On the Oriented Coloring of the Disjoint Union of Graphs

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Abstract. Let $\overrightarrow{G} = (V, A)$ be an oriented graph and G the underlying graph of $\overrightarrow{G}$. An *oriented k-coloring* of $\overrightarrow{G}$ is a partition of V into k subsets such that there are no two adjacent vertices belonging to the same subset, and all the arcs between a pair of subsets have the same orientation. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of $\overrightarrow{G}$ is the smallest k, such that $\overrightarrow{G}$ admits an oriented k-coloring. The *oriented chromatic number* of G, denoted by $\chi_o(G)$, is the maximum of $\chi_o(\overrightarrow{G})$ for all orientations $\overrightarrow{G}$ of G. Oriented chromatic number of product of graphs were widely studied, but the disjoint union has not being considered. In this article we study oriented coloring for the disjoint union of graphs. We establish the exact values of the union: of two complete graphs, of one complete with a forest graph, and of one complete and one cycle. Given a positive integer k, we denote by $\mathcal{CN}_k$ the class of graphs G such that $\chi_o(G) \leq k$. We use those results to characterize the class of graphs $\mathcal{CN}_3$. We evaluate, as far as we know for the first time, the value of $\chi_o(W_n)$ and we yield with this value an upper bound for the union of one complete and one wheel graph W_n.

Keywords: Oriented graph · Oriented chromatic number · Disconnected graphs · Graph classes · Disjoint union of graphs

1 Introduction

Given a graph $G = (V, E)$, the *orientation* of an edge $e = \{u, v\} \in E$ is one of the two possible ordered pairs uv or vu called *arcs*. If $uv \in E$ we say that u *dominates* v. An *oriented graph* $\overrightarrow{G}$ is obtained from G by orienting each edge of E, $\overrightarrow{G}$ is called an *orientation* of G, and G is called the *underlying graph* of $\overrightarrow{G}$.

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Note that an oriented graph is a digraph without opposite arcs or loops. Given an arc $uv \in E(\overrightarrow{G})$, v is called the *successor* of u and u is called the *predecessor* of v. A vertex without predecessors is called *source* and a vertex without successors is called *sink*. Let G and H be a pair of graphs. If H is a *subgraph* of G we say that G *contains* H as a subgraph, otherwise we say that G is *H-free*. Two graphs are *disjoint* if they have no vertex in common. If G and H are disjoint, their *disjoint union* graph denoted by $G \cup H$, has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

A *directed path* is the orientation of a path, a *directed cycle* is the orientation of a cycle. If for each pair u, v of consecutive vertices in a directed cycle we have the arc uv, then this orientation called *cyclic*, otherwise is called *acyclic*. A *tournament* $\overrightarrow{K}_n$ with n vertices is an orientation of a complete graph K_n. A tournament is called *transitive* if and only if whenever uv and vw are arcs, uw is also an arc. The complete bipartite graph $G = K_{1,n}$ is a *star*. A *wheel* graph W_n has $V(W_n) = \{v_1, v_2, \ldots, v_n, c\}$ and $E(W_n) = \big\{v_i v_{i+1} : i \in \{1, 2, \ldots, n-1\}\big\} \cup \{v_n v_1\} \cup \big\{v_i c : i \in \{1, 2, \ldots, n\}\big\}$. We say that $\overrightarrow{G}$ is an *oriented star* (the same for a tree, forest, cycle, and wheel).

Let $\overrightarrow{G}$ be an oriented graph, $xy, zt \in E(\overrightarrow{G})$ and $C = \{1, 2, \ldots, k\}$ be a set of colors. An *oriented k-coloring* of $\overrightarrow{G}$ is a function $c : V(\overrightarrow{G}) \to C$, such that $c(x) \neq c(y)$, and if $c(x) = c(t)$, then $c(y) \neq c(z)$. The *oriented chromatic number* of $\overrightarrow{G}$ denoted by $\chi_o(\overrightarrow{G})$ is the smallest k such that $\overrightarrow{G}$ admits an oriented k-coloring. An *oriented absolute clique* or *o-clique* [7] is an oriented graph $\overrightarrow{G}$ for which $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{G})|$.

Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be oriented graphs, a *homomorphism* of $\overrightarrow{G}$ into $\overrightarrow{H}$ is a mapping $f : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that $f(u)f(v) \in E(\overrightarrow{H})$ for all $uv \in E(\overrightarrow{G})$. When $\overrightarrow{H}$ is an oriented graph on k vertices, a homomorphism from $\overrightarrow{G}$ into $\overrightarrow{H}$ is an oriented k-coloring of $\overrightarrow{G}$.

We can extend the definition of oriented chromatic number to graphs. The oriented chromatic number of a graph G denoted by $\chi_o(G)$, is the maximum $\chi_o(\overrightarrow{G})$ for all orientations $\overrightarrow{G}$ of G. Given a positive integer k, we denote by $\mathcal{CN}_k$ the class of graphs G such that $\chi_o(G) \leq k$.

Oriented coloring has been studied by many authors. A survey on oriented coloring can be seen in [13]. Subsequently, many other papers have been published on oriented coloring. See for instance [3] and [7] on complexity aspects and approximation algorithms, and [8–10] for bounds on oriented coloring.

