Chapter 5
Intersection Homology

Jean-Paul Brasselet

Abstract The famous duality theorems for compact oriented manifolds: Poincaré duality between cohomology and homology, and Poincaré-Lefschetz duality, intersection between cycles, are no longer true for a singular variety. A huge and fantastic step forward was taken by Mark Goresky and Robert MacPherson by the simple but brilliant idea of rediscovering duality by restricting oneself to chains only meeting the singular part of a stratified singular variety in controlled dimensions. Intersection homology was born. In this survey, we recall the first geometric definition as well as the theoretical sheaf definition allowing to describe the main properties of the intersection homology. Fruitful and unexpected developpments have been obtained in the context of singular varieties. For instance de Rham's theorem and Lefschetz's fixed point theorem find their place in the theory of intersection homology. The same is true for Morse's theory (see the Mark Goresky's survey in this Handbook, Chap. 5, Vol. I). In the last section, we provide some applications of intersection homology, for example concerning toric varieties or asymptotic sets. It must be said that the main application and source, itself, of innumerable applications is the fascinating and fruitful topic of perverse sheaves, which unfortunately it is not possible to develop in such a survey.

Contents

J.-P. Brasselet (✉)
I2M, CNRS and Université d'Aix-Marseille, Marseille, France
e-mail: jean-paul.brasselet@univ-amu.fr

J. L. Cisneros-Molina et al. (eds.), *Handbook of Geometry and Topology of Singularities II*, https://doi.org/10.1007/978-3-030-78024-1_5

5.1 Introduction

In the case of manifolds, global homological invariants like Betti numbers enjoy remarkable duality properties as stated by Poincaré (1893) and Lefschetz (1926). For smooth manifolds, the de Rham theorem (1931) and Morse theory (1934) show that it is possible to compute such topological invariants using differential forms and smooth functions. Unfortunately, all these beautiful results fail to hold for singular varieties. In an attempt to generalize the powerful theory of characteristic numbers to the singular case, Mark Goresky and Robert MacPherson noticed about 1973 that the failure of Poincaré duality is caused by the lack of transversal intersection of cycles on the singular locus. As a remedy, they introduced chains with well controlled intersection behaviour on the singular locus. These "intersection chains" form a complex that yields a new (co-) homology theory, called “intersection homology”. As a key point, the new theory yields an intersection product with suitable duality properties.

That new approach turned out to be extremely fruitful, far beyond the original purpose at hand, and stimulated a whole wealth of unexpected developments. Already at an early stage in the development of intersection homology, the original geometric construction of intersection chains has been recast into the formalism of sheaf theory and (hyper-) cohomology. The powerful machinery thus made available has been indispensable for the development of the theory; yet it bears the risk to hide the beautiful geometry that lies at the bottom.

The article is divided into five main sections: In the first Sect. 5.2 the classical results in the manifold case are presented and examples show their failure for singular varieties.

Section 5.3 is devoted to the main tools in the frameworks of sheaf theory and derived categories. Definitions are provided and notations are fixed.

Section 5.4 is devoted to the definition of intersection homology, both in the PL and in the topological situations. The local calculus eventually leads to “sheafify” the original geometric approach, thus obtaining the intersection sheaf complexes. In this context, the Deligne sheaf complex is of fundamental importance.

Section 5.5 shows how several important concepts and results carry over from the usual (co-) homology of manifolds to intersection homology of singular varieties: first basic properties Sect. 5.5.1, functoriality Sect. 5.5.2, the Lefschetz fixed point theorem Sect. 5.5.3, Morse theory Sect. 5.5.4, de Rham theorem Sect. 5.5.5, and cohomology operations like Steenrod squares, cobordism and Wu classes Sect. 5.5.6.

Section 5.6 is a supplement and thus of a different nature: Here are collected various applications and generalizations that deserve mention, but where an appropriate introduction would by far exceed the scope of the present survey. Therefore brief sketches and suitable references are presented.

There is a vast literature consisting of research articles, conference papers, surveys, books, course notes etc. dealing with intersection homology and its implications and generalizations, some including historical comments. The first mention is for the surveys by MacPherson [127, 128], Goresky [89], Kleiman [117], the conferences in the Bourbaki Seminar by Brylinski [45] and Springer [171], and surveys by Friedman [79], Klinger [118]. This short list is far from being exhaustive.

Among the books dedicated to intersection homology and perverse sheaves, let mention those by Borel et al. [18], Kirwan [116], Goresky-MacPherson [96], Schürmann [167], Maxim [130] Dimca [66].

More specialized surveys are for instance: on de Rham theorem [24], on Morse theory [126], on combinatorial toric intersection homology [70], on perverse sheaves [125, 161], etc. This list presents only a small selection.

This concise overview of such an extensive theory is of mainly introductory character and remains thus necessarily incomplete; yet the author hopes that the reader will deepen the interest in this fascinating subject.

The author is very grateful to Gottfried Barthel, Karl-Hienz Fieseler, Ludger Kaup, Alice Libardi, Thủy Nguyễn Thị Bích, Eliris Rizziolli, Marcelo Saia, and to the anonymous referee for many helpful corrections and suggestions on an earlier version of this paper.

Important note: The concept of intersection homology was first defined by Goresky and MacPherson in the framework of PL-spaces and PL-stratifications ([92] see also [18, Chaps. I–III]). In this framework, definitions are intuitive and geometric, it is possible to make explicit figures and proofs are nice, ingenious, delicate but often technical. In their paper [94] Goresky and MacPherson consider the more general framework of topological spaces and topological stratifications. The use of tools such as sheaf theory and derived categories, together with the notion of Deligne sheaf, makes proofs easier and opens the door to deeper results and to more applications (see also [18, Chaps. IV and V]).

Both viewpoints are important and have their own advantages and disadvantages and it would be a mistake to hide one of them. The first one provides the motivation and the meaning of the concept, the second one provides simpler proofs, as well as a huge amount of extensions and applications. The introduction of Habegger's chapter in the Borel book [18, Chap. II] explicits these roles.

5.2 Classical Results—Poincaré and Poincaré-Lefchetz

In order to understand the introduction of intersection homology it is useful to recall some elementary properties for manifolds.

An *n-manifold* (or n-topological manifold) is a (non-empty, Hausdorff) topological space X such that each point admits a neighborhood homeomorphic with an open subset of the Euclidean space $\mathbb{R}^n$.

Let X be an n-dimensional compact, connected, oriented and without boundary smooth manifold. The Poincaré duality says that the p and $(n-p)$ Betti numbers of X agree. It first stated without proof in [152] then Poincaré gave a proof of the theorem using topological intersection theory in his 1895 paper Analysis Situs [153]. Heegaard [104], provided a counter-example to Poincaré's formula and, finally, Poincaré provided a new proof performed in terms of dual cell decompositions [154, 155]. For historical details, see for example [157]. In order to simplify, in this introduction, homology and cohomology groups are with $\mathbb{Z}$ coefficients.

The Poincaré result is presented in two ways: using intersection of cycles, i.e. showing the existence of an intersection morphism

$$H_{n-p}(X) \otimes H_{n-q}(X) \xrightarrow{\bullet} H_{n-(p+q)}(X)$$

(see Sect. 5.2.5) and showing the existence of a Poincaré duality isomorphism

$$PD : H^p(X) \xrightarrow{\cong} H_{n-p}(X),$$

using dual cells (see Sect. 5.2.6). The two definitions are linked by the commutative diagram, where $\cup$ denotes the usual cup-product.

$$\begin{array}{ccc} H^p(X) \otimes H^q(X) & \xrightarrow{\cup} & H^{p+q}(X) \\ \downarrow{\scriptstyle PD\otimes PD} & & \downarrow{\scriptstyle PD} \\ H_{n-p}(X) \otimes H_{n-q}(X) & \xrightarrow{\bullet} & H_{n-(p+q)}(X). \end{array} \tag{5.1}$$

The first part of this section consists of some useful definitions and notations. Then the classical dualities (Poincaré and Poincaré-Lefschetz) for manifolds are recalled and counter-examples in the situation of singular varieties are provided.

5.2.1 *PL*-Structures

A piecewise-linear structure, PL-structure on a topological space X is a class of locally finite (simplicial) triangulations such that any subdivision of one of them belongs to the class and two of them admit a common subdivision.

When endowed with a PL-structure, the space is said a PL-space. Not all topological space admit a PL-structure and when such structure exists it is, in general, not unique.

A triangulation of a PL-space X is a triangulation of the corresponding class. That is a simplicial complex K whose geometric realization $|K|$ is homeomorphic to X. The space is said triangulated and one writes $X = |K|$.

The advantage of having a whole class of triangulations is that any open subset $U \subset X$ inherits a PL-structure. This property is convenient for the construction of sheaves (see in particular Example 5.3.2).

A manifold equipped with a structure of PL-space is called PL-manifold. In a triangulation K of a PL-manifold, every $(n-1)$-simplex is a face of exactly two n-simplices. This property is one of the conditions for a PL-space to be a PL-pseudomanifold.

5.2.2 *Pseudomanifolds*

Definition 5.2.1 The (non-empty, paracompact, Hausdorff) topological space X is an *n-pseudomanifold* if there is a closed subspace $\Sigma \subset X$ such that:

1. $X \setminus \Sigma$ is an n-dimensional manifold dense in X.
2. $\dim \Sigma \leq n - 2$.

The subspace Σ of the pseudomanifold X contains the subset of singular points of X i.e. the points which do not admit a neighborhood homeomorphic to a ball and whose boundary is homeomorphic to a sphere.

A PL-pseudomanifold X of dimension n is an n-dimensional PL-space X containing a closed PL-subspace Σ of codimension at least 2 such that $X - \Sigma$ is an n-dimensional PL-manifold dense in X.

Equivalently, given a triangulation $X = |K|$, then $|K|$ is the union of the n-simplices and each $n-1$-simplex is face of exactly two n-simplices.

A connectivity condition of the set $X - \Sigma$ is sometimes added. The connected PL-pseudomanifold X is oriented if there exists a compatible orientation of all n-simplices. In the connected and oriented situation of a PL-pseudomanifold, the conditions ensure existence of a fundamental class $[X]$. Namely, given a triangulation $X = |K|$ of an n-dimensional connected and oriented PL-pseudomanifold, the sum of all (oriented) n-simplices is a cycle whose class is the fundamental class. The original article by Goresky and MacPherson suppose the PL-pseudomanifold to be compact and oriented. These hypothesis are dropped in the further articles.

The pinched torus (Fig. 5.1 and Example 5.2.6) and the suspension of the torus (Example 5.5.10) are examples of connected and oriented PL-pseudomanifolds.

5.2.3 Stratifications

Dealing with singular spaces, the notion of stratification is one of the most important tool. The main reference for the definitions and results is the Trotman's survey in this Handbook, vol I [179, Chap. 4] (see also [128]).

A (topological) stratification $\mathcal{S}$ of the n-dimensional pseudomanifold X is the data of a filtration

$$(\mathcal{S}) \qquad X = X_n \supset X_{n-1} = X_{n-2} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset \tag{5.2}$$

by closed subspaces such that

- every stratum $S_i = X_i - X_{i-1}$ is either empty or a finite union of i-dimensional smooth submanifolds of X,
- each point x in S_i admits a *distinguished neighborhood* $U_x \subset X$ together with a homeomorphism
$$\phi_x : U_x \to \mathbb{B}^i \times \mathring{c}(L) \tag{5.3}$$
(local triviality property) where:
  - $\mathbb{B}^i$ is an open ball in $\mathbb{R}^i$,
  - the "link" L (called the link of the stratum S_i) is a compact $(n-i-1)$-dimensional pseudomanifold independent (up to homeomorphism) of the point x in the stratum S_i and filtered by:

$$L = L_{n-i-1} \supset L_{n-i-3} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset,$$

– $\mathring{c}(L)$ is the open cone over L defined by $\mathring{c}(L) = L \times [0, 1[/(x, 0) \sim (x', 0)$, and filtred by

$$(\mathring{c}(L))_0 = \{0\} \quad \text{and} \quad (\mathring{c}(L))_k = \mathring{c}(L_{k-1}) \quad \text{if } k > 0.$$

By definition, one has $\mathring{c}(\emptyset) = \{point\}$.
Moreover, the homeomorphism ϕ_x preserves the stratifications of U_x and $\mathbb{B}^i \times \mathring{c}(L)$ respectively, that is there are restriction homeomorphisms

$$\phi_x|_{X_j} : U_x \cap X_j \to \mathbb{B}^i \times \mathring{c}(L_{j-i-1}), \quad \text{for } j \geq i.$$

In particular stratifications which satisfy the Whitney conditions (see [89], [179]) satisfy the topological local triviality property (A' Campo [18, Chap. IV]):

A PL-stratification $\mathcal{S}$ of the n-dimensional PL-pseudomanifold X is a stratification such that all involved subspaces are PL-subspaces and the local triviality property holds in the PL-category.

5.2.4 Borel-Moore Homology

In the following, G will denote an R-module, for R a PID. For example, G can be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In this (sub)section, X is a connected, oriented, not necessarily compact n-dimensional PL-manifold or PL-pseudomanifold.

Given a triangulation $X = |K|$, the complex of possibly infinite simplicial chains of K with coefficients in G is denoted by $C_*(K; G)$. A chain ξ in $C_i(K; G)$ is written $\xi = \sum \xi_\sigma \sigma$ where σ are oriented i-simplices in K and ξ_σ are elements of G. It has a canonical image in $C_i(K'; G)$ for any subdivision K' of K. Two chains in $C_i(K_1; G)$ and $C_i(K_2; G)$ are identified if their image in a common subdivision coincide. The group $C_i(X; G)$ of PL-geometric chains with closed supports of X is the direct limit under refinement of the groups $C_i(K; G)$ over all triangulations of X.

The support of $\xi \in C_i(K; G)$ is the union of the closed simplices such that $\xi_\sigma \neq 0$ and is denoted by $|\xi|$, it does not depend on subdivision, thus the support of $\xi \in C_i(X; G)$ is well defined.

Using the usual boundary operator, the complex of chains $C_*(X; G)$ is well defined and its homology, denoted by $H_*(X; G)$ is called homology with closed supports of X, or Borel-Moore homology of X [17] (see also "Homologie de deuxième espèce" in Cartan [48, Exposé 5, Sect. 6]).

The subcomplex of chains with compact supports is denoted by $C_*^c(X; G)$ and its homology $H_*^c(X; G)$ is the homology with compact support. If X is compact, the two homology groups coincide.

5.2.5 Poincaré Duality Homomorphism

The idea, for defining Poincaré duality "à la Poincaré", is to associate to a given triangulation K of an n-dimensional manifold X a decomposition of X into "cells", in such a way that there is a one-to-one correspondence between p-dimensional simplices and $(n-p)$-dimensional dual cells (called "Polyèdre réciproque" in [154, Sect. VII]).

Let X be a triangulated, compact and oriented n-dimensional PL-pseudomanifold $X = |K|$ such that the triangulation K itself is the first barycentric subdivision of a triangulation of X.

A p-elementary cochain is an (oriented) p-simplex σ, denoted by σ^* when considered as a p-cochain. A p-cochain with coefficients in G, element of $C^p(K;G)$, is a formal sum $\sum g_i \sigma_i^*$ where $g_i \in G$ and the σ_i are (oriented) p-simplices in K.

The coboundary $\delta\sigma^*$ of the p-elementary cochain σ^* is defined to be the $(p+1)$-cochain

$$\delta\sigma^* = \sum [\sigma : \tau]\tau^*$$

where the sum involves all $(p+1)$-simplices τ such that σ is a face of τ (denoted $\sigma < \tau$). The incidence number $[\sigma : \tau]$ is $+1$ if the orientation of σ is the one as boundary of τ and -1 otherwise. This defines the homomorphism

$$\delta^p : C^p(K;G) \to C^{p+1}(K;G)$$

by linearity.

Considering a first barycentric subdivision K' of K, the barycenter of every simplex σ in K is denoted by $\hat{\sigma}$. The simplices in K' whose first vertex is $\hat{\sigma}$ are all simplices on the form $(\hat{\sigma}, \hat{\sigma}_{i_1}, \dots, \hat{\sigma}_{i_q})$ with $\sigma < \sigma_{i_1} < \cdots < \sigma_{i_q}$. The union of these simplices, is called the the dual block of σ and is denoted by $D(\sigma)$. One has

$$D(\sigma) = \{\tau \in K' \ : \ \tau \cap \sigma = \{\hat{\sigma}\}\}. \tag{5.4}$$

The dual block $D(\sigma)$ has dimension $(n-p)$, it is endowed with an orientation such that the orientation of $D(\sigma)$ followed by the orientation of σ is the orientation of X (see [23, 177]).

The Poincaré homomorphism (at the level of chains and cochains) is the map

$$PD : C^p(K;G) \to C_{n-p}(K';G)$$

defined by $PD(\sigma^*) = D(\sigma)$ and extended by linearity. One has

$$PD(\delta\sigma^*) = \partial PD(\sigma^*).$$

The correspondence "simplex" $\to$ "dual block" sends K-cochains to K'-chains. By this correspondence, cocycles are sent to cycles and coboundaries to boundaries.

The Poincaré homomorphism is then well defined:

$$PD : H^p(K; G) \longrightarrow H_{n-p}(K'G).$$

As it is well known, the homology and cohomology groups of X do not depend on the given triangulation. The Poincaré duality morphism is realised by the cap-product by the fundamental class $[X]$:

$$PD : H^p(X; G) \xrightarrow{\bullet \cap [X]} H_{n-p}(X; G)$$

The main Poincaré's result is:

Theorem 5.2.2 • *In a compact oriented manifold, the dual blocks are cells, i.e. the dual block of a p-simplex σ is homeomophic to an $(n-p)$-ball and its boundary is homeomorphic to an $(n-p-1)$-sphere.*

- *In a compact oriented manifold, the Poincaré morphism is an isomorphism.*

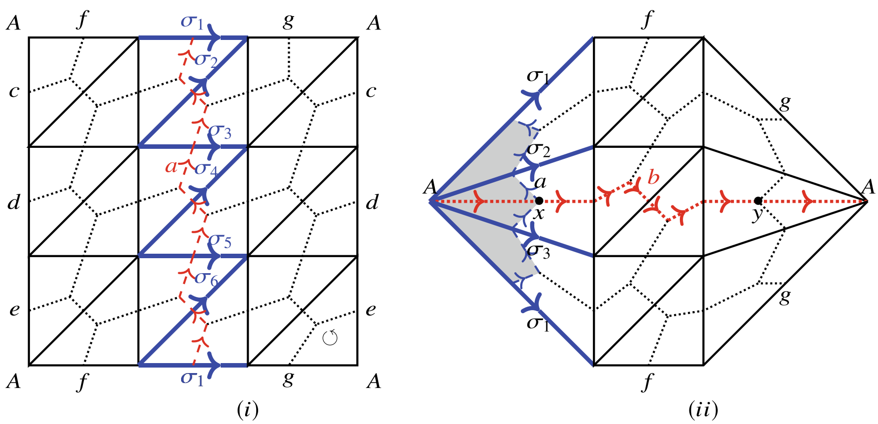


Fig. 5.1 Cycles and cocycles on the torus (i) and the pinched torus (ii)

Example 5.2.3 Examples of computations for a manifold and a pseudomanifold.

Figure 5.1i and ii are planar representations of the torus and the pinched torus, with suitable identification of the simplices of the boundary and with given compatible orientation of all 2-simplices.

In the torus, the cochain $\alpha = \sigma_1^* + \sigma_2^* + \ldots + \sigma_6^*$ is a cocycle, not a coboundary. The dual chain $a = PD(\alpha)$ is a cycle, not a boundary. In the same way, by symmetry with respect to the first diagonal, one has a "horizontal" cocycle β and dual cycle $b = PD(\beta)$. In the torus, the Poincaré homomorphism is an isomorphism (here $G = \mathbb{Z}$):

$$H^1(T; \mathbb{Z}) \to H_1(T; \mathbb{Z}), \qquad \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \to \mathbb{Z}a \oplus \mathbb{Z}b.$$

On the pinched torus, which is a singular variety, the Poincaré homomorphism is no longer an isomorphism.

On the one hand, the cocycle $\alpha = \sigma_1^* + \sigma_2^* + \sigma_3^*$ is not a coboundary. Indeed, the coboundary of the vertex A consists of all 1-dimensional simplices of which A is a vertex and α is only half of it. The cohomology class of α is not zero. The dual of the cocycle α is the cycle a. This cycle is a boundary (the boundary of the gray part), its homology class is zero: the Poincaré morphism maps the non-zero class of α on the zero class of a. It is not injective.

On the other hand, the (red) cycle b going from A to A is not a boundary, its homology class is not zero and generates the 1-dimensional homology. But it is easy to see that b is not the dual of a cochain. The Poincaré morphism is not surjective.

The Poincaré morphism of the pinched torus is neither injective, nor surjective, although it is a morphism from $\mathbb{Z}$ to $\mathbb{Z}$:

$$\mathbb{Z}\alpha \cong H^1(X;\mathbb{Z}) \xrightarrow{\bullet\cap[X]} H_1(X;\mathbb{Z}) \cong \mathbb{Z}b.$$

5.2.6 Poincaré—Lefschetz Homomorphism

In his 1895 paper [153, Sect. 9] (corrected in [154, 155]), Poincaré gave a definition of intersection of two oriented and complementary dimensional cycles in a compact oriented manifold. Lefschetz, in 1936 [122] defined the intersection of an i-chain a and a j-chain b in a compact oriented n-manifold M whenever $|a| \cap |b|$ contains simplices of dimension at most $i + j - n$, and gave a formula for the multiplicity in $a \cap b$ of an $i + j - n$-simplex $\sigma \subset |a| \cap |b|$ which is local, in the sense that it depends only on the behavior of a and b near an interior point of σ.

Let X be a smooth PL-manifold, two cycles a and b are said *dimensionally transverse* if either they do not meet or their dimensions satisfy the formula:

$$\text{codim } (|a| \cap |b|) = \text{codim } |a| + \text{codim } |b|.$$

Theorem 5.2.4 (See [120–122] and [176, Sect. 5] for a summary) *In a compact oriented smooth PL-manifold, the intersection of two dimensionally transverse cycles with appropriately defined orientations and multiplicities is a cycle.*

In a compact oriented PL-manifold X, if two dimensionally transverse cycles a and b have complementary dimensions, then the intersection $a \cap b$ is a finite number of points $\{x_i\}$. The cycles being oriented, in each of the points x_i Lefschetz defines the local intersection index $I(a, b; x_i)$, depending on orientations and multiplicities [122, 153, 154]. For elementary cycles (i.e. with multiplicities $+1$), the index $I(a, b; x_i)$ is $+1$ if the orientation of a followed by the orientation of b is the orientation of X and -1 otherwise, then extend by linearity. The intersection index

$$I(a, b) = \sum_{x_i \in a \cap b} I(a, b; x_i).$$

defines an intersection product

$$\begin{array}{ccc} C_{n-p}(X; \mathbb{Z}) \times C_p(X; \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ a \quad , \quad b & \rightsquigarrow & I(a, b) \end{array} \tag{5.5}$$

which associates to each pair of oriented, dimensionally transverse and complementary dimensional cycles (a, b) the intersection index $I(a, b)$.

The intersection index $I(a, b)$ does not depend on the representative of the homology classes of the cycles a and b.

Theorem 5.2.5 (Poincaré-Lefschetz duality [120, 122, 153, 154]) *In a compact oriented smooth PL-manifold, the intersection product (5.5) induces a bilinear map*

$$H_{n-p}(X; \mathbb{Z}) \times H_p(X; \mathbb{Z}) \longrightarrow H_0(X; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z}$$

which is non-degenerate when tensored by the rational numbers. Here ϵ is the evaluation map $\epsilon : \sum n_i \{x_i\} \mapsto \sum n_i$.

Let X be a singular variety, then the Poincaré-Lefschetz duality is no longer true. The pinched torus is a classical example:

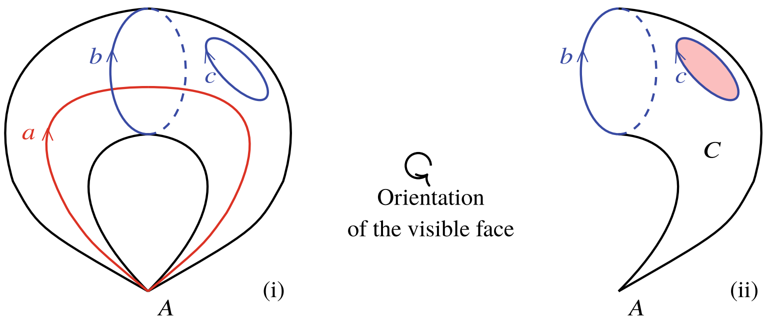


Fig. 5.2 Transverse cycles in the pinched torus

Example 5.2.6 (The pinched torus) Consider the pinched torus.

The area C in Fig. 5.2ii is a chain whose boundary is $c - b$. The cycles b and c are homologous. However, the intersection indices are $I(a, b) = +1$ and $I(a, c) = 0$. There is no intersection product at the level of homology classes.

5.3 The Useful Tools: Sheaves—Derived Category

In the previous section, the spaces considered were mainly compact and non singular. The notions of duality and intersection can be generalized to singular spaces using Borel-Moore homology on the one hand and sheaf theory on the other hand.

General useful references for these sections are Godement and Bredon (see [86, Chapitre II] and [36]). The interested reader will find a history of sheaf theory in the Christian Houzel article [106], in particular the passage from closed supports (Jean Leray) to open supports (Henri Cartan).

5.3.1 Sheaves

Let X be a topological PL-pseudomanifold. Let R be a PID, that may be sometimes $\mathbb{Z}$ or even a field such as $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$. A sheaf on X will be a sheaf $\mathcal{A}$ of R-modules. The category of sheaves on X is denoted by $Sh(X)$. The constant sheaf is denoted by $\mathbf{R}_X$.

The set of sections of the sheaf $\mathcal{A}$ over an open subset U of X is denoted by $\Gamma(U, \mathcal{A})$. Given a family of supports Φ, the subset of elements $s \in \Gamma(X, \mathcal{A})$ for which support of s belongs to Φ is denoted by $\Gamma_\Phi(X, \mathcal{A})$. The families considered are mainly the family of closed supports, the family of compact supports c and for a subspace $A \subset X$ the family (A) of supports whose elements are closed subsets contained in A.

The stalk at a point $x \in X$ of the sheaf $\mathcal{A}$ is denoted by $\mathcal{A}_x$. The restriction of $\mathcal{A}$ to a subspace $Y \subset X$ is denoted by $\mathcal{A}|_Y$ or simply $\mathcal{A}_Y$.

A sheaf $\mathcal{L}$ on the topological space X is called *locally constant* if there is an open covering $\{U_i\}$ of X and a family of R-modules $\{L_i\}$ such that $\mathcal{L}|_{U_i}$ is the constant sheaf on U_i represented by the R-module $\{L_i\}$. Equivalently, every point $x \in X$ has a neighborhood U such that the restriction maps

$$\mathcal{L}_x \leftarrow \Gamma(U; \mathcal{L}) \rightarrow \mathcal{L}_y$$

are isomorphisms for all $y \in U$.

5.3.2 System of Local Coefficients

The notion of system of local coefficients comes from Steenrod [174]. In fact in his introduction, Steenrod wrote that he generalizes an idea originating from Whitney (1940), who, in turn credits the idea to de Rham (1932). Also Steenrod claims that the notion is equivalent to the Reidemeister Überdeckung (1935) [159]. Steenrod provides applications of the notion, in particular full duality and intersection theory in a non-orientable manifold [174, Sect. 14]. Later, Steenrod applied that notion in 1951 in his book "The Topology of Fibre Bundles" when defining characteristic classes (Stiefel-Whitney and Chern classes) by obstruction theory (see [175, Sect. 30–31]).

The interest of local systems is well demonstrated by the example given by MacPherson [128, p. 19] of a local system which makes intersection homology interesting even when the space is nonsingular.

A *local coefficient system* (or local system) of R-modules on a topological space X is a locally constant sheaf $\mathcal{L}$ of R-modules.

If X is connected, then it is possible to use a single R-module L instead of a family L_i. If X is not connected, a local coefficient system is determined by the data of a base point x_i in C_i and a representation $\rho : \pi_1(X, x_i) \to Aut(L_i)$ for each connected component C_i of X.

Example 5.3.1 An exemple of local system is given by the orientation sheaf $\mathcal{O}_X$ on a (not necessarily orientable) n-dimensional manifold. That is the sheaf associated to the presheaf

$$U \mapsto H_n(X, X \setminus U; R).$$

If $\partial X = \emptyset$, then $\mathcal{O}_X$ is a locally constant sheaf with stalks isomorphic to R. It is constant if X is orientable.

5.3.3 Complexes of Sheaves

A bounded complex of sheaves $\mathcal{A}^\bullet$ is a sequence

$$\ldots \longrightarrow \mathcal{A}^{p-1} \xrightarrow{d^{p-1}} \mathcal{A}^p \xrightarrow{d^p} \mathcal{A}^{p+1} \longrightarrow \ldots \qquad p \in \mathbb{Z}$$

such that $d^p \circ d^{p-1} = 0$ for all p and $\mathcal{A}^p = 0$ for $|p|$ sufficiently large. If necessary to specify the complex, the differential will be denoted by $d^p_{\mathcal{A}^\bullet}$.

A sheaf $\mathcal{A}$ can be regarded as a complex of sheaves $\mathcal{A}^\bullet$ with $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^p = 0$ for $p \neq 0$, and $d^p = 0$ for all p. In this case, the complex $\mathcal{A}^\bullet$ is said to be concentrated in degree 0.

Given a complex of sheaves, the shifted complex $\mathcal{A}[n]^\bullet$ is defined by $\mathcal{A}[n]^p = \mathcal{A}^{n+p}$ and $d_{\mathcal{A}[n]^\bullet} = (-1)^n d_{\mathcal{A}^\bullet}$.

