Chapter 11
Mathematical Foundations of VR/AR

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Abstract In Virtual Reality and Augmented Reality, mathematical methods offer fundamental principles to model three-dimensional space. This makes it possible to provide exact information and perform calculations, e.g., to determine distances or to describe the effects of transformations such as rotations or translations exactly. This chapter compiles the most important mathematical methods, especially from linear algebra, that are frequently used in VR and AR. For this purpose, the term *vector space* is defined and extended to a Euclidean space. Afterwards, some basics of analytic geometry are introduced, especially the mathematical description of lines and planes. Finally, changes of coordinate systems as well as affine transformations are discussed and their computation with matrices in homogeneous coordinates is explained.

11.1 Vector Spaces

In Virtual Reality, we are concerned with the real space that surrounds us. It is helpful to model this space with methods of mathematics, e.g., to be able to make exact, formal, mathematically provable statements or to perform computations. In VR, we use a *vector space,* a construct of linear algebra (a branch of mathematics), for this modeling.

Each vector space is formed over a *field* G. The elements of G are called *scalars* and we denote them by small Latin letters. Being a *field* in the sense of algebra means that G is a set with the two binary operations "+" (addition) and "·" (multiplication), which combine two elements of G and as a result give an element of

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G. Moreover, there is an element 0 in G, called the *additive identity*, and an element 1 in G, called the *multiplicative identity*. Finally, the elements of G satisfy the following field axioms. For any scalar a, b, c, d (with $d \neq 0$):

$$a+(b+c)=(a+b)+c \quad \text{(associativity of addition)}$$

$$a+b=b+a \quad \text{(commutativity of addition)}$$

$$0+a=a \quad \text{(commutativity of addition)}$$

For each $a \in G$ there exists a $-a \in G$ with $-a + a = 0$ (additive inverses)

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c \quad \text{(associativity of multiplication)}$$

$$a\cdot b=b\cdot a \quad \text{(commutativity of multiplication)}$$

$$1\cdot d=d \quad \text{(multiplicative identity)}$$

For each $d \in G\backslash\{0\}$ there exists a $d^{-1} \in G$ with $d^{-1} \cdot d = 1$ (multiplicative inverses)

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad \text{(distributivity)}$$

The set of *real numbers* R, which comprises the set of natural numbers (e.g., 1, 2, 3, …), integers, rational numbers and irrational numbers (e.g., π), fulfills the field axioms and is usually chosen in VR.

The set of elements of a vector space V over a field G is called *vectors*. We denote them by Latin letters, over which an arrow is placed. Two operations are defined on vectors. First, *vector addition* takes two vectors and assigns them a third vector. We write this operation as "+" (not to be confused with addition in scalars). The vector addition adheres to the associativity of addition and the commutativity of addition. There exists also an identity element of addition, the *zero vector* $\vec{0}$. For each vector $\vec{u}$ there exists an additive inverse $-\vec{u}$ in V. Secondly, *scalar multiplication* takes a scalar and a vector and assigns them a vector. we write it as "·". Scalar multiplication adheres to distributivity:

$$\forall a,b \in G, \forall \vec{u},\vec{v} \in V : a\cdot(\vec{u}+\vec{v}) = a\cdot\vec{u}+a\cdot\vec{v} \text{ and } (a+b)\cdot\vec{u} = a\cdot\vec{u}+b\cdot\vec{u}$$

An example of a set V that fulfills these properties of a vector space is the set of 3-tuples over the real numbers, i.e., the set of all lists of real numbers of length 3. We call this set R^3. The 3-tuple (5, –2, 3), for example, is an element from the set R^3. In the following, we will not write the elements of R^3 as a list next to each other but on top of each other:

$$\vec{u} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

To specify the set R^3 completely as a vector space, we still have to specify the two operations "+" and "·" of the vector space. We do this by defining these operations based on the addition and multiplication of the real numbers (i.e., the field over which R^3 was formed).