It is NP-complete [3,6,7] to decide whether a graph belongs to $\mathcal{CN}_k$ for all $k \geq 4$. In [2] it was shown that $\mathcal{CN}_k$ for all $k \geq 4$ is NP-complete even for acyclic oriented graph such that the underlying graph has maximum degree 3 and it is at the same time connected, planar and bipartite. Already, it can be decided in polynomial time [7] whether a graph belongs to $\mathcal{CN}_k$. So, in the Sect. 2 we characterize the class of connected and disconnected graphs that belong to the $\mathcal{CN}_3$ class.

The works of [1,4,5,12] presents various bounds for oriented chromatic number on the product of graphs. In spite of the vast amount of literature dedicated to the product of graphs, we don't have many results on the disjoint union.

Assume the 3-oriented coloring for $\overrightarrow{K}_3$ in Fig. 1 (a), where colors 1, 2 and 3 are assigned respectively, to vertices a, b and c. Notice that, by definition of oriented coloring, if the P_4 in Fig. 1 (b) is colored with the three colors 1, 2 and 3, then necessarily to vertices d, e, f are assigned, respectively, colors 1, 2 and 3. Hence, it is required a fourth color to assign to vertex g.

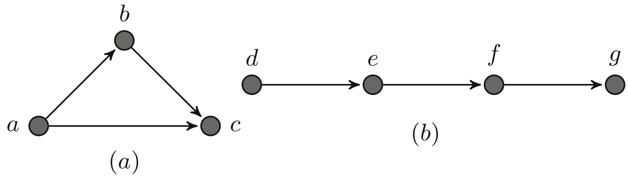

Fig. 1. Graph $\overrightarrow{K}_3 \cup \overrightarrow{P}_4$.

From the coloring given to the graph of Fig. 1 we can notice that, different from the usual coloring, the oriented coloring given to a connected component interferes in the coloring of another connected component in graphs formed by the disjoint union of two other graphs. Motivated by this fact, in Sect. 4 we determine the oriented chromatic number of the disjoint union between complete graphs and others graphs, such as stars, trees, forests, cycles and an upper bound for the union of one complete and one wheel. In Sect. 3 we show the oriented chromatic number of wheel graphs, for the first time as far as we know.

2 The Chromatic Number of the Class $\mathcal{CN}_3$

In this section, we characterize the class of graphs $\mathcal{CN}_3 = \{G; \chi_o(G) \leq 3\}$. First, we consider the case when the graph G is connected.

Lemma 1. *Let $G = (V, E)$ be a connected graph, $|V| \geq 4$. If G contains a K_3 as a subgraph, then $\chi_o(G) \geq 4$.*

Proof. Let $G = (V, E)$ be a connected graph with $|V| \geq 4$ and u, v, w be the vertices of a K_3 subgraph of G. As G is connect there is a vertex $t \notin \{u, v, w\}$ in V such that t is adjacent to a vertex in $\{u, v, w\}$. Assume $\{t, u\} \in E$. Consider an orientation $\overrightarrow{G} = (V, \overrightarrow{E})$ of G where $uv, vw, uw, tu \in \overrightarrow{E}$. We need 3 different colors to vertices u, v, w since u, v, w belong to K_3. As there is a path of size at most 2 from t to each vertex in $\{u, v, w\}$ by the oriented k-coloring definition, an additional fourth color is necessary to t. Hence, $\chi_o(G) \geq 4$. □

From Lemma 1, we know that the connected not K_3-free graphs on 4 vertices or more do not belong to the class $\mathcal{CN}_3$. Sopena [11] proved that for oriented graphs with maximum degree 2, the oriented chromatic number is at most 5. He also proved that the cycle on 5 vertices has oriented chromatic number 5, this result is presented in Lemma 2. We use these results to propose Lemma 3.

Lemma 2 ([11]). *If C_5 is the cycle on 5 vertices, then $\chi_o(C_5) = 5$.*

Lemma 3. *If a connected graph G contains C_k as a subgraph, with $k \geq 4$, then $\chi_o(G) \geq 4$. In particular, if G contains C_5 as a subgraph, then $\chi_o(G) \geq 5$.*

Now we can describe the class of connected graphs that belongs to $\mathcal{CN}_3$.

Theorem 1. *The connected graph $G \in \mathcal{CN}_3$ if and only if, G is either a K_3 or a tree.*

Proof. Let $G \in \mathcal{CN}_3$ be a connected graph. If G is acyclic, then G is a tree. If G is not acyclic, then from Lemmas 1 and 3, it follows that $G = K_3$. We conclude that G is either a K_3 or a tree. Suppose that G is a K_3 or a tree. If G is a K_3 then $\chi_o(G) = 3$. If G is a tree, then $\chi_o(G) = 3$ by [3]. Therefore $G \in \mathcal{CN}_3$. □

Now will consider the case when G is a disconnected graph.

Lemma 4. *Let G be a graph with q connected components $X_1, X_2, \ldots, X_q$, $q \geq 2$, such that X_i contains K_3 as a subgraph, for some $i \in \{1, 2, \ldots, q\}$. If there is a component X_j, $i \neq j$, containing K_3 or P_4 as a subgraph, then $\chi_o(G) \geq 4$.*

Proof. Consider a graph G with q connected components $X_1, X_2, \ldots, X_q$, $q \geq 2$. Suppose there are two connected components X_i and X_j, $i \neq j$, such that both contains K_3 as a subgraph.

