Chapter 16
Theory of Critical Distances as a Method of Failure Prediction Under Dynamic Loading

Oleg A. Plekhov, Alena I. Vedernikova, and Anastasiia A. Kostina

Abstract The linear-elastic Theory of Critical Distances (TCD) is reformulated to make it suitable for estimating the strength of notched components subjected to dynamic loading. The theory modification in case of elasto-plastic stress–strain behavior to enhance accuracy of strength assessment is presented. The efficiency of the proposed methodologies is demonstrated for the experimental data on notched Grade 2 specimens that were subjected to uniaxial tensile loads within the rate range of 10^{-3}–$10^{4}\ s^{-1}$. The obtained results showed that the modification of the TCD based on elasto-plastic analysis gives estimates that fall within an error interval of ±5–10%, more accurate predictions than the linear-elastic solution. The physical meaning of the critical distance theory, in particular, the values of the critical distances L and inherent material strength σ_0, on the base of the original statistical thermo-dynamical model of evolution of an ensemble of defects in metals developed by Naimark (2003) in ICMM UB RAS is proposed. It has been observed that the critical distance value can be considered as a fundamental length scale of dissipative structure developing in a blow-up regime.

16.1 Introduction

The Theory of Critical Distances (TCD) is an effective tool developed by Taylor [12], allowing the strength of components with geometrical discontinuities (cracks, notches, holes) to be estimated accurately by directly post-processing the entire linear-elastic stress fields in the vicinity of the stress concentrators. According to the Theory of Critical Distances, failure occurs when an equivalent stress calculated either at a certain distance from the notch tip, either averaged at the some distance or area, becomes larger than the inherent material strength σ_0. These are the central ideas of TCD that are an extension of the approaches developed by Neuber [4], Novozhilov [5], Peterson [6], Whitney and Nuismer [14], and Pluvinage [7] to estimate the

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strength of notched metallic materials. Recently, it was also proven that the TCD is successful in estimating the static [10, 11], fatigue [1, 9, 13], and dynamic strength [15, 16] of both ductile and brittle notched components.

The first part of this work is devoted to the verification TCD reformulation for notched Grade 2 specimens at the strain rates of 10^{-3}–$10^{4}\,s^{-1}$ [16]. The dynamic TCD based on the simple power laws expression for the inherent strength and critical distance with regard to the strain rate and uses the post-process the linear-elastic stress fields near the assumed crack initiation locations.

The second part of work is aimed to modification of the Theory of Critical Distance in case of elasto-plastic material behavior to enhance the accuracy of the fracture assessment of notched components.

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The final part of the work is devoted to the physical interpretation of the effective length and inherent strength parameters, which is still an issue of fracture mechanics and generally found empirically.

16.2 Extending TCD to Dynamic Loading

16.2.1 Critical Distance Concept for the Dynamic Loading

The TCD postulates that the notched component under Mode I static loading being designed does not fail as long as the following condition is assured [12]

$$\sigma_{eff} \leq \sigma_0, \tag{16.1}$$

where σ_{eff} is the effective stress determined according to one of the methods of the theory of critical distances, σ_0 is the inherent material strength. If the TCD is used to perform the assessment of brittle notched materials, σ_0 can be taken equal to the material ultimate tensile strength σ_{UTS} [12], as far as for ductile materials, σ_0 is determined by testing of specimens with different notch sharpness [10].

Much experimental evidence [15] suggests that the dependence of the dynamic strength of metal alloys on the strain rate can be summarized by adopting simple power laws. The reformulation of the Theory of Critical Distances to dynamic loading is based on the following hypothesis: since both the dynamic failure stress σ_f and dynamic fracture toughness K_{Id} vary as applied strain rate $\dot{\varepsilon}$ increases, we assume that in the same way the also the inherent material stress σ_0 depends on the strain rate, and hence the value of the critical distance L. Mathematically, the hypothesis is formulated as follows:

