different ways to divide up the triangle because it deviated from the focus of the lesson.

Key Episode 3: Returning to the Day 1 Function (Lines 247–272)

This episode began with Ms. Bell completing the last entry in a table of the class's findings. She then asked students to remind her of the equation for determining the sum of the interior angles of a polygon from day 1. Students volunteered these equations, which Ms. Bell wrote on the board and verified as being equivalent by the distributive property. Ms. Bell asked where the n-2 from the formula "shows up" in the table. Kyle answered, "The number of sides minus two is the number of triangles." Ms. Bell connected his response to the rows in the table.

Newton viewed the work done in this episode as contributing to the various methods that all led to the same conclusion. According to Newton, this work of connecting the two representations, along with the other strategies of using guess-and-check, separating polygons into triangles, and making a table, seemingly persuaded the class of the accuracy of the formula found on day 1. Overall, Newton concluded that, in spite of the collaborative nature of the classroom activity, Ms. Bell controlled the agenda to a considerable extent.

For Hähkiöniemi, most of this entire episode was contained in Step 9: connecting the table to the formulas in Step 0 (i.e., day 1). Hähkiöniemi highlighted Kyle's contribution in Line 268 as important. He viewed the exchange between Kyle and Ms. Bell as being about Kyle posing a claim and Ms. Bell supporting that claim by showing where $n-2$ was calculated in the first two rows of the table. Because the teacher, not the student, supported the claim, this step did not include justifying moves. Hähkiöniemi classified the interaction as mainly dialogic/interactive because the teacher let the students propose a connection between the table and the $n-2$ term in the formula. Overall, however, Hähkiöniemi concluded that, while there were several incidents of dialogic argumentation as the students engaged constructively with their peers' argumentation, the 2 days of classroom activity were "dominated by the teacher's view" (see "Dialogic Argumentation in Studying Mathematical Argumentation" in the chapter by Hähkiöniemi, this volume).

Morselli viewed the start of this episode, beginning with Line 247, as suggesting the need for generalization (i.e., "*I don't want to keep going any further. Tired.*"). Although the need for generalization could suggest a need for proof, Morselli claimed that Ms. Bell "force[d]" the class in another direction (see "Part 3: Towards a Formula (Again)" in the chapter by Morselli, this volume). In particular, she steered the conversation away from the geometric representation (i.e., the one that could lead to a proof) and back towards the algebraic representation. According to Morselli, calling for both forms of the equation confirmed that Ms. Bell's focus was on algebraic functions and manipulation, rather than proof. Although the $n-2$ term was connected to the minimum number of triangles, Ms. Bell made no effort to reflect on the geometric meaning of the second formula: $F(n) = 180n–360$. Consequently, Morselli concluded that despite the "great potential" of the task for fostering proof, many opportunities provided through the students' reasoning were not "caught" or "managed" by Ms. Bell (see "Conclusions" in the chapter by Morselli, this volume).

Looking Across the Key Episodes

Looking across the analyses of these three key episodes, we found some convergence and some divergence among the researchers' perspectives. In the first episode, Newton concluded that the co-constructed norms were indicative of the class's understanding that justification was required by students presenting their ideas, and Hähkiöniemi concluded that an inductive argument was put forth through an interactive/dialogic approach. In contrast, through the lens of proof, Morselli claimed that Ms. Bell was mainly focused on revealing the formula but concluded that justification[1] of the formula was not addressed at all.

All three researchers honed in on Ms. Bell's first statement about the requirement of the smallest number of triangles, which began in Episode 2. The three scholars seemed to be in agreement that this idea was crucial to the reasoning. They also identified opportunities that were missed by Ms. Bell. Newton claimed that no attention was given to *why* this process worked to get the "correct" answer. Hähkiöniemi stated that despite the episode including multiple dialogic moves, the interactions were "controlled by the teacher" in a series of steps. A consequence of sequencing the reasoning in steps, according to Hähkiöniemi, was that doing so made it difficult for students to see the big picture of where the reasoning was headed. Morselli asserted that because the requirement of the smallest number of triangles was so crucial to the proof, Ms. Bell should have stated it up front.

The analyses of the third episode varied again, particularly when it came to drawing conclusions about the reasoning that occurred across the 2 days. Noting that Ms.

[1] The term justification is used by Morselli to reference the task prompt "Justify your answer" where the answer is the student's function for the sum of the interior angles of an *n*-sided polygon.

Bell controlled the agenda to a considerable extent, Newton concluded that the class was ultimately persuaded by the various approaches of guess-and-check, dividing up the polygons into triangles, and constructing a table. Hähkiöniemi argued that even though a connection was made between the term $n-2$ in the formula and the number of triangles that divided the polygon, it was Ms. Bell, not the students, who supported this claim. Consequently, despite several incidents of dialogic argumentation, the 2 days of activity were "dominated by the teacher's view." Although they both pointed out Ms. Bell's controlling agenda, Newton and Hähkiöniemi indicated that justification and argumentation, respectively, did occur to some degree. In contrast, Morselli concluded that despite the great potential of the task, Ms. Bell did not manage the reasoning in a way that would lead to proof. In fact, Ms. Bell steered the class away from the geometric connections that could have led to a proof. Accordingly, Morselli found that these actions resulted in Ms. Bell not addressing the characteristics of proof. The extent to which the three researchers found evidence of their respective type of reasoning suggests that the threshold for evidence of proof is harder to cross than for argumentation or justification.

Considering the Non-common (Gray) Episodes

It is interesting to note that only a little more than 50% of the entire transcript resulted in common episodes. This suggests that the three researchers converged on particular excerpts of the transcript they deemed as being important for their analyses of reasoning in this classroom. When looking across the non-common episodes (i.e., the gray rows in Table 2), we noticed that the non-common episodes tended to be more about procedures and arithmetic than about reasoning.

There was, however, one non-common episode that stood out to us as surprising. We might have expected *all* researchers to highlight Lines 102–110, *A Student Asks About Conjecture and Proof*, as being important. In particular, at the beginning of day 2, after Ms. Bell suggested that they would consider an "easier way" to think about the task, Adam asked, "What are they talking about when they say conjecture?" (Line 102). After Ms. Bell asked if anyone knew what the word "conjecture" meant, Martin added, "it says 'prove your conjecture.'" Ms. Bell seemed to cut off this line of questioning when she steered the class back to her agenda of exploring another way to find the sum of the interior angles in a polygon. Although we are not given any information about what prompted students to ask these questions, we found it surprising that this episode was only taken up in a significant way by Morselli, given that proof could be considered to be a form of justification or a mathematical argument. She noted that not only did Ms. Bell not take this opportunity to clarify what a proof was, she seemed to communicate to the students that the proof was just another way to approach the problem.

Concluding Discussion

Historically, geometry, the content focus in this case, has been the area of school mathematics where students have had opportunities to engage in more sophisticated levels of reasoning. In fact, in 1972, van Akin claimed that the objectives for high school geometry are twofold: (a) to develop knowledge of geometric facts and relations and (b) to develop the ability to reason logically. In other words, he saw the goal of work in geometry as being about teaching content and teaching reasoning, with both being equally important. One could argue that Ms. Bell's focus during the lesson in this integrated course seemed to privilege having students develop the formula (i.e., learn the content) rather than justifying *why* the formula made sense (i.e., engage in reasoning). To be clear, neither we nor the chapter authors had access to information about Ms. Bell's learning goals for the lesson.

In the lesson, Ms. Bell prioritized executing the strategy of finding the least number of triangles to divide up the polygons. All three researchers concluded that regardless of her intentions, Ms. Bell did not sufficiently steer the conversation in the direction of *why* the formula made sense or connect it to the dividing-up-triangles approach explicitly. This was true even though, as pointed out by Morselli, a student explicitly asked about conjecturing and proving. According to the researchers, there were multiple events in the lesson that were deemed situations for justification or signaled the need for generalization. Instead, Ms. Bell seemed motivated to make connections across different representations, as evidenced by her highlighting the distributive property to connect the two different versions of the formula and inputting the class's findings into a table to help students recognize the pattern. The researchers pointed out that the class ended day 2 with the same conclusion as day 1, albeit with perhaps a new appreciation for a strategy to find this sum of the angles in any polygon, which was inductively generated. This is important because the teacher's emphasis influences whether the "residue" (Hiebert et al., 1997) that remains from the lesson is related to the formula, the reasoning process used to validate the formula, or both. The case and the researchers' analyses raise many questions and implications for research and teacher education related to reasoning at the secondary level.

With respect to teacher preparation, we wonder how to support teachers to differentiate between and plan opportunities for different types of reasoning. As an example, a teacher's learning goals might differ depending on the kind of reasoning they aspire to (i.e., justification versus argumentation versus proof). As we prepare teachers to nimbly steer classroom discussions informed by student thinking and provide opportunities for various kinds of reasoning, how do we also prepare teachers to honor student agency and allow students to reason in ways they deem necessary and interesting? What might teaching that supports students to deem it necessary to understand *why* a formula is true look like? The researchers in the case simultaneously concluded that the teacher heavily controlled the lesson *and* that a key idea was never justified. The analyses caused us to ask: What is the role of the teacher and how "heavy-handed" must the teacher be to ensure that the intended reasoning occurs? What would be a healthy balance of responsibility placed on the students versus the teacher for reasoning to occur? Finally, how might we swing the political pendulum away from an emphasis on content-driven performance standards and towards a healthier balance between content and processes? As van Akin (1972) suggested nearly 50 years ago, both are important.

With respect to classroom-based research and the examination of student reasoning, we recommend careful consideration of the implications of applying a particular reasoning framework to data when that particular type of reasoning (e.g., proof) did not occur and perhaps was not part of the teacher's learning goals. We wondered if engaging in such analyses is fair to the students or the teachers, particularly novice teachers. At the same time, there is something to gain from conducting such analyses. Observations, such as those made in Bieda's (2010) study, are important if we are to understand and work to improve the current situation. This is also to say that if we believe it is important for reasoning to occur in classrooms, how bluntly should we share conclusions about its absence or about the potential of tasks not being realized? Classroom research is as complex as teaching mathematics. As we pull transcripts for our own work and for teacher preparation, we must be cognizant of how our framings position teachers and students; ideally, we frame them as people from whom we can learn.

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Part IV
Argumentation, Justification and Proof at the Tertiary Level

Overview of Tertiary Level Data

David Plaxco

The following three chapters include analyses of the same transcript and classroom artifacts from an undergraduate, junior-level, inquiry-oriented abstract algebra course. The class included approximately 25 STEM majors from a large eastern US university. These data were collected as part of a project investigating undergraduate students' proof activity in abstract algebra. The instructor was an assistant professor at the university whose research focused on mathematics education; she had taught the course from these materials multiple times before the semester. The data used in these chapters were collected during the second week of a 15-week semester. The curriculum used in the course was Teaching Abstract Algebra for Understanding (TAAFU) (Larsen, 2013). TAAFU, an inquiry-oriented, RME-based curriculum, relies on Local Instructional Theories (LITs) that anticipate students' development of conceptual understanding of ideas in group theory. The curriculum is intended to allow students to feel as though the mathematics developed is a product developed from their own informal ways of thinking. Because of this, the students engage frequently in the practice of building and communicating mathematical arguments based on their current ways of understanding.

The classroom format consisted of students working in small groups, periodically breaking into whole-class discussion. The transcript provided comes from a video clip of the third meeting of the class. In it, the instructor invites students to share their work exploring the group of symmetries of an equilateral triangle and to explain their understanding of how compositions of various symmetries can be thought of as equivalent. During this segment, the classroom discourse shifts between whole-class discussion led by the teacher and small group discussions in which the students shared strategies for determining equivalence of symmetries.

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This discussion leverages an established notation system for the symmetries, including two basic symmetries and an additive notation for composing these symmetries. The purpose of the activity in the clip is to draw out more general properties of group structures, including associativity, the equivalence of different combinations of Rs and Fs, and point toward a development of some of the group axioms, most specifically the existence of an identity symmetry. Throughout the discussion, contributions from different members of the class (often) move the discussion forward. The data set provided to the authors of these chapters included transcripts of approximately 20 min of class discussion; some screen captures of images projected in the classroom at various times during the discussion were included in the transcripts.

The Mathematics Involved in the Lesson

The first unit of the TAAFU curriculum is centered around the group of symmetries of the equilateral triangle. Based on the curriculum design theory of Realistic Mathematics Education (RME; Gravemeijer, 1999), the tasks of the unit are sequences intended to support students along a trajectory from situational activity through reflective activity and on to generalizing and more formal activity. The excerpt of classroom data captured a large portion of a discussion about an activity intended to support students in generalizing their situational and reflective activity toward characteristics of a structure that students had been exploring for more than a week. Prior to this lesson, the students in the course had explored symmetries of equilateral triangles (formally known as S_3), identified and named the six symmetries, developed an algebraic notation for compositions of symmetries, and completed and discussed a worksheet intended to help organize the equivalent compositions of symmetries (Fig. 1).

In this class community, the students had decided to name two symmetries—R (for a 120° clockwise rotation) and F (for a reflection or "flip" across the triangle's vertical line of symmetry)—and developed a naming system for all six symmetries that used only combinations of Rs and Fs (formally, R and F are called the generators of the group). Consistent with RME, the TAAFU curriculum encourages instructors to have students develop their own notation for representing the group of the symmetries. In this class, the students used additive notation to represent composition of symmetries, as denoted in Fig. 1. This organizing tool was referred to throughout the data excerpt as "The Chart." In order to make sense of the compositions of these symmetries, it helps to envision the effect of each symmetry on the vertices of the triangle as multiple symmetries occur in sequence.

The data excerpt comes from the last 20 min of a class meeting during which students were asked to use The Chart (Fig. 1) to complete an addition table for the six symmetries (formally, this is called the Cayley table of the group). The table is filled in by composing the symmetries of a given row and column in that order. So,

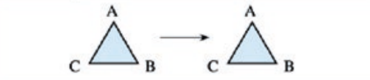
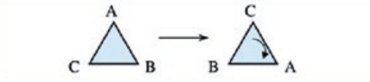
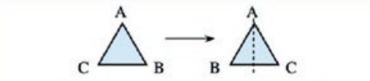
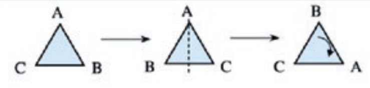
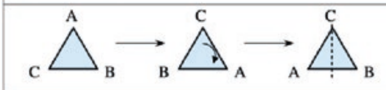
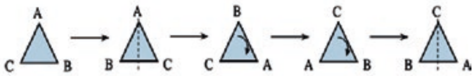
Symmetries Under Composition	Combinations of F and R	Alternate Interpretation
	R+R+R (or 3R), F+F, R+R+R+F+F, F+R+R+R+F, 0R(?)	3R: The identity symmetry – no change occurs to the triangle
	R, R+R+R+R, F+F+R, R+F+F, R+R+R+R+R+R+R	R: 120° clockwise rotation
	R+R, F+F+R+R, R+R+F+F, R+R+R+R+R, R+R+R+R+R+R+R+R	R+R: 240° clockwise rotation or a 120° counterclockwise rotation
	F, F+R+R+R, F+F+F, F+F+F+F+F, F+F+F+R+R+R+F+F	F: Reflection across the line of symmetry through A
	F+R, R+R+F, R+F+R+R, R+R+R+F+R, R+F+F+R+F	F+R: Reflection across the line of symmetry through C
	R+F, F+R+R, R+R+F+R, R+F+R+R+R, F+R+F+F+R	R+F: Reflection across the line of symmetry through B

Fig. 1 Augmented version of "The Chart" showing compositions of R and F symmetries of an equilateral triangle and equivalent combinations with an additional column indicating alternate ways of thinking about these combinations

+	3R	R	R+R	F	F+R	R+F
3R	3R	R	R+R	F	F+R	R+F
R	R	R+R	3R	R+F	F	F+R
R+R	R+R	3R	R	F+R	R+F	F
F	F	F+R	R+F	3R	R	R+R
F+R	F+R	R+F	F	R+R	3R	R
R+F	R+F	F	F+R	R	R+R	3R

(a)



(b)

(2-sided) Identity: There exists some element e such that $e*g = g*e = g$ for all g in the group.	(2-sided) Inverse: For each element g, there exists an element g^{-1} such that $g*g^{-1} = g^{-1}*g = e$.
Closure: For any two elements, g and h in the group, $g*h$ is also in the group.	Associativity: For any three elements, g, h, and k in the group, $g*(h*k) = (g*h)*k$.

(c)

Fig. 2 (**a**) Cayley table for S_3, (**b**) composition of F + R and R + F, (**c**) the group axioms

for instance, the cell in the F + R row and R + F column (Fig. 2a, highlighted in gray) shows the end result of carrying out the F + R symmetry, then the R + F symmetry (Fig. 2b). The purpose of the discussion during the excerpt was to have the students generalize the methods that they used to fill in the table in order to develop structural rules for the set of symmetries under composition. These rules would then be leveraged to generate the group axioms and, thus, define the construct of a group (Fig. 2c).

Outline of the Class Discussion with Selected Excerpts of Transcript

The general format of the class discussion centered around the instructor soliciting student strategies for filling in the Cayley table and then organizing those strategies on the overhead. Because the discussion took on this format, I am providing excerpts of the transcript that highlight the more substantive contributions of the students and significant organizing discussions from the instructor. The excerpt began with the instructor soliciting students' rules for filling in the Cayley table. The student responses align almost exactly with my discussion of the symmetry group above, except that the students never describe the existence of inverse elements. This is consistent with Larsen's (2013) description of inverses as one of the more difficult axioms to motivate from students' situated activity in the TAAFU lessons.

This segment of class began with the instructor asking, "Who has a rule they want to share with me?" A brief summary of the methods students contributed (in their own words) is listed in the following paragraphs.

S1: I used the Cayley Table shortcut[1] which, after like I saw the pattern, I knew that each one appeared in each column and row only once. So around the end of my table, I just started seeing which one was missing. [Instructor writes "Each symmetry appeared exactly once in each row and column" on projector.]

S2: You could give the 3R column and row as identities. … 3R plus any other equal whatever that any other was. … Yeah, and the same with any other plus 3R.

S3: Yeah, I did that, but then I also, for another one, before I could do filling it in Sudoku-style or like, after I did that, on the chart we did, I would just do one motion and then find it as a triangle and then do the next motion and then find it until I got to the last one. I did the side first, when I was at Row R, on this thing [holds up paper and motions toward what is written on it], I went to R, looked at the triangle, and then, say it was plus F, I flipped that, found where it was on here, and did the next thing, and then found where it was, so I that I could keep track of it.

The instructor summarized S3's contribution and highlighted an important aspect of it, drawing attention to parentheses in a representation of S3's activity:

Inst: Does that make sense? So she used these pictures, doing one at a time. If I was going to write that out algebraically [underlines R + (R + F)], according to the table I need to do R first [draws a box around R], and then I need to do this motion: R + F [circles (R + F) on the projector]. Which is this. But she didn't think of this [points at (R + F)], as this combined motion. What you kind of did is R, and then R, and then F. Right? What I'm trying to get at is that this is how it was asked: R, and then the combination, the grouping, of R + F. If I had to write that with parentheses and symbols, how could I capture what she was doing? Would those parentheses stay in the same place? Are you shaking your head no?

After a brief conversation about parentheses, the instructor asked another question, and students related their experiences with the table to previous mathematical experiences with the associative and commutative properties.

[1] The terms "Cayley table shortcut" and "Sudoku-style" were used synonymously.

Inst: In the table it's the same, if I go to where 2R and F are in the table, I get the same answer if I go from R and RF. Is that true? So these two gave me the same answer. By saying 2R plus F, I am saying R plus R plus F [writes = R + R + F]. I'm regrouping which ones I'm considering associated with each other. Does that sound like something you've seen before in math?

S8: The associative property?

S3: The way I kind of see it is, just because I just took Proofs last semester, when you had like an A or B representing different things the order mattered for that and the order matters for this, so you should be able to do the same types of things to it, so the associative property should work.

Inst: It's like something she's done before where order still mattered, but in that case, even though order mattered, I can still associate differently. … Order mattered. Do you guys know the word for that? What property that is related to?

S5: Commutativity.

Inst: Commutativity. Right. So there are two properties we see in math: commutativity and associativity. Commutativity says three times four is the same as four times three. Order doesn't matter if it's commutative. Here we don't have that one. Order matters. F plus 2R is different than 2R plus F. It looks like we can drop the parentheses all together, and we still know what we're doing and that's not changing answers, or we can regroup things, and that's not changing answers. Keep an eye on that as you go through. Did you guys just drop parentheses like they weren't there?

After the conversation about the associative and commutative properties, the instructor continued the conversation about methods for completing the table: "Anything else you guys were using while you were filling this out?"

S9: A couple of them are just intuitive, like R plus R equals 2R, and 2F equals 3R.

Inst: R plus R equals 2R, and what [writes R + R = 2R]?

S9: F plus F equals 3R.

Inst (line 48): F plus F equals 3R [writes F + F = 3R]. Yes. Good. It's important to note here, because if we're working with a square, it wouldn't be the case that F plus F equaled 3R. Right? So these rules are dependent on where we are, and how the triangle works. What else? Does someone want to explain to me the rule behind what's going on over there on the board?

S8: F plus R is the same as R plus R plus F.

S10: An R on one side of an F is equal to two Rs on the other side, in any case.

Inst: An R on one side of an F--

S10: Is equal to two Rs on the other. If you have two Rs on the other side it's equivalent to one on the other, or you could represent that as you move one R over and it becomes two Rs, and the other R over and it becomes another two Rs. Then you have four R, which is one R.

The instructor illustrated the rule contributed by S10 to simplify F + (R + F).

Inst: So let's just break it down for this guy [points at F + (R + F)]. I think the motivation here is that if I can get my two Fs together, I know they're going to cancel. I know I'm going to get nothing, right? And then I can ignore them. What I wanted to do is get this F so he lives over here, by that F [draws arrow from second F to first F; see Fig. 3]. To do that, I can move this R to the other side, but then I pick up an extra R [writes = F + (F + 2R)]. I can write this rule either like this [underlines R + F = F + R + R], or as R plus F equals F plus

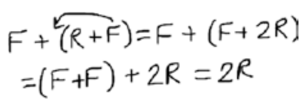


Fig. 3 Recreation of instructor's work on board with F + (R + F)

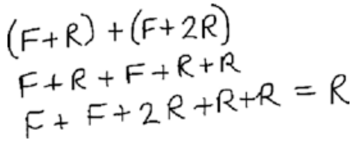


Fig. 4 Recreation of student's solution displayed on board for (F + R) + (F + 2R)

> 2R [writes R + F = F + 2R]. It's kind of like substitution. He knows these two things are equal, so any time he sees one version he can switch it out for an equivalent version. So when I do that, then I can regroup those Fs [writes (F + F) + 2R], I know what F plus F is, so I can solve this as just 2R [writes = 2R]. Does that make sense?

The instructor then set a problem for students to work, asking students to use the rule initiated by S10 to simplify (F + R) + (F + 2R). Students worked briefly in small groups, and then the instructor displayed a student's solution on the board (Fig. 4).

The instructor described S10's method, and then two students questioned that method and described their own solution methods.

> Inst: She replaced that R plus F with F plus 2R, so we see those ones showing up right here. Then the Fs are on one side, and the Rs are on one side, and somehow all those Fs and all those Rs just turned into one R.
>
> S3: Do we have to use that kind of stuff? Because I didn't do that. I just thought "F plus 2R" so I looked at my little graph and looked at what F plus 2R was, substitute that, and then it was just F plus R plus R plus F and you could get that. Were you expecting us to use those kind of substitutions from the first homework?
>
> S3: Oh, I just kept it as F plus R in parentheses plus R plus F in parentheses, and looked at what that was on the graph, or on our table.
>
> Inst: Yeah, but what if you didn't have your table?
>
> S8: If you wanted to go further then you could just say F plus 2R, combine the Rs, now that they're together.
>
> Inst: Combine these?
>
> S8: Yeah. So then you get F plus 2R plus F and we just identified that at the top, so that's F plus 2R is R plus F.
>
> Inst: This is R plus F.
>
> S8: Then you have R plus 2F, which is just R.
>
> Inst: It's the same kind of substitution though, right? We're taking this guy, and we're saying that's the same as this one. Then we're regrouping these three into this, and we know what that is. It's the same as this. It's still substituting in these equivalent things, right? Both of these used different versions of the R plus F or the F plus R, and in different places. Either case I group them together, I figure out "Oh, I know what that is!" and if I use this version of it, I get my Fs together and Rs together and then things simplify.

The transcript given to the authors ended at this point.

Conclusion

The authors in this section conducted secondary analyses of the transcribed data. This represents only a snapshot of the instructor's practice from early in the semester. As such, the authors did not have access to data that might illustrate the formation of normative behaviors or typical tasks set by the instructor. However, this snapshot allowed for analysis using different definitions and theoretical perspectives. The following three chapters offer perspectives on this classroom data, examining argumentation, justification, and proof. In the final chapter, Dawkins looks across the three analyses to provide a broader understanding of how using different constructs from different perspectives adds to our overall understanding of argumentation, justification, and proof in the field.

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Argumentation in the Context of Tertiary Mathematics: A Case Study of Classroom Argumentation and the Role of Instructor Moves

David Plaxco and Megan Wawro

In this chapter, we focus on argumentation that occurred in the modern algebra excerpt described in the "Overview of Tertiary Level Data" chapter by Plaxco, this volume. The definition of argumentation used in this book is "the process of making mathematical claims and providing evidence to support them." (see "Argumentation" in the introductory chapter by Staples and Conner, this volume) In an 18-min episode from a student-centered classroom, we gain access to the unfolding of rules that students co-developed for completing the Cayley table and for determining equivalences for symmetries of equilateral triangles. Thus, we examine the nature of students' argumentation as they explore this, through presenting their own – and responding to others'—mathematical ideas. The curriculum enacted in this classroom community (Larsen, 2013) relies on the central role of the instructor in guiding the reinvention (Gravemeijer, 1999) of key mathematical concepts. Thus, we investigate the following research questions: (1) What is the nature of students' argumentation as they explore algebraic properties of the set of symmetries (of the equilateral triangle) under the operation of combining symmetries? (2) In what ways can instructor moves leverage student argumentation to advance the mathematical agenda of a classroom?

In answering these questions, we aim to understand the mechanisms by which mathematical reasoning unfolded and moved forward for this classroom community. To operationalize our first research question, we employ Toulmin's model of

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argumentation (Toulmin, 1969) to analyze the classroom discussion. For the second research question, we analyze instructor contributions using the Teacher Moves for Supporting Student Reasoning (TMSSR) framework by Ellis et al. (2019), which we discuss in the next section.

Theoretical Frameworks and Analysis Methods

In our analysis of argumentation, we focus on the development of mathematical meaning through classroom interaction between students and the instructor. The TAAFU curriculum promotes the joint construction of meaning for both the informal notion and formal definition of the group structure. To study how students interact with and reason about this structure (itself developing over a few class days), the researcher must necessarily deal with how mathematical ideas arise in the classroom setting. In a study investigating students' developing ideas regarding equivalent fractions, Saxe et al. (2009) state:

> Teachers pose problems, solicit students' solutions and orchestrate discussions to guide students to make their mathematical ideas public and reflect on relationships among their ideas. In the social context of classroom lessons, students' ideas may be taken up or rejected, valued or devalued, and interpreted in various ways, and, in this process, students' mathematical ideas become elaborated and often transformed. (p. 203)

Thus, much of the genesis of the group definition is reflected in how students and the instructor construct arguments about its emerging constituent parts and characteristics. When considering the analysis of mathematical arguments produced by students and mathematicians, Inglis et al. (2007) state, "Generally these types of analysis are of two kinds: those that concentrate on the argument's *content* and those that concentrate on the argument's *structure*" (p. 3). These aspects of an argument are reflexive and inseparable, so we chose the framing of Toulmin (1969) to analyze the classroom argumentation in this episode more holistically.

