Ambiguity Hierarchies for Weighted Tree Automata

Andreas Maletti, Teodora Nasz, Kevin Stier(✉), and Markus Ulbricht

Institute of Computer Science, Universität Leipzig,
PO box 100 920, 04009 Leipzig, Germany
{maletti,nasz,stier,mulbricht}@informatik.uni-leipzig.de

Abstract. Weighted tree automata (WTA) extend classical weighted automata (WA) to the non-linear structure of trees. The expressive power of WA with varying degrees of ambiguity has been extensively studied. Unambiguous, finitely ambiguous, and polynomially ambiguous WA over the tropical (as well as the arctic) semiring strictly increase in expressive power. The recently developed pumping results of Mazowiecki and Riveros (STACS 2018) are lifted to trees in order to achieve the same strict hierarchy for WTA over the tropical (as well as the arctic) semiring.

1 Introduction

Trees are a fundamental data structure in computer science and are used in many application areas like natural language processing, database theory, and compiler construction. All the mentioned applications require effective representations of sets of trees. These requirements triggered detailed investigations of various classes of such sets since the 1960s [11,12] and yielded an abundance of representations [6]. The most robust class is the class of regular tree languages [7,27]. It is generated by finite-state tree automata, which are a natural generalization of finite-state automata, which generate the regular languages [28]. Finite-state tree automata are a very effective representation and most standard decision problems remain decidable and the problem complexity is often similar to that of the corresponding problem for finite-state automata [6].

Quantitative extensions of finite-state automata, called weighted automata (WA) [25], as well as finite-state tree automata, called weighted tree automata (WTA) [8], have been proposed and thoroughly investigated. The weights are usually taken from a semiring like the nonnegative integers $\mathbb{N}$, the tropical semiring $\mathbb{T}$ [26] or the arctic semiring $\mathbb{A}$.

It is well-known that the computational properties improve dramatically for deterministic devices. While deterministic finite-state automata are as expressive as general finite-state automata, this equivalence breaks down for weighted automata over relevant semirings [2]. Thus, less restricted devices have been investigated as well. While general finite-state automata might allow exponentially many successful runs (in the length of the input) on a given input,

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deterministic finite-state automata naturally permit at most one successful run for each input that is additionally locally determined. We obtain polynomially ambiguous, finitely ambiguous, and unambiguous automata by requiring that for each input the number of successful runs is restricted by a polynomial, by a uniform bound, and by 1, respectively. The expressive power of weighted automata and weighted tree automata of limited ambiguity is actively investigated [17,18,21,22], but essential questions remain open.

Recently, it was established that unambiguous, finitely ambiguous, and polynomially ambiguous WA over the tropical semiring $\mathbb{T}$ strictly increase in expressive power [19]. This result was achieved with the help of pumping lemmas, which were also used to derive the same result [5] for the arctic semiring $\mathbb{A}$. The inclusion is obvious, but for the strictness results, pumping lemmas for the smaller classes are developed [19, Theorems 7, 14, and 18] and [5, Theorems 6.1 and 6.5]. These together with specific examples from the larger class that do not obey the pumping conditions establish the strictness.

Our goal is the development of a similar hierarchy for WTA. To this end, we utilize the same approach and develop the corresponding pumping results for WTA over $\mathbb{T}$ (Theorems 3, 7, and 11) and over $\mathbb{A}$ (Theorems 9 and 13). The main ingredient is a matrix representation of the behavior of a WTA along a tree decomposition into contexts (see Section 3) since it allows us to consider WTA as special weighted automata and apply the theorems of [5,19]. Along the way we prove that unambiguous WTA over $\mathbb{T}$ and $\mathbb{A}$ can be expressed as WTA over $\mathbb{N}_\infty$ (Lemma 1). In the end, we achieve the desired results and thus prove that finitely ambiguous WTA over $\mathbb{T}$ and $\mathbb{A}$ are strictly more expressive than unambiguous WTA (Theorems 4 and 5) and strictly less expressive than polynomially ambiguous WTA (Theorems 8 and 10). Finally, Theorems 12 and 14 illustrate that polynomially ambiguous WTA over those semirings are strictly less expressive than general WTA.

2 Preliminaries

Basic Notation. We denote the set of nonnegative integers (including 0) by $\mathbb{N}$. For every $k \in \mathbb{N}$ we use the subset $[k] = \{i \in \mathbb{N} \mid 1 \le i \le k\}$. For any set S the set of all finite words over S is $S^* = \bigcup_{k \in \mathbb{N}} S^k$, where $S^k = S \times \cdots \times S$ containing k factors S and $S^0 = \{\varepsilon\}$ contains just the *empty word* ε. The *length* $|w|$ of a word $w = s_1 \cdots s_k \in S^*$ with $s_1, \dots, s_k \in S$ is $|w| = k$; i.e., the number of occurrences of symbols in w. Given words $v, w \in S^*$, their concatenation is written $v.w$ or simply vw.

Trees. A *ranked alphabet* (Σ, rk) is a pair consisting of a finite set Σ and a mapping $\mathrm{rk}\colon \Sigma \to \mathbb{N}$ that assigns a rank to each symbol of Σ. If there is no risk of confusion, we denote a ranked alphabet (Σ, rk) by just Σ. We also write $\sigma^{(k)}$ to indicate that $\mathrm{rk}(\sigma) = k$. Moreover, for every $k \in \mathbb{N}$ we let $\Sigma^{(k)} = \{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k\}$. Given a ranked alphabet Σ and a set Z, the set $\mathrm{T}_\Sigma(Z)$ of Σ" trees indexed by Z is the smallest set T such that $Z \subseteq T$ and $\sigma(t_1, \dots, t_k) \in T$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \dots, t_k \in T$. We abbreviate $\mathrm{T}_\Sigma(\emptyset)$ simply to T_Σ, and any subset $L \subseteq \mathrm{T}_\Sigma$ is called a *tree language*.

