





Splitting Recursion Schemes into Reversible and Classical Interacting Threads

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Abstract. Given a simple recursive function, we show how to extract from it a reversible and an classical iterative part. Those parts can synchronously cooperate under a Producer/Consumer pattern in order to implement the original recursive function. The reversible producer is meant to run on reversible hardware. We also discuss how to extend the extraction to a more general compilation scheme.

1 Introduction

Our goal is to compile a class of recursive functions in a way that parts of the object code produced can leverage the promised green foot-print of truly reversible hardware. This work illustrates preliminary steps towards that goal. We focus on a basic class of recursive functions in order to demonstrate its feasibility.

Contributions. Let `recF[p,b,h]` be a recursive function defined in some programming formalism, where `p` is a *predecessor* function, `h` a *step* function, and `b` a *base* function. We show how to compile `recF[p,b,h]` into `itFClS[b,h]` and `itFRev[p,pInv]` such that:

$$\text{recF}[p,b,h] \simeq \text{itFClS}[b,h] \parallel \text{itFRev}[p,pInv] \quad , \quad (1)$$

where: (i) “ \simeq ” stands for “*equivalent to*”; (ii) `itFClS[b,h]` is a classical `for`-loop that, starting from a value produced by `b`, iteratively applies `h`; (iii) `itFRev[p,pInv]` is a reversible code with two `for`-loops in it one iterating `p`, the other its inverse `pInv`; (iv) “ \parallel ” is interpreted as an *interaction* between `itFClS[b,h]` and `itFRev[p,pInv]`, according to a Producer/Consumer pattern, where `itFRev[p,pInv]` produces the values that `itFClS[b,h]` consumes to implement the initially given recursion `recF[p,b,h]`. In principle, `itFRev[p,pInv]` can drive a real reversible hardware to exploit its low energy consumption features.

In this work we limit the compilation scheme (1) to use: (i) a predecessor `p` such that the value `p(x)-x` is any *constant* Δ_p equal to, or smaller than, `-1`;

(ii) recursion functions `recF[p,b,h]` whose *condition* identifying the base case is `x<=0` instead than the more standard `x==0`; this means that more than one base *non positive* value for `recF[p,b,h]` exists in the interval $[\Delta_p + 1, 0]$. This slight generalization will require a careful management of the reversible behavior of `itFRev[p,pInv]` and its interaction with `itFCls[b,h]` in order to reconstruct `recF[p,b,h]`.

Contents. Sect. 2 sets the stage to develop the main ideas about (1), restricting `recF[p,b,h]` to a recursive function that identifies its base case by means of the standard condition `x==0`; this ease the description of how `itFRev[p,pInv]` and `itFCls[b,h]` interact. Section 3 extends (1) to deal with `recF[p,b,h]` having `x<=0`, and not `x==0`, to identify its base case(s); this impacts on how `itFRev[p,pInv]` must work. In both cases, the programming syntax we use can be interpreted into the reversible languages SRL [3,4] and RPP [4-6], up to minor syntactic details. Section 4 addresses future work.

```

1  Fix recF(x)                                     {
2      if    (c(x)) { b(x);                       }
3      else   { h(x,recF(p(x))); } }
    
```

Fig. 1. The recursive function `recF`.

```

1  /*** Assumption: the initial value of x is 3 */
2  x = p(x)           // ==2
3  x = p(x)           // ==1
4  x = p(x)           // ==0
5  y = b(x)           // ==b(p(p(p(3))))
6  y = h(x,y)         // ==h(p(p(p(3))),b(p(p(p(3))))
7  x = pInv(x)        // ==pInv(p(p(p(3))))==p(3)
8  y = h(x,y)         // ==h(p(p(3)),h(p(p(p(3))),b(p(p(p(3))))
9  x = pInv(x)        // ==pInv(p(p(3)))==p(3)
10 y = h(x,y)         // ==h(p(3),h(p(p(3)))
11                               //           ,h(p(p(p(3))),b(p(p(p(3))))
12 x = pInv(x)        // ==pInv(p(3))==3
13 y = h(x,y)         // ==h(3,h(p(3),h(p(p(3)))
14                               //           ,h(p(p(p(3))),b(p(p(p(3))))
    
```

Fig. 2. Iterative unfolding `recF(3)`: the bottom-up part.