We can obtain an oriented graph $\overrightarrow{G}$ from G with $\chi_o(\overrightarrow{G}) \geq 4$, by defining the orientation of the subgraph K_3 of component X_i as a directed cycle and the subgraph K_3 of the component X_j as a transitive tournament. Let c be an oriented coloring for the subgraph K_3 of the component X_i, c has 3 colors, suppose $\{1, 2, 3\}$ and the property that no color dominates the two others. Let c_1 be an oriented coloring of K_3 of the component X_j. In c one color dominates the two others, thus one fourth color is required in the component X_j and therefore $\chi_o(G) \geq 4$.

Now suppose that the component X_j contains P_4 as a subgraph. In the oriented graph $\overrightarrow{G}$ obtained from G, we choose the transitive orientation $\overrightarrow{K}_3$ for the subgraph K_3 of the component X_i and the directed path $\overrightarrow{P}_4$ for the subgraph P_4 of the component X_j. We know that $\chi_o(\overrightarrow{K}_3) = 3$, we use colors 1, 2 and 3 in the oriented coloring of $\overrightarrow{K}_3$ of the component X_i. We choose the oriented coloring of $\overrightarrow{K}_3$ such as the vertex with color 1 is the source and the vertex with color 2 is the sink. We will show that, using the constraints obtained in the oriented coloring of the subgraph $\overrightarrow{K}_3$ in the component X_i, we cannot color the subgraph $\overrightarrow{P}_4$ of the component X_j only with colors 1, 2 and 3.

We consider three cases:

Case 1: (Assign color 1 to the source of $\overrightarrow{P}_4$) Since the vertex with color 1 is a predecessor of the vertex with color 2 in the oriented coloring of $\overrightarrow{K}_3$, we can assign color 2 to the successor of the source in $\overrightarrow{P}_4$. The vertex with color 2 in $\overrightarrow{K}_3$ is the sink, so we cannot assign any of the colors 1, 2 or 3 to the successor of the vertex with color 2 in $\overrightarrow{P}_4$. A fourth color is needed in component X_j.

Another sub-case is to assign color 3 to the successor of the source in $\overrightarrow{P}_4$, because the vertex with the color 1 also precedes a vertex with color 3 in an oriented coloring of $\overrightarrow{K}_3$. We can assign color 2 to the successor of the vertex with color 3 in $\overrightarrow{P}_4$, but again the color 2 is assigned to a vertex that is not sink in $\overrightarrow{P}_4$ and a fourth color is needed in component X_j.

Case 2: (Assign color 2 to the source of $\overrightarrow{P}_4$) The vertex with color 2 in the oriented coloring of $\overrightarrow{K}_3$ is a sink, so none of the colors 1, 2 or 3 can be assigned to the successor of the source in $\overrightarrow{P}_4$. A fourth color is required in component X_j.

Case 3: (Assign color 3 to the source of $\overrightarrow{P}_4$) Respecting the constraints on the coloring of $\overrightarrow{K}_3$, we can assign color 2 to the successor of the source in $\overrightarrow{P}_4$. Again, the successor of the vertex with color 2 in $\overrightarrow{P}_4$ cannot be colored with any color used in $\overrightarrow{K}_3$. A fourth color is required in component X_j.

We conclude that $\chi_o(G) \geq 4$. □

It follows from Lemma 4 that the graph $G = K_3 \cup P_4 \notin \mathcal{CN}_3$. In Fig. 1 we have an orientation of graph G such that $\chi_o(G) = 4$. If we consider the graph G to be a forest, we have the following results.

Lemma 5. *Let F be a forest with a collection $\{T_1, T_2, \ldots, T_q\}$ of q disjoint trees, then $\chi_o(F) = \max\{\chi_o(T_i); i = 1, 2, \ldots, q\}$.*

From Lemma 5 we can show that every oriented forest has a homomorphism to a directed cycle, as we show on Corollary 1.

Corollary 1. *Every oriented forest $\overrightarrow{F}$ has a homomorphism into a directed cycle $\overrightarrow{C_3}$.*

Finally in Theorem 2 we can characterize the class $\mathcal{CN}_3$.

Theorem 2. *Let G be a graph. $G \in \mathcal{CN}_3$ if and only if, G is either a forest or a $K_3 \cup S$, where S is a forest of stars.*

Proof. Suppose that $G \in \mathcal{CN}_3$. If G has a cycle, then by Lemmas 1, 3 and 4 there is at most one connected component G_i of G which has a cycle as a subgraph, and in this case $G_i = K_3$. Still by Lemma 4 the remaining components have a diameter that is less than 3, and hence G is a disjoint union of K_3 and a forest of stars.

If G is acyclic, then G is a forest and by Lemma 5 and [3] we have $\chi_o(\overrightarrow{G}) \leq 3$. Conversely, first suppose that G is a forest. For every tree T_i of G we know that $\chi_o(G_i) \leq 3$, by [3]. By Lemma 5 we conclude that $\chi_o(G) \leq 3$.

