Chapter 23
Wave Attenuation in a Pre-tensioned String with Periodic Spring Supports

Y.-B. Yang, J. D. Yau, and S. Urushadze

Abstract The overhead catenary system is a crucial conductor for delivering steady electric power to the trains running on modern electrified railways. The propagation of vibration waves in the catenary system is of interest to railway engineers due to the pantograph-catenary interaction. To explore the wave transmission via the contact wires of a catenary system supported by hanging devices offered by the bracket structures, a simplified model composed of a pre-tensioned string periodically suspended by hanging springs is adopted. For a periodic structure with wider band gaps, also known as stop bands, a wider cluster of frequencies of waves propagating in the periodic structure can be attenuated (or filtered out). This will be beneficial to the maintenance of the catenary system. To take advantage of such a feature, a resonator is usually equipped on each of the hanging spring supports so as to widen band gaps for better attenuation of the waves transmitted in the pre-tensioned string. In this study, a unit cell conceived as a spring-resonator-string unit is adopted to formulate the closed-form dispersion equation, from which the key condition for widening the band gaps is derived. From the exemplar study, it was shown that the installation of adjustable resonators on a catenary system can increase the band gap width, serving as a wave filter for attenuating the pantograph-induced wave transmission in the contact wires of the pantograph-catenary system.

Y.-B. Yang (✉)
School of Civil Engineering, Chongqing University, Chongqing 400044, China
e-mail: ybyang@ntu.edu.tw

Department of Construction Engineering, National Yunlin University of Science and Technology, Yunlin 64002, Taiwan

J. D. Yau
Department of Architecture, Tamkang University, New Taipei City 25137, Taiwan
e-mail: jdyau@mail.tku.edu.tw

S. Urushadze
Institute of Applied and Theoretical Mechanics, Czech Academy of Sciences, Prague, Czech Republic
e-mail: urushadze@itam.cas.cz

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23.1 Introduction

High-speed railways have become one of the most efficient ground transportation tools for passengers traveling between major cities in many countries. For high-speed or traditional railways, the overhead catenary system contains contact wires for transmitting the electrical current to the pantographs equipped on each train, thereby supplying the power to the electrical engines of the locomotives of the train. In the past decades, many studies were conducted on the dynamic behaviors of the pantograph and catenary, considering their interaction. Sophisticated models have been developed to carry out the response analysis of the pantograph-catenary system, by which the effect of the locomotive motion was taken into account [6]. However, few research has been conducted to explore the problem of wave attenuation via the contact wires of a catenary system.

For the theoretical formulation aimed at obtaining a closed-form solution, the overhead catenary system is simplified as a pre-tensioned string supported periodically by hanging springs in this study. Using the Floquet-Bloch theory [1, 5] to account for the periodicity of a periodic structure, the dispersion relation between the wavenumber and frequency of the pre-tensioned string will be derived. Moreover, a resonator is installed on each of the hanging spring supports to widen the band gaps (stop bands) of the pre-tensioned string for better attenuation of the wave components transmitted via the string. Then, the key condition for determining the critical resonator is identified from the closed-form dispersion equation of the pre-tensioned string with resonators.

23.2 Problem Formulation of Overhead Catenary System

For the present purposes, the catenary system is simplified as a pre-tensioned string supported by periodic hanging springs with identical interval L, as shown in Fig. 23.1.

To derive the closed-form solution for the dispersion equation of the pre-tensioned string, the following assumptions are adopted:

1. The main contact wire of the catenary system is modeled as a horizontal pre-tensioned string supported by periodic hanging springs of uniform interval L [6];
2. The tensioned force T in the string is assumed to be constant during vibration.

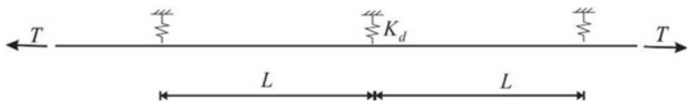

Fig. 23.1 Schematic of a pre-tensioned string suspended by equal-distance hanging springs

With the above tensioned string model, the dispersion relation between the wavenumber and frequency of the periodically supported tensile string will be derived and presented in closed form.

