Chapter 6
Abstract Generality, Simplicity, Forgetting, and Discovery

Colin McLarty

Abstract The paper contrasts two ways of generalizing and gives examples: probably most people think of examples like generalizing Cartesian coordinate geometry to differential manifolds. One kind of structure is replaced by another more complicated but more flexible kind. Call this *articulating generalization* as it articulates some general assumptions behind an earlier concept. On the other hand, by *unifying generalization*, I mean simply dropping some assumptions from an earlier concept or theorem. Hilbert, Noether, and Grothendieck were all known for highly non-trivial unifying generalizations.

> It is an error to believe that rigor in the proof is the enemy of simplicity.... The very effort for rigor forces us to find simpler methods of proof. [8, p. 257]

Complaints about "abstract generality" in mathematics, like complaints about "kids these days", have always been around and always will. In both cases, the complaints are not wholly wrong but are wrong on the whole. This paper looks at a kind of generalizing abstraction Hilbert and many others have used to find simpler methods of proof. Call it *unifying generalization*, in contrast to *articulating generalization*, which expands some earlier kind of structure into another more complicated but more flexible kind. Both have been extremely productive and they often depend on each other. But articulating generalization typically does not simplify earlier concepts or proofs.[1]

This paper gives several examples and then focusses on Grothendieck's theory of *schemes* in algebraic geometry as a unifying generalization. This may be Grothendieck's single most influential generalization of any kind.[2]

[1]The very fruitfully simplifying work of Fields Medalist Peter Scholz might be an argument against this claim. The reader may want to explore that possibility.

[2]The other candidate for his most influential generalization is his *derived functor cohomology*, which is sketched from the same viewpoint as this article in [14].

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The various examples will use various bits of mathematics that philosophers of mathematics will not normally know much about. This is an opportunity for philosophers to do what mathematicians frequently do at convention talks: get the gist of the math without all the details. The specifically relevant details are explained, and references are given.

6.1 An Articulating Generalization: Riemannian Manifolds

High schools teach Cartesian coordinate geometry on the xy-plane $\mathbb{R}^2$. A small step leads to all the finite dimensional coordinate spaces $\mathbb{R}^n$. Bernhard Riemann took a very large step when he set out, at Gauss's request, to describe "The Hypotheses which lie at the Bases of Geometry" [20]. He found the basic geometric concepts and constructions can be fitted together in many more ways than just the original spaces $\mathbb{R}^n$. Today, these more general spaces are called *Riemannian manifolds*. The idea was immediately intuitive, and productive, for a handful of important mathematicians. But it took decades before this innovation was explicated clearly and concisely enough to become a standard part of advanced mathematics.

Today, just to name some classics, the definition of Riemannian manifold can be handled in a concise form such as [10], a leisurely highly illustrated form such as [22], or a thoroughly physically motivated form such as [16].

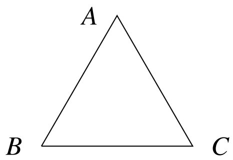
The point for us here is that all of these accounts presuppose the coordinate spaces $\mathbb{R}^n$, and use calculus on those spaces to define manifolds. The general case is not just motivated or introduced via the special case, but depends logically on first defining the special case. And then the general case of manifolds needs a lot of apparatus beyond the special case of coordinate spaces.

Without going into detail, the general case needs the idea of *local coordinate systems* which are *smoothly compatible* with each other. Calculus on $\mathbb{R}^n$ can simply take vectors to be directed line segments in $\mathbb{R}^n$, but calculus on a manifold M needs more refined apparatus. Today that apparatus often takes the form of a *tangent bundle* $\mathbb{T}M \to M$ mapping to M. Generalization by adding new apparatus can work beautifully. It does in this case. But it is not the only kind of generalization.

6.2 A Hypothetical Example of a Unifying Generalization

Many people throughout time have known that when the three sides of a triangle are equal, then so are the three angles. Maybe some of the first people to theorize on this (be it Thales or some others) proved it this way[3]:

[3] Existing historical data are too thin to support or refute this parable as a historical conjecture. For the extant Greek sources on this theorem, see [7, vol. 1, p. 252ff.].