$$\begin{cases} \sigma_f = f_{\sigma_f}(\dot{\varepsilon}) = a_f \dot{\varepsilon}^{b_f} \\ K_{Id} = f_{K_{Id}}(\dot{\varepsilon}) = \alpha \dot{\varepsilon}^{\beta} \end{cases} \Rightarrow \begin{cases} \sigma_0 = f_{\sigma_0}(\dot{\varepsilon}) = a_0 \dot{\varepsilon}^{b_0} \\ L = f_L(\dot{\varepsilon}) = \frac{1}{\pi}\left[\frac{K_{Id}}{\sigma_0}\right] = \frac{1}{\pi}\left[\frac{\alpha\dot{\varepsilon}^{\beta}}{a_0\dot{\varepsilon}^{b_0}}\right] = M\dot{\varepsilon}^{N} \end{cases}, \quad (16.2)$$

where $\dot{\varepsilon}$ is strain rate, $a_f, b_f, \alpha, \beta, a_0, b_0, M, N$ are material constants to be determined by running appropriate experiments.

According to the Theory of Critical Distances, the dynamic effective stress σ_{eff} to perform the dynamic assessment has to be determined according to the Point method (PM), the Line method (LM) or the Area method (AM) [16]

$$\sigma_{eff} = \sigma_y\left(\theta = 0, r = \frac{L}{2}\right), \quad (16.3)$$

$$\sigma_{eff} = \frac{1}{2L}\int_0^{2L} \sigma_y(\theta = 0, r)\mathrm{d}r, \quad (16.4)$$

$$\sigma_{eff} = \frac{2}{\pi L^2}\int_{-\pi/2}^{\pi/2}\int_0^{L} \sigma_1(\theta, r) r \mathrm{d}r\mathrm{d}\theta, \quad (16.5)$$

where σ_y is stress parallel to axis y, σ_1 is maximum principal stress, L is critical distance, (θ, r) are polar coordinates.

16.2.1.1 Experimental Details

The accuracy and reliability of the proposed reformulation of the TCD was checked against a set of experimental results generated by testing, under different strain rates, specimens of titanium alloy Grade 2 containing notches of different sharpness. Quasi-static tensile tests were carried out with an electromechanical testing machine Shimadzu AG-X Plus (300 kN). A Hopkinson-Kolsky Split Bar was used to study the high strain-rate material properties. The tensile tests were carried out in the strain-rate range of 10^{-3}–$10^4 s^{-1}$. Measurement of strain during materials testing was carried out using video extensometer TRViewX240S f12.5. Three types of cylindrical specimens with different stress concentrators such as semi-circular edge notches with radius 1 and 2 mm, V-shaped notches (notch root radius 0.1 mm), and un-notched (plain) specimens were used.

16.2.1.2 Validation by Experimental Data

The linear-elastic stress fields in the vicinity of the notches being investigated were determined numerically by using commercial Finite Element (FE) software ABAQUS.

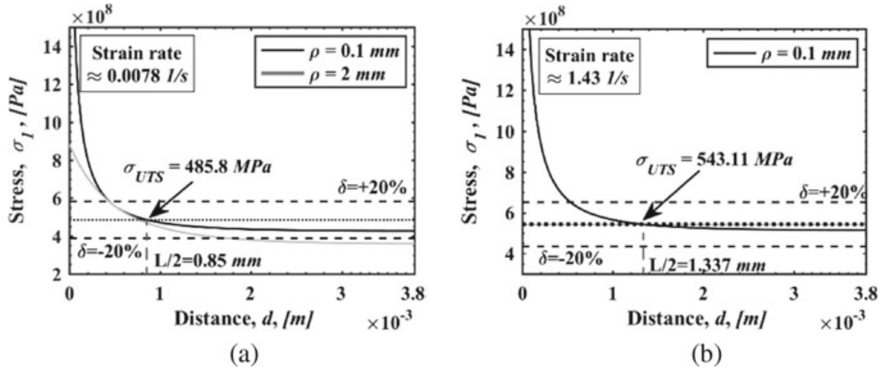

Fig. 16.1 Local linear-elastic stress fields under 0.0078/s (**a**) and 1.43/s (**b**) for notched Grade 2