This framework aims to characterize what Toulmin called substantial arguments – arguments presented in discourse in some sociohistorical context for a particular purpose. Substantial arguments "often are structurally distinct from the formal, deductive structure of logical arguments thought to be inherent and absolute. In other words, some aspects of socially presented arguments are context dependent yet still function as acceptable justifications for particular claims" (Wawro, 2015). Toulmin (1969) proposed six main components of a substantial argument: claim, data, warrant, backing, qualifier, and rebuttal. The claim is the conclusion that is being justified, and the data is the evidence being given to demonstrate the claim's validity. The warrant supports how the data justifies the claim, and the backing, if provided, demonstrates why the warrant has authority to support the data-claim pair. A qualifier describes the degree of confidence in the claim, and a rebuttal pairs with the qualifier to note under what conditions the claim would be refuted (Inglis et al., 2007). Toulmin's model of argumentation has been adapted by many in mathematics education research (e.g., Conner et al., 2014a, 2014b; Krummheuer, 1995; Nardi et al., 2012; Rasmussen & Stephan, 2008). Krummheuer

(1995) considered argumentation to be a "specific feature of social interaction" (p. 226), and Aberdein (2009) defined an argument as "an act of communication intended to lend support to a claim" (p. 1). These are consistent with this book's aforementioned adopted definition of argumentation.

In our initial analysis of the transcript, we focused on identifying distinct arguments throughout the class discussion. We first identified claims that emerged through interactions and then identified utterances that served to support and situate said claims. Each utterance was analyzed as one of the five remaining component parts of a Toulmin scheme (data, warrant, backing, qualifier, or rebuttal); together with the supported claim, these comprised an argument. From the 81 lines of transcript, we identified 15 distinct arguments, each situated in the classroom discourse via the instructor's direction, guiding comments, probing comments, and explanations or rephrasings of various mathematical ideas put forward in the discussion. The organization and structure of the Toulmin schemes helped us document how the various classroom members' utterances informed each argument. We also identified how each argument was distinguished from prior and subsequent discussion. Accordingly, we grouped the classroom argumentation into four vignettes, each characterized by the nature of the instructor's moves and each serving its role in the unfolding classroom mathematical agenda.

As we organized the Toulmin schemes, we recognized roles the instructor's actions played in the argumentation. Specifically, we consistently found that although the instructor was often contributing to an argument (by providing utterances that were analyzed as claim, data, warrant, etc.), some utterances situated various nodes of an argument. That is, some instructor utterances served to facilitate student reasoning at the organizational level, rather than as parts of the argument. This is the focus of our second research question, which we operationalized by identifying teacher moves that seemed to "live beyond" the Toulmin schemes we created through our analysis toward answering the first research question.

To investigate our second research question, we focused on each teacher move we identified. We view this as grounded theory in that we had no a priori coding scheme, other than being influenced by our own prior knowledge of inquiry-oriented instruction. After an initial round of open coding the transcript together, we searched the literature for existing frameworks that would help further characterize the nuanced instructor moves. This led us to adopt the Teacher Moves for Supporting Student Reasoning (TMSSR) framework by Ellis et al. (2019) as our primary coding scheme for teacher moves. The authors posit that teachers' actions in the classroom are "mediated by their beliefs about their students as learners of mathematics, by the students they teach, by the instructional tasks and manipulatives in use, and by their own mathematical goals for the tasks and the unit" (p. 112). According to activity theory, activities in which people are engaged are meditated by their experiences, how they use and transform tools, and by context (Engeström, 1999). Ellis et al. leverage this theory to investigate teachers' moves for supporting students' reasoning and the factors that mediate those moves.

The TMSSR framework, which is specifically for inquiry-oriented instruction, was created by analyzing middle grades instruction of a research-based unit on ratio and linear functions. The framework has four main categories of pedagogical moves: eliciting, responding, facilitating, and extending student reasoning. Specific moves comprise the main categories and are placed along a continuum (from low to high) according to their potential to support student reasoning. For instance, some teacher moves may center students as doers of mathematics or center the teacher as the mathematical authority. A teacher move should be considered in light of its impact on "students' opportunities to reason (as inferred by their responses to tasks, their written work, and their participation in class discussions)," although "each move may differ in its effect on student reasoning depending on a variety of elements in the activity system" (Ellis et al., 2019, p. 116). See Fig. 1 for the 32 teacher moves, arranged in the four main categories of the TMSSR framework.

After completing our Toulmin schemes, we revisited our initial open codes of instructor moves and determined how well they aligned with TMSSR. We found that the TMSSR framework not only satisfactorily aligned with the intent of our initial coding, it also was sufficiently nuanced and detailed enough to capture the tone of nearly all of our initial codes. Due to our data coming from inquiry-oriented instruction, albeit at the undergraduate rather than middle-school level, we find this alignment sensible and reliable. In total, our teacher moves were captured by 17 of the 32 TMSSR codes. These are discussed in detail throughout the "Results" section.

Eliciting Student Reasoning		Responding to Student Reasoning	
Low	High	Low	High
Eliciting Answer	Eliciting Ideas	Correcting Student Error	Prompting Error Correction
Eliciting Facts or Procedures	Eliciting Understanding	Re-voicing	Re-Representing
Asking for Clarification	Pressing for Explanation	Encouraging Student Re-voicing	
Figuring Out Student Reasoning		Validating a Correct Answer	
Checking for Understanding			

	Facilitating Student Reasoning		Extending Student Reasoning	
	Low	High	Low	High
Guiding	Cueing	Providing Guidance	Encouraging Evaluation	Encouraging Reflection
Guiding	Funneling	Encouraging Multiple Solution Strategies	Pressing for Precision	Encouraging Reasoning
Guiding	Topaze Effect	Building	Topaze for Justification	Pressing for Justification
Providing	Providing Information	Providing Alternative Solution Strategies		Pressing for Generalization
Providing	Providing Procedural Explanation	Providing Conceptual Explanation		
Providing	Providing Summary Explanation			

Fig. 1 The TMSSR framework, from Ellis et al. (2019) Reprinted by permission from Springer Nature: Ellis, A., Özgür, Z., & Reiten, L. (2019). Teacher moves for supporting student reasoning, *Mathematics Education Research Journal, 31*, p. 117. Copyright 2019 by Springer Nature.

Results

We organize the results according to four vignettes that convey the nature of the students' argumentation and how the teacher moves seemed to tie students' mathematics together and mold the direction of their mathematical development.

The first vignette is comprised of four arguments that occurred in response to the instructor soliciting student rules for filling a Cayley table for the symmetry group of an equilateral triangle. In Vignette 2 (four arguments), the instructor unpacks and leverages one student's (S3) activity using the Chart (see Fig. 1 in the section "The Mathematics Involved in the Lesson" in the chapter by Plaxco, this volume) to algebratize S3's activity and lead to general discussion about associativity and commutativity. In Vignette 3, the instructor solicits more rules from the class. This vignette is comprised of three arguments about equivalence relationships between specific compositions of symmetries. Across Vignettes 1–3, we coded teacher moves that tended to shift from soliciting student thinking to providing summary explanations, encouraging reflection, and encouraging generalization.

In Vignette 4, the instructor again solicits student reasoning by posing computation tasks, intending for students to use the group rules from Vignettes 2 and 3. This supports an overarching instructional goal of generalizing the students' activity toward more flexible and generalized use of algebraic reasoning, which is consistent with the Local Instructional Theories of the TAAFU curriculum (Larsen, 2013) and the design heuristics of RME (Gravemeijer, 1999; Larsen, 2018). During the discussion in Vignette 4, S3 continued to leverage the Chart as they had in Vignette 1, which contrasts with the more general algebraic relationships discussed and used in Vignettes 2 and 3. This prompted the instructor to surmise the absence of the Chart in an effort to push students toward using substitution. Over the episode, the mathematical activity shifts from students sharing their ideas (Vignette 1) toward interpreting situated actions through algebraic representations and a focus on more general algebraic properties (Vignette 2), back to soliciting students' rules for combining symmetries with a focus on simplifying algebraic expressions with equivalences (Vignette 3), and finally to applying, practicing, and discussing the algebraic techniques in an exercise (Vignette 4).

Vignette 1: How to Fill in the Cayley Table

In the first vignette, students share their strategies for filling in the Cayley table symmetries of a triangle. First, the instructor invited students to share their "rules" for filling in the table. Three students provided rules as the instructor reacted to the rules, occasionally posing questions. Throughout this process, the instructor verbally paraphrased student contributions and projected a written record on a document camera. The vignette is composed of four arguments, listed in Table 1.

Table 1 List of arguments composing Vignette 1

Argument	(Our) Title: Primary Claim
1	Sudoku Rule: You can complete the table by figuring out "which one is missing" in a row or column
2	Identity Property, pt. 1: You could give the 3R column and row as identities
3	Identity Property, pt. 2: You can "fill in" the 3R row and the 3R column
4	The Chart: To get to a place where you can complete the table "Sudoku style," you can do one motion, find it as a triangle on the chart, do the next motion, and then find it until you get to the last one.

This vignette begins with the first teacher move of this classroom episode. The instructor asked the whole class, "Who has a rule they want to share with me?" We code this as *eliciting ideas* from the TMSSR framework. Ellis et al. (2019) defined this teacher move as "asking a question or questions to elicit students' ideas for a solution strategy or about a mathematical idea" (p. 119). This utterance was open-ended and nonprescriptive in nature, it shifted the students from small group into whole class interaction, and it invited students to share their ideas with the class. In response, three students provided contributions to describe their processes for filling in the Cayley table.

Argument 1 In the first Toulmin argument (Fig. 2) we identified, a student (S1) stated their approach for completing a row or column in the table was figuring out "which one is missing" (line 2). We coded this as the claim of the first argument (C1 in Fig. 2). S1 supported their assertion, stating "each appeared in each column and row only once" (line 2), which constitutes data for the claim (D1, Fig. 2). As S1 gave their justification, the instructor re-represented the student's verbal justification by writing "each symmetry appeared exactly once in each row and column" on the overhead. This is a restatement of D1; however, the instructor slightly changed the language in the student's original utterance by saying "each symmetry" rather than "each" and by using the word "exactly" instead of "only." This move is consistent with Ellis et al.'s (2019) definition of re-representing ("Form of re-voicing in which one provides a representation as a way to publicly share a student's idea or strategy. The teacher may organize, re-frame, or formalize the student's statement or work," p. 120). We contend that the subtle change in language, while formalizing the student's statement to be more precise, introduced a nontrivial mathematical distinction between the student's utterance and the instructor's restatement of it. Specifically, the instructor's re-representation aligns the students' ideas with conventional mathematical language. Although we cannot know if the instructor intended for the change to shift the logical implications of S1's statement, such intention is consistent with the TAAFU curriculum (Larsen, 2013).

S1 concluded their justification stating, "So around the end of my table, I just started seeing which one was missing" (line 2). We coded this utterance as a warrant (W1, Fig. 2) because it served to help support how the data (D1: each element appears once) supported the claim (C1: figuring out the missing entries helps complete the table). This was colloquially called "the *Sudoku* property" in the class—S3

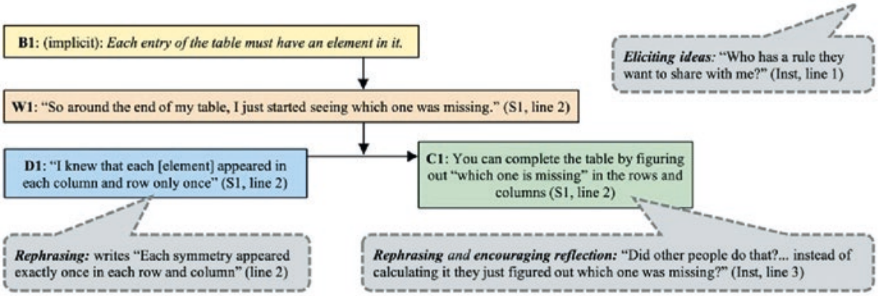

Fig. 2 Toulmin scheme for Argument 1

later refers to this idea using that phrase. Based on the lack of pushback from class members, it seems this rationale was previously established in the classroom and so here is functioning-as-if-shared (Rasmussen & Stephan, 2008). Argument 1 ends when the instructor asks, "Did other people do that?… instead of calculating it they just figured out which one was missing?" (line 3). We code this as re-voicing and encouraging reflection. Ellis et al. (2019) defined the latter as "asking students to reflect on provided answers or explanations (p.123)."

Arguments 2 and 3 The next arguments immediately followed the instructor's question above. S2 stated that the 3R column and row in the Cayley table could be given as identities, which we coded as a claim for Argument 2. The instructor wrote "3R as identities" and pressed S2 to clarify what they meant by "identities," which we coded as the instructional moves of rephrasing and figuring out student reasoning, respectively. Ellis et al. (2019) defined figuring out student reasoning as "attempting to understand a student's solution, explanation, or reasoning" (p. 119), and their exemplar includes the phrase "what do you mean?" S2 responded to this, stating, "3R plus any other equal whatever that any other was," which we coded in the Toulmin scheme as data in support of the initial claim. The instructor re-represented this statement by writing "any other + 3R = any other" on the overhead, which we also coded as data supporting the initial claim.

S2 added, without prompting, "the same with any other plus 3R." We coded this as additional data in support of their initial claim. This is a mathematically significant inclusion because it emphasizes the distinction between a left and right identity (the 3R row and the 3R column of the Cayley table). S2's extension prompted the instructor to write "any other + 3R = any other." This aligned the re-representing TMSSR with S2's second statement. We coded these data in Argument 2 in a linked structure (Aberdein, 2006), wherein multiple independent pieces of evidence are offered to support the same claim.

The instructor continued, asking, "Were other people doing that? For that row, you just went down and filled it all in? For that column you just filled it? Column and row?" (line 9). These questions explicitly describe the consequences of S2's identity relation for filling in the Cayley table. We coded the first question as encouraging reflection and the latter three collectively as the teacher move re-representing. The instructor re-represented S2's statement by shifting the language toward ways of filling in the Cayley table, which also tacitly shifted S2's claim that 3R is a two-sided identity into a process for filling in the table. Thus, two statements together form the claim for Argument 3: "For that row you [could] just [go] down and fill it all in" and "For that column you [could] just [fill] it." This claim is supported by S2's prior statements in Argument 2 that 3R is an identity. The instructor's written equations – "3R + any other = any other" and "any other + 3R = any other" —serve as an implicit warrant in Argument 3.

Arguments 2 and 3 establish 3R is a two-sided identity. It is significant that S2 stated both relationships and that the instructor re-represented those statements by turning them into algebraic equations. Each equation the instructor wrote connects S2's statements to the Cayley table. The first equation is about filling in the 3R row and the latter is about filling in the 3R column. This instance of re-representing is important for pushing forward the mathematical progression of the classroom. It is consistent with what Rasmussen and Marrongelle (2006) call a transformational record—a teacher's use of notation or diagrams to initially record student thinking that students later use to solve new problems. As we continue to discuss throughout this "Results" section, this shift from the students' contributions toward algebraic representations is consistent with a broader arc across the entirety of this classroom episode that focuses on transitioning from reasoning about the symmetries of the triangle with representations of the triangle (as in the Chart) toward reasoning about the symmetries with algebraic rules.

Argument 4 In response to the instructor's question from Argument 3, S3 contributed ideas that we analyzed as extending Argument 3, incorporating Argument 1, and also including their own new claim:

> Yeah, I did that, but then I also, for another one, before I could do filling it in Sudoku-style or like, after I did that, on the chart we did, I would just do one motion and then find it as a triangle and then do the next motion and then find it until I got to the last one. (line 10)

These comments comprise three separate means of filling in the Cayley table. In this speaking turn, we identify the claim that S3 used the Chart (see Fig. 1 in the chapter by Plaxco, this volume) to determine combinations of symmetries. In their statement, S3 situated this activity chronologically after filling in the 3R row and column (analyzed as Arguments 2 and 3), but before using the Sudoku property to finish the Cayley table (Argument 1). Accordingly, Arguments 1, 2, and 3 serve as Data 4 contextualizing the use of the Chart. S3 went on to describe using the Chart to determine the combination of R and F, which serves as additional data in support of Claim 4. S3 added that this process helps "keep track of it," which serves to warrant how this approach supported S3's ability to fill in the Cayley table. We see Argument 4 as an important moment for mathematical development in this classroom excerpt. Consistent with RME design principles, S3 is using an artifact from their prior activity (the Chart) to further organize the symmetries of a triangle in the Cayley table. This is a critical aspect of the theory of emergent models—that students will refer to their models–of situated activity that will serve as models–for their more general activity without having to refer to the original activity context (Gravemeijer, 1999).

Summary of Vignette 1 In this vignette, the instructor facilitated a discussion of the general rules that students used to fill in the Cayley table. Overall, this vignette consisted of two primary types of TMSSR—eliciting and responding to student reasoning—with some additional instances of extending student reasoning. Generally, throughout the vignette, a student or the instructor would contribute a statement that served as a claim and provide data in support of that claim, occasionally providing a warrant. This relatively straightforward argumentation pattern supports a sense that the mathematics within the arguments was functioning-as-if-shared within the classroom. In several cases, the instructor's re-voicing and re-representing moves served to bolster initial claims and data provided by the students and to document the discussion in an algebraic form in view of the entire class. Based on the TAAFU curricular goals and the trajectory of the discussion in this class, it is reasonable to assume the instructor's goal during Vignette 1 was to solicit student contributions about their activity using the Chart so that the instructor could leverage this toward a discussion of the more general algebraic relationships among the symmetries of a triangle. Following S3's contribution in Argument 4, the instructor began probing how S3 specifically used the Chart.

Vignette 2: Leveraging Specific Activity Toward General Discussion About Associativity

In Vignette 2, Arguments 5–8 leverage S3's activity from Argument 4 to lead toward a more general discussion about group properties, specifically associativity. Vignette 2 is composed of four arguments (Table 2).

Argument 5 In Argument 5, the instructor and S3 jointly unpacked S3's activity using the Chart to simplify the expression R+(R+F). This began when the instructor asked S3 to explain to the class how to simplify R+(R+F) in a way that was consistent with their description in Argument 4: "So I think what [S3] was saying, let's do a make believe one that [S3 was] doing R, and [they were] combining that with RF [writes R+(R+F)]. [S3] would first look here and see where the first R got [them], and then what would you do?" (line 12). In a back-and-forth exchange, S3 explained their reasoning by explicitly referring to the pictures in the Chart, which is consistent with the situated activity from previous TAAFU lessons. The instructor then

Table 2 List of arguments comprising Vignette 2

Argument	Title: Primary Claim
5	Actions vs. Algebra: But S3 didn't think of this as a combined motion
6	Focus on Parentheses: To capture what S3 was doing, those parentheses don't stay in the same place
7	Associative Property: R+ (R+F) could be 2R plus F
8	Affirming Associativity: We can drop the parentheses all together

shifted to focusing on an *algebraic* interpretation of S3's situated activity with the Chart, saying,

> So [S3] used these pictures, doing one at a time. If I was going to write that out algebraically [underlines R+(R+F)] according to the table I need to do R first [draws a box around R], and then I need to do this motion: R+F [circles (R+F) on the projector]. Which is this. But [S3] didn't think of this [points at (R+F)] as this combined motion. What you kind of did is R, and then R, and then F. Right? What I'm trying to get at is that this is how it was asked: R, and then the combination, the grouping, of R+F. If I had to write that with parentheses and symbols, how could I capture what [S3] did? Would those parentheses stay in the same place? (line 16)

In this excerpt, we coded five teacher moves: encouraging reflection, providing summary explanation, re-voicing, re-representing, and cueing. For instance, the instructor re-voiced S3's rationales for using the table to find equivalences and re-represented them as algebraic statements. By asking how to capture S3's activity, the instructor was encouraging reflection; we interpret this as provoking the students to recognize what the instructor saw as a difference between the exercise as originally posed and S3's activity as they described it. The instructor then engaged in cueing by asking whether the parentheses would stay in the same place.

Throughout line 16, the instructor referred back to S3's activity. Accordingly, when coding the Toulmin scheme for Argument 5, the initial exchange between S3 and the instructor served as data in support of the instructor's eventual claim for Argument 5: "But [S3] didn't think of [R+F] as a combined motion." We posit (based on the TAAFU task sequence and the direction the class discussion takes) that the instructor was intentionally using S3's activity to begin a more general conversation about the associativity of the symmetries of the triangle. The instructor initiated this by focusing on the parentheses needed to accurately algebraically describe S3's activity. This afforded them the opportunity to press for additional justification for why S3's actions were consistent with the mathematics of the class. Because of the class's prior experiences working with the symmetries, S3's situated activity was reasonable. Focusing on the algebraic representation of that activity, however, necessitated students to justify more generally why reassociating should be allowed. Accordingly, the instructor's re-representing moves during this argument serve a critical role of establishing a shared artifact around which the argumentation of the class will develop.

Argument 6 Near the end of line 16, the instructor asked, "Would those parentheses stay in the same place?" S4 responded by saying "I don't think so" (line 17). Together, these statements serve as the claim for Argument 6: that the parentheses do not "stay in the same place" when comparing the written R+(R+F) to how S3's activity should be represented algebraically. S3 immediately asked, "Wasn't it just that same thing foiled out because there's an imaginary one in front of those parentheses there?" (line 18), which we interpret as S3's justification for acting on the symmetries as they had. The instructor then asked for clarification, "So you see it as just taking the parentheses off all together?" (line 19), which S3 confirmed (data for Argument 6) and warranted a priori by their assertion that it is the "same thing foiled out." The instructor then re-voiced S3's activity and pointed out that "foiling out by 1" did not have established meaning within the class when dealing with symmetries, stating, "…we don't have ones. I don't know what foiled by one means in symmetry" (line 21). This served as a counterargument to S3's justification because it questioned the means by which S3 rationalized the equivalence of R+(R+F) and R+R+F. S5 responded to the instructor's counterargument by proposing a different inscription to notate S3's activity: "gradual outward parentheses," accompanied by a cupped hands gesture. The instructor asked for clarification while trying to write it algebraically on the overhead (re-representing) and asking if S5 thought what S3 did was (R+R)+F. S5 responded negatively and clarified that they thought "there's three different sets of parentheses." S6, S5, and S3 respectively offered several descriptions, including a reference to nested Russian dolls. The instructor re-represented their ideas by writing (((R)+R)+F) and commented, "Something crazy has happened to our parentheses" (line 29). We coded this cascade as additional data to support the claim that capturing S3's activity as an algebraic statement would require parentheses in a different place than the statement R+(R+F), such as (((R)+R)+F). This provided a moment for students to reflect on their thinking about the differences in the two inscriptions without any direct explanation from the instructor.

Arguments 7 and 8 Next, without prompting, S7 asked, "Could that [meaning R+(R+F)] be 2R+F?" (line 30), which we coded as Argument 7's claim. This led to a short exchange; to support S7's data that the two algebraic statements are equivalent according to the Chart, the instructor mentioned "regrouping which ones I'm considered associated with each other" and asked if this "sounds like something [the students have] seen before in math" (line 33), to which S8 offered "the associative property." We coded this TMSSR as funneling, which Ellis et al. (2019) define as "asking leading questions to direct students down a specific path" (p. 121). S3 then stated that they had thought to volunteer that phrase earlier in the conversation, and the instructor re-voiced and extended it, adding, "That's a great question to ask. Can I regroup?" This led to a dialogue, which the instructor summarized as, "It looks like we can drop the parentheses all together, and we still know what we're doing and that's not changing answers, or we can regroup things, and that's not changing answers. Keep an eye on that as you go through" (line 42) and "That's definitely a rule that we're using. Something about dropping parentheses …Or regrouping. Something like that was going on" (line 44). Here, the instructor was cueing the students to focus on this aspect of their algebraic manipulation. Finally, the instructor focused on the notion that their operations with symmetries were not commutative but were associative (dual claims for Argument 8). The data for this argument is entirely comprised of instructor statements supporting those claims, which we coded as the TMSSR move providing information. We see this excerpt as concluding Vignette 2 because the instructor has leveraged S3's specific activity toward the introduction and discussion of associativity. After speaking generally about associativity, the instructor then shifted to more specific language of "dropping parentheses" and to asking students to reflect on their own activity with the triangle and this task.

Vignette 3: Instructor Solicits More Rules, but These Are Specific to the Relationships Between R and F as Symmetries of a Triangle

After the Vignette 2 interactions, the instructor solicited more rules that the students used to complete the Cayley table. We identify this as a shift to a third vignette, in which the instructor solicited and emphasized important relationships specific to the group of symmetries of the triangle. This vignette, composed of three arguments (Table 3), serves as an important transition in the classroom discourse because the conversation shifts toward a focus on equivalence relationships useful for manipulating algebraic statements via substitution, rather than relying on the Chart.

Table 3 List of arguments composing Vignette 3

Argument	(Our) Title: Primary Claim
9	"Intuitive" Relationships: R plus R equals 2R, and 2F equals 3R
10	F + R = R + R + F rule: F plus R is the same as R plus R plus F
11	Moving R past F Doubles It: An R on one side of an F is equal to two Rs on the other side, in any case

Arguments 9 and 10 S9 was first to offer some rules, citing the rules' intuitive nature to explain how they found them (Argument 9 claim and data). This includes R+R=2R and 3R=2F, relationships that the class had previously discussed and written on the Chart. As S9 spoke, the instructor re-represented S9's rules on the document camera, shifting from spoken symbol references to written inscriptions for them. The instructor also said, "Yes. Good." which we coded as validating a correct answer. Unlike with the previous rules such as identity (Arguments 2–3) and associativity (Arguments 7–8), the instructor noted these relationships are context dependent and do not generally hold in other scenarios, which we coded as the TMSSR of providing information. The instructor cited the symmetries of a square to contrast the identity equivalences between the two groups; together, these serve as a qualifier and rebuttal in Argument 9.

Following this, the instructor pointed out one group's whiteboard work and asked one of the group members to explain "what's going on." We see this as a blend of the teacher moves of cueing and eliciting ideas—moves that, in this case, prompted a student to state a mathematical relationship from their group's whiteboard work that serves as the claim in the ensuing argument: "F plus R is the same as R plus R plus F" (line 51). The instructor re-voiced and re-represented this statement, extending it by saying, "Did anyone else do that? Notice that? It's even on our little piece of paper, right? F plus R is the same as 2R plus F" (line 52). We coded this as the teacher move encouraging reflection. In support of S8's initial claim, the instructor continued with a series of statements and algebraic manipulation, which we coded as the teacher move providing procedural explanation. In this speaking turn,

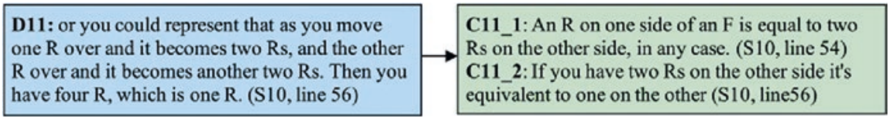

Fig. 3 Toulmin scheme for Argument 11

Argument 8 served to warrant algebraic actions in the current argument. This explanation halted abruptly as the instructor realized a small mismatch between the rule generated and the argument they were developing. In any case, this mismatch in the discussion led to the next argument involving a generalization offered by S10 (Fig. 3).

Argument 11 S10 offered a restatement of the rule, saying "An R on one side of an F is equal to two Rs on the other side, in any case" (line 54), then clarified as "If you have two Rs on the other side it's equivalent to one on the other" (line 56). In this claim, S10 focused on the reciprocal nature of the relationship, including the prepositional phrase "in any case." S10 then justified this equality: "You could represent that as you move one R over and it becomes two Rs, and the other R over and it becomes another two Rs. Then you have four R, which is one R." This language of a "move" had not yet been used in this class discussion when describing manipulation of algebraic statements. Together, these statements form a concise data-claim chain that lead to the teacher moves in Vignette 4.