Let ABC be an equilateral triangle. Lift and turn it over, putting the side $\overline{AB}$ down in the direction $\overline{AC}$ originally took. Because $\overline{AB} = \overline{AC}$ in length, line $\overline{AB}$ will now lie exactly where $\overline{AC}$ did. Because the angle $\angle A$ has not changed, the line $\overline{AC}$ will now lie in the direction $\overline{AB}$ originally took, and because they are the same length $\overline{AC}$ will now lie exactly where $\overline{AB}$ did. So the line $\overline{BC}$ will lie just where it originally did, but with reversed direction. The moved angle $\angle B$ will coincide with the original place of the angle $\angle C$, and vice versa, proving $\angle B = \angle C$. Those could have been any two angles of ABC, so all the angles are all equal.

Anyone interested in this argument would quickly realize only the very last step requires an equilateral triangle. Merely assuming $\overline{AB} = \overline{AC}$ the reasoning already shows $\angle B = \angle C$. The theorem on equilateral triangles generalizes to isosceles triangles simply by ignoring the length of $\overline{BC}$. And even if you only care about the case of equilateral triangles, the quickest way to prove it is via this proof for isosceles triangles.

6.3 Generalization at the Origin of Abstract Algebra

A brilliant unifying generalization lies at the base of modern mathematics, namely, Hilbert's solution to Gordan's problem. Hilbert proved every *algebraic form* has a *finite complete system of invariants*. I will not define any of those terms because Hilbert's strategy was to ignore them almost entirely. He just kept in mind that the problem deals with "things" that can be added or multiplied by each other and can be multiplied by numbers. He did not define any new abstract kind of algebraic system to do this. He worked explicitly with polynomials over the complex numbers, but explicitly ignored nearly everything specific to them! Then he and others immediately saw this proof would apply to many other problems.[4]

Closely related to that example, the origin of commutative algebra illustrates both kinds of generalization and forms the basis for Grothendieck's theory of schemes.[5] By the late nineteenth century, unique prime factorization for the ordinary integers $\mathbb{Z}$ had two important articulating generalizations: to *algebraic integers* and to *complex function theory*. They extended the theory of factorization from the integers to other equally specific (and more complicated) domains.

[4] For a fuller treatment, see [13].

[5] A concise history from this viewpoint is [4, pp. 21–26].

The ring of *Gaussian integers* $\mathbb{Z}[\sqrt{-1}]$ contains numbers of the form

$$a + b \cdot \sqrt{-1} \qquad \text{with } a, b \in \mathbb{Z}.$$

A proof very like the familiar one for the integers $\mathbb{Z}$ shows Gaussian integers have unique prime factorization. But a different algebraic equation, $x^2 = -5$, gives a different ring $\mathbb{Z}[\sqrt{-5}]$

$$a + b \cdot \sqrt{-5} \qquad \text{with } a, b \in \mathbb{Z}$$

which does not have unique prime factorization. These numbers have unique prime *ideal* factorization where *ideals* are a concept not used in basic arithmetic. Dedekind defined *rings of algebraic integers* where the ordinary integers $\mathbb{Z}$ are combined with irrational numbers defined by any integer polynomial equations. The Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ are one example, $\mathbb{Z}[\sqrt{-5}]$ is another. Dedekind used rather special features of integer polynomials to prove every such ring has unique prime ideal factorization [2].[6]

In a further articulating generalization, important to Kronecker and especially to Hilbert, integer polynomials are replaced by polynomials with coefficients in some ring of algebraic integers. Again the details are not important except to say Emmy Noether would remove them in her startling unified general theory.[7]

As to complex function theory, consider the two-variable polynomial equation

$$y^2 = x^3 - x.$$

Its complex-number solutions define a curve

$$C \ = \ \{\langle x, y\rangle \in \mathbb{C} \times \mathbb{C} \mid y = \pm\sqrt{x^3 - x}\}.$$

A *regular function* on the curve C is any (complex coefficient) polynomial in x and y, with the provision that two polynomials define the same function on C if they take the same value at every point of C. So, for example, the polynomial x^4 defines the same function on C as $xy^2 + x^2$ because

$$x^4 = xy^2 + x^2 \quad \text{whenever} \quad y^2 - x^3 + x = 0.$$

It was crucial to nineteenth-century algebra and to Grothendieck's scheme theory that the equation follows from simply factoring the difference between the polynomials:

$$(x^4) - (xy^2 + x^2) \ = \ -x \cdot (y^2 - x^3 + x),$$

and so each side equals 0 when $y^2 - x^3 + x = 0$.

[6] A quick introduction is in [11].

[7] [9, p. 13] correctly says "Noether's proofs [...] were (and remain) startling in their simplicity". Compare [15].

