Chapter 8
A Derivation of the Beam Theory of Second-Order with Shear, Starting from a Continuum Mechanics-Based Extension of the Reissner Finite-Strain Beam Theory

Hans Irschik

Abstract A consistent derivation of the beam theory of second order with shear is presented. The geometrically exact Reissner finite-strain beam theory is taken as the starting point, utilizing a continuum mechanics-based extension with respect to stress–strain constitutive formulations. Corresponding incremental relations for small deformations superimposed upon an intermediate configuration with (possibly) finite deformations are presented, from which the second-order beam theory with shear eventually is derived using two slight approximations.

8.1 Introduction

The present contribution is concerned with a consistent derivation of the beam theory of second order, taking into account the effect of shear, see Rubin and Vogel [9], Rubin and Schneider [8]. The increase of the practical usage of the beam theory of second order generally is due to both, economic and safety reasons, e.g. Petersen [6] for steel structures. However, the fundamental relations of the beam theory of second order have been stated in a somewhat ad hoc manner in the literature. Our present derivation attempts to provide a consistent connection to non-linear structural and continuum mechanics. From space restrictions, we consider plane deformations of originally straight shear-deformable beams, utilizing the Timoshenko hypothesis of cross sections remaining plane and un-deformed in the deformed configuration, but, due to the effect of shear, not necessarily perpendicular to the deformed beam axis, see Ziegler [12]. The geometrically exact Reissner finite-strain beam theory [7] is taken as a starting point. In Sect. 8.2 below, we shortly recall the corresponding fundamental relations of this theory, which is geometrically exact within the Timo-

H. Irschik (✉)
Institute of Technical Mechanics, Johannes Kepler University Linz, Linz, Austria
e-mail: hans.irschik@jku.at

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shenko hypothesis. Using virtual work considerations, Reissner in [7] showed that the constitutive relations for the stress resultants must be formulated as functions of certain generalized strains. In the Reissner finite-strain theory [7], the mathematical form of the constitutive relations however needs to be stipulated. In order to overcome this problem, Irschik and Gerstmayr [5] presented an extension of the Reissner finite-strain beam theory, where local stress–strain relations can be utilized, the formulation being also geometrically exact within the Timoshenko hypothesis. A three-dimensional hyperelastic stress-strain constitutive relation, dating back to Ciarlet [1] and proposed by Simo and Hughes [10], was treated exemplarily in [5], see Sect. 8.3 below for an overview. Based on this continuum mechanics-based extension, the case of (infinitesimally) small deformations superimposed upon an intermediate configuration with (possibly) large deformations are studied in Sect. 8.4, where a method by DaDeppo [2] for beams rigid in shear is adopted, see Irschik [3] for the Reissner finite-strain beam theory. Correspondingly, the fundamental relations are differentiated with respect to a non-dimensional generalized time, e.g. a characteristic load parameter, from which the incremental (rate) forms of the equilibrium relations, of the kinematic relations, as well as of the hyperelastic constitutive beam relations are obtained directly. These relations form a system of linear algebraic and ordinary differential equations for the rates, where generalized strains and stress resultants of the intermediate configuration serve as generally non-constant but known coefficients. We particularly study a straight intermediate configuration under the single action of a constant normal (axial) force. In Sect. 8.5 below, we eventually show that two slight approximations are needed only to approach the beam theory of second order from the exact linearization of the extended Reissner finite-strain beam theory given in Sect. 8.4. The notational transitions necessary to obtain coincidence with the relations introduced by Rubin and Vogel [9] are stated in some detail at the end of the paper.

8.2 The Reissner Finite-Strain Beam Theory

In this Section, we shortly recall the Reissner shear-deformable, finite-strain beam theory [7], which is geometrically exact within the Timoshenko hypothesis. We study an initially straight beam, see Fig. 8.1 for the meaning of the subsequently used static and kinematic entities.