Summarizing Vignette 3 In these three arguments, the instructor solicited more rules that students found helpful toward filling in the Cayley table. In Argument 9, S9 described what they saw as some of the more straightforward relationships between R and F. In Arguments 10 and 11, S8 claimed F+R=2R+F and S10 generalized that to also include R+F=F+2R. Together with the discussion of the Sudoku property, associativity, and identity from Vignettes 1–3, these rules can be seen as providing a rationale for a calculus for manipulating the elements R and F in algebraic statements (e.g., S10's description as a "move"). Additionally, as shown in Vignette 4, these relationships can be leveraged toward manipulating and simplifying algebraic statements through substitution.

Vignette 4: Demonstrating the New Rule and Moving Toward Incorporation of New Rules

The fourth vignette is composed of four arguments (Table 4). It builds from S10's mathematics in Argument 11 and begins with the instructor attempting to support incorporating algebraic relationships to manipulate and simplify algebraic statements, primarily through substitution. The instructor first worked an example on the overhead, which constitutes Argument 12. Then the class shifted to small group

Table 4 List of arguments composing Vignette 4

Argument	(Our) Title: Primary Claim
12	Instructor's example: (F + R) + (R + F) = 2R
13	Instructor explains student work: (F + R) + (F + 2R) = R
14	S3 returns to the table: (F + R) + (F + 2R) = R
15	A different algebraic solution: (F + R) + (F + 2R) = R

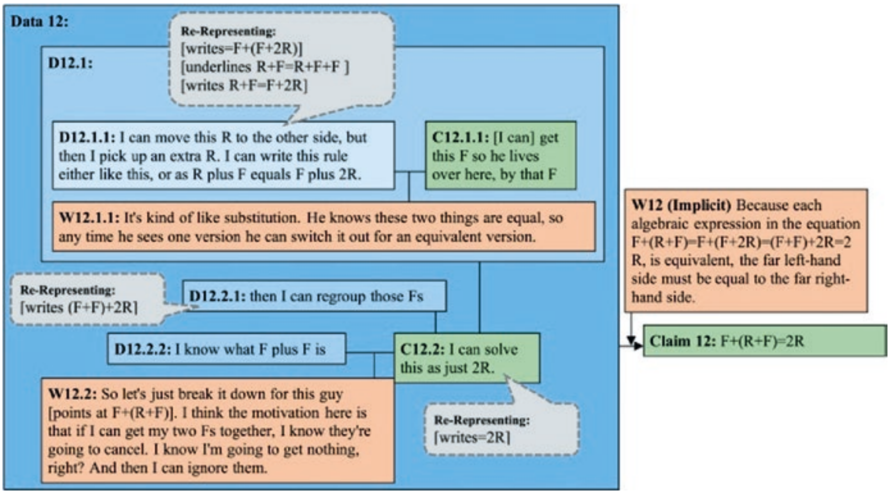

Fig. 4 Toulmin scheme for Argument 12; all quotes from the instructor, line 57

work in which the students worked to simplify a new algebraic statement. Following that, the instructor narrated one student's written algebraic solution (Argument 13), which prompted S3 to ask whether the students are expected to complete such exercises that way rather than by referring to the Chart to simplify the expression. The instructor responded by surmising the absence of the Chart and discussing the utility of the algebraic methods for manipulating statements.

Argument 12 Here, the instructor revisited simplifying F+(R+F). She began an extended speaking turn (line 57), which comprises an entire argument (Fig. 4). The instructor motivated her work in advance, laying out a strategy for altering the expression so that instances of each element are adjacent, which would allow some elements to cancel. These advance statements serve as warrants to the verbal and written algebraic activity that followed. First, the instructor removed the parentheses from the statement, which uses Argument 8's claim.

The instructor then described moving an R from one side of an F to the other, saying, "I pick up an extra R," which incorporates the rule from Argument 11's claim. The instructor used a consequence of Argument 8's claim when they "regroup[ed] the F's." Finally, they said, "I know what F plus F is," relating to S9's intuitive relationships from Argument 9. Throughout this excerpt, the instructor implicitly relied on the results of prior relationships or rules, which allowed them to manipulate the algebraic statement until it equaled 2R (claim, Fig. 4). This argument consists of various instantiations of the teacher moves providing information, providing procedural explanation, and providing summary explanation.

Argument 13 Next, the instructor gave the class time to simplify (F+R)+(F+2R) individually or with a partner. After the students finished, the instructor solicited S11's work and re-voiced the students' written solution process. This was similar to Argument 12, with changes from one algebraic statement to another reflecting the claims of prior arguments. Throughout this explanation, the instructor referred to specific algebraic statements in S11's work.

> So it looks like [S11] took this guy - first, you just dropped the parentheses, and made them all Rs, instead of 2R. Then, is it this guy, that you rewrote? (S11: Yeah, I guess so.) [S11] replaced that R plus F with F plus 2R, so we see those ones showing up right here. Then the Fs are on one side, and the Rs are on one side, and somehow all those Fs and all those Rs just turned into one R. (lines 61 and 64)

We view these statements as data supporting Argument 13's overarching claim that the beginning algebraic statement, (F+R)+(F+2R), is equivalent to 2R. As with Argument 12, S11's algebraic manipulation reflects claims from prior arguments. Specifically, S11 removed parentheses (Argument 8), substituted F+2R for R+F (Arguments 10 and 11), and removed F+F and R+R+R from the algebraic statement (Arguments 2 and 9). Again, throughout the instructor's narration of S11's written work, she is primarily *re-voicing*, *providing information*, *providing procedural explanation*, and *providing summary explanation*.

Argument 14 Without prompting, S3 then shifted the discussion, asking whether it was necessary to use that same process for simplifying the expression ("Do we have to use that kind of stuff? Because I didn't do that" (line 65)). S3 explained how they reached the same answer as S11 (so Argument 14's claim is also (F+R)+(F+2R)=R). During this process, S8 interjected comments that their work began similarly to S3. S3 explained that to substitute R+F for the statement F+2R, they "looked at [the Chart] and looked at what F plus 2R was." As S8 interjected, the two students rehashed the process that resulted in the expression (F+R)+(R+F). Together, these statements constitute two sub-arguments that serve as data to support the overarching claim (see Fig. 5). Throughout this exchange, the instructor engaged the students by asking questions coded as figuring out student reasoning and eliciting facts or procedures.

Next, S3 explained that, at this point, they kept the statement as (F+R)+(R+F) and referenced the Chart (line 74), which serves as Argument 14's warrant. Recalling S3's activity in Argument 4, this explanation is consistent with their prior reasoning. However, the instructor responded by asking, "Yeah, but what if you didn't have your table?" (line 75). With this question, the instructor is surmising a scenario that had not been previously mentioned in this class discussion—the possibility that the students might, at some point, not have access to the Chart some of them were using to simplify statements (as exemplified by S3 in Argument 4). We view this

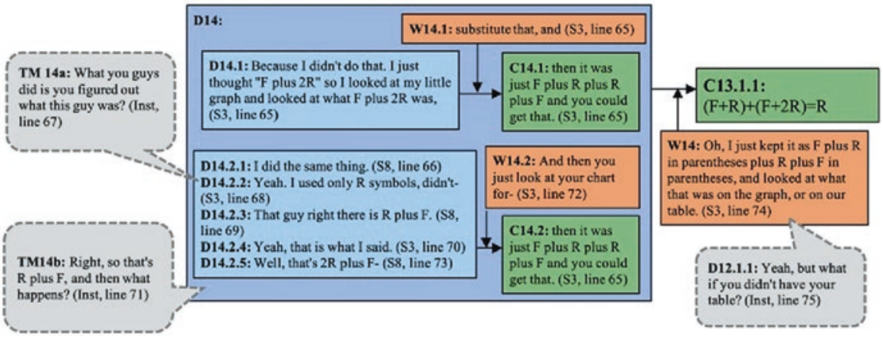

Fig. 5 Toulmin scheme for Argument 14

important teacher move as *pressing for generalization* because the instructor pushed back on S3's explanation, emphasizing the more general relationships discussed in Arguments 10–13. It is reasonable to assume, based on the TAAFU curricular materials and the general task sequencing of RME-based materials, that the instructor is attempting to support S3 to shift *from* referential mathematical activity situated in an artifact based directly on the students' prior exploration of the symmetries (the Chart) *to* more generalized activity of using algebraic relationships to manipulate the algebraic expression.

Argument 15 After the instructor pushed back on S3's reliance on the Chart, S8 offered a strategy to extend S3's activity by using the rules as the instructor had demonstrated. In this process, S8 began with (F+R)+(R+F) and regrouped the elements, tacitly using associativity (Arguments 7 and 8), then substituted R+F for F+2R (Arguments 10–13), then again tacitly relied on associativity by regrouping the two Fs together, and finally removed the 2F term. Following this, the instructor rephrased the entire approach, aligning it with their discussion for making substitutions such as R+F for F+2R (Arguments 10–12). In this episode, S8 provided the data supporting the claim that (F+R)+(F+2R)=R. During this argument, the instructor briefly asked for clarification and re-voiced S8's substitution step, emphasizing its importance in the argument. Following S8's explanation, the instructor provided conceptual explanation, emphasizing the reasons for S8's activity and also the general strategy for the algebraic manipulation.

Summarizing Vignette 4 Across these four arguments, the instructor pressed for generalized manipulation of algebraic statements, specifically focusing on using the equivalence relationships from Vignettes 1–3 to substitute one statement for another. Accordingly, the claims of prior arguments often served as tacit and explicit data and warrants in these arguments. The instructor's TMSSR codes first centered heavily on providing information, providing procedural explanation, and providing conceptual explanation (Arguments 12 and 13). In Argument 14, the teacher moves included eliciting student reasoning, as when they asked S3 about their reasoning.

When S3 referred to the Chart, the instructor then pressed for generalization, re-voiced S8's activity, and provided conceptual explanation for how the algebraic equivalences could be used to leverage algebraic statements.

Discussion

In this chapter, we used Toulmin's model of argumentation and the TMSSR framework to analyze classroom-level argumentation and teacher moves guiding it. Throughout the episode, the instructor primarily elicits, responds to, and facilitates student reasoning; the most common teacher moves in our data were re-voicing and re-representing. This could be due to the fact that this episode occurred just after the students had extensively explored the symmetries of the triangle in context and organized that work. This is consistent with S3's description of using the Chart to simplify combinations of symmetries. In Vignette 2, we see the instructor encourage reflection and press the students to interpret the situated activity with the Chart in terms of more general algebraic representation, specifically focusing on the parentheses in the algebraic statements. After soliciting more algebraic relationships in Vignette 3, the instructor continues to extend student reasoning by leveraging the relationships from prior classroom argumentation toward a calculus for simplifying algebraic statements.

Throughout this classroom discussion, the skillful teacher moves allowed students to contribute their thinking to the classroom discussion and also gave the instructor the means to document that thinking, push for clarification, and encourage other students to reflect on the mathematics of the class. While student contributions generally comprised the main parts of each argument, the TMSSRs of re-voicing and re-representing often allowed the instructor to provide additional data and claims to those arguments. Further, the re-representing TMSSR served to algebratize the mathematical relationships that students described. This supports a trajectory from situated activity with symmetries of triangles to more general activity manipulating algebraic statements based on equivalence relationships.

In Vignette 4, the instructor shifted first toward providing a procedural explanation, providing conceptual explanation, and checking for understanding. When S3 persisted in relying on the Chart to simplify an algebraic expression, the instructor pressed for generalization by asking S3 what they would do if they did not have the Chart. In surmising the absence of the Chart, the instructor imposed a condition that would require S3 to use some other means for simplification. It is in the balance of these forces – the instructional objective to generalize situated student activity and the meaningfulness of said situated activity – into which the dual focus on argumentation and TMSSR provide insight. The argumentation is both a means for students to communicate their reasoning about concepts and a mechanism for advancing the mathematics of the classroom toward more general activity. The TMSSR provides a lens through which to examine the instructor moves that both solicit, develop, and support the argumentation and push the mathematical agenda of the class.

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Justification in the Context of Tertiary Mathematics: Undergraduate Students Exploring the Properties and Relations of the Dihedral Group

Shiv Smith Karunakaran and Mariana Levin

Introduction

In mathematics, justification has a multiplicity of purposes: validation of claims, providing insight into a result, and systematizing one's knowledge, to name just a few (Bell, 1976; de Villiers, 1990, 1999, 2002; Hanna, 1990, 2000; Staples et al., 2012). While the field acknowledges these different functions of justification in the practice of mathematicians, this volume seeks to shed light on the role of justification (along with argumentation and proof) in K-16 classrooms. It has long been argued (cf. Cohen & Ball, 2001) that in addition to justification serving a role in contributing new understanding to the field of mathematics, engaging students in justification practices plays an important and generative role in deepening their understanding of mathematics. In the K-12 context, empirical data backs up this claim, showing that classrooms in which students are pressed for their mathematical rationales and to articulate their mathematical sense-making processes result in students who have deeper understanding of the mathematical concepts they are learning (Hiebert et al., 1997; Wood et al., 2006). Major curricular reform projects have been working to focus on strengthening students' engagement with justification practices by offering opportunities to learn these practices through the tasks posed by the curriculum. While an important first step is making argumentation, justification, and proving practices accessible to all students, the reality in classrooms still

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lags behind. For instance, Bieda's (2010) study of middle grades teachers' implementation of the *Connected Mathematics* curriculum showed that across 49 lesson observations in which proof-related tasks were part of the lesson, only 28 student responses included proof-related arguments. The field is still unpacking the reasons for this discrepancy between written and enacted curricula with respect to proof and justification. Staples et al. (2012) discuss teachers' conceptions of mathematics and conceptions of the processes of learning mathematics, which are factors that shape how they interact with their students around mathematics, and around proof, in particular. Indeed, this perspective is corroborated by the study of Knuth (2002) that found that secondary mathematics teachers may have limited conceptions about the role of proof in mathematics, particularly the role of proof in *learning* mathematics.

In this chapter, we explore the landscape at the tertiary level, and more specifically with respect to justification, as opposed to "proof" or "proof-related activities" more generally. At the tertiary level, there is little explicit empirical evidence in the form of research studies of "typical" instructional practice, but there are several working groups organized by professional organizations on increasing active learning in classes and disrupting teacher-centered models of instruction, even in large lecture courses. Still, there remain substantial barriers to implementing student-centered instruction that focuses on justification practices at the tertiary level. Some of the same issues that arise in the K-12 context pertain to the tertiary context. For example, even if the mathematical knowledge of university professors of mathematics and their understanding of the role of justification in mathematics as a discipline is not at issue, university professors may not believe that bringing these practices into the classroom is the most effective method of deepening students' understanding of the material. They may be even more resistant to trying instruction different from their own instructional experience as students because they do not see their task as reaching all students—students who need such "extra" help to understand the mathematics maybe should not be learning this content. In any case, university professors' beliefs about the nature of mathematics and the nature of learning still shape how they interpret and implement curricular materials (Küchle & Karunakaran, 2019).

Defining and Operationalizing Justification

Within the context and goal of this volume, we started from the definition given in Bieda and Staples (2020): "Mathematical justification is the process of supporting your mathematical claims and choices and explaining why your claim or answer makes sense" (p. 103). This definition immediately brought to mind the dichotomy between a focus on justification described as a process of justifying and justification described as the product emerging from engaging in justifying. At the top level, we point out that the definition the editors of this volume asked us to work with focuses on justification as a process and that this diverges from how justification can be used to label the product of the process of justification (e.g., a justification is not something that can be labeled valid or not valid—it is simply a process). We note further that even though we are interested in understanding the process of supporting mathematical claims, the reasons given for that support do not need to be themselves mathematical. In order to distinguish justifying/justification from argumentation/arguments, we bring attention to the fact that while argumentation involves the process of making claims, justification is more narrowly focused on the activity that follows claims that are already on the table for consideration.

In contrast, Staples et al. (2012) use the term "justification" to mean "an argument that demonstrates (or refutes) the truth of a claim and that uses accepted statements and mathematical forms of reasoning" (p. 448). Note how different that definition is from the one that is used in the current volume. Staples et al. (2012) highlight similarities between their definition and Stylianides' (2007) definition of arguments that function as proofs in classroom communities. Staples et al. do not focus on the community aspect, and they do not make judgments about forms of reasoning or representations that are within the conceptual reach of the community. However, they do stipulate that the type of reasoning they are concerned with must be mathematical (excluding possibilities like "because that's what John said"). Appeals to previously established results "That's what we proved yesterday" are allowed. Empirical and example-based reasoning is also considered a mathematical form of reasoning (even if it may not, in a given context, lead to a demonstration of the truth of a claim). Consider the following to help illustrate the process and product distinction: in a proof (which is the result of a process), the arguments and ideas that lead nowhere do not appear. They do, however, play a functional role in the proving process. Staples et al. (2012) connect their notion of justification with the Common Core (CCSSO & NGA Center, 2010) Mathematical Practice 3: "Construct viable arguments and critique the reasoning of others." This practice calls for students to "understand and use stated assumptions, definitions, and previously established results in constructing arguments." The authors describe justification as a learning practice that promotes understanding, and they discuss how justification may be more appropriate for middle school students than, say, formal proof. They worry that "proof" might narrow focus on formality of expression or specific form of argument and instead what is of interest to them is the examination of the truth of a claim.

Context for the Data Provided to Us

As with every other chapter examining the tertiary mathematics level in this volume, we were given certain information about the data we were to analyze. The complete details of this data are found in the introductory chapter of the tertiary section of this book. However, we chose to highlight the following details of the data collected as they provide relevant context to the analyses we conducted.

The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU) (IOAA Team, 2016). TAAFU, an inquiry-oriented, RME-based curriculum, relies on Local Instructional Theories (LITs) that anticipate students' development of conceptual understanding of ideas in group theory. The curriculum is intended to allow students to feel as though the mathematics developed is a product developed from their own informal ways of thinking. Because of this, the students engage frequently in the practice of building and communicating mathematical arguments based on their current ways of understanding.

The classroom format consisted of students working in small groups, periodically breaking into whole-class discussion. The clip we analyzed comes from the third meeting of the class. The data set provided to the authors of these chapters included transcripts of approximately 20 min of class discussion; some screen captures of images projected in the classroom at various times during the discussion were included in the transcripts. For additional information about the instructional goals and pedagogical approaches used in the lesson, see "Overview of the Tertiary Level Data" (Plaxco, this volume).

The data we analyzed took place at the beginning of the year, and thus the norms of justification/proving/argumentation were just in the process of being established. Rich in the transcript were examples of the instructor revoicing student contributions and tossing them back to the class for consideration, at least at some level (e.g., what do you think?), and requests for a multiplicity of viewpoints to be shared (e.g., did anyone else have a way to do X/think about Y?). This breaks a typical classroom norm of the teacher presenting mathematical theorems and specific applications without space in class discussions for students to make sense of the topic under study.

The very activity that the students were engaged in—systematically exploring symmetries of a triangle for the purposes of understanding the emergent structure in how they combine with one another—is a pedagogical choice that is based on having students explore an idea through concrete examples and experiences before giving it a formal name, such as "group" or even the specific object name in this case of "dihedral group."

Research Questions

Given the context of the data provided to us and the perspective on justification, we chose to examine the following research question: *What is the nature of participants' justifications as they explore algebraic properties of the set of symmetries (of the equilateral triangle) under the operation of combining symmetries?*

Method of Analysis

In operationalizing justification as a process or an activity, we started from the premise that we first needed to be able to identify the claims that were being considered in the class discussion in question. That said, we do not work from the assumption that the person who made a claim must have some rationale or justification in mind. This is likely to be true, at least at some level, but not necessary.

In order to explore both facets of justification (process/product), we first identified the mathematical claims in the data and any answers provided by the students or teacher. This involved examining the entire transcript segment line by line for evidence of a claim. We use our colleagues' (Fukawa-Connelly & Karahoca, this volume) characterization of a mathematical claim: "Mathematical claims that have a definite truth value," by which they mean, "a statement whose status as true or false can be known independent of an individual, and which are formulated in sufficient detail to make a determination about the truth value" (see the introduction in the chapter by Fukawa-Connelly & Karahoca, this volume).

Two things struck us as we did this preliminary pass through the data. The first was that the flow of the discussion among the teacher and the students seemed choppy with little mathematical depth for most of the rationales given for the voiced mathematical claims. The second immediate perception was that while many claims were being introduced, the pathway to becoming a claim went through the instructor. That is, it seemed to us that many statements or answers were voiced by the students in the class and then "claimified" by the teacher through discourse moves such as revoicing.

Thus, this suggested to us two additional passes through the data could be illuminating, resulting in the following three broad steps as our methodology:

1. Identify all claims and answers; code the transcript segments that were related to engaging in the process of justification (providing a rationale) for that claim, choice, or answer; and then examine the products of the process of justification, i.e., "justifications."
2. Look at the range of justifications that were identified and sort them according to their level of sophistication.
3. Finally, and acknowledging that the products observed in this class do not occur in a vacuum and instead the norms for justification were still being established at this special time in the semester (day 3 of the course), look back at what conditions were supporting the establishment of norms around justifications.

We were further interested in the *source* of the discussion around claims, choices, and answers. For example, we asked the following questions of the data: were a claim and an ensuing discussion something that typically originated in a student contribution, a teacher contribution, and a teacher revoicing of a student contribution, or was a justification pieced together by multiple participants including both students and the teacher? In this way, we also looked for patterns of engagement among the teacher and students in the classroom in order to attempt to gain insight into the source of the justifications present.

Results

Given that we were asked to consider Bieda and Staples' (2020) definition of justification in relation to a whole-class discussion of the results of students' sense-making, we often found our attention coming to the depth or sophistication of the justifications provided. A pattern that almost immediately emerged was that participants appeared to be explaining *what* they were doing as opposed to *why* it was justified or made sense. In other words, rather than expressing mathematical rationales for why their claims, choices, or answers made sense, we routinely observed the participants recounting the process they used to arrive at the said claim, choice, or answer.

To understand what we were observing in the data relative to the nature of the students' justifications, we identified three categories of justifications as a way to sort the kinds of reasoning participants were providing to support answers, choices, and claims. Before describing the categories that emerged, we note that we do not mean to privilege any of the categories over others. As we discuss later in this chapter, we recognize the need for and value of each of the categories of justifications.

We will briefly describe each category and present an illustrative narrative or two from the data in order to exemplify our description for each category.

Category 1: Perceptual Pattern Recognition Justifications

Justifications within this category are characterized by the participants offering perceptual pattern recognition as the rationale for why their claims, answers, or choices make sense.

At the very outset of the class period, the students had been asked to generate the Cayley table for the dihedral group of degree 3 (see Table 1). This dihedral group

Table 1 Filled out Cayley table for the dihedral group of degree 3

	I	**R**	**2R**	**F**	**F+R**	**F+2R**
I	I	R	2R	F	F+R	F+2R
R	R	2R	I	F+R	F+2R	F
2R	2R	I	R	F+2R	F	F+R
F	F	F+2R	F+R	I	2R	R
F+R	F+R	F	F+2R	R	I	2R
F+2R	F+2R	F+R	F	2R	R	I

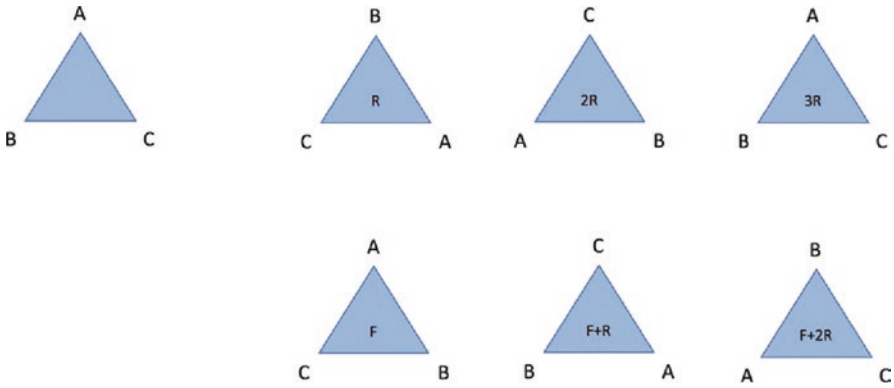

Fig. 1 The symmetries of the equilateral triangle ABC, where R is a clockwise rotation of 120° and F is a flip along the vertical axis. We follow the convention that one does the operation in the column first and then follows this with the operation in the row. Note in the figure above illustrating the rigid motions of an equilateral triangle, one reads the operations from left to right (i.e., F+R means that one does a flip first and then a rotation)

(D_3) can also be thought about as the various symmetries of an equilateral triangle with a clockwise rotation of 120° and a flip along the vertical axis (see Fig. 1).

The teacher (presumably having given some time to the students to fill out the table) then asked the students to conjecture rules for the symmetries of the equilateral triangle that could be produced from either filling out the table or by manipulating the symmetries of the triangle. In response to the teacher's question, one of the students (S1) answered, "I used the Cayley Table shortcut which, after like I saw the pattern, I knew that each one appeared in each column and row only once. So, around the end of my table, I just started seeing which one was missing." The rationale presented by the student for how ze[1] filled out of the Cayley table is centered around the perceptually recognized pattern that each of the symmetries appears in each row and column exactly once. This pattern seems to have been referred to by the students and the teacher as the Sudoku-style of filling out the table. Note that there is no mathematical reasoning presented as support for S1's answer. Instead, the rest of the participants in the classroom seem to accept S1's answer and zir justification for the answer.

[1] We chose to use the Ze/Zir pronouns for all the participants in this classroom, since we were not provided with preferred gender pronouns for the participants.

Category 2: Recounting of Process Justifications

Justifications within this category are characterized by the participants offering a blow-by-blow recounting of the actual process they undertook in order to arrive at the answer, claim, or choice as a justification for the said answer, claim, or choice.

After student S1 described zir justification for how ze filled out the table, the teacher further probed the other students for how they filled out their Cayley tables. Another student (S3) responds:

> … before I could do filling it in Sudoku-style … on the chart we did, I would just do one motion and then find it as a triangle and then do the next motion and then find it until I got to the last one. I did the side first, when I was at Row R, … I went to R, looked at the triangle, and then, say it was plus F, I flipped that, found where it was on here, and did the next thing, and then found where it was, so I that I could keep track of it.

S3 seems to be recounting the exact process by which S3 arrived at zir own answers for the cells of the Cayley table. S3 describes how ze seems to have manipulated the triangle using the requisite symmetries in order, and then ze seems to have used previously drawn out symmetries (as in Fig. 1) to identify the resultant symmetry as the answer to the missing cell of the Cayley table. In other words, this justification is a recounting of the step-by-step process undertaken by S3 as ze arrived at the answer. Note that this justification did not contain any mathematical rationale for why the process works, and rather focuses on how the process arrives at the answer.

Category 3: Mathematically Rationalized Justifications

Justifications within this category are characterized by the participants offering a predominantly mathematical rationale for the process by which they arrived at a claim, answer, or choice.

While there were a multitude of instances of category 1 and 2 justifications, in the segment analyzed, we found just one instance of a category 3 justification. Moreover, the one instance of a category 3 justification was the case of the teacher trying to extrapolate or extend to a category 3 justification from various category 2 justifications provided to the teacher by students.

Immediately after the category 2 justification previously provided by S3, the teacher said:

> Does that make sense? So, <ze> used these pictures, doing one at a time. If I was going to write that out algebraically [underlines R + (R + F)], according to the table I need to do R first [draws a box around R], and then I need to do this motion: R + F [circles (R + F) on the projector]. Which is this. But <ze> didn't think of this [points at (R + F)], as this combined motion. What you kind of did is R, and then R, and then F. Right? What I'm trying to get at is that this is how it was asked: R, and then the combination, the grouping, of R + F. If I had to write that with parentheses and symbols, how could I capture what she was doing? Would those parentheses stay in the same place? Are you shaking your head no?