Nineteenth-century complex analysts worked extensively with functions on curves such as C using all the tools of calculus and infinite series expansions plus algebraic means of reducing equations. In two (or more) variables, the reduction was not so simple as prime factorization. It led to *ideal factors* as in arithmetic but beyond the prime ideal factors it used a more general notion of *primary* ideal which will not be explained here.[8] And besides multiplying ideals by each other, it used intersections of primary ideals.

6.4 Forgetting the Details, for a Time

Emmy Noether came on this scene in the 1920s known to all leading German mathematicians and physicists for her work on conservation laws in Lagrangian mechanics. She took Dedekind's lead, but went far beyond him, seeing how huge amounts of the theory of factorization could be unified *by* forgetting the details.

Arithmeticians could forget their polynomials had integer coefficients, or coefficients in some ring of algebraic integers. They could even forget they were working with polynomials! They only had to remember the commutative, associative, and unit laws for addition and for multiplication, and the distributive law relating them, plus a few general conditions such as the *ascending chain condition* or *integral closedness* which were known and used in all the special cases already. Complex analysts could forget they were using the complex numbers, and forget about polynomials—and keep just the same general facts as the arithmeticians.

Obviously, when they forget these things they lose many specific theorems. But Noether saw the quickest way to those specific theorems was to start without the details and fill them in later as needed. This was faster than the classical approaches, simpler, more unified, and immediately gave strong new results about the special cases. It also made the theorems more general in principle. At first this generality held only *in principle* as the theorems were really only used for the previously known cases. But the general theory quickly led to the creation of useful new specific cases unimagined before.

Grothendieck was in a different situation. For one thing, Noether's methods though only 20 years old were a well-established success—so well established they were rather taken for granted, hardly associated with her in Paris where she could be regarded as having produced mere "generalities". Mathematicians were accustomed to this level of abstract generality as they had not been when Hilbert did his or Noether did hers. For another thing, Grothendieck aimed from the start of his work on this to produce a specific, new, yet-unknown case, namely, an arithmetic algebraic geometry to state and prove the Weil conjectures.[9] But he did not attack the conjectures directly. He set out to unify all the tools bearing on the problem so a solution would appear naturally.

[8] See, e.g. [19, Chap. 7].

[9] The Weil conjectures are much discussed elsewhere. See [12] and references there.

6.5 Schemes

Around 1950 algebraic geometry worked very well for cases meeting some conditions. The first of these should be reasonably familiar to philosophers of mathematics while the second is technical:

- for varieties defined by the complex-number solutions to polynomials with complex coefficients;
- as long as the defining polynomial is *reduced* (no factor divides it more than once) or the defining set of polynomials generates a reduced ideal.

These two together are sufficient, but *not necessary*, conditions to give the classical treatment of functions on varieties. Two different considerations were invoked above to show x^4 defines the same function on the curve C as $xy^2 + x^2$:

VALUES The two polynomials take the same value at every point $\langle x, y \rangle$ of the curve C.

FACTORS The difference of the two polynomials is divisible by the defining polynomial of C.

The reason classical algebraic geometry over the complex numbers insisted on using reduced defining polynomials was precisely to make these two clauses equivalent: any two polynomials on any variety V take the same values at every point of V if and only if their difference is divisible by the defining polynomial (or ideal) of V. For non-reduced defining polynomials, divisibility still implies equality of values but not conversely.

It may seem plausible that the clause VALUES gives the right criterion for identity of functions: a function is determined by its values. And the clause FACTORS explains why this part of geometry is *algebraic*: it reduces to polynomial algebra. And indeed these two clauses are equivalent in much more general settings than the classical. For example, the complex numbers can be replaced by any algebraically closed field K and not only do the clauses remain equivalent but the proof of their equivalence is nearly unchanged. This is the Hilbert Nullstellensatz described at the start of any modern textbook on algebraic geometry.[10]

But André Weil's vision of geometrized arithmetic required much more:

- varieties defined over the integers, *without* looking at integers as embedded in the complex numbers;
- non-reduced ideals are of central interest.

This was the start of an extremely valuable generalization of algebraic geometry to include arithmetic. Already in 1950 leading mathematicians saw that if it could possibly be made to work it would produce huge progress in number theory—as, in fact, it has.

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[10] Hilbert found it when he went to work replying to Gordan's objections to Hilbert's original non-constructive solution of Gordan's problem.

