On Bertelson-Gromov Dynamical Morse Entropy

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Abstract In this mainly expository paper we present a detailed proof of several results contained in a paper by M. Bertelson and M. Gromov on Dynamical Morse Entropy. This is an introduction to the ideas presented in that work. Suppose M is compact oriented connected C^∞ manifold of finite dimension. Assume that $f_0 : M \to [0, 1]$ is a surjective Morse function. For a given natural number n, consider the set M^n and for $x = (x_0, x_1, \dots, x_{n-1}) \in M^n$, denote $f_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} f_0(x_j)$. The Dynamical Morse Entropy describes for a fixed interval $I \subset [0, 1]$ the asymptotic growth of the number of critical points of f_n in I, when $n \to \infty$. The part related to the Betti number entropy does not requires the differentiable structure. One can describe generic properties of potentials defined in the XY model of Statistical Mechanics with this machinery.

Keywords Bertelson-Gromov dynamical morse entropy · Asymptotic growth of critical points · Singular homology · Betti number entropy

1 Introduction

We follow the main guidelines and notation of [1].

A Morse function is a smooth function such all critical points are not degenerate (see [2]).

Suppose M is compact oriented C^∞ manifold of dimension $q \geq 1$. Assume that $f_0 : M \to [0, 1]$ is a surjective Morse function and Γ is a free group with basis $\gamma_1, \dots, \gamma_n$. We assume that f_0 has p critical points ($p \geq 2$).

Suppose $\Omega \subset \Gamma$ is a finite non-empty set. If $x \in M^\Omega$ we denote $x_\gamma \in M, \gamma \in \Omega$, the corresponding coordinate.

Then, we define $f_\Omega : M^\Omega \to [0, 1]$ by the expression

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$$f_\Omega(x) = \frac{1}{|\Omega|} \sum_{\gamma \in \Omega} f_0(x_\gamma),$$

where $|\Omega|$ is the cardinality of Ω. This function f_Ω is also a surjective Morse function.

2 The XY Model

As a particular case we can consider $\Gamma = \mathbb{Z}$, the set $M^{\mathbb{Z}}$ and for $x = (x_j)_{j\in\mathbb{Z}} \in M^{\mathbb{Z}}$, $n > 0$, $f_0 : M \to \mathbb{R}$, and

$$f_n(x) = -\frac{1}{n} \sum_{j=0}^{n-1} f_0(x_j).$$

We mention this case because it is a more well known model in the literature and we want to trace a parallel to what will be done here.

The question about the minus sign in front of the sum is not important but if we want that f_0 represents a kind of Hamiltonian (energy) we will keep the—(at least in this section).

In this model it is natural to consider that adjacent molecules in the lattice interact via a potential (an Hamiltonian) which is described by the smooth function of two variables f_0. The mean energy up to position n is described by f_n. The points $x \in M^n$ where the mean n-energy is lower or higher are of special importance. We are interested here, among other things, in the growth of the number of critical values, when $n \to \infty$. The critical points are called the stationary states (see [1]).

Denote by $\mathrm{Cri}_n(I)$ the number of critical points of f_n in a certain interval $f^{-1}(I)$. Roughly speaking the purpose of [1] is to provide for a fixed value $c \in [0, 1]$ a topological lower bound for

$$\lim_{\delta\to 0} \lim_{n\to\infty} \frac{\log(\mathrm{Cri}_n(I))}{n}, \quad \text{where } I = (c - \delta, c + \delta),$$

in terms of a certain strictly positive concave function (a special kind of entropy). This is done by taking into account the homological behavior of the functions f_n.

The so called classical XY model consider the case where $M = S^1$ (see for instance [3–9] or [10]). A function $A : (S^1)^{\mathbb{Z}} \to \mathbb{R}$ describes an interaction between sites on the lattice $\mathbb{Z}$ where the spins are on S^1. One is interested in equilibrium probabilities $\hat{\mu}$ on $(S^1)^{\mathbb{Z}}$ which are invariant for the shift $\hat{\sigma} : (S^1)^{\mathbb{Z}} \to (S^1)^{\mathbb{Z}}$. A point x on $(S^1)^{\mathbb{Z}}$ is denoted by $x = (\ldots, x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, \ldots)$.

In the case the potential A depend just on the first coordinate $x_0 \in S^1$, that is $A(x) = f_0(x_0)$, then the setting described above applies.

In the case the potential A depend just on the two first coordinate $x_0, x_1 \in S^1$, that is $A(x) = f_0(x_0, x_1)$, then, we claim that the setting described above in the introduction applies. This is the case when $f_0 : S^1 \times S^1 \to \mathbb{R}$. Indeed, in this case one can take $M = S^1 \times S^1$ and consider that f_0 acts on M. In this case we can say that f_0 depends just in the first coordinate on $M^{\mathbb{Z}} = (S^1 \times S^1)^{\mathbb{Z}}$ and adapt the general formalism we describe here.

Therefore, we will state all results for $f_0 : M \to \mathbb{R}$, that is, the case the potential on $M^{\mathbb{Z}}$ depends just on the first coordinate.

In the case $\hat{\mu}$ is ergodic the sequence f_n describes Birkhoff means which are $\hat{\mu}$ almost everywhere constant. We are here interested more in the topological and not in the measure theoretical point of view.

In the measure theoretical (or Statistical Mechanics) point of view, if one is interested in equilibrium states at positive temperature $T = 1/\beta$, then, is natural to consider expressions like $\int e^{\sum_{j=0}^{n-1} -\beta f_0(x_j)}\, dx_0\, dx_1 \ldots d_{x_{n-1}}$ (or, when the set of spins is finite: $\sum e^{\sum_{j=0}^{n-1} -\beta f_0(x_j)}$) and its normalization (see [11–13]) which defines the partition function.

By the other hand if one is interested in the zero temperature case (see for instance [14]), then, expressions like $-\sum_{j=0}^{n-1} f_0(x_j)$ are the main focus. For instance, if f_0 has a unique point of minimum $x^- \in S^1$, then $\delta_{(x^-)^\infty}$ defines the ground state (maximizing probability). In the generic case the function f_0 has indeed a unique point of minimum.

Given $f_0 : M \times M \to \mathbb{R}$ and n one can also consider periodic conditions. In this case we are interested in sums like

$$\tilde{f}_n(x) = -\frac{1}{n}\,(f_0(x_0) + f_0(x_1) + \cdots + f_0(x_{n-2}) + f_0(x_0)),$$

or

$$-(f_0(x_0) + f_0(x_1) + \cdots + f_0(x_{n-2}) + f_0(x_0)).$$

In the case we want to get Gibbs states via the Thermodynamic Limit (see for instance [11] or [13]), given a natural number n, we have to look for the probability μ on M^n (absolutely continuous with respect to Lebesgue probability) which maximizes

$$\int e^{-\sum_{j=0}^{n-1} \beta f_0(x_j)}\, d\,\mu(dx_0,\, dx_1, \ldots, d_{x_{n-1}}),$$

or, at zero temperature the periodic probability μ on M^n which maximizes

$$-\int \sum_{j=0}^{n-1} f_0(x_j)\, d\,\mu(dx_0,\, dx_1, \ldots, d_{x_{n-1}}).$$

One can easily adapt the reasoning of [15] to show that for a generic f_0 we get that $\tilde{f}_n$ is a Morse function for all n.

When f_0 is not generic several pathologies can occur (see for instance [3, 5, 10]).

Suppose the case when there is a unique point x^- of minimum for f_0. For each $\beta > 0$ and n denote by $\mu_{n,\beta}$ the absolutely continuous with respect to Lebesgue probability which maximizes

$$\int e^{-\sum_{j=0}^{n-1} \beta\, f_0(x_j)}\, d\,\mu(dx_0,\, dx_1, \ldots, d_{x_{n-1}}).$$

By the Laplace method (adapting Proposition 3 in [7] or Lemma 4 in [8]) we get that when $\beta \to \infty$ and $n \to \infty$ the probability $\mu_{n,\beta}$ converges to the Dirac delta on $(x^-)^\infty$. Therefore, in the generic case this last probability is the ground state (zero temperature limit).

3 The General Model—The Dynamical Morse Entropy

From now we forget the—sign in front of f_0. For instance, $f_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} f_0(x_j, x_{j+1})$.