Even though the teacher acknowledges S3's answer, the teacher continues on to attempt to reacquire S3's answer algebraically. Instead of focusing on the step-by-step process undertaken by S3, the teacher chooses to consider (R+F) as one symmetry, as opposed to a combination of two symmetries. By doing so, ze claims to have arrived at the same answer as S3, but without using the drawn out triangles.

The teacher pushed the discussion further in order for the students to perhaps notice that the symmetry (R+R)+F can be regrouped as R+(R+F), and thereby introducing the associative property of D_3. However, in the snippet of data from the classroom we analyzed, the students do not actually arrive at that realization.

A Note Regarding Source of Justifications in the Classroom

As described previously, we were interested in the *source* of the discussion around claims, choices, and answers. To examine the source of each instance of justification we identified, we attempted to pinpoint the verbal utterance or written prompt that initiated any statement of claims, choices, or answers. We included both verbal and written forms in recognition of the fact that students (generally speaking) are required to respond to prompts provided to them through worksheets or textbooks.

In doing so, we noticed that all of the instances of justifications found within this data were exclusively initiated by the teacher. Even though we do have instances of students reacting to their peers' rationales (see instance described previously in Category 2), all of these instances were prompted by the instructor asking the students to report their answers and to recount their choices and process to arrive at their answers. This is unsurprising given the nature of the data that was provided to us. The data describes the teacher's debriefing of the students after the students have already worked on the questions/worksheet. As such, it is natural to expect recounting of the justifications (as a process) and to expect the presentation of justifications (as a product) by the students. Given the debriefing, it is also predictable that the source of the justification instances in this data is exclusively the teacher. We will discuss the differentiation of justifications as a process and as a product in more detail in the discussion section.

Discussion

In attempting to answer our research question, we categorized the justifications found in the data into three categories. As mentioned previously, the vast majority of justifications we found were of the first two categories. Even though it may be tempting to characterize the noticeable lack of category 3 justifications (mathematically rationalized justifications) as a deficiency in the mathematical sophistication of the students, we would urge the reader to resist this temptation. These kinds of discussions that are punctuated by category 1 and 2 justifications are essential to the success of courses such as the one we examined here. As mentioned previously, this abstract algebra course was using TAAFU, which is an inquiry-based curriculum (Larsen, 2013). As such, this curriculum relies on the development of students' understanding of the content by sustained discussions among the teacher and the students surrounding the content. These discussions are routinely characterized by increasing levels of mathematical sophistication as the semester passes.

Moreover, the data focused on here is from the very beginning of the course (day 3). We posit that the students and the teacher are still establishing the sociomathematical norms (Yackel & Cobb, 1996) that are essential to the success of inquiry-based courses. In fact, classroom activity structures at the college level, especially in upper-division courses, are mostly dominated by a default of teacher-led lectures. Thus, the way this course was organized and the materials used was likely to be a strong disruption of students' expectations and previous experiences in college-level mathematics courses, where more passive activity is permitted (or indeed encouraged) by the accountability structures in the course. The teacher in this course more than likely was aware of this expectation and as such was continually encouraging of and enthusiastically accepted any category of justification from the students.

However, this acceptance of category 1 and 2 justifications raised the question of what category or categories of justification would serve as acceptable to this classroom community. The mathematical authority in this classroom still understandably seems to be the teacher. As such, the students seem to present justifications for their claims, choices, or answers solely as a response to the teacher's prompts. There does not seem to be any inherent need from the students' perspective to provide justifications for their answers, claims, or choices. Moreover, we wondered about the categories of justifications we would have witnessed if the teacher were not present in the classroom.

Using this definition of justification raised a few questions for us. First, the definition does not seem to take the community into account. That is, there does not seem to be any discussion in this definition of justification about who will take the justifications as valid. The definition describes justification as the process by which you can explain or state why your answer makes sense. However, this raises the question of to whom or to which community should your answer make sense? Methodologically, we do acknowledge that this question of community may not be as relevant when a participant is solicited for zir justifications within a clinical interview setting. In that context, the interviewer may accept any category of justification provided by the participant as convincing to the participant. However, in the classroom context, we posit that justification is about discourse and communication. The point is to try to convince someone else. More specifically, within the context of an inquiry-based classroom, justifications are used for deepening understanding, and one of the goals is to engage students in the construction of justifications for themselves.

Second, another thing that stuck out immediately was the intentional focus on *justification as a process* in the definition. We appreciate the focus on the process of justification and agree that we need to understand more about that process. In particular, this move pushed us to think about when we saw justification processes in the data (and when we did not) and also the nature of the justifications (as a product) produced from the justification (as a process). In particular, the focus in the definition on justification as a process highlighted for us the social aspect of justification in the classroom context and the norms that may be forming around this process (as evidenced by this discussion at the beginning of the semester). However, given the particular data that we were given to analyze, the more natural kind of response in the interchanges between the students and the teacher was that of reporting already compiled reasoning by the students. That is, students were not being asked to reason on the spot and to display their justification process; they were being asked to report on what they had already done to fill in a Cayley table. Thus, the justifications produced were often restricted to the level of detail needed to explain how they produced the entries in the chart. Even though we acknowledge that this definition does seem to include justification both as a process and as a product, and that our use of this definition allowed us to think about the product of justification and the process of justification as separate, we think that it may be more productive to explicitly separate these two notions. From before our first reading of this definition, we were engaged in conversations about justification as a product and as a process. So, we may have been predisposed to engaging with these are separate entities. However, we suggest that explicitly separating the notion of justification as a product and perhaps justifying as a process will cue the users of these constructs onto the important ways that the product and the process are closely related, yet separate.

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Proof in the Context of Tertiary Mathematics: Undergraduate Inquiry-Based Learning in Abstract Algebra as a Precursor to Mathematical Proof

Timothy Fukawa-Connelly and Sera Karahoca

As explained in this book's introduction, proof is important in mathematics for a variety of reasons. At the second-year and higher undergraduate levels in the United States, proof-proficiency, often drawing on specific content (e.g., group theory), is commonly an explicitly stated learning goal for courses. The teaching of proof in the traditional lecture style—the most common approach (Fukawa-Connelly et al., 2016)—is by way of instructor modeling at the board, solving predetermined and carefully selected proofs that are canonically accepted elements of the particular course. Little has been articulated about the development of student proof-argumentation in undergraduate classes using inquiry-oriented approaches, especially those not designed for the purpose of proof development. This chapter explores the proving activities and the ways that proof might be understood to be made part of the classroom discourse in a student-centered abstract algebra class at a public research university in the eastern United States. Weber's (2014) definition of proof as a cluster concept serves as the organizing definition of proof for this chapter. For Weber, a proof is a product that has either all or a significant subset of the following aspects:

1. A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true.

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2. A proof is a deductive argument that does not admit possible rebuttals. The lack of potential rebuttals provides the proof with the psychological perception of being timeless. Proven theorems remain proven.
3. A proof is a transparent argument where a mathematician can fill in every gap (given sufficient time and motivation), perhaps to the level of being a formal derivation. In essence, the proof is a blueprint for the mathematician to develop an argument that he or she feels is complete. This gives a proof the psychological perception of being impersonal. Theorems are objectively true.
4. A proof is a perspicuous argument that provides the reader with an understanding of why a theorem is true.
5. A proof is an argument within a representation system satisfying communal norms. That is, there are certain ways of transforming mathematical propositions to deduce statements that are accepted as unproblematic by a community and all other steps need to be justified.
6. A proof is an argument that has been sanctioned by the mathematical community (Weber, 2014, p. 357)

While Weber has focused on aspects of arguments that might, when considered together, or singly, be called a proof, in a classroom, we do need to give some consideration to what types of statements would need arguments in an undergraduate mathematics class. In particular, the types of arguments that might fulfill any of Weber's aspects would be made in support of some type of claim, and, for the context under consideration, a mathematical claim.

To operationalize the notion of a claim, we might start by considering markers that are traditionally accepted, for example, whether or not a statement can even be assigned a truth-value in the mathematical system (e.g., it must concern some sort of mathematical object). Mathematical claims have a definite truth-value. By truth-value we mean that a statement whose status as true or false can be known independent of an individual, and which are formulated in sufficient detail to make a determination about the truth-value. An example of this is "I currently have a dollar in my pocket," in that it could be verified as true or false. We rule out, for example, a narration about a problem-solving process in which a student might explain their steps taken to arrive at an answer. The truth-value of such a narration cannot be known independent of the individual and could never be understood as timeless or objective. When we then read across Weber's (2014) aspects of proof, we might synthesize them and state that proof is needed for a mathematical claim that is not transparently true and for which the classroom community desires justification. Broadly, we use this chapter to explore all of the arguments that are made part of the classroom discourse, how they relate to the claims they support, and how each argument aligns with the aspects of Weber's definition of proof. The alignment of arguments to Weber's definition can be used to explore the proof-development practices in this student-centered classroom. To accomplish this task, we take as our goals for this chapter the following:

- Cataloging and analyzing all the mathematical arguments made part of the classroom discourse during the classroom episode and what claims they attempt/are intended to support

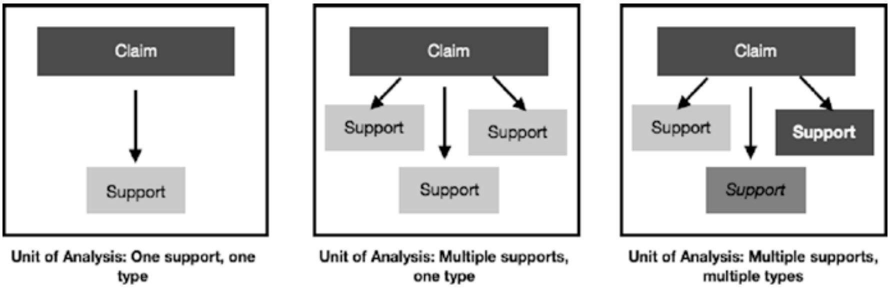

Fig. 1 Units of analysis

- Interrogating the types of arguments using Weber's (2014) cluster concept of proof
- Exploring questions of when proof might be necessary from the perspective of mathematical norms, but not useful in a particular classroom context either for communal or pedagogical reasons

For the purposes of this inquiry, we will treat a claim and any (including multiple) lines of argumentation offered in support as a single analytical unit (e.g., one argument), which may have the effect of allowing multiple lines of argumentation in support of the same claim, or one argument might be treated as supporting multiple claims (see Fig. 1). While analytically more complicated, it better reflects the "messy" data (or student-driven conversation) in an inquiry-based mathematics class.

The data we draw on is an inquiry-based abstract algebra class using a curriculum developed via a multi-year, NSF-sponsored process (cf. Larsen, 2013). We believe an inquiry-based class will provide an interesting set of data to explore given the promise of an inquiry-based classroom for making visible student claims and arguments, and including more "in-progress" thinking, conjecturing, and revision (cf. Larsen & Zandieh, 2008). This is of interest because it might allow insight into both how the classroom community understands proof, but also the types of claims that might require proof in the classroom community. Second, the inclusion of student voice means that students might be more explicit about lack of understanding or certainty both of a claim and the offered justification. These should, in theory, be prompts for making additional justification part of the classroom community.

Methods

We first note that we might take a number of different approaches to analyzing classroom data. For example, we could identify the claims and reasoning of any individual student or the professor, or, in the case of the data we are working with here, the reasoning of a particular group of students. Instead, we have chosen to treat the collective classroom discourse as the object of study. Wawro (2014) provided a rationale for doing so, arguing that Cobb's notion of learning (Cobb, 1999; Cobb & Yackel, 1996; Stephan et al., 2003) views the activity of the classroom community as inseparable from that of the learning of the individual, where each is shaped by the other. That is, taking the argumentation of the classroom community as a whole is a reasonable way to investigate the ways of thinking and reasoning about justification and proof that both the community takes-as-shared and that any particular student is likely to draw on individually.

In our first pass through the data, we identified any arguments in support of mathematical claims that students were making for further analysis. As needed, we rewrote the mathematical claims into mathematically equivalent statements, only clarifying grammatical or conversational structures if necessary. We used a clear formulation (e.g., in if-then format with common forms of quantification) in order to make them easier to analyze, and when a participant made a claim that was unclear, we wrote the most plausible mathematical version. When either students or the professor altered a claim (for instance, adding or deleting a qualifier), we treated that as a new claim. After identifying a total of 13 original and rewritten claims, we described the scope of each claim, determining whether it was local to a specific context (e.g., only in D3) or the level of generality (e.g., all dihedral groups vs. all groups). When there were multiple related claims, we listed them in order from most specific to most general. Our primary purpose in cataloging claims was to use them to explore the types of arguments provided in support of them and how those arguments might fulfill some or all of the aspects of proof that Weber (2014) identified.

We then linked all of the arguments and claims that were made part of the classroom discussion. Drawing on the transcript, we grouped sequential collections of lines of transcript that could be plausibly interpreted as supporting the claim and which appeared to draw on the same ideas. For example, in justifying that a structure is commutative, a group of students might reference a particular idea, such as a symmetric group table. In this instance, we would group all sequential lines containing the claim and supports. For each claim, we identified all relevant chunks of transcript. As needed, we also aggregated chunks when members of the community were describing similar mathematical ideas or appeared to be in conversation, even if interrupted by another idea. As needed, we noted when multiple arguments were offered in support of the same claim. Finally, we interrogated the justification offered against each of the aspects of the cluster concept of proof. In doing so, we had to wrestle with the differences between the ways that Weber (2014) had operationalized the definition in the context of mathematicians writing for mathematicians versus the types of claims and justifications in the undergraduate classroom. When multiple arguments were presented for the same claim, we noted which aspects were shared and which were not and categorized the argument that would "best" support the claim in terms of the scope and the alignment with Weber's definition. We note that the student data does not provide clear warranting for any of the aspects of Weber's definition. While the first three aspects relate to what a mathematician can or might do, including be convinced and fill gaps, the second three are about the mathematics community. As outsiders, we cannot judge whether some "readers" within the classroom might believe a particular argument provided them with understanding of why a theorem is true, nor can we make definitive claims that the argument was in a representation system satisfying communal norms. As a result, we have made such interpretations via our own perspective as mathematically savvy readers and educators.

Data and Results

For homework, the students were asked to translate a Cayley table for an equilateral triangle from their nontraditional mathematical notation into the traditional mathematical notation. The class started with the professor engaging them in a discussion about the processes they used to do so, including any shortcuts that they might have drawn on. In the subsequent discussion, the professor and students repeatedly discussed the results of binary operations (and sometimes a sequence of binary operations).

As a first pass, we note that neither the professor nor the students used any of the terms that might typically denote a claim to be proven, including *theorem*, *proposition*, *lemma*, or *claim*. Similarly, the word *proof* does not appear in the transcript in any form, including *prove* or *proven*. As a result, given the importance different types of knowing, including certainty, understanding, and truth are to Weber's (2014) definition, we similarly searched the transcript for words like true, truth, understand, know (or any other words that we thought might describe those notions) and found relatively few such uses, and few uses of "think" more generally. There is only one use of the word true, in Line 33, in which the instructor is asking a question about a collection of operations, probing whether these elements of D3 are associative.

33	Inst:	In the table it's the same, if I go to where 2R and F are in the table, I get the same answer if I go from R and RF. Is that true? So these two gave me the same answer. By saying 2R plus F, I am saying R plus R plus F [writes = R + R + F]. I'm regrouping which ones I'm considering associated with each other. Does that sound like something you've seen before in math?

There is a related, follow-up set of claims about whether D3 is associative that we explore further below. There are no uses of *certain*, *convince*, or *understand*—the other critical verbs from Weber's definition. We did find one use of "sense," where the instructor asked, "Did that make sense?" although there was not a chance for students to respond. In short, there is nothing in this class excerpt that we suggest would be unquestionably called a proof using Weber's definition. However, there is nothing that is labeled as a theorem or proposition and thus signals the need for a proof. Instead, claims are all hedged as observations, and even the versions that the professor records are hedged in particular ways. In the following sections, we will explore three categories of justification and the claims for which the justification is offered. In doing so, we consider the domain of the claim, the type(s) of argumentation offered, and the aspects of Weber's definition that the argument might fulfill for that claim. These are the only categories of argumentation that illustrated any aspects of Weber's definition.

Category 1: Perceptual evidence that would likely be accepted as proofs

The first example of a category of argument that we highlight is a perceptual argument. These were typically deployed in support of a claim about the outcome of a binary or series of binary operations. This example, which we use as illustrative of the category, is drawn from the beginning of class and includes an assertion from S3 and the instructor's follow-up, which includes an extension and graphical support.

11	S3:	I did the side first, when I was at Row R, on this thing [holds up paper and motions toward what is written on it], I went to R, looked at the triangle, and then, say it was plus F, I flipped that, found where it was on here, and did the next thing, and then found where it was, so I that I could keep track of it.
12	Inst:	So I think what she was saying [places a new handout on projector]... Let's do a make believe one that she was doing R, and she was combining that with RF [writes R + (R + F) on projector]. She would first look here and see where the first R got her [circles the triangle on current paper on projector (see Fig. 2)], and then what would you do?

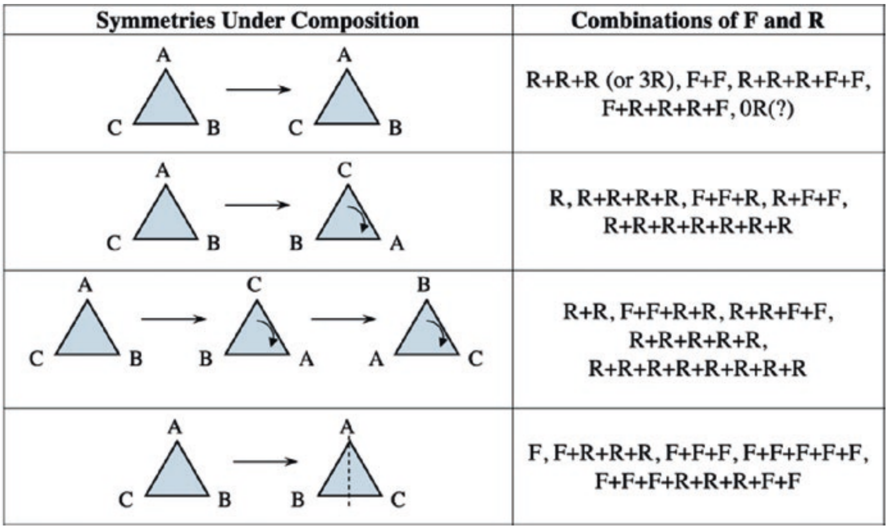

Fig. 2 Illustrated transformations (recreated from what was written on the board)

13	S3:	And then what was the next? Then, okay. Rotate that so that it was B, C, A, and find which one that was under.
14	Inst:	Okay, so that would take her to B, C, A.
15	S3:	Which takes me to 2R, and then the next step was to flip it, so I flipped the A and the C, so I have B, A, C, and then find where that is which is the F + R. And then that's the answer.
16	Inst:	Does that make sense? So she used these pictures, doing one at a time. If I was going to write that out algebraically [underlines R + (R + F)], according to the table I need to do R first [draws a box around R], and then I need to do this motion: R + F [circles (R + F) on the projector]. Which is this. But she didn't think of this [points at (R + F)], as this combined motion. What you kind of did is R, and then R, and then F. Right? What I'm trying to get at is that this is how it was asked: R, and then the combination, the grouping, of R + F. If I had to write that with parentheses and symbols, how could I capture what she was doing? Would those parentheses stay in the same place? Are you shaking your head no?

We interpreted S3's comment in Line 11 as indicating that ze first rotated the triangle (applied the R) and then performed the next operation (in this case F), using a visual that ze had previously constructed. We believe that this was previously constructed due to the fact the instructor had immediately shown an annotated projection in Fig. 2. In Lines 13 and 15, S3 continued the explanation describing a multistep process and the position of the vertices after each step. The instructor, in Line 16, summarizes S3's work as one rotation, followed by another rotation, followed by a reflection. We note that the instructor did ask, "Does this make sense?" but did not allow students a chance to respond. That is, S3, and the instructor's summary, appeared to rely on a set of perceptual claims about the relative location of the vertices of the triangle after each move, supported by visual illustrations of the triangle. We note that all claims in this category are of a similar form, that Transformation_1 composed with Transformation_2 gives a specified result, Transformation_3, and are local to this particular set and operation. The most common type of argument presented in support of these claims is a type of perceptual evidence in which the position of the vertices of the equilateral triangle is visually tracked and recorded. That is, the justification offered appears to be visual/perceptual. We make no claims about whether the students would consider this a *proof*, but instead explore how this perceptual justification does or does not fulfill aspects of Weber's definition.

In our experience as students and researchers, it is relatively common for claims about composition of transformations to be justified via perceptual evidence. For example, Gallian (9th edition), the most commonly used text for first semester abstract algebra courses (Melhuish, 2019), introduces symmetry groups in the text of Chapter 1 via the group D4, and justifies the result of operations via visual means before presenting a completed Cayley table (cf. p. 32). The exercises at the end of the chapter ask students to "with pictures and words, describe each symmetry in D3" and then complete a Cayley table (p. 37). Thus, the perceptual justification provided seems to be an argument that would be sanctioned by the mathematics community for this particular type of claim in this specific context. Moreover, given perceptual exercises are included in texts and in pedagogical explanations (the professor studied in Fukawa-Connelly and Newton (2014) suggested students imagine a child's toy while drawing and illustrating symmetries on day 22), we suggest that mathematicians behave as if this is a convincing argument that provides the reader with an understanding and/or assurance of why the claim is true. As Dawkins and Weber argued.

> There are situations in which empirical and authoritative evidence provide legitimate grounds for believing mathematical claims (and some mathematicians indeed believe mathematical claims on these grounds, as documented in Weber et al., 2014). Demonstrations on dynamic geometry software are common classroom occurrences where empirical evidence provides adequate grounds for believing a claim for both students and mathematicians (De Villiers, 2004)" (2017, p. 135).

And this example of perceptual proof appears to be similar.

At the same time, this type of argument is neither deductive, nor we would argue transparent. While the perceptual argument *can be* translated into a collection of 2 × 2 matrices that each record on symmetry and, together, form a group, the visual itself gives little clue of the conceptual handle required to create these matrices. Czocher and Weber (2020) noted that (a) visual proofs are sometimes impossible to translate into a formal system, yet (b) that they have data in a different publication (Weber & Czocher, 2019) in which mathematicians disagreed, and acknowledged that there would be disagreement in the field, about whether a "simple visual justification qualified as a proof" (p. 14). Yet, the graphic might be perceived as Weber's "timeless" in the sense that once created, it is independent of the viewer. That is, perceptually justified things remain perceptually justified as "proven things remain proven." This perception might contribute to the dissent among mathematicians about whether a visual argument might qualify as a proof.

The proffered definition of proving requires that an argument have "a significant subset" of the defined characteristics; this type of perceptual argument in support of a claim about operations appears to satisfy three of the seven (1, 2, 5), but in ways that are likely psychologically unsatisfying (they may not "feel" like a proof to a mathematician) and could generate disagreements among mathematicians (cf. Czocher & Weber, 2020). That is, this appears to be an instance where mathematicians award equal or greater value to a perceptual justification than something that is more symbolic and "rigorous."

Category 2: Computation evidence might sufficiently warrant some claims and fulfill some of the criteria so as to be considered a proof by some

The second example category is that of computational argument. Computational arguments were common in the classroom discourse and were typically deployed in support of local claims about particular structures. To illustrate this type of argument, we highlight the claim that S1 expressed in Line 2 and the instructor recorded in print and restated as a subsequent argument. S1 was responding to Prompt 1.3.2, which asked, "Did you figure all of them out by moving the triangle around or did you use some shortcuts to do the calculations?"

2	S1:	I used the Cayley Table shortcut which, after like I saw the pattern, I knew that each one appeared in each column and row only once. So around the end of my table, I just started seeing which one was missing. [Instructor writes "Each symmetry appeared exactly once in each row and column" on projector.]
3	Inst:	Each row and column. Did other people do that? They go to the very last one and instead of calculating it they just figured out which one was missing? StudentName? Is your hand up? Oh, who else wants to raise their hand? Yes?

In Line 2, S1 claimed that ze "used the Cayley Table shortcut" which ze then explained without actually referencing the context of D3, only the generic claim that "each one appeared in each column and row only once." We interpreted the notion of "the Cayley Table shortcut" to be a general claim, that this is a process that works for any Cayley table, because the phrasing and instructor's follow-up suggested to us that this is a pattern that the students had previously discussed. We note that we interpreted the instructor's written record, as recorded in Line 2, as local to the D3 Cayley table based on the past-tense phrasing, suggesting that it only described what the student had observed in this instance. We acknowledge that the evidence for either the limited or general claim is relatively sparse, yet as we are treating this as an analysis of the classroom reasoning and proving, we will consider both forms of the claim, without being overly concerned about the intentions of S1 or the instructor with regard to articulating either a local or general claim. We think that exploring both will help us explore the different ways that we might parse the classroom activity using the proffered definition of proof. That is, we treat this as two separate but related claims and analyze the cited evidence in light of the definition of proof for each of the two claims.

- **Claim 1**: The Cayley table for D3 has each element represented exactly once in each column and each row.
- **Claim 2**: All Cayley tables (representing a group) have the property that each element of the group is represented exactly once in each column and each row.

The evidence presented for both claims is the students' completed operation table. While only one student suggested the pattern, no student spoke in opposition to the claim. We first explore the relationship between Claim 1 and the presented data. Because the claim is localized to the Cayley table for D3, we might judge the claimed evidence as meeting characteristics 1, 3, and 6 of the definition of proof and parts of characteristic 2. In short, the completed (correct) Cayley table for D3 would be judged to be true and allows a mathematician to fill in every gap formally, even if the students did not create the table formally. Moreover, the offered support is somewhat deductive in that it is, essentially, an argument by exhaustion of considering every possible case (although S1 appears to have assumed it to be true after a certain point in the project). We might claim that for the very localized Claim 1, the mathematical community would agree that a completed Cayley table is a sufficient argument. Critically, this is the type of evidence, along with another few completed Cayley tables (in the TAAFU curriculum, this is the second Cayley table students complete), that would form the basis for a general conjecture (cf. Lakatos, 1976) that would take the form of Claim 2. In summary, the proffered Cayley table would likely generate disagreement among mathematicians about whether it is a proof because it appears to fulfill multiple of the needed criteria, and partially fulfills others without being a verbal-symbolic proof, starting from assumptions and containing algebraic manipulations.

In terms of the more general version in Claim 2, that any Cayley table for a group has the property that each element appears exactly once in each row and column, the proffered evidence appears to fail the test for each of the criteria for a proof. While in the specific context (Claim 1), completing the table is essentially a proof-by-cases, in the generalized claim (Claim 2), the proof-route is different. For some proofs, a generic example (cf. Leron & Zaslavsky, 2013) provides both insight into the reasons that the claim is true and a route to an algebraic proof of the claim. Critically, we see no obvious reason that if the students completed the Cayley table by computing individual binary operations, it would provide insight into why Claim 2 would be true or how to create an argument that would satisfy multiple aspects of the proof concept. That is, a completed Cayley table does not seem likely to function as a generic example for this particular claim because in this situation the proof requires a generalized consideration of the structure of a group, rather than an individual calculation. That is, we argue that completing the Cayley tables does not provide a prover with an understanding of why a theorem is true and certainly someone who is only looking at a completed Cayley table would not derive the benefit of the repeated computations for gaining insight.