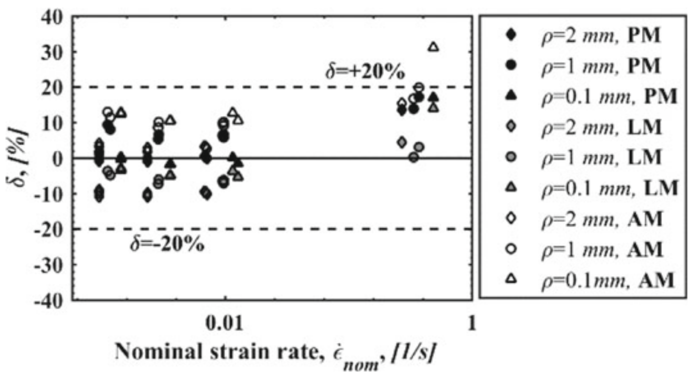

Fig. 16.2 Accuracy of the TCD reformulation in the strength predicting for notched Grade 2

Since the tested titanium alloy Grade 2 specimen were characterized by a mechanical behavior that was predominantly brittle, the hypothesis was formed that inherent material strength could be taken equal to the ultimate tensile stress. After that, the σ_0 versus $\dot{\varepsilon}_{nom}$ relationship was expressed by adopting a simple power law

$$\sigma_0 = 538.968\dot{\varepsilon}_{nom}^{0.0214}[\text{MPa}], \tag{16.6}$$

Figure 16.1a shows the linear-elastic stress-distance curves plotted under quasi-static loading for Grade 2 specimens. This diagram fully confirms that, for this material, the inherent material strength σ_0 could be taken as equal to σ_{UTS} with little loss of accuracy. Figure 16.1b resulted a critical distance value for Grade 2 at a higher strain rate. The function $L = f_L(\dot{\varepsilon})$ was derived

$$L = 2.592\dot{\varepsilon}_{nom}^{0.0869}[\text{mm}], \tag{16.7}$$

By making use of power laws (16.6), (16.7), the effective stress σ_{eff} was evaluated according to the Eqs. 16.3–16.5. The results are summarized in the Fig. 16.2, where error was calculated as

$$\delta = \frac{\sigma_{eff} - \sigma_0}{\sigma_0}[\%], \tag{16.8}$$

This validation exercise has demonstrated that the proposed reformulation of the TCD is capable of accurately assessing the static and dynamic strength of notched specimens from titanium alloy Grade 2, with the estimates falling within an error interval of ±20%.

16.3 Theory of Critical Distances Based on Elasto-plastic Analysis

16.3.1 The Simplified Johnson-Cook Model

In this part of work, a simplified Johnson-Cook law in a form of (16.9) is used to model the material response, taking into consideration the changes in the strain rate. To determine the value of the critical distance, elasto-plastic stress fields will be used. The adoption of these measures is conditioned by the fact that the material behavior being, by nature, highly nonlinear and cannot be described in the framework of the linear theory of elasticity.

$$\sigma = (A + B\varepsilon^n)\left(1 + Cln\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right), \tag{16.9}$$

where A, B, n, and C are material constants, $\dot{\varepsilon}_0$ is reference strain rate. In the Johnson-Cook constitutive model, the combined two key material responses are strain hardening and strain-rate effects. The adiabatic heating effect is considered negligible for the tension tests, as the material necks down at relatively low strains before any significant adiabatic heating. All the materials constants could be obtained from the fitting equations (16.9) with experimental data obtained under different strain rates. The material parameters for titanium alloy Grade 2: A = 363.1 MPa, B = 389.89 MPa, n = 0.435, and C = 0.0176.