Several candidates for such a theory before Grothendieck took the route of articulating generalization. They tried to replace the complex numbers by other specific kinds of suitable rings, with suitable restrictions on the defining polynomials. In particular, most of them aimed to preserve the equivalence of VALUES and FACTORS. For that reason, among others, the ring of integers $\mathbb{Z}$ itself did not seem suitable. The appendix below uses a little pretty arithmetic to show how the equivalence of VALUES and FACTORS fails on a naive approach to algebraic geometry over the integers (even for reduced defining polynomials). When one thought only suitable rings could define varieties, there were constant innovations in just which rings those were. That is, people constantly sought larger and larger classes of rings that could count as "geometrical".

Grothendieck, though, took the route of unifying abstraction: do not introduce new kinds of rings. Forget about complex numbers or integers. Forget the Nullstellensatz and Noetherian conditions. Forget varieties are defined by polynomials. Forget that functions on varieties are polynomials. Forget polynomials.

Of course, these things will come back as needed for specific results later on. But forget them all in the basic theory.

Grothendieck insisted *every ring* should be the ring of functions on a space. He named these spaces *schemes*. He immediately knew there was just one way to do it and that way is quite naively intuitive. It is "of infantile simplicity" to quote [5, p. P32]. It is exactly what classical algebraic geometry always did, except forgetting that some things are complex numbers, and others are polynomials, and so on. Indeed, mathematicians had long known there would be one and only one way to do it.[11] It just meant abandoning VALUES in favour of FACTORS. Grothendieck did it.

Here a ring is any set R with selected zero and unit elements 0,1, and selected operations called addition, additive inverse, and multiplication satisfying the familiar laws. For all elements $x, y, z \in R$:

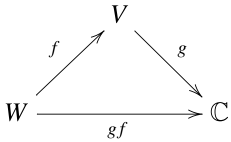
$$\begin{aligned} 0+x=x, \quad x+(-x)=0, \quad 1\cdot x=x,\\ x+y=y+x, \quad x+(y+z)=(x+y)+z,\\ x\cdot y=y\cdot x, \quad x\cdot(y\cdot z)=(x\cdot y)\cdot z,\\ x\cdot(y+z)=(x\cdot y)+(x\cdot z). \end{aligned}$$

The integers $\mathbb{Z}$ are a familiar example as are the complex numbers $\mathbb{C}$ but the whole point is to forget those specifics. A ring homomorphism $h: R \to R'$ is a function from R to R' that preserves 0, 1, and addition, additive inverse, and multiplication.

Each ring R is taken as the ring of regular functions on a space, a scheme, called the *spectrum* of R and written $\mathrm{Spec}(R)$. Classical algebraic geometry defined *regular maps* $f: W \to V$ between varieties in terms of polynomials. Scheme theory defines maps between schemes exactly the same way—only forgetting about polynomials.

[11] This includes Emmy Noether's school following van der Waerden, plus Weil and all the leading Parisian algebraic geometers, as documented in [12, p. 313ff.].

To spell this out: in classical, complex algebraic geometry a regular function g on any variety V will compose with any regular map $f: W \to V$ between varieties to give a regular function gf on W



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Composition obviously takes the constant 0 function on V to the constant 0 function on W, and similarly for the constant function 1. It also preserves pointwise addition of functions of V, and pointwise multiplication. That is, composition with a regular map produces a homomorphism from the ring of regular functions on V to that on W. Conversely every homomorphism (preserving also the complex coefficients) corresponds to a regular map.

Altogether a regular map of varieties $W \to V$ corresponds to a ring homomorphism, preserving complex coefficients, from the regular functions on V to those on W. And this is no by-the-way fact. It is central to the techniques of algebraic geometry.

A map of schemes $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ corresponds to a ring homomorphism in the other direction $R \to R'$. Just forget about complex numbers and polynomials.

In classical algebraic geometry, every reduced polynomial f on a variety V defines a subvariety $V' \subseteq V$, namely,

$$V' \;=\; \{p \in V \mid f(p) = 0\}.$$

More generally every reduced ideal I of regular functions on V defines a subvariety

$$V' \;=\; \{p \in V \mid f(p) = 0 \text{ for all } f \in I\}.$$

The powerful algebraic fact is that two subvarieties V', V'' of V have $V' \subseteq V''$ if and only if the ideal of V' contains the ideal of V''. In particular, the points of V correspond to the maximal ideals of the ring of regular functions.