As with the above category, we note that there are multiple cases in which mathematicians find such justifications sufficient to warrant conviction (Dawkins & Weber, 2017). Moreover, while we can certainly apply the cluster category of proof to these first two types of justification, neither seems likely to yield a definitive answer, instead, raising additional questions about what constitutes "significant" and whether, in classrooms, proof should or could be considered something else (e.g., a pre-proof). As a result, we close this example with Czocher and Weber's (2020) suggestion regarding classroom proofs—that, instead of treating them as a binary—this is/is not a proof, that we specify what aspects of proof the particular justification fulfills. In this case, the justification would be both clear (perspicuous) and convincing, and we could label it a clear and convincing proof. That is, when considering the pedagogical actions of teaching proof-based mathematics and the practices and processes of proving, the questions we should be focusing on are:

- What is good in the work of the students—what elements of proof are present in the work?
- What is a reasonable next step for this student in moving toward the "ideal" proof? We might recognize that the ideal proof for pedagogical purposes might not be the canonical proof for mathematicians.
- Is the student improving with regard to proof-proficiency over time and across content areas by using appropriate elements of Weber's conception of proof with greater accuracy, frequency, command or less prompting?

For a professor, we might ask them to consider the pedagogical values of proofs that align with different aspects of the cluster concept and which and what values they want to emphasize. Some proofs give little insight into the structure of concepts and would have little or no pedagogical utility, while proofs like generic examples (Leron & Zaslavsky, 2013) might have more pedagogical value than an algebraic, general proof (e.g., by reducing the level of abstraction for students). Professors might consider the amount of insight that a proof offers or how commonly the structure of the argument is used in making a decision about the type(s) of justification offered in support of a claim as part of their class (either in a lecture or a student-centered class like this). Lai et al. (2012) claimed that professors might present less formal, less complete proofs in a lecture than they would want presented in a text. As a result, we would suggest, much like Czocher and Weber (2020) claim on naming the positives in students' justifications, professors should regularly be naming both the aspects of proof that are met and those that are not by their own proof presentations.

Category 3: Evidence that does not meet any of the criteria for proof

There is a third category of argument, one in which the argument does not meet any of the criteria for proof of the claim. There were multiple instances of such an argument; here, we have chosen to highlight an example where the student asserts, based on prior experience with similar structures, that D3 is associative. This claim was unique in a different way, in that it is the only claim for which a student explicitly expressed doubt about the truth-value during the classroom discussion. In Line 36, a student stated "I didn't know if it was…" and the instructor completed the idea, that the student did not know if the associative property would hold in D3.

34	S8:	The associative property?
35	Inst:	The associative property. Heard that before?
36	S3:	I was gonna say that before, but I didn't know if it was ...
37	Inst:	You didn't know if it would work for this! That's a great question to ask. Can I regroup? There's a whole string of things on the board and there's no parentheses whatsoever. Is the first one three motions, and then the next one four motions? Or are those all representing two motions?

That is, in Line 36, S3 created an opening for proof, essentially asking, "Is this true?" This is the only time we found in the transcript in which a student expressed doubt about a universally quantified claim.

In Line 38, S3 makes an explicit claim that D3 is associative and provides an argument based on a past experience with structures in a different class. S3 perceived structural similarities between D3 and the structures in another class. Approximately 1 min later, in Line 44, the professor agrees that it is "a rule that we're using."

38	S3:	The way I kind of see it is, just because I just took Proofs last semester, when you had like an A or B representing different things the order mattered for that and the order matters for this, so you should be able to do the same types of things to it, so the associative property should work.
...		...
44	Inst:	That's definitely a rule that we're using.

We summarized S3's argument in the following way: in ze's previous experience with noncommutative structures, the structure was associative, and so D3 should be as well. There were other points in the transcript where either a student or the instructor noted that the operation, on a particular collection of elements, obeyed the associative property. At no point in the transcript is there any other type of evidence supplied in support of this claim. We note that the instructor did not assert a truth-value for the claim, only acknowledged that the class was using the rule, so we did not categorize that as a mathematical argument.

Checking for associativity (with the tools that students would have developed at this point in the TAAFU curriculum) would require the students to check all triplets, which they do not appear to have done, a cumbersome task which provides no meaningful insight into the structure. Thus, we suggest that the instructor's acknowledgment that the students are relying on associativity without proof is correct and pedagogically appropriate. The instructor might make more explicit that the class is relying on it without proof and ask what some implications are for doing so. For example, the instructor might ask, "What happens if we later discover that we were wrong?" Alternatively, the presentation could be further developed by stating that if D3 is associative, a proof would be required eventually, but that the class could continue to rely on the rule for the sake of examination/investigation.

Discussion

There are a few points to take away from our investigation of this classroom discourse. First, while nothing was labeled a proposition or a proof, there were numerous instances of claims with accompanying arguments, and some of those arguments satisfied some or all of the criteria for proof of the claims they were deployed to support. Thus, even though nothing was explicitly called a proof, that does not mean that an observer could suggest that proofs were not made part of the classroom discourse. Second, we only noted one explicit claim that was warranted with evidence that fulfilled none of the criteria of Weber's (2014) definition, and there were pedagogical reasons that not offering or pushing for a proof was a sensible decision on the part of the instructor. Most interesting was the case where there were multiple related claims, at different levels of generality, developed from the same set of observations. These claims exemplified how evidence that fully warrants one version of the claim may fulfill fewer of the criteria of proof for another, more general, version of the claim. Similarly, our exploration gave further instances in which graphical evidence is treated as sufficient to warrant a claim beyond the dynamic geometry software discussed by Dawkins and Weber (2017). Finally, we return to Czocher and Weber's (2020) claim that, in terms of pedagogical practice, we should not attempt to categorize justifications on the proof/not-proof binary, but rather tell students, explicitly, what criteria of the proof-cluster definition their provided justification fulfills and challenge them to try to fulfill additional aspects.

The second major idea that we raise is that a closer consideration of the types and range of claims that are made part of the classroom discourse should both be explored via research and be part of the pedagogical considerations in curriculum design. As shown in this chapter, the structure and form of claims determines what types of evidence might fulfill a sufficient number of Weber's (2014) aspects of proof. At the same time, while claims are typically stated, but in none of the instances above was there discussion of how general each claim was or whether it was stated in the form that the class felt was most appropriate. In having such a discussion, it then would feel reasonably natural to examine the types of evidence that are sufficient to warrant those claims. Moreover, as shown in the data above, there was an instance of a student expressing uncertainty in the truth-value of a claim, which even without offering a proof would be an opening for such a discussion. Similarly, making explicit the kinds of moves that mathematicians and mathematics instructors use to make claims more general and then discussing how changing the claim would change the needed evidence would make the concept of proof more transparent and illuminate how those needs change as aspects of the claim change. Moreover, by changing the claims and making evident how those claims have been changed, it might provoke greater intellectual need for the first aspect of Weber's definition—providing certainty about the truth-value of a claim. Relatedly, we might ask what good is "proof" as a tool for conveying certainty, if students never use it to confront false claims. In some texts, the item prompts are typically "prove or show a counterexample" but not the process of generating and revising claims.

Finally, we raise a number of questions that we might use to shape future explorations. In particular, who is responsible for prompting for proof in an IBL-type classroom? How do we know and teach when proof by perception is acceptable, warranted, or rejected? And how do we inculcate in students a desire for proof (in either IBL or lecture)? For example, we would like students to think proof is necessary for all claims, but if they perceive a claim to be true (and it is), should we attempt to perturb that and ask for proof? If proof is not psychologically necessary, and a mathematician knows a fact to be true, then what is the value in asking students for a proof? If all that remains is "the communal norm of proof," that is a much weaker foundation to rest on than might be desired. We should want a stronger rationale: one that goes beyond the "trick" statements meant to illustrate a problem with empirical evidence as a justification for universally quantified claims (a common claim is that n^2 + n + 41 always gives a prime output; this is true for the first 40 cases, 0–39). We want a justification that relies on the differences between mathematical proof, argumentation, and justification. What would such a rationale, appropriate for the classroom, look like?

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Mathematics Educators as Polymaths, Brokers, and Learners: Commentary on the Tertiary Chapters on Argumentation, Justification, and Proof

Paul Christian Dawkins

I frequently encounter opportunities to explain to people why mathematics education as a research field skews toward qualitative inquiry, which I enjoy in large part for the sake of irony. There are many such ways in which learning about people learning mathematics differs starkly from mathematics itself, and the preceding chapters put this on display quite dramatically with regard to *defining*. The project documented in this book is an interesting experiment in how mathematics education definitions influence, facilitate, and even constrain our inquiry. The question behind the question in this project might be stated: what are definitions in mathematics education and what role do they play in our research? Mathematical definitions are taken to be precise, unambiguous, general(izable), abstract, stipulated, value-free, and Boolean. What we see in the preceding chapters is that mathematics education definitions instead have natural histories, require interpretation and contextualization, adapt and morph with use, entail value judgments and priorities, and do not afford precise extension (examples and non-examples). I judge that this is completely appropriate because, as I shall develop in this chapter, mathematics education is an interdisciplinary field in which we draw on diverse traditions of inquiry to pursue complex and contextualized goals to advance our science. Accordingly, I organize my reflections on these three chapters around the themes of mathematics educators as *polymaths* in terms of disciplinary knowledge and tools, *brokers* between the various communities vested in mathematics instruction and educational research, and *learners* who are advancing our knowledge base. Reflecting on these aspects of mathematics education work will occasion my reflections on this book's central question about how the definitions of argumentation, justification, and proof influence that work.

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Mathematics Educators as Polymaths: Definitions Have Natural Histories

In the tertiary chapter on argumentation, Plaxco and Wawro clarify the treatment of arguments as used by philosopher Stephen Toulmin (1969) as follows:

> This framework aims to characterize what Toulmin called substantial arguments – arguments presented in discourse in some socio-historical context for a particular purpose. Substantial arguments "often are structurally distinct from the formal, deductive structure of logical arguments thought to be inherent and absolute. In other words, some aspects of socially presented arguments are context dependent yet still function as acceptable justifications for particular claims" (Wawro, 2015). (See "Theoretical Frameworks and Analysis Methods" in the chapter by Plaxco & Wawro, this volume)

This characterization is quite important inasmuch as Toulmin developed his framework for characterizing arguments (in various sociohistorical contexts) as a rival to the dominant use of mathematical logic, which he found ill-suited to the purpose. Indeed, one of Toulmin's driving points was that arguments and rationales for human choice were often quite contextual and specific, contrary to the logical frame that entails a certain amount of abstractness and generality (he is one of mathematics educators' favorite critics of mathematics). His point is parallel to the one that Plaxco and Wawro implicitly make about student arguments and the one I am making about mathematics education definitions. Plaxco and Wawro highlight how the students' arguments in the data analyzed were rooted in their enacted activity with triangles, which is the intent of the guided reinvention-based curriculum. I am arguing that the definition of argumentation that Plaxco and Wawro used has a natural history and they extended that history by pairing it with Toulmin's framework.

I applaud Plaxco and Wawro for their ongoing attention to the natural histories of the terms they use in their chapter. They further acknowledge that the TMSSR framework they employ to describe teacher moves draws upon Activity Theory (another sociological body of theory with its own inherited set of perspectives and paradigms) and arose from use in inquiry-oriented middle grades classrooms. This attention to the contexts and intentions from which research tools arise is not unique to their chapter, but their exposition nicely portrays the complex understandings necessary to do such interdisciplinary research well.

Mathematics education is a social science often drawing upon psychology, sociology, communications, linguistics, education, philosophy, statistics, and many blends thereof. Not only does good mathematics education research (which I think these chapters are) require appreciation of these various fields—their knowledge base, methods, limitations, and orientations—but it also requires sensitivity to the natural history behind the definition of a construct. This is not to say that we are not allowed to modify those tools for our own use. Indeed, Plaxco and Wawro include the all but ubiquitous invocation of grounded theory, which is our field's favorite imported piece of methodology that is usually used in manners only loosely compatible with nature and intent of the source material. Plaxco and Wawro explain:

> We view this as grounded theory in that we had no a priori coding scheme, other than being influenced by our own prior knowledge of inquiry-oriented instruction. After an initial

> round of open coding the transcript together, we searched the literature for existing frameworks that would help further characterize the nuanced instructor moves. (See "Theoretical Frameworks and Analysis Methods" in the chapter by Plaxco & Wawro, this volume)

This is an honest description of how grounded theory is used in our field (though it runs against some of the spirit of the method). It describes that we begin our analysis with an open interpretive stance to get a sense of what is occurring in a complex and multifaceted space. The key point is that, as often happens in mathematics education, once Plaxco and Wawro oriented themselves to what appeared relevant for making sense of their data, they found that someone else has already provided an analytical tool for describing what they intend to capture. To adopt an existing framework after the fact is entirely appropriate, lest we perpetuate Dreyfus' (2006) observation that mathematics educators "tend to invent theories, or at least theoretical ideas, at a pace faster than we produce data to possibly refute our theories" (p. 78). My point is less that Plaxco and Wawro were not entirely faithful to grounded theory, but rather that our community has transformed grounded theory for our own purposes. Those authors maintain what I think is the essential awareness as we import these various tools from other disciplines—even as we adapt mathematics education findings and frameworks across contexts, classrooms, and cultures—that we must attend to the nature and viability of these recontextualizations. The definitions of the terms we use have natural histories, and it is safe to say that they evolve. Like biological evolution, some resulting species end up being more fit than others.

Similarly, in the tertiary justification chapter, Karunakaran and Levin trace some of the history of defining justification behind the particular definition they were assigned. They also attend carefully to the goals inherent to the original defining:

> The authors [of the definition of justification] describe justification as a learning practice that promotes understanding and they discuss how justification may be more appropriate for middle school students than say, formal proof. They worry that "proof" might narrow focus on formality of expression or specific form of argument and instead what is of interest to them is the examination of the truth of a claim. (See "Defining and Operationalizing Justification" in the chapter by Karunakaran & Levin, this volume)

To divorce this definition of justification from its history and context of use is to lose part of what it intends to capture. Naturally, we can recontextualize these definitions (in word, intent, or both), but we also must not trivialize the adaptations necessary.

Mathematics Educators as Brokers: Adapting Definitions to Balance Obligations

A broker is someone who has some level of membership in two different communities and thereby can act as a connection between those communities. This term has been used to describe the way mathematics teachers operate as both members of the mathematical community and the classroom community (and more), maintaining obligations to each. What I observe in the preceding three chapters is how the researchers balanced various obligations in their efforts to use the definitions.

Brokers Between Mathematics Instruction and Mathematics Research

Inasmuch as I have spoken of mathematics education as a social science, one unique feature of tertiary mathematics education is that instruction is most often carried out by people with heavy mathematical training and comparatively little pedagogical training. As a result, the obligations of mathematics instruction to the discipline of mathematics are most pronounced. Accordingly, uses of the term "proof" carry much weight (if not baggage) since it occupies the seemingly paradoxical status of being the hallmark of mathematics for many professional mathematicians that is largely absent from the first 13–15 years of many children's mathematics instruction (a blatant overgeneralization I intend largely for provocation). The authors of the tertiary proof chapter, Fukawa-Connelly and Karahoca, do a commendable job wrestling with the provided definition of proof. They acknowledge that the definition of proof is meant to open the term by shifting proof away from a binary category, and they found that only one of the classroom arguments they coded failed to meet at least one of the criteria in the multipart definition. However, this definition was developed as a way to categorize the proofs produced by mathematicians and thus makes repeated reference to the judgments and practices of the professional mathematical community. Accordingly, the proof chapter authors made recourse to scholarly accounts of how proof operates in the mathematical community to provide context and warrant for their categorizations of student activity. They explained, "We had to wrestle with the differences between the ways that Weber (2014) had operationalized the definition in the context of mathematicians writing for mathematicians versus the types of claims and justifications in the undergraduate classroom" (see "Methods" in the chapter by Fukawa-Connelly & Karahoca, this volume). While locally this shows the work necessitated by the recontextualization of the definition, it more generally reflects the work of the mathematics educator as a broker between the classroom and mathematical communities.

Brokers Between Mathematics Education and Psychology

The authors of the justification chapter attended carefully to the fact that their definition of justification focused on the (cognitive) process in which students engage and not upon the product of that process. They reported the challenge they experienced in inferring about that process since the data analyzed stemmed from a whole-class session reporting on mathematical work that students had previously carried out. I sense this tension reflects the justification chapter authors' role as brokers between the classroom community and the psychological community. As psychologists, they were keenly aware of the trouble inferring cognitive processes from later reports thereof. I also infer that their psychological orientation led them to focus on individual justifications, while the argumentation and proof chapter researchers chose to analyze the emergent products of the classroom community as a whole. Regarding our guiding question about the implicit constraints from our construct definitions, I observe that many other relevant constraints on our inquiry arise from our sense of obligation to the various professional communities in which we understand ourselves to be participating.

Brokers Between Researchers and Instructors

A pattern that emerged among all of the chapters reflects the authors' positions as brokers between the community of educational researchers and in the community of mathematics instructors. While the stated research questions invite documenting the nature of the argumentation, justification, and proof observed within the classroom episode, all of the chapters contextualize their observations against what they judged to be reasonable and appropriate within the instructional context. For instance, Karunakaran and Levin describe three kinds of justifications observed in the data, the most common of which depend upon observed patterns or sequences of action rather than upon underlying mathematical properties or theory. They note about one category, "this justification did not contain any mathematical rationale for why the process works, and rather focuses on how the process arrives at the answer" (see "Category 2: Recounting of Process Justifications" in the chapter by Karunakaran & Levin, this volume). They further noted that "all of the instances of justifications found within this data were exclusively initiated by the teacher" (see "A Note Regarding Source of Justifications in the Classroom" in the chapter by Karunakaran & Levin, this chapter). Recognizing how these observations they made as researchers could reflect negatively upon the quality of the instruction in a manner they did not endorse, they framed their findings as justifiable from their position as instructors. Karunakaran and Levin noted that the observed lesson occurred early in the semester, that the class was seeking to set up norms for justification that were unlikely to be familiar to students, and that this curriculum is structured around increasing levels of mathematical sophistication over time. All of these mitigate any implicit sense of criticism in their coding of the justifications.

Both the justification and proof chapters included such clarifications of their findings to fend off undesired negative interpretations of their research observations. They also explored this tension to propose productive next steps for inquiry and instruction. Both pondered how instruction could produce more intellectual need for justification beyond simply leveraging of the instructor's authority. The justification chapter questioned how the definition of justification might be improved by including some reference to the community in which the justifications take place. The proof chapter posed a number of questions about how that increasing sophistication in proving might emerge and develop over time. I understand them to suggest that the questions they ponder behind their stated research questions are simply not addressed by the data they analyzed. This again is a trend in mathematics education research that so many of our studies are "part of a larger project" and our research questions are merely the small steps we could manage to advance toward much larger questions that cannot be addressed in single studies.

The argumentation chapter took this brokering role between research and instruction one step further by analyzing with tools integral to the design of the curriculum used in the classroom. They cited both emergent models heuristics (accounting for the intended development of mathematical structures from within students' situated activity) and local instructional theories (accounting for how abstract algebra ideas develop in particular). These tools allowed them to make inferences about how the observed student activity should progress along the trajectories discussed only indirectly in the justification and proof chapters. Indeed, their second research question focused on how instructor moves advanced the mathematical agenda. Their analysis thus entailed a sense of foresight regarding the nascent student arguments and allowed Plaxco and Wawro to make complex claims about instructional intentionality such as, "Although we cannot know if the instructor intended for the change to shift the logical implications of S1's statement, such intention is consistent with the TAAFU curriculum" (see "Vignette 1: How to Fill in the Cayley Table" in the chapter by Plaxco & Wawro, this volume). The authors thus maintained enough analytical caution not to step beyond what the data supports about the instructor's intentions while rooting their claims in the intentions behind the curriculum design and instructional design heuristics. I claim this reflects those authors' brokering role between positions as researchers, instructors, and curriculum designers.

Managing insights, goals, and constraints across the various influences at play in the mathematics classroom is one of the unique challenges and affordances of mathematics education work. We operate at the interface of a scientific field of inquiry and a profession with its accompanying institutions and necessities. All three chapters contextualized their research findings along this interface. The argumentation authors found appropriate tools to bring these contextual factors into their inquiry, which is not always possible when the curriculum was not designed using mathematics education theories.

Regarding our central question about the role of definitions in mathematics education inquiry, I observe how those definitions become operationalized in tandem with a range of other analytical tools that make it hard to really consider the influence of that definition by itself. What might Plaxco and Wawro have found had they attempted to look at the argumentation in the classroom without Toulmin schemes, the TMSSR framework, or RME design heuristics? Such a question would ultimately miss the point, regarding both the nature of definitions in our research field and the nature of our mode of inquiry. Coordinating various tools to account for the manifold complexities of a mathematics classroom is at the heart of what it means to be a mathematics educator: to be a broker among a myriad of perspectives, value systems, as well as professional and scientific obligations. The argumentation authors claim as much when they endorsed the value they derived from coordinating multiple elements from their analytical toolbox. They claim that the multiple analytical tools correspond to different features of the classroom argumentation being studied:

> It is in the balance of these forces – the instructional objective to generalize situated student activity and the meaningfulness of said situated activity – into which the dual focus on argumentation and TMSSR provide insight. The argumentation is both a means for students to communicate their reasoning about concepts and also a mechanism for advancing the mathematics of the classroom toward more general activity. (See "Discussion" in the chapter by Plaxco & Wawro, this volume)

Mathematics Educators as Learners: Accumulating Knowledge of Complex Phenomena

Our research field faces a very real challenge to our ability to accumulate knowledge from the interdisciplinary and multifaceted work we do. Indeed, this provides some motivation for this project comparing the influence of definitions on our research. As the abstract for the Argumentation, Justification, and Proof working group in 2017 stated:

> Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. (Conner et al., p. 1464)

This constitutes a real problem for constructing a meaningful knowledge base as a science. Inglis (2011) noted this quite powerfully in a review of Reid and Knipping's (2010) book surveying the literature on the teaching and learning of proof in mathematics. He said:

> [My] second point of disagreement was with R&K's argument, presented throughout the book, that the multiplicity of research perspectives which exist in the literature should be considered a strength, because "a diversity of perspectives offers opportunities to make sense of phenomena that might be seen in a limited way from a single perspective" (p. xiv). Unfortunately the authors failed to convince me of this argument, and indeed I found some of their (very competently arranged) summaries of the literature to be rather depressing. Take, for example, their review of argument classification schemes: R&K point out that what they would call an "empirical-perceptual" argument would be classified by Bell as "extrapolation", by the preformalists as "experimental", by Balacheff as "naive empiricism" and by van Dormolen as an argument of the "first level". Similarly they explain that, depending on which perspective is adopted, the same argument might be classified as "symbolic", "a complete deductive explanation", "formal scientific", "mathematical proof", or of the "third level". Are these numerous labels for the same (or at least extremely similar) phenomena really a sign of the strength of the literature? Or are they a sign that as a discipline we have failed to build a cumulative body of knowledge? (p. 318)

On one hand, I agree with Inglis' judgment that the proliferation of minor distinctions is a major vice in our field. Also, I agree with the organizers of this book that we need better tools for putting various research findings in dialogue across our disparate and contradictory meanings for central terms (such as argumentation, justification, and proof). On the other, I recognize how "findings" in our field always come (by necessity) with a myriad of contextual clarifications. I argued above that definitions in our field are highly complex in that they have natural histories of development and are contextually operationalized by the array of other tools they are paired with. The same may be said of the research findings that result from using those definitions. The lurking problem is to contextualize our findings to the point that they lack any meaningful interface with other research. While I recognize that one could propose a list of methodological rigors that might constrain our inquiry toward standardization and thus comparability, I do not anticipate this as a productive way forward.

How then will we develop a knowledge base using our conflicting and highly contextualized definitions of central terms? My rather simple answer to this conundrum is to recognize mathematics educators as reflective learners and mathematics education definitions as part of the outcome of our learning process, not strictly its inputs. In the three preceding chapters, I observe all of the authors engaging in the assigned research task with productive nuance, self-reflection, and intellectual constraint. Like any other science, we thrive through peer and self-critique. In these chapters, the authors drew upon interdisciplinary tools to add analytical power to their investigations. They also traced changes in our own definitions. Defining these constructs and putting them to use, much like any defining activity in which we engage our students, is a part of our process of learning.

This brings me to at least one way we might say mathematics education definitions are like mathematical ones. A key shift has occurred in how mathematical definitions are understood. Mathematicians have shifted away from definitions capturing some sort of natural essence toward definitions identifying properties that foster proving. Defining continuity in terms of epsilons and deltas is far less intuitive than the idea of a function's graph having no jumps or gaps, but the former allows us to prove theorems about continuity much more precisely and fruitfully. In short, among a range of possible mathematical definitions, we will prioritize the one that does the most work for us in proving. In many ways, I think the same should be said of mathematics education. Definitions should help us to do our work and advance our inquiry. Over time, we will continue to learn and refine our definitions in light of our learning. This entire book project is an interesting exercise in that reflective process. While I am not sure I can say much regarding how the definitions themselves afford or constrain the inquiry, being that they are operationalized in terms of a number of other analytical decisions beyond the definition itself (the definitions underdetermine the inquiry), I think we learn something about their true nature in the process.

In closing, I propose some other pathways forward that I think will help advance our research on argumentation, justification, and proof in addition to the other insights afforded in this book's experiment in comparative research:

1. *Instrumentation* – some researchers are starting to take on the hard work of developing validated measures related to comprehending, validating, and constructing proofs. This is another important way to clarify our constructs and put our research findings in dialogue. Our field would benefit greatly by more of this work being done.

2. *Clear documentation of learning* – the three preceding chapters all attempted to contextualize the student reasoning observed in light of the trajectory it might follow in the coming weeks and months of instruction. Even though it is useful at times to investigate student reasoning using particular snapshots, our ultimate goal is to promote learning, in this case advancement in student arguing, justifying, and proving. It is impressively easy to find negative evidence regarding the extent and quality of student engagement in these processes. Well-documented accounts of meaningful learning are much more productive.
3. *Research on intellectual need* – both the justification and proof chapters considered how those activities could be incentivized in the classroom without relying purely on the instructor's authority. This requires considering students' intellectual aims within their ongoing inquiry and how these might shift through their educational experiences. While this book has problematized how we as researchers and instructors define these key processes, better insight into how students might experience need for these processes (or how K-12 teachers experience need to promote these in their classrooms) would provide a means of inducing definitions of these constructs for the "end users" we care about most.

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Part V
Lenses on Researching Argumentation, Justification and Proof Across the Grade Levels

Participation in Argumentation: Teacher and Student Roles Across the Grades

AnnaMarie Conner

The chosen definition of argumentation, “the process of making mathematical claims and providing evidence to support them,” offered authors of the argumentation chapters across the grade bands quite a bit of latitude in applying theoretical lenses and constructs to the provided data. Within the application of these different lenses, authors of the argumentation chapters highlighted several similar aspects of argumentation. Primary among these were the roles of the teachers and the students as they interacted in the classrooms. While each chapter focused on students and the teacher to a different extent, all four chapters provoke thought about what it means to participate in argumentation. In this chapter, I look across the four perspectives on argumentation to highlight four themes: opening spaces for student agency and ownership, responding to others’ ideas by building and critiquing, using multiple frameworks to examine teacher actions, and considering argumentation as larger than mathematics. In doing so, I attend to the consequences of using the same definition of argumentation to examine classroom data across four different grade levels.

Opening Spaces for Student Agency and Ownership

One of the consistent themes in the argumentation chapters across grade levels was the importance of opening spaces for students to engage in and with mathematical ideas. At the elementary level, Rumsey and colleagues (this volume) focused their analysis by using Horn’s (2008) conceptualization of accountable argumentation, in which students are expected to contribute to argumentation in meaningful ways. In

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order for these kinds of contributions to occur, Rumsey and colleagues cited the importance of social and sociomathematical norms that foster shared authority among students and teachers (Cobb et al., 2009; Yackel & Cobb, 1996). Rumsey and colleagues illustrated this in Ms. Kirk's second-grade classroom in their description of how Ms. Kirk began the lesson: She asked students for the meaning of the equal sign (written on the board) without naming the sign. However, Rumsey et al. concluded that some opportunities for student agency and ownership were missed by Ms. Kirk, identifying some of the complexities of the interaction of tools, task selection, and task implementation. They highlighted the importance of the kinds of tasks chosen by Ms. Kirk, including the larger topic of equality in elementary mathematics, but pointed to specific aspects of the task and implementation, such as the insistence on use of Cuisenaire rods to solve the problems, as leading to a focus on certainty rather than an opportunity for argumentation.