2 The Driving Idea

Let `recF[p,b,h]` in (1) have a structure as in Fig. 1 where `b(x)` is the *base* function, `h(x,y)` the *step* function, `p(x)` the *predecessor* `x-1`, and `c(x)` the *condition* `x==0` to identify a unique base case.

Figure 2 details out $h(3, h(p(3), h(p(p(3))), h(p(p(p(3))), b(p(p(p(3))))))$, unfolding of `recF(3)`. Every comment asserts a property of the values that `x` or `y` stores. Lines 2–4 unfold an iteration that computes $p(p(p(3)))$, which eventually sets the value of `x` to 0. Line 5 starts the construction of the final value of `recF(3)` by applying the base case of `recF`, i.e. $b(x)$. By definition, let `pInv` denote the inverse of `p`, i.e. $pInv(p(z))=p(pInv(z))=z$, for any z . Clearly, in our running example, the function `pInv(x)` is $x+1$. Lines 6–13 alternate $h(x, y)$, whose result `y`, step by step, gets closer to the final value `recF(3)`, and `pInv(x)`, which produces a new value for `x`.

```

1  s = 0, e = 0, g = 0, w = 0
2  w = w + x;
3  for (i = 0; i<=w; i++)      {
4      if      (x> 0) { g++; }
5      else if (x==0) { e++; }
6      else           { s++; }
7      x = p(x);           }
8
9  for (i = 0; i<=w; i++)      {
10     x = pInv(x);
11     if      (x> 0) { g--; y = h(x,y); }
12     else if (x==0) { e--; y = b(x);   }
13     else           { s--;           } }
14  w = w - x;

```

Fig. 3. Iterative `itF` equivalent to `recF`.

Let us call `itF` the code in Fig. 3. It implements `recF` by means of finite iterations only. Continuing with our running example, if we run `itF` here above starting with $x==3$, then $x==0$ holds at line 8, just after the first `for`-loop; after the second `for`-loop $y==recF(3)$ holds at line 14.

The code of `itF` has two parts. Through lines 2–7 the variable `g` counts how many times `x` remains positive, the variable `e` how many it stays equal to 0, and the variable `s` how many it becomes negative. In this running example we notice that `x` never becomes negative, because the iteration at lines 3–7 is driven by the value of `x` which, initially, we can assume non negative, and which `p(x)` decreases of a single unity. We shall clarify the role of `s` later. Lines 9–13 undo what lines 2–7 do by executing `pInv(x)`, `g--`, `e--`, `s--`, i.e. the inverses, in reversed order, of `p(x)`, `g++`, `e++`, `s++`. So the correct values of `x` are available at lines 12, and 11, ready to be used as arguments of `b(x)` and `h(x,y)` to update `y` as in Fig. 3, according to the results we obtain by the recursive calls to `recF`.

Now, let us focus on the main difference between Figs. 4 and 3.

Both $x=b(x)$ and $y=h(x,y)$ at lines 12, and 11 of Fig. 3 are missing from lines 12, and 11 of Fig. 4. Dropping them let Fig. 4 be the *reversible side* of `itF`; calling `b(x)` and `h(x,y)` in it generates `y`, which is the result we need, so preventing

```

1  s = 0, e = 0, g = 0, w = 0
2  w = w + x;
3  for (i=0; i<=w; i++)      {
4      if      (x> 0) { g++; } //number of times x is 'g'reater than 0
5      else if (x==0) { e++; } //number of times x is 'e'qual to 0
6      else      { s++; } //number of times x is 's'maller than 0
7      x = p(x);              }
8
9  for (i=0; i<=w; i++)      {
10     x = pInv(x);
11     if      (x> 0) { g--; /* Value of x for h availabe here */ }
12     else if (x==0) { e--; /* Value of x for b availabe here */ }
13     else      { s--; }
14 w = w - x;

```

Fig. 4. Reversible side of `itF`.

the possibility to reset the value of every variable dealt with in Fig. 4 to their initial value. This is why we also need a *classical side* of `itF` that generates `y` in collaboration with the *reversible side* in order to implement the initial `recF` correctly.

```

1  /** Assumption. The value of the input x is available here */
2  /* Inject the current x at line 2 of itFRev to let it start */
3  iterations = /* Probe line 9 of itFRev to get the
4               number of iterations to execute */
5  y = b(/* Probe line 14 of itFRev to get the argument */);
6  for (i = 0; i<iterations; i++)      {
7      y = h(/* Probe line 12 itFRev to get
8            the first argument of h    */ , y); }

```

Fig. 5. Classical side of `itF`: the consumer `itFCLs`.