Given $c \in [0, 1]$ and $\delta > 0$, take $N_\Omega(c, \delta)$ the number of critical points of f_Ω in $f_\Omega^{-1}[c-\delta, c+\delta]$. Note that if f_0 has p critical points then f_Ω has $p^{|\Omega|}$ critical points.

Consider the cylinder sets

$$\Omega_i = \{a_1\gamma_1 + \cdots + a_n\gamma_n\,;\ |a_j| \le i,\ 1 \le j \le n\},\ i = 1, 2, \ldots,,$$

where a_j are integers.

Denote $N_i(c, \delta) = N_{\Omega_i}(c, \delta)$. Then, of course, $N_i(c, \delta)$ for c fixed decrease with δ.

For a fixed $0 \le c \le 1$, we denote the entropy by

$$\varepsilon(c) = \lim_{\delta \to 0} \left(\liminf_{i \to +\infty} \frac{\log(N_i(c, \delta))}{|\Omega_i|} \right).$$

The above limit exists and it is bounded by $\log p$ but in principle could take the value $-\infty$. We call $\varepsilon(c)$ the **dynamical Morse entropy** on the value c.

In the case $\Gamma = \mathbb{Z}$ as we mentioned before $\varepsilon(c)$ is described by

$$\varepsilon(c) = \lim_{\delta \to 0} \left(\liminf_{n \to +\infty} \frac{\log(\text{number of critical points of } f_n \text{ in } f_n^{-1}[c-\delta, c+\delta])}{n} \right).$$

Later we introduce a function $b(c)$ (see Definition 3 and also Definition 2), which will be a topological invariant of f_0. The function $b(c)$ is defined in terms of rank of linear operators and Cohomology groups.

We will show later that

(1) $0 \le b(c) \le \varepsilon(c),\ 0 \le c \le 1$;

(2) $b(c)$ is continuous and concave;

(3) $b(c)$ is not constant equal to 0.

Finally, in the case $M = S^1$ (the unitary circle) and f_0 has just two critical points, we show in Sect. 7 that

$$\varepsilon(c) = b(c) = -c \log c - (1-c)\log(1-c).$$

$b(c)$ is sometimes called the **Betti entropy** of f_0.

Our definition of $b(c)$ is different from the one in [1] but we will show later (see Sect. 8) that is indeed the same.

A key result in the understanding of the main reasoning of the paper is Lemma 6 which claims that for any Morse function f, given $a, b \in \mathbb{R}$, $a < b$, the number of critical points of f in $f^{-1}[a, b]$ is bigger or equal to the dimension of the vector space

$$\frac{H^*(f^{-1}(\infty, b))}{H^*(f^{-1}(-\infty, a))},$$

where H^* denotes the corresponding cohomology groups which will be defined in the following paragraphs (see also [16] for basic definitions and properties).

$H^*(X, \mathbb{R})$ denotes the usual cohomology. Note that H^* will have another meaning (see Definition 1).

4 Cohomology

Suppose X is a metrizable, compact, oriented topological manifold C^∞ manifold. We will consider the singular homology. Suppose $U \subset X$ is an open set and $a \in H^*(X, \mathbb{R})$. The meaning of the statement supp $a \subset U$ is: there exist an open set $V \subset X$, such that, $X = U \cup V$, and $a|_V = 0$.

Definition 1 $H^*_X(U) = \{a \in H^*(X, \mathbb{R}) \,:\, \text{supp } a \subset U\}$, where U is an open subset of X. When X is fixed we denote $H^*_X(U) = H^*(U)$.

Remember (see for instance [16]) that when $U \subset X$ is open we get the exact cohomology sequence:

$$... \to H^{k-1}(X-U, \mathbb{R}) \to H^k_c(U, \mathbb{R}) \to H^k(X, \mathbb{R}) \to H^k(X-U, \mathbb{R}) \to H^{k+1}_c(U, \mathbb{R}) \to ... \tag{1}$$

where H^*_c denotes the compact support cohomology.

Lemma 1 *If U is an open set, then*

$$H^*(U) = Im(\ H^*_c(U, \mathbb{R}) \to H^*(X, \mathbb{R})\ ) = \ Ker(\ H^*(X, \mathbb{R}) \to H^*(X-U, \mathbb{R})\ ).$$

Proof The second equality follows from the fact that the above sequence is exact.

We will prove that

$$\mathrm{Im}(\ H_c^*(U,\mathbb{R}) \to H^*(X,\mathbb{R})\ ) \subset H^*(U) \subset \ \mathrm{Ker}\,(\ H^*(X,\mathbb{R}) \to H^*(X-U,\mathbb{R})\ ).$$

Let $a \in \mathrm{Im}(\ H_c^*(U,\mathbb{R}) \to H^*(X,\mathbb{R})\ )$. Then, a is represented by a cocycle α with compact support $K \subset U$. Therefore, $a\,|(X-K) = 0$.

Defining $V = X - K$ we have that $U \cup V = X$ and $a\,|V = 0$. Then, $a \in H^*(U)$.

Let be $\alpha \in H^*(U)$. Let $V \subset X$ be an open set such that $U \cup V = X$ and $\alpha\,|V = 0$.

Since $X - U \subset V$, we have $\alpha\,|(X-U) = 0$.

Then, $\alpha \in \mathrm{Ker}\,(\,H^*(X,\mathbb{R}) \to H^*(X-U,\mathbb{R})\,)$. □

Lemma 2 *If U is an open set then $H^*(U)$ is a graded ideal of the ring of cohomology of X.*

Proof This follows at once from Lemma 1. □

Now we consider a continuous function $f : X \to \mathbb{R}$.

Definition 2 Given $\delta > 0$ and $c \in \mathbb{R}$ we define

$$b'_{c,\delta} = Dim\left(\frac{H^*(f^{-1}(-\infty, c+\delta)\,)\,)}{H^*(f^{-1}(-\infty, c-\delta)\,)}\right).$$

Proposition 1 *Suppose X and Y are metrizable compact, oriented topological manifolds, moreover take $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ continuous functions. If we define $f \oplus g : X \times Y \to \mathbb{R}$, by $(f \oplus g)(x, y) = f(x) + g(y)$, then, if $c, c' \in \mathbb{R}$, $\delta, \delta' > 0$, we get*

$$b'_{c,\,\delta}(f)\ b'_{c',\,\delta'}(g)\ \le b'_{c+c',\,\delta+\delta'}(f \oplus g). \tag{2}$$

Before the proof of this important proposition we need two more lemmas.

As it is known (see [16]) the cup product $\vee$ defines an isomorphism

$$\mu : H^*(X,\mathbb{R}) \otimes H^*(Y,\mathbb{R}) \ \to\ H^*(X \times Y,\mathbb{R}).$$

Lemma 3 *If $U \subset X$ and $V \subset Y$ are open sets, then*

$$\mu(\,H_X^*(U) \otimes H^*(Y,\mathbb{R}) + H^*(X,\mathbb{R}) \otimes H_Y^*(V)\,)\ = H_{X\times Y}^*(\,(U \times Y) \cup (X \times V)\,).$$

Proof By Lemma 1 we get

$$H_{X\times Y}^*(\,(U \times Y) \cup (X \times V)\,) = \mathrm{Ker}\,(\,H^*(X \times Y,\mathbb{R}) \to H^*((X-U) \times (Y-V),\mathbb{R}).$$

Then,

$$H_{X\times Y}^*(\,(U \times Y) \cup (X \times V)\,) =$$

$$\mu(\ \mathrm{Ker}\,(\,H^*(X,\mathbb{R}) \otimes H^*(Y,\mathbb{R})\ \to H^*(X-U,\mathbb{R}) \otimes H^*(Y-V,\mathbb{R})\,)\ ).$$