In the middle grades chapter, Gomez Marchant and colleagues also highlighted the importance of student engagement in argumentation, albeit through the analysis of the role of the teacher in the classroom. Gomez Marchant and colleagues provided evidence for the centrality of student thinking in this classroom by recounting the number of components of arguments contributed by students and contrasting that with Mr. MC's contributions. They interpreted several of Mr. MC's teaching actions as opening spaces for student agency and ownership: His questioning patterns indicated valuing student thinking; he assisted students in putting claims together to reach a final conclusion; and he consistently positioned students as contributors to mathematical activity.

Hähkiöniemi also examined the teacher's role in the high school classroom, concluding that she had exhibited a range of communicative approaches. These communicative approaches included a "dialogic/interactive" approach, in which the teacher invites students to contribute ideas and different points of view. Hähkiöniemi concluded that the teacher exhibited this communicative approach primarily in the first part of the lesson, in which she invited a student to share his solution and then connected it to other students' ideas. He indicated that, on the second day, she missed opportunities for student agency and ownership by asking questions that directed students to focus on small parts of the argument rather than on the bigger picture. "The sequence of the steps seemed to be strictly controlled by the teacher so that the steps funnelled (Wood, 1998) students toward the teacher's aim" (see "Dialogic Argumentation and Teacher Support" in the chapter by Hähkiöniemi, this volume). Hähkiöniemi found that students responded to the teacher's communicative approaches by engaging in dialogic moves when she used a dialogic/interactive approach but providing simple answers when she used an authoritative/interactive approach. This suggests that the teacher can open spaces for student agency and authority at different grain sizes. The teacher in the high school chapter provided opportunities for agency and authority on a large scale on the first day, but restricted those opportunities to exist within a teacher-directed structure on the second day. Hähkiöniemi suggested that, within the small steps in the second part of the lesson, there may have been opportunities for the teacher to open spaces for student agency by exploring a pattern contributed by a student rather than continuing in her pre-planned sequence. This opportunity would have existed at a smaller grain size, within a smaller part of the argument, than the originally identified opportunities for student contributions.

In the tertiary chapter, Plaxco and Wawro gave several illustrations of the teacher inviting students to participate in the argumentation. Most of the teacher's actions seem to be described as leading the students toward the teacher's goals for the activity. For instance, in their description of Vignette 1, Plaxco and Wawro related the teacher's actions that might have provided student agency to the purpose of recording an algebraic representation.

> The instructor's re-voicing and re-representing moves served to bolster initial claims and data provided by the students and to document the discussion in an algebraic form in view of the entire class…so that the instructor could leverage this toward a discussion of the more general algebraic relationships among the symmetries of a triangle. (See "Summary of Vignette 1" in the chapter by Plaxco and Wawro, this volume)

In this chapter, Plaxco and Wawro identified multiple teacher moves as the teacher interacted with the students in the abstract algebra class. However, their description of the teacher's actions consistently focused on the teacher's goals and how she invited student contributions and shaped those contributions into forms that could be used to achieve her goals. This raises a question about student agency and authority in argumentation within inquiry classrooms. Do the students experience invitations to participate as contributing to agency and authority when their contributions are re-represented into forms preferred (or needed) by the teacher to foreground the mathematical agenda of the classroom?

In each chapter, the authors illustrated how engaging students in argumentation can open spaces for student agency and ownership of mathematical ideas. Correspondingly, they demonstrated the crucial role of the teacher in student participation. That is, in facilitating collective argumentation, the teacher coordinates opportunities for student agency and ownership of mathematical ideas by, for example, how she structures the argument (Hähkiöniemi, this volume) or how she poses tasks and coordinates tasks and tools (Rumsey et al., this volume).

Responding to Others' Ideas by Building and Critiquing

The importance of building on other people's ideas as well as appropriate critique was another common theme across the grades. Hähkiöniemi, in the high school chapter, focused his analysis of student contributions on the extent to which the argumentation was dialogic. Dialogic argumentation is concerned with aspects of collaboration, including how students engage with each other's argumentation. Hähkiöniemi's coding scheme for the dialogic aspects of argumentation captured when students question, challenge, or elaborate on another student's contribution (high levels of dialogicity) or when they comment or respond to another student's contribution (lower levels of dialogicity). As an example, Hähkiöniemi described a "high-level engagement with another student's idea" (see "Students' Dialogic and Justifying Moves" in the chapter by Hähkiöniemi, this volume) when a student elaborated another student's drawing by introducing the idea of infinitely many triangles to describe what she had drawn. Hähkiöniemi described multiple instances of elaborating (high level) and commenting (lower level) in his analysis of dialogicity in the high school chapter. However, he noted no instances of questioning or challenging within the argumentation observed in that transcript. Similarly, Plaxco and Wawro provided multiple examples of student contributions, including students explicitly acknowledging (and extending) other students' contributions, but provide only one example of a student questioning another's solution. This one example occurred when a student questioned if they had to use the other student's method when completing a certain kind of problem and was not an example of questioning or challenging a peer's argument.

Likewise, Gomez Marchant and colleagues explicitly recorded Mr. MC's support for students' arguments while noting no examples of students supporting or critiquing their peers' arguments in the given data.

> It was, however, always Mr. MC who supported the mathematical arguments [in the data provided]. There did not seem to be a norm established where students could question or support other students in the construction of mathematical arguments. Mr. MC empowered students to construct arguments, but with the exception of one rebuttal, did not seem to give space to critique the reasoning of others. (See "What We Learned from Mr. MC's Facilitation of Argumentation" in the chapter by Gomez Marchant et al., this volume)

Rumsey and colleagues similarly argued for the importance of critique in facilitating accountable argumentation. They note that Ms. Kirk elicited ideas from multiple students, but she then used her own mathematical authority to establish whether ideas were correct or not. In one case, she did not attempt to make sense of a student's argument, instead steering that student into her own solution; in another, rather than convening the class to critique an argument, she said, "Can't really argue with that. That is some hard concrete evidence" (see "The Role of Participants" in the chapter by Rumsey et al., this volume). Rumsey and colleagues concluded, "Opening up the discussion to the whole class would help the community realize their role in critiquing arguments and could encourage them away from an appeal to authority justification in the future" (see "The Role of Participants" in the chapter by Rumsey et al., this volume).

In the United States, the standard most often appealed to as a justification for attention to argumentation is "construct viable arguments and critique the reasoning of others" (MP3, National Governors Association Center for Best Practices & Council of Chief State Officers, 2010). As highlighted by this volume's argumentation authors across the grades, students are being given opportunities to contribute to and construct arguments. However, these chapters suggest opportunities to critique the reasoning of others are scarcer. Beyond proposing a connection to sociomathematical norms (Yackel & Cobb, 1996), the authors do not speculate on the source of this scarcity. They do, however, emphasize the importance of critiquing arguments in the development of student agency and authority. Engaging students in critiquing arguments is an intriguing area of growth for teachers and a needed area of investigation for researchers. What additional knowledge, skills, or dispositions might be needed for critiquing arguments to become a common and nonthreatening occurrence in classrooms?

Multiple Frameworks for Teacher Actions

The authors of the argumentation chapters chose different frameworks to examine teacher actions with respect to argumentation. Three of the frameworks are specific to examining argumentation and facilitated analyses that generated results focused on teachers that consider student participation in argumentation. The fourth (in Plaxco and Wawro's chapter) was generated from observations of inquiry instruction and led to results focused on how the argumentation pushed the mathematical agenda of the class. All of the authors acknowledged the given definition of argumentation, and all explicitly identified claims and evidence within their chapters. Three of the chapters included Toulmin's (1958/2003) conceptualization of argumentation, and two used some modification of his diagrams in their analysis, evidence of the prevalence of Toulmin's model in mathematics education research involving argumentation.

Plaxco and Wawro chose Ellis et al.'s (2019) framework to examine the teacher's moves with respect to argumentation. According to Plaxco and Wawro, this framework gave insight into how the teacher advanced the mathematical agenda of the classroom, while the use of Toulmin's model revealed the nature of students' argumentation. They concluded that "the argumentation is both a means for students to communicate their reasoning about concepts and also a mechanism for advancing the mathematics of the classroom toward more general activity" (see "Discussion" in the chapter by Plaxco and Wawro, this volume). Ellis et al.'s framework guided the researchers in this chapter toward the progression of mathematical ideas, and their results were framed accordingly.

Hähkiöniemi also coordinated two frameworks in his analysis, one to look at the students' argumentative work (Hähkiöniemi et al., under review) and the other to examine the teacher's communicative approaches (Mortimer & Scott, 2003). His use of the two related frameworks generated explanations for how the students participated in argumentation as related to the teachers' stance with respect to authority and activity. Additionally, Hähkiöniemi's analyses foregrounded issues of grain size when examining argumentation. He noted the teacher's overall structure of the larger argument constrained the larger conversation while individual steps allowed for more student agency and participation.

Gomez Marchant and colleagues relied on Conner et al.'s (2014) framework, which includes references to Toulmin's (1958/2003) diagrams, to examine the teacher's support for argumentation. By examining the contributions of argument components and the actions of the teacher surrounding those components, they identified Mr. MC's focus on student reasoning as well as his scaffolding of smaller student claims into larger and more global ones. In so doing, Gomez Marchant and colleagues concluded that Mr. MC's support for argumentation gives us insights into how students can be empowered "in developing their identities as doers-of-mathematics" (see "What We Learned from Mr. MC's Facilitation of Argumentation" in the chapter by Gomez Marchant et al., this volume).

Rumsey and colleagues' use of Horn's (2008) accountable argumentation to frame their analysis led to a focus on language, participants, task, and tools. In particular, Horn's framework focused Rumsey and colleagues on the role of the teacher in modeling accountability for making sense of student contributions and choosing combinations of task and tools to create opportunities for multiple claims and possible generalizations. Rumsey and colleagues concluded that Ms. Kirk invited students to participate in argumentation, but ultimately "avoided [the] messiness" of argumentation (see "Conclusion" in the chapter by Rumsey et al., this volume). Using Horn's framework, Rumsey and colleagues proposed alternate choices Ms. Kirk could make to engage her students in accountable argumentation; they acknowledged these choices will likely increase a teacher's discomfort as they increase the messiness of argumentation. However, Rumsey and colleagues make a compelling case for the utility of messiness and uncertainty in student participation in argumentation.

Considering Argumentation as Larger than Mathematics

The authors of the middle grades chapter, Gomez Marchant and colleagues, explicitly asked the field to consider argumentation as larger than mathematics. In their concluding section, Gomez Marchant and colleagues suggested that, within a mathematics classroom, people make two additional kinds of arguments: social arguments and sociomathematical arguments. Social arguments have to do with how a person wishes to be seen by others. These arguments may add to discussions about how (and why) students participate in class as well as larger issues such as "how dominant narratives are perpetuated in mathematics education" (see "Conclusion: Larger Spheres of Argumentation" in the chapter by Gomez Marchant et al., this volume).

Sociomathematical arguments, according to Gomez Marchant and colleagues, are inextricably linked to Yackel and Cobb's (1996) construct of sociomathematical norms, a construct explicitly mentioned in each of the argumentation chapters. These kinds of arguments address how individuals do mathematics. For instance, Gomez Marchant and colleagues refer to Mr. MC making an argument about "what makes a strong mathematical argument" and "the importance of checking the boundary cases" within his classroom as a *sociomathematical argument* (see "Conclusion: Larger Spheres of Argumentation" in the chapter by Gomez Marchant et al., this volume). Rumsey and colleagues' description of using uncertainty to create a need for argument (see "The Role of the Tool in Terms of Evidence and Potential Connections" in the chapter by Rumsey et al., this volume) may fit into this category of sociomathematical argument. Hähkiöniemi's implicit claim about students not experiencing a need to ask questions or challenge other students' statements (see "Dialogic Argumentation and Teacher Support" in the chapter by Hähkiöniemi, this volume) may also fit into this category. As teachers and students participate in argumentation, they may engage in arguments that transcend the immediate mathematical ideas: that address what an argument entails, how to do mathematics, or even how to participate in the classroom community. Attending to these kinds of arguments in addition to the mathematical ones described in this book has the potential to enrich our understanding of participation in mathematics.

Consequences of the Definition

The given definition of argumentation, the process of making mathematical claims and providing evidence to support them, neither prescribed a framework for analysis nor suggested any of the themes that were common across the argumentation chapters. However, the explicit mention of claims and evidence in the definition may have brought to mind Toulmin's (1958/2003) model for arguments, which was at least referenced in each of the argumentation chapters. In operationalizing the definition within their analyses, each author group, regardless of grade level or additional frameworks used, identified claims and evidence within the classroom discourse and, in doing so, attended to the *process* of argumentation. Additionally, all of the chapters include at least one representation of an argument, in which the authors notate different parts of arguments (including claims and evidence, even if different terms were used) they reconstructed based on the data they were given.

The given definition did not define claims or evidence, nor did it suggest how one might identify these, who might contribute them, or even why they might be important. Perhaps this definition was specific enough to define the construct and general enough to allow identification of argumentation across multiple grade levels. The arguments identified by the authors address different topics and contain different kinds of evidence (as would be expected from the different datasets provided). This definition was useful in that it allowed researchers to identify argumentation (and arguments) within data from four different classrooms (generated by four different research projects, which were originally focused on different questions). It could be usefully combined with a variety of theoretical perspectives to understand aspects of the mathematical discourse within these classrooms from different grade levels. That is, this definition constrained the construct of interest, argumentation, in a way that it could be identified, but it allowed for a range of theoretical perspectives to be usefully applied to the given data.

All four chapters revealed patterns of participation in argumentation that are not obvious consequences of the definition. The given definition does not specify that it matters who contributes claims and evidence, but the researchers who used the definition attended to patterns in participation. Other potential definitions may foreground the participants in argumentation more than the chosen definition. For instance, Wood (1999) defined argumentation as "discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (p. 172). Arguably, Wood's definition foregrounds participation more than the one used in this volume. Likewise, Mueller et al. (2012) defined argumentation as "mathematical explanation intended to convince oneself or others about the truth of a mathematical idea" (p. 376). Again, the audience and participant(s) in argumentation are foregrounded in this definition. It is impossible to know how the use of a different definition would have impacted the analyses conducted by the authors of these chapters. However, it is interesting to note that these groups of researchers attended to participation and sociomathematical norms (as well as issues of agency and interaction) given a definition that did not include these constructs.

Concluding Thoughts

As the four chapters addressing argumentation in this volume illustrate, there are a multitude of frameworks that can be used in examining argumentation in classrooms. Given the same definition of argumentation and a snippet of classroom transcript as data, the authors used different methods to obtain their results. Perhaps surprisingly, given the large range of grade levels in the data analyzed, the authors did not stray far from the given definition in their analyses, using the ideas of claims and evidence in their analyses of argumentation from grade 2 to college-level abstract algebra. The different frameworks illuminated nuances of practice with respect to argumentation, and the different mathematical contexts and tasks provided different opportunities for argumentation, but these four analyses resulted in several common aspects that deserve consideration. In each of these classrooms, the teacher's role was essential in posing tasks and structuring arguments. Each author identified times and ways that students and the teacher built on student arguments, but overall the authors identified a scarcity of critiquing arguments within the given data. Finally, all authors acknowledged the importance of sociomathematical norms (Yackel & Cobb, 1996) in coordinating the students' and teacher's roles in participation in argumentation. As the field moves forward in examining argumentation, can we capture arguments that go beyond a single problem and address aspects of how the classroom community participates in mathematics?

The scarcity of examples of questioning or critiquing students' arguments, an idea brought up by several of the authors, struck me as an important piece of collective argumentation that is not easily captured by looking purely at the content of arguments using, for instance, Toulmin's (1958/2003) model by itself. The authors' choices to use different or additional frameworks allowed the observation about critiquing arguments to be made, and their examination of both teacher and student roles made this observation more salient. All of the authors who observed the scarcity of critiquing arguments seemed concerned about this phenomenon, and this caused me to consider why critiquing arguments might matter in students' experiences in collective argumentation, particularly in mathematics classrooms. One answer, which is implicit in Hähkiöniemi's chapter, is that in order to question or critique an argument, one has to engage with and attempt to understand that argument. This is one of the hallmarks of dialogicity, and it involves both a disposition (to desire to understand another's perspective) and a level of mathematical understanding (one has to understand something about the mathematics in the argument to engage with it). I believe both of these are important in building a mathematics learning community. But the utility of engaging with and critiquing mathematical arguments is also related to Rasmussen et al.'s (2020) finding that engaging with another person's reasoning creates opportunities for growth in mathematical understanding. That is, in a study that combined classroom data with individual interviews, Rasmussen et al. found that engaging with another person's reasoning provided students with (at a minimum) opportunities to grow in understanding if not provoking actual growth in understanding. In the individual interviews, Rasmussen et al. engaged students in critiquing a hypothetical student's argument, after which participants articulated different understandings of a mathematical idea. Rasmussen et al. are careful not to attribute causality to the situation, but their evidence for the utility of engaging with another person's reasoning is compelling.

As our field continues to examine collective argumentation, our frameworks and methodologies must continue to grow. Frameworks, such as Conner et al. (2014), which competently capture many aspects of the teacher's role in argumentation, may need to be expanded to be useful in examining how the teacher engages students in critiquing arguments. New frameworks, such as Hähkiöniemi et al. (under review), will need to be examined and adapted. And, as we consider twenty-first-century classrooms, we may be able to use the tools of argumentation to examine arguments beyond mathematics.

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Justification Across the Grade Bands

Amy B. Ellis, Megan Staples, and Kristen N. Bieda

Introduction

In this synthesis chapter, we discuss three themes that emerge when looking across these four chapters that analyzed data from different grade levels and drew on different perspectives. Specifically, we address (1) the authors' attempts to operationalize the definition of justification; (2) their use of additional frameworks to help capture key features of the justification activity; and (3) the focus on justification as a social process and practice, which includes attending to sociomathematical norms. We conclude by reflecting on the question – why define justification? – a question that arose in response to the various ways the authors positioned justification with respect to proof and other valued mathematical and learning practices.

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Using the Definition of Justification

The intended goal for the four sets of authors was to analyze justification in a given data set using a common definition: "The process of supporting your mathematical claims and choices when solving problems or explaining why your claim makes sense" (Bieda & Staples, 2020). None of the authors used the definition as is. We see evidence of a range of interpretations of the definition across the four justification chapters. The authors focused on different key phrases of the definition, and they chose to supplement the definition to help them highlight certain important mathematical or interactional work in the classrooms. Take, for instance, the phrase "the process of supporting mathematical claims and choices." On the one hand, this phrase could be broadly interpreted as "recounting what you did," perhaps while explaining a problem solution or describing a strategy. On the other hand, its interpretation could be restricted to the activity of "providing a deductive chain of reasoning that begins with given and accepted statements and proceeds to the claim." The interpretation chosen will influence what is considered a justification. These uses and interpretations of the definition for the purposes of data analysis suggest different priorities for these researchers. We examine some of these uses and interpretations and then reflect on the consequences of how the authors chose to interpret and operationalize the definition in their analysis.

In the elementary chapter, Thanheiser and Sugimoto (this volume) used the given definition and gave it further shape by distinguishing among three types of justifications: validation justifications, experiential justifications, and elaboration justifications. This stance toward justification, which is compatible with Bieda and Staples's use of the terms *supporting* and *explaining why*, explicitly includes contributions such as describing a process, providing a chain of reasoning, or even demonstrating with an example. Newton (this volume), while similarly casting a broad net, differed in that she did not admit to the category of justification the simple recounting of mathematical activity (i.e., explaining one's process or steps). This is, however, a justification for Thanheiser and Sugimoto (called "elaboration justification"). Generally speaking, the authors for both chapters considered justification a broad class of ways to support and make sense of claims.

By contrast, the tertiary chapter authors (Karunakaran & Levin, this volume) turned to Staples et al. (2012), who defined a justification as "an argument that demonstrates (or refutes) the truth of a claim and that uses accepted statements and mathematical forms of reasoning" (p. 448). This definition does not address the *supporting choices* part that is admitted in the given Bieda and Staples (2020) definition. Consequently, some of the activities that Thanheiser and Sugimoto might consider to be justifying would not fall under the umbrella of justification for Karunakaran and Levin. For example, Thanheiser and Sugimoto included experiential justification as a category, in which students can draw a picture to support a claim – a type of contribution not likely to reflect a mathematical form of reasoning, and thus not admitted as a justification for Karunakaran and Levin. As Karunakaran and Levin point out, critical here is what is considered a *mathematical* form of reasoning. Staples and colleagues did not precisely define this term, but they did give examples of what does and does not count as mathematical. Arguments that appeal to authority are not mathematical, whereas arguments that appeal to previously established results are. They also considered empirical or example-based arguments to be mathematical, provided the relevant claim is one that can be demonstrated empirically (such as in proof by exhaustion). Karunakaran and Levin noted that they preferred this definition precisely because it stipulated mathematical forms of reasoning, which are not specified in the Bieda and Staples (2020) definition.

The middle grades chapter authors (Lesseig & Lepak, this volume) explicitly addressed what it means to "support your mathematical claims and choices." Lesseig and Lepak offered that justification is a process in which individuals, working collectively with others, explain why a claim is true. Like Karunakaran and Levin, the authors operationalized the definition to focus on mathematical claims with truth values. Consequently, they did not consider support for the choices students made when doing the mathematical work as justification work. In addition, by using the Teacher Moves for Supporting Student Reasoning (TMSSR) framework (Ellis et al., 2019), the authors' analysis attended significantly to how the teacher supported reasoning that could be extended toward a generalization. With these choices, Lesseig and Lepak centralized the nature of the student reasoning offered in support of various claims and the degree to which teacher moves encouraged students' reasoning toward generalizations.

As pointed out by Karunakaran and Levin ("Defining and Operationalizing Justification," this volume), Bieda and Staples's definition of justification focused more on defining justification as a process than on specifying the characteristics of a justification as a product. On the one hand, foregrounding the process aspect of the practice may sidestep some controversy that can emerge when scholars try to pin down whether or not something is a justification. This controversy has been well documented, particularly with regard to what counts as mathematical proof (Balacheff, 2002; Cai & Cirillo, 2014). In fact, scholars such as Weber (2014) and Czocher and Weber (2020) have introduced the linguistic idea of "cluster concept" (Lakoff, 1987) to advance a more ecumenical approach to defining a proof – a definition used for other chapters in this book. On the other hand, Bieda and Staples's definition suggests, with phrases such as "supporting *your* mathematical claims" and "explaining why *your* claim makes sense" (italics added for emphasis), the importance of attending to students' perceptions of justifying activity rather than an overreliance on an observer's beliefs about when students are engaged in the justification process and what counts. This potentially shifts the locus of the authority in determining justification activity from the discipline of mathematics (e.g., with terms like mathematical reasoning) to the students – an idea present in Newton's framing and analyses with the concept of a situation for justification. In such cases, what students do in response to such situations is quite revealing of the classroom context and sociomathematical norms, which we discuss in the next section.

We see across the chapters that the authors grappled with many facets of understanding the meaning of the given definition, as well as how to best use it to understand the justifications and justifying activity evident in the data sets they were provided. In doing so, they all supplemented the definition with additional frameworks or definitions, which shaped their analyses and ultimately led to what may be understood as a fairly disparate capturing of justification activity. It is probably not a coincidence that it is the elementary authors who took the broadest stance on justification and the tertiary authors who took the narrowest. Reflecting on the question of how the grade level might influence the use or interpretation of terms, it seems that there is a potential effect suggested here. A benefit of thinking about justification as different from proof is that it admits a larger category of mathematical activity and thus gives us a way to consider the argumentation and deductive reasoning of younger children who may not be developmentally ready for formal proof. It allows us to see what mathematical work students *are* doing in response to supporting claims and choices in their mathematical activity, even in the earlier grades.

Frameworks and Classification Schemes: Capturing and Making Sense of Classroom Activity

A second theme emerged in relation to frameworks and classification schemes. In rendering the definition useful for the given data set, most of the authors chose to conceptualize either the types of justifications that occurred in the respective classrooms (Karunakaran & Levin, this volume; Thanheiser & Sugimoto, this volume) or the types of classroom situations in which justifying activity was occurring (Newton, this volume). (Lesseig and Lepak's chapter is the exception; they used the TMSSR framework, which provided some additional structure and a way to highlight some aspects of justification activity.) For instance, Thanheiser and Sugimoto distinguished *validation justifications* from *experiential justifications* from *elaboration justifications*. These categorizations connect justification to opportunities for justifying in the classroom context, including those prompted by task design. Contrast this categorization with Karunakaran and Levin's approach, which used categories that describe the mode of reasoning of the justification: *perceptual pattern recognition justifications*, *recounting of process justifications*, and *mathematically rationalized justifications*.

There are implications involved in devising any classification scheme. Firstly, a classification scheme acts as a lens that can bring particular aspects of justification into focus, and blur, or obscure, others. For example, in classifying justifications according to their type, Thanheiser and Sugimoto chose a frame that described students' responses to classroom prompts, which – by necessity – de-emphasized other aspects of justification. A useful contrast is the approach taken in Newton's chapter ("Methods," this volume), with her focus on the construct of justification as a disciplinary practice (drawing on Staples et al., 2012). This is defined to include justification "for the purpose of validation," but single instances of validation justifications are not sufficient evidence to claim the emergence of justification as a disciplinary practice. Newton's practice approach necessarily pushes scholars to take a more holistic view of students' engagement during a classroom episode. It is informative to contrast this holistic framework with a construct such as Thanheiser and Sugimoto's justification types, which highlights discrete episodes of justification activity and analyzes each episode for its features.

The question of grain size is relevant here. For researchers who aim to explore the justifications shared in a second-grade classroom and how they were or were not leveraged to provide access to students' mathematical reasoning (Thanheiser & Sugimoto, "Abstract," this volume), a smaller grain size is more informative. By contrast, when one seeks to "examine the co-construction and enactment of justification norms and practices …, which mathematical ideas were justified, and the extent to which members of the classroom community were convinced by the justifications" (Newton, "Abstract," this volume), a broader accounting of the nature of justifications and justification activity is appropriate.

A second implication of researchers' choice of a classification scheme, particularly a scheme that is distinct from students' schemes, is the degree to which the conclusions researchers make position students as logical, coherent thinkers and doers of mathematics who bring significant knowledge to bear in their classroom activity. Positioning students as productive thinkers whose knowledge is an asset for justification activity is critical for supporting equitable participation in a mathematics learning community, a goal rightly emphasized by Thanheiser and Sugimoto. We note that some classification schemes, such as Harel and Sowder's (1998) proof schemes taxonomy, provide a way to recognize, characterize, and build on students' nascent forms of reasoning that are seen as powerful and productive, even if they are incomplete or informal. Such an approach makes sense of students' justifications in a manner that affords the development of models of students' mathematics (Steffe & Olive, 2010), rather than considering a classroom solely from the observer's perspective.

In reconsidering the definition for justification these authors were charged to use, the language centers the student's perspective in the process of justification ("supporting your mathematical claims" and "explaining why your claim makes sense"). Yet, as discussed in Lesseig and Lepak ("Theoretical Framework," this volume), using evidence from classroom discourse to infer students' perceptions while they engage in justifying activity within a classroom context must be done with care. Data that are transcripts of classroom activity are, by necessity, limited in their "observer-focused" lens. Unlike the data analyzed by Harel and Sowder (1998) when constructing their definitions of proof schemes, the authors of these chapters did not have the opportunity to conduct follow-up interviews with students, nor, in some cases, even see examples of students' written work. This highlights an important consideration regarding the choices we make as researchers about how we define the constructs that are the focus of our study and the methods we choose to conduct our research – namely, the tension that researchers can only "see" what the data allow them to find.