The differential forms of the local equilibrium relations read

$$\mathbf{R}' + \mathbf{p} = \mathbf{0}, \tag{8.1}$$

$$\mathbf{M}' + \mathbf{r}' \times \mathbf{R} + \mathbf{m} = \mathbf{0}, \tag{8.2}$$

where

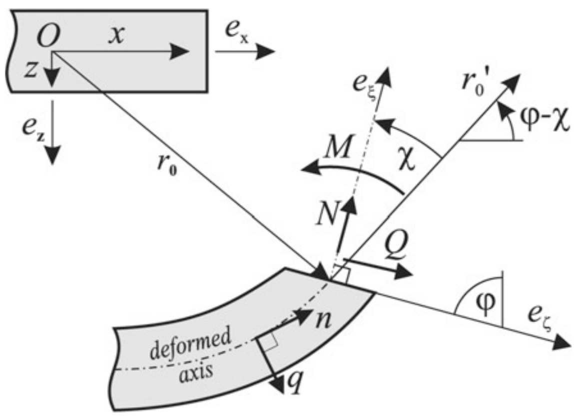

Fig. 8.1 Kinematic and static entities in the finite-strain beam theory

$$\mathbf{R} = R_x \mathbf{e}_x + R_z \mathbf{e}_z = N\mathbf{e}_1 + Q\mathbf{e}_3, \tag{8.3}$$

$$\mathbf{M} = M\mathbf{e}_y. \tag{8.4}$$

In the Reissner formulation [7], the normal force N is taken as perpendicular to the cross section in the deformed configuration, the shear force Q being in the direction of the cross section, see Fig. 8.1. The y-axis is perpendicular to the plane of deformation, the corresponding moment M representing the bending moment. The position vector of the deformed axis is

$$\mathbf{r}_0 = \mathbf{r} = x\mathbf{e}_x + \mathbf{u} \tag{8.5}$$

with the displacement vector of the axis

$$\mathbf{u} = u_x \mathbf{e}_x + u_z \mathbf{e}_z = u\mathbf{e}_x + w\mathbf{e}_z. \tag{8.6}$$

We utilize the Lagrange description of continuum mechanics; hence, every entity is understood as a function of the axial coordinate x in the un-deformed configuration. The global (x, z)- and the local (ξ, η)-coordinate systems are related by, see Fig. 8.1,

$$\mathbf{e}_\xi = \mathbf{e}_1 = \mathbf{e}_x \cos\varphi - \mathbf{e}_z \sin\varphi, \qquad \mathbf{e}_\zeta = \mathbf{e}_3 = \mathbf{e}_x \sin\varphi + \mathbf{e}_z \cos\varphi. \tag{8.7}$$

Derivatives with respect to the coordinate x and rates with respect to a generalized non-dimensional time t are indicated by superimposed primes and dots, respectively:

$$f' = \frac{\partial f}{\partial x}, \quad \dot{f} = \frac{\partial f}{\partial t}: \quad \mathbf{e}_1' = -\varphi'\mathbf{e}_3, \quad \dot{\mathbf{e}}_1 = -\dot{\varphi}\mathbf{e}_3, \quad \mathbf{e}_3' = \varphi'\mathbf{e}_1, \quad \dot{\mathbf{e}}_3 = \dot{\varphi}\mathbf{e}_1. \tag{8.8}$$

Particularly, the deformation gradient vector of the axis, being tangential to the latter, is

$$\mathbf{r}' = (1 + u')\mathbf{e}_x + w'\mathbf{e}_z = \Lambda(\cos\chi\,\mathbf{e}_1 - \sin\chi\,\mathbf{e}_3), \tag{8.9}$$

where Λ denotes the axial stretch,

$$\Lambda = \|\mathbf{r}'\|. \tag{8.10}$$

Substituting Eq. (8.7), we find that

$$1 + u' = \Lambda\ \cos(\varphi - \chi)\,, \qquad w' = -\Lambda \sin(\varphi - \chi)\,. \tag{8.11}$$

External forces and moments per unit axial length in the un-deformed configuration are

$$\mathbf{p} = p_x\mathbf{e}_x + p_z\mathbf{e}_z = n\mathbf{e}_1 + q\mathbf{e}_3, \tag{8.12}$$

$$\mathbf{m} = m\mathbf{e}_y. \tag{8.13}$$

The local equilibrium relations of the Reissner theory eventually can be written as

$$\mathbf{R}' + \mathbf{p} = \left(N' + \varphi' Q + n\right)\mathbf{e}_1 + \left(Q' - \varphi' N + q\right)\mathbf{e}_3 = \mathbf{0}, \tag{8.14}$$

$$M' - \Lambda\,(Q\cos\chi - N\sin\chi) + m = 0. \tag{8.15}$$

The problem must be closed by constitutive relations for normal force N, shear force Q and bending moment M. Using virtual work arguments, Reissner [7] showed that these constitutive relations must be formulated as functions of corresponding generalized strains:

$$\varepsilon = \Lambda\cos\chi - 1, \qquad \gamma = \Lambda\sin\chi, \qquad \kappa = \varphi'. \tag{8.16}$$

Reissner [7] exemplarily discussed the appropriateness of linear matrix-type constitutive relations, where, in an example, he simplified to a decoupled form:

$$N = C^{-1}\varepsilon, \qquad Q = B^{-1}\gamma, \qquad M = D\kappa. \tag{8.17}$$

In Eqs. (8.17), D represents a cross-sectional bending stiffness, and C and B are cross-sectional extensional and shear compliances, respectively.