23.2.1 Dynamic Stiffness Matrix of a Tensioned String

As shown in Fig. 23.1, the governing equation for the transverse motion of a tensioned string can be written as follows [3]:

$$m_s \frac{\partial^2 u(x,t)}{\partial t^2} - T \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{23.1}$$

where m_s is the mass of the string per unit length and $u(x,t)$ the vertical displacement of the string. By letting $a^2 = m_s \omega^2 / T$, with ω denoting the frequency, and solving Eq. (23.1), one can obtain the following solution:

$$u(x,t) = [C_1 \sin(ax) + C_2 \sin(a(L-x))]\, e^{i\omega t}. \tag{23.2}$$

By introducing the dynamic equilibrium conditions at the two ends of a string of length L, the well-known dynamic stiffness matrix of the pre-tensioned string element with length L can be written as [4]

$$\mathbf{D}(aL)_{string} = \frac{aT}{\sin(aL)} \begin{bmatrix} \cos(aL) & -1 \\ -1/\cos(aL) & \end{bmatrix}. \tag{23.3}$$

With the dynamic stiffness matrix given in Eq. (23.3), one can conceive a unit cell of the periodic spring-supported string as in Fig. 23.2, and derive from this the closed-form expression for the dispersion relation of the wavenumber and the frequency, as in the section to follow.

23.2.2 Dispersion Equation of a Tensioned String with Periodical Spring Supports

With the dynamic stiffness matrix given in Eq. (23.3), the dynamic stiffness equation of the unit cell in Fig. 23.2 for the pre-tensioned string with periodic spring supports can be expressed as

$$\begin{bmatrix} d_{LL} & d_{LR} \\ d_{LR} & d_{RR} \end{bmatrix} \begin{Bmatrix} u_L \\ u_R \end{Bmatrix} = \begin{Bmatrix} f_L \\ f_R \end{Bmatrix} \tag{23.4}$$

Fig. 23.2 A spring-string unit cell

where $\{f\}$ and $\{u\}$ are the element forces and displacements, respectively, and

$$d_{LR} = d_{RL} = \frac{-aT\csc(aL/2)}{2\left(\cos(aL/2) + \frac{K_d}{2aT}\sin(aL/2)\right)}, \tag{23.5}$$

$$d_{LL} = d_{RR} = aT\cot(aL/2) + d_{LR}. \tag{23.6}$$

By adopting the periodic boundary conditions of ($u_R = e^{-i\kappa L}u_L, f_R + e^{-i\kappa L}f_L = 0$) of the Floquet-Bloch theory [2] for the string element, the following dispersion equation can be obtained [1, 5]:

$$\cos(\kappa L) = \cos(aL) + \frac{K_d L}{2aL \times T}\sin(aL). \tag{23.7}$$

For the pass-band condition of $|\cos(\kappa L)| \le 1$ in Eq. (23.7), the bounding frequencies are defined by $|\cos(\kappa L)| = \pm 1$. Clearly, the condition of bounding frequencies listed in Table 23.1 depends on the stiffness parameter ($K_d L/T$), which is related to the tensile force T in the string, spring stiffness K_d, and uniform interval L of the hanging supports offered by the bracket structures. In real electrified railways, the span interval L of the bracket structures and the pre-tensioned force T in a catenary system are determined by the regulations or provisions suggested by railway codes. Consequently, a change in the hanging spring stiffness K_d may lead to a spectral band gap that allows certain frequency components to be attenuated (or filtered out) during the wave transmission in the periodically spring-supported string. To further this consideration, a resonator will be equipped in the hanging spring support for the purpose of widening the band gaps for attenuating certain frequency components in the pre-tensioned string.

Table 23.1 Bounding frequencies of the unit cell

Modes	Bounding frequencies (aL)
Symmetrical Mode ($u_R = u_L, \cos(\kappa L) = 1$) $u(x,t) = \sin\left(\frac{aL}{2}\right)\cos\left(\frac{aL}{2}\left(1 - \frac{2x}{L}\right)\right)$	$\frac{aL}{2}\tan\left(\frac{aL}{2}\right) = \frac{1}{2}\frac{K_d L}{2T}$
Anti-symmetrical Mode ($u_R = -u_L, \cos(\kappa L) = -1$) $u(x,t) = \cos\left(\frac{aL}{2}\right)\sin\left(\frac{aL}{2}\left(1 - \frac{2x}{L}\right)\right)$	$\frac{aL}{2}\cot\left(\frac{aL}{2}\right) + \frac{1}{2}\frac{K_d L}{2T} = 0$

23.3 Dispersion Equation of the Unit Cell with Resonator

For a periodic structure with wider band gaps, more frequencies of waves propagating in the periodic structure can be attenuated (or filtered out). To make use of this feature, a resonator is equipped on each of the hanging spring supports (Fig. 23.3) so that a widened band gap (stop band) can be achieved, so as to attenuate a wider range of frequencies of waves transmitted via the pre-tensioned string. By using the element assemblage procedure, the spectral equation of the unit cell with two string elements each of length $L/2$ and an intermediate resonator at the mid-node (see Fig. 23.3) can be expressed as follows:

$$\begin{bmatrix} aT\cot(aL/2) & -aT\csc(aL/2) & 0 & 0 \\ -aT\csc(aL/2) & 2aT\cot(aL/2)+K_d+k_r & -k_r & -aT\csc(aL/2) \\ 0 & -k_r & k_r - m_r\omega^2 & 0 \\ 0 & -aT\csc(aL/2) & 0 & aT\cot(aL/2) \end{bmatrix} \begin{Bmatrix} u_L \\ u_m \\ u_r \\ u_R \end{Bmatrix} = \begin{Bmatrix} f_L \\ 0 \\ 0 \\ f_R \end{Bmatrix}. \tag{23.8}$$