Now suppose that $G = K_3 \cup S$. The connected component K_3 can be oriented in two different ways, with circular orientation or transitive orientation. If the component K_3 have a circular orientation $\overrightarrow{K}_3$, we know by Corollary 1 that there is a homomorphism from $\overrightarrow{S}$ into $\overrightarrow{K}_3$ and $\chi_o(G) \leq 3$. Now consider the component K_3 with a transitive orientation $\overrightarrow{K}'_3$. We choose the oriented coloring of $\overrightarrow{K}'_3$ with the colors 1, 2 and 3, so that the vertices with color 1 are predecessors of vertices with color 2 and the vertices with color 2 are predecessors of vertices with color 3.

We define a homomorphism from $\overrightarrow{S}$ into $\overrightarrow{K}'_3$ where all sources in $\overrightarrow{S}$ are mapped into the vertex with color 1 in $\overrightarrow{K}'_3$, and all sinks in $\overrightarrow{S}$ are mapped into the vertex with color 3 in $\overrightarrow{K}'_3$, if the vertex is neither a source nor a sink in $\overrightarrow{S}$, then it is mapped into a vertex with color 2 in $\overrightarrow{K}'_3$. This homomorphism is easily verified, since only one vertex that has more than one neighbor in $\overrightarrow{S}$ can be mapped into the vertex with color 2 in $\overrightarrow{K}'_3$. □

3 The Oriented Chromatic Number of Wheel Graphs

In this section we establish that the family of wheel graphs W_q with $q \geq 8$ has its oriented chromatic number 8. We use this value, in Sect. 4, in order to establish an upper bound for the disjoint union of a wheel with a complete graph.

Theorem 3. *Let $q \geq 8$, be a positive integer. Then $\chi_o(W_q) = 8$.*

Proof. We consider $q \mod 3$, i.e., $q = 3k+1, 3k+2, k \geq 2$ and $q = 3k, k \geq 3$. We prove first that 8 colors are sufficient to color every orientation ω of W_q. Consider an orientation ω for W_q. We construct an 8–oriented color for this orientation. Let $V(W_q) = \{v_1, v_2, \ldots, v_q, c\}$ and $E(W_q) = \big\{v_i v_{i+1}, v_i c : i \in \{1, 2, 3, \ldots, q-1\}\big\} \cup \big\{v_q v_1, v_q c\big\}$.

In order to yield an 8–oriented coloring for ω we consider a key property of an orientation ω that is when there is a 4-oriented coloring for the corresponding C_q, such that there is one color, say color 4, that occurs just in one vertex $v \in V(C_q)$.

From this 4–oriented coloring of C_q, we give the following recipe to color C_q in W_q, with at most 7 colors, and hence W_q with 8 colors. For each $x \in \{1, 2, 3\}$ of the 3 colors that can be repeated, consider the oriented bipartite graph B_x induced of W_q by the vertices with color x and vertex c. If there are sinks and sources in $B_x \setminus \{c\}$, then If $v \in V(B_x) \setminus \{c\}$ and v is a sink, set to $x+4$ the color of v. If the orientation ω in C_q is acyclic, then there is a sink vertex v_i, hence we color the path $v_{i+1}, \ldots, v_n, v_1, \ldots, v_{i-1}$ with 6 colors in $\{1, 2, 3, 5, 6, 7\}$, color v_i with color 4, and c with color 8. Hence, when ω is acyclic, there is an 8–oriented coloring for W_q.

The remaining case is when the orientation ω is cyclic in C_q. Next we consider $q = 3k, k \geq 3$ and $q = 3k+1, k \geq 2$, and prove that there is a 4–oriented coloring for the corresponding C_q where there is a color class with at most one vertex $v \in C_q$, say color 4.

1. If $q = 3k, k \geq 3$, in this case we color $v_1, v_2, \ldots, v_n$, respectively, with colors $1, 2, 3, \ldots, 1, 2, 3$.
2. If $q = 3k + 1, k \geq 2$, in this case we color $v_1, v_2, \ldots, v_{n-1}$, respectively, with colors $1, 2, 3, \ldots, 1, 2, 3$, and color v_n with color 4.

Hence, when ω is cyclic in C_q, $q = 3k, k \geq 3$ or $q = 3k + 1, k \geq 2$, there is an 8–oriented coloring for W_q.

We prove that if the orientation is cyclic, and $q = 3k+2, k \geq 3$, then there is a 5–oriented coloring such that exactly 2 colors appear once. For that we color $v_1, v_2, \ldots, v_{n-2}$, respectively, with colors $1, 2, 3, \ldots, 1, 2, 3$, color v_{n-1} with color 4, and v_n with color 5. From this 5–oriented coloring of C_q, we give the following recipe to color C_q in W_q, with at most 7 colors, and hence W_q with 8 colors.

We consider 2 cases:

1. Vertex c is a sink or a source of W_q. In this case we can color W_q with 6 colors.
2. Vertex c is neither a sink nor a source of W_q. In this case we assume that $cv_n, v_1c \in \omega$. We can assume that because the orientation of C_q is cyclic.
   First, for each $x \in \{1, 2, 3\}$ of the 3 colors that can be repeated in C_q, consider the oriented bipartite graph B_x induced by the vertices with color x and vertex c. If there are sinks and sources in $B_x \setminus \{c\}$, then If $v \in V(B_x) \setminus \{c\}$ and v is a sink, set to $x + 5$ the color of v. Hence, we have an 8-oriented coloring of C_q in W_q, which is an 9-oriented coloring of W_q, that we will reduce to a 8-oriented coloring of W_q.
   Hence, we set to 6 the color of vertex v_n. This can be done, since v_1 has color 1, and every other vertex in C_q with color 6, has a distance to v_n of at least 3. And thus, we have a coloring of C_q with colors $1, 2, 3, 4, 6, 7, 8$, and we can give the color 5 to vertex c.