8.3 Extension of the Reissner Theory with Respect to Stress-Strain-Based Constitutive Relations

A problem associated with the phenomenological constitutive approach discussed in [7] is that the mathematical form of the constitutive relations must be stipulated. In order to allow a formulation utilizing the continuum mechanics level of stress–strain constitutive relations, the Reissner theory [7] was extended by Irschik and Gerstmayr [5], see also [4] for beams rigid in shear, $\chi = 0$. It was shown in [5] that the following constitutive relations are work equivalent to the ones derived in [7]:

$$N = \int_A S_{xx} J \, dA, \quad Q = \int_A (S_{xz} + S_{xx}\gamma) \, dA, \quad M = \int_A S_{xx} z J \, dA. \tag{8.18}$$

In Eq. (8.18), integration is with respect to the beam cross section A in the un-deformed reference configuration, and S_{xx} and S_{xz} represent respective components of the matrix of the second Piola–Kirchhoff stress tensor in the global (x, z)-coordinate system, see Washizu [11] for an enlightening geometric interpretation. The Jacobian determinant of the deformation gradient tensor is denoted by J. For the Timoshenko-type deformation assumed here, J reads

$$J = \Lambda \cos \chi + z\varphi' = 1 + \varepsilon + z\kappa. \tag{8.19}$$

In order to exemplary illustrate the use of Eq. (8.18) in a non-linear context, we address a hyperelastic constitutive relation dating back to Ciarlet [1], see also Simo and Hughes [10]:

$$\underline{S} = \frac{\lambda}{2} \left(J^2 - 1 \right) \underline{C}^{-1} + \mu \left(\underline{1} - \underline{C}^{-1} \right). \tag{8.20}$$

The second Piola–Kirchhoff stress tensor is denoted by $\underline{S}$, and $\underline{C}$ is the right Cauchy–Green tensor. The Lamé parameters are λ and μ, and the unit tensor is written as $\underline{1}$. Within the Timoshenko-type kinematic hypothesis, the matrix of the inverse tensor $\underline{C}^{-1}$ is, see [5]:

$$\left[\underline{C}^{-1} \right] = \begin{bmatrix} \frac{1}{J^2} & 0 & -\frac{\gamma}{J^2} \\ 0 & 1 & 0 \\ -\frac{\gamma}{J^2} & 0 & 1 + \frac{\gamma^2}{J^2} \end{bmatrix}. \tag{8.21}$$

Substituting Eqs. (8.20) and (8.21) into Eqs. (8.18), the following constitutive relations for normal force N, shear force Q, and bending moment M is obtained, see again [5]:

$$N = \frac{1}{2} (2\mu + \lambda) \int_A \left(J - \frac{1}{J} \right) dA, \tag{8.22}$$

$$Q = \mu A \gamma, \tag{8.23}$$

$$M = \frac{1}{2} (2\mu + \lambda) \int_A \left(J - \frac{1}{J} \right) z \, dA. \tag{8.24}$$

In Eqs. (8.22)–(8.24), it has been assumed that the beam is homogeneous in the un-deformed reference configuration, such that the material parameters can be put in front of the cross-sectional integrals, and that the beam axis is formed by the cross-sectional centroids, such that

$$\int_A z\,dA = 0. \tag{8.25}$$

Note that Eqs. (8.22)–(8.24) indeed represent functions of the generalized strains introduced by Reissner [7], as this should be, cf. Eqs. (8.16) and (8.19) above.

