



Injective Colouring for H -Free Graphs

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Abstract. A function $c : V(G) \rightarrow \{1, 2, \dots, k\}$ is a k -colouring of a graph G if $c(u) \neq c(v)$ whenever u and v are adjacent. If any two colour classes induce the disjoint union of vertices and edges, then c is called injective. Injective colourings are also known as $L(1, 1)$ -labellings and distance 2-colourings. The corresponding decision problem is denoted INJECTIVE COLOURING. A graph is H -free if it does not contain H as an induced subgraph. We prove a dichotomy for INJECTIVE COLOURING for graphs with bounded independence number. Then, by combining known with further new results, we determine the complexity of INJECTIVE COLOURING on H -free graphs for every H except for one missing case.

1 Introduction

Graph colouring is a well-studied topic in Computer Science and Discrete Mathematics, both for theoretical and practical reasons. The classical variant is to give each vertex of a graph a colour in such a way that two adjacent vertices are not coloured alike while using as few colours as possible. Formally, a *colouring* of a graph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, 2, \dots\}$ such that $c(u) \neq c(v)$ for each pair of vertices u, v with $uv \in E$. If $c(V) \subseteq \{1, \dots, k\}$, then c is also called a k -colouring. The integer $c(u)$ is the *colour* of u , and the set of all vertices of the same colour i is a *colour class* of c . The problem COLOURING is to decide if a given graph G has a k -colouring for some given integer k . It is well-known that COLOURING is NP-complete even if $k = 3$.

We prove new results for a well-studied variant of graph colouring. We denote the disjoint union two graphs F and G by $F + G = (V(F) \cup V(G), E(F) \cup E(G))$; the disjoint union of s copies of G by sG and the n -vertex path by P_n . A colouring is *injective* if the union of any two colour classes of c induces an $sP_1 + tP_2$ for some integers $s \geq 0$ and $t \geq 0$. Injective colourings are also known as $L(1, 1)$ -labellings or *distance-2 colourings*. An equivalent formulation is to say that a colouring of a graph G is injective if all the neighbours of every vertex of G are coloured differently. The problem INJECTIVE COLOURING is to decide whether a given

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graph has an injective k -colouring for some given integer k . In the literature (see, for example, [6, 8, 10]) it is sometimes allowed that injective colourings give adjacent vertices the same colour. This is *in contrast to our paper*: we emphasize that all colourings in our paper are proper (as can be seen from their definitions).

It is known that INJECTIVE COLOURING is NP-complete for split graphs [1], unit disk graphs [15] and planar graphs [13], respectively. Moreover, INJECTIVE COLOURING is NP-complete for line graphs of bipartite graphs of girth at least g for any fixed integer $g \geq 3$ [14] and cubic graphs [7] even when $k = 4$. On the positive side, INJECTIVE COLOURING is polynomial-time solvable for graphs of bounded treewidth [16]. Injective colourings, viewed as $L(1, 1)$ -labellings, belong to the distance constrained labelling framework; we refer to the $L(h, k)$ -labelling survey of Calamoneri [3] for more algorithmic and structural results.

In the above results of [1, 7, 13–16] the input is restricted to some special graph class. Restricting the input is also the focus in our paper. Our goal is to obtain *complexity dichotomies*, which we compare with similar dichotomies for COLOURING. In particular we consider classes of H -free graphs, that is, graphs that do not contain some fixed graph H as an induced subgraph, or equivalently, cannot be modified into H by a sequence of vertex deletions. On a side note, the difference between the *chromatic number* χ and *injective chromatic number* χ_i of a graph can be arbitrarily large as illustrated by the $(n + 1)$ -vertex star $K_{1,n}$, which has chromatic number 2 but injective chromatic number $n + 1$.

1.1 Dichotomies for Colouring

Kráľ et al. [12] completely classified the complexity of COLOURING for H -free graphs. We write $H_1 \subseteq_i H_2$ to say that H_1 is an induced subgraph of H_2 .

Theorem 1 ([12]). *Let H be a graph. Then COLOURING for H -free graphs is polynomial-time solvable if $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$ and NP-complete otherwise.*

The complexity of k -COLOURING, the variant of COLOURING where k is *fixed* (in other words, not part of the input) has not been classified yet; in particular there are infinite families of open cases when H is a *linear forest* (disjoint union of paths); for example, the complexities of 3-COLOURING for P_t -free graphs for $t \geq 8$ and k -COLOURING for sP_3 -free graphs for $s \geq 2$ and $k \geq 4$ are still open; see also [4, 5, 11].

A set of vertices in a graph G is *independent* if each pair of vertices is non-adjacent. The *independence number* $\alpha(G)$ of G is the size of a largest independent set of G . The class of sP_1 -free graphs coincides with the class of graphs with $\alpha \leq s - 1$. Hence, Theorem 1 immediately implies the following dichotomy.

Theorem 2. *Let $s \geq 1$ be an integer. Then COLOURING for graphs with $\alpha \leq s$ is polynomial-time solvable if $s \leq 2$ but NP-complete if $s \geq 3$.*

The *complement* of a graph G is the graph \overline{G} with vertex set $V(G)$ and an edge between two distinct vertices u and v if and only if $uv \notin E(G)$. A k -colouring of G is a partition of $V(G)$ into at most k independent sets. Hence, a

(k -)colouring of G corresponds to a (k -)clique-covering of \overline{G} , which is a partition of $V(\overline{G}) = V(G)$ into (at most k) cliques. The *clique covering number* $\overline{\chi}(G)$ of G is the smallest number of cliques in a clique-covering of G . Note that $\overline{\chi}(G) = \chi(\overline{G})$. The following theorem follows from a known result of [9] and a standard trick (see Sect. 2).