$$a \in \mathbb{R}, \vec{u}, \vec{v} \in \mathbb{R}^3 :$$

$$\vec{u} + \vec{v} := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \text{ and } a \cdot \vec{u} := a \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a \cdot u_1 \\ a \cdot u_2 \\ a \cdot u_3 \end{pmatrix}$$

In vector spaces, vector addition and scalar multiplication are generally used to define a *linear combination* of a number of n scalars and n vectors:

$$\vec{u} = a_1 \cdot \vec{u}_1 + a_2 \cdot \vec{u}_2 + \ldots + a_n \cdot \vec{u}_n$$

If all n scalars must have the value 0 for the linear combination to yield the zero vector, the n vectors of the linear combination are called *linearly independent.* If one finds a maximum of d linearly independent vectors in a vector space V, then d is the *dimension* of the vector space V. In our example, the vector space R^3 has dimension 3. By the way, it is not only the set of all 3-tuples that forms a vector space. If k is a natural number, then the set of all *k-tuples* of real numbers forms a vector space R^k, which has dimension k.

If V is a vector space of dimension n and we find n linearly independent vectors, these vectors are called a *base* of V. We can then represent each vector of V by a linear combination of these base vectors. The n scalars that occur in this linear combination are called the components or *coordinates* of a vector.

11.2 Geometry and Vector Spaces

In geometry, *directed line segments* are called *geometric vectors.* You can visualize them with an arrow, having a length and a direction. The beginning of the geometric vector is called the *tail*, and the end of the geometric vector is called the *tip.* We define an addition operation of two geometric vectors as follows. We place the tail of the second vector at the tip of the first vector – the result of the addition is a geometric vector that then runs from the tail of the first vector to the tip of the second vector. We also define a *scalar multiplication*, where we choose the real numbers R as scalars (see Fig. 11.1). If we multiply the scalar a by a geometric vector, we get

Fig. 11.1 Vector addition and scalar multiplication of geometric vectors

as a result a geometric vector with $a \times$ the length of the original geometric vector. If a is positive, the resulting vector points in the same direction; if not, the result vector points in the opposite direction. With these two operations the set of geometric vectors forms a vector space over R.

Directed line segments are useful constructs when we want to model the space surrounding us. However, performing computations with them directly proves to be difficult. Therefore, we take a base from the space of geometric vectors – if we are in the three-dimensional space, it consists of three base vectors. We can represent each geometric vector as a linear combination of these three base vectors. The coordinates in this linear combination are three real numbers – which in turn we can understand as 3-tuples, i.e., an element of the vector space R^3.

We can proceed as follows. We assign a vector from R^3 to each directed line segment, i.e., to each geometric vector, with the help of a base. In R^3 we can calculate with vectors based on the addition and multiplication of real numbers. The result of the calculation is then transferred into the space of the geometric vectors by inserting the calculated result as a scalar into the linear combination of the base vectors. If, for example, we want to add two geometric vectors, then we assign two vectors from R^3, the "world of numbers", to these two vectors from the "world of geometry". In the "number world" we can calculate the result vector. We transfer this result vector back into the "world of geometry" and thus we have determined the geometric vector resulting from the addition by computation.

11.3 Points and Affine Spaces

However, the usefulness of our mathematical model is still limited: geometric vectors possess only *length* and *direction,* but no fixed *position* in space. This also means that we cannot model essential concepts from the real world, such as distances. Therefore, we introduce the term *point* in addition to scalar and vector. We write points with capital Latin letters. Points have no length and no direction, but a position. Let P and Q be two elements from the set of points. Then we define an operation "–", called *point-point subtraction,* which connects two points and results in a vector:

$$P - Q = \vec{u} \Leftrightarrow P = \vec{u} + Q$$

With this we also define an addition between a point and a vector (called *point-vector addition*), where the result is a point. Thus, we can represent any point P in three-dimensional space as an addition of a point O (called the *origin*) and a linear combination of three linearly independent geometric vectors $\vec{u}, \vec{v}, \vec{w}$, the base vectors:

$$P = O + a \cdot \vec{u} + b \cdot \vec{v} + c \cdot \vec{w} = O + \vec{p}$$

We call these three base vectors, together with O, a *coordinate system* K. We call the 3-tuple (a, b, c) the coordinates of P with respect to K. Thus, every point in our "world of geometry" for a given K can be represented by an element from R^3, our "world of numbers". So, we can "calculate" not only with vectors, but also with points, i.e., with fixed positions in our world. We call $\vec{p}$ the *position vector* belonging to P.