Here, m_r is the lumped mass and k_r the spring constant of the resonator, and (u_m, u_r) denote the vertical displacements of the mid-node of the string and the lumped mass. By the matrix condensation method, one can condense the slaved displacements (u_m, u_r) into the corresponding master displacements (u_L, u_R) of the unit cell, as shown in Fig. 23.3. Then the condensed stiffness equation becomes

$$\begin{bmatrix} \bar{d}_{LL} & \bar{d}_{LR} \\ \bar{d}_{LR} & \bar{d}_{RR} \end{bmatrix} \begin{Bmatrix} u_L \\ u_R \end{Bmatrix} = \begin{Bmatrix} f_L \\ f_R \end{Bmatrix} \tag{23.9}$$

where

$$\bar{d}_{LR} = \bar{d}_{RL} = \frac{-aT\csc(aL/2)}{2\left(\cos(aL/2) + \frac{\bar{K}_d L}{2aL\times T}\sin(aL/2)\right)}, \tag{23.10}$$

$$\bar{d}_{LL} = \bar{d}_{RR} = aT\cot(aL/2) + \bar{d}_{LR}, \tag{23.11}$$

$$\bar{K}_d = K_d\left(1 + \frac{k_r}{K_d}\frac{(\omega/\omega_r)^2}{(\omega/\omega_r)^2 - 1}\right) \tag{23.12}$$

with $\omega_r = \sqrt{k_r/m_r}$ denoting the frequency of the resonator. Clearly, the effect of the resonator was taken into account in the expression for the condensed spring stiffness

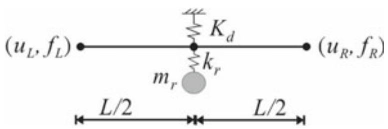

Fig. 23.3 A spring-resonator-string unit cell

in Eq. (23.12). For the special case when the spring constant k_r of the resonator is zero, the condensed stiffness equation of Eq. (23.9) reduces to Eq. (23.4) as expected. By introducing the periodic boundary conditions mentioned above to Eq. (23.9), the following dispersion equation can be derived

$$\cos(\kappa L) = \cos(aL) + \frac{\bar{K}_d L}{2aL \times T} \sin(aL). \tag{23.13}$$

The second term on the right side of the preceding characteristic equation describes the dispersive feature of a resonator to attenuate the wave components transmitting from one span to the next one of the periodic pre-tensioned string. For the calculation to follow, the frequency ratio ω/ω_r is introduced:

$$\left(\frac{\omega}{\omega_r}\right)^2 = \frac{\mu T}{k_r L}(aL)^2 \tag{23.14}$$

where μ is the mass ratio defined as $\mu = m_r/m_s L$. Then the second term in Eq. (23.13) can be rewritten as

$$\frac{\bar{K}_d L}{2aL \times T} \sin(aL) = \frac{K_d L}{2aL \times T} \sin(aL) + \frac{\mu}{2} \frac{aL \times \sin(aL)}{(aL)^2 \mu T/k_r L - 1}. \tag{23.15}$$

Let us consider the critical condition by letting $aL \to N\pi|_{N=1,2,3...}$ and $\omega \to \omega_r$ (or $(aL)^2 \mu T/k_r L \to 1$) in Eq. (23.15), that is,

$$\lim_{aL \to N\pi, \omega \to \omega_r} \frac{\bar{K}_d L}{2aL \times T} \sin(aL) = (-1)^N \frac{\mu}{2} N\pi. \tag{23.16}$$

With this, the dispersion relation in Eq. (23.13) reduces to

$$\cos(\kappa L) = (-1)^N \left(1 + \mu \left(\frac{N\pi}{2}\right)^2\right) \tag{23.17}$$

or

$$|\cos(\kappa L)| = 1 + \mu \left(\frac{N\pi}{2}\right)^2 > 1. \tag{23.18}$$

As can be seen from Eq. (23.18), the resonator provides a widening mechanism to increase the band gap of wave transmission in a pre-tensioned string supported by the hanging springs. Concerning the band gap for attenuating the wave transmission with specific frequencies in the string, some numerical analyses will be carried out in the following section.