8.4 Infinitesimally Small Deformations Superimposed Upon an Intermediate Configuration

We now study (infinitesimally) small deformations that are superimposed upon some intermediate configuration with a (possibly) large pre-deformation. For this sake, we perform derivatives of the relations in Sect. 8.3 with respect to a generalized time, following DaDeppo [2] and Irschik [3], see Eqs. (8.8). This yields the following set of relations:

$$\dot{\varepsilon} = \dot{\Lambda}\cos\chi - \Lambda\dot{\chi}\sin\chi, \tag{8.26}$$

$$\dot{\gamma} = \dot{\Lambda}\sin\chi + \Lambda\dot{\chi}\cos\chi, \tag{8.27}$$

$$\dot{\kappa} = \dot{\varphi}', \tag{8.28}$$

$$\dot{J} = \dot{\varepsilon} + z\dot{\kappa}, \tag{8.29}$$

$$\dot{u}' = \dot{\Lambda}\cos(\varphi - \chi) - \Lambda(\dot{\varphi} - \dot{\chi})\sin(\varphi - \chi), \tag{8.30}$$

$$\dot{w}' = -\dot{\Lambda}\sin(\varphi - \chi) - \Lambda(\dot{\varphi} - \dot{\chi})\cos(\varphi - \chi), \tag{8.31}$$

$$\dot{N}' + \dot{\kappa}Q + \kappa\dot{Q} + \dot{\varphi}Q' - \dot{\varphi}\kappa N + \dot{n} + \dot{\varphi}q = 0, \tag{8.32}$$

$$\dot{Q}' - \dot{\kappa}N - \kappa\dot{N} - \dot{\varphi}N' - \dot{\varphi}\kappa Q + \dot{q} - \dot{\varphi}n = 0, \tag{8.33}$$

$$\dot{M}' - Q\dot{\varepsilon} - \dot{Q}(\varepsilon + 1) + N\dot{\gamma} + \dot{N}\gamma + \dot{m} = 0, \tag{8.34}$$

$$\dot{N} = \frac{1}{2}(2\mu + \lambda)\int_A \dot{J}\left(1 + \frac{1}{J^2}\right) dA, \tag{8.35}$$

$$\dot{Q} = \mu A\dot{\gamma}, \tag{8.36}$$

$$\dot{M} = \frac{1}{2}(2\mu + \lambda)\int_A \dot{J}\left(1 + \frac{1}{J^2}\right) z\,dA. \tag{8.37}$$

We particularly are interested in a straight intermediate configuration without shear deformation, $\varphi = 0$, $\chi = 0$. From Eq. (8.16), it follows that $\gamma = 0$, $\kappa = 0$, and Eq. (8.19) implies that $J = \Lambda = 1 + \varepsilon$ does not depend on the transverse coordinate z. From Eqs. (8.22)–(8.25), we yield that $M = 0$, $Q = 0$, while $N = const$. Moreover, Eqs. (8.14) and (8.15) clarify that there must be no distributed loadings then, $n = 0, q = 0, m = 0$. The kinematic relations for small deformations from this straight intermediate deformation become, see Eqs. (8.26)–(8.31):

$$\dot{\varepsilon} = \dot{\Lambda}, \qquad \dot{\gamma} = \Lambda \dot{\chi} = (1+\varepsilon)\dot{\chi}, \qquad \dot{\kappa} = \dot{\varphi}', \tag{8.38}$$

$$\dot{J} = \dot{\varepsilon} + z\dot{\kappa}, \tag{8.39}$$

$$\dot{u}' = \dot{\Lambda} = \dot{\varepsilon}, \qquad \dot{w}' = -\Lambda(\dot{\varphi} - \dot{\chi}) = -(1+\varepsilon)\dot{\varphi} + \dot{\gamma}. \tag{8.40}$$

The incremental equilibrium relations, Eqs. (8.32)–(8.34), read

$$\dot{N}' + \dot{n} = 0, \tag{8.41}$$

$$\dot{Q}' - N\dot{\kappa} + \dot{q} = \dot{Q}' - N\dot{\varphi}' + \dot{q} = 0, \tag{8.42}$$

$$\dot{M}' - \dot{Q}(\varepsilon + 1) + N\dot{\gamma} + \dot{m} = 0. \tag{8.43}$$