From simple Linear Algebra arguments the claim follows from Lemma 1. □

Lemma 4 *If $U \subset X$ and $V \subset Y$ are open sets then*

$$\mu(\, H_X^*(U) \otimes H_Y^*(V)\,) = H_{X\times Y}^*(U \times V).$$

Proof The $\vee$ product defines a natural isomorphism

$$H^*(X, X-U, \mathbb{R}) \otimes H^*(Y, Y-V, \mathbb{R}) \;\to\; H^*(X\times Y, (X \times (Y-V) \cup (X-U) \times Y, \mathbb{R}) =$$

$$H^*(\, X \times Y, (X \times Y) - (U \times V, \mathbb{R})\,).$$

By Lemma 1 and the exact relative cohomology sequence we get:

$$H_X^*(U) = \mathrm{Im}\,(\, H^*(X, X-U, \mathbb{R}) \;\to\; H^*(X, \mathbb{R})\,),$$

$$H_Y^*(V) = \mathrm{Im}\,(\, H^*(Y, Y-V, \mathbb{R}) \;\to\; H^*(Y, \mathbb{R})\,),$$

and

$$H_{X\times Y}^*(U \times V) = \mathrm{Im}\,(\, H^*(X \times Y, (X \times Y) - (U \times V, \mathbb{R})) \;\to\; H^*(X \times Y, \mathbb{R})\,).$$

From this the claims follows at once. □

Now we will present the proof of Proposition 1.
Take $h = f \oplus g$ and denote

$$A^- = f^{-1}(-\infty, c-\delta),\;\; B^- = g^{-1}(-\infty, c'-\delta'),\;\; C^- = h^{-1}(-\infty, (c+c') - (\delta+\delta'))\;,$$

and

$$A^+ = f^{-1}(-\infty, c+\delta),\;\; B^+ = g^{-1}(-\infty, c'+\delta'),\;\; C^+ = h^{-1}(-\infty, (c+c') + (\delta+\delta'))\;.$$

Note that

$$A^+ \times B^+ \subset C^+ \subset (A^+ \times Y) \cup (X \times B^+)$$

$$A^- \times B^- \subset C^- \subset (A^- \times Y) \cup (X \times B^-).$$

Consider the commutative diagram

$$
\begin{array}{ccc}
H^*(X,\mathbb{R}) \otimes H^*(Y) & \to \text{ (using } \mu \text{)} & H^*(X \times Y, \mathbb{R}) \\
\cup & & \cup \\
H^*_X(A^+) \otimes H^*_Y(B^+) \to H^*_{X\times Y}(C^+) & \subset & H^*_{X\times Y}((A^+ \times Y) \cup (X \times B^+)) \\
\cup & & \cup \\
H^*_X(A^+) \otimes H^*_Y(B^-) + H^*_X(A^-) \otimes H^*_Y(B^+) & \to & H^*_{X\times Y}((A^- \times Y) \cup (X \times B^-)) \\
\cup & & \cup \\
H^*_X(A^-) \otimes H^*_Y(B^-) & \to & H^*_{X\times Y}(C^-).
\end{array}
$$

From this follows the linear transformation

$$\tilde{\mu} : \frac{H^*_X(A^+) \otimes H^*_Y(B^+)}{H^*_X(A^-) \otimes H^*_Y(B^-)} \to \frac{H^*_{X\times Y}(C^+)}{H^*_{X\times Y}(C^-)}.$$

By the other hand

$$(H^*_X(A^+) \otimes H^*_Y(B^+) \cap \mu^{-1}(H^*_{X\times Y}(C^-)) \subset$$

$$(H^*_X(A^+) \otimes H^*_Y(B^+) \cap \mu^{-1}(H^*_{X\times Y}((A^- \times Y) \cup (X \times B^-))) =$$

$$(H^*_X(A^+) \otimes H^*_Y(B^+)) \cap (H^*_X(A^-) \otimes H^*(Y,\mathbb{R}) + H^*(X,\mathbb{R}) \otimes H^*_Y(B^-)) =$$

$$H^*_X(A^-) \otimes H^*_Y(B^+) + H^*_X(A^+) \otimes H^*_Y(B^-).$$

The first equality above follows from Lemma 3; the second follows from Linear Algebra; namely, if $E_2 \subset E_1 \subset E$ and $F_2 \subset F_1 \subset F$, then

$$(E_1 \otimes F_1) \cap (E_2 \otimes F + E \otimes F_2) = E_2 \otimes F_1 + E_1 \otimes F_2.$$

From the above it follows that

$$\operatorname{Ker} \tilde{\mu} \subset \frac{H^*_X(A^-) \otimes H^*_Y(B^+) + H^*_X(A^+) \otimes H^*_Y(B^-)}{H^*_X(A^-) \otimes H^*_Y(B^-)}.$$

Therefore,

$$b'_{c+c',\delta+\delta'} = \dim \frac{H^*_{X\times Y}(C^+)}{H^*_{X\times Y}(C^-)} \geq \dim (\operatorname{Im} \tilde{\mu}) \geq$$

$$\dim \frac{H_X^*(A^+) \otimes H_Y^*(B^+)}{H_X^*(A^-) \otimes H_Y^*(B^+) \; + \; H_X^*(A^+) \otimes H_Y^*(B^-)} =$$

$$\dim \left(\frac{H_X^*(A^+)}{H_Y^*(A^-)} \otimes \frac{H_Y^*(B^+)}{H_Y^*(B^-)} \right) = b'_{c,\delta}(f)\; b'_{c',\delta'}(g). \qquad \square$$

5 Critical Points

In what follows X is a compact, oriented C^∞ manifold and $f : X \to \mathbb{R}$ is a Morse function.

Lemma 5 *Suppose X is a compact, oriented C^∞ manifold and $U \subset X$ is an open set. If $a \in H^*(X, \mathbb{R})$, then, supp $a \subset U$, if and only if, there exists a closed C^∞ differentiable form w such that supp $w \subset U$, and a is the de Rham cohomological class of w.*

Proof If there exists $w \in a$, such that supp $w \subset U$, then

$$a|_{(X - \text{supp }w)} = 0 \text{ and } U \;\cup\; (X - \text{supp } w) \;=\; X.$$

If there exists an open set $V \subset X$ such that $U \cup V = X$ and $a|_V = 0$, then, there exist a C^∞ form η on V such that $d\eta = w|_V$ where $w \in a$.

Let W be an open set such that $\overline{W} \subset V$ and $W \cup U = X$. Take a C^∞ function $\varphi : X \to [0, 1]$ such that $\varphi|_{\overline{W}} = 1$ and $\varphi|_{X-K} = 0$, where K is compact set such that $\overline{W} \subset K \subset V$. Then, $\varphi\,\eta$ has an extension to X and $(w - d\,(\varphi\,\eta)\,) \in a$. But,

$$\text{supp} \;\; (w - d\,(\varphi\,\eta)\,) \subset X - W \subset U. \qquad \square$$

Lemma 6 *Given $a, b \in \mathbb{R}$, $a < b$, then, the number of critical points of f in $f^{-1}[a, b]$ is bigger or equal that*

$$dim \;\; \frac{H^*(f^{-1}(\infty, b)\,)}{H^*(f^{-1}(-\infty, a)\,)}.$$

Proof Without lost of generality we can assume that a and b are regular values of f (decrease a and increase b a little bit).

Given $c_1 < c_2 < \cdots < c_m$, the critical values of f in (a, b), take

$$a = d_0 < c_1 < d_1 < c_2 < d_2 < \cdots < d_{m-1} < c_m < d_m = b.$$

By Proposition 3 and Lemma 8, the number of critical points in $f^{-1}(c_i)$, $i = 1 =$ $, 2, \ldots, m$, is bigger or equal to

$$dim \ \frac{H^*(f^{-1}(\infty, d_i))}{H^*(f^{-1}(-\infty, d_{i-1}))}.$$

Finally consider the filtration

$$H^*(f^{-1}(\infty, a) \ = \ H^*(f^{-1}(-\infty, d_0)) \subset \ H^*(f^{-1}(-\infty, d_1)) \subset ...$$

$$\subset \ H^*(f^{-1}(-\infty, d_{m-1}) \ \subset \ H^*(f^{-1}(-\infty, d_m)) = \ H^*(f^{-1}(-\infty, b)).$$

□

Now we denote $b'_\Omega(c, \delta) = b'_{c,\delta}(f_\Omega)$ and $b'_i(c, \delta) = b'_{\Omega_i}(c, \delta)$, $0 \le c \le 1$, $\delta > 0$.

Corollary 1 $b'_i(c, \delta) \le N_i(c, \delta)$ *for all* $i = 1, 2, 3, \ldots$ *and* $0 \le c \le 1$, $\delta > 0$.