Now we prove that 8 colors are necessary. For that we show an example of W_8 that requires 8 colors. For the convenience of the reader we exhibit this example in Fig. 2 and ask the reader to follow the Figure with the next items. Let ϕ be an 8-coloring of W_8. The set of vertices $\{v_1, v_2, v_4, v_5, v_6, v_8, c\}$ is an o-clique, thus the colors of this vertices are different, respectively $\{0, 1, 2, 3, 4, 5, 6\}$. Hence, we know from the orientation of W_8 that $\phi(v_3) \notin \{0, 1, 2, 3, 5, 6\}$ because all of the vertices with these colors are adjacent or have a path of size two to v_3. We can color v_3 with the color 4. Again from the orientation of W_8 we have that $\phi(v_7) \notin \{0, 2, 3, 4, 5, 6\}$ because all of the vertices with these colors are adjacent or have a path of size two to v_7. We also can not color v_7 with the color 1 because we have $v_3v_2 \in E(\overrightarrow{W_8})$ and $\phi(v_3) = 4$, so we need an eighth color for v_7. $\square$

4 On the Oriented Chromatic Number of the Union of Graphs

The study of the class $\mathcal{CN}_3$ motivated us to study the oriented chromatic number of disconnected graphs. We show an example in Fig. 1, where the oriented

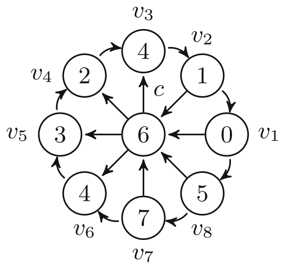

Fig. 2. An orientation of W_8 that has $\chi_o(\overrightarrow{W_8}) = 8$.

chromatic number of a graph $G = K_3 \cup P_4$ is greater than the oriented chromatic number of each of its connected components separately.

In Fig. 3, where $G = K_4 \cup P_5$, consider the orientation $\overrightarrow{G}$ of G in which $\overrightarrow{K}_4$ is the transitive tournament and $\overrightarrow{P}_5$ is the directed path.

So we have another example in which $\chi_o(G) > \max\{K_4; P_5\}$, where K_4 and P_5 are components of G. Since $\chi_o(\overrightarrow{K}_4) = 4$, we assign a 4-oriented coloring of $\overrightarrow{K}_4$. Using the constraints of 4-oriented coloring of $\overrightarrow{K}_4$ in the component $\overrightarrow{P}_5$, we prove that $\overrightarrow{P}_5$ cannot be colored only with four colors and one fifth color is required, so the graph $G = K_4 \cup P_5 \notin \mathcal{CN}_4$.

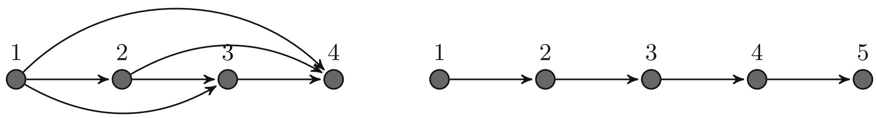

Fig. 3. Graph $\overrightarrow{K}_4 \cup \overrightarrow{P}_5$.

Now, we will obtain the oriented chromatic number of the disjoint union between the complete graph and others graphs, such as graphs that can be colored by the path $\overrightarrow{P_3}$ or the cycle $\overrightarrow{C_3}$, stars, trees, forests and cycles. First we analyse the case of graphs that have a homomorphism to the path $\overrightarrow{P_3}$ or the cycle $\overrightarrow{C_3}$

Theorem 4. *Let G be a graph with two connected components G_1 and G_2, where G_1 is a complete graph K_p, $p \geq 3$, and G_2 is a graph such that all oriented graphs $\overrightarrow{G}_2$ have a homomorphism f into a directed path $\overrightarrow{P}_3$, then $\chi_o(G) = p$.*

Proof. Since $\chi_o(\overrightarrow{P}_3) = 3$ (by definition of oriented coloring), considering an oriented coloring c of $\overrightarrow{P}_3$, in which we assign color 1 to the source, color 3 to the sink, and color 2 to the remaining vertex (successor of color 1 and predecessor of color 3). By hypothesis, all oriented graphs $\overrightarrow{G}_2$ have a homomorphism f

into a directed path $\overrightarrow{P}_3$. Thus, we can assign an oriented coloring for $\overrightarrow{G}_2$ using $c \circ f : V(\overrightarrow{G}_2) \rightarrow \{1, 2, 3\}$.