Theorem 3. *Let $s \geq 1$ be an integer. Then COLOURING for graphs with $\overline{\chi} \leq s$ is polynomial-time solvable if $s \leq 2$ but NP-complete if $s \geq 3$.*

Note that Theorems 2 and 3 are incomparable: for $p \geq 2$, the class of graphs with $\alpha \leq p$ is a proper superclass of the class of graphs with $\overline{\chi} \leq p$. For example, as the class of K_3 -free graphs properly contains the class of bipartite graphs, the class of $3P_1$ -free graphs, that is, graphs with $\alpha \leq 2$, properly contains the class of co-bipartite graphs, that is, graphs with $\overline{\chi} \leq 2$. Hence, the polynomial part of Theorem 2 is stronger than the polynomial part of Theorem 3, and the reverse statement holds for the hardness parts of these two dichotomies.

1.2 Dichotomies for Injective Colouring

In contrast to the situation for k -COLOURING, the variant of INJECTIVE COLOURING where the number k of colours is fixed has been completely settled: in [2] we proved the following dichotomy for INJECTIVE k -COLOURING; note that INJECTIVE 3-COLOURING is polynomial-time solvable, as the only yes-instances are graphs of maximum degree 2 (disjoint unions of cycles and paths).

Theorem 4. *Let H be a graph and $k \geq 4$ be an integer. Then INJECTIVE k -COLOURING for H -free graphs is polynomial-time solvable if H is a linear forest and NP-complete otherwise.*

Our first result, proven in Sect. 2, is a similar dichotomy as Theorem 2.

Theorem 5. *Let $s \geq 1$ be an integer. Then INJECTIVE COLOURING for graphs with $\alpha \leq s$ is polynomial-time solvable if $s \leq 3$ but NP-complete if $s \geq 4$.*

Comparing Theorem 2 with Theorem 5, we see that the jump in complexity for COLOURING happens when $s = 3$ instead of $s = 4$. This has the following consequence. It is well known and easy to see that an injective colouring of a graph G is a colouring of the *square* G^2 of G . The latter graph is obtained from G by adding an edge between every pair of non-adjacent vertices in G that have a common neighbour in G . If G has $\alpha \leq 3$, then G^2 has $\alpha \leq 3$ as well. However, COLOURING for graphs with $\alpha \leq 3$ is NP-complete by Theorem 2. Hence, we do not have any polynomial-time algorithm for COLOURING that we can use. Instead of this, we will develop a direct approach to obtain polynomial-time solvability of INJECTIVE COLOURING for $4P_1$ -free graphs (on a side note, it can be observed that the hard instances with $\alpha = 3$ of COLOURING are not squares of graphs).

The hardness part of Theorem 5 follows from the hardness part of the following analogue of Theorem 3 for INJECTIVE COLOURING, which we prove in Sect. 2.

Theorem 6. *Let $s \geq 1$ be an integer. Then INJECTIVE COLOURING for graphs with $\bar{\chi} \leq s$ is polynomial-time solvable if $s \leq 3$ but NP-complete if $s \geq 4$.*

In [2] we combined known results from [1, 14] with some new results to obtain a partial dichotomy for INJECTIVE COLOURING where we left open ten cases of linear forests H , including the cases $H = 4P_1$ and $H = 5P_1$, which we now have solved (these cases correspond to graphs with $\alpha \leq 3$ and $\alpha \leq 4$, respectively, so follow from Theorem 5). In this paper we solve a total of nine open cases. In Sect. 3 we use our result for $4P_1$ -free graphs from Theorem 5 to obtain new polynomial-time algorithms for two superclasses of $4P_1$ -free graphs, namely when $H = 2P_1 + P_3$ and $H = 3P_1 + P_2$. In the same section we also prove polynomial-time solvability if $H = P_1 + P_4$. As a consequence of our results we can update the partial dichotomy of [2] as follows, leaving only one missing case.

Theorem 7. *Let $H \neq 2P_1 + P_4$ be a graph. Then INJECTIVE COLOURING for H -free graphs is polynomial-time solvable if $H \subseteq_i P_1 + P_4$ or $H \subseteq_i 2P_1 + P_3$ or $H \subseteq_i 3P_1 + P_2$ and NP-complete otherwise.*

Compared to Theorem 1 we notice that INJECTIVE COLOURING is polynomial-time solvable for larger classes of H -free graphs than COLOURING. We prove Theorem 7 in Sect. 3 as well and state some relevant open problems in Sect. 4.

2 The Proofs of Theorems 3, 5 and 6

An injective colouring c of a graph G is *optimal* if G has no injective colouring using fewer colours than c . An injective colouring c is ℓ -*injective* if every colour class of c has size at most ℓ . An ℓ -injective colouring c of a graph G is ℓ -*optimal* if G has no ℓ -injective colouring that uses fewer colours than c . We start with a useful lemma for the case where $\ell = 2$ that we will also use in the next section.

Lemma 1. *An optimal 2-injective colouring of a graph G can be found in polynomial time.*

Proof. Let c be a 2-injective colouring of G . Then each colour class of size 2 in G corresponds to a *dominating* edge of \overline{G} (an edge uv of a graph is dominating if every other vertex in the graph is adjacent to at least one of u, v). Hence, the end-vertices of every non-dominating edge in \overline{G} have different colours in G . Algorithmically, this means we may delete every non-dominating edge of \overline{G} from \overline{G} ; note that we do not delete the end-vertices of such an edge.