Now we define the function b using Proposition 3(a)

Definition 3

$$b(c) = \lim_{\delta \to 0} \liminf_{i \to \infty} \ \frac{\log(b'_i(c, \delta))}{|\, \Omega_i \,|}, \ \ 0 \le c \le 1.$$

We will show that in above definition we can change the lim inf by lim.

Lemma 7

$$b(c) \le \varepsilon(c) \le \log\left(\textit{the number of critical points of } \ f_0\right).$$

Proof The first inequality follows from Corollary 1. From the definition is easy to see that $\varepsilon(c)$ is smaller than log of the number of critical points of f_0. □

We denote $B(\Gamma)$ a family of finite subsets of Γ and $B_N(\Gamma)$, $N \in \mathbb{N}$, the family of sets $\Omega \in B(\Gamma)$ such that $|\Omega| > N$.

Proposition 2 *Suppose* $\Omega', \Omega'' \in B(\Gamma)$ *are disjoint not empty sets. Then,*

$$b'_{\Omega \cup \Omega''}(\alpha c_1 + (1-\alpha)c_2, \delta) \ge b'_{\Omega'}(c_1, \delta)\, b'_{\Omega''}(c_2, \delta),$$

where $0 \le c_1, c_2 \le 1$, $\delta > 0$ *and* $\alpha = \frac{|\,\Omega'\,|}{|\,\Omega'\,|+|\,\Omega''\,|}$.

Proof By definition

$$f_{\Omega' \cup \Omega''} = \alpha f_{\Omega'} \oplus (1-\alpha) f_{\Omega''}.$$

By Proposition 1, as $\delta = \alpha\, \delta + (1-\alpha)\delta$, then

$$b'_{\alpha\, c_1 + (1-\alpha)\, c_2, \delta}(f_{\Omega' \cup \Omega''}) \ge b'_{\alpha\, c_1, \alpha\, \delta}(\alpha\, f_{\Omega'})\, b'_{(1-\alpha)\, c_2, (1-\alpha)\, \delta}((1-\alpha)\, f_{\Omega''}) =$$

$$b'_{c_1, \delta}(f_{\Omega'}) \ \ b'_{c_2, \delta}(f_{\Omega''}).$$

□

Lemma 8 *Suppose the interval* $[a, b]$ *does no contains critical values of* f. *Then,*

$$H^*(f^{-1}(-\infty, a)) = H^*(f^{-1}(-\infty, b)).$$

Proof This follows from Lemma 1 and the fact that $f^{-1}[b, \infty)$ is a deformation retract of $f^{-1}[a, \infty)$. □

Definition 4 Given $c \in \mathbb{R}$ we define

$$\tilde{b}_c(f) = \lim_{\delta \to 0} b'_{c,\delta}(f).$$

Proposition 3 *For a fixed c we have*

(a) $b'_{c,\delta}(f)$ *decreases with* δ *and* $b'_{c,\delta}(f) = \tilde{b}_c(f)$ *for all* δ *small enough.*

(b) $\tilde{b}_c(f) = 0$ *if c is not a critical value of f*

(c) $\tilde{b}_c(f)$ *is smaller than the number of critical points of* f *in* $f^{-1}(c)$

(d) $\sum_c \tilde{b}_c(f) = Dim\ H^*(X)$.

Proof (a) follows from the above definitions and Lemma 8.

(b) follows from Lemma 8

For the proof of (c) consider the exact diagram

$$\begin{array}{ccc} & H^*(X,\mathbb{R}) & \\ & \downarrow r_1 \qquad\qquad r_2 \searrow & \\ H^*(f^{-1}[c-\delta,\infty)), f^{-1}(c+\delta,\infty),\mathbb{R}) \to & H^*[f^{-1}(c-\delta,\infty),\mathbb{R}) & \to H^*(f^{-1}[c+\delta,\infty),\mathbb{R}), \end{array}$$

where r_1 and r_2 are the restriction homomorphisms.

By Lemma 1

$$H^*(f^{-1}(-\infty, c+\delta)) = \operatorname{Ker} r_2 \quad \text{and} \quad H^*(f^{-1}(-\infty, c-\delta)) = \operatorname{Ker} r_1.$$

From this follows that

$$b'_{c,\delta}(f) = \ \text{Dim}\ (r_1(\text{Ker}\,(r_2))\ \leq\ \text{Dim}\,(H^*(f^{-1}(c-\delta,\infty)), f^{-1}(c+\delta,\infty)), \mathbb{R})$$

because the above sequence is exact.

In order to finish the proof we apply Morse Theory (see [2]) with δ small enough.

For the proof of (d) suppose $c_1 < c_2 < \cdots < c_m$ are the critical values of f. Now, consider

$$d_0 < c_1 < d_1 < c_2 < d_2 < \cdots < d_{m-1} < c_m < d_m.$$

Now, from (a) and Lemma 8 we have

$$\tilde{b}_{c_i}(f) = \text{Dim}\,(\frac{H^*(f^{-1}(-\infty, d_i))}{H^*(f^{-1}(-\infty, d_{i-1}))}), \quad i = 1, 2, \ldots, m.$$

Finally, note that

$$0 = H^*(f^{-1}(-\infty, d_0)) \subset H^*(f^{-1}(-\infty, d_1)) \subset ... \subset$$

$$H^*(f^{-1}(-\infty, d_m)) = H^*(X).$$

□

Lemma 9 *Given $\delta > 0$, there exists an integer N such that: $b'_{\Omega}(c, \delta) \geq 1$ for all $c \in [0, 1]$ and all $\Omega \in B_N(\Gamma)$. Therefore, $b(c) \geq 0$, for all $0 \leq c \leq 1$.*

Before the Proof of Lemma 9 we need two more lemmas.

Lemma 10 *Suppose X is a compact oriented C^∞ manifold and $f : X \to \mathbb{R}$ is a Morse function. Then, for all $\delta > 0$*

$$b'_{a_1,\delta}(f) \geq 1 \text{ and } b'_{a_2,\delta}(f) \geq 1,$$

where a_1 and a_2 are respectively the maximum and minimum of f.

Proof If δ is small enough, $f^{-1}(-\infty, a_2 + \delta)$ is the disjoint union of a finite number of open discs and $f^{-1}(-\infty, a_2 - \delta) = \emptyset$.

If n is the dimension of X, then, it follows from Lemma 1 that

$$H^n(X, \mathbb{R}) \subset H^*(f^{-1}(-\infty, a_2 + \delta)) \neq 0$$

and

$$H^*(f^{-1}(-\infty, a_2 - \delta)) = 0.$$

Then, $b'_{a_2,\delta}(f) \geq 1$, if $\delta > 0$ is small enough. Therefore, this claim is also true for any $\delta > 0$ by Proposition 3(a).

In a similar way we have that for small $\delta > 0$

$$H^0(X, \mathbb{R}) \subset H^*(f^{-1}(-\infty, a_1 + \delta))$$

and

$$H^0(X, \mathbb{R}) \text{ is not contained } H^*(f^{-1}(-\infty, a_1 - \delta)).$$

From this the final claim is proved. □

Lemma 11 *Consider $\Omega \in B(\Gamma)$ where $|\Omega| = m \geq 1$, then, $b'_{\Omega}(k/m, \delta) \geq 1$, for all $\delta > 0$ and $k = 0, 1, 2, \ldots, m$.*

Proof If $k = 0$, or m, the claim follows from Lemma 10 with $X = M^{\Omega}$, $f = f_{\Omega}$.

Given $0, k, m$, $0 < k < m$, take $\Omega = \Omega' \cup \Omega''$, where Ω', Ω'' are disjoints and $k = |\Omega'|$.

By Proposition 2 with $c_1 = 1$ and $c_2 = 0$ we get

$$b'_{\Omega}(k/m,\ \delta) \geq b'_{\Omega'}(1,\ \delta)\, b'_{\Omega''}(0,\ \delta) \geq 1.$$

Yet from last lemma. □

Now we will prove Lemma 9.