16.3.2 Using the TCD by Post-processing the Elasto-plastic Stress Fields

The strength estimation algorithm remained similar to the previous case. Mises stress field distributions at the cross section away from the notch tip at the time requiring to calculation of the value of the critical distance and the effective stress according to the TCD were determined. The cylindrical un-notched specimens and specimens with sharp stress concentrators under different strain rates were used for determining the values of the critical distance. The value of the critical distance for different strain rates is constant, which is equal to 0.24 mm (Fig. 16.3), while with linear-elastic analysis, the critical distance is a function of the strain rate.

Using value of critical distance equal to 0.24 mm, the effective stress for notched specimens under different strain rates according to the Point and Line Methods of the Theory of Critical Distances were calculated. The results of this analysis are summarized in Fig. 16.4.

The results showed that the use of modification of the TCD based on elasto-plastic analysis gives us estimates falling within an error band of ±5–10%, that more accurate predictions than the linear-elastic TCD solution. The use of an improved

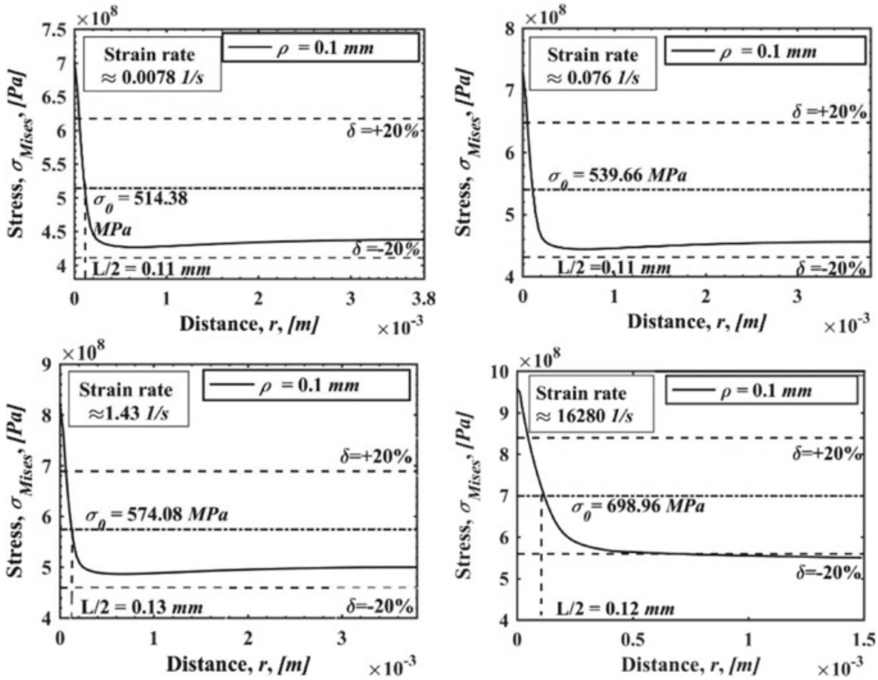

Fig. 16.3 Local elasto-plastic stress fields under different strain rates

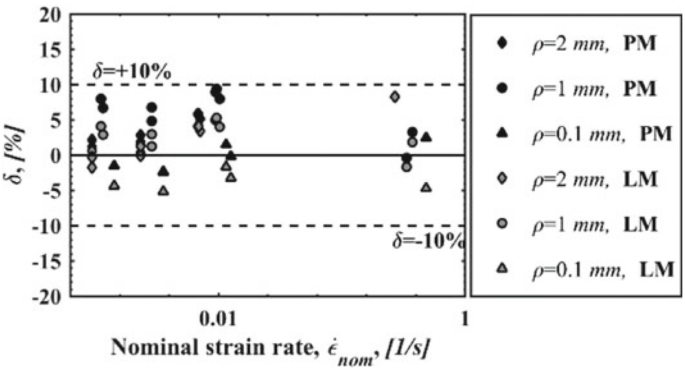

Fig. 16.4 Accuracy of the TCD based on elasto-plastic analysis in predicting the strength

description of the stress–strain state at the notch tip allows introducing the critical distances as a material parameter.