Let μ^* be the size of a maximum matching in the graph obtained from \overline{G} after deleting all non-dominating edges of \overline{G} . The edges in such a matching will form exactly the colour classes of size 2 of an optimal 2-injective colouring of G . Hence, the injective chromatic number of G is equal to $\mu^* + (|V(G)| - 2\mu^*)$. It remains to observe that we can find a maximum matching in a graph in polynomial time by using a standard algorithm. \square

We are now ready to prove the following result.

Lemma 2. INJECTIVE COLOURING is polynomial-time solvable for $4P_1$ -free graphs, or equivalently, graphs with $\alpha \leq 3$, and thus for graphs with $\bar{\chi} \leq 3$.

Proof. Let $G = (V, E)$ be a $4P_1$ -free graph on n vertices. We first analyze the structure of injective colourings of G . Let c be an optimal injective colouring of G . As G is $4P_1$ -free, every colour class of c has size at most 3. From all optimal injective colourings, we choose c such that the number of size-3 colour classes is as small as possible. We say that c is *class-3-optimal*.

Suppose c contains a colour class of size 3, say colour 1 appears on three distinct vertices u_1, u_2 and u_3 of G . As G is $4P_1$ -free, $\{u_1, u_2, u_3\}$ dominates G . As c is injective, this means that every vertex in $G - \{u_1, u_2, u_3\}$ is adjacent to exactly one vertex of $\{u_1, u_2, u_3\}$. Hence, we can partition $V \setminus \{u_1, u_2, u_3\}$ into three sets T_1, T_2 and T_3 , such that for $i \in \{1, 2, 3\}$, every vertex of T_i is adjacent to u_i and not to any other vertex of $\{u_1, u_2, u_3\}$. If two vertices t, t' in the same T_i , say T_1 , are non-adjacent, then $\{t, t', u_2, u_3\}$ induces a $4P_1$, a contradiction. Hence, we partitioned V into three cliques $T_i \cup \{u_i\}$. We call the cliques T_1, T_2, T_3 , the *T-cliques* of the triple $\{u_1, u_2, u_3\}$.

Let $t \in T_i$ for some $i \in \{1, 2, 3\}$. For $i \in \{0, 1, 2\}$ we say that t is *i-clique-adjacent* if t has a neighbour in zero, one or two cliques of $\{T_1, T_2, T_3\} \setminus T_i$, respectively. By the definition of an injective colouring and the fact that every T_i is a clique, a 1-clique-adjacent vertex of $T_1 \cup T_2 \cup T_3$ belongs to a colour class of size at most 2, and a 2-clique-adjacent vertex of $T_1 \cup T_2 \cup T_3$ belongs to a colour class of size 1. Hence, all the vertices that belong to a colour class of size 3 are 0-clique-adjacent. The partition of $V(G)$ is illustrated in Fig. 1.

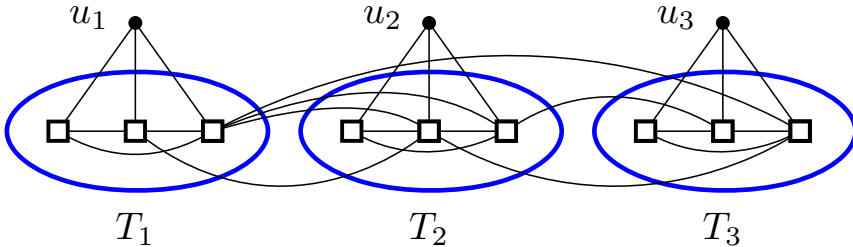


Fig. 1. The partition of $V(G)$ from Lemma 2. The squares inside each $T_i, i \in \{1, 2, 3\}$, represent the sets of 0-clique-adjacent, 1-clique-adjacent and 2-clique-adjacent vertices in T_i , respectively.

We now use the fact that c is class-3-optimal. Let $t \in V \setminus \{u_1, u_2, u_3\}$, say $t \in T_1$, be i -clique-adjacent for $i = 0$ or $i = 1$. Then we may assume without loss of generality that t has no neighbours in T_2 . If t belongs to a colour class of size 1, then we can set $c(u_2) := c(t)$ to obtain an optimal injective colouring with fewer size-3 colour classes, contradicting our choice of c .

We now consider the 0-clique-adjacent vertices again. Recall that these are the only vertices, other than u_1, u_2 and u_3 , that may belong to a colour class of

size 3. As every T_i is a clique, every colour class of size 3 (other than $\{u_1, u_2, u_3\}$) has exactly one vertex of each T_i . Let $\{w_1, w_2, w_3\}$ be another colour class of size 3 with $w_i \in T_i$ for every $i \in \{1, 2, 3\}$. Let $x \in T_1 \setminus \{w_1\}$ be another 0-clique-adjacent vertex. Then swapping the colours of w_1 and x yields another class-3-optimal injective colouring of G . Hence, we derived the following claim, which summarizes the discussion above and where statement (iv) follows from (i)–(iii).