Proof Take $N > \frac{2}{\delta}$, $\Omega \in B_N(\Gamma)$, $|\Omega| = m > N$ and k such that $\frac{k}{m} \leq c < \frac{k+1}{m}$,
By definition,

$$b'_{c,\delta}(f_\Omega) \geq b'_{k/m,\,\delta/2}(f_\Omega),$$

since $c - \delta < k/m - \delta/2$ and $c + \delta > k/m + \delta/2$.
Therefore, $b'_{\Omega}(c, \delta) \geq b'_{\Omega}(k/m,\ \delta/2) \geq 1$ by Lemma 11. □

Proposition 4

$$0 \leq b(c) \leq \varepsilon(c) \leq \ \log(\textit{number of critical points of } f_0\,),\ \ 0 \leq c \leq 1.$$

Proof This follows from Lemmas 7 and 9 □

Lemma 12 *Given $c \in [0, 1]$ and $\delta > 0$, consider a non-empty set $\Omega \in B(\Gamma)$ and $\gamma \in \Gamma$. Then,*

$$b'_{\Omega}(c, \delta) = b'_{\Omega+\gamma}(c, \delta).$$

In the case $\Gamma = \mathbb{Z}$ we have that for any $\Omega = \{1, 2, \ldots, k\}$

$$b'_{\Omega}(c, \delta) = b'_{\hat{\sigma}(\Omega)}(c, \delta),$$

where $\hat{\sigma}$ is the shift acting on $M^{\mathbb{Z}}$.

Proof For fixed γ consider the transformation $x \in M^{\Omega} \to y \in M^{\Omega+\gamma}$, such that $y_w = x_{w-\gamma}$, which is a diffeomorphism which commutes $f_{\Omega+\gamma}$ with f_{Ω}.
The result it follows from this fact. □

We will show now that indeed one can change lim inf by inf in Definition 3. In order to do that we need the following proposition which describes a kind of subadditivity.

Proposition 5 *Given an integer number $N > 0$ take $h : B_N(\Gamma) \to \mathbb{R}$, $h \geq 0$, which is invariant by Γ and such that*

$$h(\Omega' \cup \Omega'') \geq h(\Omega') + h(\Omega''),$$

if $\Omega',\ \Omega'' \in B_N(\Gamma)$, are disjoint. Then, the limiit

$$\lim_{i\to\infty} \frac{h(\Omega_i)}{|\Omega_i|} \geq 0 \ \ \textit{exists: finite or } +\infty\,.$$

From this follows:

Corollary 2 *For $c \in [0, 1]$ and $\delta > 0$,*
(a) there exist the limit

$$\lim_{i\to\infty} \frac{\log b'_i(c,\delta)}{|\,\Omega_i\,|} = b'(c,\delta).$$

(b) $0 \le b'(c,\delta) \le \log$ (number of critical points of f_0),
(c) $b(c) = \lim_{\delta\to 0} b'(c,\delta)$

Proof The claim (a) follows from last proposition applied to $h(\Omega) = \log b'_{\Omega}(c,\delta)$, by Lemma 9, Proposition 2 taking $c_1 = c_2 = c$ and also by Lemma 12.
Item (b) follows from Lemma 2 and Corollary 1.
Item (c) follows from item (a) and the definition of $b(c)$. □

Before the proof of Proposition 5 we need two lemmas.

Lemma 13 *Given an integer positive number k, then for each $i > (3\,k + 1)$ there exists $\Omega_{k,i} \in B(\Gamma)$ such that: (a) $\Omega_{k,i} \subset \Omega_i$; (b) $\Omega_{k,i}$ is a disjoint union of a finite number of translates of Ω_k ; (c) $\lim_{i\to\infty} \frac{|\,\Omega_{k,i}|}{|\,\Omega_i\,|} = 1$; (d) $|\Omega_i| - |\Omega_{k,i}| \ge (2k+1)^n$, where n is the number of generators of Γ.*

Proof For the purpose of the proof we can assume that $\Gamma = \underbrace{\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n}$ and take $\gamma_1, \gamma_2, \ldots, \gamma_n$ the canonical basis.
Take $m \ge 1$ an integer such that

$$k + m\,(2\,k + 1) \le i < k = (m+1)\,(2\,k+1),$$

and

$$\Omega_{k,i} = \cup\,\{\ \Omega_k + (\,j_1(2k+1), \ldots, j_n(2k+1)\,)\,|$$

$$-m \le j_1, \ldots, j_n \le m,\ (j_1, \ldots, j_n) \ne (0, \ldots, 0)\ \}.$$

It is easy to see that the sets $\Omega_{k,i}$ satisfy all the above claims. □

Lemma 14 *Given real numbers $x_i \ge 0$ $i = 1, 2, 3, \ldots$, suppose that for each k and each $\varepsilon > 0$ there exist $N_{k,\varepsilon}$ such that*

$$x_i \ge x_k(1-\varepsilon) \ \textit{if}\ \ i \ge N_{k,\varepsilon}.$$

Then, the $\lim_{i\to\infty} x_i$ exists (can finite or $+\infty$).

Proof Take $L = \limsup_{i\to\infty} x_i$ and $a \in \mathbb{R}$, $a < L$. Then, there exists $x_k > a$. Therefore, $x_i \ge a$, if i is very large. Then, $\liminf_{i\to\infty} x_i \ge a$. From this follows the claim. □

Now we will prove Proposition 5.

Proof Suppose k is such that $(2k+1)^n > N$. Take $i > 3k+1$, then, $|\,\Omega_{k,i}\,| \geq (2k+1)^n > N$ and $|\Omega_i - \Omega_{k,i}| \geq (2k+1)^n > N$.

Then, $h(\Omega_i) = h(\,\Omega_{k,i} \cup (\,\Omega_i - \Omega_{k,i})\,) \geq h(\Omega_{k,i})$.

Moreover, each translate of Ω_k has cardinality $(2k+1)^n$. Therefore,

$$h(\Omega_{k,i}) \;\geq\; \frac{|\,\Omega_{k,i}\,|}{|\,\Omega_k\,|} h(\Omega_k).$$

From this follows that

$$\frac{h(\Omega_i)}{|\,\Omega_i\,|} \geq \frac{h(\Omega_{k,i})}{|\,\Omega_{k,i}\,|}\,\frac{|\,\Omega_{k,i}\,|}{|\,\Omega_i\,|} \geq \frac{h(\Omega_k)}{|\,\Omega_k\,|}\,\frac{|\,\Omega_{k,i}\,|}{|\,\Omega_i\,|},$$

and the claim is a consequence of Lemmas 13 and 14. □

The next lemma will be used later

Lemma 15 *Under the hypothesis of Proposition 5 consider*

$$\Omega'_i = (\,\Omega_i + (2i+1)\,\gamma_1\,) \cup \Omega_i, \quad i = 1, 2, 3, \ldots$$

Then,

$$\lim_{i\to\infty} \frac{h(\Omega'_i)}{|\,\Omega'_i\,|} = \lim_{i\to\infty} \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

Proof If $i > N$, then $|\Omega_i| > N$. Therefore,

$$h(\Omega'_i) \geq h(\,\Omega_i + (\,2\,i+1\,)\,\gamma_1\,) + h(\Omega_i) = 2\,h(\Omega_i).$$

From this follows

$$\frac{h(\Omega'_i)}{|\,\Omega'_i\,|} \;\geq\; \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

Therefore,

$$\liminf_{i\to\infty} \frac{h(\Omega'_i)}{|\,\Omega'_i\,|} \;\geq\; \liminf_{i\to\infty} \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

We assume that $\Gamma = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n}$ and $\gamma_1, \gamma_2, .., \gamma_n$ is the canonical basis.

Take k such that $(2k+1)^n > N$. For $i > 5\,k+2$, take $m > 1$ such that $k + m\,(2k+1) \leq i \leq k + (m+1)\,(2\,k+1)$.