The sheaf of sections associated with a complex of sheaves $\mathcal{A}^\bullet$ assigns to every open subset $U \subset X$ the chain complex

$$\ldots \to \Gamma(U; \mathcal{A}^{p-1}) \to \Gamma(U; \mathcal{A}^p) \to \Gamma(U; \mathcal{A}^{p+1}) \to \ldots.$$

The p-th cohomology sheaf $\mathcal{H}^p(\mathcal{A}^\bullet)$ associated with $\mathcal{A}^\bullet$ is the sheafification ([86, Chapitre II, Sect. 1.2]) of the presheaf whose group of sections over U is the p-th homology group of $\mathcal{A}^\bullet$. The stalk at a point $x \in X$ of the sheaf $\mathcal{H}^p(\mathcal{A}^\bullet)$ is $\mathcal{H}^p(\mathcal{A}^\bullet)_x \cong \mathcal{H}^p(\mathcal{A}^\bullet_x)$.

Borel and Moore [17] define *cohomologically locally constant* (denoted CLC) complex of sheaves if the associated cohomology sheaves are locally constant. A complex of sheaves $\mathcal{A}^\bullet$ is said *(cohomologically) constructible* with respect of a filtration (5.2) of X if all $\mathcal{A}^\bullet|_{(X_i - X_{i-1})}$ are CLC and their stalk cohomology is finitely generated.

The complex $\mathcal{A}^\bullet$ is said *PL-(cohomologically) constructible* if it is bounded and (cohomologically) constructible with respect of a filtration of X by closed PL-subsets. Finally, the complex $\mathcal{A}^\bullet$ is said *topologically (cohomologically) constructible* if it is bounded and (cohomologically) constructible with respect to a topological filtration of X.

Henceforth, we will adopt the modern shorthand of replacing the words "(cohomologically) constructible" simply with "constructible". As in [94], all complexes of sheaves considered will be topologically constructible.

Example 5.3.2 The *sheaf complex of PL-chains with closed suports.*

Let X be a connected, oriented, not necessarily compact n-dimensional PL-manifold or PL-pseudomanifold. In Sect. 5.2.4 the Borel-Moore homology chains $C_i(X; G)$ with coefficients in an abelian group have been defined. The same definition applies with coefficients in a local system $\mathcal{L}$, denoted by $C_i(X; \mathcal{L})$.

In a first step the presheaf $U \mapsto C_i(U; \mathcal{L})$ for U open in X, is defined as follows. Let $V \subset U$ be two open subsets in X, the natural restriction maps

$$\rho_{VU} : C_i(U; \mathcal{L}) \to C_i(V; \mathcal{L}) \tag{5.6}$$

are defined in the following way: (see also [94, Sect. 2.1]) For a chain $\xi \in C_i(U; \mathcal{L})$, there is a locally finite triangulation K_U of U such that ξ can be written $\sum_{\sigma \in K_U} \xi_\sigma \sigma$ with $\xi_\sigma \in \mathcal{L}_\sigma$ and $\mathcal{L}_\sigma$ is the (constant) value of $\mathcal{L}$ on σ. Any triangulation of V admits a subdivision K_V such that every simplex υ in K_V is contained in a simplex $\sigma(\upsilon)$ of a subdivision of K_U and such that $\dim \upsilon = \dim \sigma(\upsilon)$. Considering orientations of all simplices of the triangulations, the chain $\rho_{VU}(\xi) \in C_i(V; \mathcal{L})$ is defined by

$$\rho_{VU}(\xi) = \sum_{\upsilon \in K_V} (-1)^{(\upsilon : \sigma(\upsilon))} \xi_{\sigma(\upsilon)} \upsilon$$

where the sign is $+1$ if υ and $\sigma(\upsilon)$ have the same orientation and -1 otherwise.

The boundary $\partial_i : C_i(U; \mathcal{L}) \to C_{i-1}(U; \mathcal{L})$ is defined in the following way: A chain $\xi \in C_i(U; \mathcal{L})$ is written $\sum_{\sigma \in K_U} \xi_\sigma \sigma$ for a locally finite triangulation K_U of U and $\xi_\sigma \in \mathcal{L}_\sigma$. Let τ be a face of σ and $\rho^\sigma_\tau : \mathcal{L}_\sigma \to \mathcal{L}_\tau$ the natural morphism, then

$$\partial_i(\xi) = \sum_\sigma \sum_{\tau < \sigma} [\tau : \sigma] \rho^\sigma_\tau(\xi_\sigma).\tau.$$

where the incidence number $[\tau : \sigma]$ is $+1$ if the orientation of τ is the one as boundary of σ and -1 otherwise.

Definition 5.3.3 ([94, Sect. 2.1], *see Remark* 5.4.10 *for the notation*) The Borel-Moore complex of sheaves of PL-chains $\mathcal{C}^\bullet$ on X with coefficients in $\mathcal{L}$ is defined by

$$\Gamma(U; \mathcal{C}^{-i}) = C_i(U; \mathcal{L})$$

with the above boundary.

5.3.4 Injective Resolutions

The injective resolutions are particularly important, by the fact that for any abelian category with enough injective objects, each R-module admits an injective resolution.

Definition 5.3.4 ([94, Sect. 1.5], [18, II, Sect. 5]) A map of complexes of sheaves $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ which commutes with the differentials

$$\varphi^{i+1} \circ d^i_{\mathcal{A}^\bullet} = d^i_{\mathcal{B}^\bullet} \circ \varphi^i$$

is called a *quasi-isomorphism* if it induces isomorphisms $\mathcal{H}^i(\varphi^\bullet) : \mathcal{H}^i(\mathcal{A}^\bullet) \to \mathcal{H}^i(\mathcal{B}^\bullet)$ of the cohomology sheaves of the complexes.

Definition 5.3.5 ([49, IV, Sect. 3]) Two morphisms of complexes $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ and $\psi^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ are *homotopic* if there exists a collection $\{h^i : \mathcal{A}^i \to \mathcal{B}^{i-1}\}$, $i \in \mathbb{Z}$ of sheaf maps, called a homotopy, so that:

$$d^{i-1}_{\mathcal{B}^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathcal{A}^\bullet} = \phi^i - \psi^i$$

for all $i \in \mathbb{Z}$.

Definition 5.3.6 Let $K(X)$ denote the category whose objects $\mathcal{A}^\bullet$ are topologically constructible bounded complexes of sheaves on X and whose morphisms $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ are homotopy classes of sheaf maps which commute with the differentials (Definition 5.3.5).

Definition 5.3.7 ([86, II, Sect. 7.1], [49, V, Sect. 1], [17, Sect. 1], [36, II, Sect. 9]) A sheaf $\mathcal{I}$ is *injective* if, for any sheaf monomorphism $\mathcal{F} \to \mathcal{G}$ and any sheaf map $\mathcal{F} \to \mathcal{I}$ there exists an extension $\mathcal{G} \to \mathcal{I}$.

Coefficients are important in the definition: the sheaf of integers $\mathbb{Z}$ on a point is fine (see Definition 5.3.10) but not injective, because $\mathbb{Z}$ is not injective over itself.

If $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of complex of sheaves on X and if each $\mathcal{I}^\bullet$ is injective, then $\mathcal{I}^\bullet$ is called an *injective resolution* of $\mathcal{A}^\bullet$.

Main properties of injective resolutions are (see [17, Sect. 1], [36, II, Sect. 9], [86, II Sect. 7.1], [89, Sect. 4.2], and [94, Sect. 1.5]):

Proposition 5.3.8 *1. If $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of complex of sheaves on X and $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{A}^\bullet$, then there exists a morphism $\psi^\bullet : \mathcal{I}^\bullet \to \mathcal{A}^\bullet$ that is a homotopy inverse to $\varphi^\bullet$. Therefore $\varphi^\bullet$ is invertible in the category $K(X)$.*

2. Injective resolutions exist for any complex of sheaves of R-modules and are uniquely determined up to chain homotopy.

3. Every bounded complex of sheaves admits a canonical bounded injective resolution (see given references).

5.3.5 Hypercohomology

The p-th *hypercohomology group* $\mathbb{H}^p(X; \mathcal{A}^\bullet)$ of a complex of sheaves $\mathcal{A}^\bullet$ is the p-th cohomology group of the cochain complex

$$\ldots \to \Gamma(X; \mathcal{I}^{p-1}) \to \Gamma(X; \mathcal{I}^p) \to \Gamma(X; \mathcal{I}^{p+1}) \to \ldots,$$

where $\mathcal{I}^\bullet$ is the canonical injective resolution of $\mathcal{A}^\bullet$ ([49, XVII, Sect. 2], [94, 1.6]).

Considering sections with supports in a family of supports Φ, one defines *hypercohomology* $\mathbb{H}^p_\Phi(X; \mathcal{A}^\bullet)$ *with support in the family* Φ as

$$\mathbb{H}^p_\Phi(X; \mathcal{A}^\bullet) = H^p(\Gamma_\Phi(X; \mathcal{I}^\bullet)).$$

A quasi-isomorphism induces an isomorphism on hypercohomology. In particular, the hypercohomology groups are naturally isomorphic to the cohomology group of the single complex which is associated to the double complex $C^p(X; \mathcal{A}^q)$ (see [86, II Sect. 4.6]).

Definition 5.3.9 ([86, II, Sect. 3.5 and 3.6]) A sheaf $\mathcal{A}$ is called *soft* ("faisceau mou" in french) if any section over any closed subset of X can be extended to a global section, i.e. the restriction maps

$$\Gamma(U, \mathcal{A}) \to \Gamma(B, \mathcal{A})$$

are surjective for all open $U \subset X$ and closed subset $B \subset U$.

Definition 5.3.10 ([86, II, Sect. 3.7]) A sheaf $\mathcal{A}$ over a paracompact Hausdorff space X is called *fine* ("faisceau fin" in french) if for every locally finite open cover $\{U_i\}$ of X there are endomorphisms φ_i of $\mathcal{A}$ such that:

- for every i, φ_i is zero outside a closed subset contained in U_i,
- one has $\sum_i \varphi_i = id$.

Here, locally finite means that every point $x \in X$ admits an open neighborhood which meets a finite number of elements U_i.

Every fine sheaf is soft, but the converse is not true [86, II, Sect. 3.7].

Proposition 5.3.11 ([18, II, Sect. 5], [86]) *Let X be a paracompact topological space, if the sheaf complex $\mathcal{A}^\bullet$ consists of injective, fine or soft sheaves, then*

$$\mathbb{H}^i(X; \mathcal{A}^\bullet) = H^i(\Gamma(X; \mathcal{A}^\bullet)). \tag{5.7}$$

Examples of Hypercohomology

Let X be a n-dimensional PL-space, the following examples provide particular cases of hypercohomology groups which will be useful for considering the properties of intersection homology. Coefficients are either a R-module G or a sheaf of local coefficients $\mathcal{L}$.

Example 5.3.12 (a) Hypercohomology of the sheaf complex of PL-chains with closed supports.
The complex of sheaves of PL-chains $\mathcal{C}^\bullet$ is a complex of fine sheaves on X (see [18, Sect. 5, Note]). Hence the complex $\mathcal{C}^\bullet$ satisfies Proposition 5.3.11. One has:

$$\mathbb{H}^{-i}(X; \mathcal{C}^\bullet) \cong H^{-i}(\Gamma(X; \mathcal{C}^\bullet)) = H_i(X).$$

For every family of supports Φ one has:

$$\mathbb{H}^{-i}_\Phi(X; \mathcal{C}^\bullet) = H_i^\Phi(X). \tag{5.8}$$

(b) Hypercohomology of the constant sheaf.
Consider the *constant sheaf* $\mathbf{R}_X$ on X, viewed as a complex concentrated in degree 0, then,

$$\mathbb{H}^i(X; \mathbf{R}_X) = H^i(X)$$

is cohomology of X with closed supports. For every family of supports Φ one has:

$$\mathbb{H}^i_\Phi(X; \mathbf{R}_X) = H^i_\Phi(X). \tag{5.9}$$

5.3.6 The (Constructible) Derived Category

The derived category was defined by Verdier [183, 184]. An object in the derived category is a complex of sheaves. In this category, new morphisms are added so that every quasi-isomorphism has an inverse and, consequently, every quasi-isomorphism becomes an isomorphism in the derived category (Property 5.3.13). Verdier found he was able to prove his duality theorems only for complexes of sheaves $\mathcal{A}^\bullet$ whose cohomology sheaves are constructible. Since then, it has become common to focus on the *constructible* derived category, in which each object is a complex of sheaves with constructible cohomology.

The reader is assumed to be familiar with the notions of categories and functors [100, Chapitre 1]. Text books which provide useful notions are, for instance, Kashiwara-Shapira [112]) or Gelfand-Manin [85]. Classsical references for this section are [103, 183], [184, Chap. 8]. The useful tools are well presented in [94, Sects. 1.8–1.15] and the reader will find there all necessary tools and material.

For the convenience of the reader, as far as possible, conventions and notations of main references [92, 94] and [18] are used. However in the case of possible doubt the notations of [94] are privileged (see Remark 5.4.10).

The Derived Category

Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves on X, let $Hom(\mathcal{A}, \mathcal{B})$ denote the abelian group of all sheaf maps $\mathcal{A} \to \mathcal{B}$. Let $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ be the sheaf whose sections over an open set U are the sections $\Gamma(U; \mathcal{H}om(\mathcal{A}, \mathcal{B})) = Hom(\mathcal{A}|_U, \mathcal{B}|_U)$. If $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ are complexes of sheaves, $\mathcal{H}om(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ is the single complex of sheaves which is obtained from the double complex $\mathcal{H}om^{p,q}(\mathcal{A}^\bullet, \mathcal{B}^\bullet) = \mathcal{H}om(\mathcal{A}^p, \mathcal{B}^q)$.

The *derived category* $D^b(X)$ was introduced by J.L. Verdier by localization of $K(X)$. The objects in $D^b(X)$ are still topologically constructible bounded complexes of sheaves on X but morphisms $\mathcal{A}^\bullet \to \mathcal{B}^\bullet$ are defined as equivalence class of diagrams of chain maps:

$$\mathcal{A}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}^\bullet \longrightarrow \mathcal{B}^\bullet$$

where "q.i." means a quasi-isomorphism. Two such diagrams

$$\mathcal{A}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}_1^\bullet \longrightarrow \mathcal{B}^\bullet, \qquad \mathcal{A}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}_2^\bullet \longrightarrow \mathcal{B}^\bullet$$

are equivalent is there is a commutative diagram in $K(X)$ (meaning a diagram that commutes up to homotopy).

$$\begin{array}{ccccc} & & \mathcal{E}_1^\bullet & & \\ & \overset{\text{q.i.}}{\swarrow} & \uparrow & \nwarrow & \\ \mathcal{A}^\bullet & \xleftarrow{\text{q.i.}} & \mathcal{E}_3^\bullet & \longrightarrow & \mathcal{B}^\bullet. \\ & \underset{\text{q.i.}}{\nwarrow} & \downarrow & \nearrow & \\ & & \mathcal{E}_2^\bullet & & \end{array}$$

Property 5.3.13 The derived category converts quasi-isomorphisms to isomorphisms: If $\varphi^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism (that is, a morphism of bounded complexes of sheaves whose induced map on cohomology is an isomorphism) then it has an inverse in the derived category $D^b(X)$.

This because $\varphi^\bullet$ may be composed with an injective resolution $\psi^\bullet : \mathcal{B}^\bullet \to \mathcal{I}^\bullet$. Then Proposition 5.3.8 (1) implies that $\psi^\bullet \circ \varphi^\bullet$ has a homotopy inverse $\theta^\bullet : \mathcal{I}^\bullet \to \mathcal{A}^\bullet$ which is therefore also a quasi-isomorphism, so $\theta^\bullet \circ \psi^\bullet$ is an inverse to $\varphi^\bullet$ in the derived category.

5.3.7 Derived Functors

An exact functor $F : Sh(X) \to Sh(Y)$ gives rise to a functor $D^b(X) \to D^b(Y)$ on derived categories. In this case, the homotopy category functor $F : K(X) \to K(Y)$ transforms quasi-isomorphisms into quasi-isomorphisms. However important functors such as $\mathcal{H}om(\mathcal{A}, \bullet)$, $\mathcal{A} \otimes \bullet$, $\Gamma(X, \bullet)$, direct image f_*, are not exact. The way to extend such functors in D^b is the Verdier's notion of derived functor. That will be very useful to express properties of intersection homology, in particular using formulae (5.12) and (5.13).

A covariant additive functor T from complexes of sheaves to an abelian category gives rise to its *right derived functor* RT defined on $D^b(X)$ by defining

$$RT(\mathcal{A}^\bullet) = T(\mathcal{I}^\bullet).$$

where $\mathcal{I}^\bullet$ is the canonical injective resolution of $\mathcal{A}^\bullet$ (see references in (5.3.5) in particular [49, Chap. V] and [94, Sect. 1.5]).

Classical Derived Functors

(a) The functor $\mathcal{H}om(\mathcal{A}, \bullet) \to K(X)$ has a (right) derived functor $R\mathcal{H}om^\bullet$ [49, Ch. VI]. Let $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ be bounded complexes of sheaves on X. To define $R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, consider $\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ as a functor of $\mathcal{B}^\bullet$, and take its right derived functor. The functor

$$\mathcal{A}^\bullet \to R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$$

is a functor from $D^b(X)$ into itself [18, V.5.17].

(b) The (left) derived tensor product functor $\mathcal{A}^\bullet \overset{L}{\otimes} \bullet : D^b(X) \to D^b(X)$ is defined in a similar way to the right derived functors by the formula

$$\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet = \mathcal{A}^\bullet \otimes \mathcal{J}^\bullet$$

where $\mathcal{J}^\bullet \to \mathcal{B}^\bullet$ is a resolution of $\mathcal{B}^\bullet$ whose stalks are flat R-modules (see [49, Ch. VI], [168]). If R is a field then $\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet = \mathcal{A}^\bullet \otimes \mathcal{B}^\bullet$.

If $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet \in D^b(X)$ are constructible with respect to a given stratification of X then so are $R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ and $\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet$ ([94, 1.9]).

(c) By definition, the i-th hypercohomology group $\mathbb{H}^i(X; \mathcal{A}^\bullet)$ of $\mathcal{A}^\bullet \in D^b(X)$ is the i-th derived functor of the global section functor $\Gamma(X, \bullet)$.

Functors associated to a map

Consider now a continuous map $f : X \to Y$ between locally compact topological spaces. Complete definitions and properties of the following functors are presented in the Grivel chapter in [18, Chapter VI].

(d) The functor direct image $f_* : Sh(X) \to Sh(Y)$.

If $\mathcal{A}$ is a sheaf on X, the presheaf defined by

$$\Gamma(V, f_*\mathcal{A}) = \Gamma(f^{-1}(V); \mathcal{A}) \qquad \text{for all } V \text{ open in } Y$$

is a sheaf on Y denoted by $f_*\mathcal{A}$. If $j : X \hookrightarrow Y$ is a closed immersion and $\mathcal{A}$ a sheaf on X, then $j_*\mathcal{A} = \mathcal{A}^Y$ is the extension of the sheaf $\mathcal{A}$ by zero. One defines the derived functors $Rf_* : D^b(X) \to D^b(Y)$ as in the definition.

(e) The functor $f_! : Sh(X) \to Sh(Y)$ direct image with proper supports.

If $V \subset Y$ is open, the family of subsets $C \subset X$ which are closed in $f^{-1}(V)$ and such that the map $f|_C : C \to V$ is proper is a family of supports in $f^{-1}(V)$ denoted by Φ_V. If $\mathcal{A}$ is a sheaf on X, the presheaf defined by

$$\Gamma(V; f_!\mathcal{A}) = \Gamma_{\Phi_V}(f^{-1}(V); \mathcal{A}) \qquad \text{for all } V \text{ open in } Y$$

is a sheaf on Y [18, VI, 2.2].

If Y is a point, then $f_!\mathcal{A} = \Gamma_c(X; \mathcal{A})$, where c denotes the family of compact subsets in X. If $j : X \hookrightarrow Y$ is an open (or closed) immersion, then $j_!\mathcal{A} = \mathcal{A}^Y$ and the functor $j_!$ is exact, one has $Rj_!\mathcal{A}^\bullet = j_!\mathcal{A}^\bullet$. Finally, the functor $f_!$ is exact in the subcategory of injective sheaves on X.

The right derived functor of $f_!$ denoted $Rf_! : D^b(X) \to D^b(Y)$ has stalks

$$\mathcal{H}^\bullet(Rf_!\mathcal{A}^\bullet)_y \cong \mathbb{H}^\bullet_c(f^{-1}(y); \mathcal{A}^\bullet) \qquad \forall y \in Y.$$

If $f : X \to Y$ is stratified with respect of stratifications of X and Y ([179, Sect. 4.4]), then $Rf_*\mathcal{A}^\bullet$ and $Rf_!\mathcal{A}^\bullet$ are constructible with respect to the stratification of Y. This is a consequence of the topological locally trivial nature of a stratification, see (5.3).

(f) The functor pull-back $f^* : Sh(Y) \to Sh(X)$.

The functor $f_* : Sh(X) \to Sh(Y)$ admits a left adjoint, denoted by $f^* : Sh(Y) \to Sh(X)$. There is an isomorphism

$$\mathcal{H}om_{Sh(X)}(f^*\mathcal{B}, \mathcal{A}) \cong \mathcal{H}om_{Sh(Y)}(\mathcal{B}, f_*\mathcal{A}). \quad (5.10)$$

for $\mathcal{A} \in Sh(X)$ and $\mathcal{B} \in Sh(Y)$. The functor f^* is exact and $Rf^*\mathcal{B} = f^*\mathcal{B}$ for all $\mathcal{B} \in Sh(Y)$.

For every point $x \in X$ and $\mathcal{B} \in Sh(Y)$, there is an isomorphism at the level of stalks

$$(f^*\mathcal{B})_x = \mathcal{B}_{f(x)} \qquad \forall x \in X.$$

For an inclusion $j : X \hookrightarrow Y$, then $j^*\mathcal{B} = \mathcal{B}|_X$ is the restriction of the sheaf $\mathcal{B}$ to X.

Denote by $p : X \to \{\text{pt}\}$ the map to a point. The constant sheaf $\mathbf{R}_X$ is equal to

$$\mathbf{R}_X = p^*\mathbf{R}_{\text{pt}}. \quad (5.11)$$

(g) Unlike the adjonction (5.10) between the functors f_* and f^*, in general there is no functor $f^! : Sh(Y) \to Sh(X)$ with a sheaf isomorphism $\mathcal{H}om(f_!\mathcal{A}, \mathcal{B}) \cong f_*\mathcal{H}om(\mathcal{A}, f^!\mathcal{B})$.

The functor $f^! : D^b(Y) \longrightarrow D^b(X)$ is defined at the level of derived categories (see [94, 1.12] and [18, V, 5.12]). If $\mathcal{I}^\bullet$ is a complex of injective sheaves on Y, then $f^!(\mathcal{I}^\bullet)$ is defined to be the sheaf associated to the presheaf whose sections over an open set $U \subset X$ are $\Gamma(U; f^!\mathcal{I}^\bullet) = Hom^\bullet(f_!\mathcal{K}^\bullet_U, \mathcal{I}^\bullet)$ where $\mathcal{K}^\bullet_U$ is the canonical injective resolution of the constant sheaf $\mathbf{R}_U$.

Example 5.3.14 • For an open immersion $j : X \hookrightarrow Y$, one has $j^! = j^*$.
• For a closed immersion $j : X \hookrightarrow Y$, one has

$$j^!(\mathcal{G}^\bullet)(U) = \Gamma_{(X)}(V; \mathcal{G}^\bullet)$$

where V is an open subset in Y such that $U = V \cap X$ Here (X) denotes the family of supports whose elements are closed subsets contained in X.

The (local) Verdier duality theorem ([94, 1.12]) is a canonical isomorphism in $D^b(Y)$,

$$Rf_*R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, f^!\mathcal{B}^\bullet) \cong R\mathcal{H}om^\bullet(Rf_!\mathcal{A}^\bullet, \mathcal{B}^\bullet)$$

for any $\mathcal{A}^\bullet \in D^b(X)$ and $\mathcal{B}^\bullet \in D^b(Y)$.

5.3.8 Dualizing Complex ([94, 1.12], [18, V, 7.1])

Borel and Moore first defined the dual $\mathcal{D}(\mathcal{A}^\bullet)$ of a complex of sheaves $\mathcal{A}^\bullet$ [17], and they showed that for any open set $U \subset X$ the hypercohomology groups $\mathbb{H}^i_c(U; \mathcal{A}^\bullet)$ and $\mathbb{H}^i(U; \mathcal{D}(\mathcal{A}^\bullet))$ are dual. Here, $\mathbb{H}^i_c$ denotes the hypercohomology with compact supports, i.e., $R^i\Gamma_c$. This property characterizes $\mathcal{D}(\mathcal{A}^\bullet)$ up to quasi-isomorphism. It implies, for example, that if X is compact and R is a field, then

$$\mathbb{H}^i(X;\mathcal{A}^\bullet) = Hom(\mathbb{H}^{-i}(X;\mathcal{D}(\mathcal{A}^\bullet)), R).$$

On a PL-manifold, Borel and Moore considered the dual of the constant sheaf, and showed that this is the Borel-Moore sheaf of chains $\mathcal{C}^\bullet$ (cf. Definition 5.3.3).

Later, Verdier [183, 184] defined a complex of sheaves $\mathbf{D}^\bullet_X$, the *dualizing complex* such that

$$\mathcal{D}(\mathcal{A}^\bullet) \cong R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, \mathbf{D}^\bullet_X)$$

for any bounded complex $\mathcal{A}^\bullet$. Verdier identified $\mathbf{D}^\bullet_X = \mathcal{D}(\mathbf{R}_X)$ and showed that, in D^b, the sheaf $\mathcal{C}^\bullet$ is isomorphic to the dualizing sheaf. Therefore, the Borel-Moore dual of $\mathcal{A}^\bullet$ may be identified with $\mathcal{H}om(\mathcal{A}^\bullet, \mathcal{C}^\bullet)$. While defining the dualizing sheaf, Verdier provided the good language to express the duality and showed an isomorphism in the derived category between $\mathcal{A}^\bullet$ and the double dual of $\mathcal{A}^\bullet$, i.e. if $\mathcal{A}^\bullet$ is a bounded topologically constructible complex of sheaves on X, then there is a natural isomorphism in $D^b(X)$

$$\mathcal{A}^\bullet \cong \mathcal{D}(\mathcal{D}(\mathcal{A}^\bullet)).$$

If $\mathcal{B}^\bullet \cong \mathcal{D}(\mathcal{A}^\bullet)$, then the corresponding pairing

$$\mathcal{B}^\bullet \overset{L}{\otimes} \mathcal{A}^\bullet \to \mathbf{D}^\bullet_X$$

is called a Verdier dual pairing.

The associated cohomology sheaves of $\mathbf{D}^\bullet_X$ are nonzero in negative degree only, with stalks $\mathcal{H}^{-i}(\mathbf{D}^\bullet_X)_x = H_i(XX - \{x\}; R)$. If X is an n-dimensional PL-manifold, the shifted complex $\mathbf{D}^\bullet_X[-n]$ is naturally isomorphic to (in fact, an injective resolution of) the orientation sheaf of X (see Example 5.3.1 and [94, Sect. 1.12], also [18, V, Sect. 7.3] but taking care of notations cf. Remark 5.4.10).

The hypercohomology groups $\mathbb{H}^{-i}(X;\mathbf{D}^\bullet_X)$ equal the ordinary homology groups with closed support $H_i(X;R)$ and $\mathbb{H}^{-i}_\Phi(X;\mathbf{D}^\bullet_X) = H_i^\Phi(X;R)$ for any family of supports Φ on X [18, V,7.1-2-3].

Consider the projection $p : X \to \{\mathrm{pt}\}$ and the sheaf $\mathbf{R}_{\mathrm{pt}}$. The dualizing sheaf satisfies [18, V,7.18]:

$$\mathbf{D}^\bullet_X \cong p^!\mathbf{R}_{\mathrm{pt}}.$$

Hence for every map $f : X \to Y$ one has a canonical isomorphism $\mathbf{D}^\bullet_X \cong f^!\mathbf{D}^\bullet_Y$ (compare with (5.11)).

Let $f : X \to Y$ be a continuous map between topological manifolds, $\mathcal{A}^\bullet \in D^b(X)$ and $\mathcal{B}^\bullet \in D^b(Y)$. The functors satisfy the following duality formulae

$$f^!\mathcal{B}^\bullet \cong \mathcal{D}_X(f^*\mathcal{D}_Y(\mathcal{B}^\bullet))$$
$$Rf_!\mathcal{A}^\bullet \cong \mathcal{D}_Y(Rf_*\mathcal{D}_X(\mathcal{A}^\bullet)).$$

If $\mathcal{A}^\bullet$ is a topologically constructible complex of sheaves on X, $f_x \colon \{x\} \to X$ is the inclusion of a point, and U_x is a distinguished neighborhood of x (see 5.3), then [94, p.91] (see also [18, V, Sect. 4.5])

$$H^j(f_x^* \mathcal{A}^\bullet) \cong \mathbb{IH}^j(U_x; \mathcal{A}^\bullet) \tag{5.12}$$

$$H^j(f_x^! \mathcal{A}^\bullet) \cong \mathbb{IH}_c^j(U_x; \mathcal{A}^\bullet). \tag{5.13}$$

These groups are respectively called the stalk homology and the costalk homology of $\mathcal{A}^\bullet$ at x.

The following geometric interpretations are taken from [94, Sect. 4, p. 106] and will be useful to interpreting the Theorem 5.4.9.

If a class $\xi \in \mathbb{IH}^j(X; \mathcal{A}^\bullet)$ does not vanish under the homomorphism

$$\mathbb{IH}^j(X; \mathcal{A}^\bullet) \to H^j(f_x^* \mathcal{A}^\bullet)$$

then any cycle representative of ξ must contain the point x. Thus, $H^j(f_x^* \mathcal{A}^\bullet)$ represents local classes which "cannot be pulled away from the point x". The set

$\{x \in X | H^j(f_x^* \mathcal{A}^\bullet) \neq 0\}$ is called the *local j-support of the complex* $\mathcal{A}^\bullet$.