Claim. *Let c be a class-3-optimal injective colouring of G with $c(u_1) = c(u_2) = c(u_3)$ for three distinct vertices u_1, u_2, u_3 and with $p \geq 0$ other colour classes of size 3. Then the following four statements hold:*

- (i) *All 0-clique-adjacent and 1-clique-adjacent vertices belong to a colour class of size at least 2.*
- (ii) *Let $S = \{y_1, \dots, y_s\}$ be the set of 2-clique-adjacent vertices. Then $\{y_1\}, \dots, \{y_s\}$ are exactly the size-1 colour classes.*
- (iii) *For $i \in \{1, 2, 3\}$, let $x_1^i, \dots, x_{q_i}^i$ be the 0-clique-adjacent vertices of T_i and assume without loss of generality that $q_1 \leq q_2 \leq q_3$. Then $p \leq q_1$ and if $p \geq 1$, we may assume without loss of generality that the size-3 classes, other than $\{u_1, u_2, u_3\}$, are $\{x_1^1, x_1^2, x_1^3\}, \dots, \{x_p^1, x_p^2, x_p^3\}$.*
- (iv) *The number of colours used by c , or equivalently, the number of colour classes of c is equal to $1 + s + p + \frac{1}{2}(n - s - 3(p + 1)) = \frac{1}{2}n + \frac{1}{2}s - \frac{1}{2}p - \frac{1}{2}$.*

We are now ready to present our algorithm. We first find, in polynomial time, an optimal 2-injective colouring of G by Lemma 1. We remember the number of colours used. Recall that the colour classes of every injective colouring of G have size at most 3. So, it remains to compute an optimal injective colouring for which at least one colour class has size 3.

We consider each triple u_1, u_2, u_3 of vertices of G and check if $\{u_1, u_2, u_3\}$ can be a colour class. That is, we check if $\{u_1, u_2, u_3\}$ is an independent set and has corresponding T -cliques T_1, T_2, T_3 . This takes polynomial time. If not, then we discard $\{u_1, u_2, u_3\}$. Otherwise we continue as follows. Let $S = \{y_1, \dots, y_s\}$ be the set of 2-clique adjacent vertices in $T_1 \cup T_2 \cup T_3$. Exactly the vertices of S will form the size-1 colour classes by Claim (ii). For $i \in \{1, 2, 3\}$, let $x_1^i, \dots, x_{q_i}^i$ be the 0-clique-adjacent vertices of T_i , where we assume without loss of generality that $q_1 \leq q_2 \leq q_3$. By Claim (iii), any injective colouring of G which has $\{u_1, u_2, u_3\}$ as one of its colour classes has at most q_1 other colour classes of size 3 besides $\{u_1, u_2, u_3\}$. As can be seen from Claim (iv), the value $\frac{1}{2}n + \frac{1}{2}s - \frac{1}{2}p - \frac{1}{2}$ is minimized if the number p of size-3 colour classes is maximum.

From the above we can now do as follows. For $p = q_1, \dots, 1$, we check if G has an injective colouring with exactly p colour classes of size 3. We stop as soon as we find a yes-answer or if p is set to 0. We first set $\{x_1^1, x_1^2, x_1^3\}, \dots, \{x_p^1, x_p^2, x_p^3\}$ as the colour classes of size 3 by Claim (iii). Let Z be the set of remaining 0-clique-adjacent and 1-clique-adjacent vertices. We use Lemma 1 to check in polynomial time if the subgraph of G induced by $S \cup Z$ has an injective colouring that uses $s + \frac{1}{2}(n - s - 3(p + 1))$ colours (which is the minimum number of colours

possible). If so, then we stop and note that after adding the size-3 colour classes we obtained an injective colouring of G that uses $\frac{1}{2}n + \frac{1}{2}s - \frac{1}{2}p - \frac{1}{2}$ colours, which we remember. Otherwise we repeat this step after first setting $p := p - 1$.

As the above procedure for a triple u_1, u_2, u_3 takes polynomial time and the number of triples we must check is $O(n^3)$, our algorithm runs in polynomial time. We take the 3-injective colouring that uses the smallest number of colours and compare it with the number of colours used by the optimal 2-injective colouring that we computed at the start. Our algorithm then returns a colouring with the smallest of these two values as its output. \square

We now show two hardness results.

Lemma 3. COLOURING is NP-complete for graphs with $\bar{\chi} \leq 3$.

Proof. The LIST COLOURING problem takes as input a graph G and a list assignment L that assigns each vertex $u \in V(G)$ a list $L(u) \subseteq \{1, 2, \dots\}$. The question is whether G admits a colouring c with $c(u) \in L(u)$ for every $u \in V(G)$. Jansen [9] proved that LIST COLOURING is NP-complete for unions of two complete graphs. This is the problem we reduce from.

Let G be a graph with a list assignment L and assume that $V(G)$ can be split into two (not necessarily disjoint) cliques K and K' . We set $A_1 := K$ and $A_2 := K \setminus K'$. As both A_1 and A_2 are cliques, we have that $\bar{\chi}(G) \leq 2$. We may assume without loss of generality that the union of all the lists $L(u)$ is $\{1, \dots, k\}$ for some integer k . We now extend G by adding a clique A_3 of k new vertices v_1, \dots, v_k and by adding an edge between a vertex x_ℓ and a vertex $u \in V(G)$ if and only if $\ell \notin L(u)$. This yields a new graph G' with $\bar{\chi}(G') \leq 3$. It is readily seen that G has a colouring c with $c(u) \in L(u)$ for every $u \in V(G)$ if and only if G' has a k -colouring. \square

We use Lemma 3 to prove the next lemma, which uses the same arguments as the proof of our NP-hardness result for $6P_1$ -free graphs [2]. The only difference is that we now reduce from COLOURING for graphs with $\bar{\chi} \leq 3$ instead of COLOURING for graphs with $\bar{\chi} \leq 4$ as we did in [2].