Consider

$$\Omega'_{k,i} = \cup \{ \; \Omega'_k + (\, j_1(2k+1), \ldots, j_n(2k+1)\,) \mid j_1 \text{ is even}\,, -m \leq j_1 \leq m-1,$$

$$-m \leq j_2, .., j_n \leq m, \; (j_1, j_2, \ldots, j_n) \neq (0, \ldots, 0) \; \}.$$

Then, $\Omega'_{k,i} \subset \Omega_i$, and $\Omega'_{k,i}$ is a finite union of disjoints translates of Ω'_k. Moreover $\lim_{i\to\infty} \frac{|\Omega'_{k,i}|}{|\Omega_i|} = 1$,

$$|\, \Omega'_{k,i} \,| \geq 2\,(2\,k+1)^n > N \;\text{ and }\; |\, \Omega_i - \Omega'_{k,i} \,| \geq 2\,(2\,k+1)^n > N \;.$$

From this follows that

$$h(\Omega_i) = h(\Omega'_{k,i} \, \cup \, (\, \Omega_i - \Omega'_{k,i})\,) \geq h(\, \Omega'_{k,i}\,),$$

By the other hand, all translate of Ω'_k has cardinality bigger than N.
Therefore,

$$h(\Omega'_{k,i}) \geq \frac{|\, \Omega'_{k,i} \,|}{|\, \Omega'_k \,|} \, h(\Omega'_k).$$

Then,

$$\frac{h(\Omega_i\,)}{|\, \Omega_i \,|} \geq \frac{h(\Omega'_{k,i}\,)}{|\, \Omega_i \,|} \geq \frac{1}{|\Omega_i|} \; \frac{|\, \Omega'_{k,i} \,|\, h(\Omega'_k\,)}{|\, \Omega'_k \,|} = \frac{|\, \Omega'_{k,i} \,|}{|\, \Omega_i \,|} \; \frac{h(\Omega'_k\,)}{|\, \Omega'_k \,|} \,.$$

Now, for a fixed k, taking $i \to \infty$ in the above inequality we get

$$\lim_{i\to\infty} \frac{h(\Omega_i\,)}{|\, \Omega_i \,|} \geq \frac{h(\Omega'_k\,)}{|\, \Omega'_k \,|} \,.$$

From this follows that

$$\lim_{i\to\infty} \frac{h(\Omega_i\,)}{|\, \Omega_i \,|} \geq \limsup_{k\to\infty} \frac{h(\Omega'_k\,)}{|\, \Omega'_k \,|} \,.$$

□

6 Properties of $b(c)$

Lemma 16 *There exists* $c \in [0, 1]$ *such that*

$$b(c) \geq \log(\, dim\, H^*(M, \mathbb{R})\,) > 0$$

Proof Note that dim $(H^*(M)) \geq 2$ because dim $M \geq 1$. Let q be the number of connected components of M.

If $|\Omega_i| = m_i$, take $0 = t_0 < t_1 < \cdots < t_{m_i} = 1$, a partition of $[0, 1]$ in m_i intervals of the same size. By Lemma 1

$$H^*(f_{\Omega_i}^{-1}(-\infty, t_{m_i})) = \oplus_{r>0} H^r(M^{\Omega_i}, \mathbb{R}),$$

Denote A_{ij} a supplement of $H^*(f_{\Omega_i}^{-1}(-\infty, t_{j-1}))$ in $H^*(f_{\Omega_i}^{-1}(-\infty, t_j))$, $1 \leq j \leq m_i$. Then,

$$\sum_{j=1}^{m_i} \dim A_{ij} = \dim H^*(f_{\Omega_i}^{-1}(-\infty, t_{m_i})) = \dim H^*(M^{\Omega_i}, \mathbb{R}) - q.$$

Therefore, there exists a certain $A_{i\,j_i} = A_i$, such that,

$$\dim A_i \geq \frac{(\dim H^*(M, \mathbb{R}))^{m_i} - q}{m_i}.$$

Denote s_i the middle point of $(t_{j_i-1}, t_{j_i}]$ and $\delta_i = \frac{1}{2m_i}$.

Then, by definition of $b_i'(s_i, \delta_i) = \dim A_i$.

There exists a subsequence $s_{i_k} \to c \in [0, 1]$, when $k \to \infty$.

Given $\delta > 0$, there exists a $K > 0$ such that $\delta_{i_k} < \delta/2$ and $|s_{i_k} - c| < \delta/2$, if $k > K$.

This means $c - \delta < s_{i_k} - \delta_{i_k}$ and $s_{i_k} + \delta_{i_k} < c + \delta$.

From this follows that $b_{i_k}'(c, \delta) \geq b_{i_k}'(s_{i_k}, \delta_{i_k}) = \dim A_{i_k}$.

Finally, we get

$$\frac{\log(b_{i_k}'(c, \delta))}{|\Omega_{i_k}|} \geq \frac{1}{m_{i_k}} \log \frac{(\dim H^*(M, \mathbb{R}))^{m_{i_k}} - q}{m_{i_k}}.$$

Now, taking limit in $k \to \infty$ in the above expression we get

$$b'(c, \delta) \geq \log(\dim(H^*(M, \mathbb{R})).$$

□

Lemma 17 *The function $b(c)$ is upper semicontinuous.*

Proof Suppose $c_k, k \in \mathbb{N}$ is a sequence of points in $[0, 1]$ such that, $c_k \to c$.

Given $\varepsilon > 0$, take $\delta > 0$, such that, $b'(c, \delta) < b(c) + \varepsilon$. There exists a $N > 0$ such that $|c - c_k| < \delta/2$, if $k \geq N$. Then, $c - \delta < c_k - \delta/2$ and $c_k + \delta/2 < c + \delta$, if $k \geq N$.

Then, $b_i'(c, \delta) \geq b_i'(c_k, \delta/2)$, if $k \geq N$, for all $i = 1, 2, 3, \ldots$

From this follows that $b'(c,\delta) \geq b'(c_k, \delta/2)$. Therefore,

$$b(c) + \varepsilon > b'(c,\delta) \geq b'(c_k,\delta/2) \geq b(c_k), \text{ if } k \geq N.$$

Therefore

$$\limsup_{k\to\infty} b(c_k) \leq b(c) + \varepsilon,$$

for any $\varepsilon > 0$. From this it follows the claim. □

Lemma 18 *The function $b(c)$ is concave.*

Proof Consider $0 \leq c_1 < c_2 \leq 1$ and $0 \leq t \leq 1$, we will show that

$$b(t\,c_1 + (1-t)\,c_2) \geq t\,b(c_1) + (1-t)\,b(c_2).$$

First we will show the claim for $t = 1/2$. Denote $\tilde{\Omega}_i = \Omega_i + (2\,i+1)\gamma_1$ and $\Omega'_i = \Omega_i \cup \tilde{\Omega}_i$.

By Proposition 2 and Lemma 12 we get:

$$b'_{\Omega'_i}(1/2\,c_1 + 1/2\,c_2,\ \delta) \geq b'_{\Omega_i}(c_1,\delta)\,b'_{\tilde{\Omega}_i}(c_2,\delta) = b'_i(c_1,\delta)\,b'_i(c_2,\delta),$$

for all $\delta > 0$.

Now, applying Lemma 15 to $h(\Omega) = \log b'_\Omega(1/2\,c_1 + 1/2\,c_2,\delta)$, we get $b'(1/2\,c_1 + 1/2\,c_2,\delta) \geq 1/2\,b'(c_1,\delta) + 1/2\,b'(c_2,\delta)$.

Now, taking $\delta \to 0$, we get $b(1/2\,c_1 + 1/2\,c_2) \geq 1/2 b(c_1) + 1/2\,b(c_2)$.

The inequality we have to prove is true for a dense set of values of t in $[0, 1]$. Then, by Lemma 17 is true for all $t \in [0, 1]$. □

Corollary 3 *The function $b(c)$ is continuous for $c \in [0,]$.*

Proof This follows from Lemmas 17 and 18. □

We summarize the above results in the following theorem.

Theorem 1 *(a)* $0 \leq b(c) \leq \varepsilon(c) \leq \log($ *number of critical points of* f_0*), for all* $0 \leq c \leq 1$.

(b) $b(c)$ *is continuous on* $[0, 1]$

(c) $b(c)$ *is concave, that is, its graph is always above the cord*

(d) $b(c)$ *is not constant equal zero. Moreover, there exists a point c where* $b(c) \geq \log\,($*dim* $H^*(M,\mathbb{R})\,) > 0$.

7 An Example

The next example shows that the item (d) in the above theorem can not be improved.

Take $M = S^n$, $n \geq 1$, and a Morse function $f_0 : M \to [0, 1]$ which is surjective with only two critical points. Suppose x_- is the minimum and x_+ the maximum of f_0. We will compute $b(c)$ and $\varepsilon(c)$.