A vector space that has been extended by a set of points and an operation, the point-point subtraction, is called an *affine space* in mathematics. Geometrically, we can interpret point-point subtraction like this: $P - Q$ is a vector that we get when we choose a directional path with starting point Q and final point P.

11.4 Euclidean Space

We add the concept of *distance* to our existing mathematical model of the space surrounding us. For this purpose, we introduce another operation, which we denote by "·" and which takes two vectors and results in a scalar. We call this operation the *scalar product* (not to be confused with scalar multiplication, which takes a scalar and a vector and results in one vector – even if we write both operations with "·", we always know which operation is meant because of the types of the two operands). The scalar product must adhere to commutativity of multiplication and the following axioms for scalars a, b, vectors $\vec{u}, \vec{v}, \vec{w}$ and the null vector $\vec{0}$:

$$\left(a \cdot \vec{u} + b \cdot \vec{v}\right) \cdot \vec{w} = a \cdot \vec{u} \cdot \vec{w} + b \cdot \vec{v} \cdot \vec{w}$$

$$\vec{u} \cdot \vec{u} > 0 \text{ if } \vec{u} \neq \vec{0}$$

$$\vec{0} \cdot \vec{0} = 0$$

In our vector space R^3, we can define a scalar product as follows so that all the above conditions are fulfilled:

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$$

In honor of the Ancient Greek mathematician Euclid of Alexandria, an affine space supplemented by the scalar product operation is called a *Euclidean point space.* Using the scalar product, we define the *amount* of a vector as follows:

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

In our three-dimensional space, the amount of a vector is equal to its length. Thus, we can also determine the *distance d* between two points *P* and *Q* as

$$d = |P - Q| = \sqrt{(P - Q) \cdot (P - Q)}$$

The *angle* α enclosed by two vectors can be determined from the following equation:

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\alpha$$

In the case α = 90° (i.e., the two vectors are perpendicular to each other) the scalar product of the two vectors is 0. Two vectors whose scalar product is 0 are called *orthogonal.* If the two vectors also have length 1, they are called *orthonormal.* For the base in our space, we want to use orthonormal vectors in the following. A corresponding coordinate system (base vectors are perpendicular to each other and have length 1) is called a *Cartesian coordinate system.* In the case of R^3, we take the three unit vectors

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in the given order and the point *O* as the origin point, whose position vector is the zero vector.

To be able to easily find a vector orthogonal to two vectors in R^3, we define an operator "×", which we call the *cross product* and which takes two vectors and results in one vector:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} u_2 \cdot v_3 - u_3 \cdot v_2 \\ u_3 \cdot v_1 - u_1 \cdot v_3 \\ u_1 \cdot v_2 - u_2 \cdot v_1 \end{pmatrix} = -1 \cdot (\vec{v} \times \vec{u})$$

The resulting vector is called a *normal vector*. In this order, the vectors $\vec{u}, \vec{v}, \vec{n}$ form a *right-handed system*, i.e., if you take them as geometric vectors and place their tail on a common point, the vectors are oriented like the thumb, index finger and middle finger of the right hand. The vector product is not commutative. While one can generalize our definition of the scalar product from R^3 to R^n and thus obtain Euclidean point spaces of dimension n, the cross product is defined exclusively in R^3.