The corresponding incremental constitutive relations, Eqs. (8.35)–(8.37), take on the form

$$\dot{N} = C^{-1}\,\dot{\varepsilon} = C^{-1}\,\dot{u}', \qquad C^{-1} = \frac{1}{2}\,(2\,\mu + \lambda)\;A\,\left(1 + \frac{1}{(1+\varepsilon)^2}\right) \tag{8.44}$$

$$\dot{Q} = B^{-1}\dot{\gamma}, \qquad B^{-1} = \mu A, \tag{8.45}$$

$$\dot{M} = D\,\dot{\varphi}' = D\,\dot{\kappa}, \qquad D = \frac{1}{2}\,(2\,\mu + \lambda)\;I\,\left(1 + \frac{1}{(1+\varepsilon)^2}\right), \tag{8.46}$$

with the cross-sectional moment of inertia about the y-axis:

$$I = \int_A z^2\,dA. \tag{8.47}$$

The constitutive relations presented in Eqs. (8.44)–(8.46) are of the linear form that was stipulated by Reissner in [7], see Eqs. (8.17). However, since we started from the Ciarlet non-linear hyperelastic stress-strain relation stated in Eq. (8.20), our formulation reflects the influence of the deformation ε in the intermediate configuration upon stiffness and compliances.

Equations (8.40)–(8.46) form a set of eight linear relations for eight unknown entities, namely the three incremental stress resultants $\dot{N}$, $\dot{Q}$ and $\dot{M}$, the two incremental displacements $\dot{u}$ and $\dot{w}$, and the three generalized strains $\dot{\varepsilon}$, $\dot{\gamma}$ and $\dot{\varphi}$. This set can be considered as an exact linearization of the extended Reissner finite-strain theory that has been discussed in Sect. 8.3 above.

By proper elimination, this set of eight relations can be decoupled into a linear differential equation of second order for $\dot{u}$, and a linear differential equation of fourth order for $\dot{w}$. This demonstrates that prescribing the usual three static or kinematic boundary conditions at each beam end is sufficient to obtain a complete linear boundary value problem; e.g., at a clamped end, $\dot{u}$, $\dot{w}$, and $\dot{\varphi}$ must be prescribed, while at a free end, one has to prescribe the two components of the internal force $\dot{\mathbf{R}}$, as well as the bending moment $\dot{M}$. Note that there is

$$\dot{R}_x = \dot{N} + Q\dot{\varphi}, \qquad \dot{R}_z = -N\dot{\varphi} + \dot{Q}. \tag{8.48}$$

The linear boundary value problem under consideration conveniently can be solved in closed form using symbolic computer codes, given N, Λ, and ε in the intermediate configuration.

8.5 Beam Theory of Second Order with Shear

By inspection of the relations presented in Sect. 8.4, it becomes evident that two slight approximations only are necessary for approaching the fundamental relations of the beam theory of second order with shear as stated by Rubin and Vogel [9]. The first approximation is that the extensional strain ε in the intermediate configuration can be neglected

$$\varepsilon = 0 \rightarrow \Lambda = 1, \qquad \dot{\gamma} = \dot{\chi}, \qquad \dot{w}' = -\dot{\varphi} + \dot{\gamma}, \tag{8.49}$$

see Eqs. (8.38) and (8.40). Using Eq. (8.49), the incremental moment equilibrium relation, Eq. (8.43), simplifies to

$$\dot{M}' - \dot{Q} + N(\dot{w}' + \dot{\varphi}) + \dot{m} = 0. \tag{8.50}$$

The constitutive relations, Eqs. (8.44)–(8.50) become

$$\dot{N} = C^{-1}\dot{u}', \qquad C^{-1} = (2\mu + \lambda)A, \tag{8.51}$$

$$\dot{Q} = B^{-1}(\dot{w}' + \dot{\varphi}), \tag{8.52}$$

$$\dot{M} = D\dot{\varphi}', \qquad D = (2\mu + \lambda)I. \tag{8.53}$$

Equations (8.50)–(8.53), together with the unchanged force equilibrium relations, Eqs. (8.41) and (8.42), form a set of six relations for the six unknowns $\dot{N}$, $\dot{Q}$, $\dot{M}$, $\dot{u}$, $\dot{w}$ and $\dot{\varphi}$. We note that, although the approximation of inextensibility in the intermediate configuration, Eq. (8.49), appears to be reasonable under many circumstances, it does not result in considerable mathematical simplifications, when compared to the formulation in Sect. 8.4.