Take $\Omega \in B(\Gamma)$ with $|\Omega| = m \geq 1$. For each $\Omega' \subset \Omega$ consider the canonical projection $p_{\Omega'} : M^{\Omega} \to M^{\Omega'}$. Now, take

$$\mu^{\Omega'} = p^*_{\Omega'}([\, M^{\Omega'}\,]) \in H^{n\,|\Omega'|}(M^{\Omega}, \mathbb{R}),$$

where [ ] represents fundamental class. Then,

$$\{\, \mu^{\Omega'} \;:\; \Omega' \subset \Omega\}$$

is a $\mathbb{R}$-homogeneous basis of $H^*(M^{\Omega}, \mathbb{R})$.

For $0 \leq d \leq 1$ denote

$$L_d = \{x \in M^{\Omega} \;:\; f_{\Omega}(x) < d\,\} \subset M^{\Omega}.$$

For $x \in M^{\Omega}$ we denote by x_γ the corresponding coordinate, where $\gamma \in \Gamma$.

Lemma 19 *If* $0 \leq d \leq 1$, *where* d *is not rational, then*

$$\{\mu^{\Omega'} \;:\; |\Omega'| > m\,(1-d)\,\}$$

is a basis of $H^*(L_d)$.

Proof Take $K_d = M^{\Omega} - L_d$. By Lemma 1

$$H^*(L_d) = \mathrm{Ker}(\, H^*(M^{\Omega}, \mathbb{R}) \;\to\; H^*(K_d, \mathbb{R})\,)\ \text{(natural restriction)}.$$

The claim follows from

(1) $H^k(M^{\Omega}, \mathbb{R}) \to H^k(K_d, \mathbb{R})$ is zero if $k > m\,(1-d)\,n$, and

(2) $H^k(M^{\Omega}, \mathbb{R}) \to H^k(K_d, \mathbb{R})$ is injective if $k < m\,(1-d)\,n$.

Now we prove (1) and (2).

(1) Suppose $\Omega' \subset \Omega$ is such that $\mu^{\Omega'} \in H^k(M^{\Omega})$ where $k > m\,(1-d)\,n$. Then, $|\Omega'| > m\,(1-d)$. Suppose

$$F_{\Omega'} = \{x \in M^{\Omega} \;:\; x_\gamma = x_-\,,\ \text{if}\, \gamma \in \Omega'\}.$$

If $x \in F_{\Omega'}$, then $f_{\Omega}(x) \leq \frac{1}{m}\,(m - |\Omega'|) < d$. Then, $F_{\Omega'} \cap K_d = \emptyset$. This means that: if $x \in K_d \;\to\; x_\gamma \neq x_-$ for some $\gamma \in \Omega'$. Then, $K_d \subset p^{-1}_{\Omega'}(M^{\Omega'} - \{z\})$ where $z_\gamma = x_-$ for all $\gamma \in \Omega'$.

From this follows

$$\mu^{\Omega'} \mid K_d = p^*_{\Omega'}([M^{\Omega'}]) \mid K_d = 0, \text{ because } [M^{\Omega'}] \mid ([M^{\Omega'}] - \{z\}) = 0.$$

(2) Denote $T = \{x \in M^{\Omega} : \text{cardinality}(\{\gamma : x_\gamma = x^+\}) > m\, d\}$. The set T is closed.

If $x \in T$, then $f_\Omega(x) > \frac{1}{m} m\, d = d$. Then, $T \subset K_d$.

We have to show that

$$H^k(M^\Omega, \mathbb{R}) \to H^k(T, \mathbb{R}) \text{ is injective if } k < m\,(1-d)\,n.$$

As we had seen before $H^k(M^\Omega, \mathbb{R}) = 0$ if k is not multiple of n. Then, we can assume that $k = q\, n$, if $q = 0, 1, 2, \ldots$. The claim follows from the next lemma, taking s the integer part of $m\, d$, by the exact sequence of homology, given that $U = U_s(\Omega)$.

Lemma 20 *Suppose* $s = 0, 1, 2, .., m$. *Suppose*

$$U_s(\Omega) = \{x \in M^\Omega : card(\{\gamma : x_\gamma = x^+\}) \leq s\},$$

then, $H^k_c(U_s(\Omega), \mathbb{R}) = 0$, *if* $k < (m-s)\,n$.

Proof The claim is trivial for $s = 0$ or $s = m$ ($U_0(\Omega)$ is homeomorphic to $(\mathbb{R}^n)^m$).

The proof is by induction in m. The claim for $m = 1$ is trivial. Suppose is true for $m - 1 \geq 1$. Take $0 < s < m$. Fix $w \in \Omega$ and take $\Omega' = \Omega - \{w\}$.

Consider $\varphi : M^{\Omega'} \to M^\Omega$ and $\psi : M^{\Omega'} \times (M - \{x^+\}) \to M^\Omega$, where for a given x we define $\varphi(x)$ by $x_\omega = x^+$ if $x \in M^{\Omega'}$, and $\psi(x, u)$ is defined by $x_w = u$ if $x \in M^{\Omega'}$ and $u \in M$, $u \neq x^+$.

ψ identifies $U_s(\Omega') \times (M - \{x^+\})$ with an open set A contained in $U_s(\Omega)$.

Moreover, φ identifies $U_{s-1}(\Omega')$ with the complement of this open set A in $U_s(\Omega)$.

As $M - \{x^+\}$ is homeomorphic to $\mathbb{R}^n$ and by recurrence we get that

$$H^k_c(U_s(\Omega') \times (M - \{x^+\}), \mathbb{R}) = 0,$$

if $k < (m-1-s)\,n + n = (m-s)\,n$ and, moreover, $H^k_c(U_{s-1}(\Omega', \mathbb{R})) = 0$, if $k < ((m-1) - (s-1))\,n = (m-s)\,n$.

Now, using the exact sequence of homology we finish the proof. □

Now we fix irrationals d_1, d_2, $0 < d_1 < d_2 < 1$. Denote $a_m = m\,(1-d_1)$, $b_m = m(1-d_2)$, and, $c_m = \dim\,(H^*(L_{d_2})/H^*(L_{d_1}))$.

By Lemma 19 we get

$$c_m = \sum \{\binom{m}{j} : b_m < j < a_m\}.$$

Assume m is much bigger than $(d_2 - d_1)$.

Take an integer j_m, such that $b_m < j_m < a_m$,

$$\binom{m}{j_m} = \sup\{\binom{m}{j} : b_m < j < a_m\}.$$

Then,

$$\binom{m}{j_m} \leq c_m \leq (a_m - b_m + 1) \binom{m}{j_m} .$$

By Stirling formula:

$$\frac{1}{m} \log \binom{m}{j} \sim \frac{1}{m} \log \left(\frac{m^{m+\ 1/2}}{j^{j+\ 1/2} \, (m-j)^{\, m-j+\ 1/2}} \right) =$$

$$\frac{1}{m} \log \left(m^{-1/2} \left(\frac{j}{m}\right)^{-1/2} \left(1 - \frac{j}{m}\right)^{-1/2} \left(\frac{j}{m}\right)^{-j} \left(1 - \frac{j}{m}\right)^{-m+j} \right).$$

Therefore,

$$\frac{1}{m} \log \binom{m}{j_m} \sim \frac{1}{m} \log \left(\left(\frac{j_m}{m}\right)^{-j_m} \left(1 - \frac{j_m}{m}\right)^{-m+j_m} \right) =$$

$$-\frac{j_m}{m} \log \left(\frac{j_m}{m}\right) - \left(1 - \frac{j_m}{m}\right) \log \left(1 - \frac{j_m}{m}\right),$$

when $m \sim \infty$.