The previous observations lead to Fig. 5 which defines the *classical side* `itFCLs` of `recF`, and to Fig. 6 which defines the *reversible side* `itFCRev` of `recF`.

So, here below we can illustrate how `itFCLs` and `itFRev` synchronously interact, `itFRev` producing values, `itFCLs` consuming them as arguments of `b(x)` and `h(x,y)`.

Line 2 of `itFCLs` is the starting point of the synchronous interaction between `itFCLs` and `itFRev`; its comment:

```

/* Inject the current x at line 2 of itFRev to let it start */

```

describes what, in a fully implemented version of `itFCLs`, we expect in that line of code. The comment says that `itFCLs` injects (sends, puts) its input value `x` to

```

1  s = 0, e = 0, g = 0, w = 0;
2  x = /* Inject here the value of x at line 2 of itFClS */
3  w = w + x;
4  for (i = 0; i<=w; i++)      {
5      if      (x> 0) { g++; }
6      else if (x==0) { e++; }
7      else          { s++; }
8      x = p(x);              }
9  /* itFClS probes here g which has the number of iterations */
10 for (i = 0; i<=w; i++)      {
11     x = pInv(x);
12     if      (x> 0) { g--; /* itFClS probes here the
13                          first argument value of h */ }
14     else if (x==0) { e--; /* itFClS probes here the
15                          argument value of b          */ }
16     else          { s--; } }
17 w = w - x;

```

Fig. 6. Reversible side of `itF` updated to be the producer `itFRev` of the values that the consumer `itFClS` needs.

line 2 of the *reversible side* `itFRev` (cf. Fig. 6). Once `itFRev` obtains that value at line 2, as outlined by:

```
/* Inject here the value of x from line 2 of itFClS */
```

its `for`-loop at lines 4–8 executes.

After line 2, `itFClS` stops at line 3. It waits for `itFRev` to produce the number of times that `itFClS` has to iterate line 7. Accordingly to:

```
/* Probe line 9 of itFRev to get the number of iterations to execute */
```

`itFRev` makes that value available in its variable `g` at line 9:

```
/* itFClS probes here g which has the number of iterations */ .
```

Once gotten the value in `iterations`, `itFClS` proceeds to line 5 and stops, waiting for `itFRev` to produce the argument of `b` which is eventually available for probing at line 14 of `itFRev`.

Once the argument becomes available `b` is applied, and `itFClS` enters its `for`-loop, stopping at line 7 at every iteration. The reason is that `itFClS` waits for line 12 in `itFRev` to produce the value of the first argument of `h(x,y)`. This interleaved dialog between line 7 of `itFClS` and line 12 of `itFRev` lasts `iterations` times.

3 From Recursion to Iteration

We now generalize what we have seen in Sect. 2. Inside (1) we use `recG` of Fig. 7 instead than `recF` of Fig. 1. This requires to generalize Fig. 6.

```

1  Fix recG(x)                                     {
2    if (x<=0) { b(x);                             }
3    else     { h(x,recG(p(x))); } }

```

Fig. 7. The generic structure of `recG`.

From the introduction we recall that, given a *predecessor* $p(x)$, we define $\Delta_p = p(x)-x$, which is a negative value. In this section Δ_p can be any *constant* $k \leq -1$, not only $k = -1$; this requires to consider the slightly more general *condition* $x \leq 0$ in `recG`. For example, let $p(x)$ be $x-2$. The computation of `recG(3)` is $h(3, h(p(3), h(p(p(3)), b(p(p(3))))))$ which looks for the least n of iterated applications of $p(x)$ such that $p(\dots p(3) \dots) \leq 0$; in our case we have $2 = n < 3$.

Figure 8 introduces `itG` which generalizes `itF` in Fig. 3.

The scheme `itG` iteratively implements any recursive function whose structure can be brought back to the one of `recG`. We remark that line 1 in Fig. 8 initializes ancillae s, e, g , and w , like Fig. 3 initializes the namesake variables of `itF`, but line 2 of `itG` has new ancillae $z, \text{predDivX}$, and predNotDivX .

We also assume an initial *non negative* value for x . The reason is twofold. Firstly, it keeps our discussion as simple as possible, with no need to use the absolute value of x to set the upper limit of every index i in the `for`-loops that occur in the code. Second, negative values of x would widen our discussion about what a classical recursive function on negative values is and about what its reversible equivalent iteration has to be; we see this as a very interesting subject connected to [1], which is much more oriented than us to optimization issues of recursively defined functions.