Next, we recall some notions for trees. Let $t \in \mathrm{T}_\Sigma(Z)$ be a tree for a ranked alphabet Σ and a set Z. The set $\mathrm{pos}(t)$ of *positions* of t is inductively defined for all $z \in Z$ by $\mathrm{pos}(z) = \{\varepsilon\}$ and for all $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in \mathrm{T}_\Sigma(Z)$ by $\mathrm{pos}(\sigma(t_1, \ldots, t_k)) = \{\varepsilon\} \cup \{ip \mid i \in [k], p \in \mathrm{pos}(t_i)\}$. The *height* of t is $\mathrm{height}(t) = \max_{p \in \mathrm{pos}(t)} |p|$, and the *size* of t is $\mathrm{size}(t) = |\mathrm{pos}(t)|$. A *leaf* of t is a position $p \in \mathrm{pos}(t)$ such that $p1 \notin \mathrm{pos}(t)$. We denote the set of all leaves of t by $\mathrm{leaf}(t)$. Given a position $p \in \mathrm{pos}(t)$, the label $t(p)$ of t at p and the subtree $t|_p$ of t at p are given by $z(\varepsilon) = z|_\varepsilon = z$ for all $z \in Z$ and

$$\big(\sigma(t_1, \ldots, t_k)\big)(p) = \begin{cases} \sigma & \text{if } p = \varepsilon \\ t_i(p') & \text{if } p = ip' \text{ with } i \in \mathbb{N} \text{ and } p' \in \mathrm{pos}(t_i) \end{cases}$$

$$\sigma(t_1, \ldots, t_k)|_p = \begin{cases} \sigma(t_1, \ldots, t_k) & \text{if } p = \varepsilon \\ t_i|_{p'} & \text{if } p = ip' \text{ with } i \in \mathbb{N} \text{ and } p' \in \mathrm{pos}(t_i) \end{cases}$$

for all $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in \mathrm{T}_\Sigma(Z)$. Finally, the *replacement* $t[t']_p$ of the subtree at position $p \in \mathrm{pos}(t)$ by a tree $t' \in \mathrm{T}_\Sigma(Z)$ is given by $z[t']_\varepsilon = t'$ for all $z \in Z$ and

$$\sigma(t_1, \ldots, t_k)[t']_\varepsilon = t'$$
$$\sigma(t_1, \ldots, t_k)[t']_{ip'} = \sigma(t_1, \ldots, t_{i-1}, t_i[t']_{p'}, t_{i+1}, \ldots, t_k)$$

for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in \mathrm{T}_\Sigma(Z)$, $i \in [k]$, and $p' \in \mathrm{pos}(t_i)$.

We reserve the use of the special symbol $\square$. A tree $t \in \mathrm{T}_\Sigma(\{\square\})$ is a *context*, if there exists exactly one $p \in \mathrm{pos}(t)$ with $t(p) = \square$; i.e., there is exactly one occurrence of $\square$ in t. The set of all such contexts is denoted by C_Σ. Given a context $C \in \mathrm{C}_\Sigma$ and a tree $t \in \mathrm{T}_\Sigma(\{\square\})$, the substitution $C[t]$ of t into C yields the tree $C[t]_p$, where p is the unique position $p \in \mathrm{pos}(C)$ with $C(p) = \square$. Note that $C[C'] \in \mathrm{C}_\Sigma$ for $C, C' \in \mathrm{C}_\Sigma$. Similarly, we write C^k for $C[\cdots C[C] \cdots]$ containing k times the context C. The set of *decompositions* of $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$ is

$$D(\xi) = \bigcup_{\substack{k \geq 1 \\ C_1, \ldots, C_{k-1} \in \mathrm{C}_\Sigma \\ \xi' \in \mathrm{C}_\Sigma \cup \mathrm{T}_\Sigma}} \big\{(C_1, \ldots, C_{k-1}, \xi') \mid \xi = C_1[\cdots C_{k-1}[\xi'] \cdots]\big\}.$$

Note that $\xi \in \mathrm{T}_\Sigma$ iff $\xi' \in \mathrm{T}_\Sigma$ for every $(C_1, \ldots, C_{k-1}, \xi') \in D(\xi)$. The *depth* $\mathrm{depth}(C)$ of a context $C \in \mathrm{C}_\Sigma$ is $\mathrm{depth}(C) = |p|$, where $p \in \mathrm{pos}(C)$ is the unique position with $C(p) = \square$. A context $c \in \mathrm{C}_\Sigma$ of depth 1 is elementary, and the set of all such elementary contexts is denoted by E_Σ. A decomposition $(E_1, \ldots, E_k) \in D(C)$ of a context $C \in \mathrm{C}_\Sigma$ is *elementary* if $E_1, \ldots, E_k \in \mathrm{E}_\Sigma$. In fact, the monoid $(\mathrm{C}_\Sigma, \cdot[\cdot], \square)$ is freely generated by E_Σ [4], which proves the existence of an elementary decomposition for each context. Finally, let $t \in \mathrm{T}_\Sigma$, $\mathcal{C} = (D_n, C_n, \ldots, D_1, C_1, s) \in D(t)$ and $\mathcal{D} = (D'_n, C'_n, \ldots, D'_1, C'_1, s') \in D(t)$ be decompositions of the tree t. We call $\mathcal{D}$ a *refinement* of $\mathcal{C}$ (refining the occurrences of C_i) if for every $i \in [n]$ there exist $L_i, R_i \in \mathrm{C}_\Sigma$ such that $D'_i = R_{i+1}[D_i[L_i]]$, $s' = R_1[s]$, and $C_i = L_i\big[C'_i[R_i]\big]$, where $R_{n+1} = \square$.

Weighted Automata. A *commutative semiring* [13,15] is a tuple $(S, +, \cdot, 0, 1)$ such that both $(S, +, 1)$ and $(S, \cdot, 1)$ are commutative monoids, $\cdot$ distributes over $+$, and $0 \cdot s = 0$ for all $s \in S$. More specifically we consider

- the Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$,
- the extended Boolean semiring $\mathbb{B}_\infty = (\{0, 1, \infty\}, \vee, \wedge, 0, 1)$ with $\infty \vee n = \infty$ for all $n \in \{0, 1, \infty\}$ and $\infty \wedge 0 = 0$ and $\infty \wedge 1 = \infty \wedge \infty = \infty$,
- the tropical semiring $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$,
- the arctic semiring $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$, and
- the extended semiring $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ of nonnegative integers.