Lemma 4. INJECTIVE COLOURING is NP-complete for graphs with $\bar{\chi} \leq 4$, and thus for $5P_1$ -free graphs, or equivalently, graphs with $\alpha \leq 4$.

We are now ready to prove three theorems.

Proofs of Theorems 3, 5 and 6. Theorem 3 follows from combining Theorem 1, after observing that graphs with $\bar{\chi} \leq 2$ are $3P_1$ -free and thus $(P_1 + P_3)$ -free, with Lemma 3, whereas Theorems 5 and 6 follow from combining Lemmas 2 and 4.

3 The Proof of Theorem 7

In this section we prove our final theorem. We start by showing three new polynomial results. We shall use Lemma 2, on $4P_1$ -free graphs, in the proofs of Lemmas 6 and 7.

Lemma 5. INJECTIVE COLOURING is polynomial-time solvable for $(P_1 + P_4)$ -free graphs.

Proof. Let G be a $(P_1 + P_4)$ -free graph. Since connected P_4 -free graphs have diameter at most 2, no two vertices can be coloured alike in an injective colouring. Hence, the injective chromatic number of a P_4 -free graph is equal to the number of its vertices. Consequently, INJECTIVE COLOURING is polynomial-time solvable for P_4 -free graphs. From now on, we assume that G is not P_4 -free.

We first show that any colour class in any injective colouring of G has size at most 2. For contradiction, assume that c is an injective colouring of G such that there exists some colour, say colour 1, that has a colour class of size at least 3. Let $P = x_1x_2x_3x_4$ be some induced P_4 of G .

We first consider the case where colour 1 appears at least twice on P . As no vertex has two neighbours coloured with the same colour, the only way in which this can happen is when $c(x_1) = c(x_4) = 1$. By our assumption, $G - P$ contains a vertex u with $c(u) = 1$. As G is $(P_1 + P_4)$ -free, u has a neighbour on P . As every colour class is an independent set, this means that u must be adjacent to at least one of x_2 and x_3 . Consequently, either x_2 or x_3 has two neighbours with colour 1, a contradiction.

Now we consider the case where colour 1 appears exactly once on P , say $c(x_h) = 1$ for some $h \in \{1, 2, 3, 4\}$. Then, by our assumption, $G - P$ contains two vertices u_1 and u_2 with colour 1. As G is $(P_1 + P_4)$ -free, both u_1 and u_2 must be adjacent to at least one vertex of P , say u_1 is adjacent to x_i and u_2 is adjacent to x_j . Then $x_i \neq x_j$, as otherwise G has a vertex with two neighbours coloured 1. As every colour class is an independent set, we have that $x_h \notin \{x_i, x_j\}$, and hence, x_h, x_i, x_j are distinct vertices. Moreover, x_h is not a neighbour of x_i or x_j , as otherwise x_i or x_j has two neighbours coloured 1. Hence, we may assume without loss of generality that $h = 1, i = 3$ and $j = 4$. As every colour class is an independent set, u_1 and u_2 are non-adjacent. However, now $\{x_1, u_1, x_3, x_4, u_2\}$ induces a $P_1 + P_4$, a contradiction.

Finally, we consider the case where colour 1 does not appear on P . Let u_1, u_2, u_3 be three vertices of $G - P$ coloured 1. As before, $\{u_1, u_2, u_3\}$ is an independent set and each u_i has a different neighbour on P . We first consider the case where x_1 or x_4 , say x_4 is not adjacent to any u_i . Then we may assume without loss of generality that u_1x_1 and u_2x_2 are edges. However, now $\{x_4, u_1, x_1, x_2, u_2\}$ induces a $P_1 + P_4$, which is not possible. Hence, we may assume without loss of generality that u_1x_1, u_2x_2 and u_4x_4 are edges of G . Again we find that $\{x_4, u_1, x_1, x_2, u_2\}$ induces a $P_1 + P_4$, a contradiction.

From the above, we find that each colour class in an injective colouring of G has size at most 2. This means we can use Lemma 1. \square

Lemma 6. INJECTIVE COLOURING is polynomial-time solvable for $(2P_1 + P_3)$ -free graphs.

Proof. Let $G = (V, E)$ be a $(2P_1 + P_3)$ -free graph. We may assume without loss of generality that G is connected and by Lemma 2 that G has an induced $4P_1$. We first show that any colour class in any injective colouring of G has size at

most 2. For contradiction, assume that c is an injective colouring of G such that there exists some colour, say colour 1, that has a colour class of size at least 3. Let $U = \{u_1, \dots, u_p\}$ for some $p \geq 3$ be the set of vertices of G with $c(u_i) = 1$ for $i \in \{1, \dots, p\}$.

As c is injective, every vertex in $G - U$ has at most one neighbour in U . Hence, we can partition $G - U$ into (possibly empty) sets T_0, \dots, T_p , where T_0 is the set of vertices with no neighbour in U and for $i \in \{1, \dots, p\}$, T_i is the set of vertices of $G - U$ adjacent to u_i .

We first claim that T_0 is empty. For contradiction, assume $v \in T_0$. As G is connected, we may assume without loss of generality that v is adjacent to some vertex $t \in T_1$. Then $\{u_2, u_3, u_1, t, v\}$ induces a $2P_1 + P_3$, a contradiction. Hence, $T_0 = \emptyset$.

We now prove that every T_i is a clique. For contradiction, assume that t and t' are non-adjacent vertices of T_1 . Then $\{u_2, u_3, t, u_1, t'\}$ induces a $2P_1 + P_3$, a contradiction. Hence, every T_i and thus every $T_i \cup \{u_i\}$ is a clique.