16.4 Physical Explanation of the Critical Distance Theory

16.4.1 Mathematical Model of Damage to Fracture Transition

Accounting of plastic deformation, which allows one to switch from the function of the critical distance of the strain rate to the material constant, makes it possible to introduce the hypothesis about critical distance as the fundamental length of the dissipative structure in the ensemble of defects, which develops in the blow-up regime. The description of the evolution of the ensemble of defects near the stress concentrator based on original statistical thermo-dynamical model of evolution of an ensemble of defects in metals, developed at ICMM UB RAS [3].

The constitutive equation for structural strain (deformation caused by the appearance of defects) has the form

$$\dot{\mathbf{p}} = \Gamma_p \left(\boldsymbol{\sigma} - \rho \frac{\delta F}{\delta \mathbf{p}} \right) + \Gamma_{p\sigma} \boldsymbol{\sigma}, \tag{16.10}$$

where $\Gamma_p, \Gamma_{p\sigma}$ are kinetic coefficients, F is specific Helmholtz free energy, $\mathbf{p}$ is defect density tensor (structural-sensitive parameter), $\boldsymbol{\sigma}$ is stress tensor, and ρ is volumetric mass.

The approximation of function $\boldsymbol{\sigma} - \rho \frac{\delta F}{\delta \mathbf{p}}$, which determines equilibrium state of material with defects in the one-dimensional case [2]

$$Z - \frac{\delta \Psi}{\delta p} = \frac{\sigma}{\sigma_{\max}} - ap + qp^{\beta} + \bar{\nabla} \cdot \left(kp^{s} \left(\bar{\nabla} p \right) \right), \tag{16.11}$$

where $\sigma_{\max}$ is maximum value of the stress tensor component near the concentrator, β, s are degree of polynomials that determines the character of generation and the rate of diffusion of defects, q, k, a—material parameters, Z, Ψ—dimensionless stress and free energy corresponding.

The self-similar solution of Eq. (16.10) in the one-dimensional case with approximation (16.11) for constant stress values and parameters $\beta = s + 1$ can be written as [8]

$$p(x, t) = (q(t - t_c))^{-\frac{1}{s}} \left(\frac{2(s + 1)}{s(s + 2)} \sin^2 \left[\frac{\pi x}{L_c} \right] \right)^{\frac{1}{s}}, \tag{16.12}$$

where t_c—the critical time of and L_c—the fundamental length scale.

The time of structure localization is estimated according to the relation

$$t_c = \frac{2(s + 1)}{s(s + 2)} \frac{1}{p^s q}, \tag{16.13}$$

The fundamental length scale is defined by the following expression:

$$L_c = 2 \frac{\pi}{s} \sqrt{s + 1} \sqrt{\frac{k}{q}}. \tag{16.14}$$

Equations (16.10)–(16.14) used for the explanation of the fracture mechanisms near stress concentrators of the titanium alloy Grade 2 under tensile loading.

16.4.2 Application of the Proposed Model in One-Dimensional Case

For the explanation of the fracture mechanisms near stress concentrators, consider an analytical solution for stress at round-tip notch. In case of infinite plate with semi-circular notch (notch root radius r_n) under tensile, the stress components can be evaluated as [17]

$$\sigma_y(x, 0) = \frac{K_t \sigma_\infty}{3} \left(1 + \frac{1}{2} \left(\frac{x}{r_n} + 1 \right)^{-2} + \frac{3}{2} \left(\frac{x}{r_n} + 1 \right)^{-4} \right), \tag{16.15}$$

where K_t—stress concentration factor, σ_∞—nominal stress.