Scheme theory preserves all of this only forgetting about polynomials and reducedness: Every element $f \in R$ determines a *closed subscheme* of $\mathrm{Spec}(R)$. Indeed every ideal $I \subset R$ determines a closed subscheme, while inclusion of ideals is equivalent to inclusion in the opposite direction of closed subschemes. Closed points correspond to maximal ideals.

Obviously, a detailed account of either classical or scheme theoretic algebraic geometry is impossible here. The comparison is made brilliantly in the famous *Red Book* [17]. To summarize: take any account of classical, complex algebraic varieties stated using the now classical ideas of *Zariski topology* and *generic points*, scratch out all reference to the complex numbers, replace all reference to "polynomials" or "regular functions" by "ring elements", scratch out the word "reduced". What remains is a reasonably clear account of scheme theory. Of course, it will only be the basics.

In practice, these turn out to be the right basics even for classical algebraic geometry. Thus, the introductory textbook [18] and the survey lectures [21] present basic classical algebraic geometry in a form pervasively shaped by schemes even though Reid mentions schemes only a few times—and Smith mentions them more often but never actually defines them.

Schemes are a huge extension of varieties. Most rings are not at all like the coordinate ring of any classical variety and so most schemes are not at all like varieties. In this sense, Deligne says "the decision to let every commutative ring define a scheme gives standing to bizarre *schemes*". Yet schemes are standard in today's algebraic geometry for two reasons. One is the unifying effect of taking restrictions out of the basic definitions and theory. And Deligne emphasizes the practical value of working in the full, larger domain: "it gives a *category of* schemes with nice properties" [3, p. 13]. That is also how Grothendieck and Dieudonné explain the success of schemes versus other attempted mergers of arithmetic with algebraic geometry. Arithmetical and geometrical constructions "get easy mathematical expression thanks to the functorial language (whose absence no doubt explains the timidity of earlier attempts)" [6, p. 6].[12]

Appendix: Naive Algebraic Geometry over the Integers

Let $y^2 - x^3 + x = 0$ define a "curve" of integer points:

$$C_{\mathbb{Z}} = \{\langle x, y\rangle \in \mathbb{Z} \times \mathbb{Z} \mid y^2 - x^3 + x = 0\}.$$

Efficient use of prime factorization, given below, shows $C_{\mathbb{Z}}$ has exactly three points:

$$C_{\mathbb{Z}} = \{\langle -1, 0\rangle, \langle 0, 0\rangle, \langle 1, 0\rangle\}.$$

All these points have $y = 0$ so the polynomials y and 0 give the same regular function on $C_{\mathbb{Z}}$ by criterion VALUES. Yet their difference y is obviously not divisible by the defining polynomial $y^2 - x^3 + x$ so they are not the same by criterion FACTORS. To study $C_{\mathbb{Z}}$ by tools of algebraic geometry, we must restore the role of FACTORS.

There are two ways to do this: 1) find new kinds of points for the "curve" $C_{\mathbb{Z}}$ so that the polynomials y and 0 do not agree at all these points or 2) give up the idea that a function is determined by its values at points. Scheme theory actually does both.

[12] Pierre has explained some of the reasons behind this absence [1, p. 398].

To keep the argument grounded in some mathematics, here is a proof that $C_{\mathbb{Z}}$ has just the three integer points.

Theorem 6.1 *The curve $C_{\mathbb{Z}}$ has just three integer points: $\langle -1, 0\rangle$, $\langle 0, 0\rangle$, $\langle 1, 0\rangle$.*

Proof Setting $y = 0$ gives three trivial solutions to $y^2 = 0 = x^3 - x$, namely:

$$x = 0,\ y = 0 \quad \text{or} \quad x = 1,\ y = 0 \quad \text{or} \quad x = -1,\ y = 0.$$

This follows from factoring $x^3 - x$ as $x \cdot (x - 1) \cdot (x + 1)$.

There are no solutions with $y \neq 0$. To see this, suppose $x^3 - x$ is a non-zero square. Then $x \neq 0$ and $x \neq \pm 1$. And notice a prime factor of x cannot also be a prime factor of $x^2 - 1$. So, since their product $x^3 - x$ is a square, both x and $x^2 - 1$ must be squares. (This is where the proof uses unique prime factorization of non-zero integers.) But then the successive integers $x^2 - 1$ and x^2 would both be squares. And this contradicts $x \neq \pm 1$ since the only successive integer squares are 0, 1.

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