We start observing that line 3 of `itG` sets w to the initial value of x ; the reason is that every `for`-loop, but the one at lines 10–12, has to last $x+1$ iterations, and x changes in the course of the computation; so, w stores the initial value of x and stays constant from line 4 through line 21. In fact it can change at lines 22–33. We will see why, but w is eventually reset to its initial value 0 at line 36.

With the here above assumptions, given a non negative x , and in analogy to `itF`, the `for`-loop at lines 4–8 of `itG` iterates the application of $p(x)$ as many times as $w+1$, i.e. the initial value of x plus 1. So, the value of x at line 9 is equal to $w+(w+1)*\Delta_p$ which cannot be positive. In particular, all the values that x assumes in the `for`-loop at lines 4–8 belong to the following interval:

$$I(w) \triangleq [w+(w+1)*\Delta_p, w+w*\Delta_p, \dots, w+\Delta_p, w] \tag{2}$$

from the least to the greatest; the counters g, e, s say how many elements of $I(x)$ are *greater, equal or smaller* than 0, respectively. Depending on 0 to belong to $I(x)$ determines the behavior of the reminder part of `itG`, i.e. lines 10–36.

We distinguish two cases in order to illustrate them.

```

1  s = 0, e = 0, g = 0, w = 0;
2  z = 0, predDivX = 0, predNotDivX = 1;
3  w = w + x; /* x is assumed to be the input */
4  for (i = 0; i <= w; i++) {
5      if (x > 0) { g++; }
6      else if (x == 0) { e++; }
7      else { s++; }
8      x = p(x); }
9
10 for (i = 0; i < e; i++) {
11     predDivX = predDivX + predNotDivX;
12     predNotDivX = predDivX - predNotDivX; }
13
14 for (j = 0; j < predDivX; j++) {
15     for (i = 0; i <= w; i++) {
16         x = pInv(x);
17         if (x > 0) { g--; y = h(x,y); }
18         else if (x == 0) { e--; y = b(x); }
19         else { s--; }}
20
21 for (j = 0; j < predNotDivX; j++) {
22     w++;
23     for (i = 0; i <= w; i++) {
24         x = pInv(x);
25         if (x > 0) { g--;
26                     x = p(x);
27                     if (z < 0) { }
28                     else if (z == 0) { y = b(x); z++; }
29                     else { y = h(x,y); }
30                     x = pInv(x); }
31         else if (x == 0) { e--; }
32         else { s--; }}
33     w--; }
34 for (i = 0; i < predNotDivX; i++) {
35     z--; }
36 w = w - x;
37 /* y carries the output */

```

Fig. 8. The iterative function `itG`.

First Case. Let $w\% \Delta_p = 0$, i.e. the integer value Δ_p divides with no remainder the initial value of x that we find in w . So, $0 \in I(x)$, which implies the following relations hold at line 9:

$$e == 1 \qquad g == -\frac{w}{\Delta_p} \qquad s == (w+1)-g-e . \qquad (3)$$

```

1  if      (e < 0) {
2  else if (e == 0) { predDivX = predDivX+predNotDivX;
3                      predNotDivX = predDivX - predNotDivX; }
4  else      {

```

Fig. 9. A possible replacement of lines 10–12 in Fig. 8.

Lines 10–12 execute exactly once, swapping `predDivX` and `predNotDivX`. As a remark, we could have well used the `if`-selection in Fig. 9 (a construct of RPP) in place of the `for`-loop at lines 10–12, but we opt for a more compact code.

Swapping `predDivX` and `predNotDivX` sets `predDivX==1` and `predNotDivX==0`, computationally exploiting that Δ_p divides `w` with no remainder: the `for`-loop body at lines 15–19 becomes accessible, while lines 22–33, with `for`-loops among them, do not. Lines 15–19 are identical to lines 10–16 of `itF` in Fig. 4 which we already know to correctly apply `b(x)` and `h(x,y)` in order to simulate the recursive function we start from.