11.5 Analytical Geometry in $\mathbb{R}^3$

In R^3, our mathematical model of the space surrounding us, we can solve geometric problems by computation, e.g., finding an intersection of lines or determining the distance of a point to a plane. A *line* is the generalization of a directed line segment: it has no direction and has infinite length. A line is defined by two points. Mathematically we model a line g through points P and Q as a subset of R^3 that includes all points X whose position vector $\vec{x}$ satisfies the equation of the line, using the position vectors associated with P and Q:

$$g = \left\{ \vec{x} \in \mathbb{R}^3 \,|\, \exists t \in \mathbb{R}, \vec{x} = \vec{p} + t \cdot \left(\vec{q} - \vec{p} \right) \right\}$$

The scalar t is called the *parameter* and the equation above is also called the *vector equation* of a line. The vector that is multiplied by t is called the *directional vector* of the line g. Similarly, we can model a *plane* E as a subset of R^3. It is defined by three points P, Q, R and the equation of the plane contains two parameters and two directional vectors:

$$E = \left\{ \vec{x} \in \mathbb{R}^3 \,|\, \exists t, s \in \mathbb{R},\ \vec{x} = \vec{p} + t \cdot \left(\vec{q} - \vec{p} \right) + s \cdot \left(\vec{r} - \vec{p} \right) \right\}$$

By means of the cross product, we can compute the normal vector $\vec{n}$ from the directional vectors, which is perpendicular to E. For the distance d of a point X to a plane E we know the following equation in linear algebra, where the sign of the scalar product indicates on which side of E the point X is located:

$$d = \left| \frac{\vec{n}}{|\vec{n}|} \cdot \left(\vec{x} - \vec{p} \right) \right|$$

Thus, we can reformulate the condition that points X belong to the subset E. This is because all points X that have the distance 0 from E lie on the plane E. Thus, we obtain the *point-normal form* of a plane:

$$E = \left\{ \vec{x} \in \mathbb{R}^3 \,|\, \vec{n} \cdot \left(\vec{x} - \vec{p} \right) = 0 \right\}$$

With these definitions you can compute intersections between lines and between a line and a plane as well as intersections between planes. The first step is to equate the equations that define the set of points that form a line or a plane. Alternatively, substitution can sometimes be used. This results in either an equation to be solved or a linear system of equations, the solution of which can be computed by mathematical methods (for example, Gaussian elimination).

11.6 Matrices

In virtual reality, another mathematical construct is often used to compute transformations such as rotations or translations in three-dimensional space: the *matrix* (plural: *matrices*). A matrix is a table of n rows and m columns where each entry is a scalar. In the following, we will always assume that entries are real numbers. We find the scalar a_{ij} in row i and column j of the matrix. It is called the *entry* in place (i, j). We write matrices with bold capital letters: $\mathbf{A} = [\ a_{ij}\ ]$ and say $\mathbf{A}$ is an $n\times m$ matrix. The matrix $\mathbf{M}$ in our example has two rows and four columns, so it is a 2×4 matrix, and the entry $m_{1,3}$ has the value 5:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 1 & 9 & 2 & 0 \end{bmatrix}$$

For matrices, we define three operations. First, the *scalar-matrix multiplication,* denoted by "·", which combines a scalar s and a $n\times m$ matrix $\mathbf{A} = [a_{ij}]$ to form an $n\times m$ matrix: $s\cdot\mathbf{A} = s\cdot[a_{ij}]:=[s\cdot a_{ij}]$. This operation adheres to associativity. Secondly, *matrix-matrix addition*, denoted by "+", links two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size $n\times m$ to form a matrix of size $n\times m$: $\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] := [a_{ij} + b_{ij}]$. This operation adheres to associativity and commutativity. Third, *matrix-matrix multiplication*, denoted by "·", combines a matrix $\mathbf{A}$ of size $n \times k$ and a matrix $\mathbf{B}$ of size $k \times m$ to form a matrix of size $n \times m$:

$$\boldsymbol{A}\cdot\boldsymbol{B} := \left[c_{ij}\right] \quad \text{with} \quad c_{ij} = \sum_{l=1}^{k} a_{il}\cdot b_{lj}$$

This operation adheres to associativity. It should be emphasized that commutativity does not apply to matrix-matrix multiplication: $\mathbf{A}\cdot\mathbf{B}$ does not always equal $\mathbf{B}\cdot\mathbf{A}$.