We will refer to a semiring $(S, +, \cdot, 0, 1)$ by its carrier set S.

A *weighted automaton* (WA) [24] over S is a tuple $\mathcal{A} = (Q, A, I, (M_a)_{a \in A}, F)$, where Q is a finite set of *states*, A is a finite set of *symbols*, $M_a \in S^{Q \times Q}$ is a *transition weight* matrix for every $a \in A$, and $I, F \in S^Q$ are *initial* and *final weight* vectors, respectively. Given a word $w = a_1 \cdots a_n$ with $a_1, \ldots, a_n \in A$, we let $M_w = M_{a_1} \cdot \ldots \cdot M_{a_n}$ with standard matrix multiplication using the semiring operations.

Finally, the weighted language $[\![\mathcal{A}]\!] \colon A^* \to S$ recognized by $\mathcal{A}$ is defined for every $w \in A^*$ by $[\![\mathcal{A}]\!](w) = I^T \cdot M_w \cdot F$. A weighted language $f \colon A^* \to S$ is *recognizable* if there exists a WA recognizing it.

A *weighted tree automaton* (WTA) [10] over S is a tuple $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$, where Q is a finite set of *states*, Σ is a ranked alphabet, $\Delta \subseteq \bigcup_{k \in \mathbb{N}} Q^k \times \Sigma^{(k)} \times Q$ is a set of *transitions*, $\mathrm{wt} \colon \Delta \to S$ is a *transition weight* function, and $F \in S^Q$ is a *root weight* vector. We generally assume that $\mathrm{wt}(\tau) \neq 0$ for all $\tau \in \Delta$, and we write $\sigma(q_1, \ldots, q_k) \xrightarrow{s} q$ for a transition $\tau = (q_1, \ldots, q_k, \sigma, q) \in \Delta$ with $\mathrm{wt}(\tau) = s$. Weighted tree automata over the Boolean semiring (i.e., for $S = \mathbb{B}$) are also called *tree automata* (TA) and their weight function 'wt' is superfluous. Given $t \in \mathrm{T}_\Sigma$, a mapping $r \colon \mathrm{pos}(t) \to Q$ is called *run* of $\mathcal{A}$ on t, if $(r(p1), \ldots, r(pk), t(p), r(p)) \in \Delta$ for all $p \in \mathrm{pos}(t)$, where $k = \mathrm{rk}(t(p))$. The run is *accepting* if $F_{r(\varepsilon)} \neq 0$. We denote the set of all accepting runs of $\mathcal{T}$ on t by $\mathrm{Run}_\mathcal{T}(t)$. Moreover, for every $q \in Q$ let $\mathrm{Run}^q_\mathcal{T}(t) = \{r \in \mathrm{Run}_\mathcal{T}(t) \mid r(\varepsilon) = q\}$ be the set of runs with root label q. The *weight* of a run $r \in \mathrm{Run}_\mathcal{T}(t)$ is

$$\mathrm{wt}_\mathcal{T}(r) = \prod_{\substack{p \in \mathrm{pos}(t) \\ k = \mathrm{rk}(t(p))}} \mathrm{wt}\big(r(p1), \ldots, r(pk), t(p), r(p)\big).$$

The weighted tree language $[\![\mathcal{T}]\!] \colon \mathrm{T}_\Sigma \to S$ *recognized* by $\mathcal{T}$ is defined for every tree $t \in \mathrm{T}_\Sigma$ by $[\![\mathcal{T}]\!](t) = \sum_{r \in \mathrm{Run}_\mathcal{T}(t)} \mathrm{wt}_\mathcal{T}(r) \cdot F_{r(\varepsilon)}$. A weighted tree language $f \colon \mathrm{T}_\Sigma \to S$ is *recognizable* if there exists a WTArecognizing it. The class of recognizable weighted tree languages over S is denoted by $\mathrm{RTL}(S)$.

3 Matrix Representation and Ambiguity

For our pumping arguments we first need a matrix-like representation for the weighted tree language recognized by a WTA $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$ that is similar to that of weighted automata. Since processing a symbol σ of rank k requires k vectors from the subtrees, we can directly utilize KRONECKER products [9] or tensor products [23], but a simpler approach [3] using contexts, whose processing again will only require a single vector for the subtree replacing $\square$, actually suffices for our purposes. Our run semantics is rather unsuitable for this purpose, so let us recall the equivalent initial algebra semantics [1]. We immediately present the extended variant that can handle contexts as well. For every $\xi \in \mathrm{T}_\Sigma(\{\square\})$ we inductively define the weight matrix $\mathrm{wt}_\mathcal{T}(\xi) \in S^{Q \times Q}$ by

– $\mathrm{wt}_\mathcal{T}(\square)_{q,q} = 1$ and $\mathrm{wt}_\mathcal{T}(\square)_{q,q'} = 0$ for all $q, q' \in Q$ with $q \neq q'$, and
– for all $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in \mathrm{T}_\Sigma(\{\square\})$, and $q, q' \in Q$

$$\mathrm{wt}_\mathcal{T}\big(\sigma(t_1, \ldots, t_k)\big)_{q,q'} = \sum_{(q_1,\ldots,q_k,\sigma,q') \in \Delta} \mathrm{wt}(q_1, \ldots, q_k, \sigma, q') \cdot \prod_{i=1}^{k} \mathrm{wt}_\mathcal{T}(t_i)_{q,q_i} .$$

Note that $\mathrm{wt}_\mathcal{T}(t)_{q_1,q'} = \mathrm{wt}_\mathcal{T}(t)_{q_2,q'}$ for all $q_1, q_2, q' \in Q$ and $t \in \mathrm{T}_\Sigma$. Hence we identify $\mathrm{wt}_\mathcal{T}(t)$ with a vector of S^Q and obtain $[\![\mathcal{T}]\!](t) = \mathrm{wt}_\mathcal{T}(t)^T \cdot F$ for all $t \in \mathrm{T}_\Sigma$ by [3, Lemma 4.1.13] as well as

$$\mathrm{wt}_\mathcal{T}(c[\xi]) = \mathrm{wt}_\mathcal{T}(\xi) \cdot \mathrm{wt}_\mathcal{T}(c) \tag{1}$$

for all contexts $c \in \mathrm{C}_\Sigma$ and $\xi \in \mathrm{T}_\Sigma \cup \mathrm{C}_\Sigma$ by [3, Lemma 4.1.8].