We now claim that $p = 3$. For contradiction, assume that $p \geq 4$. As G is connected and U is an independent set, we may assume without loss of generality that there exist vertices $t_1 \in T_1$ and $t_2 \in T_2$ with $t_1 t_2 \in E$. Then $\{u_3, u_4, u_1, t_1, t_2\}$ induces a $2P_1 + P_3$, a contradiction. Hence, $p = 3$.

Now we know that V can be partitioned into three cliques $T_1 \cup \{u_1\}$, $T_2 \cup \{u_2\}$ and $T_3 \cup \{u_3\}$. However, then G is $4P_1$ -free, a contradiction. We conclude that every colour class of every injective colouring of G has size at most 2. This means we can use Lemma 1. \square

Lemma 7. INJECTIVE COLOURING is polynomial-time solvable for $(3P_1 + P_2)$ -free graphs.

Proof. Let G be a $(3P_1 + P_2)$ -free graph on n vertices. We may assume without loss of generality that G is connected and by Lemma 2 that G has an induced $4P_1$. As before, we will first analyze the structure of injective colourings of G . We will then exploit the properties found algorithmically.

Let c be an injective colouring of G that has a colour class U of size at least 3. So let $U = \{u_1, \dots, u_p\}$ for some $p \geq 3$ be the set of vertices of G with, say colour 1. As c is injective, every vertex in $G - U$ has at most one neighbour in U . Hence, we can partition $G - U$ into (possibly empty) sets T_0, \dots, T_p , where T_0 is the set of vertices with no neighbour in U and for $i \in \{1, \dots, p\}$, T_i is the set of vertices of $G - U$ adjacent to u_i .

Assume that $p \geq 4$. As G is connected, there exists a vertex $v \notin U$ but that has a neighbour in U , say $v \in T_1$. Then $\{u_2, u_3, u_4, u_1, v\}$ induces a $3P_1 + P_2$, a contradiction. Hence, we have shown the following claim.

Claim 1. Every injective colouring of G is ℓ -injective for some $\ell \in \{1, 2, 3\}$.

We continue as follows. As $p = 3$ by Claim 1, we have $V(G) = U \cup T_0 \cup T_1 \cup T_2 \cup T_3$. Suppose T_0 contains two adjacent vertices x and y . Then $\{u_1, u_2, u_3, x, y\}$ induces a $3P_1 + P_2$, a contradiction. Hence, T_0 is an independent set. As G is connected, this means each vertex in T_0 has a neighbour in $T_1 \cup T_2 \cup T_3$.

Suppose T_0 contains two vertices x and y with the same colour, say $c(x) = c(y) = 2$. Let $v \in T_1 \cup T_2 \cup T_3$, say $v \in T_1$ be a neighbour of x . Then, as $c(x) = c(y)$ and c is injective, v is not adjacent to y . As T_0 is independent, x and y are not adjacent. However, now $\{u_2, u_3, y, x, v\}$ induces a $3P_1 + P_2$, a contradiction. Hence, every vertex in T_0 has a unique colour. Suppose T_0 contains a vertex x and $T_1 \cup T_2 \cup T_3$ contains a vertex v such that $c(x) = c(v)$. We may assume without loss of generality that $v \in T_1$. Then $\{u_2, u_3, x, v, u_1\}$ induces a $3P_1 + P_2$, a contradiction.

Finally, suppose that $T_1 \cup T_2 \cup T_3$ contain two distinct vertices v and v' with $c(v) = c(v')$. Let $x \in T_0$. Then x is not adjacent to at least one of v, v' , say $xv \notin E$ and also assume that $v \in T_1$. Then $\{u_2, u_3, x, v, u_1\}$ induces a $3P_1 + P_2$. Hence, we have shown the following claim.

Claim 2. If c is 3-injective and U is a size-3 colour class such that G has a vertex not adjacent to any vertex of U , then all colour classes not equal to U have size 1.

We note that the injective colouring c in Claim 2 uses $n - 2$ distinct colours.

We continue as follows. From now on we assume that $T_0 = \emptyset$. Every T_i is $(P_1 + P_2)$ -free, as otherwise, if say T_1 contains an induced $P_1 + P_2$, then this $P_1 + P_2$, together with u_2 and u_3 , forms an induced $3P_1 + P_2$, which is not possible. Hence, each T_i induces a complete r_i -partite graph for some integer r_i (that is, the complement of a disjoint union of r_i complete graphs). Hence, we can partition each T_i into r_i independent sets $T_i^1, \dots, T_i^{r_i}$ such that there exists an edge between every vertex in T_i^a and every vertex in T_i^b if $a \neq b$. See also Fig. 2.

Suppose G contains another colour class of size 3, say v_1, v_2 and v_3 are three distinct vertices coloured 2. If two of these vertices, say v_1 and v_2 , belong to the same T_i , say T_1 , then u_1 has two neighbours with the same colour. This is not possible, as c is injective. Hence, we may assume without loss of generality that $v_i \in T_i^1$ for $i \in \{1, 2, 3\}$.

Suppose that T_1^2 contains two vertices s and t . Then, as s and t are adjacent to v_1 , both of them are not adjacent to v_2 (recall that $c(v_1) = c(v_2)$ and c is injective). Hence, $\{s, t, u_3, v_2, u_2\}$ induces a $3P_1 + P_2$ (see Fig. 2). We conclude that for every $i \in \{1, 2, 3\}$, the sets $T_i^2, \dots, T_i^{r_i}$ have size 1.