In order to approach the particular formulas published in Rubin and Vogel [9], a skew decomposition must be utilized for the internal force $\mathbf{R}$. This decomposition is to be performed into a normal force $\bar{N}$ tangential to the deformed axis, i.e., in the direction of $\mathbf{r}'$, and into a shear force $\bar{Q}$ in the direction of the deformed cross section:

$$N = \bar{N}\cos\chi, \qquad Q = \bar{Q} + \bar{N}\sin\chi. \tag{8.54}$$

The corresponding rate forms are

$$\dot{N} = \dot{\bar{N}}\cos\chi + \bar{N}\dot{\chi} sin\chi, \qquad \dot{Q} = \dot{\bar{Q}} + \dot{\bar{N}}\sin\chi + \bar{N}\dot{\chi}\cos\chi. \tag{8.55}$$

Since there is no shear in the intermediate configuration, $\chi = 0$, we obtain that

$$N = \bar{N}, \quad \dot{N} = \dot{\bar{N}}, \quad \dot{Q} = \dot{\bar{Q}} + \bar{N}\,\dot{\chi} = \dot{\bar{Q}} + N\,\dot{\gamma} = \dot{\bar{Q}} + N\,\left(\dot{w}' + \dot{\varphi}\right). \tag{8.56}$$

We thus may write, instead of Eqs. (8.41), (8.42) and (8.50):

$$\dot{\bar{N}}' + \dot{n} = 0, \qquad \dot{\bar{Q}}' + \bar{N}\dot{w}'' + \dot{q} = 0, \qquad \dot{M}' - \dot{\bar{Q}} + \dot{m} = 0. \tag{8.57}$$

The constitutive force relations, Eqs. (8.51) and (8.52), become

$$\dot{\bar{N}} = C^{-1}\dot{u}', \qquad \dot{\bar{Q}} = (B^{-1} + \bar{N})(\dot{w}' + \dot{\varphi}). \tag{8.58}$$

Moreover, we consider the sign conventions introduced by Rubin and Vogel [9], see Fig. 3.2-2 of [9]. Since rates were not introduced explicitly in [9], we subsequently avoid superimposed dots for the entities in the formulas of [9], but we indicate the latter by the index RV:

$$\dot{n} = -n_{RV}, \qquad \dot{q} = q_{RV}, \qquad \dot{m} = -m_{RV}, \tag{8.59}$$

$$\bar{N} = -N_{RV}, \qquad \dot{\bar{N}}' = -N'_{RV}, \qquad \dot{\bar{Q}} = Q_{RV}, \qquad \dot{M} = M_{RV}, \tag{8.60}$$

$$\dot{w} = w_{RV}, \qquad \dot{u} = u_{RV}, \qquad \dot{\varphi} = -\varphi_{RV}. \tag{8.61}$$

Substituting Eqs. (8.59)–(8.61) into the above relation (8.57), we first obtain Eq. (3.2-14) of [9]

$$N'_{RV} + n_{RV} = 0. \tag{8.62}$$

Using Eqs. (8.57) above, we get

$$Q'_{RV} - N_{RV}w''_{RV} + q_{RV} = 0, \tag{8.63}$$

which coincides with the result of substituting Eq. (3.2.-12) of [9].

From Eq. (8.57), one obtains Eq. (3.2.-13) of [9]

$$M'_{RV} - Q_{RV} - m_{RV} = 0. \tag{8.64}$$

Concerning the constitutive relations, replace C^{-1} by D_{RV} in Eq. (8.58) above. This gives Eq. (3.2-17) of [9]

$$N_{RV} = -D_{RV} u'_{RV}. \tag{8.65}$$

Similarly, replacing $(B^{-1} + \bar{N})$ in Eq. (8.58) by S_{RV} yields Eq. (3.2-16) of [9]:

$$Q_{RV} = S_{RV}(w'_{RV} - \varphi_{RV}). \tag{8.66}$$

This clarifies that, when using B^{-1} for S_{RV} directly, the influence of the normal force in the intermediate configuration is neglected. Since this influence often will be small, this second approximation appears to be reasonable, but again brings no computational advantage.

Finally, replacement of D in Eq. (8.53) above by B_{RV} finally results in Eq. (3.2-15) of [9]:

$$M_{RV} = -B_{RV} \varphi'_{RV}. \tag{8.67}$$

This closes our derivation of the fundamental relations of the beam theory of second order with shear, as stated in [9].

8.6 Conclusion

In Sects. 8.4 and 8.5 above, the fundamental relations of the beam theory of second order, see Rubin and Vogel [9], have been shown to represent a slight simplification of an exactly linearized version of the continuum mechanics-based extension of the Reissner finite-strain beam theory discussed before in Sect. 8.3. The more involved case of beams with shear and initial imperfections, which also was treated in [9], will be studied in a further contribution.

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