23.4 Illustrative Example

Let us consider the simplified catenary system shown in Fig. 23.1. Using the empirical data given by Ref. [6], the pre-tensioned force in the contact wire is set to be $T = 15$ kN, the mass per unit length of the contact wire is $m_s = 0.925$ kg/m, the suspension stiffness of the registration arm assembly is $K_d = 130$ N/m, and the span length of bracket structures is $L = 65$ m. See Table 23.2 for a list of the properties

Table 23.2 Properties of the pre-tensioned string

L (m)	m_s (kg/m)	T (N)	K_d (N/m)	$v_c = \sqrt{T/m_s}$ (m/s)
65	0.925	15 000	130	127[a]

[a]Critical velocity of the pre-tensioned string

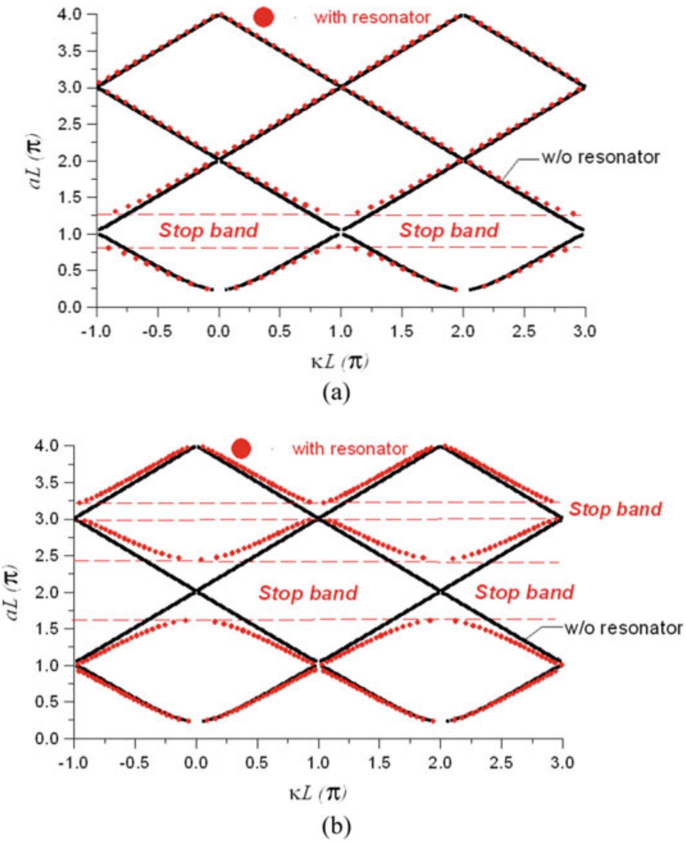

Fig. 23.4 Dispersion curves of the string with critical resonators with **a** $k_{r,cr} = \pi^2 \mu T/L$; **b** $k_{r,cr} = 4\pi^2 \mu T/L$

adopted in the analysis. Correspondingly, the stiffness parameter $(K_d L/T)$ is equal to 0.564. Let us adopt a resonator with the mass of $m_r = 0.1$ kg, $m_s L = 6$ kg. If the nondimensional frequency aL is selected as $aL = \pi$ for attenuating the frequencies, then the critical spring stiffness ($k_{r,cr} = \pi^2 \mu T/L$) of the resonator can be designed as 228 N/m.

With these data, the dispersion curves of the pre-tensioned string derived have been plotted in Fig. 23.4a, in which the black lines represent the dispersion curve of the string without resonator and the lines with red dots the curve with resonators.

As indicated, the band gap (stop band) at $aL = \pi$ is significantly widened once the critical resonator is taken into account. Similarly, if the nondimensional frequency is set at $aL = 2\pi$, the critical spring stiffness is $k_{r,cr} = 911$ N/m. Figure 23.4b shows the corresponding dispersion curves of the pre-tensioned string, in which the band gaps (stop bands) at $aL = 2\pi$ and $aL = 3\pi$ have been significantly widened.

23.5 Concluding Remarks

In this study, a simplified model composed of pre-tensioned string suspended periodically by equal-distance springs is used to simulate the pantograph-catenary interaction encountered in railway engineering. With this, the dispersion relation between the wavenumber and frequency of the pre-tensioned string hung by periodical spring supports is derived in closed form. For wave attenuation, a resonator was attached to each hanging spring support, for which the closed-form solution was also derived from the corresponding dispersion relation. The numerical results indicate that the installation of resonators can widen the band gaps (or stop bands) of the dispersion curves of the pre-tensioned string. With this conclusion, a further pantograph/catenary interaction model will be carried out to study the overall pantograph-catenary interaction dynamics.

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