As $1 - d_2 < \frac{j_m}{m} < 1 - d_1$, then (changing x by $(1 - x)$) we get

$$\limsup_{m \to \infty} \frac{1}{m} \log \binom{m}{j_m} \leq \sup_{d_1 < x < d_2} (- x \log(x) - (1 - x) \, \log(1 - x)),$$

and

$$\liminf_{m \to \infty} \frac{1}{m} \log \binom{m}{j_m} \geq \inf_{d_1 < x < d_2} (- x \log(x) - (1 - x) \, \log(1 - x)).$$

From this follows

$$\limsup_{m \to \infty} \frac{\log c_m}{m} \leq \sup_{d_1 < x < d_2} (- x \log(x) - (1 - x) \, \log(1 - x)),$$

and

$$\liminf_{m \to \infty} \frac{\log c_m}{m} \geq \inf_{d_1 < x < d_2} (- x \log(x) - (1 - x) \, \log(1 - x)).$$

Proposition 6

$$\varepsilon(c) = b(c) = -c\,\log c - (1-c)\,\log(1-c),\;\; 0 \le c \le 1.$$

Proof Given $0 < c < 1$, there exists small $\delta > 0$ such that

$$0 < c-\delta < c < c+\delta < 1 \;\text{ and }\; c-\delta, c+\delta \text{ are not in } \mathbb{Q}.$$

From the above for $d_1 = c - \delta$ and $d_2 = c + \delta$ we get

$$\inf_{d_1 < x < d_2} (-x\log(x) - (1-x)\,\log(1-x)) \le b'(c,\delta) \le$$

$$\sup_{d_1 < x < d_2} (-x\log(x) - (1-x)\,\log(1-x)).$$

Now, taking $\delta \to 0$, we get

$$b(c) = (-c\log(c) - (1-c)\,\log(1-c)).$$

For $c = 0$ or $c = 1$ the result follows from continuity.
Now we will estimate $\varepsilon(c)$.
The critical values of f_Ω are $0, \frac{1}{m}, \frac{2}{m}, \ldots, 1$.
To the critical values $\frac{j}{m}$ $(j = 0, 1, 2, \ldots, m)$ corresponds $\binom{m}{j}$ critical points.
Therefore, given $d_1, d_2 \in \mathbb{R}$ $d_1 < d_2$, the number c'_m of critical points of f_Ω in $f_\Omega^{-1}(d_1, d_2)$ is

$$c'_m = \sum \{ \binom{m}{j} : d_1 < \frac{j}{m} < d_2 \} =$$

$$\sum \{ \binom{m}{j} : m\,(1-d_2) < j < m\,(1-d_1) \}.$$

The computation of $\varepsilon(c)$ is analogous to the one for $b(c)$. This also follows from the last Theorem and the fact that $H^*(M)$ = number critical points of f_0 in the present case. □

8 About the Definition of $b(c)$

We will show that the definition of $b(c)$ presented here coincides with the one in [1].
Suppose X is a compact connected oriented C^∞ manifold.

Lemma 21 *Given an open set V in X consider $\alpha \in H^*(X, \mathbb{R})$ such that $\alpha|_V \neq 0$. Then, there exists $\beta \in H^*(V)$ such that $\alpha \wedge \beta \neq 0$.*

Proof Take $w \in \alpha$. As $\alpha|_V \neq 0$, then there exists a cycle z on V such that $\int_z w \neq 0$.

Suppose w' is a closed form with compact support on V such that its cohomology class in $H_c^*(V, \mathbb{R})$ is the Poincare dual of the homology class of z in $H_*(V, \mathbb{R})$.

w' can be extended to a closed form on X (putting 0 where needed) and by Poincare duality:

$$0 \neq \int_z w = \int_V w \wedge w' = \int_X w \wedge w'.$$

Therefore, $w \wedge w'$ is not exact on X.

Denote $\beta \in H^*(X, \mathbb{R})$ the cohomology class of w'. By Lemma 1 we have that $\beta \in H^*(V)$. As $w \wedge w'$ is not exact we get that $\alpha \wedge \beta \neq 0$. □

Notation: if $S \subset X$, then $\mathcal{H}^*(S) = \cap \{H^*(W) : W \subset X$ is an open set and $S \subset W\}$.

Lemma 22 *Suppose $U, V \subset X$ are open sets and $X = U \cup V$. Take $K = U - V$ and $\alpha \in H^*(U)$. Then, $\alpha \wedge \beta = 0$ for all $\beta \in H^*(V)$, if and only if, $\alpha \in \mathcal{H}^*(K)$.*

Proof Suppose $\alpha \in \mathcal{H}^*(K)$ and take $\beta \in H^*(V)$. By Lemma 5 there exists $w \in \beta$ such that supp $w \subset V$.

Take $W = X-$ supp w (which contains K). By definition we get that $\alpha \in H^*(W)$. Then, by Lemma 5, there exists $w' \in \alpha$ such that supp $w' \subset W$. Therefore, $w \wedge w' = 0$, and finally it follows that $\alpha \wedge \beta = 0$.

Reciprocally, suppose that $\alpha \wedge \beta = 0$ for all $\beta \in H^*(V)$. By Lemma 21 we have that $\alpha|V = 0$. Take $W \supset K$, then $V \cup W = X$. Therefore, by definition $\alpha \in H^*(W)$. □

Lemma 23 *Take $K \subset X$ a compact submanifold with boundary such that $K - \delta K$ is an open subset of X.*

Then,

$$\mathcal{H}(K) = \textit{Ker}\ (H^*(X, \mathbb{R}) \to H^*(X - K, \mathbb{R}))\ \textit{restriction}.$$

WE leave the rest of the proof for the reader.

Proof Take W an open set by adding a necklace to K. Then, $X - K$ can be retracted by deformation over $X - W$.

Then, if $\alpha \in H^*(X, \mathbb{R})$, we get that $\alpha|_{X-K} = 0$ is equivalent to $\alpha|_{X-W} = 0$.

Now, the claim follows from Lemma 1 and by the definition of $\mathcal{H}(K)$. □

Corollary 4 *Under the same hypothesis of last lemma it also follows that $\mathcal{H}(K) = H^*(\textit{ int } (K))$.*

Proof This follows from the fact that $H^*(X - \text{int}\,(K)\,,\mathbb{R}) \to H^*(X - K, \mathbb{R})$ is an isomorphism. □

Proposition 7 *Suppose U, V are open sets such that $X = U \cup V$ and moreover that $\overline{U}$, $\overline{V}$ are submanifolds with boundary of X.*

Consider the linear transformation L such that

$$L : H^*(U) \to Hom\,(\,H^*(V), H^*(U \cap V)\,),$$

where, $a \to \ (b \to a \wedge b)$.

Then, the rank of L is dim $(\,H^*(U)/H^*(M - \overline{V})\,)$.

Proof By Lemma 22 we get that Ker $L = H^*(X - V)$. Finally, by the last corollary $H^*(X - V) = H^*(M - \overline{V})$. □

Consider now a Morse function $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$, $\delta > 0$.

Definition 5 $b_{c,\delta}(f)$ is the rank of the linear transformation

$$H^*(\,f^{-1}(-\infty, c+\delta)\,) \ \to \ \text{Hom}\,(\,H^*(f^{-1}(\,c-\delta, \infty\,), H^*(f^{-1}(c-\delta, c+\delta)\,)\,),$$

where $a \to (b \ \to a \wedge b)$.

Note that $b_{c,\delta}(f)$ decreases with δ.

Lemma 24 *If $c - \delta$ and $c + \delta$ are regular values of f, then*

$$b_{c,\delta}(f) = b'_{c,\delta}(f).$$

Proof Just apply Proposition 7 to $U = f^{-1}(-\infty, c+\delta)$ and $V = f^{-1}(\,c-\delta, \infty\,)$. □

Note that $b_\Omega(c, \delta) = b_{c,\delta}(f_\Omega)$, where $\Omega \in B(\Gamma)$ and $\Omega \neq \emptyset$, and moreover that $b_i(c, \delta) = b_{\Omega_i}(c, \delta)$. The next limit exists (see [1]).

Definition 6

$$b(c, \delta) = \lim_{i \to \infty} \frac{\log(\,b_i(c, \delta)\,)}{|\,\Omega_i\,|}.$$

The set $S \subset [0, 1]$ of all critical values of all f_Ω is countable. By Lemma 24 we get that $b'_i(c, \delta) = b_i(c, \delta)$ if $c - \delta \notin S$ and $c + \delta \notin S$. Therefore, $b'(c, \delta) = b(c, \delta)$ if $c - \delta \notin S$ and $c + \delta \notin S$.

Finally,

$$\lim_{\delta \to 0} b'(c, \delta) = \lim_{\delta \to 0} b(c, \delta)$$

because both limits exist.

Therefore the function $b(c)$ we define coincides with the one presented in [1].

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