Next we recall the relevant notions of *ambiguity*. Let $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$ be a WTA. For a given $\ell \in \mathbb{N}$, the WTA $\mathcal{T}$ is *ℓ-ambiguous* if every tree $t \in \mathrm{T}_\Sigma$ has at most ℓ accepting runs; i.e., $|\mathrm{Run}_\mathcal{T}(t)| \leq \ell$. It is *unambiguous* (or a UA-WTA) if it is 1-ambiguous, and it is *finitely ambiguous* (or an FA-WTA) if there exists $\ell \in \mathbb{N}$ such that $\mathcal{T}$ is ℓ-ambiguous. For the notions of '*polynomially ambiguous*' and '*exponentially ambiguous*' we distinguish two variants: one based on the size and another based on the height of the input tree. More precisely, $\mathcal{T}$ is *polynomially ambiguous* in $f\colon \mathrm{T}_\Sigma \to \mathbb{N}$ if there exists a polynomial P such that $|\mathrm{Run}_\mathcal{T}(t)| \leq P(f(t))$ for all $t \in \mathrm{T}_\Sigma$. We say that $\mathcal{T}$ is a PA-WTA (respectively, a PAH" WTA) if it is polynomially ambiguous in 'size' (respectively, in 'height'). Similarly, $\mathcal{T}$ is *exponentially ambiguous* in $f\colon \mathrm{T}_\Sigma \to \mathbb{N}$ if there exists an exponential e such that $|\mathrm{Run}_\mathcal{T}(t)| \leq e(f(t))$. We say that $\mathcal{T}$ is an EA-WTA(respectively, an EAH-WTA) if it is exponentially ambiguous in 'size' (respectively, in 'height'). Note that every WTA $(Q, \Sigma, \Delta, \mathrm{wt}, F)$ is an EA-WTAbecause there are naturally at most $|Q|^{\mathrm{size}(t)}$ runs for every input tree t. We use the same prefixes π in front of $\mathrm{RTL}(S)$ for the class of weighted tree languages over S that are recognizable by π-WTA. For example, PA-RTL(S) is the class of those weighted tree languages over S that are recognizable by WTAthat are polynomially ambiguous in size.

4 Unambiguous vs. Finitely Ambiguous

In this section we present a weighted tree language over the tropical semiring $\mathbb{T}$ that is recognized by a FA-WTA but cannot be recognized by any UA-WTA, which proves that UA-RTL($\mathbb{T}$) $\subsetneq$ FA-RTL($\mathbb{T}$). The main component of this result is a pumping result for recognizable weighted tree languages over $\mathbb{N}_\infty$, which is applicable due to the folklore result UA-RTL($\mathbb{T}$) $\subseteq$ RTL($\mathbb{N}_\infty$), which we recall first. The inclusion follows from the well-known construction that is used to show that size $\in$ RTL($\mathbb{N}_\infty$).

Lemma 1. UA-RTL($\mathbb{T}$) $\subseteq$ RTL($\mathbb{N}_\infty$).

The matrix representation allows us to apply a well-known result of idempotent elements, which we recall next. Given a monoid $(M, \cdot, 1)$ an element $m \in M$ is *idempotent* if $m \cdot m = m$. The following well-known result for finite monoids, which states that any sequence of sufficiently many factors contains a nonempty subsequence of factors whose product is idempotent, is the main tool for our first pumping result.

Lemma 2 (e.g. [14, Theorem 3.1]). *Let M be a finite monoid. There exists a constant $N > 0$ such that for all $n \geq N$ and $x_1, \ldots, x_n \in M$ there exist $\ell, u \in \mathbb{N}$ with $\ell < u \leq n$ such that $\prod_{i=\ell+1}^{u} x_i$ is idempotent.*

Theorem 3 (Pumping Lemma for RTL($\mathbb{N}_\infty$)**).** *Let $f \in$ RTL($\mathbb{N}_\infty$). There exists $N \in \mathbb{N}$ such that for each tree $t \in \mathrm{T}_\Sigma$ and decomposition $\mathcal{C} = (D, C, s) \in D(t)$ with $\mathrm{depth}(C) \geq N$ there is a refinement $(D', B, s') \in D(t)$ of $\mathcal{C}$ with $B \neq \Box$ such that*

- $f\big(D'[B^h[s']]\big) = f\big(D'[B^{h+1}[s']]\big)$ *for all $h \geq N$ or*
- $f\big(D'[B^h[s']]\big) < f\big(D'[B^{h+1}[s']]\big)$ *for all $h \geq N$.*

Next we present a weighted tree language $f \in$ FA-RTL($\mathbb{T}$) $\setminus$ UA-RTL($\mathbb{T}$) inspired by [19, Examples 2 and 8]. We explicitly show $f \in$ FA-RTL($\mathbb{T}$) as well as $f \notin$ RTL($\mathbb{N}_\infty$) using Theorem 3. The latter result yields $f \notin$ UA-RTL($\mathbb{T}$) by Lemma 1. For those particular weighted tree languages in the differences we use a ranked alphabet with a single binary symbol and a single nullary symbol. By various encodings (e.g., first-child-next-sibling [6, Proposition 8.3.2]) these results apply to essentially any ranked alphabet. This correspondence extends to weighted tree languages (see [16, Lemma 4.2]).

Theorem 4. UA-RTL($\mathbb{T}$) $\subsetneq$ FA-RTL($\mathbb{T}$).