We will now make use of the fact that G contains an induced $4P_1$. We note that each $T_i \cup \{u_i\}$ is a clique, unless $|T_i^1| \geq 2$. As $V(G) = T_1 \cup T_2 \cup T_3 \cup \{u_1, u_2, u_3\}$ and G contains an induced $4P_1$, we may assume without loss of generality that T_1^1 has size at least 2. Recall that $v_1 \in T_1^1$. Let $z \neq v_1$ be some further vertex of T_1^1 . If z is not adjacent to v_2 , then $\{z, v_1, u_3, v_2, u_2\}$ induces a $3P_1 + P_2$, which is not possible. Hence, z is adjacent to v_2 . For the same reason, z is adjacent to v_3 . This is not possible, as c is injective and v_2 and v_3 both have colour 2. Hence, we have proven the following claim.

Claim 3. If c is 3-injective and U is a size-3 colour class such that each vertex of $G - U$ is adjacent to a vertex of U , then c has no other colour class of size 3.

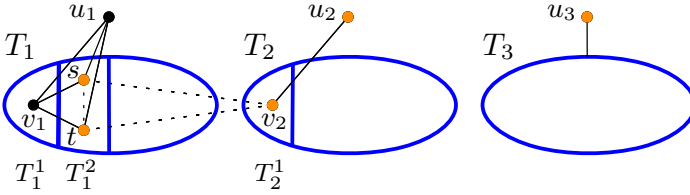


Fig. 2. The situation in Lemma 7 where T_1^2 contains two vertices s and t . We show that this situation cannot happen, as it would lead to a forbidden induced $3P_1 + P_2$. Note that each u_i is adjacent to all vertices of T_i and not to any vertices of T_j for $j \neq i$. There may exist edges between vertices of different sets, but these are not drawn.

We are now ready to present our polynomial-time algorithm. We first use Lemma 1 to find in polynomial time an optimal 2-injective colouring of G . We remember the number of colours it uses.

By Claim 1, it remains to find an optimal 3-injective colouring with at least one colour class of size 3. We now consider each set $\{u_1, u_2, u_3\}$ of three vertices. We discard our choice if u_1, u_2, u_3 do not form an independent set or if $V(G) \setminus \{u_1, u_2, u_3\}$ cannot be partitioned into sets T_0, \dots, T_4 as described above. Suppose we have not discarded our choice of vertices u_1, u_2, u_3 . We continue as follows.

If $T_0 \neq \emptyset$, then by Claim 2 the only 3-injective colouring of G (subject to colour permutation) with colour class $\{u_1, u_2, u_3\}$ is the colouring that gives each u_i the same colour and a unique colour to all the other vertices of G . This colouring uses $n - 2$ colours and we remember this number of colours.

Now suppose $T_0 = \emptyset$. By Claim 3, we find that $\{u_1, u_2, u_3\}$ is the only colour class of size 3. Recall that no vertex in $G - \{u_1, u_2, u_3\} = T_1 \cup T_2 \cup T_3$ is adjacent to more than one vertex of $\{u_1, u_2, u_3\}$. Hence, we can apply Lemma 1 on $G - \{u_1, u_2, u_3\}$. This yields an optimal 2-injective colouring of $G - \{u_1, u_2, u_3\}$. We colour u_1, u_2, u_3 with the same colour and choose a colour that is not used in the colouring of $G - \{u_1, u_2, u_3\}$. This yields a 3-injective colouring of G that is optimal over all 3-injective colourings with colour class $\{u_1, u_2, u_3\}$. We remember the number of colours.

As the above procedure takes polynomial time and there are $O(n^3)$ triples to consider, we find in polynomial time an optimal 3-injective colouring of G that has at least one colour class of size 3 (should it exist). We compare the number of colours used with the number of colours of the optimal 2-injective colouring of G that we found earlier. Our algorithm returns the minimum of the two values as the output. Since both colourings are found in polynomial time, we conclude that our algorithm runs in polynomial time. \square

We are now ready to prove Theorem 7.

Theorem 7 (restated). *Let $H \neq 2P_1 + P_4$ be a graph. Then INJECTIVE COLOURING for H -free graphs is polynomial-time solvable if $H \subseteq_i P_1 + P_4$ or $H \subseteq_i 2P_1 + P_3$ or $H \subseteq_i 3P_1 + P_2$ and NP-complete otherwise.*

Proof. The problem is readily seen to be in NP. Let C_n denote the n -vertex cycle. In [2], we proved that INJECTIVE 4-COLOURING is NP-complete for C_3 -free graphs (in fact, the proof of [2] shows NP-completeness for bipartite graphs). Hence, if $C_3 \subseteq_i H$, then INJECTIVE COLOURING is NP-complete for H -free graphs. If $C_p \subseteq_i H$ for some $p \geq 4$ or $K_{1,3} \subseteq_i H$, then we use a result of Mahdian [14], namely that for every $g \geq 4$ and $k \geq 4$, INJECTIVE k -COLOURING is NP-complete for line graphs of bipartite graphs of girth at least g : by setting $g = p + 1$ we obtain a class of $(C_p, K_{1,3})$ -free graphs.