Justification as a Social Process That Serves a Purpose in the Community

One common element across all four chapters was a lens on justifying as a social, even collective, activity. In part, this was afforded by the Bieda and Staples (2020) definition highlighting justification as a *process*. Both the elementary and middle school authors addressed justification as both a cognitive and social process. Justification, they noted, relies on previously established mathematical knowledge, satisfies community-specific requirements for accepted forms, and depends on collectively determined notions of what is mathematically valid (Thanheiser & Sugimoto, this volume; Lesseig & Lepak, this volume). The tertiary authors also mentioned the need to consider who takes justifications as valid, asking, "to whom or to which community should your answer make sense?" (Karunakaran & Levin, "Discussion," this volume). Finally, the high school author addressed justification as a collective activity, citing Blumer (1969) as well as Cobb et al.'s (1992; Yackel & Cobb, 1996) micro-level analysis of teacher and student turns (Newton, this volume). By framing justification as a collective process, all of the authors addressed the need to attend to the social and sociomathematical norms in the classroom that foster justification, that determine what types and forms of justification were acceptable and seen as convincing, and that encourage the development of particular forms of interaction and engagement surrounding justification. Given the limitations of the data sets – 1 or 2 days of classroom transcript – we saw few explicit identifications of relevant social or sociomathematical norms across the chapters, but the importance of this construct in helping to make sense of the justification activity was clear.

Closely related to social and sociomathematical norms are the *purposes* of justification within a classroom, as the sociomathematical norms governing justification in a classroom can reflect the purpose(s) of engaging in justification. Justification – as revealed in these chapters – does not serve a singular purpose. Notably, it was positioned as an important learning practice in each of the chapters. Lesseig and Lepak anchored justification by framing it as an opportunity to promote an understanding of proof (in the long run) and specifically the practice of generalization. Karunakaran and Levin were concerned with moving toward proof as well. A key purpose of justification for these chapter authors, then, is as a developmental step on the road to proof. Justification is understood and positioned in relation to proof.

Complementing this, Karunakaran and Levin also explicitly acknowledged the value of their (non-proof) categories of justification, such as pattern recognition and recounting, for learning content. Regarding the nature of the classroom activity, they noted: "discussions that are punctuated by category 1 and 2 justifications are essential to the success of courses such as the one we examined here," in no small part because "this curriculum relies on the development of students' understanding of the content by sustained discussions" (see "Discussion" in the chapter by Karunakaran & Levin, this volume). Thus, Karunakaran and Levin valued justification work for how it supports engagement with course content.

The chapters by Newton and Thanheiser and Sugimoto positioned justification in the classroom in relation to learning content as well. Thanheiser and Sugimoto, for instance, considered the opportunities to come to new understanding about the meaning of the equals sign. Citing Jaffe (1997), Thanheiser and Sugimoto explicitly stated that justifications do not have to be logically complete. Newton, who focused on the co-construction of justifications, attended to which ideas were justified, which were left unjustified, as well as when the class was convinced of a result (i.e., when they accepted the formula for the sum of the interior angles of an *n*-sided polygon), even if they had not developed a complete argument to support the claim. (Note also the connection to sociomathematical norms here.) Newton observed that the class did not explore why one triangulates the polygon, nor how one ensures the minimum number of triangles are drawn. Nevertheless, the class engaged exploratory work, motivated the relevant formula, and seemed convinced by the formula at the end of the lesson. Although mathematicians and mathematics educators might desire a more complete or mathematically reasoned justification or even a proof of the formula, which appeared to be within the conceptual reach of the students, this outcome may not be the goal in a classroom that is organized to meet content standards. Rather, the activity in this classroom aligns with what Stein et al. (2009) have called *procedures with connections*. In Newton's chapter, as it was in Thanheiser and Sugimoto's chapter, justification is not positioned in relation to proof, but rather it is considered to be a co-constructed community activity valued for its role in developing mathematical ideas and helping students learn content.

Thanheiser and Sugimoto extended these ideas to explicitly discuss how justification provides *access* for students to one another's thinking and opportunities to build their understanding. In doing so, they described justification as an equity tool. They elaborated:

> By asking students to justify, the teacher asks the students to build on their own understanding, reason from those understandings, and make their reasoning public for the class. Asking several students to justify and connecting across those justifications, allows students with varying levels of prior knowledge and understanding to access and connect to each other's reasoning. (Thanheiser & Sugimoto, "Summary," this volume)

In this case, Thanheiser and Sugimoto consider justification in light of the ability to create opportunities to make sense of what the equals sign means. By positioning justification differently, Thanheiser and Sugimoto and Newton make different kinds of claims about the justification activity in their data sets. Justification is not positioned on the road to other proof-related practices, but for how it supports access to learning opportunities.

It is interesting to reflect on how justification is positioned as a knowledge building activity in each of these chapters. As a developmental step on the way to proof, justification is a practice that helps students get better at the "knowledge building" practices for the discipline. As a classroom teaching and learning practice (Staples et al., 2012), justification helps students develop conceptual understanding of mathematical ideas – another kind of knowledge building practice, but focused on the learners' knowledge. Both goals/purposes can be pursued and are valuable. As revealed in the Lesseig and Lepak chapter, however, there are times when it may be challenging to pursue both simultaneously. In the middle grades chapter, the teacher's efforts to emphasize the valid reasoning that established the number trick for the numbers 1 through 10 may have simultaneously reduced attention to sense-making for how the number trick "worked," which would provide opportunities to engage the conceptual basis of the distributive property and a generalized argument demonstrating that the trick must work for all values.

Our discussion of the purposes of justification in these chapters has primarily divided them into two "camps" – anchoring justification in relation to the valued goal of *proof* and anchoring justification in relation to the valued goal of *access* and *opportunities to learn content*. These two camps of course are not two camps but two sides of a coin. The role of both in knowledge building is unique and needed in mathematics. The authors, however, also mentioned many other potential purposes of justification. Looking across the chapters, we see justification positioned as a process that accomplishes a variety of goals: it can be a tool for supporting students' sense-making (Thanheiser & Sugimoto), a way to foster reform-based practices (Lesseig & Lepak), a vehicle for promoting conceptual understanding (Karunakaran & Levin; Newton; Thanheiser & Sugimoto), a method for increasing learning across diverse student populations (Newton; Thanheiser & Sugimoto), a practice that fosters mathematical and lifelong dispositions (Newton), and finally, a tool to "co-construct meaning and develop mathematical understandings" (Newton, "Methods," this volume).

Unique Contributions of Justification as a Construct

Although it seems like an odd question to ask in a book that has spent many pages discussing justification, these four chapters made us pause to ask: Why do we need justification? What do we need justification to do for us as researchers and mathematics educators? Alternately phrased: Why would it be important to highlight justification activity as distinct from proof activity in mathematics classrooms? Why is it valued? One reason this question is important is because it is not clear that there is consensus, or even relative consensus, in the answers. As scholars, we might analyze classrooms to determine evidence of justification activity, but perhaps not with a shared vision of *why* it is a construct that deserves attention.

The field has wrestled with this issue with respect to proof, and it seems that the rationale for attending to proof and proving in classrooms is more established. Commonly held reasons to incorporate proof and proving into classrooms include because it can (a) support students' understanding of the role of proof in the discipline; (b) foster students' abilities to develop, understand, and critique proofs (Knuth, 2002; Dickerson & Doerr, 2014); and (c) support students' development of mathematical understanding and knowledge (Stylianides, 2007). Proving is widely understood to be a core disciplinary practice in mathematics for establishing and advancing knowledge. Several researchers do argue that we must not overlook other key purposes of proof (e.g., Hanna, 2000; de Villiers, 1990), but they all require the notion that proof is for establishing truths. The practice benefits from a level of consensus around its purpose, which reflects its established position within the discipline.

Justification, by contrast, does not share the same strong disciplinary grounding, and its role does not have such a focused rationale. As seen across these chapters, the authors looked to justification to accomplish different end goals, and they admitted fundamentally different sets of contributions as justifications, as discussed above. This state of affairs may result from a strength of justification as a practice: it is powerful and well suited to accomplish many different ends. At the same time, this state of affairs may result in a weakness for the field, as it is difficult to compile a strong research base when a construct has such flexibility, unless great care is taken with defining, operationalizing, and specifying the purposes and value of the construct in research studies.

Why, then, do we need justification? We need justification to capture reasoning and sense-making that is different from proof. In general, as mathematics educators and mathematics education researchers, we need terms to help us name and discuss valued elements in a teaching and learning environment. The role/purpose we ascribe for those elements (beyond their definition) shapes how the term is used and what we can learn by examining the term (as we have seen in these chapters). Indeed, the two foci outlined for justification help us attend to valuable aspects of classrooms. As a field, we have terms that are well positioned to describe and analyze the more disciplinary-focused truth-establishing elements of this work (i.e., proof, frameworks, and proof schemes). By contrast, we do not have as well developed a set of ideas or terminology to capture this sense-making mathematical work that provides *support for claims and choices* that is so crucial and powerful in our classrooms in providing access and opportunity to learn. By defining justification to capture elements of mathematical activity that can support reasoning and proof but do not need to be positioned in relation to proof, we can attend to aspects of student reasoning that might otherwise remain unexamined. As a field, in addition to examining classrooms for opportunities to provide a foundation for proof and proving, we also must examine classrooms for opportunities to access and make sense of mathematical concepts, ideas, and processes in a co-constructed learning environment.

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Lens, Blinders, or Kaleidoscope? Using a Definition of Proof to Make Sense of Classroom Activity

Sean Larsen and Tenchita Alzaga Elizondo

Introduction

In the previous chapters, four sets of authors were tasked with making sense of classroom data (representing four different educational levels) using the same definition of proof (Weber, 2014). The authors did not select the definition of proof, nor did they select the data set that they were asked to analyze. The authors were selected based on their expertise related to argumentation, justification, and proof at the education level corresponding to the data set they were assigned. This task presented each author (or author team) with a challenging problem to solve: How can the provided definition and data set be used to generate useful insight related to proof in the given educational context? As the authors of this synthesis chapter, we have a dual challenge/opportunity. Our objective is to consider this collection of chapters and (1) determine what general insights we can extract related to proof in mathematics classrooms across education levels and (2) determine what insight we can extract related to the challenges and affordances of using a definition of proof to make sense of classroom data.

We first briefly characterize the approach taken in each chapter to operationalizing the given definition of proof. We compare and contrast these approaches both to support our subsequent discussion of the researcher(s)' findings and to highlight very interesting differences in how this task of operationalizing the definition was approached. Then, we briefly characterize the conclusions reported in each chapter, drawing links between the nature of these findings and the approach to operationalizing the definition. Finally, we take a step back in order to directly address our two

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main goals. These goals are to (1) synthesize the four sets of findings to extract some insights related to proof in the context of teaching and learning and (2) draw some general conclusions related to the constraints and affordances that come along with using a definition of proof in this way and share some thoughts about how these opportunities might be realized and the constraints overcome.

Operationalizing the Definition

Each of the four chapters took a different approach to operationalizing Weber's (2014) cluster definition of proof. The author teams explicitly explained and justified their approach. Here, we will briefly characterize the approaches used and the authors' explanations for them. For convenience, we first recall Weber's definition.

Mathematical proving is a process by which the prover generates a product that has either all or a significant subset of the following characteristics:

1. A proof is a convincing argument that convinces a knowledgeable mathematician that a claim is true.
2. A proof is a deductive argument that does not admit possible rebuttals. The lack of potential rebuttals provides the proof with the psychological perception of being timeless. Proven theorems remain proven.
3. A proof is a transparent argument where a mathematician can fill in every gap (given sufficient time and motivation), perhaps to the level of being a formal derivation. In essence, the proof is a blueprint for the mathematician to develop an argument that he or she feels is complete. This gives a proof the psychological perception of being impersonal. Theorems are objectively true.
4. A proof is a perspicuous argument that provides the reader with an understanding of why a theorem is true.
5. A proof is an argument within a representation system satisfying communal norms. That is, there are certain ways of transforming mathematical propositions to deduce statements that are accepted as unproblematic by a community and all other steps need to be justified.
6. A proof is an argument that has been sanctioned by the mathematical community. (Weber, 2014, p. 537)

Elementary Grades Operationalization

Walkington and Woods were tasked with using the cluster definition to guide their analysis of transcript data from a second-grade classroom. Rather than operationalize the cluster definition in its entirety, the authors chose to focus on two aspects, one that they saw as missing from the definition and one that they connected to a particular characteristic included in the cluster. First, while Walkington and Woods note that the generality of an argument may be implicitly addressed by Characteristic 2 (as a deductive argument for a mathematician would necessarily address the generality of the claim), they argue that because children often reason with examples, this aspect of proof is particularly important in the elementary grades context. Second, the authors, citing Characteristic 4, argue that "the act of proving may encompass more than an argument that may be read by a 'reader'" and so it is particularly important to conceptualize proof as a multimodal endeavor.

Middle Grades Operationalization

Yopp, Adams, and Ely were tasked with the analysis of a seventh-grade lesson and chose to focus on the collected written work. The authors conduct an explicit operationalization of the definition in two steps. First, each characteristic is adapted to address the fact that applying them depends a great deal on subjectivity. For example, the authors note that whether an argument successfully explains why a theorem is true (Characteristic 4) depends on the reader, so this characteristic is adapted and framed in terms of whether an argument reveals conceptual insights that provide structural links between the hypothesis and conclusion of an argument. After adapting each of the six characteristics, the authors developed a three-level rubric where a score of 2 indicated that most of the six adapted characteristics were met by the student's argument while a score of 1 indicated that these were not met *but* the proof did reveal conceptual insight which the authors took as evidence of partial alignment with all of the characteristics. A score of 0 indicated that the argument did not address or indicate awareness of the need to address all cases (generality).

Secondary Grades Operationalization

Morselli's approach to operationalizing Weber's (2014) cluster definition of proof was to augment and coordinate it with a framework of principles for task design (Lin et al., 2012) and a framework (Cusi & Malara, 2016) that characterizes the roles of the teacher in mediating the students' approach to proof. Morselli justified this decision by arguing that although the cluster definition of proof offers the opportunity to view proving as a practice which happened in a social environment, it does not address the ways in which proof can be introduced in the classroom and the different roles the teachers and students play in the proving activity. The analysis process primarily leveraged Weber's definition to identify opportunities to support the improvement of the students' arguments.

Postsecondary Operationalization

Fukawa-Connelly and Karahoca were tasked with using Weber's cluster definition to analyze a lesson in an introductory university group theory course. Their operationalization of the definition focused on two issues that they felt directly impacted their analysis. First, the authors note that Characteristics 4–6 each are framed in terms of the specific community in which the argument is presented as a proof. They note that they do not have access to the norms and standards of the classroom community and so their interpretations were based on their “own perspective as a mathematically savvy reader and educator” (see “Methods” in the chapter by Fukawa-Connelly & Karahoca, this volume). Second, the authors specifically made explicit their need (1) to identify and characterize the claims that were being supported by the argumentation in the classroom and (2) to link the claims with the arguments. This acknowledges the fact that in a classroom setting (unlike a textbook or research article), it is highly nontrivial to identify and characterize what the claims are that students are making arguments to support. The authors then proceeded by trying to assess (from their perspective) how each of the identified arguments satisfied the characteristics of the cluster definition.

Two Tensions Reflected in These Operationalizations

We identify two primary tensions that the authors confronted in operationalizing Weber’s definition for use with their classroom data sets. First, each author team addressed tensions related to the nature of the classroom community and how it differs from the community of mathematicians that is featured in the definition. (Additional tensions emerged as they put these operationalizations to work, and we will address those later in the chapter.) Second, the definition characterizes proving in terms of the product (a proof) that arises from that activity. This created tension because most of the classroom data featured students engaged in proving activity but did not include any arguments that the authors could be sure were meant to be considered as the resulting products of this activity. Before comparing the ways that the different author teams addressed these tensions with their operationalizations of the definition, it is important to note the apparent purpose of the analysis varied from chapter to chapter. Two of the chapters (middle grades and postsecondary) were primarily focused on analyzing students’ arguments in light of the definition, while the other two chapters (elementary and secondary) were primarily focused on the overall trajectory of the mathematical activity and the teacher’s moves to support or constrain the students’ proving activity. It seems likely that the two tensions we observed are reflected in selection of these foci and not just in the operationalizations themselves.

Community Tension

Interestingly, the two author teams that primarily focused on analyzing student arguments took significantly different approaches to dealing with the tension involved in adapting the definition to the classroom environment. In Weber’s original definition, one considers the prover and the reader to both be members of the community of mathematicians. Yopp et al. operationalized the definition by repositioning the prover as a member of the classroom community while continuing to view the reader as a member of the community of mathematicians. Then as representatives of the mathematician community, they compared the students' arguments to those that would be offered by mathematicians. In contrast, Fukawa-Connelly and Karahoca conceptualized both prover and reader as members of the classroom community. They identified themselves as "mathematically savvy readers and educators" and attempted to draw inferences about how the classroom community was (or could have been) making sense of the arguments offered by students.

The other two author teams took much more extreme approaches to operationalizing Weber's definition. Walkington and Woods conceptualize both the prover and the reader as members of the classroom community. From this perspective, they focused on two constructs that are connected to but not explicit in Weber's definition. Specifically, they leverage the notion of generality (which they suggest may be implicit in the deductive characteristic from Weber's definition) to identify opportunities for progress in the students' proving activity and leverage the notion of multimodality (which they argue is an aspect directly connected to the "perspicuous argument" characteristic) as a crucial factor related to supporting this progress. Together, these constructs are used to analyze the teachers' actions in supporting (or limiting) the students' progress. Instead of adapting Weber's definition, Morselli assigns it a supporting role in her analysis (identifying opportunities for moving an argument closer to one that could be considered a proof), while drawing on other conceptual tools to accomplish the primary goal of analyzing the teacher's actions. This is explained by her observation that the definition "doesn't refer to the ways of introducing proof in the classroom via specific activities (such as class discussion) and doesn't discuss specificities for the different actors that are involved in the class discussion" (Morselli, "Introduction and Background," this volume). This can be taken as an argument that while the definition might help to characterize a proof, it does not contribute to understanding aspects of the community in which the proof emerged.

Process/Product Tension

The second tension the authors dealt with was the fact that the definition, while ostensibly about proving activity, essentially only provides characteristics of a product (proof) resulting from this activity. The authors did not explicitly discuss this tension when explaining their operationalizations; however, we can see it reflected in their choices.

The authors (Yopp et al.) that took the fewest liberties when operationalizing the definition were also the only team that focused on analyzing written work. This choice may reflect an awareness of the process/product tension in that it can be seen as a choice to analyze the part of the data set that could most reasonably be considered to reflect the product of proving activity. Similarly, Fukawa-Connelly and Karahoca used an early pass of their data analysis to identify the claims and linked arguments that emerged during the lesson, resulting in collection of transcript excerpts that could be analyzed as if they were the products (proofs) of a proving process.

The other two chapters also used the definition to analyze arguments presented by students during the lesson as if they were products; however, these were treated as snapshots taken during a larger process and the evaluations were conducted not to determine whether the arguments should count as proofs but rather to identify opportunities where the teacher could have supported the students in developing the arguments to better reflect one or more of the characteristics included in the definition. Additionally, one could argue that Walkington and Woods explicitly operationalize the definition to allow them to analyze proving processes. Their notion of multimodality emphasizes moving between and making connections among different modalities while developing a proof, while their notion of generality focuses on "opportunities for generalization" (p. 5).

Insights Drawn from the Analyses

With their operationalizations of Weber's (2014) definition in hand, each of the author teams put them to work in order to draw insights from the data excerpt they had been assigned. The reader can refer to the specific chapters to learn more about the methods used to conduct the analyses. Here, we will consider the findings shared in each chapter and make some brief comments about how these connect to the authors' operationalizations. The purpose of this section is to provide the foundation for the commentary to follow in which we directly address our two goals of our chapter which are to (1) extract some general insights related to proof in the context of teaching and learning and (2) draw some conclusions related to the constraints and affordances that come along with using this kind of definition of proof to conduct analyses of classroom data and share some thoughts about how these opportunities might be realized and the constraints overcome.

Elementary Grades Insights

Walkington and Woods, like all of the author teams, experienced difficulties in using their operationalization of Weber's (2014) definition to analyze the given data set. While their operationalization provided promising foci for their analysis (generality and multimodality), certain aspects of the data set still constrained their efforts. Specifically, the authors found that even when the teacher asked for justifications, her goal was actually to have the students characterize the equal sign as "the same as." However, the authors were able to use the notion of generality to analyze some student contributions and find opportunities for supporting the development of more general arguments. Then the authors were able to explain, in part, why these opportunities failed to be taken up by drawing on their notion of multimodality. In particular, they found that while there were a number of other modalities present in the classroom milieu, the teacher persistently emphasized the use of Cuisenaire rods over other modalities, including some that were used spontaneously by students and were more suited to supporting a general argument by explaining *why* the equation was true. The authors also noted missed opportunities to support students' learning and their proving activity by making connections between modalities.

In summary, the authors argue that these episodes provide evidence for the importance of encouraging multimodal forms of communication when proving. Drawing on the work of Nathan et al. (2013), they highlight missed opportunities by the teacher for connecting the different modalities that were used in the classroom. Overall, the authors found that the teacher's adamant use of a singular modality and lack of opportunities for generalizability created a conceptualization of proof within the classroom community that did not align with Weber's definition.

Middle Grades Insights

Recall that Yopp et al. adapted the six characteristics included in Weber's (2014) definition to account for the multifaceted subjectivity involved in applying the definition in a classroom context. Based on these adaptations, they developed a rubric and used it to analyze the written work of nine students. They found that only one of the students produced an argument that would be considered a proof according to their operationalization of Weber's definition. However, in most of the samples, the authors found evidence of a *search* for conceptual insight or a structural link between the two equivalent expressions involved in the Number Trick task.

Based on their analysis, the authors argue that students need explicit instruction regarding the "rules of the game" when it comes to proof. The authors note that they cannot make any claims about what prior instruction the students from this data had received regarding proofs. So, to support their argument, they share data from their own project, offering it as evidence that presenting the students the "rules of the game" can empower students to produce more rigorous and general proofs.

High School Insights

Morselli first used her operationalization of Weber's definition to analyze the task that was featured in the secondary grades data set. This analysis indicated that the task presented various opportunities to promote conjecturing and proof according to the task design principles outlined by Lin et al. (2012), and it afforded opportunities to address the different characteristics of Weber's cluster. However, Morselli's analysis of the classroom transcript revealed that the teacher did not take up many of these opportunities and did not support the students in producing an argument that would count as a proof as defined by Weber. For each of Weber's characteristics, the author identified moments in the whole class discussion in which the teacher missed an opportunity. For example, in reference to the fourth characteristic (explaining why), when the teacher engaged the students in a discussion about decomposing a polygon into triangles, she presented summing up the interior angles as an alternative solution rather than a justification for the formulas they had previously found. The author argues that as a result, the students viewed the sum of the triangle's angles as a "neat trick" for getting the same answer rather than seeing it as a way to prove that the formula will hold in all cases.

Drawing on Cusi and Malara's (2016) framework, Morselli also argued that the teacher could have better supported the students' proving activity if she had enacted the roles of strategic guide (e.g., by linking the local goals like triangulating a polygon to the overarching goal of creating a general argument in support of the formula for the sum of the interior angles) and reflective guide (e.g., by asking how the observation of the triangulation could be leveraged to produce a proof) the teacher could have better supported her students in generating arguments that would count as proofs as defined by Weber's cluster definition.

Postsecondary Insights

Fukawa-Connelly and Karahoca were able to find several claims for which students offered support. The claims were often not articulated as general claims, but tended to appear situated in the specific context of the group of symmetries of an equilateral triangle. The authors were able to identify a number of aspects of students' warranting that reflected aspects of Weber's cluster definition, but they were unable to identify any arguments that would count as a proof with respect to that definition. Based on this, the authors suggest two general insights. First, the authors observed that the kind of argument one makes (and whether it counts as a proof) is related to the nature of the claim being warranted. More general claims are more likely to suggest the need for a general proof. Second, the authors argued that this suggests that the instructor has an important role in making claims transparent and general if the goal is to promote the production of a general proof. Finally, the authors conclude by raising questions about how and when a teacher should press students for more rigorous proofs in an inquiry-based course.

Our Synthesis of the Four Chapters

In this section, we share some of our takeaways from the collection of chapters on proof. The first three subsections address insights related to proof in the context of teaching and learning, and the final two subsections address methodological insights related to the operationalization of a definition of proof in the context of classroom-based educational research.

The Multifaceted Nature of Proof in the Classroom

Weber's (2014) definition is in the form of a cluster of characteristics because the nature of proof in the mathematics community is multifaceted. A proof in the mathematics community is expected to possess most of the aspects included in Weber's cluster; however, the relative importance of the characteristics is a function of the context in which the proof is being offered and the role of the proof in that context. The characteristics that would be most valued when a proof is initially created are different than those that would be most valued in a proof that appears in a journal article, which in turn would be different than those that would be most valued in a proof that appears in a textbook.

The analysis offered in the four chapters makes it abundantly clear that proof in the mathematics classroom is at least as multifaceted as in the mathematics community. Issues such as representational modality, community norms, generality, and conceptual insight were consistent themes across the chapters (and grade levels). Each of these issues is related to the question of whether an argument should count as a proof, and each was derived from an adaptation of one or more of Weber's characteristics to the classroom context.

While it is no surprise that proof in the classroom is multifaceted, the impact of this fact on the authors' findings is pervasive. Each of the authors reported that their analysis found evidence of students producing arguments that reflected *some* aspects of the characteristics included in Weber's cluster. One way to synthesize this collection of findings is to hypothesize that while one would likely observe (in a student-centered mathematics classroom) students engaging in argumentation that reflects at least one of the characteristics of a proof as captured by Weber's definition, one would likely *not* observe them producing arguments that would count as proofs.

The Emergent Nature of Proof in the Classroom

The second insight that we draw from these four chapters also relates to the finding that most of the arguments analyzed could not be characterized as proofs. We argue that this reflects the fact that specific proofs and the notion of proof are emergent phenomena in a mathematics classroom. What we mean by this is that proofs and the notion of proof evolve over time in a classroom community. The authors found very few examples of proofs because most of the things that were analyzed as proofs were not products of proving activity, they were snapshots of proving in action. Note that none of the data sets included arguments that were purported to be proofs, and often it was not even clear what claims were being supported. This can be seen in the findings reported by Walkington and Woods and Morselli that identified potential proofs in the students' work and missed opportunities for the teachers to realize this potential. It can be seen in the frustrated efforts of Fukawa-Connelly and Karahoca to attempt to ascertain the intended level of generality of often implicit claims and arguments in the context of an exploratory lesson designed to provide an experiential foundation for *future* conjecturing and proving. Summarizing, the students were engaged in making observations that later would be treated as conjectures to be proven using definitions that would also be formulated later. Finally, we argue that this emergent nature of proving in the middle school lesson (together with the authors' strong commitment to using the framework mostly as is) is behind Yopp et al.'s decision to complement their analysis by bringing in data from another project that reveals an explicit commitment to proof as a product.