As a Second Case. Let $w \% \Delta_p \neq 0$, i.e. the integer value Δ_p divides the initial value of `x` that `w` stores, but with some remainder. So, $0 \notin I(x)$, which imply:

$$e == 0 \qquad g == - \left\lfloor \frac{w}{\Delta_p} \right\rfloor \qquad s == (w+1)-g-e \qquad (4)$$

hold at line 9. Lines 11–12 cannot execute, leaving `predDivX` and `predNotDivX` as they are: lines 22–33 become accessible and the `for`-loop at lines 15–19 does not. Line 22 increments `w` to balance the information loss that the rounding of `g` in (4) introduces; line 33 recovers the value of `w` when the outer `for`-loop starts. The `if`-selection at lines 25–32 identifies when to apply `b(x)`, which must be followed by the required applications of `h(x,y)`. We know that $0 \notin I(x)$, so `x==0` can never hold. Clearly, `s--` is executed until `x>0`. But the *first* time `x>0` holds true we must compute `b(p(x))`, because the *base* function `b(x)` must be used the *last time* `x` assumes a negative value, *not the first time* it gets positive; lines 26–30 implement our needs. Whenever `x>0` is true, the value of `x` is one step ahead the required one: we get one step back with line 26 and, if it is the first time we step back, i.e. `z==0` holds, then we must execute line 28. If not, i.e. `z!=0`, we must apply the *step* function at line 29. Line 30, restores the right value of `x`. Finally, the `for`-loop at line 34 sets `z` to its initial value.

At this point, in order to obtain the fully reversible version of Fig. 8 we must think of replacing the calls to `h(x,y)` and `b(x)` at lines in 28, and 29 by means of actions that probe the value of `x`, in analogy to Fig. 6, lines 12 and 14. The full details are in [7] which we look as a playground with Java classes that implement Fig. 8 and Fig. 5 as synchronous and parallel threads, acting as a producer and a consumer.

4 Future Work

We have shown that we can decompose every classical recursive function, based on a *predecessor* that decreases every of its input by a constant value, into reversible and classical components that cooperate to implement the original recursive functions under a Producer/Consumer pattern (see (1)).

Firstly, we plan to extend (1) to recursive functions `recF` based on predecessors `p` not limited to a constant Δ_p not greater than `-1`. A predecessor `p` should be at least such that:

1. Δ_p is not necessarily a constant. For example, $\Delta_p == -3$ on even arguments, and `-2` on odd ones can be useful;
2. the predecessor can be an integer division `x/k`, for some given `k>0`, like in a dichotomic search, which has `k==2`.

Secondly, we aim at generalizing (1) to a compiler `[[·]]`:

$$\begin{aligned}
 \llbracket p \rrbracket &= \text{some implementation code} \\
 \llbracket p\text{Inv} \rrbracket &= !\llbracket p \rrbracket, \text{ i.e. implementation that inverts } \llbracket p \rrbracket \\
 \llbracket \text{recF}[p, b, h] \rrbracket &= \text{itFClS}[\llbracket b \rrbracket, \llbracket h \rrbracket] \parallel \text{itFRev}[\llbracket p \rrbracket, \llbracket p\text{Inv} \rrbracket].
 \end{aligned} \tag{5}$$

The domain of `[[·]]` should be a class `R` of recursive functions built by means of standard composition schemes, starting from a class of predecessors `p1`, `p2`, ... each of which must have the corresponding inverse function `p1Inv`, `p2Inv`, ...

In these lines we want to explore interpretations of `||` more liberal than the essentially obvious synchronous Producer/Consumer that we implement in [7]. We shall very likely take advantage of parallel discrete events simulators as described in [8,9] in order to get rid of any explicit synchronization between the pairs of reversible-producer/classical-consumer that (5) would recursively generate when applied to an element in `R`.

We also plan to follow a more abstract line of research. The compilation scheme (5) recalls Girard's decomposition $A \rightarrow B \simeq !A \multimap B$ of a classical computation into a linear one that can erase/duplicate computational resources. Decomposing `recF[p, b, h]` in terms of `itFClS[b, h]` and `itFRev[p, pInv]` suggests that the relation between reversible and classical computations can be formalized by a linear isomorphism $A^n \multimap B^n$ between tensor products A^n , and B^n of A , and B , in analogy to [2]. Then we can think of recovering classical computations by some functor, say γ , whose purpose is, at least, to forget, or to inject replicas, of parts of A^n , and B^n in a way that $(\gamma A^n \rightarrow \gamma A^n) \uplus (\gamma A^n \leftarrow \gamma A^n)$ can be their type. The type says that we move from a reversible computation to a classical one by choosing which is input and which is output, so recovering the freedom to manage computational resources as we are used to when writing classical programs.

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