Proof. Let $\Sigma = \{\sigma^{(2)}\alpha^{(0)}\}$ be a ranked alphabet and $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$ a WTA over $\mathbb{T}$ with $Q = \{q_\ell, q_r, q_\alpha\}$, $F(q) = 0$ for each $q \in Q$, and the following transitions and weights

$$\big\{\alpha \xrightarrow{0} q_\alpha \quad \sigma(q_\alpha, q_\ell) \xrightarrow{1} q_\ell \quad \sigma(q_\ell, q_\alpha) \xrightarrow{0} q_\ell \quad \sigma(q_\ell, q_\ell) \xrightarrow{0} q_\ell \quad \sigma(q_\alpha, q_\alpha) \xrightarrow{1} q_\ell$$
$$\sigma(q_r, q_\alpha) \xrightarrow{1} q_r \quad \sigma(q_\alpha, q_r) \xrightarrow{0} q_r \quad \sigma(q_r, q_r) \xrightarrow{0} q_r \quad \sigma(q_\alpha, q_\alpha) \xrightarrow{1} q_r\big\}.$$

Clearly, $\mathcal{T}$ has two runs for each input tree t, in which we mark all leaves by q_α and then proceed to count either occurrences of $L = \sigma(\alpha, \square)$ using q_ℓ or occurrences of $R = \sigma(\square, \alpha)$ using q_r. We thus calculate the minimum of occurrences of L and R, and $f = [\![\mathcal{T}]\!] \in$ FA-RTL($\mathbb{T}$).

Now let us apply our pumping lemma in order to prove that no WTAover $\mathbb{N}_\infty$ can recognize f. We observe that $f(R^n[L^m[\alpha]]) = \min(m, n)$ for all $m, n \in \mathbb{N}$. Assume that $f \in$ RTL($\mathbb{N}_\infty$). Let N be the constant of Theorem 3 applied to f, and let $t = R^{(N+1)^2}[L^N[\alpha]]$ and $\mathcal{C} = (D, C, \alpha) \in D(t)$ be a decomposition, where $D = R^{(N+1)^2}$ and $C = L^N$. Theorem 3 yields a refinement $(D', B, s') \in D(t)$ of $\mathcal{C}$; i.e., $B = L^n$ for some $0 < n < N$. However,

$$f\big(D'[B^N[s']]\big) = (n+1)N - n \;<\; (n+1)N = f\big(D'[B^{N+1}[s']]\big)$$
$$f\big(D'[B^{(N+1)^2}[s']]\big) = (N+1)^2 \;=\; (N+1)^2 = f\big(D'[B^{2(N+1)^2}[s']]\big),$$

contradicting Theorem 3. Hence $f \notin$ RTL($\mathbb{N}_\infty$). Since UA-RTL($\mathbb{T}$) $\subseteq$ RTL($\mathbb{N}_\infty$) by Lemma 1 we obtain $f \notin$ UA-RTL($\mathbb{T}$) as desired. □

Since clearly UA-RTL($\mathbb{T}$) = UA-RTL($\mathbb{A}$), we may replace the minimum in the proof of Theorem 4 with a maximum and similar calculations show that this language is not in RTL($\mathbb{N}_\infty$), either.

Theorem 5. UA-RTL($\mathbb{A}$) $\subsetneq$ FA-RTL($\mathbb{A}$).

5 Finitely Vs. Polynomially Ambiguous

The second pumping lemma will allow us to give a weighted tree language over $\mathbb{T}$, which can be recognized by a PA-WTA, but cannot be recognized by any FA-WTA. The theorem itself works on point-wise minima of recognizable weighted tree languages over $\mathbb{N}_\infty$. We call $f \colon \mathrm{T}_\Sigma \to \mathbb{N}_\infty$ a *point-wise recognizable minimum* if there exist $k \in \mathbb{N}$ and recognizable weighted tree languages $f_1, \ldots, f_k \in$ RTL($\mathbb{N}_\infty$) of type $f_1, \ldots, f_k \colon \mathrm{T}_\Sigma \to \mathbb{N}_\infty$ such that for all $t \in \mathrm{T}_\Sigma$ it holds that $f(t) = \min\{f_1(t), \ldots, f_k(t)\}$. To relate this notion to finitely ambiguous weighted tree languages over $\mathbb{T}$, we recall the following result.

Theorem 6 ([20, Theorem 2]). *Let $\ell \in \mathbb{N}$ and $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$ be an ℓ-ambiguous* WTA *over the commutative semiring S. Then there exist ℓ unambiguous* WTA *$\mathcal{U}_1, \ldots, \mathcal{U}_\ell$ over S such that $[\![\mathcal{T}]\!] = \sum_{i=1}^{\ell} [\![\mathcal{U}_i]\!]$.*

Let us now move on to the pumping lemma. To this end, let $t \in \mathrm{T}_\Sigma$ be a tree and $\mathcal{D} = (D_n, C_n, \ldots, D_1, C_1, s) \in D(t)$ be a decomposition of t. Additionally, let $h \in \mathbb{N}$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{B}^n$ be a selector. Then we let

$$\mathcal{D}^h_\varphi = D_n\big[C_n^{\varphi_n \cdot h}\big[\cdots D_1[C_1^{\varphi_1 \cdot h}[s]] \cdots\big]\big]. \tag{2}$$

Theorem 7 (Pumping Lemma for Point-wise Minima). *Let $f\colon \mathrm{T}_\Sigma \to \mathbb{N}_\infty$ be a point-wise recognizable minimum. Then there exists $N \in \mathbb{N}$ such that for each tree $t \in \mathrm{T}_\Sigma$ and decomposition $\mathcal{C} = (D_n, C_n, \dots, D_1, C_1, s) \in D(t)$ of t with $n \geq N$ and $\operatorname{depth}(C_j) \geq N$ for all $j \in [n]$ the following holds. There is a refinement $\mathcal{D} = (D'_n, B_n, \dots, D'_1, B_1, s') \in D(t)$ of $\mathcal{C}$ with $B_1, \dots, B_n \neq \Box$ such that for every subset $\Phi \subseteq \mathbb{B}^n$ with $|\Phi| \geq N$*

- *there exists $\varphi \in \Phi$ such that $f(\mathcal{D}^h_\varphi) < f(\mathcal{D}^{h+1}_\varphi)$ for all h sufficiently large or*
- *there are $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$ such that $f(\mathcal{D}^h_{\varphi\vee\psi}) = f(\mathcal{D}^{h+1}_{\varphi\vee\psi})$ for all h sufficiently large.*

Finally, we give a weighted tree language $f \in \text{PA-RTL}(\mathbb{T}) \setminus \text{FA-RTL}(\mathbb{T})$ inspired by [19, Examples 3 and 15]. To this end, we show that $f \in \text{PA-RTL}(\mathbb{T})$ and that f is not a point-wise recognizable minimum over $\mathbb{T}$ using Theorem 7. By Theorem 6 we can then conclude that $f \notin \text{FA-RTL}(\mathbb{T})$.