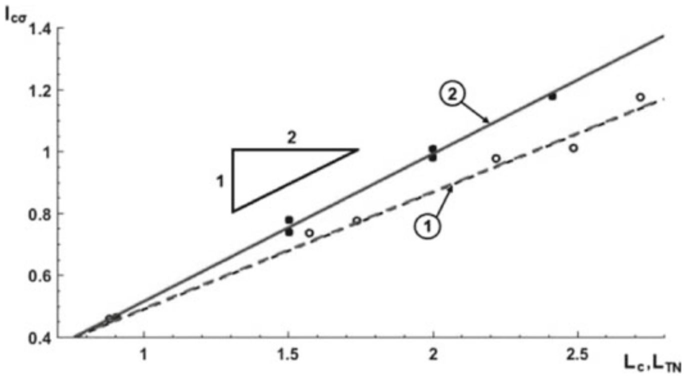

Fig. 16.5 Relations between analytically estimated fundamental length of dissipative structures L_c and spatial scale $l_{c\sigma}$ (line 1), numerically estimated fundamental length L_{TN} and spatial scale $l_{c\sigma}$

The Eqs. (16.11), (16.15) with initial condition $p_y(x,t)\mid_{t=0}= 0$ and boundary conditions $p_y(x,t)\mid_{x=0}= 0$, $\frac{\delta p_{yy}(x,t)}{\delta x}\mid_{x\to+\infty}$ were solved numerically for different value of nominal stress (model material).

The analysis of numerical simulation allows to conclude that the failure process (initiation of dissipative structure) requests a simultaneous fulfillment of two conditions: the stress should be bigger than critical value σ_0 (inherent material strength) in some area near the stress concentrator and the length of this area should be bigger than some critical spatial scale $l_{c\sigma}$ $((\exists l \geq l_{c\sigma})) : (\forall x \in [0,1]), \sigma_y(x) > \sigma_0$.

Figure 16.5 presents a relations between the analytical estimation of spatial scale of dissipative structure by Eq. (16.14) (L_c) and scale $l_{c\sigma}$, numerically obtained value of fundamental length (L_{TN}) and scale $l_{c\sigma}$. The points represent simulation results for set of parameters β and s. Analysis of the data presented in Fig. 16.5 allow to conclude that the analytical assessment gives an overestimation of the localization scale. The estimation of the fundamental length scale by the results of numerical simulation gives the exact ratio corresponding to the result of the Theory of Critical Distances: the critical stress must be achieved at the half of the fundamental length of the dissipative structure

$$l_{c\sigma} = \frac{1}{2} L_{TN} \tag{16.16}$$

16.4.3 Application of the Proposed Model in Three-Dimensional Case

Consider the link between the critical distance and fundamental length scale of dissipative structure using quasi-static tension of the U-notched specimen of titanium alloy Grade 2 (notch root radius 1 mm) as an example.

Figures 16.6 and 16.7 show the values of the defect density tensor at the notch tip for two cases: $\sigma_y < \sigma_0$ and $\sigma_y > \sigma_0$, $1 < L_{TN/2}$, respectively. In both cases, there is a stable situation with an equilibrium concentration of defects in the notched area.

Figure 16.8 shows numerical results of the defect density along the line characterizing distance to the notch in the plane with the maximum normal stress. It can be seen that when $\sigma_y > \sigma_0$ and $1 = L_{TN/2}$ there is no equilibrium defect concentration and the dissipative structure is localized on the spatial scale that is equal to the half of the critical distance obtained for Grade 2 specimen under quasi-static loading.

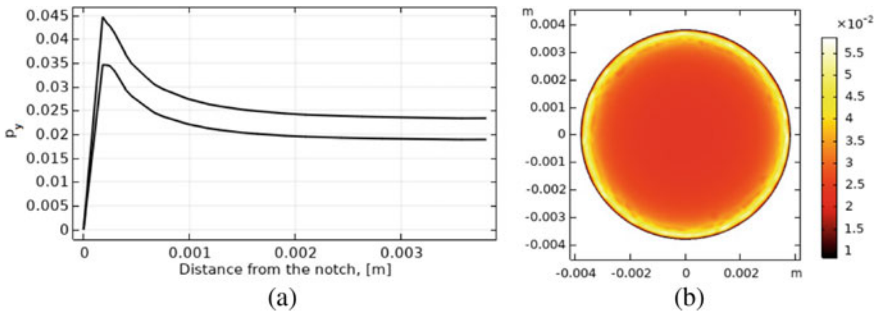

Fig. 16.6 **a** Values of p_y versus distance from the notch ($\sigma_y < \sigma_0$). **b** Spatial distribution of p_y component in the cross-sectional area perpendicular to the loading direction at the end-point to evolution