If we swap the rows and columns in a matrix, we get the *transposed matrix.* The transposed matrix of matrix $\mathbf{M} = [a_{ij}]$ is $\mathbf{M}^T=[a_{ji}]$. The following applies: $(\mathbf{A}\cdot\mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$. A special case are matrices that have the same number of rows and columns. These are called *square matrices.* The square matrix $\mathbf{I}$ for which the following applies

$$\boldsymbol{I} = \left[a_{ij}\right], \quad a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

is called the *unit matrix*. The following applies: **A**·**I** = **I**·**A** = **A**, where **A** and **I** are both $n \times n$ matrices If a matrix $\mathbf{A}^{-1}$ of the same size exists for an $n \times n$ matrix **A** and the equation $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$ applies, then $\mathbf{A}^{-1}$ is called the *inverse matrix* of **A**. **A** is then called *invertible.* The following applies: $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$. If the following applies to a matrix **A**: $\mathbf{A}^{-1} = \mathbf{A}^{T}$, then **A** is called *orthogonal.*

11.7 Affine Transformations

Assume that the point P has coordinates (x, y, z) with respect to a Cartesian coordinate system. If we translate P by t_x in the x-direction, by t_y in the y-direction and by t_z in the z-direction, we map point P to a new point P′. What are its coordinates? To calculate such *transformations*, we utilize matrices. We introduce a special notation for matrices that consist of only one column: we write them with small bold letters and call them *column matrices.* Now we want to represent the point P by the column matrix **p.** We do this as follows:

$$\mathbf{p} = \begin{bmatrix} w \cdot x \\ w \cdot y \\ w \cdot z \\ w \end{bmatrix}, \text{for any real number } w \text{ with } w \neq 0$$

We call $(w \cdot x, w \cdot y, w \cdot z, w)$ the *homogeneous coordinates* of P. In practice, for the sake of simplicity, usually $w = 1$ is chosen. If one chooses $w = 0$, one can represent a vector in a column matrix instead of a point by means of homogeneous coordinates:

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

The translation from P to P' can be described by a matrix **M**. The following simple equation applies:

$$\boldsymbol{p}' = \boldsymbol{M} \cdot \boldsymbol{p}$$

In our translation example, this equation looks like this:

$$\boldsymbol{p}' = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w \cdot x \\ w \cdot y \\ w \cdot z \\ w \end{bmatrix} = \begin{bmatrix} w \cdot (x + t_x) \\ w \cdot (y + t_y) \\ w \cdot (z + t_z) \\ w \end{bmatrix}$$

From the resulting column matrix **p′**, we can obtain the coordinates of point P' after division by w: $(x + t_x, y + t_y, z + t_z)$. If instead of **p,** which represents a point, we were to use the column matrix **v**, which represents a vector, in the above equation, then **v** would be mapped exactly back to **v.** This is also what we expect: since a vector has no fixed position in space, it is not changed by a displacement. As we will see below, the transformation of a vector by a more complex transformation is slightly more complicated.

Let us take a closer look at the matrix **M** that represents this translation. You can think of its four columns as column matrices. The first three columns represent vectors, because the value in the fourth row is zero. In fact, these are the base vectors of our three-dimensional space if we apply the translation to them. They do not change, because a translation does not change the length or the direction of a vector. The fourth column vector represents a point, because the value in the fourth row is not zero. This column vector represents the origin when the translation is applied to it. As a result of the translation, the origin (0, 0, 0) is mapped to (t_x, t_y, t_z). Therefore, this transformation can be seen as a change from one coordinate system of our three-dimensional space to another coordinate system. In fact, mathematicians have been able to show that each change of coordinate systems can be represented as a matrix **M.** With 4×4 matrices **M,** not only can translations be computed, but also other affine transformations that map one affine space into another. Besides translation, the following geometric transformations are also included: rotation, scaling, reflection and shearing. If you invert the matrix **M**, you get the matrix $\mathbf{M}^{-1}$, which represents the inverse mapping of **M**, i.e., it reverses the mapping represented by **M**.