We will assign an oriented coloring with p colors to any oriented graph $\overrightarrow{G}_1$ from G_1 respecting the constraints used in $\overrightarrow{G}_2$. As $G_1 = K_p$, $p \geq 3$, $\chi_o(G_1) = p$ and all oriented graphs $\overrightarrow{G}_1$ from G_1 contains either a transitive or a circular $\overrightarrow{K}_3$. In both cases, there exists a directed path P_3 as a subgraph. This directed path can be colored with the same constraints used in G_1. There are no restrictions for the remaining $p - 3$ colors and therefore we can assign these colors to the other vertices not yet colored without conflict. □

Theorem 5. *Let G be a graph with two connected components G_1 and G_2, where G_1 is a complete graph K_p, $p \geq 3$, and G_2 is a graph such that all oriented graphs $\overrightarrow{G}_2$ have a homomorphism into a directed cycle $\overrightarrow{C}_3$ and diameter greater than p. Then $\chi_o(G) = p + 1$.*

Proof. By hypothesis, $\overrightarrow{G}_2$ requires three colors $1, 2, 3$ to an oriented coloring, with the property that no color dominates the two others. We can obtain an oriented graph $\overrightarrow{G}$ from G with $\chi_o(\overrightarrow{G}) \geq p + 1$, in the following way: orient $\overrightarrow{G}_1$ as a transitive tournament. It follows that all subgraphs $\overrightarrow{K}_3$ of $\overrightarrow{G}_1$ are transitive. As in an oriented coloring of $\overrightarrow{G}_1$, for all $\overrightarrow{K}_3$ of $\overrightarrow{G}_1$ one color dominates the two others, at least one different color from $1, 2, 3$ is required in some component $\overrightarrow{K}_3$. Then $\chi_o(G) \geq p + 1$.

Conversely, we show that $\chi_o(G) \leq p + 1$. Let $\overrightarrow{K}_p$ be any orientation for G_1. As $\chi_o(\overrightarrow{K}_p) = p$, without loss of generality, we admit a coloring of $\overrightarrow{K}_p$ using the colors from 1 to p. We add a vertex v to the graph $\overrightarrow{K}_p$, and if there is source f or sink s in $\overrightarrow{K}_p$ we add the arcs vf and sv, we call the resulting graph of $\overrightarrow{K}'_{p+1}$, the remaining edges assume any orientation so that v is neither source nor sink in the new graph. We assign the color $p+1$ to the vertex v. Note that $\overrightarrow{K}'_{p+1}$ has neither sources nor sinks. On the other hand, considers the directed cycle $\overrightarrow{C}_3$. We assign an oriented coloring of $\overrightarrow{C}_3$ respecting the constraint on the coloring of $\overrightarrow{K}'_{p+1}$.

We start by assigning a color $p+1$ to any vertex v_1 of $\overrightarrow{C}_3$. By the construction of $\overrightarrow{K}'_{p+1}$ the vertex v with color $p+1$ is neither source nor sink, so we divide the neighbors of v into two disjoint sets, a set of successors of v denoted by $Suc(v)$ and a set of predecessors of v denoted by $Pred(v)$. We will assign the same color as the successor v_2 of v_1 in $\overrightarrow{C}_3$ of a vertex $r \in Suc(v)$ who has a successor in $t \in Pred(v)$, the same color for predecessor v_3 of v_1 in $\overrightarrow{C}_3$ of the vertex of $t \in Pred(v)$.

By construction, there exists at least one vertex in $r \in Suc(v)$ such that rt is an arc in $\overrightarrow{K}'_{p+1}$, where $t \in Pred(v)$. So we can assign colors to $\overrightarrow{C}_3$ with the $p+1$ colors of $\overrightarrow{K}'_{p+1}$ and as $\overrightarrow{K}_p$ is a subgraph of $\overrightarrow{K}'_{p+1}$ then $\chi_o(G) \leq \chi_o(\overrightarrow{K}'_{p+1} \cup \overrightarrow{C}_3) = p + 1$. □

Corollary 2 follows directly from Theorem 5 and Corollary 1. We also show an upper bound for the disjoint union of complete graphs and stars on Corollary 3.

Corollary 2. *Given $G = K_p \cup P_q$ or $G = K_p \cup T_q$ or $G = K_p \cup F_q$, then $\chi_o(G) = p+1$. Where $p \geq 3$ and P_q, T_q, F_q be respectively a path, a tree and a forest on q vertices and diameter greater than 2.*

Corollary 3. *Given $G = K_p \cup S_q$, then $\chi_o(G) = p$, where $p \geq 3$ and S_q is a star on q vertices.*

Now we define a special tournament on 5 vertices that we will use to describe the union of cycles and a few other graph classes. Let T_5^U be the tournament where $V(T_5^U) = \{v_1, v_2, v_3, v_4, v_5\}$, and $E(T_5^U) = \{v_1v_2, v_2v_3, v_2v_5, v_3v_1, v_3v_4, v_3v_5, v_4v_2, v_4v_5, v_1v_4, v_5v_1\}$. Also for this purpose we show that every tournament in 4 vertices has a sub-tournament which has a homomorphism to the acyclic tournament in 3 vertices.

Lemma 6. *Every tournament with 4 vertices has a homomorphism into T_5^U.*

Proof. We can verify by exhaustion that every 4-vertex tournament has a homomorphism into T_5^U .

Corollary 4. *Every tournament in 4 vertices has a sub-tournament which has a homomorphism to the acyclic tournament in 3 vertices.*

Now we define the chromatic number of the disjoint union of graphs that belongs to the class $\mathcal{CN}_4$ and cycles.