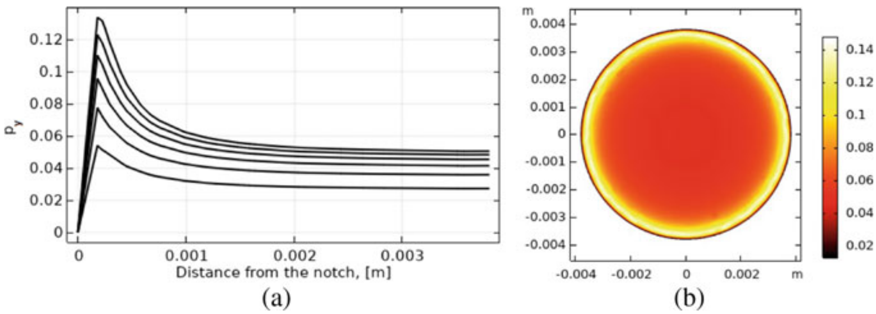

Fig. 16.7 **a** Values of p_y versus distance from the notch ($\sigma_y > \sigma_0, l_{c\sigma} < L_{TN}/2$). **b** Spatial distribution of p_y component in the cross-sectional area perpendicular to the loading direction at the end-point to evolution

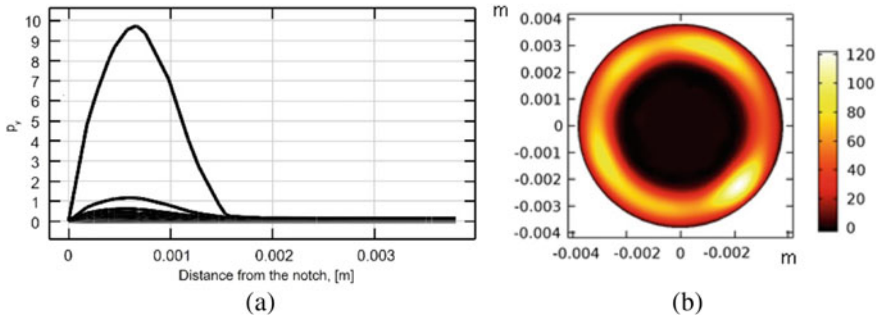

Fig. 16.8 **a** Values of p_y versus distance from the notch ($\sigma_y > \sigma_0, l_{c\sigma} = L_{TN}/2$). **b** Spatial distribution of component in the cross-sectional area perpendicular to the loading direction at the end-point to evolution

To get this scale equal to the 0.85 mm, we have used the material parameters included in Eq. (16.11): $q = 50.9, k = 2 \cdot 10^{-6}, a = 5.1$. Initial uniform distribution of p_{yy} is replaced by the heterogeneous p_{yy} with the localization zones where we can observe sharp increase in the defect density (Fig. 16.8b).

16.5 Conclusion

The aim of this work is to verify a reformulation of the linear-elastic TCD proposed in [15, 16] for notched specimens from titanium alloy Grade 2 under dynamic loading. It was shown that the TCD is capable of accurately assessing the static and dynamic strength of notched specimens within an error interval of ±20%.

The modification of the TCD in cases of elasto-plastic material behavior for dynamic loading was proposed. The use of an improved description of the stress–strain state at the notch tip allows estimating the fall within an error up to 10% and introducing the critical distances as a material parameter.

Last part of the work presents one of the possible physical explanation of the critical distance theory based on the statistical theory of defect evolution. As a result, it was shown that localization of the defect ensemble can be observed when two requirements are fulfilled: existence of the area where stresses are higher than inherent material strength and the spatial size of this area should be equal to the half of the critical distance. The critical distance mainly depends on the microstructural material morphology and can be considered as a fundamental length scale of dissipative structure developing in a blow-up regime.

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