Similarly, a class $\eta \in \mathbb{IH}^j(X; \mathcal{A}^\bullet)$ is in the image of the homomorphism

$$H^j(f_x^! \mathcal{A}^\bullet) \to \mathbb{IH}^j(X; \mathcal{A}^\bullet).$$

if some cycle representative of η is completely contained in a neighborhood of x. Thus $H^i(f_x^! \mathcal{A}^\bullet)$ represents local classes which are "supported near x". The set

$\{x \in X | H^j(f_x^! \mathcal{A}^\bullet) \neq 0\}$ is called the *local j-cosupport of the complex* $\mathcal{A}^\bullet$.

5.4 Intersection Homology—Geometric and Sheaf Definitions

In order to recover duality properties for singular varieties, the idea of intersection homology, due to Mark Goresky and Robert MacPherson, is to restrict the consideration to cycles which meet the singular part of the variety with a "controlled" dimension. That makes sense if the variety is endowed with a suitable stratification. The considered singular varieties are pseudomanifolds.

As observed by Goresky and MacPherson [92], in a PL-pseudomanifold of dimension n, if two cycles of respective dimensions i and j are in general position, then their intersection can be given canonically the structure of an $i + j - n$

chain. However, their intersection is (in general) no longer a cycle and Theorems 5.2.4 and 5.2.5 do not hold. That is the motivation for the following definitions.

The first definition has been given by Goresky and MacPherson in the framework of stratified compact oriented PL-pseudomanifolds (see [91, 92]) and also [18, Chaps. I–IV]). The compactness is not required here and the considered chains are the PL-geometric chains (Sect. 5.2.4). The second definition, using sheaves and, in particular the Deligne sheaf complex, has been given by the same authors in [94] (see also [18, Chaps. V–IX]).

5.4.1 The Definition for PL-Stratified Pseudomanifolds ([91], 53)

Let X be a PL-stratified pseudomanifold. If a chain ξ meets transversaly an element $X_{n-\alpha}$ of the PL-filtration, then one has

$$\dim(|\xi| \cap X_{n-\alpha}) = i - \alpha.$$

The allowed chains and cycles will be those which meet each element $X_{n-\alpha}$ of the singular part with a controlled and fixed transversality defect p_α. This defect is called the *perversity* (in French: *Perversité*, in German: *Toleranz*).

A *perversity*, also called GM-perversity for Goresky-MacPherson perversity, is an integer value function

$$\bar{p} : [0, \dim X] \cap \mathbb{Z} \to \mathbb{N}, \qquad p_\alpha := \bar{p}(\alpha)$$

such that $p_0 = p_1 = p_2 = 0$ and

$$p_\alpha \leq p_{\alpha+1} \leq p_\alpha + 1 \quad \text{for} \quad \alpha \geq 2. \tag{5.14}$$

This condition is the one given originally by Goresky and MacPherson in order to ensure the main properties of the theory. More general perversities have been considered by various authors (see [128] and Sect. 5.6.4) providing other aspects for the theory (Fig. 5.3).

Example 5.4.1 Examples of perversities are

- the zero perversity $\bar{0} = (0, 0, \ldots, 0)$,
- the maximal (or top) perversity $\bar{t} = (0, 0, 0, 1, 2, \ldots, n-2)$,
- for n even, $n \geq 4$, the upper middle $\bar{n} = (0, 0, 0, 1, 1, 2, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} - 1)$ and the lower middle perversities $\bar{m} = (0, 0, 0, 0, 1, 1, \ldots, \frac{n}{2} - 2, \frac{n}{2} - 1)$.

Let $\bar{p} = (p_0, p_1, p_2, \ldots, p_n)$ be a perversity, the complementary perversity $\bar{q} = (q_0, q_1, q_2, \ldots, q_n)$ is defined by $p_\alpha + q_\alpha = t_\alpha$ for all $\alpha \geq 2$.

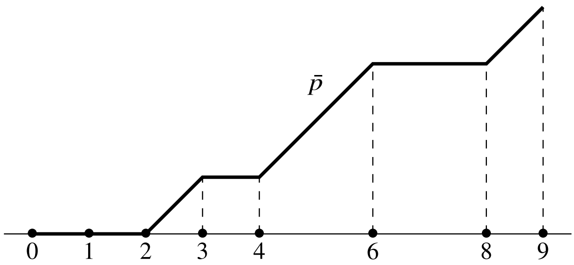


Fig. 5.3 The perversity $\bar{p} = (0, 0, 0, 1, 1, 2, 3, 3, 3, 4)$

Given a stratification (5.2) of a n-dimensional pseudomanifold X, Goresky and MacPherson [92, Sect. 1.3] call $(\bar{p}, i)$-*allowable* an i-chain $\xi \in C_i(X; G)$ such that

$$\dim(|\xi| \cap X_{n-\alpha}) \leq i - \alpha + p_\alpha \qquad \forall \alpha \geq 0$$

The condition means that the perversity is the maximum admissible defect of transversality. The boundary of a $\bar{p}$-allowable chain is not necessarily $\bar{p}$-allowable (easy examples). In order to define a complex of chains, one has to set:

Definition 5.4.2 The intersection chains $IC_i^{\bar{p}}(X; G)$ is the subset of $C_i(X; G)$ consisting of chains ξ such that ξ and $\partial\xi$ are $\bar{p}$-allowable, that is

$$IC_i^{\bar{p}}(X; G) = \left\{ \xi \in C_i(X; G) \middle| \begin{array}{c} \dim(|\xi| \cap X_{n-\alpha}) \leq i - \alpha + p_\alpha \\ \dim(|\partial\xi| \cap X_{n-\alpha}) \leq (i-1) - \alpha + p_\alpha \end{array} \quad \forall \alpha \geq 2 \right\}$$

Using the usual boundary of chains, the obtained chain complex is denoted by $(IC_*^{\bar{p}}(X; G), \partial_*)$.

Definition 5.4.3 The *intersection homology* groups $IH_*^{\bar{p}}(X; G)$ are the homology groups of the complex $(IC_*^{\bar{p}}(X; G), \partial_*)$.

Using, in the definition, the subcomplex $C_*^c(X; G)$ of chains with compact supports (see Sect. 5.2.4) provides the intersection homology groups with compact supports, denoted by $IH_*^{\bar{p},c}(X; G)$. Notice that on the one hand, the intersection homology defined in [92] agrees with the intersection homology with compact supports as defined in [94]. On the other hand, the intersection homology defined in [94] agrees with the Borel-Moore intersection homology (with closed supports) of [92].

5.4.2 Definition with Local Systems

To make the construction of homology with coefficients in a local system, work in intersection homology, one only needs the local system $\mathcal{L}$ to be defined on the dense open part $X - \Sigma$ of X.

Let $\mathcal{L}$ be a local coefficient system of R-modules on $X - \Sigma$. Given an open subset $U \subset X$ and a locally finite triangulation K of U, since $\mathcal{L}$ may not be defined on all of U, it is impossible to define a group $C_i^K(U, \mathcal{L})$ of i-chains ξ with coefficients in $\mathcal{L}$. Nevertheless, [94, 2.2] or [89, 9.4] observes that, for any perversity $\bar{p}$, and for any $(\bar{p}, i)$-allowable chain ξ, if σ is any i-simplex with nonzero coefficient in ξ, both the interior of σ and the interiors of all the $i - 1$ dimensional faces of σ lie entirely in $X - \Sigma$ by the allowability conditions. That justifies the definition:

$$IC_i^{\bar{p},K}(U;\mathcal{L}) = \left\{ \xi \in C_i^K(U;\mathcal{L}) \middle| \begin{array}{c} \dim(|\xi| \cap U \cap X_{n-\alpha}) \leq i - \alpha + p_\alpha \\ \\ \dim(|\partial\xi| \cap U \cap X_{n-\alpha}) \leq (i-1) - \alpha + p_\alpha \end{array} \forall \alpha \geq 0 \right\}$$

The map $IC_i^{\bar{p},K}(U;\mathcal{L}) \to IC_{i-1}^{\bar{p},K}(U;\mathcal{L})$ is well defined and the intersection homology groups $IH_*^{\bar{p}}(X;\mathcal{L})$ are defined as in Sect. 5.2.4 (with $U = X$).

Many examples of computation of intersection homology groups can be found for instance in the Goresky-MacPherson's chapter [18, Chap. III] and in [27, 63, 71, 128]. See also Example 5.5.10. Here are two elementary examples, where the coefficient sheaf $\mathcal{L}$ is the constant sheaf $\mathbb{Z}_X$.

Example 5.4.4 (The pinched torus) (see Fig. 5.1) The singular set is a point: the pinched point $\{0\}$. The considered stratification is given by the filtration

$$X \supset \Sigma = \{0\} \supset \emptyset$$

The only possible perversity is the perversity $\bar{0}$. The 1-dimensional intersection homology of the pinched torus is zero, while its 1-dimensional homology does not vanish.

$$IH_0^{\bar{0}}(X) = \mathbb{Z}[pt] \qquad IH_1^{\bar{0}}(X) = 0 \qquad IH_2^{\bar{0}}(X) = \mathbb{Z}[X].$$

(compare with Example 5.2.3).

Example 5.4.5 (The double cone) Though it is similar to the previous example and it is not a connected pseudomanifold, the example of the double cone is instructive. One may compare with the example of the suspension of two circles [128].

The double cone X is obtained by pinching the cylinder $\mathbb{S}^1 \times \mathbb{R}$ at level $\{0\}$ into a point $\{a\}$. The line ℓ (see Fig. 5.4) goes through the singular point $\{a\}$ and C_1 and C_2 are the two 2-dimensional components of the double cone. Poincaré duality fails for the double cone X. The only possible perversity is the perversity $\bar{0}$. Two points x_1 and x_2 contained in different connected components of $X \backslash \{a\}$ are not homologous, in intersection homology, as any 1-chain linking these two points contain the vertex $\{a\}$ and is not permitted. Poincaré duality is recovered with intersection homology (see 5.23) (Table 5.1).

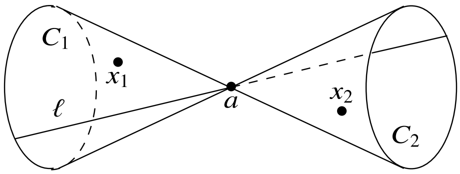


Fig. 5.4 The double cone

Table 5.1 Homology and intersection homology of the (double) cone

i	$H_i(X)$	$H_i^c(X)$	$IH_i(X)$	$IH_i^c(X)$
0	0	$\mathbb{Z}\{pt\}$	0	$\mathbb{Z}\{x_1\} \oplus \mathbb{Z}\{x_2\}$
1	$\mathbb{Z}[\ell]$	0	0	0
2	$\mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2]$	0	$\mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2]$	0

5.4.3 Witt Spaces

For many applications, the class of spaces with even dimension strata is too restrictive. The largest class of Witt spaces still enjoys Poincaré duality of the middle intersection homology, but allows for some strata of odd dimension (see for instance the following references (5.26), (5.5.3), (5.5.3), (5.5.46)).

Definition 5.4.6 A stratified pseudomanifold X is a *R-Witt space* ([94, Sect. 5.6.1] [169]) if, for each stratum of odd codimension $\alpha = 2k + 1$, then $IH_k^{\bar{m}}(L_\alpha; R) = 0$, where L_α is the link of the stratum (5.3) For such a space, the intersection homology groups of the two middle perversities coincide (see Example 5.4.1):

$$IH_*^{\bar{m}}(X; R) \cong IH_*^{\bar{n}}(X; R).$$

5.4.4 The Intersection Homology Sheaf Complex

The intersection homology sheaf complex is defined in the context of PL-pseudomanifolds. In the following section, the Deligne complex will be defined in the more general context of topological pseudomanifolds. When both defined, the intersection homology sheaf complex and the Deligne complex are quasi-isomorphic and their hypercohomology computes intersection homology. That is made precise in the following sections.

Definition 5.4.7 Let $\mathcal{IC}_{\bar{p}}^{-i}$ be the subsheaf of $\mathcal{C}^{-i}$ (see Example 5.3.2) whose sections over an open subset $U \subset X$ consist of all locally finite PL-chains $\xi \in \Gamma(U; \mathcal{C}^{-i})$ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable with respect to the filtration of U

$$U \supset U \cap X_{n-2} \supset \cdots \supset U \cap X_1 \supset U \cap X_0.$$

That is

$$\Gamma(U;\mathcal{IC}_{\bar{p}}^{-i}) = \left\{ \xi \in \Gamma(U;\mathcal{C}^{-i}) \middle| \begin{array}{c} \dim(|\xi| \cap U \cap X_{n-\alpha}) \leq i - \alpha + p_\alpha \\ \dim(|\partial\xi| \cap U \cap X_{n-\alpha}) \leq (i-1) - \alpha + p_\alpha \end{array} \quad \forall \alpha \geq 0 \right\}$$

The sheaf $\mathcal{IC}_{\bar{p}}^{-i}$ is well defined and $\Gamma(U;\mathcal{IC}_{\bar{p}}^{-i}) = IC_i^{\bar{p}}(U)$. If $V \subset U$ are two open subsets in X, then for every perversity $\bar{p}$, there are natural restriction maps $\rho_{VU} : IC_i^{\bar{p}}(U) \to IC_i^{\bar{p}}(V)$ as in (5.6) (see [94, Sect. 2.1]).

Using the restriction of the usual boundary, one obtains a complex of sheaves $\mathcal{IC}_{\bar{p}}^{\bullet}$ on X. This complex is soft [18, II, Sect. 5] so that the complex satisfies Proposition 5.3.11. The hypercohomology groups $\mathbb{H}^{-i}(X;\mathcal{IC}_{\bar{p}}^{\bullet})$ are canonically isomorphic to the intersection homology groups $IH_i^{\bar{p}}(X;R)$ defined in [92, Sect. 1.3] for $R = \mathbb{Z}$. Also, one has

$$\mathbb{H}_c^{-i}(X;\mathcal{IC}_{\bar{p}}^{\bullet}) = IH_i^{\bar{p},c}(X;R),$$

intersection homology with compact supports, and more generally

$$\mathbb{H}_\Phi^{-i}(X;\mathcal{IC}_{\bar{p}}^{\bullet}) = IH_i^{\bar{p},\Phi}(X;R) \tag{5.15}$$

for any family of supports Φ on X (see [18, II, 5]).

The associated cohomology sheaves $\mathcal{H}^{-i}(\mathcal{IC}_{\bar{p}}^{\bullet})$ are called the local intersection homology sheaf. The stalk at $x \in X$ of this sheaf is $IH_i^{\bar{p}}(X, X - \{x\}; R)$.

Definition with Local Systems

Considering local systems provide many useful examples as well as powerful applications.

Let $\mathcal{L}$ be a local coefficient system of R-modules on $X - \Sigma$. Given an open subset $U \subset X$ and a locally finite triangulation K of U, the group $IC_i^{\bar{p},K}(U;\mathcal{L})$ is well defined (see Sect. 5.4.2) as well as maps $IC_i^{\bar{p},K}(U;\mathcal{L}) \to IC_{i-1}^{\bar{p},K}(U;\mathcal{L})$.

Definition 5.4.8 ([94, Sect. 2.2], [89, 9.4]) Let X be a PL-stratified PL-pseudomanifold and $\bar{p}$ a perversity, the sheaf complex $\mathcal{IC}_{\bar{p}}^{\bullet}(\mathcal{L})$ of intersection chains with local coefficients in $\mathcal{L}$ is defined by

$$\Gamma(U,\mathcal{IC}_{\bar{p}}^{-i}(\mathcal{L})) = \lim_{K} IC_i^{\bar{p},K}(U;\mathcal{L})$$

where the limit is taken over locally finite compatible triangulations of U. The intersection homology groups of X with coefficients in $\mathcal{L}$, denoted $IH_i^{\bar{p}}(X;\mathcal{L})$, are the hypercohomology groups $\mathbb{H}^{-i}(X;\mathcal{IC}_{\bar{p}}^{\bullet}(\mathcal{L}))$.

Combining formula (5.15) with (5.12) and (5.13), one obtains the following theorem:

Theorem 5.4.9 *Let $f_x \colon \{x\} \to X$ be the inclusion of a point x in X, U_x a distinguished neighborhood of x and let $\mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})$ be the sheaf complex of intersection chains, one has:*

$$H^j(f_x^* \mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})) \cong \mathbb{H}^j(U_x; \mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})) = IH_{-j}^{\bar{p}}(U_x; \mathcal{L}) \tag{5.16}$$

$$H^j(f_x^! \mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})) \cong \mathbb{H}_c^j(U_x; \mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})) = IH_{-j}^{\bar{p},c}(U_x; \mathcal{L}). \tag{5.17}$$

Remark 5.4.10 An important remark is that the index for the dimension of the homology and intersection homology groups differs according to the authors. That can be considered as unfortunate but shows the diversity of the theory and diversity of applications.

In [94, Sect. 2.3] Goresky and MacPherson explicit four different indices in the literature and the reader has to take care of the convention used in the concerning article.

(a) Homology subscripts, as in [92] or [63]: a subscript k indicates chains of dimension k.
(b) Homology superscripts, as in [94] and this survey: a superscript $-k$ indicates chains of dimension k.
(c) Cohomology superscripts, as in [18, 19, 84]: a superscript j indicates chains of dimension $n - j$.
(d) The Beilinson-Bernstein-Deligne-Gabber scheme [10]: a superscript j indicates chains of codimension $\frac{n}{2} + j$.

For an n–dimensional compact oriented pseudomanifold these schemes compare as follows: $H_k(X)$ in scheme (a) is isomorphic to $H^{-k}(X)$ in scheme (b), $H^{n-k}(X)$ in scheme (c), and $H^{\frac{n}{2}-k}(X)$ in scheme (d).

5.4.5 *The Deligne Construction*

In a conversation at the IHES, in the fall of 1976, R. MacPherson explained about intersection homology to P. Deligne. P. Deligne had been thinking about variation of Hodge structures on a smooth algebraic curve where truncation arises naturally. When R. MacPherson explained the intersection homology of a cone, it looked like this truncation so P. Deligne conjectured that perhaps intersection homology might be explained by repeated truncation. His conjecture was proven by Goresky and MacPherson [94], who pointed out that this construction could be used to prove the topological invariance of intersection homology, and to give a definition that works in characteristic p. In the meantime, on 20 April 1979, P. Deligne had written to D. Kazhdan and G. Lusztig about the theory, describing his interpretation using truncation, and conjecturing that intersection homology might be pure [94, Sect. 3] or [18, V, Sect. 2.2].

The Deligne idea is to start with the constant sheaf (or a local system of coefficients) on the non-singular part and extending stratum by stratum by alternate operations of "pushing" and "truncating". While requiring technical tools, the idea of Deligne construction is relatively simple. One starts with (all) chains on the regular stratum, then pushing the complex on the "next" stratum, and then "cutting" (truncating) according to the perversity in order to retain only allowed chains. One continues the process by induction on decreasing dimension of the strata.

Therefore, the Deligne construction uses two tools: the "pushing" attaching property and the "truncating" operation.

The Attaching Map

Let Y be a closed subspace of X and i the inclusion of $U = X - Y$ into X. For a sheaf $\mathcal{A}^\bullet$, the composition of the natural morphisms

$$\mathcal{A}^\bullet \to i_* i^* \mathcal{A}^\bullet \to R i_* i^* \mathcal{A}^\bullet$$

is the *attaching map*.

Consider a stratification (5.2)

$$X = X_n \supset X_{n-1} = X_{n-2} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

and, denoting by $U_k = X - X_{n-k}$ the complementary open subsets, consider the filtration

$$U_1 = U_2 \subset U_3 \subset \cdots \subset U_{n+1} = X.$$

One has $U_2 = X - \Sigma$. Denote by $i_k : U_k \hookrightarrow U_{k+1}$ the inclusion. Then

$$\mathcal{A}^\bullet_k = \mathcal{A}^\bullet|_{U_k} = i_k^* \mathcal{A}^\bullet|_{U_{k+1}}.$$

The following result is one of the main ingredients of the sheaf axiomatic construction of intersection homology (see Sect. 5.4.7).

Theorem 5.4.11 ([94, Proposition 2.5], [18, II, Theorem 6.1]) *The natural homomorphism*

$$\mathcal{IC}^\bullet|_{U_{k+1}} \to R i_{k*} \mathcal{IC}^\bullet|_{U_k} = R i_{k*} i_k^* \mathcal{IC}^\bullet|_{U_{k+1}}$$

induces an isomorphism

$$\mathcal{H}^j(\mathcal{IC}_{n-\bullet})_x \to \mathcal{H}^j(R i_{k*} \mathcal{IC}_{n-\bullet})_x \qquad \textit{for } x \in U_{k+1} - U_k$$

for all $j \leq p(k) - n$.

The Deligne Truncation Functor

If $k \in \mathbb{Z}$, the *truncation* of a complex of sheaves $\mathcal{A}^\bullet$ on X is a new complex ([94, 1.14], [18, V,1.10]):

$$(\tau_{\leq k}\mathcal{A}^\bullet)^i = \begin{cases} \mathcal{A}^i & \text{if } i < k \\ \ker d^i & \text{if } i = k \\ 0 & \text{if } i > k. \end{cases}$$

The functor $\tau_{\leq k}$ determine a truncation functor on the derived category $D^b(X)$. See [94, 1.14] for more detailed properties.

The Deligne Sheaf [94, Sect. 3.1], [18, V, Sect. 2.2]

In this section X is a topological pseudomanifold and $\mathcal{L}$ denotes a system of local coefficients on the regular part $X - \Sigma$.

Let $\bar{p}$ a fixed perversity, the Deligne complex of sheaves (or Deligne sheaf) $\mathbb{P}^\bullet_k(\mathcal{L}) \in D^b(U_k)$ is defined inductively by

$$\mathbb{P}^\bullet_2(\mathcal{L}) = \mathcal{L}[n]$$
$$\mathbb{P}^\bullet_{k+1}(\mathcal{L}) = \tau_{\leq p(k)-n} Ri_{k*}\mathbb{P}^\bullet_k(\mathcal{L}) \qquad \text{for } k \geq 2.$$

The resulting complex $\mathbb{P}^\bullet(\mathcal{L}) = \mathbb{P}^\bullet_{n+1}(\mathcal{L})$ is called the Deligne intersection homology chain complex with coeffcients in $\mathcal{L}$.

Starting with a regular Noetherian ring R of finite Krull dimension and the constant sheaf $\mathbf{R}$ on $X - \Sigma$ instead of $\mathcal{L}$, i.e. starting with

$$\mathbb{P}^\bullet_2 = \mathbf{D}^\bullet_{U_2} \cong \mathbf{R}_{U_2}[n]$$

the complex $\mathbb{P}^\bullet = \mathbb{P}^\bullet_{n+1}$ is written

$$\mathbb{P}^\bullet = \tau_{\leq p(n)-n} Ri_{n*} \cdots \tau_{\leq p(3)-n} Ri_{3*} \tau_{\leq p(2)-n} Ri_{2*}\mathbf{R}_{X-\Sigma}[n].$$

5.4.6 Local Calculus and Consequences

The local calculus, and precisely the computation in formulae (5.18) and (5.19) below, are the starting points for the characterization of intersection homology.

Let $x \in X$ be a point in the stratum $S_{n-\alpha}$ with codimension α in X. Let U be a neighborhood of x homeomorphic to $\mathbb{B}^{n-\alpha} \times \mathring{c}(L_x)$, where $\dim L_x = \alpha - 1$ (see 5.3). The following result is proved in [18, II, Sect. 3–4] in the context of PL-

pseudomanifolds and in [94, Sect. 2.4], [93, 1.7] in the context of topological pseudomanifolds.

Proposition 5.4.12 *Let X be a locally compact stratified pseudomanifold and $\bar{p}$ any perversity. Let x be a point in a stratum with codimension α in X and let U be a neighborhood of x homeomorphic to $\mathbb{B}^{n-\alpha} \times \mathring{c}(L_x)$, then one has:*

$$IH_i^{\bar{p}}(U) \cong IH_{i-(n-\alpha)}^{\bar{p}}(\mathring{c}(L_x)) \cong \begin{cases} 0 & i < n - p_\alpha \\ IH_{i-(n-\alpha)-1}^{\bar{p}}(L_x) & i \geq n - p_\alpha . \end{cases} \tag{5.18}$$

$$IH_i^{\bar{p},c}(U) \cong IH_i^{\bar{p},c}(\mathring{c}(L_x)) \cong \begin{cases} IH_i^{\bar{p}}(L_x) & i < \alpha - p_\alpha - 1 \\ 0 & i \geq \alpha - p_\alpha - 1 \end{cases} \tag{5.19}$$

The link L_x is compact and its homology groups, with compact and closed supports coincide.

Here is an useful and important notation for the sequel:

Denoting by $\bar{p}$ a perversity and $\bar{q}$ the complementary perversity, one recalls that $p_k + q_k = k - 2$ for all $k \geq 2$. If $j \in \mathbb{N}$, one defines the inverse perversity function

$$p^{-1}(j) = \min\{ k \mid p_k \geq j\}$$

and $p^{-1}(j) = \infty$ if $j > p_n$ (Fig. 5.5).

Using the "inverse perversity function"; the properties (5.18) and (5.19) are written

$$\dim\{x \in X \mid IH_i^{\bar{p}}(U_x) \neq 0\} \leq n - p^{-1}(n-i) \qquad \text{for } i \leq n-1,$$
$$\dim\{x \in X \mid IH_i^{c,\bar{p}}(U_x) \neq 0\} \leq n - q^{-1}(i) \qquad \text{for } i \geq 1,$$

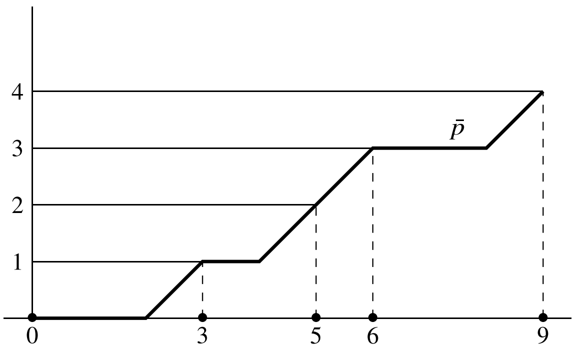


Fig. 5.5 An example of the function p^{-1} (for the perversity of Fig. 4) One has $p^{-1}(1) = 3$, $p^{-1}(2) = 5$, $p^{-1}(3) = 6$, $p^{-1}(4) = 9$, $p^{-1}(5) = +\infty$

and in terms of hypercohomology (see (5.15) and with $j = -i$)

$$\dim\{x \in X \mid \mathbb{IH}^j(U_x; \mathcal{IC}_X^{\bar{p},\bullet}) \neq 0\} \leq n - p^{-1}(j+n) \qquad \text{for } j \geq 1-n. \quad (5.20)$$

$$\dim\{x \in X \mid \mathbb{IH}_c^j(U_x; \mathcal{IC}_X^{\bar{p},\bullet}) \neq 0\} \leq n - q^{-1}(-j). \qquad \text{for } j \leq -1. \quad (5.21)$$

As observed by Goresky and MacPherson [93, 1.8, Theorem] the results of this section are valid for intersection homology with coefficients in a local system.

5.4.7 Characterizations of the Intersection Complex

In the introduction of their Sect. 3 [94], Goresky and MacPherson provide the motivation for the characterizations of the intersection complex, in particular topological invariance of intersection homology. The theorem [94, Theorem 3.5] shows that if X has a PL structure and is stratified by a PL stratification and if $\bar{p}$ denotes a fixed perversity then the Deligne complex $\mathbb{P}^\bullet(\mathcal{L})$ and the complex of PL intersection chains $\mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})$ are canonically isomorphic in $D^b(X)$ whenever they are both defined. That justifies the use of the notation $\mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})$ to denote this isomorphism class of objects, for any topological pseudomanifold.

In this section, X is a topological pseudomanifold. The first characterization of the intersection complex, as a system of axioms called $[AX_1]_{\bar{p}}$, is given in [94, Sect. 3.3] and [18, V, Sect. 4]. In [18, V, Sect. 4.20], Borel discusses some points concerning the "differences" between [18, 94], in particular the usefulness of the hypothesis "topologically constructible".

If $\mathcal{S}$ denotes a filtration (5.2) of the space X, let $U_k = X - X_{n-k}$ denote the complementary increasing filtration by open sets. There are inclusions

$$i_k : U_k \hookrightarrow U_{k+1} \quad \text{and} \quad j_k : S_{n-k} = (U_{k+1} - U_k) \hookrightarrow U_{k+1}.$$

Definition 5.4.13 ([94, Sect. 3.3]) Let $\bar{p}$ be a perversity and $\mathcal{L}$ is a local system defined on the regular part of X. A complex of sheaves $\mathcal{A}^\bullet$ on X satisfies axioms $[AX_1]_{\bar{p}}(\mathcal{L})$ if it satisfies:

(1a) $\mathcal{A}^\bullet$ is constructible with respect to the given stratification and $\mathcal{A}^\bullet|_{U_2}$ is quasi-isomorphic to $\mathcal{L}[n]$.
(1b) $\mathcal{H}^i(\mathcal{A}^\bullet) = 0$ for $i < -n$.
(1c) $\mathcal{H}^i(\mathcal{A}^\bullet|_{U_{k+1}}) = 0$ for $i > p(k) - n$.
(1d) The attaching maps (see Theorem 5.4.11) induce isomorphisms

$$\mathcal{H}^i(j_k^*\mathcal{A}^\bullet|_{U_{k+1}}) \to \mathcal{H}^i(j_k^* Ri_{k*}i_k^*\mathcal{A}^\bullet|_{U_{k+1}})$$

for all $k \geq 2$ and $i \leq p(k) - n$.