Let us assume that we perform n geometric transformations of the point P. We represent the transformation performed first by $\mathbf{M}_1$, the second by $\mathbf{M}_2$ and so on, until finally the transformation performed last is represented by $\mathbf{M}_n$. This allows us to determine the coordinates of the point P' resulting from the back-to-back execution (*concatenation*) of these transformations as follows:

$$\boldsymbol{p}' = (\boldsymbol{M}_n \cdot \ldots \cdot \boldsymbol{M}_3 \cdot \boldsymbol{M}_2 \cdot \boldsymbol{M}_1) \cdot \boldsymbol{p}$$

Note the order of the matrices and keep in mind that matrix multiplication is not commutative. If you perform the computation as indicated by the brackets, you only need to compute the product of all n matrices once, even if you transform hundreds of points with the same transformation. For a large number of points to be transformed, this results in a considerable saving of computing time. Matrix operations for 4×4 matrices are implemented directly in hardware in graphics processors, which leads to another reduction in computing time.

Besides points, vectors can also be transformed by a matrix **M** that describes an affine transformation. If we want to know where the vector $\vec{v}$ is mapped to after the transformation described by **M**, we represent the vector in the column matrix **v**. We compute $\mathbf{v}' = (\mathbf{M}^{-1})^{T} \cdot \mathbf{v}$ and the first three rows of the column matrix $\mathbf{v}'$ contain the coordinates of the transformed vector.

11.8 Determination of Transformation Matrices

To calculate geometric transformations or to perform a change between coordinate systems, we need a matrix **M** that represents this transformation, as described in the last section. But how do we determine this matrix **M**? In principle there are two ways.

The first alternative is to know formulas for these matrices for certain standard cases. The formula for translation has already been given in Sect. 11.7. For rotation by an angle α around the *x*-axis around the origin point, the following formula can be found for the matrix **M**:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Accordingly, one can also find formulas for transformation matrices for rotation around the *y*-axis, around the *z*-axis or around any other axis, for reflection, or for scaling in computer graphics textbooks. From these standard cases, more complex transformations can be computed by concatenation (see Sect. 11.7). For example, if you want to calculate a rotation of 30° around the *x*-axis around the center of rotation (1, 2, 3), you divide this transformation into three transformations for which a formula is known: first, you perform a translation by (–1, –2, –3), which takes the center of rotation to the origin (because we only know the formula for rotations around the origin). Then you rotate 30° around the *x*-axis around the origin point and reverse the first translation performed with the inverse translation. The matrix for the entire transformation is obtained by multiplying the three matrices for the standard cases (note the order):

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ & 0 \\ 0 & \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The second alternative to determine the matrix **M**, which we need according to the formula $\mathbf{p}' = \mathbf{M} \cdot \mathbf{p}$ to compute a transformation or to change coordinate systems, is to construct **M** directly:

- We start with our coordinate system *K*, which consists of three base vectors and the origin point. We also need to know the target coordinate system K' after the transformation, which results from the geometrical transformation of the three base vectors and the origin point of *K*. Let **M** be the matrix that changes coordinates from coordinate system *K* to K', i.e., **M** computes the geometric transformation from *K* to K'.
- We represent the first base vector of K' as a column matrix of size 4 by entering its three coordinates with respect to *K* in the first three rows of the column matrix and a zero in the fourth row. Analogously, we obtain column matrices for the second and third base vector of K'. We represent the origin point of K' by entering its coordinates with respect to *K* in the first three rows of a column matrix of size 4 and a one in the fourth column. From these four column matrices, we form the matrix $\mathbf{M}^{-1}$ of size 4×4 by writing them next to each other according to the above order. By inverting $\mathbf{M}^{-1}$ we obtain the matrix **M** that we are looking for.

If a point *P* has coordinates (x, y, z) with respect to the old coordinate system *K*, its new coordinates with respect to K' are calculated with the matrix **M** as follows:

- We represent *P* as a column matrix **p** with the homogeneous coordinates $(x, y, z, 1)$.
- We calculate the matrix product $\mathbf{p}' = \mathbf{M} \cdot \mathbf{p}$
- The values in the first three rows of $\mathbf{p}'$ are the coordinates of *P* with respect to the new coordinate system K'