Theorem 8. $\text{FA-RTL}(\mathbb{T}) \subsetneq \text{PA-RTL}(\mathbb{T})$.

Proof. We consider the ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and the two elementary contexts $R = \sigma(\Box, \alpha)$ and $L = \sigma(\alpha, \Box)$. Additionally, we consider the WTA $\mathcal{T} = (Q, \Sigma, \Delta, \text{wt}, F)$ over $\mathbb{T}$ with $Q = \{q_\ell, q_r, q_\alpha\}$, $F(q_\ell) = F(q_\alpha) = \infty$ and $F(q_r) = 0$, and the following transitions and weights

$$\begin{aligned}
\big\{\alpha \xrightarrow{0} q_\alpha \quad \sigma(q_\alpha, q_\ell) \xrightarrow{1} q_\ell \quad \sigma(q_\ell, q_\alpha) \xrightarrow{0} q_\ell \quad \sigma(q_\alpha, q_\alpha) \xrightarrow{0} q_\ell \qquad &(\text{counting } L)\\
\sigma(q_r, q_\alpha) \xrightarrow{1} q_r \quad \sigma(q_\alpha, q_r) \xrightarrow{0} q_r \quad \sigma(q_\alpha, q_\alpha) \xrightarrow{0} q_r \qquad &(\text{counting } R)\\
\sigma(q_\ell, q_\alpha) \xrightarrow{0} q_r \quad \sigma(q_\alpha, q_\ell) \xrightarrow{1} q_r\big\}. \qquad &(\text{switch})
\end{aligned}$$

In each run, reading the input tree bottom-up the WTA $\mathcal{T}$ first counts occurrences of L in state q_ℓ, then nondeterministically switches to q_r, and finally counts occurrences of R in state q_r. Thus, $\mathcal{T}$ has at most $\operatorname{height}(t)$ runs for each $t \in \mathrm{T}_\Sigma$, which proves that $f = [\![\mathcal{T}]\!] \in \text{PA-RTL}(\mathbb{T})$. Additionally, we have

$$f(t) = \min_{i \in [n]} \Big\{ |\{j \in [i] \mid C_j = R\}| + |\{j \in [n] \setminus [i] \mid C_j = L\}| \Big\}$$

for a tree of the form $t = C_1[\cdots C_n[\sigma(\alpha, \alpha)] \cdots]$ with $C_1, \dots, C_n \in \{L, R\}$ and $n \in \mathbb{N}$, and $f(t) = \infty$ otherwise.

It remains to show that $f \notin \text{FA-RTL}(\mathbb{T})$, which we prove by showing that f cannot be a point-wise recognizable minimum. For the sake of a contradiction, suppose that it is. Let N be the constant of Theorem 7 and consider the decomposition $\mathcal{C} = \big(\Box, L^N, \Box, R^N, \dots, \Box, L^N, \Box, R^N, \sigma(\alpha, \alpha)\big) \in D(t)$ of $t = (L^N[R^N])^N[\sigma(\alpha, \alpha)]$. Moreover, for every $j \in [N]$, let $\varphi^j \in \mathbb{B}^{2N}$ be such that $\varphi^j(2j-1) = \varphi^j(2j) = 1$ and 0 otherwise. Finally, let $\Phi = \{\varphi^1, \dots, \varphi^N\}$. We first claim that $f(\mathcal{D}^h_\varphi) = N(N-1)$ for each refinement $\mathcal{D} \in D(t)$ of $\mathcal{C}$, $\varphi \in \Phi$, and $h > N$. To see this, let $j \in [N]$ be such that $\varphi = \varphi^j$. Then $\mathcal{D}^h_\varphi$ only pumps one part of the j-th block of $L^N[R^N]$ to $L^{\ell_1}[R^{\ell_2}]$ for some $\ell_1, \ell_2 > N$. Thus for the minimum, the WTA $\mathcal{T}$ should switch from q_ℓ to q_r after processing the segment R^{ℓ_2}, which yields $f(\mathcal{D}^h_\varphi) = N(N-1)$. For the second item of Theorem 7, let $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$ and $\phi = \varphi \vee \psi$. Since φ and ψ select different blocks, no matter where the WTA $\mathcal{T}$ switches from q_ℓ to q_r we will either count the pumped occurrences of L or the pumped occurrences of R in at least one block. Thus, $f(\mathcal{D}^h_\phi) < f(\mathcal{D}^{h+1}_\phi)$ for all $h > N$. Thus, f is not a point-wise recognizable minimum, which together with Theorem 6 proves $f \notin \text{FA-RTL}(\mathbb{T})$. □

The height and size of the input trees in Theorem 8, for which accepting runs exist, are linearly related, so we also obtain $\text{FA-RTL}(\mathbb{T}) \subsetneq \text{PAH-RTL}(\mathbb{T})$.

In fact, using [5, Theorem 6.1] we are able to present similar results for the arctic semiring $\mathbb{A}$. Let $\mathcal{C} = (D_1, C_1, \ldots, D_n, C_n, s) \in D(t)$ be a decomposition of a tree $t \in \mathrm{T}_\Sigma$ and $f \colon \mathrm{T}_\Sigma \to \mathbb{A}$ be a weighted tree language. The decomposition $\mathcal{C}$ is *linear* if for all $\varphi \in \mathbb{B}^n$ there is a constant K_φ such that $f(\mathcal{C}^{h+1}_\varphi) = K_\varphi + f(\mathcal{C}^h_\varphi)$ for all sufficiently large h. Given a linear decomposition $\mathcal{C}$, a selector $\phi \in \mathbb{B}^n$ is *elementarily linear* for $\mathcal{C}$ if $K_\phi = \sum_{j=1}^n \phi_j \cdot K_{\mathbb{1}_j}$, where $\mathbb{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 occurring in the j-th component.