Theorem 5.4.14 ([94, Sect. 3.5] [18, V, Theorem 2.5]), *The sheaf* $\mathbb{P}^\bullet_{\bar{p}}(\mathcal{L})$ *satisfies properties* $[AX_1]_{\bar{p}}(\mathcal{L})$. *Any complex of sheaves* $\mathcal{A}^\bullet$ *satisfying* $[AX_1]_{\bar{p}}(\mathcal{L})$ *is quasi-isomorphic to* $\mathbb{P}^\bullet_{\bar{p}}(\mathcal{L})$.

Theorem 5.4.15 ([94, Sect. 3.5, Corollary]) *Let* $\bar{p}$ *be a perversity and* $\mathcal{A}^\bullet$ *be a constructible complex of fine (or soft) sheaves on* X *satisfying axioms* $[AX_1]_{\bar{p}}(R)$, *then the cohomology groups of the complex*

$$\ldots \to \Gamma(X; \mathcal{A}^{j-1}) \to \Gamma(X; \mathcal{A}^j) \to \Gamma(X; \mathcal{A}^{j+1}) \to \ldots$$

i.e., the hypercohomology groups $\mathbb{H}^j_c(X; \mathcal{A}^\bullet)$, *are naturally isomorphic to the intersection homology groups* $IH^{\bar{p}}_{n-j}(X; R)$.

In fact, Goresky and MacPherson prove the following main result, of which follow the main properties of intersection homology (Sect. 5.5.1).

Theorem 5.4.16 ([94, Sect. 3.5]) *The functor* $\mathbb{P}^\bullet_{\bar{p}}$ *which assigns to any locally trivial sheaf* $\mathbb{F}$ *on* $X^0 = X - \Sigma$, *the complex*

$$\mathbb{P}^\bullet_{\bar{p}}(\mathbb{F}) = \tau_{\leq p(n)-n} Ri_{n*} \cdots \tau_{\leq p(3)-n} Ri_{3*} \tau_{\leq p(2)-n} Ri_{2*} \mathbb{F}[n].$$

defines an equivalence of categories between

(a) the category of locally constant sheaves on $X^0 = X - \Sigma$ *and*
(b) the full subcategory of $D^b(X)$ *whose objects are all complexes of sheaves which satisfy the axioms* $[AX_1]_{\bar{p}}$.

Example 5.4.17 The orientation sheaf $\mathcal{O}$ on X^0 is quasi-isomorphic to the dualizing sheaf $\mathbf{D}^\bullet_{X^0}[-n]$. Then $\mathbb{P}^\bullet_{\bar{p}}(\mathcal{O})$ is the intersection homology sheaf and its cohomology is:

$$H^{-i}(\mathbb{P}^\bullet(\mathcal{O})) = IH^{\bar{p}}_i(X; \mathbb{Z})$$

for any $r \geq 0$.

Example 5.4.18 Let $\mathbf{R}_{X^0}$ be the constant sheaf on X^0, placed in degree 0. Then

$$H^j(\mathbb{P}^\bullet_{\bar{p}}(\mathbf{R}_{X^0})) = IH^j_{\bar{p}}(X; R)$$

is the intersection cohomology.

Theorem 5.4.19 ([94, Sect. 3.6] [18, II, Theorem 6.1]) *Let* X *be a* PL*-pseudomanifold with a fixed* PL*-stratification then the sheaf of* PL*-intersection chains* $\mathcal{IC}^{\bar{p},\bullet}_X$ *satisfies the axioms* $[AX_1]_{\bar{p}}(R)$ *with respect to the given stratification. It is naturally quasi-isomorphic to* $\mathbb{P}^\bullet_{\bar{p}}(\mathbf{R})$.

The second characterization of the intersection complex of sheaves goes as follows, as a consequence of the local calculus (see [18, V, Sect. 2.12] and formulae (5.20), (5.21), (5.16), (5.17)).

Definition 5.4.20 ([94, Sect. 4.1], [127, Sect. 9] and [18, V Sect. 4.13].) Let $\mathcal{L}$ be a local system on an open dense submanifold U of codimension at least 2 in X and let $f_x : \{x\} \to X$ be the inclusion of a point x in X. One says that the sheaf complex $\mathcal{A}^\bullet$ satisfies the axioms $[AX_2]_{\bar{p}}(\mathcal{L})$ for the perversity $\bar{p}$ if one has:

(2a) $\mathcal{A}^\bullet$ is a topologically constructible complex and $\mathcal{A}^\bullet|_U = \mathcal{L}[n]$ for some open dense submanifold U of codimension at least 2 in X and over which the local system $\mathcal{L}$ is defined.
(2b) $\mathcal{H}^j(\mathcal{A}^\bullet) = 0$ if $j < -n$
(2c) $\dim\{x \in X | H^j(f_x^* \mathcal{A}^\bullet) \neq 0\} \leq n - p^{-1}(j+n)$ for every $j \geq -n+1$.
(2d) $\dim\{x \in X | H^j(f_x^! \mathcal{A}^\bullet) \neq 0\} \leq n - q^{-1}(-j)$ for every $j \leq -1$.

where is $\bar{q}$ the complementary perversity of $\bar{p}$

The uniqueness theorem, proved in Goresky and MacPherson [94, 4.1] (see also [18, V, 4.17]) states that up to canonical isomorphism, there exists an unique complex in $D^b(X)$ which satisfies axioms $[AX_2]_{\bar{p}}(\mathcal{L})$. It is given by the sheaf $\mathcal{IC}_X^{\bar{p},\bullet}(\mathcal{L})$, constructed as before with any stratification of X. As a corollary, the intersection homology groups $IH_*^{\bar{p}}(X)$ are topological invariant and exist independently of the choice of the stratification of X. One has:

Theorem 5.4.21 ([94, Sect. 4.1] [18, V, 4.17]) *Let $\mathcal{A}^\bullet$ be a fine (or soft) sheaf complex on X satisfying Axioms $[AX_2]_{\bar{p}}$ for a perversity $\bar{p}$ and Φ a family of supports on X, then the cohomology groups of the complex*

$$\dots \to \Gamma_\Phi(X; \mathcal{A}^{j-1}) \to \Gamma_\Phi(X; \mathcal{A}^j) \to \Gamma_\Phi(X; \mathcal{A}^{j+1}) \to \dots$$

i.e., the hypercohomology groups $\mathbb{H}_\Phi^j(X; \mathcal{A})$, are isomorphic to the intersection homology groups $IH_{-j}^{\bar{p},\Phi}(X; \mathcal{L})$.

In the common setting, equivalence of the systems of axioms $[AX_1]_{\bar{p}}(\mathcal{L})$ and $[AX_2]_{\bar{p}}(\mathcal{L})$ is proved in [94, 4.3], [18, V Sect. 4.10].

5.5 Main Properties of Intersection Homology

The first properties have been proved by Goresky and MacPherson [92] in the framework of PL-pseudomanifolds (see also [18, Chaps. I–IV]). They have been proved in the topological setting, using sheaves and in particular the Deligne sheaf complex, by the same authors in [94] (see also [18, Chaps. V–IX]).

5.5.1 First Properties

In general, results and proofs of this section can be found in various books or surveys concerning intersection homology. However, references will be given to the original papers of Goresky and MacPherson [92] for the PL situation and [94] for the case with sheaves and systems of local coefficients. References to the Borel book [18] will be provided as well, always taking care of the difference of notations.

The intersection homology groups are not homotopy invariant. The intersection homology groups of a cone do not (all) vanish (see (5.18)), while the cone is homotopic to a point, whose non-zero homology groups vanish. However, one has.

Topological Invariance

In [92, Corollary, p.148] Goresky and MacPherson show that the PL intersection homology groups are PL-invariants, i.e. independent of the PL-stratification (see also [18, V, 4.19]). In [94, 4.1] (see also [18, V, 4.18]), Goresky and MacPherson show independence of the topological stratification as a consequence of the Deligne construction performed for the canonical $\bar{p}$-filtration they defined ([94, 4.2]) and the system of axioms $[AX_2]_{\bar{p}}(\mathcal{L})$. The canonical $\bar{p}$-filtration is a homological stratification, the coarsest one for which the intersection homology sheaf is cohomologically constructible.

Theorem 5.5.1 ([94, Sect. 4, Introduction and Sect. 4.1, Corollary 1]) *Let X be a locally compact pseudomanifold and $\bar{p}$ a perversity. Let $\mathcal{L}$ be a local system on the regular part $X^0 = X - \Sigma$.*

- *The intersection homology groups $IH_*^{\bar{p}}(X;\mathcal{L})$ and $IH_*^{\bar{p},c}(X;\mathcal{L})$ are topological invariants and exist independently of the choice of a stratification of X,*
- *For any homeomorphism $f : X \to Y$, the complexes $\mathcal{IC}_X^{\bar{p},\bullet}$ and $f^*\mathcal{IC}_Y^{\bar{p},\bullet}$ are isomorphic in the derived category.*

In [113] King proves topological invariance without sheaves in the case of GM perversities. He also provides a generalization of the intersection homology groups using singular theory and general perversities ("loose perversities"). King claims that the PL intersection homology theory of [92] agrees with his singular theory for any loose perversity and PL stratified set (see the discussion [113, p. 158]). "One can define intersection homology for topological pseudomanifolds, independently of PL structures". A modification of the King's method is provided by Friedman [79, 5.6.2].

In [162] Rourke and Sanderson use homology stratifications to present a simplified version of the Goresky-MacPherson proof valid for PL-spaces.

Products in Intersection Homology

Definition 5.5.2 ([94, Sect. 5.0]) An R-orientation for X is a chosen quasi-isomorphism

$$\mathbf{R}_{X-\Sigma}[n] \to \mathbf{D}^\bullet_{X-\Sigma}.$$

If char(R) $\neq 2$ then an R-orientation of X is equivalent to an orientation of $X - \Sigma$ in the usual topological sense.

Suppose X is a (not necessarily orientable) n-dimensional pseudomanifold. For any local system $\mathcal{L}$ on the regular part $X^0 = X - \Sigma$ and any perversity $\bar{p}$ the Deligne's sheaf $\mathbb{P}^{\bullet}_{\bar{p}}(\mathcal{L})$ is defined on X. In [94, Sect. 5.2] and [18, V, Sect. 9, C] it is shown that any pairing of local systems $\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_3$ induces a pairing

$$\mathbb{P}^{\bullet}_{\bar{p}}(\mathcal{L}_1) \otimes \mathbb{P}^{\bullet}_{\bar{q}}(\mathcal{L}_2) \to \mathbb{P}^{\bullet}_{\bar{r}}(\mathcal{L}_3) \qquad \text{for } \bar{r} \geq \bar{p} + \bar{q}. \tag{5.22}$$

by induction using the construction of Deligne "attaching–truncating" from the multiplication on the regular subset X^0.

The generalized Poincaré duality, Poincaré-Lefschetz theorem as well as intersection pairing, and cup and cap-products follow from particular cases of formula 5.22, mainly in the case of Examples 5.4.17 and 5.4.18 (see also [18, V, Sect. 9.15] taking care of difference of indices—see Remark 5.4.10).

For instance, cup products $IH^a \otimes IH^b \to IH^{a+b}$ and cap products $IH^a \otimes IH_b \to IH_{b-a}$ in intersection cohomology follow from the canonical pairings

$$\mathbf{R}_X \overset{L}{\otimes} \mathbf{R}_X \to \mathbf{R}_X \qquad \text{and} \qquad \mathbf{R}_X \overset{L}{\otimes} \mathbf{D}^{\bullet}_X \to \mathbf{D}^{\bullet}_X.$$

This constructions works over any commutative ring R of finite cohomological dimension.

Intersection Pairing

One of the most important properties of intersection homology is the generalization of the Poincaré-Lefschetz duality, i.e. the intersection pairing (Sect. 5.2.6).

The following Proposition ([92, Sect. 2]) has been first stated by Goresky and MacPherson in the PL setting, using a McCrory Lemma [131, 132], itself using the Zeeman technique to move cycles into general position (see [92, Sect. 2.2]).

Proposition 5.5.3 ([92, Sect. 2.3]) *Let X a compact oriented PL-pseudomanifold and let $\bar{p}$, $\bar{q}$ and $\bar{r}$ perversities such that $\bar{p} + \bar{q} \leq \bar{r}$, one has canonical bilinear pairings*

$$IH^{\bar{p}}_i(X;\mathbb{Z}) \times IH^{\bar{q}}_j(X;\mathbb{Z}) \to IH^{\bar{r}}_{i+j-n}(X;\mathbb{Z}),$$

These pairings are compatible with the cup and cap products ([92, Sect. 7, Appendix]).

Note that, in the non compact situation, the preceding construction gives rise to the pairings

$$IH^{\bar{p}}_i(X) \times IH^{\bar{q},c}_j(X) \to IH^{\bar{r},c}_{i+j-n}(X).$$

Goresky and MacPherson generalized the results in the topological setting [94, Sects. 5.2 and 5.3], using the intersection sheaf complex (see also [18, V, 9.15] taking care of difference of indices—see Remark 5.4.10).

Starting with local coefficient systems $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ on $X - \Sigma$, a product $\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_3$ gives rise to intersection pairings (cf 5.22) [94, 5.2]

$$\mathcal{IC}^\bullet_{\bar{p}}(\mathcal{L}_1) \overset{L}{\otimes} \mathcal{IC}^\bullet_{\bar{q}}(\mathcal{L}_2) \to \mathcal{IC}^\bullet_{\bar{r}}(\mathcal{L}_3)[n].$$

and the Theorem:

Theorem 5.5.4 ([94, 5.2], [18, I, 4.2; V, 9.14]) *Let X be a topological pseudomanifold. If $\bar{p} + \bar{q} \leq \bar{r}$ there are canonical intersection pairings*

$$IH_i^{\bar{p}}(X; \mathcal{L}_1) \times IH_j^{\bar{q}}(X; \mathcal{L}_2) \to IH_{i+j-n}^{\bar{r}}(X; \mathcal{L}_3).$$

These pairings are compatible with the cup and cap products.

Goresky and MacPherson remark that it is not necessary to have an orientation in the preceding construction [94, Sect. 5.2].

Verdier Duality—The Generalized Poincaré-Lefschetz

In their original article, in a delicate and very geometric proof, using so-called "basic sets" $Q_i^{\bar{p}}$, Goresky and MacPherson prove the generalized Poincaré duality:

Theorem 5.5.5 ([92, 3.3, Theorem]) *Let X be a compact, oriented pseudomanifold and let $\bar{p}$ and $\bar{q}$ be two complementary perversities, then the pairing*

$$IH_i^{\bar{p}}(X; \mathbb{Z}) \times IH_{n-i}^{\bar{q}}(X; \mathbb{Z}) \to IH_0^{\bar{t}}(X; \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}$$

followed by the evaluation map ε (which counts points with their multiplicity order) is non-degenerate, when tensorised by the rationals $\mathbb{Q}$.

Note that, in the non compact situation, the preceding construction gives rise to the pairing (see Example 5.4.5).

$$IH_i^{\bar{p}}(X; \mathbb{Z}) \times IH_{n-i}^{\bar{q},c}(X; \mathbb{Z}) \to IH_0^{\bar{t},c}(X; \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z} \tag{5.23}$$

In a more general way, let k be a field, then the pairing

$$IH_i^{\bar{p}}(X; k) \times IH_{n-i}^{\bar{q}}(X; k) \to IH_0^{\bar{t}}(X; k) \xrightarrow{\varepsilon} k$$

is non-degenerate and induces isomorphisms

$$IH_i^{\bar{p}}(X; k) \cong \mathrm{Hom}(IH_{n-i}^{\bar{q}}(X; k), k)$$

Note that, in the non-compact case, one has isomorphisms:

$$IH_i^{\bar{p}}(X;k) \cong \mathrm{Hom}(IH_{n-i}^{\bar{q},c}(X;k),k)$$

In [94] these results have been generalized by Goresky and MacPherson, assuming that the coefficient ring R is a field k and that X is k-orientable (with a choice of k-orientation). The following results follow from the property of duality between $\mathcal{IC}^\bullet_{\bar{p}}$ and $\mathcal{IC}^\bullet_{\bar{q}}$ for complementary pervesities $\bar{p}+\bar{q}=\bar{t}$. In particular if X has even codimension strata, then $\mathcal{IC}^\bullet_{\bar{m}}$ is self dual (for example, if X is a complex analytic variety).

Definition 5.5.6 ([94, 5.3]) Let $n=\dim(X)$, a pairing $\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet \to \mathbf{D}^\bullet_X[n]$ of objects in $D^b(X)$ is called a Verdier dual pairing if it induces an isomorphism in $D^b(X)$

$$\mathcal{A}^\bullet \longrightarrow R\mathcal{H}om^\bullet(\mathcal{B}^\bullet, \mathbf{D}^\bullet_X)[n].$$

Theorem 5.5.7 ([94, 5.3, Theorem]) *Suppose $\bar{p}$ and $\bar{q}$ are complementary perversities, then the intersection pairing followed by the map to homology*

$$\mathcal{IC}^\bullet_{\bar{p}} \overset{L}{\otimes} \mathcal{IC}^\bullet_{\bar{q}} \to \mathcal{IC}^\bullet_{\bar{t}}[n] \to \mathbf{D}^\bullet_X[n]$$

is a Verdier dual pairing.

Corollary 5.5.8 ([94, 5.3, Corollary]) *Let X be a compact, oriented stratified pseudomanifold and let $\bar{p}$ and $\bar{q}$ be two complementary perversities, then the pairing*

$$IH_i^{\bar{p}}(X;k) \times IH_{n-i}^{\bar{q}}(X;k) \to IH_0^{\bar{t}}(X;k) \xrightarrow{\varepsilon} k$$

followed by the evaluation map ε (which counts points with their multiplicity order) induces isomorphisms

$$IH_i^{\bar{p}}(X;k) \cong \mathrm{Hom}(IH_{n-i}^{\bar{q}}(X;k),k)$$

Dropping the assumption that X is oriented, let $\mathcal{O}$ be the orientation local system of k-modules on $X-\Sigma$. A pairing $\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{O}$ of local systems on $X-\Sigma$ is called perfect if the induced mapping $\mathcal{L}_1 \to Hom(\mathcal{L}_2,\mathcal{O})$ is an isomorphism.

Theorem 5.5.9 ([94, 5.3, last Theorem]) *Suppose $\bar{p}$ and $\bar{q}$ are complementary perversities and the pairing $\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{O}$ is perfect. Then the intersection pairing followed by the map to homology*

$$\mathcal{IC}^\bullet_{\bar{p}}(\mathcal{L}_1) \overset{L}{\otimes} \mathcal{IC}^\bullet_{\bar{q}}(\mathcal{L}_2) \to \mathcal{IC}^\bullet_{\bar{t}}(\mathcal{O})[n] \to \mathbf{D}^\bullet_X[n]$$

is a Verdier dual pairing.

Note that, using Alexander-Whitney chains, Friedman and McClure [80] re-prove these results in the special case of field coefficients.

Example 5.5.10 (Suspension of the torus) That is the original Goresky-MacPherson example (see [92]) for which Poincaré duality fails for usual homology. The suspension of the torus (Fig. 5.6) is the join of the torus with two points A and B. It is a 3-dimensional singular variety with two singular points A and B: the link of A (or B) is a torus, not a sphere. See the alternative very nice picture of the suspension of the torus in [92].

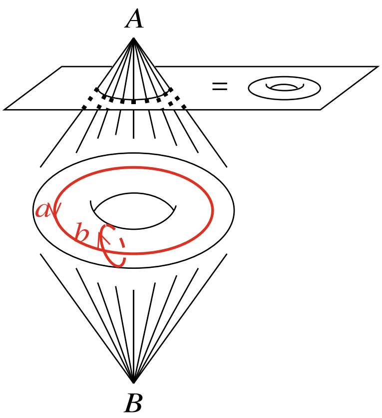


Fig. 5.6 Suspension of the circle $\mathbb{S}^1$ and of the torus

The natural stratification of the suspension of the torus is

$$X \supset X_0 = \{A, B\} \supset \emptyset.$$

There are two possible perversities:

$$\bar{p} = \bar{0} = (0, 0, 0, 0) \qquad \text{and} \qquad \bar{p} = \bar{t} = (0, 0, 0, 1)$$

An i-dimensional chain c containing one (or two) of the singular points A and B is allowable if

$$0 = \dim(|c| \cap X_0) \leq i - 3 + p_3,$$

that means, if $\bar{p} = \bar{0}$, then $i \geq 3$ and if $\bar{p} = \bar{t}$, then $i \geq 2$.

The intersection homology groups $IH_i^{\bar{p}}(X; \mathbb{Q})$ corresponding to the two perversities are easily computed and are resumed in the Table 5.2. The cycle a is boundary of the cycle $c(a)$, cone with vertex a. The suspension of a is a 2-dimensional cycle denoted by $\Sigma(a)$.

Table 5.2 Intersection homology of the suspension of the torus

Perversities	$\bar{p} = \bar{0}$	$\bar{p} = \bar{t}$
i	$IH_i^{\bar{0}}(X)$	$IH_i^{\bar{t}}(X)$
0	$\mathbb{Q}_{\{x\}}$	$\mathbb{Q}_{\{x\}}$
1	$\mathbb{Q}_a \oplus \mathbb{Q}_b$	0
2	0	$\mathbb{Q}_{\Sigma(a)} \oplus \mathbb{Q}_{\Sigma(b)}$
3	$\mathbb{Q}_{[X]}$	$\mathbb{Q}_{[X]}$

The intersection matrix of the intersection product

$$IH_1^{\bar{0}}(X; \mathbb{Q}) \times IH_1^{\bar{t}}(X : \mathbb{Q}) \to \mathbb{Q}$$

$$\begin{array}{cc} & \begin{array}{cc} \Sigma(a) & \Sigma(b) \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix} \end{array}$$

is non-degenerate.

Factorization of Poincaré Homomorphism. See [92, Sect. 1.4], [94, Sect. 5.1], [18, I, Sect. 4.1; I, Sect. 3.2]

Poincaré Duality—Return to the smooth case
The Poincaré duality can be proved by using sheaf complexes: R is still a regular Noetherian ring with finite Krull dimension, which can be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Assuming that X is an n-dimensional oriented manifold, the quasi-isomorphism of complexes of sheaves $R_X[n] \cong \mathcal{C}_X^\bullet$ induces isomorphisms of hypercohomology groups:

$$\mathbb{H}_\Phi^{-i}(X; \mathbf{R}_X[n]) \cong \mathbb{H}_\Phi^{-i}(X; \mathcal{C}_X^\bullet)$$

i.e., (5.8), (5.9)

$$H_\Phi^{n-i}(X) \cong H_i^\Phi(X).$$

In particular, one has (see Sect. 5.2.2):

$$H^{n-i}(X) \cong H_i(X) \qquad \text{and} \qquad H_c^{n-i}(X) \cong H_i^c(X).$$

Poincaré Duality—Singular case
An orientation on X is an isomorphism (5.5.2)

$$\mathbf{R}_{X-\Sigma}[n] \to \mathbf{D}_{X-\Sigma}^\bullet.$$

The (unique in D^b) canonical lift of the orientation

$$\mathbf{R}_X[n] \to Ri_*\mathbf{R}_{X-\Sigma}[n] \overset{\cong}{\to} Ri_*\mathbf{D}^\bullet_{X-\Sigma}$$

induces the cap-product with the orientation (fundamental) class

$$PD_X : \mathbf{R}_X[n] \to \mathbf{D}^\bullet_X.$$

Proposition 5.5.11 ([94, Sect. 5.1]) *Assume X oriented, and let $i : X - \Sigma \to X$ denote the inclusion. For any perversity $\bar{p}$ there is a unique morphism in D^b*

$$\mathbf{R}_X[n] \to \mathcal{IC}^\bullet_{\bar{p}} \to \mathbf{D}^\bullet_X$$

such that the induced morphism $i^*\mathbf{R}_X[n] \to i^*\mathcal{IC}^\bullet$ is the evident one and $i^*\mathcal{IC}^\bullet \to i^*\mathbf{D}^\bullet_X$ is given by the orientation. These morphisms factor the cap-product with the fundamental class $[X]$, i.e. $PD_X : \mathbf{R}_X[n] \to \mathbf{D}^\bullet_X$.

For any perversity $\bar{p}$, denote the previous morphisms by

$$\alpha_X : \mathbf{R}_X[n] \to \mathcal{IC}^\bullet_{\bar{p}} \qquad \text{and} \qquad \omega_X : \mathcal{IC}^\bullet_{\bar{p}} \to \mathbf{D}^\bullet_X.$$

By taking hypercohomology, one obtain the classical comparison morphisms

$$H^\bullet(X) \xrightarrow{\alpha_X} IH^{\bar{p}}_{n-\bullet}(X) \quad \text{and} \quad IH^{\bar{p}}_{n-\bullet}(X) \xrightarrow{\omega_X} H_{n-\bullet}(X).$$

The composition $\omega_X \circ \alpha_X : \mathbf{R}_X[n] \to \mathbf{D}^\bullet_X$

$$\begin{array}{ccc} \mathbf{R}_X[n] & \xrightarrow{\quad PD_X \quad} & \mathbf{D}^\bullet_X \\ & {\scriptstyle\alpha_X}\searrow \quad \nearrow{\scriptstyle\omega_X} & \\ & \mathcal{IC}^\bullet_{\bar{p}} & \end{array}$$

induces at the global level, i.e., taking hypercohomology, the “classical” Poincaré duality homomorphism

$$H^\bullet(X) \to H_{n-\bullet}(X)$$

that is factorized by intersection homology

$$\begin{array}{ccccc} H^{n-i}(X) & & \xrightarrow{\quad \bullet\cap[X] \quad} & & H_i(X) \\ {\scriptstyle\alpha_X^{\bar{0}}}\downarrow & \searrow^{\alpha_X^{\bar{p}}} & & \nearrow^{\omega_X^{\bar{t}}} & \uparrow{\scriptstyle\omega_X^{\bar{t}}} \\ IH_i^{\bar{0}}(X) & \longrightarrow & IH_i^{\bar{p}}(X) & \longrightarrow & IH_i^{\bar{t}}(X). \end{array} \tag{5.24}$$

Poincaré Duality—Singular case—geometry
In this (sub)-section, X is an oriented compact PL-pseudovariety.

First, remark that if $\bar{p}$ and $\bar{q}$ are two perversities such that $\bar{p} \leq \bar{q}$, that is $p_\alpha \leq q_\alpha$ for all α, then one has a natural morphism

$$IC_*^{\bar{p}}(X) \hookrightarrow IC_*^{\bar{q}}(X) \tag{5.25}$$

for every support family and it induces a morphism $IH_*^{\bar{p}}(X) \to IH_*^{\bar{q}}(X)$.

In particular, one has a morphism $IH_*^{\bar{0}}(X) \to IH_*^{\bar{p}}(X)$ and a morphism $IH_*^{\bar{p}}(X) \to IH_*^{\bar{t}}(X)$ for every perversity $\bar{p}$.

The morphism $\alpha_X^{\bar{0}} : H^{n-i}(X) \to IH_i^{\bar{0}}(X)$ can be described in the following way:

Assuming that X is embedded in a smooth m-dimensional PL oriented manifold M, the stratification of X can be extended to a stratification of M by taking $M \setminus X$ as the regular stratum. Let K be a locally finite triangulation of M compatible with the stratification. For each $p = n - i$-simplex $\sigma \in K$ contained in X, the dual cell of σ in M, denoted by $D\sigma$ has dimension $m - p$ (see 5.4) and is transverse to all strata. The Poincaré homomorphism

$$C_{(K)}^{n-i}(X) \to C_i^{(K')}(X)$$

associates to the elementary $(n - i)$-cochain σ^* which corresponds to the simplex σ in K, the i-chain $\xi = D\sigma \cap X$ of (K'), which is $\bar{0}$-allowed. Therefore, for each perversity $\bar{p}$, one has the factorisation (5.24).

Relative Homology, See [93, 1.3], [18, I, 2.2.2]

Let X be a stratified pseudomanifold, and U an open subset in X, then U inherits a structure of stratified pseudomanifold induced by the one of X. For every perversity $\bar{p}$, the complex of intersection chains of U with compact supports $IC_*^{\bar{p}}(U)$, is a sub-complex of $IC_*^{\bar{p}}(X)$.

Defining $IC_*^{\bar{p}}(X, U) = IC_*^{\bar{p}}(X)/IC_*^{\bar{p}}(U)$, one obtains a relative complex and one has a long exact sequence:

$$\cdots \to IH_i^{\bar{p}}(U) \to IH_i^{\bar{p}}(X) \to IH_i^{\bar{p}}(X, U) \to IH_{i-1}^{\bar{p}}(U) \to \cdots$$

The property is also valid with local systems (see [93, 1.8]).

Excision, See [93, 1.5]

Lemma 5.5.12 *Let X be a locally compact stratified pseudomanifold and U and V two open subsets in X, then the inclusion $(U, U \cap V) \hookrightarrow (U \cup V, V)$ induces an isomorphism of intersection homology groups with compact supports*

$$IH_i^{\bar{p}}(U, U \cap V) \cong IH_i^{\bar{p}}(U \cup V, V).$$

Proposition 5.5.13 *Let X be a locally compact stratified pseudomanifold, U an open subset in X and A a closed subset in U. Let $\bar{p}$ be any perversity, then the inclusion $(X - A, U - A) \hookrightarrow (X, U)$ induces an isomorphism of intersection homology groups with compact supports*

$$IH_i^{\bar{p}}(X, U) \cong IH_i^{\bar{p}}(X - A, U - A).$$

The property is also valid with local systems (see [93, 1.8]).

Künneth Formulae, See [94, 6.3]

Künneth formulae in homology If X and Y are topological spaces, and R a PID (principal ideal domain) the Künneth formula is written as short exact sequence (where all homology groups have R coeffivients)

$$0 \to \bigoplus_{a+b=i} H_a(X) \otimes H_b(Y) \to H_i(X \times Y) \to \bigoplus_{a+b=i-1} \mathrm{Tor}^R(H_a(X), H_b(Y)) \to 0,$$

that (not canonically) splits.