The Role of the Teacher

While all four of the chapters consider teaching and reflect the belief that the teacher has an important role in supporting students' proving activity, only two featured analyses of the teacher's activity (so, only those two can be considered to have produced findings regarding teaching). The chapters focused on the elementary and secondary levels each explicitly emphasized on the role of the teacher in the analysis process, with Morselli bringing in additional theory to support this emphasis. The two chapters articulated similar findings regarding the teacher's activity. Both reported that the teacher missed opportunities to support students, with Walkington and Woods explaining this primarily in terms of the teacher's persistent emphasis of a single modality and Morselli explaining this primarily in terms of the teacher neglecting to serve as a strategic and reflective guide. In each chapter, the authors found encouraging signs in the way of student contributions that could potentially lead to arguments that would count as proofs in the sense of Weber's definition, and they were able to hypothesize moves that the teacher could have made to support the development of these arguments. While we find the authors' analyses to be convincing, we would argue that some of the teachers' moves in these lessons could be attributed at least in part to the fact that the teacher was balancing multiple goals and not focused exclusively on supporting students in generating proofs.

Operationalizations Have a Massive Impact on Resulting Findings

It is well understood by qualitative researchers that the way constructs are operationalized has an enormous impact on the resulting findings. Nevertheless, this is the message that rings out most clearly when these four chapters are considered as a whole. Two author teams (Walkington & Woods and Morselli) operationalized the definition in ways that allowed them to make sense of the mathematical activity as a process that unfolded throughout a lesson. As a result, they were able to glean insights into the teacher's actions and how it supported (or did not support) the students' proving activity. At the same time, those operationalizations de-emphasized the characteristics included in Weber's cluster definition, so the findings do not include deep insights into the nature of these arguments and how exactly they compare to what a mathematician might consider a proof. On the other hand, the two author teams (Fukawa-Connelly & Karahoca and Yopp et al.) who operationalized the definition less liberally were able to provide detailed observations about the nature of the students' arguments, but they were unable to draw on their analysis when they considered questions of how the students could have been supported to generate arguments that would be considered proofs in light of Weber's definition.

Further, all of the authors' operationalizations represented reactions to the tension created by bringing a definition of proof that emerged from analyzing the community of mathematicians to bear in the context of a classroom community. Yopp et al. operationalized the definition in a way that placed the provers (students) in the classroom community and the readers (themselves) in the community of mathematicians. As a result, they were able to say definitively that the students almost exclusively provided arguments that would not count as proofs in the community of mathematicians, but they were unable to share any insight as to what extent these arguments functioned as proofs in the classroom community. Fukawa-Connelly and Karahoca made a different choice, which resulted in them trying to make sense of the students' arguments from the students. However, their limited access to the classroom community meant that their inferences were somewhat speculative. Walkington and Walker made a conscious choice to focus on two aspects of proof that they saw as being both related to Weber's definition and salient in the elementary classroom. As a result, they generated significant insights related to these two aspects (multimodality and generality) as they indeed turned out to be quite salient in the data set. Morselli took additional liberties, when operationalizing the definition to allow an analysis situated in a classroom community, by augmenting the definition with tools specifically designed to focus on the teacher's roles in such a community. As such, she was able to generate important insights regarding what the teacher did (and could have done) to support the students' proving activity.

All of the authors were able to generate some valuable insights related to proof in the teaching and learning of mathematics. These insights were shaped in significant ways by the decisions the authors made when operationalizing Weber's definition. We have attempted to illustrate this point without taking a position regarding the relative value of the various operationalizations. There is merit to each of the approaches represented, and given that they were assigned a rather unique task, the authors' choices (especially those of the authors who took fewer liberties when operationalizing the definition) may not be an accurate reflection of how they typically approach bringing theory to bear in analyzing the data generated by their own studies. Nevertheless, we view this exercise as being extremely valuable precisely because it highlights the impact of operationalizing constructs in ways that traditional research reports do not.

Lens, Blinder, or Kaleidoscope?

As noted above, the task assigned to the four author teams was quite unique. Typically, a researcher has much more control over the selection of data to analyze and the selection of conceptual tools. This unique exercise highlighted some of the affordances of Weber's definition in the context of classroom-based research (e.g., it casts a wide net when it comes to capturing aspects of students' arguments that resonate with the notion of proof in the mathematics community). At the same time, this exercise served to highlight and amplify some of the challenges inherent to using such a definition for classroom-based research. Specifically, we argue that analysis presented in the four chapters offers a distorted view of the classroom – an outcome that is a danger when bringing any theoretical construct to bear in analyzing a classroom community but was largely unavoidable in this case because the authors had limited access to the classroom communities they were investigating.

The use of Weber's definition led the authors to analyze students' contributions as if they were intended to be proofs regardless of whether this was the case. At the risk of being too cute, this means that a lot of oranges were evaluated to be lacking as apples. We have already observed that the definition foregrounds proof as a product, so that what may be better conceptualized as a snapshot of a (potentially very long) process of constructing an argument instead is assessed as a final product of that process. However, this issue extends beyond this process/product distinction. These arguments also exist in the context of lessons whose purpose was not simply (or sometimes not at all) to generate proofs. As the authors noted, the elementary lesson was clearly more focused on establishing the meaning of the equal sign than generating proofs while the middle school lesson was meant to motivate the distributive property rather than to produce proofs that treated it as an established fact. The secondary lesson clearly was designed to support discovery of the formula for the sum of the interior angles of a polygon and not just a proof of the resulting formula. One could also argue that the teacher may have been more focused on introducing the technique of triangulation than transitioning the students to leveraging it as a tool for proof. Finally, a look at the instructor support materials (Lockwood et al., 2013) reveals that the postsecondary lesson was not intended by the designers to include proofs, but instead computations and explorations designed to support conjecturing and proving to occur in later lessons.

The result of applying an operationalization of a definition of proof in the context of classroom activity that was not meant exclusively to produce proofs is that it served to distort the view of the researchers. In the case of the Yopp et al.'s analysis, we would argue that the definition served as a blinder. The authors chose to only analyze the part of the data that made the most sense in light of its nature and then they struggled to find much value in the nine student samples they analyzed in part because the arguments were not understood as part of a process of discovery designed to motivate the distributive property via the practice of justification. In the case of the Fukawa-Connelly and Karahoca analysis, we would say that the definition acted as a kaleidoscope. The authors sought to embrace the challenge of making sense of the arguments from the perspective of a member of the classroom community. But they did not have sufficient knowledge of the classroom community to conduct such an analysis reliably. As a result, they focused on things (matrices, ordered triples, perceptual arguments) that were not relevant to the argumentation of the community they were attempting to understand.

Insight into at least one way to limit such distortions comes from the two author teams who took extensive liberties when they operationalized the definition. Walkington and Woods and Morselli both made explicit decisions that provided a larger (or additional) window into the activity of the classroom community than did the more faithful operationalizations utilized by the other two author teams. As a result, Walkington and Woods were able to explicitly attend to the apparent commitments of the teacher and how these were not focused on proof, while Morselli was able to document that the teacher did not seem to emphasize the idea that triangulation provided a proof technique. As a result, we (as readers of their report) feel we have a better idea of the context in which students were sharing their arguments. So, while the use of Weber's definition may be resulting in an underappreciation of what the students and teachers did accomplish, we feel much more comfortable with their arguments that the teacher could have made different moves to support the students.

Conclusions

Taken as a collection, the four chapters we have explored provide interesting insights into the nature of proof in the context of the teaching and learning of mathematics and into the nature of classroom-based educational research. While the findings related to proof may not be surprising (proof is multifaceted, students often only partially succeed in producing proofs, and teachers are important), we found it particularly enlightening to reflect on the methodological issues that emerged (and were starkly highlighted) in the context of the unique task that the editors assigned to the chapter authors. We are grateful to the authors for embracing the challenge because in doing so, they made their problem solving highly transparent and surfaced important issues regarding the use of theory in educational research.

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Conclusion: Considering the Consequences of Our Conceptions of Argumentation, Justification, and Proof

Karl W. Kosko and Kristen N. Bieda

This book is based on the premise that argumentation, justification, and proof are highly valued mathematical processes and products in mathematics and mathematics education. We hope that the chapters contained therein shed light upon the consequences for mathematics education in how these constructs are defined. As noted by Staples and Conner (this volume), the origins of this book stemmed from conversations in a Psychology of Mathematics Education-North American (PME-NA) Chapter working group, where scholars from the USA and abroad debated how argumentation, justification, and proof should be defined and why attending to our definitions matters for research on these mathematical processes. One initial finding for the working group was the dramatic variation in how participants defined argumentation, justification, and proof, as well as how they considered these constructs to interact with one another (Cirillo et al., 2016). Furthermore, such variance interacted somewhat with which construct scholars prioritized in their research.

From these conversations, this book was born. The impetus for this book was the need to more directly understand the consequences of how we choose to define these constructs, moving beyond the traditional debate of "Is this piece of work/classroom episode/interview response argumentation, justification, or proof?" to a debate about questions such as "How does analyzing this piece of work/classroom episode/interview response from the lens of proof influence what we say about it?" To prompt reflection on these types of questions, we charged authors to examine common sets of data at different grade bands with given definitions for each

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particular construct. As such, this book differs from most edited volumes that feature a range of perspectives across the field. Rather than choose their own given definition, or their own data, authors were pressed to use the provided definitions and analyze unfamiliar data to engage in meaningful scholarship and knowledge generation. Despite the challenges inherent in this task, the authors succeeded in providing useful, informative insights into the constructs of argumentation, justification, and proof, as well as how these processes may be better facilitated from elementary through tertiary mathematics education.

Similarly, the authors of the seven synthesis chapters were challenged to consider the conceptions and consequences of defining these constructs either within or across grade bands. In turn, each synthesis chapter provides a unique contribution in interpreting these constructs and the implications thereof. In essence, each chapter in this book provides a useful argument for a particular case – a particular instantiation of defining a specific construct or constructs given evidence from a specific classroom context (argumentation, justification, and/or proof). But as *mathematics* educators, we are often concerned with the generalizable and, thus, are compelled to consider what lessons can be learned that generalize across the chapters in this book and, perhaps, to the field as a whole. In particular, in this chapter, we seek to touch on the consequences of definitions for argumentation, justification, and proof.

Consequences of Our Definitions of Argumentation, Justification, and Proof

Each of the chapters in the book speaks in a different way to one of the original questions guiding the research group: How does the way we define the constructs of our inquiry influence the claims we make, and ultimately the body of knowledge we collectively build in the field? As editors, we took seriously the task of ensuring that the authors made explicit how the definitions they were using for their given construct influenced the outcomes of their analysis. We reflect that this was not a trivial task; in the majority, if not all, of the initial drafts for chapters, peer reviewers commented that the consequences of the definitions for the analysis presented were unclear or not present. The reviewers noted, as did the editors, that in some cases it appeared that the analysis may have been based on different conceptions of the construct than the definitions as assigned. To further complicate the situation, the editors chose to give some license to authors to modify the assigned definitions if needed to be able to engage in the work – with the directive that the reasons behind those modifications be explicit in the chapters.

What we have learned through the experience of supporting authors in making explicit their definitions as well as the consequences of applying particular conceptions of a construct is that, put simply, this is hard to do. Perhaps the difficulty of attending to this aspect of scholarly inquiry is due to the fact that we, as scholars, always need to situate the findings of our research with findings from existing, related work in the field and make explicit our theoretical framings, but we are not necessarily expected to justify how our theoretical framings influenced our findings. This is often the task for the reviewer or reader. We have learned, and hope that the readers of this book will learn as well, that explicitly reflecting on the consequences of our theoretical frameworks, including the definitions of constructs central to our inquiry, is essential for thoughtful research that contributes to a coherent body of knowledge in the field.

The chapters also suggest that the definitions we use have specific consequences for how our work can influence practice. We can consider this consequence by asking the question: Do our definitions help us to understand and address the needs of practitioners who are working to engage students in argumentation, justification, and proof processes? As Conner notes in the argumentation synthesis chapter (Chapter "Participation in Argumentation: Teacher and Student Roles Across the Grades," this volume), each of the argumentation chapters, regardless of grade level focus, discusses more than just the nature of student work products. Rather, the central themes across the argumentation chapters include aspects of dialogue among teachers and students, ways students are positioned to be agentive, and the broader purposes of argumentation beyond its role as a mathematical practice. To attend to these themes, the authors all needed to amend the given definition or apply frameworks that could provide a lens to understand these aspects of argumentation activity beyond the specific framing of the given definition. These adaptations were not unique to the set of argumentation chapters. Ellis et al. (Chapter "Justification Across the Grade Bands," this volume) note both a central theme related to the social aspects of justifying activity and how authors of justification chapters took liberties with bringing in other definitions of justification. Larsen and Elizondo (Chapter "Lens, Blinders, or Kaleidoscope? Using a Definition of Proof to Make Sense of Classroom Activity," this volume) noted that authors significantly adapted the given definition of proof when they wished to attend to classroom interactions that supported the development of proof arguments. This suggests a potential problem in how the field of study focusing on these constructs, particularly proof, can inform practice: namely, that perhaps many of the definitions of these constructs that inform our work are overly focused on articulating the nature of products rather than the nature of argumentation, justification, or proof processes. Knuth, Zaslavsky, and Kim (this volume) draw attention to this issue in the latter part of their synthesis of the middle school chapters, claiming that attempts to parse whether students' activity is argumentation, justification, or proof may not be as productive as attending to a broader set of practices that they name "proving-related activities." They argue:

> If we want the activities associated with argumentation, justification, and proof to play a more central role in middle school classrooms, it may be more powerful and instructive to focus on all the activities involved in such practices. It is easy to get lost in trying to categorize teachers' practices or students' activities as related to argumentation, justification, or proof, when in the end what perhaps really matters is teachers' efforts to meaningfully engage students in the proving-related activities of developing, exploring, justifying

(including proving), and refuting conjectures. (See "Capturing the Breadth of Proving-Related Activity" in the chapter by Knuth, Zaslavsky, & Kim, this volume)

What is evident across all chapters of this book is how the definitions that guide scholarly inquiry into classroom practice, whether the focus be on argumentation, justification, and/or proof, influence what analytic frameworks are chosen, what parts of and extent to which the data are analyzed, and, as a result, the claims that are made. Moreover, we were surprised at how much variation there was across constructs in terms of how the definitions were operationalized. In the sections that follow, we unpack the consequences of authors' choices within the chapters focusing on the same construct, with the goal of illustrating the evident differences in how the field has used and embraced these constructs and where our field's understanding of these constructs might evolve to improve the cohesiveness, coherency, and impact of our collective body of work.

Argumentation

The editors of this book tasked authors with using a definition for argumentation that described it as *the process of making mathematical claims and providing evidence to support them*. Across all chapters on argumentation, authors attended well to elements of the definition focusing on claims and evidence. As noted by Conner's (this volume) examination across these chapters, "this definition constrained the construct of interest, argumentation, in a way that it could be identified, but it allowed for a range of theoretical perspectives to be usefully applied to the given data" (see "Consequences of the Definition" in the chapter by Conner, this volume). Yet, authors also attended to elements not explicit in the provided definition. Specifically, all authors focused on participation structures and patterns in some form. Discussing chapters in the elementary section, Stylianides and Stylianides (this volume) noted this facet of argumentation that, though absent from the provided definition, was evident in Rumsey et al.'s (this volume) description and analysis of argumentation. Similarly, Gomez Marchant et al. (this volume) and Dawkins (this volume) note a need for attending to social and sociomathematical aspects of argumentation.

Although all argumentation-focused chapters, as well as synthesis chapters, noted the importance of social and sociomathematical aspects of argumentation, it is worth observing that such emphasis may not be universal in either research or practitioner literature. For example, some scholars have attended to mathematical argument and argumentation in different mediums (e.g., mathematical writing). In such instances, attention to claims and evidence is generally maintained, but because analysis focuses on a product (written work), discussion of social and sociomathematical elements is tacitly inferred (Cohen et al., 2015; Kosko & Zimmerman, 2019). Stylianides and Stylianides (this volume) noted that the data used to examine arguments and argumentation interacts with the definition one uses. Stated differently, particular consequences of the definition one uses for argumentation will interact with the data at hand and the purposes one engages. Argumentation is a social process and mathematical arguments are social products, but there are evident consequences in attending to their social nature to a greater or lesser degree. For example, Conner (this volume) observed that "students are being given opportunities to contribute to and construct arguments. However, these chapters suggest opportunities to critique the reasoning of others are scarcer" (see "Responding to Others' Ideas by Building and Critiquing" in the chapter by Conner, this volume).

Given that mathematical argumentation and arguments are social constructs, it is worth pondering the degree to which such aspects should be incorporated into definitions used by mathematics educators. On one hand, not explicitly mentioning social facets in the definition does not preclude them from being addressed in research. Yet, exclusion of certain forms of social and sociomathematical elements (e.g., critiquing the reasoning of others) may lead to neglect of such facets within the work of practitioners (e.g., mathematics teachers). Even within analysis of arguments as products, such as mathematical writing, attention to critique has been noted as worthwhile but seldom examined (Conner, 2013; Kosko, 2016), with a majority of K-12 teachers reporting less familiarity with this mathematical practice and few having received any professional development focusing on it. In considering the consequences of definitions for argumentation used by mathematics education researchers, we must consider not only the consequences to research and theory but also how such research and theory affects professional development opportunities for teachers and resources for them to use in their classrooms (across K-12 and tertiary education).

Justification

Compared to argumentation and proof, there was a smaller set of possible definitions of justification to consider for the purposes of this book. The definition chosen – "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020, p. 103) – is a more process-oriented framing, like the argumentation definition. However, unlike the argumentation chapters, authors of the justification-focused chapters (Thanheiser & Sugimoto; Lesseig & Lepak; Newton with Yackel; Karunakaran & Levin) did not apply the definition as is. Ellis, Staples, and Bieda (this volume) note in their synthesis of justification-focused chapters that the authors who analyzed data at the elementary and high school levels applied definitions of justification that account for a variety of ways that students provide support for, and make sense of, their claims. In contrast, authors of the middle school and tertiary level chapters focused more on a notion of justification as providing support to establish claims with truth values, which required focusing on a subset of claims that can be justified in a classroom.

We argue, agreeing with Ellis, Staples, and Bieda (this volume), that it may not be surprising that authors chose to modify the definition or use a different one entirely, and that there was considerable distinction among the foci of the definitions used across the chapters, because the field has not taken up the question of "What is *justification*?" as thoroughly as the questions of "What is *argumentation*?" and "What is *proof*?" The role of proof in school mathematics has been debated and questioned, with some suggesting that structural and systemic forces present in classrooms necessarily limit how authentically students can engage in proof and proving (Herbst & Brach, 2006). Yet, the relative lack of attention to justification is surprising given that justification, while a necessary component of proof and proving, emerges in a range of mathematical activities that need not lead to a proof.

The chapters also reveal consequences of defining justification in service of the disciplinary practice of proof, rather than in service of students' engagement with making sense of mathematical ideas and processes. Karunakaran and Levin (this volume) state: "The point [of justification] is to try to convince someone else." Also, Lesseig and Lepak argued that the middle school teacher's insistence that students make sure to check all cases for the first part of the Number Trick Task before moving into generalization reduced the cognitive demand of the task. Yet, we wonder how holding conceptions that justification involves convincing others influences the ways that authors interpreted the data and advanced claims about the justification activity in the data. What if the point of justification is *not* about trying to convince someone else? Is there a role for mathematical sense-making that is not only about figuring out whether a statement is mathematically valid? If we think about justification as potentially more than informal, proof-related arguments, could we see activity where students generate examples and reason about examples as *justification*-related? We might also think about justifying a choice made during problem solving as included in the set of activities that involve justification in school mathematics.

It is interesting to consider that the only chapter of the justification-focused chapters that did not explicitly deal with the issue of who seemed to be the mathematical authority in the classroom episode was the elementary chapter written by Thanheiser and Sugimoto (this volume). An implicit goal of some definitions of justification (i.e., Bieda & Staples, 2020) and proof (i.e., Stylianides, 2009) is to shift the locus of authority for determining what is mathematically valid from being solely the teacher to students or the classroom community. We wonder if seeing justification as an intermediary between naive, empirically based ways of reasoning and mathematical proof, where justifications emerge only when students grapple with questions of veracity, may perpetuate teachers' difficulties with minimizing their authority while facilitating justification. If we think about justification as also happening in cases like justifying a choice of strategy or representation, it suggests that mathematical authority can naturally shift between individual students, the classroom community, and the teacher as the goals of the activity demand (Staples & Lesseig, 2020).

Proof

The definition for proof used in this book was Weber's (2014) cluster definition:

> (1) A proof is a *convincing argument* that convinces a knowledgeable mathematician that a claim is true. (2) A proof is a *deductive argument* that does not admit possible rebuttals. (3) A proof is a *transparent argument where a mathematician can fill in every gap* (given sufficient time and motivation), perhaps to the level of being a formal derivation. (4) A proof is a *perspicuous argument that provides the reader with an understanding of why a theorem is true*. (5) A proof is an *argument within a representation system satisfying communal norms.* (6) A proof is an *argument that has been sanctioned by the mathematical community.* (Weber, 2014, p. 357)

Weber (2014) was careful to note that while some proofs may include all criteria, many will only attend to a subset of the criteria. Indeed, authors across grade bands in this text who focused on proof generally addressed subsets of Weber's (2014) definition. Larsen and Elizondo (this volume) note that these authors' use of the definition in such ways, along with their incorporated analysis, "makes it abundantly clear that proof in the mathematics classroom is at least as multifaceted as in the mathematics community" (see "The Multifaceted Nature of Proof in the Classroom" in the chapter by Larsen & Elizondo, this volume). Yet, Weber's (2014) definition notably references proof repeatedly, not proving. This inevitably led to a tension in how the definition was operationalized, with some scholars focusing on proof as a product and others on proof as a process; "most of the things that were analyzed as proofs were not products of proving activity, they were snapshots of proving in action" (see "The Emergent Nature of Proof in the Classroom" in the chapter by Larsen & Elizondo, this volume). This tension has been noted throughout the literature in various ways. Harel and Sowder (1998) suggested that an overemphasis on proof as a product in US High School Geometry has led to a de-emphasis on the actual process of proving common to the mathematics discipline. Stylianides and Ball (2008) suggested teachers may have different forms of pedagogical content knowledge related to the different kinds of tasks for proving, as well as the relationship between the process of proving and such tasks. However, evidence suggests there are few useful proving tasks in textbooks for teachers (Bieda et al., 2014), and when students are provided opportunities to engage in such tasks, the experiences are insufficient for facilitating their experiences in the proving process (Bieda, 2010). Given such prior work, and coupled with Larsen and Elizondo's (this volume) observations, it appears that the specificity of the terms proof or proving, within a definition, is less influential than whether scholars, utilizing a given definition, attend to proof as a product and/or proving as a process. Stated differently, a definition of proof focuses on a product *iff* the scholar utilizing it examines proofs as products, and a definition of proving focuses on the process *iff* the scholar utilizing it examines the interactions associated with the process of proving. Thus, the enactment of a given definition of proof or proving contributes its own set of consequences to a given definition.

Larsen and Elizondo (this volume) note that a focus on proof as a process allowed for inferences regarding how to better support the proving process, while a focus on proof as a product allowed for inferences regarding the mathematical nature of students' proving and argumentation process. Each affordance is, in essence, a consequence of how the definition was operationalized, and each affordance is necessary for extending the field's understanding of proof both in research and in the classroom. Yet, for both product and process, analyses both within the current book and in much of the literature represent cross-sectional instantiations. As noted by Harel and Sowder (1998), a proof as a product may not necessarily be complete, and the process of proving is similarly an evolving thing. This is true for both a particular topic being proven and for students' ability to engage in proof from elementary through tertiary classrooms (Tall et al., 2012). This is to say that regardless of whether mathematics educators focus on proof as a product or process, findings can be useful to both researchers and practitioners. However, whether our conceptions of what we call proof are useful and whether such conceptions are used are quite different things. As noted by Balacheff (2008), mathematics educators' individual epistemologies influence how we conceptualize proof and may limit whether we take up the conceptualizations of others. In this book, we tasked authors to take up Weber's (2014) conceptualization. Even in this regard, there were differences, but such differences could be said to be more manageable than the wider range observed by Balacheff (2008). In saying this, we do not argue for one particular definition over another, but we suggest that when we take up the same or similar definitions that there is more common ground and a greater ability to understand the consequences of such definitions. In the case of Weber's (2014) definition used in this text, such consequences center primarily on the interplay between process and product. It is a noteworthy tension for both research and classroom practice.

Moving Forward

For decades, scholars have noted the inconsistent definitions for what mathematics educators call argumentation, justification, and proof (Balacheff, 2008; Cai & Cirillo, 2014; Reid & Knipping, 2010; Steen, 1999). Notably, when provided the same definitions for these constructs, many scholars in this text still operationalized them in ways that were different from one another. Using argumentation as an example, Stylianides and Stylianides (this volume) noted that even when the construct is defined similarly between two sets of scholars, the conclusions drawn can still differ. As noted by some scholars (Balacheff, 2008; Cai & Cirillo, 2014; Larsen & Elizondo, this volume), some of these differences are due to individual scholars' goals. Each conceptualization of mathematical argumentation, justification, and proof provides an understanding of part, but not the whole, picture (Dawkins, this volume; Cirillo & Cox, this volume). Yet, a potential consequence of various scholars holding their own variations of how argumentation, justification, and proof are defined is a lack of consistent norms in how these various pieces fit together.

Calls for consistency and shared norms in mathematics education research are not unique to the constructs of focus in this book, nor are such calls new. Calling for such norms regarding validation of quantitative measures, Krupa et al. (2019) advocate for the creation of validity arguments that connect the purpose of an instrument with its claim for how a score may be interpreted. This connection is facilitated by warrants conveyed through particular forms of validity that are established across multiple studies. “Single studies when linked together can push forth a coherent validation argument and have potential to build this cumulative knowledge for scholarship in mathematics education” (Krupa et al., 2019, p. 11). It is possible that a similar approach to argumentation, justification, and proof may be advantageous. For example, scholars utilizing Weber’s (2014) cluster definition would be advised to reference others’ applications, as well as stated affordances and constraints to the definition in particular contexts (such as provided in this volume). Perhaps the most useful example of how such efforts can manifest in mathematics education is the literature on Toulmin’s (1958/2003) scheme to examine classroom argumentation. First introduced by Krummheuer (1995), in another book focused on analysis of common sets of data (Cobb & Bauersfeld, 1995), mathematics education scholars have incorporated and adapted Toulmin’s scheme for research and practice. For example, Conner et al. (2014) adapted the scheme to examine teachers’ contributions in facilitating mathematical argumentation in high school classrooms. Remarkably, Toulmin’s scheme was used in three of four chapters focusing on argumentation in this volume and cited in all four of them. Conner (this volume) noted this resulted in significant consistency in how the definition of argumentation was attended in this volume. Much like the case of quantitative measures, analytic frames like Toulmin’s scheme facilitate shared meaning of how mathematical arguments and argumentation are defined. It need not, and should not, be the only analytic framework, but there is clear evidence from the literature and within this book that such widely adopted schemes facilitate the shared norms needed to construct knowledge of mathematical argumentation.

As noted by Cirillo and Cox (this volume), the various chapters in this book provide glimpses of part but not the entire construct(s) on which they focused. Having shared definitions did not guarantee commonality across chapters, but having similarity, or at least acknowledgment of, analytic frameworks such as Toulmin’s (1958/2003) scheme did allow for more commonality (Conner, this volume). Thus, the operationalization of definitions through the analytic tools used may serve as a potential pathway for moving forward as a field. As Dawkins (this volume) noted, “amongst a range of possible mathematical definitions, we will prioritize the one that does the most work for us” (see “Mathematics Educators as Learners: Accumulating Knowledge of Complex Phenomena” in the chapter by Dawkins, this volume). However, some of this work that we prioritize must include the work that mathematics teachers, across age bands, engage (Staples & Lesseig, 2020). In considering this doing of work, we encourage mathematics educators to attend not only to the work of research, but also the work of teaching, and the consequences of our research on the mathematics teaching profession in which we are invested.

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