In the remaining case H is a linear forest. If $5P_1 \subseteq_i H$, then we use Lemma 4. If $2P_2 \subseteq_i H$, then we obtain a superclass of $(2P_2, C_4, C_5)$ -free graphs (split graphs) for which Bodlaender et al. [1] proved NP-completeness. In all other cases, $H \subseteq_i P_1 + P_4$, $H \subseteq_i 2P_1 + P_3$ or $H \subseteq_i 3P_1 + P_2$, and we use Lemma 5, 6 or 7, respectively. \square

4 Conclusions

We extended the partial classification of INJECTIVE COLOURING for H -free graphs in [2] to a new classification that leaves open only the case where $H = 2P_1 + P_4$. Note that each of the graphs $P_1 + P_4$, $2P_1 + P_3$ and $3P_1 + P_2$ is an induced subgraph of $2P_1 + P_4$. Our strategy for solving the cases where $H = 3P_1 + P_2$ or $H = 2P_1 + P_3$ is based on the presence of an induced $4P_1$; as otherwise we can use our algorithm for $4P_1$ -free graphs from Lemma 2. We then argue in each of the two cases that the number of size-3 colour classes is small. However, such an approach no longer works for the unknown case where $H = 2P_1 + P_4$, as there exist connected $(2P_1 + P_4)$ -free graphs with an induced $4P_1$ that have an arbitrarily large number of size-3 colour classes. On the other hand, the algorithms for the cases where $H \in \{P_1 + P_4, 2P_1 + P_3, 3P_1 + P_2\}$ might be useful for a polynomial-time algorithm for the case $H = 2P_1 + P_4$ (should such an algorithm exist). However, we first need to understand the structure of these graphs better and leave the case where $H = 2P_1 + P_4$ for future work.

Open Problem 1. *Determine the complexity of INJECTIVE COLOURING for $(2P_1 + P_4)$ -free graphs.*

We recall that INJECTIVE 3-COLOURING is polynomial-time solvable for general graphs and that in [2] we proved that, for every $k \geq 4$, INJECTIVE k -COLOURING is NP-complete for bipartite graphs and thus for graphs of girth at least 4. Mahdian [14] proved that for every $g \geq 4$ and $k \geq 4$, INJECTIVE EDGE k -COLOURING is NP-complete for bipartite graphs of girth at least g . For the vertex variant, such a result does not seem to be known (except when $g = 4$). Hence, we pose the following challenging but highly interesting open problem (see also [2]).

Open Problem 2. *For every $g \geq 5$, determine the complexity of INJECTIVE COLOURING and INJECTIVE k -COLOURING ($k \geq 4$) for graphs of girth at least g .*

References

1. Bodlaender, H.L., Kloks, T., Tan, R.B., van Leeuwen, J.: Approximations for lambda-colorings of graphs. *Comput. J.* **47**, 193–204 (2004)
2. Bok, J., Jedličková, N., Martin, B., Paulusma, D., Smith, S.: Acyclic colouring, star colouring and injective colouring for H -free graphs. In: *Proceedings ESA 2020, LIPIcs*, vol. 173, pp. 22:1–22:22 (2020)
3. Calamoneri, T.: The $L(h, k)$ -labelling problem: an updated survey and annotated bibliography. *Comput. J.* **54**, 1344–1371 (2011)
4. Chudnovsky, M., Huang, S., Spirkl, S., Zhong, M.: List-three-coloring graphs with no induced $P_6 + rP_3$. *CoRR*, abs/1806.11196 (2018)
5. Golovach, P.A., Johnson, M., Paulusma, D., Song, J.: A survey on the computational complexity of colouring graphs with forbidden subgraphs. *J. Graph Theory* **84**, 331–363 (2017)
6. Hahn, G., Kratochvíl, J., Širáň, J., Sotteau, D.: On the injective chromatic number of graphs. *Discret. Math.* **256**, 179–192 (2002)
7. Heggenes, P., Telle, J.A.: Partitioning graphs into generalized dominating sets. *Nordic J. Comput.* **5**, 128–142 (1998)
8. Hell, P., Raspaud, A., Stacho, J.: On injective colourings of chordal graphs. In: *Laber, E.S., Bornstein, C., Nogueira, L.T., Faria, L. (eds.) LATIN 2008. LNCS*, vol. 4957, pp. 520–530. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-78773-0_45
9. Jansen, K.: Complexity Results for the Optimum Cost Chromatic Partition Problem, Universität Trier, Mathematik/Informatik, Forschungsbericht, pp. 96–41 (1996)
10. Jin, J., Baogang, X., Zhang, X.: On the complexity of injective colorings and its generalizations. *Theoret. Comput. Sci.* **491**, 119–126 (2013)
11. Klímošová, T., Malík, J., Masařík, T., Novotná, J., Paulusma, D., Slívová, V.: Colouring $(P_r + P_s)$ -free graphs. In: *Proceedings ISAAC 2018, LIPIcs*, pp. 123:5:1–5:13 (2018)
12. Král', D., Kratochvíl, J., Tuza, Z., Woeginger, G.J.: Complexity of coloring graphs without forbidden induced subgraphs. In: *Brandstädt, A., Le, V.B. (eds.) WG 2001. LNCS*, vol. 2204, pp. 254–262. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-45477-2_23
13. Lloyd, E.L., Ramanathan, S.: On the complexity of distance-2 coloring. *Proc. ICCI* **1992**, 71–74 (1992)
14. Mahdian, M.: On the computational complexity of strong edge coloring. *Discret. Appl. Math.* **118**, 239–248 (2002)
15. Sen, A., Huson, M.L.: A new model for scheduling packet radio networks. *Wireless Netw.* **3**, 71–82 (1997)
16. Zhou, X., Kanari, Y., Nishizeki, T.: Generalized vertex-coloring of partial k -trees. *IEICE Trans. Fundamentals Electron. Commun. Comput. Sci.* **E83-A**, 671–678 (2000)