If R is a field k, then the Künneth formula is written

$$\bigoplus_{a+b=i} H_a(X; k) \otimes H_b(Y; k) \cong H_i(X \times Y; k).$$

Künneth formula in intersection homology In general, the Künneth formula is no longer true for intersection homology (see counterexamples in [61, Sect. 5]). However, there are some situations for which the the Künneth formula is true.

(1) Cheeger [59] observes that the Künneth formula holds for the middle intersection cohomology, and for Witt spaces X, Y (see Sect. 5.4.6) with $k = \mathbb{R}$.

$$IH_i^{\bar{m}}(X \times Y; \mathbb{R}) \cong \bigoplus_{a+b=i} IH_a^{\bar{m}}(X; \mathbb{R}) \otimes IH_b^{\bar{m}}(Y; \mathbb{R}). \tag{5.26}$$

The formula is extended in [94, 6.3] in the context of "middle homology sheaves". A middle homology sheaf is a complex of sheaves $\mathcal{S}^\bullet$ such that for some local coefficient system $\mathcal{F}$ on $X^0 = X \backslash \Sigma$,

$$\mathcal{S}^\bullet = \mathcal{IC}^\bullet_{\bar{m}}(\mathcal{F}) = \mathcal{IC}^\bullet_{\bar{n}}(\mathcal{F}).$$

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local coefficients systems on the regular parts of X and Y respectively, satisfying the previous formula, then

$$IH_i^{\bar{m}}(X \times Y; \pi_1^*\mathcal{F}_1 \otimes \pi_2^*\mathcal{F}_2) \cong \bigoplus_{a+b=i} IH_a^{\bar{m}}(X; \mathcal{F}_1) \otimes IH_b^{\bar{m}}(Y; \mathcal{F}_2).$$

(2) When one of the element of the product is a smooth manifold, the formula is verified (see [93, Sect. 1.6], [61]).

Proposition 5.5.14 *Let X be a locally compact stratified pseudomanifold and M a manifold. Let $\bar{p}$ be a perversity, one has a split exact sequence:*

$$0 \to (IH_*^{\bar{p}}(X) \otimes H_*(M))_i \to IH_i^{\bar{p}}(X \times M) \to (IH_*^{\bar{p}}(X) * H_*(M))_{i-1} \to 0$$

(3) Fix a coefficient ring R which is a principal ideal domain and suppose X and Y are compact pseudomanifolds. Cohen, Goresky and Ji show [61] more general results showing, for instance that if the perversity $\bar{p}$ satisfies

$$p(a) + p(b) \leq p(a+b) \leq p(a) + p(b) + 1$$

for all a and b, then there is a split short exact sequence for intersection cohomology with the perversity $\bar{p}$ and coefficients in R:

$$0 \to \bigoplus_{a+b=i} IH^a(X) \otimes IH^b(Y) \to IH^i(X \times Y) \to \bigoplus_{a+b=i-1} \mathrm{Tor}^R(IH^a(X), IH^b(Y)) \to 0.$$

The condition on the perversity $\bar{p}$ means that the graph of the perversity function does not deviate far from some straight line through the origin (see also [28, Corollary 9.3]).

(4) G. Friedman, in [74], considers biperversities $(\bar{p}, \bar{q})$ and obtains a Künneth theorem relating $IH_*^{\bar{p},\bar{q}}(X \times Y)$ and $IH_*^{\bar{p}}(X)$ and $IH_*^{\bar{q}}(Y)$ for all choices of $\bar{p}$ and $\bar{q}$. and this recovers the result of Cohen,Goresky and Ji.

(5) Let k be a field, Friedman and McClure [80] define a perversity $\bar{Q}(\bar{p}, \bar{q})$ on the product $X \times Y$, whose value depends on regularness or not of the elements of the filtration of the product. They obtain an isomorphism

$$IH_*^{\bar{p}}(X; k) \otimes IH_*^{\bar{q}}(Y; k) \to IH_*^{\bar{Q}}(X \times Y; k).$$

Normalization and Intersection Homology, [92, Sect. 4], [18, I, 1.6, I, 3.2, V,2.8 and 2.12]

The oriented n-dimensional pseudomanifold X is *normal* if

$$H_n(X, X - \{x\}; \mathbb{Z}) = \mathbb{Z} \quad \text{for all} \quad x \in X.$$

Equivalently, each point x admits a fundamental system of neighborhoods U whose regular part $U \setminus \Sigma$ is connected.

Proposition 5.5.15 ([92, Sect. 4.3], [94, Sect. 5.6] [18, I, Sect. 4.1]) *Let X be a normal pseudomanifold, the morphisms α_X and ω_X (see Sect. 5.5.1) induce isomorphisms:*

$$\mathbf{R}_X \cong \mathcal{IC}^\bullet_{\bar{0}} \quad \textit{and} \quad \mathcal{IC}^\bullet_{\bar{t}} \cong \mathbf{D}^\bullet_X$$

respectively for the zero perversity $\bar{0}$ and the total one $\bar{t}$:

$$H^{n-i}(X) \cong IH_i^{\bar{0}}(X), \qquad IH_i^{\bar{t}}(X) \cong H_i(X).$$

The vertical arrows in diagram (5.24) are isomorphisms.

Proposition 5.5.16 ([92, Sect. 4.2], [18, I, Sect. 3.2]) *Let $\tilde{X}$ be the normalization of a pseudomanifold X, then one has:*

$$IH_i^{\bar{p}}(\tilde{X}) = IH_i^{\bar{p}}(X).$$

Homology Manifolds

Goresky and MacPherson conjecture in [92, Sect. 6.6] that if X is a normal pseudovariety such that $IH_*^{\bar{p}}(X) \to IH_*^{\bar{q}}(X)$ are isomorphisms for all $\bar{p} \leq \bar{q}$, then X is a $\mathbb{Z}$-homology manifold, i.e. there is an integer n such that for each point $\{x\}$ in X, the local homology group satisfies

$$H_i(X, X \setminus \{x\}\,; \mathbb{Z}) = \begin{cases} 0 & \text{if } \; i \neq n \\ \mathbb{Z} & \text{if } \; i = n. \end{cases} \tag{5.27}$$

The conjecture is false, by a counter-example of King [114] who shows that it is true if one considers more general perversities, so-called "loose perversities". In [32], Brasselet and Saralegi show that the conjecture is true with a supplementary hypothesis, namely if there are tubular neighborhoods of the strata without homological monodromy. On the other hand, Fieseler and Kaup define in [73, 115] invariants linked to properties of the fibers of the Deligne sheaf. Using these invariants Brasselet, Fieseler and Kaup provide computable criteria for X being a homology manifold [25].

Case of Isolated Singularities

Proposition 5.5.17 ([18, Sect. 5.1]) *Let X be an n-dimensional pseudomanifold with an isolated singularity at $\{x\}$. The integer p_n is the only one pertinent element of the perversity, one has $0 \leq p_n \leq n-2$ and:*

$$IH_i^{\bar{p},c}(X) = \begin{cases} H_i^c(X \setminus \{x\}) & i < n - p_n - 1 \\ \mathrm{Im}(H_i^c(X \setminus \{x\}) \to H_i^c(X)) & i = n - p_n - 1 \\ H_i^c(X) & i > n - p_n - 1, \end{cases} \tag{5.28}$$

$$IH_i^{\bar{p}}(X) = \begin{cases} H_i^{\phi}(X \setminus \{x\}) & i < n - p_n - 1 \\ \mathrm{Im}(H_i^{\phi}(X \setminus \{x\}) \to H_i(X)) & i = n - p_n - 1 \\ H_i(X) & i > n - p_n - 1 \end{cases} \tag{5.29}$$

where ϕ denotes the family of closed subsets in X which are contained in $X \setminus \{x\}$.
If n is even and $\bar{p}$ is the middle perversity, one has $n - p_n - 1 = n/2$.

The Proposition is valid with local coefficieny systems.

Example of Thom Spaces, [18, I, Sect. 5.3], [26]

Let B a compact $2n$-dimensional manifold and $\pi : E \to B$ a real oriented vector bundle with even rank r on B. The Thom space $\mathfrak{T}$ associated to E is the Alexandroff compactification of E by adjonction of a point at infinity. It is also the quotient $T(E)/S(E)$ where $T(E)$ and $S(E)$ are the fibre bundles associated to E whose fibers are respectively closed balls and spheres in the fibers of E. The Thom space is a pseudomanifold with an isolated singular point and its dimension is $2s = 2n + r$.

Let $[\mathfrak{T}]$ be the fundamental class of $\mathfrak{T}$ and $e \in H^r(B)$ the Euler class of the bundle E. For every i, different from 0 and $2s$, one has a commutative diagram ([27], see also [18, I, Sect. 5.3]):

$$\begin{array}{ccc} H^{2s-i}(\mathfrak{T}) & \xrightarrow{\cdot \cap [\mathfrak{T}]} & H_i(\mathfrak{T}) \\ \downarrow \cong & & \downarrow \cong \\ H_i(B) & \xrightarrow{\cdot \cap e} & H_{i-r}(B) \end{array}$$

and for the middle (lower) perversity:

$$IH_i(\mathfrak{T}) = \begin{cases} H_i(\mathfrak{T}) & i < s \\ \mathrm{Im}(H_i(B) \xrightarrow{\cdot \cap e} H_{i-r}(B)) & i = s \\ H_{i-r}(B) & i > s. \end{cases} \tag{5.30}$$

In [92, Sect. 6.3] Goresky and MacPherson illustrate the behavior of torsion in the intersection homology by the example of Thom space for which the universal coefficient theorem fails and the generalized Poincaré duality theorem is not true over $\mathbb{Z}$.

Exemples of computations of Thom spaces associated to the Segre and Veronese embeddings are provided by Brasselet and Gonzalez-Sprinberg in [27].

5.5.2 Functoriality

In general, for a map $f : X \to Y$, there is no functoriality, i.e. no maps If^* and If_* such that the diagrams below (5.32) and (5.33) commute. The functoriality problem has been proposed by Goresky and MacPherson in [18, IX, C, Problem 4]): "Find the most general category of spaces and maps (perhaps with additional data) on which intersection homology is functorial."

Goresky and MacPherson earlier proved functoriality for Normally Nonsingular Maps [94, Sect. 5.4]:

(a) A *normally nonsingular map* ([82, Sect. 4.1]) $f : X \to Y$ between oriented topological spaces, is a map such that there is a diagram

$$\begin{array}{ccc} N & \xrightarrow{\ i\ } & Y \times \mathbb{R}^n \\ s \big\uparrow \big\downarrow \pi & & \big\downarrow p \\ X & \xrightarrow[\ f\ ]{} & Y \end{array} \tag{5.31}$$

in which $\pi : N \to X$ is a rank d vector bundle with zero-section s, the map i is an open embedding, p is the first projection and $f = p \circ i \circ \pi$. The integer $d - n$ is the relative codimension of f. As said in [82], "Geometrically, that says that the singularities of X at any point x are no better or worse than the singularities of Y at $f(x)$." Topological pseudomanifotds and normally nonsingular maps form a category (see [82]).

Theorem 5.5.18 ([94, 5.4.3]) *Let $f : X \to Y$ be a proper normally nonsingular map of relative dimension ν. Then there are homomorphisms*

$$If_* : IH_k^{\bar{p}}(X) \to IH_k^{\bar{p}}(Y) \qquad \textit{and} \qquad If^* : IH_k^{\bar{p}}(Y) \to IH_{k-\nu}^{\bar{p}}(X).$$

$IH_k^{\bar{p}}$ is both a covariant functor (via If_*) and a contravariant functor (via If^*) on the category of topological pseudomanifotds and normally nonsingular maps.

In their discussion in [18, IX, C], Goresky and MacPherson give and discuss several classes of maps $f : X \to Y$ for which there are natural homomorphisms between $IH_*^{\bar{m}}(X)$ and $IH_*^{\bar{m}}(Y)$ (where $\bar{m}$ is the middle (lower) perversity). In particular, they give the following examples:

(b) The placid maps. A continuous map $f : X \to Y$ between stratified spaces is said placid if it is stratum preserving (i.e. the image of every stratum of X is contained in a single stratum of Y) and for each stratum S in Y, the inequality holds:

$$\operatorname{codim}_X f^{-1}(S) \geq \operatorname{codim}_Y(S).$$

Proposition 5.5.19 *[98, Proposition 4.1] Assume that $f : X \to Y$ is a placid map. Then pushforward of chains and pullback of generic chains induce homomorphisms on intersection homology:*

$$If_* : IH_k^{\bar{m}}(X) \to IH_k^{\bar{m}}(Y) \quad \textit{and} \quad If^* : IH_{n-k}^{\bar{m}}(Y) \to IH_{m-k}^{\bar{m}}(X).$$

where $m = \dim(X)$ and $n = \dim(Y)$.

(c) Small maps. [94, Sect. 6.2]

A proper surjective algebraic map $f : X \to Y$ between irreducible complex n-dimensional algebraic varieties is small if X is nonsingular and for all $r > 0$,

$$\operatorname{cod}_{\mathbb{C}}\{y \in Y | \ \dim_{\mathbb{C}} f^{-1}(y) \geq r\} > 2r.$$

If Y is one or two dimensional then a small map $f : X \to Y$ must be a finite map. If Y is a threefold then the fibres of a small map f must be zero dimensional except possibly over a set of isolated points in Y where the fibres may be at most curves.

(d) A more general result has been proved by Barthel, Brasselet, Fieseler, Gabber and Kaup.

Theorem 5.5.20 ([5, Théorème 2.3]) *Let $f : X \to Y$ be a map between algebraic complex varieties of respective pure (real) dimensions m and n, and consider $R = \mathbb{Q}$. Then*
(1) There are contravariant homomorphisms (with closed supports)

$$If^* : IH_{n-\bullet}(Y) \to IH_{m-\bullet}(X)$$

and covariant homomorphisms with compact supports

$$If_* : IH_\bullet^c(X) \to IH_\bullet^c(Y)$$

such that the following diagrams commute:

$$\begin{array}{ccccccc} IH_{n-\bullet}(Y) & \xrightarrow{If^*} & IH_{m-\bullet}(X) & & IH_\bullet^c(X) & \xrightarrow{If_*} & IH_\bullet^c(Y) \\ \uparrow \alpha_Y & & \uparrow \alpha_X & & \downarrow \omega_X & & \downarrow \omega_Y \\ H^\bullet(Y) & \xrightarrow{f^*} & H^\bullet(X) & & H_\bullet^c(X) & \xrightarrow{f_*} & H_\bullet^c(Y). \end{array} \tag{5.32}$$

(2) Assume that the map $f : X \to Y$ is proper, then there are contravariant homomorphisms with compact supports

$$If^* : IH_{n-\bullet}^c(Y) \to IH_{m-\bullet}^c(X)$$

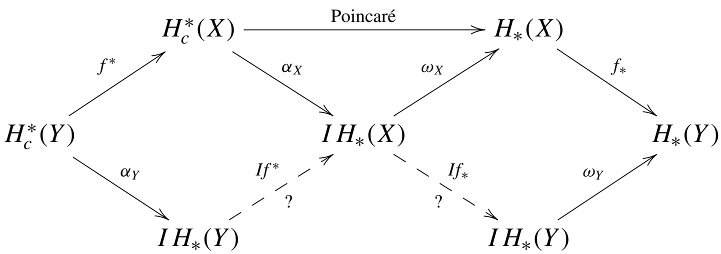
and covariant homomorphisms with closed supports

$$If_* : IH_\bullet(X) \to IH_\bullet(Y)$$

such that the following diagrams commute:

$$\begin{array}{ccccccc} IH^c_{n-\bullet}(Y) & \xrightarrow{If^*} & IH^c_{m-\bullet}(X) & \quad & IH_\bullet(X) & \xrightarrow{If_*} & IH_\bullet(Y) \\ \uparrow \alpha_Y & & \uparrow \alpha_X & & \downarrow \omega_X & & \downarrow \omega_Y \\ H_c^\bullet(Y) & \xrightarrow{f^*} & H_c^\bullet(X) & & .H_\bullet(X) & \xrightarrow{f_*} & H_\bullet(Y). \end{array} \tag{5.33}$$

The results can be summarized by the following commutative diagram (the reader is invited to write the similar diagram with compact supports):



Note that the notations used in [5] are μ^f for If^* and ν_f for If_*. In general, the associated maps If^* and If_* in intersection homology are not uniquely determined. They are uniquely determined by f in the following particular cases:

- if Y is smooth. In that case, α_Y and $If^* \circ \alpha_Y$ are isomorphisms and $If^*$is determined by α_X,
- if f is an equidimensional dominant map or, more generally, a placid map (see [5, (3.3)]),
- if f is the embedding of a closed submanifold X with codimension 1 in Y such that Y is locally analytically irreducible along X (see [5, (3.6)]),
- if f is a homologically small map in the sense of [94, Sect. 6.2].

Based on the previous results, Weber [185] assumes that a map of analytic varieties is an inclusion of codimension one. He shows that the existence of an associated morphism in intersection homology follows from Saito's decomposition theorem. For varieties with conical singularities he shows that the existence of intersection homology morphism is equivalent to the validity of the Hard Lefschetz Theorem for links.

Lifting of Algebraic Cycles

The notion of intersection homology $IH_\bullet^{(C)}(Y)$ of Y with supports in a closed subvariety C of Y, i.e. relative intersection homology $IH_\bullet(Y, Y \setminus C)$ (see Sect. 5.3.1) is useful for this section.

Theorem 5.5.21 ([5, Théorème 2.4]). *Let C be a closed subvariety of Y, with pure dimension n, then the homology class $[C]$, with rational coefficients, is in the image of the morphism*

$$\omega_Y : IH_n^{(C)}(Y) \longrightarrow H_n^{(C)}(Y)\ .$$

The classes corresponding to algebraic cycles in an algebraic variety can be lifted in intersection homology, however the lifting is not unique.

Coming to the original question asked by Goresky and MacPherson (see [18, Goresky-MacPherson, Chap. IX, Sect. H, Problem 10]) the homology Chern-Schwartz-MacPherson classes of an algebraic variety can be lifted to intersection homology, for the middle perversity and with rational coefficients [5, Corollaire 2.6]. On the one hand, Goresky constructed an example for which there is no lifting when using $\mathbb{Z}$ coefficients, on the other hand Verdier constructed an example for which the lifting is not unique even with rational coefficients (see [26, 27] for these examples). Also, using the previous results obtained for the middle perversity (and higher ones) it is not possible to multiply more than two homology classes. This gives an obstruction to the definition of general characteristic numbers for singular complex algebraic varieties.

The Classification Theorem

The morphisms If^* and If_* in intersection homology are not uniquely determined by the morphism f. The following result provides a measure of the ambiguity. It gives also a geometric meaning of the motivation and completes the principal result:

Theorem 5.5.22 ([5, Théorème 2.7]) *There is a one-to-one correspondence between the morphisms If^*, resp. If_*, such that the diagrams (5.32) and (5.33) commute and classes $\gamma \in IH_n^{(\Gamma_f)}(X \times Y)$ which are liftings of the homology class $[\Gamma_f] \in H_n^{(\Gamma_f)}(X \times Y)$ of the graph of f.*

5.5.3 *Lefschetz Fixed Points and Coincidence Theorems*

Lefschetz Fixed Points Theorem

The smooth case

Let M be an n-dimensional oriented smooth manifold, and $f : M \to M$. One of the possible definitions of the Lefschetz number $L(f)$ (known as Lefschetz fixed point formula [122]) is:

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{Trace}(f_k : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})). \tag{5.34}$$

Let $G(f) \subset M \times M$ be the graph of f. In general, $G(f)$ is not transverse to the diagonal Δ_M in $M \times M$. However, one can find a map $f' : M \to M$ homotopic to f such that the graph $G(f')$ is transverse to Δ_M. The (oriented) cycles $G(f')$ and Δ_M are transverse and complementary dimensional in $M \times M$. Moreover, there are finitely many intersection points $b_j \in G(f') \cap \Delta_M$. In such a point, the intersection number $I(G(f'), \Delta_M; b_j)$ is well defined (see Sect. 5.2.6) and one has

$$L(f) = \sum_{b_j} I(G(f'), \Delta_M; b_j). \tag{5.35}$$

That number does not depend on the map f' homotopic with f and such that $G(f')$ is transverse to Δ_M.

The main properties of the Lefschetz number are the following: If $L(f) \neq 0$, then f admits fixed points. If $f = \mathrm{id}_M$ then $L(f) = \chi(M)$. If f and g are two homotopic maps from M to M, then $L(f) = L(g)$.

The singular case

Goresky and MacPherson proved in [98] the Lefschetz fixed point theorem in the context of placid (Sect. 5.5.2 b) self maps of Witt spaces (see Sect. 5.4.6) and by using intersection homology with middle lower perversity.

The intersection homology Lefschetz number of a placid self-map $f : X \to X$ is defined by the formula [98, Sect. 4 Definition]:

$$IL(f) = \sum_{i=0}^{\dim X} (-1)^i \mathrm{Trace}(f_i : IH_i^{\bar{m}}(X; \mathbb{Z}) \to IH_i^{\bar{m}}(X; \mathbb{Z})). \tag{5.36}$$

In [98, Proposition 4.2], Goresky and MacPherson show that if $f : X \to Y$ is a placid map between two compact oriented $\mathbb{Q}$-Witt spaces, with $n = \dim X$, then the graph of f determines a canonical homology class $[G(f)] \in IH_n^{\bar{m}}(X \times Y; \mathbb{Q})$.

For a placid self map of a $\mathbb{Q}$-Witt space, both the graph of f and the diagonal carry fundamental classes in intersection homology of $X \times X$ and one has:

Theorem 5.5.23 ([98], Theorem I) *Let $f : X \to X$ be a placid self map of an n-dimensional $\mathbb{Q}$-Witt space. Let $[G(f)]$ and $[\Delta]$ be the homology classes of the graph of f and of the diagonal in $IH_n^{\bar{m}}(X \times X; \mathbb{Q})$. Then the Lefschetz number $IL(f)$ is given by*

$$IL(f) = [G(f)] \bullet [\Delta]$$

where $\bullet$ denotes the intersection product of cycles in intersection homology.

The formula 5.35 has been extended in the singular situation by Goresky and MacPherson (see [97],[98, Sects. 7–12]) in terms of local Lefschetz numbers of a placid map $f : X \to X$ at isolated fixed points.

Theorem 5.5.24 ([98, Theorem II]) *The intersection Lefschetz number is the sum of the local contributions taken over all the fixed points.*

Another way to define local Lefschetz numbers is developed by Bisi et al. [14] using Čech-de Rham theory. The coincidence of this later notion with Goresky and MacPherson ones is shown in Brasselet-Suwa [33].

The Coincidence Theorem

The smooth case

In [122] Lefschetz defined the coincidence number of two maps $f : M \to N$ and $g : M \to N$ where M and N are compact oriented smooth n-dimensional manifolds without boundaries. The coincidence set $C(f, g)$ is defined to be

$$C(f, g) = \{x \in M \mid f(x) = g(x)\}.$$

The Lefschetz coincidence number is defined as

$$L(f, g) = \sum_{k=0}^{n} (-1)^k \mathrm{Trace}(PD_M \circ g^{n-k} \circ PD_N^{-1} \circ f_k) \tag{5.37}$$

$$\begin{array}{ccc} H_k(M;\mathbb{Q}) & \xrightarrow{f_k} & H_k(N;\mathbb{Q}) \\ {\scriptstyle PD_M}\uparrow \cong & & {\scriptstyle PD_N}\uparrow \cong \\ H^{n-k}(M;\mathbb{Q}) & \xleftarrow{g^{n-k}} & H^{n-k}(N;\mathbb{Q}) \end{array}$$

where vertical arrows are Poincaré duality isomorphisms. If $L(f, g)$ is not zero, then there is at least one coincidence point: $C(f, g)$ is not empty.

The singular case

In the case of singular varieties, Goresky and MacPherson defined the notion of placid correspondences C between n-dimensional Witt spaces X and Y as beeing an n-dimensional compact oriented pseudomanifold $C \subset X \times Y$ such that each of the projections $\pi_X : C \to X$ and $\pi_Y : C \to Y$ is placid. According to the Proposition 5.5.19, one has homomorphisms on intersection homology:

$$(\pi_Y)_*(\pi_X)^* : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y) \quad \text{and} \quad (\pi_X)_*(\pi_Y)^* : IH_i^{\bar{m}}(Y) \to IH_i^{\bar{m}}(X).$$

If C_1 and C_2 are two correspondences between the Witt spaces X and Y, the Lefschetz number $IL(C_1, C_2)$ is defined to be the alternating sum of traces of the induced map

$$(\pi_X^2)_*(\pi_Y^2)^*\pi_Y^1)_*(\pi_X^1)^* : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(X).$$

Each correspondence defines a canonical intersection homology class

$$[C_i] \in IH_n^{\bar{m}}(X \times Y;\mathbb{Q})$$

and the Lefschetz number $IL(C_1, C_2)$ is equal to the intersection product $[C_1] \bullet [C_2]$ ([98, Theorem I']). Moreover, Goresky and MacPherson show ([98, Theorem II']) that it is equal to the sum of the local linking numbers suitably defined (see [98, Sect. 8]).

In the particular case of coincidences, given $f, g : X \to Y$ placid maps between n-dimensional oriented compact $\mathbb{Q}$-Witt spaces, the Lefschetz coincidence number is defined by [35]

$$IL(f, g) = \sum_i (-1)^i \mathrm{Trace}(g^i f_i),$$

where $f_i : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$ and $g^i : IH_i^{\bar{m}}(Y) \to IH_i^{\bar{m}}(X)$ are defined for the lower middle perversity $\bar{m}$ (Proposition 5.5.19).

Theorem 5.5.25 *[98, Sect. 14], [35] The Lefschetz coincidence number of (f, g) is determined by the intersection of the canonical homology classes of the graphs, $[G(f)]$ and $[G(g)]$.*

$$IL(f, g) = (-1)^n [G(f)] \bullet [G(g)].$$

If $IL(f, g) \neq 0$ then there is (at least one point) $x \in X$ such that $f(x) = g(x)$.

Examples of coincidence of maps are provided in [35] (J.-P. Brasselet, A.K.M. Libardi, T.F.M. Monis, E.C. Rizziolli and M.J. Saia) with local and global explicit computations.

5.5.4 Morse Theory

A complete history of Morse theory can be found, for instance in the Introduction of the Goresky-MacPherson's book [96], Sect. 1.7. A complete survey is given by Mark Goresky in the Chap. 5 of this Handbook (Vol. 1), see [90].

The Smooth Case

The main results of classical Morse theory for ordinary homology and for a compact smooth variety M can be summarized as follows [136]:

A critical point of a smooth function $f : M \to \mathbb{R}$ on a manifold M is a point where the differential of f vanishes, its image by f is a critical value. A non-degenerate critical point of f is a point for which the Hessian matrix of second partial derivatives of f is non-singular.

A smooth function $f : M \to \mathbb{R}$ on a manifold M is a Morse function if it has only non-degenerate critical points. According to a result by René Thom [178], the Morse functions form an open, dense subset of all smooth functions $f : M \to \mathbb{R}$ (for the $\mathcal{C}^2$- Whitney topology).

Considering a smooth function $f : M \to \mathbb{R}$ and a point $c \in \mathbb{R}$, let $M_{<c}$ denote the inverse image by f of the open interval $]-\infty, c[$.

The Morse Lemma says:

- For small enough ε, if the interval $]v-\varepsilon, v+\varepsilon[$ does not contain any critical value, then $M_{<v+\varepsilon}$ is homeomorphic to $M_{<v-\varepsilon}$.
- If p is a non-degenerate critical point of $f : M \to \mathbb{R}$, then there exists a chart $(x_1, x_2, \dots, x_n)$ in a neighborhood U_p of p such that $x_i(p) = 0$ for all i and $f(x) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$ in U_p. The integer k is the Morse index of f at p. For small enough ε, one has, with $v = f(p)$,

$$H_i(M_{<v+\varepsilon}, M_{<v-\varepsilon}) = \begin{cases} 0 & \text{for} \quad i \neq k \\ \mathbb{Z} & \text{for} \quad i = k \end{cases} \tag{5.38}$$

The Singular Case

In the case of a singular variety, there is no longer a Morse index for ordinary homology. Goresky and MacPherson [93, Sect. 4.5 (3)] provide a nice counter-example.

In fact, the concept of Morse function in the case of isolated singular varieties has been introduced by F. Lazzeri [119]. Some conditions for beeing a Morse function on a stratified space have been stated by Benedetti [12] and Pignoni [151] (see [96, Introduction, Sect. 1.4]).

Goresky and MacPherson assume that X is a purely n-dimensional complex analytic variety, endowed with a Whitney stratification (with complex analytic strata), and embedded in a complex analytic manifold M. In [96, Introduction, Sect. 1.4 What is a Morse function ?] A $\mathcal{C}^\infty$ function $f : M \to \mathbb{R}$ is called a Morse function for X provided

- For each stratum S of X, the function $f|_S$ has only nondegenerate critical points. The critical points of f are the critical points of $f|_S$ and the critical values of f are the values of f at these points.
- At each critical point $p \in X$, the differential $df(p)(\tau) \neq 0$ whenever τ is a limit of tangent planes from some larger stratum containing S in its closure.
- All critical values are distinct.

If p is a critical point in the stratum S, then the Morse index k of f at p is defined to be $c + \lambda$ where c is the complex codimension of S in X and λ is the classical Morse index of $f|_S$.

In order to recover Morse theory in the context of intersection homology, Goresky and MacPherson define the following ingredients [95, 96], see also [90].