Theorem 6. *Let $G \in \mathcal{CN}_4$ be a graph and C be a cycle. Then $\chi_o(G \cup C) = 5$.*

Proof. Let $\overrightarrow{C}_5^d$ be a directed cycle with 5 vertices, then $\chi_o(C_5) = 5$, see Lemma 2. By Lemma 3 and because any other orientation of C_5 has a 4-oriented coloring, the class $C \backslash \overrightarrow{C}_5^d \in \mathcal{CN}_4$. By Lemma 6 every $G \in \mathcal{CN}_4$ has a homomorphism into T_5^U. The cycle $\overrightarrow{C}_5^d$ also has homomorphism in T_5^U, see that T_5^U has a directed cycle $1, 2, 3, 4, 5, 1$. Therefore, $\chi_o(G \cup C) = 5$ with T_5^U as a color graph. □

Corollary 5. *Let $G = C \cup C$ or $G = C \cup P$ or $G = C \cup T$ or $G = C \cup K4$, then $\chi_o(G) = 5$, where C, P, T, K_4 be respectively a cycle, a path, a tree and the complete graph with 4 vertices.*

We also define the chromatic number of the disjoint union of complete graphs and cycles.

Theorem 7. *Let p and q be a pair of integers with $p \geq 2$ and $q \geq 3$, then*

$$\chi_o(K_p \cup C_q) = \begin{cases} 3, & \text{if } p = 2 \quad \text{and} \quad \chi_o(C_q) = 3 \\ 4, & \begin{cases} \text{if } p = 2 & \text{and} \quad \chi_o(C_q) = 4 \\ \text{if } p = 3 & \text{and} \quad (\chi_o(C_q) = 3 \text{ or } \chi_o(C_q) = 4) \end{cases} \\ 5, & \begin{cases} \text{if } p = 2 & \text{and} \quad \chi_o(C_q) = 5 \\ \text{if } p = 3 & \text{and} \quad \chi_o(C_q) = 5 \end{cases} \\ p+1, & \text{if } p \geq 4 \end{cases}$$

Finally we will analyse the chromatic number of the disjoint union of two complete graphs.

Lemma 7. *Let c be an oriented coloring of $\overrightarrow{K}_p \cup \overrightarrow{K}_q$. Given $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ subgraphs induced of $\overrightarrow{K}_p$ and $\overrightarrow{K}_q$ respectively, such that $\exists\, u \in V(\overrightarrow{G}_1)$ if and only if $\exists\, a \in V(\overrightarrow{G}_2)$ with $c(u) = c(a)$. Then $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are isomorphic.*

Proof. As $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are induced subgraphs by vertices of tournaments, then $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are also tournaments. Thus, in an oriented coloring c of $\overrightarrow{K}_p \cup \overrightarrow{K}_q$ there are no identical colors between the vertices of $\overrightarrow{G}_1$, as well as between the vertices of $\overrightarrow{G}_2$, then by hypothesis we know that $|V(\overrightarrow{G}_1)| = |V(\overrightarrow{G}_2)|$.

Case $|V(\overrightarrow{G}_1)| = |V(\overrightarrow{G}_2)| \leq 2$ then $\overrightarrow{G}_1$ and $\overrightarrow{G}_2$ are isomorphic.

Suppose that $|V(\overrightarrow{G}_1)| = |V(\overrightarrow{G}_2)| \geq 2$. Let $u, v \in V(\overrightarrow{G}_1)$ and $a, b \in V(\overrightarrow{G}_2)$ such that $c(u) = c(a)$ and $c(v) = c(b)$. We define $f : V(\overrightarrow{G}_1) \rightarrow V(\overrightarrow{G}_2)$ such that $f(u) \mapsto a$ and $f(v) \mapsto b$.

Let $f(u) = f(v)$. As $\overrightarrow{G}_2$ is a tournament then $c(f(u)) = c(f(v))$. By function f we have that $c(f(u)) = c(u)$ we get by replacing $c(u) = c(v)$. Like $\overrightarrow{G}_2$ also is a tournament, then $u = v$. We conclude that the function f is injective. As $|V(\overrightarrow{G}_1)| = |V(\overrightarrow{G}_2)|$ and $\overrightarrow{G}_1$, $\overrightarrow{G}_2$ are tournaments, then the function f is sobrejective. □

Theorem 8. *Let K_p and K_q be complete graphs, and $\overrightarrow{K}$ be the collection of all tournaments. Consider the sets P and Q consisting of all orientations of K_p and K_q respectively. Define the set $L = \{\overrightarrow{K}^l \in K; |V(\overrightarrow{K}^l)| = \max\{|V(\overrightarrow{K}^j)|; \overrightarrow{K}^j \subseteq \overrightarrow{K}'_p,\ \overrightarrow{K}^j \subseteq \overrightarrow{K}'_q\},\ \forall \overrightarrow{K}'_p \in P$ and $\overrightarrow{K}'_q \in Q\}$. Let $r = \min\{|V(\overrightarrow{K}^l)|; \forall \overrightarrow{K}^l \in L\}$. Then $\chi_o(K_p \cup K_q) = p + q - r$.*

Proof. Let $\overrightarrow{K}_r$ a tournament on r vertices, where $r = \min\{|V(\overrightarrow{K}^l)|; \forall \overrightarrow{K}^l \in L\}$. We denote by $\overrightarrow{K}^p_r$ a subgraph $\overrightarrow{K}_r$ of $\overrightarrow{K}_p$ and $\overrightarrow{K}^q_r$ a subgraph $\overrightarrow{K}_r$ of $\overrightarrow{K}_q$. Since $\overrightarrow{K}^p_r$ and $\overrightarrow{K}^q_r$ are isomorphic, we can assign identical r colors to the vertices of both graphs. As $r \leq q \leq p$ remain $p + q - r$ vertices to be colored. Then $\chi_o(K_p \cup K_q) \leq p + q - r$.