Theorem 9 (Pumping Lemma for $\text{FA-RTL}(\mathbb{A})$**).** *Let $f \in \text{FA-RTL}(\mathbb{A})$. There exists a constant $N \in \mathbb{N}$ such that for each tree $t \in \mathrm{T}_\Sigma$ and decomposition $\mathcal{C} = (D_n, C_n, \ldots, D_1, C_1, s) \in D(t)$ of t with $n \geq N$ and $\text{depth}(C_j) \geq N$ for all $j \in [n]$, there exists a linear refinement $\mathcal{D} = (D'_n, B_n, \ldots, D'_1, B_1, s') \in D(t)$ of $\mathcal{C}$ with $B_1, \ldots, B_n \neq \square$ such that for every subset $\Phi \subseteq \mathbb{B}^n$ with $|\Phi| \geq N$*

- *there exists $\varphi \in \Phi$ that is not elementarily linear for $\mathcal{D}$ or*
- *there exist $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$ such that $\mathbb{1}_i \vee \mathbb{1}_j$ is elementarily linear for $\mathcal{D}$ for all $i, j \in [n]$ with $\varphi_i = 1$ and $\psi_j = 1$.*

Theorem 10. $\text{FA-RTL}(\mathbb{A}) \subsetneq \text{PA-RTL}(\mathbb{A})$.

Proof. We reconsider the WTA $\mathcal{T} = (Q, \Sigma, \Delta, \text{wt}, F)$ of the proof of Theorem 8 over the arctic semiring $\mathbb{A}$. Clearly, $f = [\![\mathcal{T}]\!] \in \text{PA-RTL}(\mathbb{A})$. Additionally, we have

$$f(t) = \max_{i \in [n]} \Big\{ |\{j \in [i] \mid C_j = R\}| + |\{j \in [n] \setminus [i] \mid C_j = L\}| \Big\}$$

for a tree of the form $t = C_1[\cdots C_n[\sigma(\alpha, \alpha)] \cdots]$ with $C_1, \ldots, C_n \in \{L, R\}$ and $n \in \mathbb{N}$, and $f(t) = -\infty$ otherwise. For the proof of $f \notin \text{FA-RTL}(\mathbb{A})$ we use the same technique as in the proof of Theorem 8. Since now the maximum is taken, each $\varphi \in \Phi$ is elementarily linear for $\mathcal{D}$. For the second condition, let $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$ and $i, j \in [n]$ such that $\varphi_i = 1$ and $\psi_j = 1$. However, $\phi = \mathbb{1}_i \vee \mathbb{1}_j$ is not elementarily linear for $\mathcal{D}$ since $K_\phi = \max(K_{\mathbb{1}_i}, K_{\mathbb{1}_j})$. □

6 Polynomially Ambiguous Vs. Recognizable

Our last pumping lemma will allow us to present a recognizable weighted tree language over $\mathbb{T}$ that is not recognizable by any PAH-WTA. To this end, we introduce some additional notation. Let $n \in \mathbb{N}$. A set $\Phi \subseteq \mathbb{B}^n \setminus \{(0, \ldots, 0)\}$ is called a *partition* of $[n]$ if $\bigvee \Phi = (1, \ldots, 1)$ and $\varphi \wedge \psi = (0, \ldots, 0)$ for all $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$. We call $\psi \in \mathbb{B}^n$ a *cover* of Φ if $\sum_{j \in [n]} (\psi \wedge \varphi)_j = 1$ for every $\varphi \in \Phi$.

Theorem 11 (Pumping Lemma for PAH-RTL($\mathbb{T}$)**).** *Let $f \in$ PAH-RTL($\mathbb{T}$). There exists $N \in \mathbb{N}$ and a mapping $c\colon \mathbb{N} \to \mathbb{N}$ such that for each tree $t \in \mathrm{T}_\Sigma$ and decomposition $\mathcal{C} = (D_n, C_n, \ldots, D_1, C_1, s) \in D(t)$ of t with* $\operatorname{depth}(C_i) \geq N$ *for all $j \in [n]$, there exists a refinement $\mathcal{D} = (D'_n, B_n, \ldots, D'_1, B_1, s') \in D(t)$ of $\mathcal{C}$ such that for every partition Φ of $[n]$ with $|\Phi| \geq c(\sum_{j\in[n]} \varphi_j)$ for all $\varphi \in \Phi$*

- *there exists $\varphi \in \Phi$ such that $f(\mathcal{D}^h_\varphi) = f(\mathcal{D}^{h+1}_\varphi)$ for all h sufficiently large or*
- *there exists a cover ψ of Φ such that $f(\mathcal{D}^h_\psi) < f(\mathcal{D}^{h+1}_\psi)$ for all h sufficiently large.*

Now we give a recognizable weighted tree language $f \notin$ PA-RTL($\mathbb{T}$). We will actually show $f \notin$ PAH-RTL($\mathbb{T}$), but due to the special shape of f the height and size are themselves polynomially related, so $f \notin$ PAH-RTL($\mathbb{T}$) implies $f \notin$ PA-RTL($\mathbb{T}$). In contrast to the previous examples Theorem 11 operates directly on the tropical semiring.

Theorem 12. PA-RTL($\mathbb{T}$) $\subsetneq$ RTL($\mathbb{T}$).