The first one is the complex link of a stratum S. Choose a manifold N meeting S transversaly at p and a generic projection $\pi : N \cap X \to \mathbb{C}$ sending p to 0. For

$0 < \varepsilon \ll \delta \ll 1$ denote by $B(p, \delta)$ the ball of radius δ centered at p and $B_\delta = X \cap B(p, \delta)$, $\partial B_\delta = X \cap \partial B(p, \delta)$. The complex link $\mathcal{L}_\mathbb{C}$ of S is a pseudomanifold with:

$$\mathcal{L}_\mathbb{C} = \pi^{-1}(t) \cap B_\delta, \qquad \partial \mathcal{L}_\mathbb{C} = \pi^{-1}(t) \cap \partial B_\delta.$$

where $0 < |t| < \varepsilon$.

Denote by μ he monodromy transformation obtained by carrying a chain Z in $\mathcal{L}_\mathbb{C}$ with in $\partial \mathcal{L}_\mathbb{C}$, over a small loop around 0 in $\mathbb{C}$. The second ingredient is the Morse group A_p, image of the variation map (see [93, Sect. 3.7] and [96, Part II, Sect. 6.3]):

$$(1 - \mu) : IH_{c-1}(\mathcal{L}_\mathbb{C}, \partial \mathcal{L}_\mathbb{C}; \mathbb{Z}) \longrightarrow IH_{c-1}(\mathcal{L}_\mathbb{C}; \mathbb{Z})$$

where $(1 - \mu)$ vanishes on $IH_{c-1}(\partial \mathcal{L}_\mathbb{C})$.

Using intersection homology, Goresky and MacPherson recover Morse theory for a compact Whitney stratified singular complex analytic variety X analytically embedded in a smooth variety M, as follows:

Theorem 5.5.26 ([96]) *For an open dense set of Morse functions $f : M \to \mathbb{R}$ (in the sense of Lazzeri and Pignoni), all values $v \in \mathbb{R}$ have exactly one of the following properties (and only finitely many values have property 2):*

(1) For small enough ε, then $X_{<v+\varepsilon}$ is homeomorphic to $X_{<v-\varepsilon}$ in a stratum preserving way,
(2) There is a Morse index k of the critical point p with critical value v such that for small enough ε,

$$IH_i(X_{<v+\varepsilon}, X_{<v-\varepsilon}; \mathbb{Z}) = \begin{cases} 0 & \text{for} \quad i \neq k \\ A_p & \text{for} \quad i = k \end{cases} \tag{5.39}$$

In [93, 95, 96] Goresky and MacPherson provide various applications of stratified Morse theory :

- The Lefschetz hyperplane theorem holds for the intersection homology of a (singular) projective algebraic variety [93, Sect. 5.4].
- The intersection homology of a complex n-dimensional Stein space vanishes in dimensions $> n$ [93, Sect. 5.3].
- (3) The sheaf of intersection chains on a general fibre specializes (over a curve) to a perverse object on the special fibre [93, Sect. 6.1].

As the authors write on [93, Sect. 0.2], "other methods have been used to obtain some of these results ..., however the method of Morse theory has several advantages: it can be used to study homology with $\mathbb{Z}$ coefficients (as well as $\mathbb{Q}$ coefficients) and it applies to analytic (as well as algebraic) varieties."

5.5.5 *De Rham Theorems*

By relating differential geometry to topology, de Rham's theorem (1931) opened the door to "countless" new results, applications, conjectures, and many alternative proofs.

The passage from the smooth case to the singular case is due to Cheeger, Goresky and MacPherson. From "geometric" results, they mainly developed the theory within the framework of sheaves (see [88]). Several important conjectures have resulted in various fields.

Although implicit in the previous works, the explicit and geometric translation in terms of order of poles corresponding to the perversity was given in [28] (see Sect. 5.5.5).

This section is divided into four parts: de Rham's theorem in the smooth case, de Rham's theorem in the singular case, conjectures and applications, geometric translation. In this section, all intersection homology groups are written with the middle perversity the notation of which is omitted.

The Smooth Case

The de Rham Theorem (de Rham thesis [160]) provides a very useful relationship between the topology and the differentiable structure of a PL-manifold. The de Rham complex is the complex of smooth differential forms on a manifold M with exterior derivative as the differential:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

The de Rham Theorem says that the cohomology $H^j_{dR}(M)$ of the de Rham complex is isomorphic to the PL-cohomology $H^j(M; \mathbb{R})$. There are many proofs in the literature. The Whitney's book "Geometric integration theory" provides a nice geometric proof of the Theorem [186, Chap. IV, Theorem 29A].

Let M be a Riemannian (compact) oriented manifold endowed with a metric g. The metric induces an inner product on fibers $T^*_x(M)$ of the cotangent bundle and then an $\mathcal{L}^2$-metric on $\Omega^j(M) = \Gamma(\Lambda^j(T^*(M)))$. Let $\delta : \Omega^j(M) \to \Omega^{j-1}(M)$ be the formal adjoint of d relatively to the inner product and $* : \Omega^j(M) \to \Omega^{n-j}(M)$ the Hodge star operator [20]. The Hodge Theorem, first proved by Hodge (1933–1936) with final proof by Hermann Weyl and Kunihiko Kodaira, says that every de Rham cohomology class is represented by a unique harmonic form, i.e. a differential form of which the Laplacian Δ is zero:

$$\Delta(\omega) = (d\delta + \delta d)(\omega) = 0.$$

A compact complex projective manifold is a Kähler manifold. The cohomology groups admit a decomposition (pure Hodge decomposition)

$$H^r(M;\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M;\mathbb{C}),$$

as direct sum of complex vector spaces and $\overline{H^{p,q}(M;\mathbb{C})} = H^{q,p}(M;\mathbb{C})$. The (p,q) components of a harmonic form are again harmonic.

The complex of sheaves is exact and is a soft resolution of the constant sheaf $\mathbb{R}_M$. It follows that the sheaf cohomology is the singular cohomology with $\mathbb{R}$ coefficients.

The Sheaf of $\mathcal{L}^2$ Differential Forms

In order to extend the theory to singular varieties, one considers a pseudomanifold X whose regular part $X^0 = X \setminus \Sigma$ is a smooth (possibly incomplete) Riemannian manifold with a metric g ([60], Sect. 3).

One define a presheaf on X by assigning by assigning to each open set $U \subset X$ the subset $\Gamma(U, \Omega^i_{(2)})$ in $\Omega^i(U \cap X^0)$ of differential forms ω such that for any point x in U, there is a neighbourhood V of x in U such that

$$\int_{V\cap X^0} \omega \wedge *\omega < \infty \qquad \text{and} \qquad \int_{V\cap X^0} d\omega \wedge *d\omega < \infty$$

This presheaf is filtered by differential form degree and the exterior derivative makes it into a complex of presheaves. The associated sheaf complex, obtained by "sheafification" is the sheaf of $\mathcal{L}^2$ differential forms denoted by $\Omega^*_{(2)}$. It is a complex of fine sheaves whose cohomology is denoted by $H^*_{(2)}(X)$.

The definition makes sense in the case of a local system $\mathcal{L}$ on X^0 provided that $\mathcal{L}$ has a smoothly varying positive definite inner product on each fiber. The restriction of the sheaf to X^0 is the sheaf of all smooth differential forms (with arbitrary growth) on X^0 (and coefficients in $\mathcal{L}$).

The Cheeger-Goresky-MacPherson's Conjecture

The study of $\mathcal{L}^2$ cohomology on the non-singular part of a variety with conical singularities was initiated by Cheeger in the context of the study of analytic torsion. In 1976 Sullivan observes similarity between IH and $\mathcal{L}^2$, namely similarity between local results in Proposition 5.4.12 and forthcoming Lemma 5.5.30 The same year, Deligne proposes to consider variation of Hodge structure on intersection homology [60, p. 308] (see Sect. 5.5.5).

These observations led Cheeger, Goresky and MacPherson's to the famous CGM conjecture which concerns complex n-dimensional projective varieties and is in fact made up of three conjectures [60, Sect. 4].

The conjecture are written for the middle perversity (here the two middle perversities agree) which will be omitted.

Conjecture 5.5.27 states that the intersection homology groups $IH_*(X)$ satisfy the following 5 conditions of the "*Kähler package*". That is :

1. Poincaré duality (see Sect. 5.5.8). The intersection pairing

$$IH_i(X) \times IH_{2n-i}(X) \to \mathbb{C} \tag{5.40}$$

is non singular for all i.

2. Pure Hodge decomposition. There is a natural direct sum Hodge decomposition

$$IH_r(X) \cong \oplus_{p+q=r} IH_{p,q}(X)$$

such that

$$IH_{p,q}(X) \cong \overline{IH_{q,p}(X)}.$$

The decomposition is compatible with maps If_* and If_* when they exist, for example if $f : Y \to X$ is normally nonsingular with relative dimension m then

$$If_* : IH_{p,q}(Y) \to IH_{p,q}(X) \qquad \text{and} \qquad If^* : IH_{p,q}(X) \to IH_{p-m,q-m}(Y).$$

The map from cohomology $H^i(X) \to IH_{2n-i}(X)$ is a morphism of Hodge structures.

3. Hard Lefschetz. Let H be a hyperplane in the ambient projective space, which is transverse to a Whitney stratification of X. Let $N \in H^2(X)$ denote the cohomology class represented by $H \cap X$ and let $L : IH_i(X) \to IH_{i-2}(X)$ denote multiplication by this class. then the map

$$L^k : IH_{n+k}(X) \to IH_{n-k}(X)$$

is an isomorphism for each k.

Let define $P_{n+k}(X) = \ker(L^{k+1})$, then the Lefschetz decomposition

$$IH_m(X) = \oplus_k L^k(P_{m+2k}(X))$$

is compatible with the Hodge decomposition.

4. Lefschetz Hyperplane Theorem, Let H be a hyperplane in the ambient projective space, which is transverse to a Whitney stratification of X. The homomorphism induced by inclusion

$$IH_k(X \cap H; \mathbb{Z}) \to IH_k(X; \mathbb{Z})$$

is an isomorphism for $k < n - 1$ and a surjection for $k = n - 1$. (for instance see [72, 102]).

5. Hodge Signature Theorem. If $\sigma(X)$ denotes the signature of the intersection pairing (5.40) on $IH_n(X)$, then

$$\sigma(X) = \sum_{p+q \equiv 0 (\mathrm{mod} 2)} (-1)^p \dim IH_{(p,q)}(X).$$

As written in [60], Conjecture 5.5.27 follows from the stronger following Conjectures 5.5.28 and 5.5.29.

Conjecture 5.5.28 The $\mathcal{L}^2$ cohomology group $H^k_{(2)}(X)$ is finite dimensional and is isomorphic to the subspace $\mathcal{H}^k$ of $\Omega^k \cap \mathcal{L}^2$ which consists of the square summable differential k-forms which are closed and co-closed $d\omega = \delta\omega = 0$. Furthermore, the operator "integration" preserves this subspace $\mathcal{H}^k$.

Conjecture 5.5.29 For almost any chain $\xi \in C_k(X)$ and almost any differential form $\theta \in \mathcal{H}^k$, the integral $\int_\xi \theta$ is finite and $\int_{\partial\eta} \theta = \int_\eta d\theta$ whenever both sides are defined. The induced homomorphism

$$H^j_{(2)}(X) \xrightarrow{\int} \mathrm{Hom}(IH^{\bar{m}}_j(X); \mathbb{C})$$

is an isomorphism.

Cheeger, Goresky and MacPherson conjectured that each class contains an unique harmonic (closed and co-closed) representative and that splitting the harmonic forms into their (p, q)–pieces yields a (pure) Hodge decomposition, compatible with Deligne's mixed Hodge structure on the ordinary cohomology groups of X. They noted that the Hodge decomposition would exist if the metric on U were complete, and they suggested that another approach to constructing a Hodge decomposition of $IH^*(X)$ is to construct a complete (Kähler) metric. Moreover, they gave a lot of evidence for the validity of the conjectures. This fundamental work of Cheeger, Goresky, and MacPherson has lead to a great deal of work by many people.

Poincaré Lemma for $\mathcal{L}^2$-cohomology

Let L be an $n-1$-dimensional Riemannian (compact) manifold endowed with a metric g_L. For $h > 0$, the metric cone on L, denoted by $c^h(L)$, is the completion of the incomplete Riemannian manifold $L \times [0, \infty[$ endowed with the metric $g = dr \otimes dr + r^{2h} g_L$.

As before, $\Omega^\bullet_{(2)}(c^h L)$ denotes the subset of differential forms $\omega \in \Omega^\bullet(c^h(L) \setminus \{0\})$ such that

$$\int_{c^h(L)\setminus\{0\}} \omega \wedge *\omega < \infty \qquad \text{and} \qquad \int_{c^h(L)\setminus\{0\}} d\omega \wedge *d\omega < \infty$$

where $d : \Omega^i \to \Omega^{i+1}$ is induced by the external derivative and the operator $*$ is the Hodge operator [20, 58, 59].

The $\mathcal{L}^2$-cohomology groups of the cone $c(L)$, denoted by $H^j_{(2)}(c^h(L))$ are cohomology groups of the complex $\Omega^\bullet_{(2)}(c^h(L))$.

Lemma 5.5.30 ([59, Lemma 3.4]) *The $\mathcal{L}^2$-cohomology groups of the cone $c(L)$ satisfy*

$$H^j_{(2)}(c^h(L)) = \begin{cases} H^j_{\mathrm{dR}}(L) & \text{if } j < \dfrac{n-1}{2} + \dfrac{1}{2h} \\ 0 & \text{if } j \geq \dfrac{n-1}{2} + \dfrac{1}{2h}. \end{cases}$$

Varieties with isolated conical singularities

Let X be a singular variety whose singularities are isolated points $\{a_i\}$ admitting each one a neighborhood U_i in X, which is isometric to the (open) metric cone $\mathring{c}(L_i)$ whose basis is a smooth manifold L_i. In the particular case n is even, $h = 1$ and $\bar{p}$ is the middle perversity $\bar{m}$, then $\bar{p}(n) = \frac{n}{2} - 1$, and one has:

$$n - 1 - p_n = \frac{n-1}{2} + \frac{1}{2h} = \frac{n}{2}$$

Cheeger, Goresky and MacPherson study the two following cases:

(1) [59, Theorem 6.1] and [60, Sect. 3.4]. Let X be a pseudomanifold embedded as PL-subvariety in $\mathbb{R}^N$ and let Σ the singular subset in X. There is, on $X^0 = X \setminus \Sigma$, a metric $\tilde{g}$ which endows the manifold $X \setminus \Sigma$ of a structure of flat Riemannian manifold, i.e. every point x in the $n-1$-skeleton admits a neighborhood U_x isometric to an open subset in $\mathbb{R}^N$.

(2) [59, Sect. 3.5], [60, Sect. 3.5]. X is a compact analytic variety embedded in a Kählerian manifold. Then $X \setminus \Sigma$ is endowed with the metric induced by restriction of the Kählerian metric. One assume that Σ is locally analytically conical, that means the following:

A variety X is locally analytically conical if each point $p \in X$ has a neighborhood U and an analytic embedding $\rho : U \to \mathbb{C}^N$ such that $\rho(U)$ is a cone at $\rho(p)$ (see [60, Sect. 3.5 Definition and Examples]).

Theorem 5.5.31 ([60]) *In the two previous cases, the integration map induces an isomorphism:*

$$H^j_{(2)}(X) \xrightarrow{\int} \mathrm{Hom}(IH^{\bar{p}}_j(X); \mathbf{R}) \tag{5.41}$$

The idea is to prove that the direct image of the presheaf on U formed of the appropriate $\mathcal{L}^2$-forms of degree i has a "fine" associated sheaf and that, as i varies, those associated sheaves form a (de Rham) complex that satisfies the axioms that characterize $\mathcal{IC}^*(X)$; the cohomology groups of the complex are equal to its hyper-cohomology groups because the sheaves are fine.

Other proofs of CGM conjectures

The conjectures of Cheeger, Goresky and MacPherson were also treated with some success in the case that X^0 is the smooth part of a complex projective variety X with isolated singularities.

Let X be a normal singular algebraic surface (over $\mathbb{C}$) embedded in the projective space $\mathbb{P}^N(\mathbb{C})$ and let Σ be its singularity set, which consists of isolated singular points. Restricting the Fubini-Study metric of $\mathbb{P}^N(\mathbb{C})$ to $X^0 = X \setminus \Sigma$, provides an incomplete Riemannian manifold (X^0, g). Wu-Chung Hsiang and Vishwambhar Pati proved in [107] that the $\mathcal{L}^2$-cohomology $H^i_{(2)}(X^0)$ is naturally isomorphic to the dual of the middle intersection homology $IH^{\overline{m}}_i(X)$. However their proof has a certain gap corrected by Nagase [139] (see also [137, 138]). The "non- normal" case can be proved in the same way by making its normalization, as asserted in [107].

Saper [163, 164] who was inspired by the case of the Zucker conjecture (Sect. 5.5.5), constructed a complete Kähler metric on X^0 whose $\mathcal{L}^2$-cohomology groups are dual to the intersection homology groups of X.

Finally, Ohsawa [147, 148] proved the conjecture in dimension $\dim X \leq 2$: If X in $\mathbb{P}^n(\mathbb{C})$ is a projective variety of dimension $\dim X \leq 2$, then the $\mathcal{L}^2$ de Rham cohomology groups of the regular part X^0, with respect to the Fubini-Study metric are canonically isomorphic to the intersection cohomology groups of X.

The Deligne Conjecture: Variation of Hodge Structures

Over a compact Kähler manifold X, Deligne (unpublished manuscript, see [188]) has constructed canonical Hodge structures on the cohomology groups $H^p(X, \mathcal{L})$, of weight $p + k$. When the basis X is non-compact, Deligne's arguments still put Hodge structures on the $\mathcal{L}^2$ cohomology groups of the completion $\overline{X}$ provided they are finite dimensional.

Let X be a nonsingular algebraic variety and D is a divisor with normal crossings in X, which may be interpreted as giving a stratification of X whose largest stratum is $X \setminus D$. The considered local system is underlying a polarizable variation of Hodge structure.

A variation of Hodge structure, considered as local system $\mathcal{L}$ on $X \setminus D$, has an $\mathcal{IC}$ extension to all of X. Deligne conjectures that $\mathcal{IC}(X; \mathcal{L})$ is isomorphic to the sheaf of $\mathcal{L}^2$ differential forms on X, where the Riemannian metric on $X \setminus D$, is the *complete* metric that is hyperbolic near each codimension 1 divisor.

In the case of one dimensional base, Zucker [188] has obtained a natural identification

$$H^*_{(2)}(\overline{X}, \mathcal{L}) \cong H^*(\overline{X}, i_*\mathcal{L})$$

here $i_*\mathcal{L}$ is the direct image of $\mathcal{L}$ on $\overline{X}$. The $\mathcal{L}^2$ cohomology groups are then finite dimensional and come equipped with Hodge structures.

Cattani et al. [51], and independently, Kashiwara and Kawai [111] proved the Deligne conjecture, for higher dimensions:

Theorem 5.5.32 *The complex of sheaves $\mathcal{L}^2$ differential forms on X satisfies the axioms of middle intersection cohomology sheaf with values in the local system $\mathcal{L}$. In particular*

$$H^*_{(2)}(\overline{X}, \mathcal{L}) \cong IH^*(\overline{X}, \mathcal{L})$$

Corollary 5.5.33 *The intersection cohomology groups* $IH^*(\overline{X}, \mathcal{L})$ *carry canonical (pure) Hodge structures of weight* $p + k$.

The Zucker Conjecture

Zucker was aware of the work of Cheeger, Goresky, and MacPherson that appears in [59, 60] when he made the following conjecture, which first appeared in a 1980 preprint ([189]):

Conjecture 5.5.34 Let X be the Satake, Baily-Borel compactification of the quotient space U of a Hermitian symmetric domain modulo a proper action of an arithmetic group Γ. Let U be provided with the natural complete metric, then the sheaf of $\mathcal{L}^2$-differential forms on X with coefficients in a metrized local system $\mathcal{L}$ on U is isomorphic (in the derived category) to the sheaf $\mathcal{IC}(X; \mathcal{L})$ (see Introduction in [123]).

Zucker was led to this conjecture by some examples that he worked out [188, Sect. 6] of his general results [189, (3.20) and (5.6)] about the $\mathcal{L}^2$-cohomology groups of an arithmetic quotient of a symmetric space. In the examples, the compactification is obtained by adjoining a finite number of isolated singular points, and Zucker was struck by the values of the local $\mathcal{L}^2$-cohomology groups at these points: they are equal to the singular cohomology groups of the link in the bottom half dimensions and to 0 in the middle and in the top half dimensions (compare with Lemma 5.5.30).

Borel [15], Borel and Casselman [16] proved the Zucker conjecture in the particular case of a group of $\mathbb{Q}$-rank one or two (see also [50]).

The conjecture has been fully proved by Looijenga [123], Saper and Stern [165]. Looijenga uses Mumford's (1975) desingularization of X and the decomposition theorem. Saper and Stern use a more direct method, which they feel will also yield a generalization of a conjecture due to Borel (see [165]).

One reason for the great interest in Zucker's conjecture is that it makes it possible to extend the "Langlands program" to cover the important non compact case, as Zucker indicates in [190].

Other Related Results

There is a lot of results related to the previous ones. The intersted reader may consult Nagase [139], Saper [163], Pardon and Stern [150], etc.

The Geometric Viewpoint

In this section, all the homology and cohomology groups will be with real coefficients.

Shadow forms

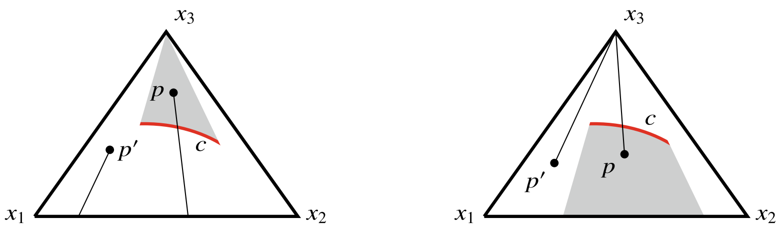
The shadow forms have been defined by Brasselet, Goresky and MacPherson [28].

The idea is to associate a differential form $\omega(\xi)$ to simplices of a barycentric subdivision K' of a given triangulation K, so that there is a clear relationship between the defect of transversality of the simplices relatively to the simplices σ of K and the order of the pole of the corresponding differential form on σ.

Various equivalent definitions of the shadow forms are provided in [28]. One of them goes as follows: Let $\Delta = \Delta^n$ be the standard n-simplex.

$$\Delta = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | \quad 0 \le x_i \le 1, \sum x_I = 1\}.$$

The shadow forms are defined for k-simplices ξ of the barycentric subdivision Δ' which do not lie in the boundary of Δ. Such a barycentric subdivision can be defined for each point p in the interior of Δ, requiring that for each pair $F' < F$ of faces of Δ, the barycenters of F, F' and of the face opposite to F' in F are collinear. The corresponding barycentric subdivision of Δ will be denoted $\Delta'(p)$. Every k-simplex ξ admits a geometrical realization $\xi(p)$ in this subdivision.



Vertices of ξ are barycenters of (x_1x_2) and of Δ Vertices of ξ are x_3 and the barycenter of Δ

Fig. 5.7 The shadow is the dotted area. The point p is in the shadow $S_\xi(c)$ but p' is not

Let c be a singular chain in the interior of Δ, the shadow $S_\xi(c)$ cast by an $(n-k)$-chain c with respect to ξ is the set of all points p such that $\xi(p)$ intersects c (Fig. 5.7).

Definition 5.5.35 The shadow form $\omega(\xi)$ is the unique differential form such that the value of its integral over any $(n-k)$-chain c is the volume of the shadow $S_\xi(c)$:

$$\int_c \omega(\xi) = \text{ volume } (S_\xi(c)).$$

An explicit equivalent definition goes as follows: denote by D_ξ the incidence variety

$$D_\xi = \{(p, x) \in int(\Delta) \times int(\Delta) : x \in \xi(p)\}.$$

Let i be the inclusion $i : D_\xi \hookrightarrow \Delta \times \Delta$ and let π_1 and π_2 be the projections on the first and second factors of $\Delta \times \Delta$.

$$\begin{array}{ccccc} D_\xi & \overset{i}{\hookrightarrow} & \Delta \times \Delta & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ \Delta & & & & \Delta \end{array}$$

If $(x_1, \dots, x_{n+1})$ are the barycentric coordinates of Δ^n, the Whitney form $W(\Delta^n)$ is the volume form of Δ^n,

$$W(\Delta^n) = W(x_1, \dots, x_{n+1}) = n! \sum_{i=1}^{n+1} (-1)^{i+1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Proposition 5.5.36 *The shadow form $\omega(\xi)$ is the $(n-k)$-differential form defined by*

$$\omega(\xi) = \int_{\pi_2} i^* \pi_1^* (W(\Delta^n))$$

where $\int_{\pi_2}$ denotes integration along the fibres of π_2 (see [20]).

The differential form $\omega(\xi)$ is C^∞ on $int(\Delta)$. Indeed D_ξ is a smooth manifold and the fibres $\pi_2^{-1}(x) \cap D_\xi$ are relatively compact.

Generalizing the definition to polyhedra provides:

Theorem 5.5.37 ([28, Corollary 9.3]) *Let X be a polyhedron in the Euclidean space $\mathbf{R}^n$. Fix q, $1 \leq q \leq \infty$, and denote by $\bar{p}(q)$ the highest perversity whose graph is situated strictly below the line from origin and with slope $1/q$. Then the intersection homology of X, for the perversity $\bar{p}(q)$, is isomorphic to $\mathcal{L}^q$-cohomology of X:*

$$IH_k^{\bar{p}(q)}(X) \cong H_{(q)}^{n-k}(X).$$

Conjecture 5.5.38 (Brasselet et al. [28]) Let X be a stratified space with a Riemannian metric and conical singularities. Let Σ be the singular set, $q \geq 2$, and $\mathcal{L}^q$-cohomology of $X \setminus \Sigma$ is finite dimensional, then it is isomorphic to intersection cohomology of X.

The conjecture has been proved by Youssin [187] who also extends the result to spaces with horn-singularities.

Belkacem Bendifallah [11] provided an explicit formula for the coefficients of shadow forms as integrals of Dirichlet type, obtaining an alternative proof of Theorem 5.5.37. He gave a duality formula and a product formula for shadow forms and constructed the correct underlying algebraic structure.

The Brasselet-Legrand approach.

J.-P. Brasselet and A. Legrand consider the situation of an n-dimensional pseudovariety X endowed with a Thom-Mather stratification, and whose strata are smooth manifolds.

The idea is to prove a de Rham type theorem by considering a complex of differential forms whose coefficients are $\mathcal{C}^\infty$ functions on the regular part of X which may have poles on the singular strata but whose behavior in a neighborhood of the strata is controlled. The control is performed through two parameters, associated with each stratum $S_{n-\alpha}$. The first control β_α corresponds to an admissible maximum order of poles of the functions on the stratum, the second c_α is related to the local conical metric in the neighborhood of the stratum. An admissible differential form can have a pole on a stratum, but the the order of the poles should not be too large for the Poincaré lemma to be verified to some degree. Also, the quotients $[\beta_\alpha/c_\alpha]$ should satisfy the same inequalities than the GM-perversities (see formula 5.14).

The obtained complex $\Omega^\bullet_{\beta,c}$ is a complex of soft sheaves satisfying axioms $[AX1]_{\bar{p}}$ with

$$p_\alpha = \alpha - 2 - \left[\frac{\beta_\alpha}{c_\alpha}\right]$$

whose hypercohomology is intersection homology for the complementary perversity.

On the one hand the complex $\Omega^\bullet_{\beta,c}$ is a generalization of the complex of shadow forms.

On the other hand, it allows to define a suitable algebra in order to generalize the Hochschild-Kostant-Rosenberg theorem to the case of singular varieties, more precisely to manifolds with boundary and to varieties with isolated singularities. The classical result of Hochschild et al. [105] asserts that the Hochschild homology of a finitely generated, smooth complex algebra A equals the space of Kähler differentials over A. In 1982, Connes [62] extended this result in a topological setting and in [31] the authors generalize the Connes's idea to the case of singular varieties with isolated singularities.

The relation between the defects of transversality (perversity) of a cycle and the order of the poles of the associated differential form are explicit in the context of shadow forms and the context of the complex $\Omega^\bullet_{\beta,c}$. The smallest is the dimension of the stratum, the greater the admissible order of the poles of the differential forms. The physicist Alain Connes (private conversation) says that "there is a higher concentration of energy in the smaller singular strata".

The Goresky-MacPherson's complex $\Omega^\bullet_{\bar{q}}$.
The Goresky-MacPherson complex has been described by Brylinski [46] (for an interpretation in terms of sheaf defined on the resolution of the stratified space see [1, Sect. 6.5]).

Let $\pi : M \to B$ be a smooth fibration of smooth manifolds. A filtration (Cartan's filtration) of the de Rham complex $\Omega^\bullet_M$ is defined as follows :

Definition 5.5.39 For $k \geq 0$, $F_k\Omega^\bullet_M$ is the sub-complex of $\Omega^\bullet_M$ consisting of the differential forms ω such that ω and $d\omega$ satisfy: if $\xi_1, \xi_2, \ldots, \xi_{k+1}$ are $k+1$ vector fields on M, tangent to the fibres of π, then $i(\xi_1) \circ \cdots \circ i(\xi_{k+1})(\omega) = 0$.

Let X be a pseudomanifold with $\mathcal{C}^\infty$-structure, equipped with a Thom-Mather stratification (see [179, 4.2.17]). If $S_i \subset \bar{S}_j$, $(\pi_i)|_{T_i \cap S_j}$ is $\mathcal{C}^\infty$ and $\pi_i \circ \pi_j = \pi_i$ on $T_i \cap T_j$. One denotes by X^0 the smooth stratum of X.

Definition 5.5.40 Let $\bar{q}$ be a perversity, denote by $\Omega^\bullet_{\bar{q}}$ the sub-complex of $\Omega^\bullet_{X^0}$ consisting of the differential forms ω such that every point of $S_{n-\alpha}$ admits a neighborhood $V \subset T_{n-\alpha}$ on which the restriction of ω is in $F_{q(\alpha)}\Omega^\bullet_{V\cap X^0}$, relative to the projection $V \cap X^0 \to S_{n-\alpha}$ induced by $\pi_{n-\alpha}$.