By Lemma 7, the maximum number of colors used in both $\overrightarrow{K}_p$ and $\overrightarrow{K}_q$ is r, otherwise we contradict the cardinality of $\overrightarrow{K}_r$. Hence $\chi_o(K_p \cup K_q) = p + q - r$. □

We also analyse some specific disjoint unions of K_5 with another K_5 and with complete graphs.

Theorem 9. *Given the union $K_5 \cup K_5$, set $L = \{\overrightarrow{K}^l \in K; |V(\overrightarrow{K}^l)| = \max\{|V(\overrightarrow{K}^j)|; \overrightarrow{K}^j \subseteq \overrightarrow{K}'_5,\ \overrightarrow{K}^j \subseteq \overrightarrow{K}'_5\},\ \forall \overrightarrow{K}'_p \in P$ and $\overrightarrow{K}'_q \in Q\}$, then $r = \min\{|V(\overrightarrow{K}^l)|; \forall \overrightarrow{K}^l \in L\} = 3$.*

Corollary 6. *Given the union* $K_p \cup K_5, p \geq 5$*, set* $L = \{\overrightarrow{K}^l \in K; |V(\overrightarrow{K}^l)| = \max\{|V(\overrightarrow{K}^j)|; \overrightarrow{K}^j \subseteq \overrightarrow{K}'_5, \overrightarrow{K}^j \subseteq \overrightarrow{K}'_5\}, \forall \overrightarrow{K}'_p \in P$ *and* $\overrightarrow{K}'_q \in Q\}$*, then* $r = \min\{|V(\overrightarrow{K}^l)|; \forall \overrightarrow{K}^l \in L\} = 3$.

We have done some computational experiments, that drove us to Conjecture 1.

Conjecture 1. Let K_p, K_q be 2 complete graphs with $p, q \geq 4$. Then $\chi_o(K_p \cup K_q) = p + q - 3$.

Lastly we show an upper bound for the disjoint union of wheel graphs and complete graphs.

Theorem 10. *Let* p, q*,* $p \geq 4, q \geq 3$ *be positive integers. Then* $\chi_o(K_p + W_q) \leq p + 5$.

Proof. Let $\overrightarrow{K_3}$ be the transitive orientation of the tournament with 3 vertices. We consider 2 cases:

1. $\overrightarrow{K_3}$ is not a subgraph of W_q. In this case we can color W_q with 3 colors. Hence, 2 colors of the graph K_p can be used with color $p + 1$ to color W_q.
2. $\overrightarrow{K_3}$ is a subgraph of W_q. In this case according to Theorem 3 we can color W_q with 8 colors. From Corollary 4 we know that we can use 3 colors of the graph K_p plus additional 5 colors to color W_q. □

5 Conclusions

In this paper, we prove that if $q \geq 8$ then $\chi_o(W_q) = 8$ and for every forest F, $\chi_o(F)$ is determined by the connected component of F with the largest oriented chromatic number of its connected components, what is an exception to the general case of disconnected graphs.

We characterized the class $\mathcal{CN}_3$ of the graphs with $\chi_o(G) \leq 3$. This characterization motivated us to study the oriented chromatic number of disconnected graphs. We have established $\chi_o(K_p \cup P_q)$, $\chi_o(K_p \cup F)$, $\chi_o(K_p \cup C_q)$, and an upper bound for $\chi_o(K_p \cup W_q)$.

We establish the oriented chromatic number of the union of two complete graphs K_p, K_q as $\chi_o(K_p \cup K_q) = p + q - r$, where r is the size of the maximum tournament contained in all orientations of K_p and K_q. We have conjectured that $r = 3$ for every pair $4 \leq p, q$.

Table 1 presents the results obtained in this paper regarding to the union of complete graphs with other graph classes. For future works we intend to expand our Table of results where most of the important classes be added in the firsts column and row of the Table, besides considering the cases when we have more than 2 components.

Table 1. Oriented chromatic number of the union $\chi_o(G \cup H)$.

$G \setminus H$	Forest diameter $d \leq 2$	Forest diameter $d \geq 3$	C_q, $q \geq 3$	K_q	W_q
K_p, $p = 2$	$d + 1$	3 (Corol. 1)	3, if $\chi_o(C_q) = 3$ 4, if $\chi_o(C_q) = 4$ 5, if $\chi_o(C_q) = 5$	$p + q - r$ (Corol. 8)	$q + 1$, if $3 \leq q \leq 6$ 8, if $q \geq 8$
K_p, $p = 3$	3 (Thm. 2)	4 (Corol. 8)	4, if $\chi_o(C_q) = 3$ 4, if $\chi_o(C_q) = 4$ 5, if $\chi_o(C_q) = 5$	$p + q - r$ (Thm. 8)	$q + 1$, if $3 \leq q \leq 6$ 8, if $q \geq 8$
K_p, $p \geq 4$	p (Corol. 3)	$p + 1$ (Corol. 2)	$p + 1$ (Thm. 7)	$p + q - r$ (Corol. 8)	$\leq p + 5$ (Thm. 10)

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