Proof. We consider the ranked alphabet $\Sigma = \{\sigma^{(2)}, \tau^{(1)}, \alpha^{(0)}\}$, $s = \sigma(\alpha, \alpha)$, and the contexts $R = \sigma(\Box, \alpha)$ and $L = \sigma(\alpha, \Box)$ as before. Additionally, we consider the weighted tree language $f\colon \mathrm{T}_\Sigma \to \mathbb{T}$ such that for every $t \in \mathrm{T}_\Sigma$

$$f(t) = \begin{cases} \sum_{\ell=1}^{k} \min(i_\ell, j_\ell) & \text{if } t = \tau\big(L^{i_1}\big[R^{j_1}\big[\cdots\big[\tau(L^{i_k}[R^{j_k}[s]])\big]\cdots\big]\big]\big) \\ & \qquad \text{for } (i_1, \ldots, i_k), (j_1, \ldots, j_k) \in \mathbb{N}^k \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} = (Q, \Sigma, \Delta, \mathrm{wt}, F)$ be the WTAover $\mathbb{T}$ with $Q = \{q_\ell, q_r, q_\alpha\}$, $F(q) = 0$ for each $q \in Q$, and the following transitions and weights

$$\begin{array}{lllll} \big\{\alpha \xrightarrow{0} q_\alpha & \sigma(q_\alpha, q_\ell) \xrightarrow{1} q_\ell & \sigma(q_\ell, q_\alpha) \xrightarrow{0} q_\ell & \sigma(q_\alpha, q_\alpha) \xrightarrow{0} q_\ell & \text{(counting } L\text{)} \\ & \sigma(q_r, q_\alpha) \xrightarrow{1} q_r & \sigma(q_\alpha, q_r) \xrightarrow{0} q_r & \sigma(q_\alpha, q_\alpha) \xrightarrow{0} q_r & \text{(counting } R\text{)} \\ \tau(q_\ell) \xrightarrow{0} q_\ell & \tau(q_\ell) \xrightarrow{0} q_r & \tau(q_r) \xrightarrow{0} q_\ell & \tau(q_r) \xrightarrow{0} q_r\big\}. & \text{(reset)} \end{array}$$

Clearly, $\mathcal{T}$ recognizes f and thus $f \in$ RTL($\mathbb{T}$). Assume now $f \in$ PAH-RTL, as mentioned above by the special shape of f, this implies $f \in$ PA-RTL. By Theorem 11 there exist a constant N and a mapping $c\colon \mathbb{N} \to \mathbb{N}$ with various properties. Let $m > c(2)$ and consider $t = (\tau[L^N[R^N]])^m[s]$ with decomposition $\mathcal{C} = (\tau(\Box), L^N, \Box, R^N, \ldots, \tau(\Box), L^N, \Box, R^N, s)$. Additionally, for each $j \in [m]$ let $\varphi^j \in \mathbb{B}^{2N}$ be such that $\varphi^j(2j-1) = \varphi^j(2j) = 1$ and 0 otherwise. Finally, let $\Phi = \{\varphi^1, \ldots, \varphi^m\}$. Clearly, $f(t) = Nm$. However, for every refinement $\mathcal{D}$ of $\mathcal{C}$ and $h > N$ we have $f(\mathcal{D}^h_\varphi) < f(\mathcal{D}^{h+1}_\varphi)$ for every $\varphi \in \Phi$ as well as $f(\mathcal{D}^h_\psi) = f(\mathcal{D}^{h+1}_\psi)$ for every cover ψ of Φ. $\Box$

As before, we collect the corresponding results for the arctic semiring $\mathbb{A}$.

Theorem 13 (Pumping Lemma for PAH-RTL($\mathbb{A}$)**).** *Let* $f \in$ PAH-RTL($\mathbb{A}$). *There exists* $N \in \mathbb{N}$ *and a mapping* $c\colon \mathbb{N} \to \mathbb{N}$ *such that for each tree* $t \in \mathrm{T}_\Sigma$ *and decomposition* $\mathcal{C} = (D_n, C_n, \dots, D_1, C_1, s) \in D(t)$ *of* t *with* $\operatorname{depth}(C_i) \geq N$ *for all* $j \in [n]$*, there exists a linear refinement* $\mathcal{D} = (D'_n, B_n, \dots, D'_1, B_1, s') \in D(t)$ *of* $\mathcal{C}$ *such that for every partition* Φ *of* $[n]$ *with* $|\Phi| \geq c(\sum_{j\in[n]} \varphi_j)$ *for all* $\varphi \in \Phi$

- *there exists* $\varphi \in \Phi$ *that is elementarily linear for* $\mathcal{D}$ *or*
- *there exists a cover* ψ *of* Φ *that is not elementarily linear for* $\mathcal{D}$*.*

Theorem 14. PA-RTL($\mathbb{A}$) $\subsetneq$ RTL($\mathbb{A}$).

Proof. Reconsider the WTA $\mathcal{T}$ as well as the other infrastructure of the proof of Theorem 12 over the arctic semiring $\mathbb{A}$ and its recognized mapping $g = [\![\mathcal{T}]\!]$, which is essentially the mapping f with the minimum replaced by the maximum. It is straightforward to see that no $\varphi \in \Phi$ is elementarily linear for $\mathcal{D}$, but each cover ψ of Φ is elementarily linear for $\mathcal{D}$ since different selectors apply to different parts of t, separated by an occurrence of τ. □

7 Conclusion

We investigated the expressive power of weighted tree automata with various amounts of ambiguity over the tropical semiring $\mathbb{T}$ as well as the arctic semiring $\mathbb{A}$. More precisely, we compared the expressive power of WTAthat are unambiguous (UA-WTA), finitely ambiguous (FA-WTA), and polynomially ambiguous (PA-WTA) and proved the strictness of the corresponding hierarchy UA-RTL(S) $\subsetneq$ FA-RTL(S) $\subsetneq$ PA-RTL(S) $\subsetneq$ RTL(S) for $S \in \{\mathbb{T}, \mathbb{A}\}$ using arguments corresponding to those of [5,19]. Moreover, we obtain a similar hierarchy UA-RTL(S) $\subsetneq$ FA-RTL(S) $\subsetneq$ PAH-RTL(S) $\subsetneq$ EAH-RTL(S) for the same ambiguity notions in the height of the input tree. Obviously it holds that PAH-RTL(S) $\subseteq$ PA-RTL(S) as well as EAH-RTL(S) $\subseteq$ RTL(S). It remains open, whether those inclusions are strict.

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