This means that, near $S_{n-\alpha}$, ω satisfies $i(\xi_1) \circ \cdots \circ i(\xi_{q(\alpha)+1})\omega = 0$ if the ξ_i are vector fields defined on X^0 and are tangent to the fibers of $\pi_{n-\alpha}$.

Proposition 5.5.41 *[46, Proposition 1.2.6] The complex of sheaves $\Omega^\bullet_{\bar{q}}$ satisfies $(AX_1)_{\bar{q}}$.*

As a corollary, the hypercohomology groups of the complex of sheaves $\Omega^\bullet_{\bar{q}}$ are isomorphic to $\mathrm{Hom}(IH^{\bar{p}}_j(X); \mathbb{R})$, where $\bar{p}$ and $\bar{q}$ are complementary perversities. See also the survey by Pollini [156].

$\mathcal{L}^\infty$-cohomology.
Let X be a subanalytic compact pseudomanifold. In [182] Valette shows a de Rham theorem for $\mathcal{L}^\infty$-cohomology forms on the nonsingular part of X. The obtained cohomology is isomorphic to the intersection cohomology of X for the top perversity. There is a Lefschetz duality theorem relating the $\mathcal{L}^\infty$-cohomology to the so-called Dirichlet $\mathcal{L}^1$-cohomology. As a corollary, the Dirichlet $\mathcal{L}^1$-cohomology is isomorphic to intersection cohomology in the zero perversity.

Morse functions.
Let X be a space with isolated conical singularities. In [124] U. Ludwig establishes, using anti-radial Morse functions on X, a combinatorial complex which computes the intersection homology of X. The complex constructed is generated by the smooth critical points of the Morse function and representatives of the de Rham cohomology (in low degree) of the link manifolds of the singularities of X. It can be seen as an analogue of the famous Thom-Smale complex for smooth Morse functions and singular homology on a compact manifold.

5.5.6 Steenrod Squares, Cobordism and Wu Classes

In this section, the coefficients are the mod 2 integers $\mathbb{Z}_2$.

Steenrod Squares and Wu Classes

Goresky and Pardon [99] define four classes of singular spaces for which they define various characteristic numbers and for which these characteristic numbers determine the cobordism groups. In the four cases, they construct characteristic numbers by lifting Wu classes to intersection homology. Then they can multiply them.

In the singular case, the mod 2 *Steenrod square operations* have been defined in intersection cohomology by Goresky in [87] (see also [99, Sect. 4]), as operations

$$Sq^i : IH^j_{\bar{c}}(X) \to IH^{i+j}_{2\bar{c}}(X)$$

for perversities $\bar{c}$ such that $2\bar{c} \leq \bar{t}$. Via Poincaré duality one has similar operations in intersection homology (with compact supports).

Definition 5.5.42 ([99, Sect. 5.1]) Let X be an n-dimensional pseudomanifold. Assume $\bar{c}$ is a perversity such that $2\bar{c} \leq \bar{t}$. Let $\bar{b} = \bar{t} - \bar{c}$ be the complementary perversity. For any i with $0 \leq i \leq [n/2]$ the Steenrod square operation

$$Sq^i : IH^{\bar{c}}_i(X) \to IH^{2\bar{c}}_0(X) \to \mathbb{Z}_2$$

is given by multiplication with the intersection cohomology i^{th}-Wu class of X:

$$v^i(X) = v^i_{\bar{b}}(X) \in IH^i_{\bar{b}}(X).$$

One defines $v^i(X) = 0$, for $i > [n/2]$.

If X is a $\mathbb{Z}_2$-Witt space (see Sect. 5.4.6), then the middle intersection homology group is self-dual, i.e., satisfies the Poincaré duality over $\mathbb{Z}_2$. Also the natural homomorphism

$$IH^i_{\bar{m}}(X) \to IH^i_{\bar{n}}(X)$$

is an isomorphism.

Definition 5.5.43 ([99, Sect. 8.1]) A stratified pseudomanifold X is locally orientable if, for each stratum, the link is an orientable pseudomanifold. A stratified pseudomanifold X is a locally orientable Witt space if it is both locally orientable and a $\mathbb{Z}_2$-Witt space.

In the situation of a locally orientable Witt space, the Wu classes which are defined to be middle intersection homology classes, can be multiplied to construct characteristic Wu numbers

$$\varepsilon(v_i(X) \cdot v_j(X)) = \langle v^{n-i}(X) \cup v^{n-j}(X), [X] \rangle \in \mathbb{Z}_2$$

where $i + j = n$. The map $\varepsilon : H_0(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the augmentation and the following diagram commutes:

$$\begin{array}{ccccc} IH_i^{\bar{m}}(X) \times IH_j^{\bar{m}}(X) & \xrightarrow{\cdot} & IH_0^{\bar{t}}(X) & \xrightarrow{\varepsilon} & \mathbb{Z}_2 \\ \cong\times\cong \uparrow & & \cong \uparrow & & \\ IH_{\bar{m}}^{n-i}(X) \times IH_{\bar{m}}^{n-j}(X) & \xrightarrow{\cup} & IH_{\bar{0}}^{n}(X). & & \end{array}$$

Theorem 5.5.44 ([99, Theorem 10.5]) *A locally orientable Witt space X of dimension n is a boundary of a locally orientable Witt space Y if and only if each of the characteristic Wu numbers*

$$v^{ij}(X) = \varepsilon(v^i(X)v^j(X)v^1(X)^{n-i-j}) \in \mathbb{Z}_2$$

vanish, where $\varepsilon : H_0(X; \mathbb{Z}_2) \to \mathbb{Z}_2$ denotes the augmentation.

Here, the class v^1 is a cohomology class and $v^i v^j$ is a (intersection) homology class, so the product is a well defined cobordism invariant.

In [99] M. Goresky and W. Pardon provide further important results concerning cobordism of singular spaces (see also [56]).

Cobordism of Maps in the Singular Case

Generalizing the results of R. Stong in the smooth case, J.-P. Brasselet, A. Libardi, E. Rizziolli and M. Saia define the cobordism of maps in the following way:

Definition 5.5.45 ([34]) Let $f : X \to Y$ be a map between pseudomanifolds of dimensions m and n respectively. The triple (f, X, Y) is null-cobordant if there exist:

1. pseudomanifolds V and W with dimensions $m+1$ and $n+1$, respectively, and $\partial V = X$ and $\partial W = Y$.
2. a map $F : V \to W$ such that the following diagram commutes.

$$\begin{array}{ccc} U_X & \xrightarrow{F_{|U_X}} & U_Y \\ \cong \downarrow \phi & & \psi \downarrow \cong \\ \partial V \times [0,1) & \xrightarrow{f \times Id} & \partial W \times [0,1), \end{array}$$

   where U_X and U_Y are collared neighborhoods of X and Y in V and W respectively, and ϕ and ψ are PL-diffeomorphisms such that $\phi(x) = (x, 0)$, $x \in \partial V$ and $\psi(y) = (y, 0)$, $y \in \partial W$.
3. $F_{|\partial V} = f : \partial V \to \partial W$.

Let $f : X \longrightarrow Y$ be a map, with X a compact locally orientable Witt space of pure dimension m and Y a closed n-dimensional smooth manifold. Then the map $f_! : IH_i^{\bar{p}}(X) \to IH_i^{\bar{p}}(Y)$ is defined in such a way that the following diagram commutes

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow \omega_X & & \omega_Y \uparrow \simeq \\ IH_i^{\bar{p}}(X) & \xrightarrow{f_!} & IH_i^{\bar{p}}(Y) \end{array}$$

i.e. $f_! = (\omega_Y)^{-1} \circ f_* \circ \omega_X$, where the map ω_Y is an isomorphism since Y is smooth.

Denote by $\widetilde{f_!}$ the composition map $\widetilde{f_!} = \alpha_Y^{-1} \circ f_!$, i.e. the composition map

$$IH_i^{\bar{p}}(X) \xrightarrow{\omega_X} H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{P_Y^{-1}} H^{n-i}(Y)$$

where the last arrow denotes the inverse Poincaré isomorphism.

Theorem 5.5.46 ([34]) *Let X be a compact locally orientable Witt space of pure dimension m and Y a closed n-dimensional smooth manifold. Given a map $f : X \longrightarrow Y$, if the triple (f, X, Y) is null-cobordant, with $(f, X, Y) = \partial(F, V, W)$ and W is a smooth manifold, then for any partition ℓ and r numbers $u_1, \ldots, u_r$ satisfying $u_i \leq [m/2]$ for all i and $(\ell_1 + \ell_2 + \ldots \ell_s) + u_1 + \ldots + u_r + r(m-n) = n$, the Stiefel-Whitney–Wu numbers*

$$\langle w_\ell(Y).\widetilde{f_!}(v_{m-u_1}(X)). \cdots .\widetilde{f_!}(v_{m-u_r}(X)), [Y]\rangle$$

are zero.

Let $f : X \to Y$ be a proper and normally nonsingular map of pseudomanifolds, there is an unique Gysin map

$$If_i : IH_i^{\bar{m}}(X) \to IH_i^{\bar{m}}(Y)$$

such that the following diagram commutes (Theorem 5.5.18, see [94, Sect. 5.4.3]).

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow \omega_X & & \omega_Y \uparrow \\ IH_i^{\bar{p}}(X) & \xrightarrow{If_i} & IH_i^{\bar{p}}(Y). \end{array} \tag{5.42}$$

The same result holds for placid maps as well (Proposition 5.5.19, see [94] and [5, Proposition 3.2]).

Theorem 5.5.47 ([34]) *Let $f : X \longrightarrow Y$ be a normally nonsingular (or placid) map, with X and Y compact locally orientable Witt spaces of pure dimension m and n respectively. If (f, X, Y) is null-cobordant, then for any u with $0 \leq u \leq n$, the following Wu numbers vanish:*

$$\langle v_{n-u}(Y).If_i(v_u(X)), [Y]\rangle = 0.$$

5.6 Supplement: More Applications and Developments

5.6.1 Toric Varieties

Max Brückner (for the octatope) [43, 44], Max Dehn (in 1905, for dimensions 4 and 5) [65] and Duncan Sommerville [170] (in 1927, in all dimensions) proved certain relations involving numbers of faces for simplicial polytopes.

Let P be an n-dimensional simplicial polytope. For $i = 0, \ldots, d-1$, let f_i denote the number of i-dimensional faces of p. The sequence

$$(f_0, f_1, \ldots, f_{d-1})$$

is called the f-vector of the polytope P. Additionally, set $f_{-1} = f_d = 1$. Then for any $k = 0, \ldots, d-2$ the following Dehn-Sommerville equation holds:

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k.$$

When $k = -1$, it expresses the fact that Euler characteristic of an $(d-1)$-dimensional simplicial sphere is equal to $1 + (-1)^{d-1}$.

For $k = 0, 1, \ldots, d+1$, let

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}.$$

The $(d+2)$-uple

$$h(P) = (h_0, h_1, \ldots, h_{d+1})$$

is called the h-vector of P.

The generalized lower bound conjecture (McMullen-Walkup) [135] is the following:

Conjecture 5.6.1 Let P be a simplicial n-dimensional polytope. Then

1. $1 = h_0 \leq h_1 \leq \cdots \leq h_{[\frac{d}{2}]}$.
2. for an integer $1 \leq r \leq \frac{d}{2}$, the following are equivalent:
   a. $h_{r-1} = h_r$.
   b. there is a triangulation K of P all of whose faces of dimension at most $d-r$ are faces of P.

In McMullen [134] conjectured that the Dehn-Sommerville relations together with the generalized lower bound conjecture provide sufficient conditions for the existence of a simplicial polytope with a given h-vector.

If k and i are positive integers, then k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$k^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i-1} + \cdots + \binom{n_j+1}{j+1}.$$

Also define $0^{<i>} = 0$. A vector $k_0, k_1, \ldots, k_d$ of integers is an M-vector if $k_0 = 1$ and $0 \leq k_{i-1} \leq k_i^{<i>}$ for $1 \leq i \leq d-1$. McMullen conjectured that a sequence $(h_0, \ldots, h_d)$ of integers is the h-vector of a simplicial convex d-polytope if and only if $h_0 = 1$, $h_i = h_{d-i}$ for $0 \leq i \leq d$ and the following sequence is an M-vector:

$$(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[\frac{d}{2}]} - h_{[\frac{d}{2}]-1}).$$

The “if” part was proven by Billera and Lee [13].

The “only if” part was proven by Stanley [173] in a very surprising paper, as a consequence of the inequalities of Betti numbers provided by the hard Lefschetz theorem, and considering the cohomology of an associated toric variety, which is non-singular. By this paper deep results from algebraic geometry are related to the study of combinatorics.

A simplicial polytope is always rational so there exists an associated toric variety. In the non-simplicial (but still rational) case the associated toric variety is singular. In 1981 R. MacPherson showed how to compute the (rational) intersection cohomology of the (possibly singular) toric variety associated to any rational convex polytope and spoke about it in many conferences. This calculation was popularized by J. Bernstein and A. Khovanski. Proofs were published by: Fieseler [68] and by Denef and Loeser [64]

In [172] Stanley used this calculation together with the hard Lefschetz theorem for IH to prove the generalized lower bound conjectures for rational convex polytopes and conjectured that the same result holds in the non-rational case as well. The calculations are simplified if one considers the torus-equivariant intersection cohomology instead.

In the case of a non-rational polytope, no toric variety exists. This led to the possibility of proving the same result for non-rational polytopes by constructing the torus-equivariant intersection cohomology, in a purely combinatorial manner, together with a proof that it satisfies the hard Lefschetz theorem. The theory was successfully developed by Barthel et al. [8, 9], Bressler [37], Bressler and Lunts [38, 39], Karu [110] (see also Fieseler [69] and Braden [21, 22]). This completed the proof of Stanley’s conjectures for non-rational polytopes.

5.6.2 *The Asymptotic Set*

Let $F : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ be a polynomial mapping. In the study of geometrical or topological properties of polynomial mappings, the set of points at which those maps fail to be proper plays an important role. The asymptotic set

$$S_F = \{a \in Y \quad s.t. \quad \exists\{\xi_k\} \subset X, |\xi_k| \to \infty, F(\xi_k) \to a\}$$

is the smallest set S_F such that the map

$$F : X \setminus F^{-1}(S_F) \to Y \setminus S_F$$

is proper. In a topological approach of the Jacobian conjecture, it reduces to show that the asymptotic set of a complex polynomial mapping with non zero constant Jacobian is empty. It is then natural to study the topology of the asymptotic set.

Define by $Sing(F)$ the singular locus of F (the zero set of its Jacobian determinant) and denote by $K_0(F)$ the set of critical values of F, i.e. the set $F(Sing(F))$. Define the Riemannian manifold M_F as $\mathbb{C}^n \setminus Sing(F)$ with the pull back of Euclidean Riemannian metric on $\mathbb{R}^{2n} = \mathbb{C}^n$. This metric is non degenerate outside the singular locus of F.

Proposition 5.6.2 ([180, Proposition 2.3]) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. There exists a real semi-algebraic pseudomanifold $N_F \subset \mathbb{R}^\nu$, for some $\nu \geq 2n$, such that*

$$Sing(N_F) \subset (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}.$$

with $p = \nu - 2n$, and there exists a semi-algebraic bi-Lipschitz map:

$$h_F : M_F \to Reg(N_F),$$

where N_F is equipped with the metric induced by $\mathbb{R}^\nu$.

Case $F : \mathbb{C}^2 \to \mathbb{C}^2$.

The first result comes from Anna and Guillaume Valette. In [180], they associate the singular pseudomanifolds N_F to polynomial mappings $F : \mathbb{C}^2 \to \mathbb{C}^2$. They prove that the map F with non-vanishing Jacobian is not proper if and only if the intersection homology of N_F is nontrivial in dimension 2 and for any (or some) perversity. The intersection homology of N_F describes the geometry of the singularities at infinity of the mapping F. This provides a new and original approach to the Jacobian conjecture.

Case $F : \mathbb{C}^n \to \mathbb{C}^n$.

Thủy Nguyễn Thị Bích, with Anna and Guillaume Valette [144] consider the leading forms $\widehat{F_i}$ of the components of a polynomial mapping

$$F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n.$$

They obtain:

Theorem 5.6.3 ([144]) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with nowhere vanishing Jacobian. If* $\mathrm{rank}(D_{\mathbb{C}}\widehat{F_i})_{i=1,\dots,n} > n-2$ *then F is not proper if and only if $IH_2^{\bar{p}}(N_F) \neq 0$ for any (or some) perversity $\bar{p}$.*

In [140] Thủy Nguyễn T.B. shows that for a class of non-proper generic dominant polynomial mappings, the results in [144, 180] hold also without hypothesis of non emptyness of the set $K_0(F)$. In her thesis, [141], she provides explicit stratifications of the asymptotic set S_F and of the critical set $K_0(F)$ of polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ by a new method, that she called the method of "façons". That method appears to be a very powerful and a promising method not only for the computation of intersection homology. A large number of examples is provided.

In [142], Thủy Nguyễn T.B. describes explicitely such a variety N_F associated to the Pinchuk's map and calculate its intersection homology. The result describes the geometry of singularities at infinity of the Pinchuk's map. She also shows that the real version of the A. and G. Valette's results in [180] does not hold.

Case $F : \mathbb{C}^n \to \mathbb{C}^{n-1}$

Given a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, with $n \geq 2$, in [143], Thủy Nguyễn T.B. and M.A. Soares Ruas construct singular varieties V_G, similarly to the previous N_F. They prove that if the intersection homology with total perversity (with compact or closed supports) in dimension two of (any of the corresponding) V_G is trivial then G is a fibration.

5.6.3 Factorization of Poincaré Morphism for Toric Varieties

In this section, all homology and cohomology groups are with $\mathbb{Z}$ coefficients. References for this section are [6–8] (see also [83]).

The Cartier and Weil divisors play an important role for normal varieties. Given a Cartier divisor of a complex n-dimensional variety X one can associate its Chern class in $H^2(X; \mathbb{Z})$. Given a Weil divisor , one can associate its class in $H_{2n-2}(X; \mathbb{Z})$. In the two cases, the class of a principal divisor is zero. Denoting by $\mathrm{Div}_{\mathrm{C}}(\mathrm{X})$ and $\mathrm{Div}_{\mathrm{W}}(\mathrm{X})$ the abelian groups of classes of algebraic divisors of Cartier and Weil modulo the principal divisors, there are homomorphisms

$$c^1 : \mathrm{Div_C(X)} \to \mathrm{H^2(X)} \qquad \text{and} \qquad \kappa : \mathrm{Div_W(X)} \to \mathrm{H_{2n-2}(X)}.$$

For $k = 2$ the Poincaré morphism,

$$P_2 : H^2(X) \longrightarrow H_{2n-2}(X),$$

is cap-product by the fundamental class $[X]$ of X. Let D be a Cartier divisor, then

$$P_2(c^1(D)) = \kappa(D).$$

In the smooth case, the Poincaré morphism is an isomorphism and the two notions of divisors coincide.

For a normal toric variety X, the divisors classes admit invariant representatives under action of the torus $\mathbb{T}$ (see [81, 3.4]); There are isomorphisms

$$\mathrm{Div_C(X)} \cong \mathrm{Div_C^{\mathbb{T}}(X)} \qquad \text{and} \qquad \mathrm{Div_W(X)} \cong \mathrm{Div_W^{\mathbb{T}}(X)}$$

where $\mathrm{Div_C^{\mathbb{T}}(X)}$ and $\mathrm{Div_W^{\mathbb{T}}(X)}$ denote the groups of invariant divisors classes.

A non-degenerated toric varieties is a toric variety which is not isomorphic to the product of a toric variety of dimension $d < n$ and an $n - d$-dimensional torus. Its fundamental group is finite.

For every perversity $\bar{p}$, let denote by $i(p)$ the highest integer $i \leq n$ such that $p(2i) \leq 1$ and V_p the invariant open subset of X union of orbits with dimension at least $n - i(p)$. Then the group $IH^{\bar{p}}_{2n-2}(X)$ is isomorphic to the group

$$\mathrm{Div}^{\mathbb{T}}_{\bar{p}}(X) = \{[D] \in \mathrm{Div_W^{\mathbb{T}}(X) : D|_{V_p} \in Div_C^{\mathbb{T}}(V_p)}\}$$

and one has.

Theorem 5.6.4 ([6, Satz 2]) *Let X be a non degenerated toric variety, then:*

$$H^1(X) \cong H^{cld}_{2n-1}(X) = 0$$

and one has a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Div_C^{\mathbb{T}}(X)} & \hookrightarrow & \mathrm{Div}^{\mathbb{T}}_{\bar{p}}(X) & \hookrightarrow & \mathrm{Div_W^{\mathbb{T}}(X)} \\
{\scriptstyle c^1}\downarrow\cong & & \downarrow\cong & & {\scriptstyle \kappa}\downarrow\cong \\
H^2(X) & \xrightarrow{\alpha_X} & IH^{\bar{p}}_{2n-2}(X) & \xrightarrow{\omega_X} & H_{2n-2}(X).
\end{array}$$

where the composition of maps in the lowest line is the Poincaré homomorphism.

In the case of degenerate toric variety, one has a more general result taking into account the torus factor [6, 8].

5.6.4 General Perversities

Several authors are breaking away from the conditions on perversities as defined by Goresky and MacPherson in their original articles; see the (non exhaustive list of) papers of Cappell and Shaneson [47], Chataur et al. [55], Friedman [75–77, 79], Habegger and Saper [101], King [113], Saralegi-Aranguren [166].

These provides some interesting generalizations and results. that are sketched out at certain points in this survey. Friedman's article [79] itself provides a very good survey on the subject. Quoting Friedman, his article is an expository survey of the different notions of perversity in intersection homology and how different perversities require different definitions of intersection homology theory.

"With more general perversities than GM-perversities, one usually loses topological invariance of intersection homology (though this should be seen not as a loss but as an opportunity to study stratification data), but duality results remain, at least if one chooses the right generalizations of intersection homology. Complicating this choice is the fact that there are a variety of approaches to intersection homology."

With previous notation of strata, perversities such that $p_\alpha \leq codim_X(S_{n-\alpha}) - 2$ have been studied in detail by Friedman (see [78]) who proved in particular Poincaré duality for general perversities, Lefschetz duality for pseudomanifolds with boundary and Mayer-Vietoris sequence.

The Lefschetz duality for pseudomanifolds with boundary is also aim of the paper [181] by G. Valette. On a pseudomanifold X with boundary, two perversities are considered, the one for X and the other for the boundary ∂X. If the difference between the chosen perversities is constant, then Lefschetz duality holds on X. Here, allowable chains of the boundary ∂X are allowable on X.

5.6.5 Equivariant Intersection Cohomology

Equivariant intersection cohomology has been mainly studied by J.L. Brylinski, M. Brion and R. Joshua, by T. Oda and, in the circle case, by J.I.T. Prieto, G. Padilla and M. Saralegi-Aranguren.

Brylinski [46] provides an explicit complex in order to compute intersection homology in the equivariant setting. T. Oda considers the situation of toric action [145, 146]. Brion [40], Brion and Joshua [41], Joshua [108] provide a relationship between the vanishing of the odd dimensional intersection cohomology sheaves and of the odd dimensional global intersection cohomology groups. The authors provide a geometric proof of the vanishing of odd dimensional local and global intersection cohomology for Schubert varieties and complex spherical varieties. For a survey on these works, see [109]. In their paper [42] the authors extend their methods to algorithmically compute the intersection cohomology Betti numbers of reductive varieties.

In the papers [149, 158] G. Padilla, J.I.T. Prieto and M. Saralegi-Aranguren study circle actions on pseudomanifolds by using intersection cohomology and equivariant intersection cohomology. The orbit space and the Euler class of the action determine the equivariant intersection cohomology of the pseudomanifold as well as its localization.

5.6.6 Intersection Spaces

In [67], Timo Essig assigns cell complexes to certain topological pseudomanifolds depending on a perversity function in the sense of intersection homology. The main property of the intersection spaces is Poincaré duality over complementary perversities for the reduced singular (co)homology groups with rational coefficients. In the paper [3] of M. Banagl, using differential forms, the resulting generalized cohomology theory for pseudomanifolds was extended to 2-strata pseudomanifolds with a geometrically flat link bundle.

The resulting homology theory HI is well-known not to be isomorphic to intersection homology (see Banagl and Hunsicker [4]) but mirror symmetry "tends to" interchange IH and HI ([3]). A new duality theory for pseudomanifolds is obtained, which addresses certain needs in string theory related to the existence of massless D-branes in the course of conifold transitions and their faithful representation as cohomology classes (see Banagl [2]).

5.6.7 Blown-Up Intersection Homology

In [29, 30] Brasselet et al. use a notion of "déplissage" apparented to blow-up in order to define integration of differential forms on simplices and to prove a de Rham theorem for stratified varieties.

A similar method has been used by D. Chataur, M. Saralegi and D. Tanré to define the so called "blown-up intersection homology". The initial aim [52] is to extend Sullivan's minimal models theory to the framework of pseudomanifolds. The authors prove also a conjecture of M. Goresky and W. Pardon on Steenrod squares in intersection homology [99]. The relation with rational homotopy has been extended in [53]. The authors work in a context of simplicial sets in the sense of Rourke and Sanderson [162]. This provides a definition of formality in the intersection setting.

In [54] the authors prove the topological invariance of the blown-up intersection cohomology with compact supports in the case of a paracompact pseudomanifold with no codimension one strata.

Based upon simplicial blow-up, Chataur and Tanré construct in [57] Eilenberg-MacLane spaces for the intersection cohomology groups of a stratified space, answering a problem asked by M. Goresky and R. MacPherson ([18, Chap. IX, Problem 11]).

5.6.8 Real Intersection Homology

Whether there is a good analog of intersection homology for real algebraic varieties was stated as a problem by Goresky and MacPherson in [18, Chap. IX, Problem 7)]. They observed that if such a theory exists then it cannot be purely topological; indeed the groups constructed by McCrory and Parusiński in [133] are not homeomorphism invariants. These authors consider a class of algebraic stratifications that have a natural general position property for semialgebraic subsets. They define the real intersection homology groups $IHS_k(X)$ and show that they are independent of the stratification. If X is nonsingular and pure dimensional then $IHS_k(X) = H_k(X; \mathbb{Z}_2)$, classical homology with $\mathbb{Z}_2$ coefficients. An intersection pairing is defined.

5.6.9 Perverse Sheaves and Applications

Perverse sheaves and applications deserve a survey for the subject itself. Various authors wrote surveys concerning perverse sheaves and applications, they are clear and informative. In the MacPherson papers [127, 128], MacPherson and Vilonen [129], Massey survey [125] and Klinger survey [118], many references and results are given concerning in particular three main applications of perverse sheaves: Decomposition theorem, Weak and Hard Lefschetz theorems.

It is fair to mention in these papers other important applications such as Kazdhan-Lusztig conjecture, D-modules and Riemann-Hilbert correpondence, characteristic p and Weil conjecture, etc.

The story is far from over. Today there are many books and papers: an extensive literature on perverse sheaves in various fields of mathematics, showing the interest and diversity of the subject. The interested reader will find in the perverse sheaves a subject of fascinating discovery and exploratory innovation.

References

1. P. Albin, *On the Hodge theory of stratified spaces.* arXiv:1603.04106 [math.DG].
2. M. Banagl, *Intersection Spaces, Spatial Homology Truncation, and String Theory*, Lecture Notes, Springer-Verlag Berlin and Heidelberg GmbH and Co. KG, Germany (2010).
3. M. Banagl, *Foliated stratified spaces and a De Rham complex describing intersection space cohomology*, J. Diff. Geom. 104 (2016) 1, 1–58.
4. M. Banagl and E. Hunsicker, *Hodge Theory For Intersection Space Cohomology,* Geometry and Topology 23 (2019) 2165–2225.
5. G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O.Gabber et L. Kaup, *Relèvement des cycles algébriques et homomorphismes associés en homologie d'intersection*, Ann. of Math. (2) 141 (1995), no. 1, 147–179.
6. G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Invariante Divisoren und Schnittho-mologie von torischen Varietäten.* Banach Center Publications, "Parameter Spaces", Vol. 36, Warszawa 1996, ed. P. Pragacz, pp. 9–23.

7. G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Equivariant Intersection Cohomology of Toric Varieties*, Algebraic Geometry, Hirzebruch 70, 45–68, Contemp. Math.241, Amer. Math. Soc., Providence, R.I., 1999.
8. G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Combinatorial Intersection Cohomology for Fans*, Tohoku Math. J. 54 (2002), 1–41.
9. G. Barthel, J.-P. Brasselet, K.-H. Fieseler and L. Kaup, *Combinatorial Duality and Intersection Product: A Direct Approach* Tohoku Math. J. (2) 57 (2003), no 2, 273–292.
10. A. Beilinson, J. Bernstein, O. Gabber and P. Deligne, *Faisceaux pervers.* Astérisque, 100, 1982.
11. B. Bendiffalah, *Shadow forms of Brasselet-Goresky-MacPherson* Transactions of the Amer. Math. Soc. Vol 347, N.12, Dec 1995, 4747–4764.
12. R. Benedetti, *Density of Morse functions on a complex space*, Math. Ann.229 (1977), pp. 135–139.
13. L. Billera and C. Lee, *Sufficiency of McMullen's condition for f-vectors of simplicial polytopes*, Bulletin AMS 2 (1980), 181–185.
14. C. Bisi, F. Bracci, T. Izawa and T. Suwa, *Localized intersection of currents and the Lefschetz coincidence point theorem*, Annali di Matematica Pura e Applicata 195 (2016) no 2, 601–621.
15. A. Borel, *$\mathcal{L}^2$-cohomology and intersection cohomology of certain arithmetic varieties.* In: Emmy Noether in Bryn Mawr, eds. B. Srinivasan and J. Sally, Springer, 1983, 119–131.
16. A. Borel and W. Casselman, *Cohomologie d'intersection et $\mathcal{L}^2$-cohomologie de variétés arithmétiques de rang rationnel 2.* CR Acad. Sci. Paris 301(1985), 369–373.
17. A. Borel and J.C. Moore, *Homology theory for locally compact spaces*, Michigan Math J. 7, 137–159 (1960).
18. A. Borel and al. *Intersection Cohomology,* Progress in Mathematics Vol 50, Birkhäuser, 1984.
19. W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes.* C.R. Acad. Sci. Paris, t. 292 Ser. I, 707–710 (1981).
20. R. Bott and L. Tu, *Differential forms in Algebraic Topolology*. Springer-Verlag, New-York, 1982.
21. T. Braden, *Remarks on the Combinatorial Intersection Cohomology of Fans*. Pure and Applied Mathematics Quarterly Volume 2, Number 4. 1149– 1186, 2006.
22. T. Braden and R.D. MacPherson, *Intersection homology of toric varieties and a conjecture of Kalai*, Comment. Math. Helv. 74 (1999), 442–455.
23. J.-P. Brasselet, *Définition combinatoire des homomorphismes de Poincaré, Alexander et Thom pour une pseudo-variété,* Astérisque no 82–83, (1981), p. 71–91.
24. J.-P. Brasselet, *de Rham theorems for singular varieties*, Contemp. Math. 161 (1994) 95–112.
25. J.-P. Brasselet K.-H. Fieseler and L. Kaup, *Caractérisation des variétés homologiques à l'aide des invariants de l'homologie d'intersection,* Banach Institut Varsovie, Vol. 33 (1996), p. 19–22.
26. J.-P. Brasselet et G.Gonzalez-Sprinberg, *Espaces de Thom et contre-exemples de J.L.Verdier et M.Goresky,* Bol. Soc. Bras. Mat. vol. 17 n°2 (1986).
27. J.-P. Brasselet et G.Gonzalez-Sprinberg, *Sur l'homologie d'intersection et les classes de Chern des variétés singulières,* Travaux en cours n°23 Hermann (1987), 5–11.
28. J.-P. Brasselet, M. Goresky and R. MacPherson, *Simplicial Differential Forms with Poles*, Amer. Journal of Maths., 113 (1991), 1019–1052.
29. J.-P. Brasselet, G. Hector et M. Saralegi, *Théorème de de Rham pour les variétés singulières,* Annals of Global Analysis and Geometry Vol. 9, no 3, 1991, 211–243.
30. J.-P. Brasselet, G. Hector et M. Saralegi, *$\mathcal{L}^2$-cohomologie des espaces stratifiés,* Manuscripta Mathematica. Vol. 76, 1992, 21–32.
31. J.-P. Brasselet and A. Legrand, *Differential forms on singular varieties and cyclic homology,* in: Singularity Theory, Liverpool 1996, in: London Math. Soc. Lecture Note Ser., vol. 263, Cambridge Univ. Press, 1999, pp. 175–187.
32. J.-P. Brasselet et M. Saralegui, *Variétés homologiques et homologie d'intersection,* C. R. Acad. Sci. Paris Sér. I, 314, (1992), 847–850.

33. J.-P. Brasselet and T. Suwa, *Local and global coincidence homology classes*, Journal of Fixed Point Theory and Applications, 23(2), 2021.
34. J.-P. Brasselet, A.K.M. Libardi, E.C. Rizziolli and M.J. Saia, *Cobordism of maps of locally orientable Witt spaces*, Publicationes Mathematicae Debrecen, 2019, 94 (3–4), pp. 299–317.
35. J.-P. Brasselet, A.K.M. Libardi, T.F.M. Monis, E.C. Rizziolli and M.J. Saia, *A Lefschetz coincidence theorem for singular varieties*. Singularities and Foliations. Geometry, Topology and Applications, pp. 513–535.
36. G. Bredon, *Sheaf theory.* New York: McGraw-Hill, 1967.
37. P. Bressler, *Intersection Cohomology on Nonrational Polytopes*. Compositio Mathematica volume 135, 245–278 (2003).
38. P. Bressler and V. Lunts, *Intersection cohomology on nonrational polytopes*, Compositio Mathematica 135(3), 2000.
39. P. Bressler and V. Lunts, *Hard Lefschetz theorem and Hodge-Riemann relations for intersection cohomology of nonrational polytopes*, Compositio Mathematica, volume 135, 245–278 (2003).
40. M. Brion, *Equivariant cohomology and equivariant intersection theory.* Feb. 2008. arXiv:math/9802063v1.
41. M. Brion and R. Joshua, *Vanishing of odd dimensional intersection cohomology II*, Mathematische Annalen volume 321, pages 399–437 (2001).
42. M. Brion and R. Joshua, *Intersection Cohomology of reductive varieties*, Journal of the European Mathematical Society, 6(4), 2003.
43. M. Brückner, *Die Elemente der vierdimensionalen Geometrie mit besonderer Beriicksichtigung der Polytope*, Jahresbericht des Vereins für Naturkunde zu Zwickau I.S. – 1893, 1–61.
44. M. Brückner, *Uber die Ableitung der allgemeinen Polytope und die nach Isomorphismus verschiedenen Typen der allgemeinen Achtzelle (Oktatope)*, Verhl. Nederl. Akad. Wetensch. Sect. I. 10, No. 1 (1909).
45. J.L. Brylinski, *(Co)-homologie d'intersection et faisceaux pervers,* Séminaire Bourbaki 589, Astérisque 92–93.
46. J.L. Brylinski, *Equivariant Intersection Cohomology,* IHES 1986 and in Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), 5–32, Contemp. Math., 139, Amer. Math. Soc., Providence, RI, 1992.
47. S.E. Cappell and J.L. Shaneson, *Singular spaces, characteristic classes, and intersection homology,* Annals of Mathematics134(1991), 325–374.
48. H. Cartan *Homologie et cohomologie singulière.*, Séminaire Henri Cartan, tome 1 (1948-1949), exposé no 5.
49. H. Cartan and S. Eilenberg, *Homological algebra,*, The Princeton University Press, Princeton, 1956.
50. W. Casselman, *Introduction to the $\mathcal{L}$-cohomology of arithmetic quotients of bounded symmetric domains.* In: Complex Analytic Singularities, Adv. Stud. Pure Math. 8 (1986), 69–93.
51. E. Cattani, A. Kaplan and W. Schmid , *$\mathcal{L}^2$ and intersection cohomologies for a polarizable variation of Hodge structure.* Inventiones mathematicae volume 87, pages 217–252 (1987).
52. D. Chataur, M. Saralegi-Aranguren and D. Tanré, *Blown-up intersection cohomology,* Contemp.Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 45–102.
53. D. Chataur, M. Saralegi-Aranguren and D. Tanré, *Intersection cohomology, simplicial blow-up and rational homotopy,* Mem. Amer. Math. Soc. 254 (2018), no. 1214, viii+108.
54. D. Chataur, M. Saralegi-Aranguren and D. Tanré, *Poincaré duality with cap products in intersection homology,* Adv. Math. 326 (2018), 314–351.
55. D. Chataur, M. Saralegi-Aranguren and D. Tanré, *Intersection homology: General perversities and topological invariance*, Ilinois J. Math. Volume 63, Number 1 (2019), 127–163.
56. D. Chataur, M. Saralegi-Aranguren and D. Tanré, *Steenrod squares on intersection cohomology and a conjecture of M Goresky and W Pardon,* Algebr. Geom. Topol.16 (2016), no. 4, p. 1851–1904.
57. D. Chataur and D. Tanré, *Natural operations in intersection homology*, arXiv:2005.04960v2.

58. J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*. Proc Natl Acad Sci U S A. 1979 May; 76(5): 2103–2106.
59. J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds.* In: Geometry of the Laplace operator. Proceedings of Symposia in pure Mathematics 36, 91–146 (1980). Amer. Math. Soc. Providence, RI.
60. J. Cheeger, M. Goresky and R.MacPherson, *$\mathcal{L}^2$-cohomology and intersection cohomology of singular algebraic varieties*, in "Seminar on differential geometry" ed. by S. T. Yau, Princeton University Press, 1982, p. 303–340.
61. D.C. Cohen, M. Goresky and L. Ji, *On the Künneth Formula for Intersection Cohomology,* Trans. AMS 333 (1992), 63–69.
62. A. Connes *Noncommutative Differential Geometry*, Inst. Hautes Etudes Sci., Publ. Math. No. 62 (1985), 257–360.
63. M.A. de Cataldo and L. Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps.* Bull. AMS Vol. 46, No. 4, 2009, 535–633.
64. J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra.* Invent. math. 106, 275–294 (1991).
65. M. Denh, *Die Eulersche Formel im Zusammenhang mit den Inhalt in der Nicht Euklidischen Geometrie.* Math. Ann. 61 (4), 561–586 (1905).
66. A. Dimca, *Sheaves in Topology*, Springer Science and Business Media, 2004.
67. T. Essig *About a De Rham Complex Describing Intersection Space Cohomology in a Three Strata Case*, in Ed. M. Banagl, Ruprecht-Karls-Universität Heidelberg, 2016. Also in arXiv:1804.06690v2.
68. K.H.Fieseler, *Rational Intersection Cohomology of Projective Toric Varieties*. Journal für die reine und angewandte Mathematik 413 (1991): 88–98.
69. K.H.Fieseler, *Towards a combinatorial intersection cohomology for fans*, C.R. Acad. Sci. Paris 300 (1999).
70. K.H.Fieseler, *Survey on combinatorial intersection homology* In: Math. Forschungsinstitut Oberwolfach, no. 20, p. 1071–1075. http://www2.math.uu.se/~khf/finis.pdf.
71. K.H.Fieseler and L. Kaup, *Singular Poincaré duality and intersection homology,* Proceedings of the 1984 Vancouver Conference in Algebraic Geometry. AMS, Providence, (1986), 113–161.
72. K. Fieseler and L. Kaup, *Theorems of Lefschetz type in intersection homology, I. The hyperplane section theorem.* Revue Roumaine de Mathématiques Pures et Appliquées 33, 175–195 (1988).
73. K. Fieseler and L. Kaup, *The vanishing invariants in intersection homology of complex spaces*, in : Singularities, Banach Center Publ. 20, 1988, 225–235.
74. G. Friedman, *Intersection homology Künneth theorems.* Mathematische Annalen volume 343, 371–395, (2009).
75. G. Friedman, *Superperverse intersection cohomology: stratification (in)dependence,* Math. Z. 252 (2006), no. 1, p. 49–70.
76. G. Friedman, *Singular chain intersection homology for traditional and super-perversities,* Trans. Amer. Math. Soc. 359 (2007), no. 5, p. 1977–2019.
77. G. Friedman, *Intersection homology with general perversities*, Geom. Dedicata 148 (2010), p. 103–135.
78. G. Friedman, *Intersection homology and Poincaré duality on homotopically stratified spaces*, Geom. Topol. Volume 13, Number 4 (2009), 2163–2204.
79. G. Friedman, *An introduction to intersection homology with general perversity functions*, in Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, p. 177–222.
80. G. Friedman and J. E. McClure, *Cup and cap product in intersection (co)homology,* Advances in mathematics, vol. 240, pp. 383–426, 2013.
81. W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies, Princeton Univ. Press, 1993.

82. W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces.*, Memoirs of the Amer. Math. Soc. vol. 243, A.M.S., Providence, RI 1981.
83. W. Fulton and B. Sturmfels, *Intersection theory on toric varieties*. Topology, 36(2):335–353, 1997.
84. S.I. Gelfand, and R. MacPherson, *Verma modules and Schubert cells: a dictionary,* Séminaire d'Algèbre. Lecture Notes in Mathematics vol. 924. Berlin-Heidelberg-New York: Springer 1982.
85. S.I. Gelfand, and Y.I. Manin, *Methods of Homological Algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, New York: Springer-Verlag (2003).
86. R. Godement, *Topologie algébrique et théorie des faisceaux.* Actualites Scientifiques et Industrielles. 1252. Publications de l'Institut de Mathématique de l'Université de Strasbourg. XIII. Paris: Hermann VIII, 283 p. (1958).
87. M.Goresky, *Intersection homology operations,* Comment. Math. Helv. 59 (1984), 485–505.
88. M.Goresky, $\mathcal{L}^2$*-cohomology is intersection cohomology,* The zeta functions of Picard modular surfaces, 47–63, Univ. Montréal, Montréal, QC, 1992.
89. M.Goresky, *Course given at Brown University in the fall of 2016* https://www.math.ias.edu/~goresky/math2710/all.pdf.
90. M. Goresky, *Morse theory, stratifications and sheaves.* In: Cisneros-Molina, J.L., Lê, D.T., Seade, J. (eds.) Handbook of Geometry and Topology of Singularities I. Springer, Cham, 261–301.
91. M. Goresky et R. MacPherson, *La dualité de Poincaré pour les espaces singuliers,* C.R. Acad. Sci. Paris 284 (Janvier 1977), 1549–1551.
92. M. Goresky and R. MacPherson, *Intersection homology theory,* Topology 19 (1980), 135–162.
93. M. Goresky and R. MacPherson, *Morse Theory and Intersection Homology*, in Analyse et topologie sur les espaces singuliers, Astérisque 101–102, Soc. Math. France, 1982, pp. 135–192.
94. M. Goresky and R. MacPherson, *Intersection homology theory II,* Inv. Math. 71 (1983), 77–129.
95. M. Goresky and R. MacPherson, *Stratified Morse theory.* In Singularities, Part1 (Arcata, Calif., 1981), volume 40 of Proc. Sympos. Pure Math., pages 517–533. Amer. Math. Soc., Providence, R.I., 1983.
96. M. Goresky and R. MacPherson, *Stratified Morse theory,* Ergebnisseder Mathematik und ihrer Grenzgebiete volume 14, Springer-Verlag, Berlin, 1988.
97. M. Goresky and R. MacPherson, *Local contribution to the Lefschetz fixed point formula*, Inv. Math. 111 (1993), 1–33. 295, 296.
98. M. Goresky and R. MacPherson, *Letschetz fixed point theorem for intersection homology*, Comment. Math. Helvetici 60 (1985) 366–391.
99. M. Goresky and W. Pardon, *Wu Numbers of Singular Spaces*, Topology 28, (1989), 325–367.
100. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Mathematical Journal, (2), 9: 119–221 (1957).
101. N. Habegger and L. Saper, *Intersection cohomology of cs-spaces and Zeeman's filtration*, Invent. Math. 105 (1991), 247–272.
102. H.A. Hamm and Lê D.T., *The Lefschetz Theorem for hyperplane sections.* In: Cisneros-Molina, J.L., Lê, D.T., Seade, J. (eds.) Handbook of Geometry and Topology of Singularities I. Springer, Cham, 467–514.
103. R. Hartshorne, *Residues and duality*. Lecture Notes in Mathematics vol. 20. Berlin-Heidelberg-New York: Springer 1966.
104. P. Heegaard, *Sur l'Analysis situs*, Bulletin de la Société Mathématique de France, vol. 44, 1916, p. 161–242 . French translation of the danish thesis : *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhæng* , Copenhague, Det Nordiske Forlag-Ernst Bojesen, 1898.
105. G. Hochschild, B. Kostant and A. Rosenberg *Differential Forms on regular affine Algebras*, Trans. AMS 102 (1962), 383–408.

106. C. Houzel, *Histoire de la théorie des faisceaux* Matériaux pour l'histoire des mathématiques au XXe siècle (Nice, 1996). Sémin. Congr., 3, Soc. Math. France, Paris, pp. 101–119, 1998. also in M. Kashiwara and P. Schapira, Sheaves on Manifolds: With a Short History "Les débuts de la théorie des faisceaux" by C. Houzel, Berlin/Heidelberg/Paris etc., Springer, 1990, 512 p.
107. W. C. Hsiang and V. Pati, *L^2-cohomology of normal algebraic surfaces I*, Invent. Math. 81 (1985), no. 3, 395–412.
108. R. Joshua, *Vanishing of odd-dimensional intersection cohomology*, Mathematische Zeitschrift 195(2) 239–253 (1987).
109. R. Joshua, *Equivariant intersection cohomology–a survey.* in Invariant Theory, Proc. of AMS Special Session, October 31–November 1, 1986, Contemporary Math. 88, Amer. Math. Soc., 1989, 25–31.
110. K. Karu, *Hard Lefschetz Theorem for Nonrational Polytopes*, (pr)e-printmath.AG/0112087(25 pages), 2002.
111. M. Kashiwara and T. Kawai, *The Poincaré lemma for variations of polarized Hodge structure.* Publ. RIMS 23 (1987), 345–407.
112. M. Kashiwara and P. Shapira, *Categories and Sheaves.* Grundlehren der Mathematischen Wissenschaften 332, Springer (2006).
113. H.C. King, *Topological invariance of intersection homology without sheaves,* Topology and its Applications, Volume 20, Issue 2, August 1985, Pages 149–160.
114. H.C. King, *Intersection homology and homology manifolds*, Topology 21 (1982), 229–234.
115. L. Kaup and K.-H. Fieseler, *Singular Poincarée duality and intersection homology,* in: Proc. 1984 Vancouver Conf. in Algebraic Geometry, Amer. Math. Soc., Providence, 1986, 113–161.
116. F. Kirwan and J. Woolf, *An introduction to intersection homology theory,* 2nd ed. Boca Raton, FL: Chapman Hall. 2006.
117. S. Kleiman, *The development of intersection homology theory*. A century of mathematics in America, Part II, Hist. Math. 2, Amer. Math. Soc., 1989, pp. 543–585.
118. B. Klinger, *Intersection homology and perverse sheaves*. http://webusers.imj-prg.fr/~bruno.klingler/cours/courspervers.pdf.
119. F. Lazzeri, *Morse Theory on singular spaces*, in Singularités à Cargèse. Publ. Soc. Math. de France, 1973 (Astérisque 7-8).
120. S. Lefschetz, *Intersection of complexes on manifolds*, Proceedings of the National Academy of Sciences Vol. 11, 287–289, 1925.
121. S. Lefschetz, *Continuous transformations of manifolds*, Proceedings of the National Academy of Sciences Vol. 9, 90–93, 1923.
122. S. Lefschetz, *Intersections and transformations of manifolds*, Trans. Amer. Math. Soc. Vol. 28 No. 1. (1926), 1–49.
123. E. Looijenga, *$\mathcal{L}^2$–cohomology of locally symmetric varieties,* Compositio Math. 67 (1988), 3–20.
124. U. Ludwig, *A complex in Morse theory computing intersection homology*, Annales de l'Institut Fourier, Tome 67, no. 1 (2017), p. 197–236.
125. D. Massey, *Notes on perverse sheaves and vanishing cycles* http://arxiv.org/abs/math/9908107 http://www.massey.math.neu.edu/Massey/Massey_preprints/perverse_notes.pdf.
126. D. Massey, *Stratified Morse Theory: Past and Present* Pure and applied mathematics quarterly 2(4) October 2006. Also in http://www.massey.math.neu.edu/Massey/Massey_preprints/SMT_survey.pdf.
127. R. MacPherson, *Global questions in the topology of singular spaces,* Proceedings of the International Congress of Mathematicians, Vol. I, Warszawa, 1983. PWN– Polish Scientific Publishers, Warszawa, and Elsevier Science Publishers B.V., Amsterdam, (1984), 213–236.
128. R. MacPherson, *Intersection Homology and Perverse Sheaves,* Report December 15, 1990. Unpublished Notes: http://faculty.tcu.edu/gfriedman/notes/ih.pdf.
129. R. MacPherson and K. Vilonen, *Elementary construction of perverse sheaves*, Invent. Math. 84 (1986), 403–435.

130. L.G. Maxim, *Intersection Homology and Perverse Sheaves: with Applications to Singularities* Graduate Texts in Mathematics, 281. Springer.
131. C. Mc Crory, *Poincaré duality in spaces with singularities.* Thesis, Branseis University. May 1972. Revised version in : https://www.maths.ed.ac.uk/~v1ranick/papers/mccrory4.pdf.
132. C. Mc Crory, *Stratified general position*. Algebraic and Geometric Topology Lecture Notes in Mathematics Volume 664, 1978, pp 142–146.
133. C. Mc Crory and A. Parusiński, *Real intersection homology*. Topology and its Applications, Elsevier, 2018, 234, pp. 108–129.
134. P. McMullen, *The numbers of faces of simplicial polytopes*, Israel J. Math. 9 (1971), 559–570.
135. P. McMullen and D. W. Walkup, *A generalized lower-bound conjecture for simplicial polytopes.* Mathematika, Lond., 18: 264–273, 1971.
136. J. Milnor, *Morse Theory*, Ann. of Math. studies 51, Princeton Univ. Press, Princeton N.J. 1963.
137. M. Nagase, *$\mathcal{L}^2$-cohomology and intersection cohomology of stratified spaces,* Duke Math. J. 50(1983), 329–368.
138. M. Nagase, *Sheaf theoretic $\mathcal{L}^2$-cohomology,* Adv. Stud. Pure Math. $n°8$ (1987), 273–279, North Holland.
139. M.Nagase, *Remarks on the $\mathcal{L}^2$-cohomology of singular algebraic surfaces.* J. Math. Soc. Japan Vol. 41, No. 1, 1989, 97–116.
140. Thủy Nguyễn T. B., *A Remark on the Topology at Infinity of a Polynomial Mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ via Intersection Homology*. Advances in Pure Mathematics. Vol. 6 No. 13, December 2016, 1037–1052.
141. Thủy Nguyễn T. B., *Etude de certains ensembles singuliers associés à une application polynomiale,* Thèse soutenue le 30 sept. 2013 Université de la Mediterranée, Marseille, France available on https://tel.archives-ouvertes.fr/tel-00875930/document.
142. Thủy Nguyễn T. B., *Geometry of singularities at infinity of the Pinchuk's map.* https://arxiv.org/pdf/1710.03318.pdf.
143. Thủy Nguyễn T. B., and M.A. Soares Ruas *On singular varieties associated to a polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*. Asian Journal of Mathematics Volume 22 (2018), Number 6, 1157–1172.
144. Thủy Nguyễn T. B., A. Valette and G. Valette, *On a Singular Variety Associated to a Polynomial Mapping*. Journal of Singularities, 7, 2013, 190–204.
145. T. Oda, *The intersection cohomology and toric varieties (Modern aspects of combinatorial structure on convex polytopes*. Kyoto university (1994).
146. T. Oda, *The intersection cohomology of toric varieties,* in Geometry of Toric Varieties and Convex Polytopes (T. Hibi, ed.), RIMS Kokyuroku 776, March 1992, 49–67.
147. T. Ohsawa, *Cheeger-Goresky-MacPherson conjecture for varieties with isolated singularities,* Math. Z. 206 (1991), 219–224.
148. T. Ohsawa, *On the $\mathcal{L}^2$-cohomology of complex spaces,* Math. Z. 209 (1992), 519–530.
149. G. Padilla, M. Saralegi-Aranguren *Intersection cohomology of the circle actions,* Topology and its Applications 154 (2007) 2764–2770.
150. W.L. Pardon and M. Stern *$\mathcal{L}^2 - \bar{\partial}$-cohomology of complex projective varieties*. J. Amer. Math. Soc. vol. 4, 1991, 603–621.
151. R. Pignoni, *Density and stability of Morse functions on a stratified space*, Ann. Scuola Nor. Sup. Pisa ser. IV vol. 6 (1979) 593–608.
152. H. Poincaré, *Sur la généralisation d'un théorème d'Euler relatif aux polyèdres,* Comptes rendus de l'Acad. des Sciences 117 (1893), 144-145. Œuvres de Poincaré XI, Gauthier-Villars Ed. Paris, 1953, 6–7.
153. H. Poincaré, *Analysis Situs,* Journal de l'Ecole Polytechnique ser 2, 1 (1895) pp. 1–123. (Reprinted in Œuvres de Poincaré VI, Gauthier-Villars Ed. Paris, 1953, VI, 193–288).
154. H. Poincaré, *Complément à l'Analysis Situs,* Rendiconti del Circolo matematico di Palermo, 13 (1899) pp. 285–343. (Reprinted in Œuvres de Poincaré VI, Gauthier-Villars Ed. Paris, 1953, VI, 290–337).

155. H. Poincaré, *Second complément à l'Analysis Situs,* Proceedings of the London Mathematical Society, 32 (1900), pp. 277–308. (Reprinted in Œuvres de Poincaré VI, Gauthier-Villars Ed. Paris, 1953, 338–370).
156. G. Pollini, *Intersection differential forms,* Rendiconti del Seminario Matematico della Università di Padova (2005) Volume: 113, page 71–97.
157. P. Popescu-Pampu, *La dualité de Poincaré,* Images des Mathématiques, CNRS, 2012 https://images.math.cnrs.fr/La-dualite-de-Poincare.html.
158. J. Prieto and M. Saralegi-Aranguren, *Equivariant intersection cohomology of the circle actions*. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas volume 108, 49–62 (2014).
159. K. Reidemeister *Topologie der Polyeder und kombinatorische Topologie der Komplexe*, Akademische Verlagsgesellschaft, Lpz., 1938-Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 17.
160. G. de Rham, *Sur l'analysis situs des variétés à n dimensions*, J. Math. Pures Appl. (9) 10 (1931), p. 115–200.
161. K. Rietsch, *An introduction to perverse sheaves*. arXiv:math/0307349v1. 26 Jul. 2003.
162. C. Rourke and B. Sanderson *Homology stratifications and intersection homology.* Geometry and Topology Monographs 2 (1999) 455–472.
163. L. Saper, *$\mathcal{L}^2$-cohomology and intersection homology of certain algebraic varieties with isolated singularities,* Invent. Math. 82 (1985), no. 2, 207–255.
164. L. Saper, *$\mathcal{L}^2$-cohomology of Kähler varieties with isolated singularities*, J. Differential Geometry, 36 (1992), 89–161.
165. L. Saper and M. Stern, *$\mathcal{L}^2$-cohomology of arithmetic varieties.* Annals of Mathematics. Second Series, Vol. 132, No. 1 (Jul., 1990), pp. 1–69.
166. M. Saralegi-Aranguren, *de Rham intersection cohomology for general perversities,* llinois J. Math. Volume 49, Number 3 (2005), 737–758.
167. J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, Monografie Matematyczne 63, Birkhäuser Verlag, Basel, 2003.
168. J.P. Serre *Géomérie algébrique et géométrie analytique*, Annales de l'Institut Fourier, vol. 6,? 1956, p. 1–42.
169. P. Siegel, *Witt Spaces: a Geometric Cycle Theory for KO-Homology at odd primes*, Amer. J. Math., **105**, (1983), no. 5, 1067–1105.
170. D. Sommerville, *The relations connecting the angle sums and volume of a polytope in space of n dimensions.* Proceedings of the Royal Society London, Ser. A, Vol. 115, (1927), 103–119.
171. T. Springer, *Quelques applications de la cohomologie d'intersection,* Séminaire Bourbaki 589, Astérisque 92–93, Soc. Math. France, 1982, pp. 249–274.
172. R. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties and related results*, M. Nagata, H. Matsumura, eds.,Commutative Algebra and Combinatorics, 187–213, Adv. Stud.Pure Math. 11, Kinokunia, Tokyo, and North Holland, Amsterdam/New York, 1987.
173. R. Stanley, *The number of faces of a simplicial convex polytope*. Adv. in Math. 35 (1980), 236–238.
174. N. Steenrod, *Homology with local coefficients*. Annals of Mathematics. 44 (4) 610–627, 1943.
175. N. Steenrod, *The Topology of Fibre Bundles* Princeton University Press, Princeton 1951. New edition 7th printing, Princeton landmarks in mathematics and physics, 1999.
176. N. Steenrod, *The Work and Influence of Professeur S. Lefschetz in Algebraic Topology.* in Algebraic Geometry and Topology: A Symposium in Honor of Solomon Lefschetz. Princeton Mathematical Series, Vol. 12. Princeton University Press, 1957.
177. T. Suwa, *Complex Analytic Geometry*, World Scientific, March 2021.
178. R. Thom, *Les singularités des applications différentiables*, Annales de l'Institut Fourier, vol. 6, 1956, p. 43–87.
179. D. Trotman, *Stratification theory* In: Cisneros-Molina, J.L., Lê, D.T., Seade, J. (eds.) Handbook of Geometry and Topology of Singularities I. Springer, Cham, 231–260.

180. A. Valette and G. Valette, *Geometry of polynomial mappings at infinity via intersection homology* Annales de l'Institut Fourier, vol. 74, 2014, p. 2147–2163.
181. G. Valette *A Lefschetz duality for intersection homology,* Geom. Dedicata 169 (2014), 283–299.
182. G. Valette $\mathcal{L}^\infty$*-cohomology is intersection cohomology*. Advances in Mathematics, Volume 231, Issues 3–4, 2012, 1818–1842.
183. J.L. Verdier, *Dualité dans la cohomologie des espaces localement compacts* Séminaire Bourbaki. Vol. 1965/1966. Exposé 300.
184. J.L. Verdier, *Catégories dérivés, état 0.* in Séminaire de Géométrie Algébrique du Bois-Marie, Cohomologie Étale, SGA 4 1/2 par P. Deligne. Lecture Notes in Mathematics vol. 569. Springer-Verlag 1977.
185. A. Weber, *A morphism of intersection homology and hard Lefschetz,* Algebraic geometry conference, 339–348, 1999.
186. H. Whitney *Geometric integration theory*, Princeton University Press, 1957.
187. B. Youssin, $\mathcal{L}^p$*-cohomology of cones and horns*, J. Differential Geom. 39 (1994), no. 3, 559–603.
188. S. Zucker, *Hodge theory with degeneration coefficients:* L_2*-cohomology in the Poincaré metric*, Ann. of Math. 109, 415–476 (1979).
189. S. Zucker, *Cohomology of warped products and arithmetic groups.* Inventiones Math. 70 (1982), 169–218.
190. S. Zucker, $\mathcal{L}^2$*-cohomology and intersection homology of locally symmetric varieties, II.* Compositio Math. 519 (1986), 339–398.