Chapter 5
Reflections on the Axiomatic Approach to Continuity

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Abstract In Hilbert's paper "Axiomatic Thinking"—the published version of his 1917 Zürich talk - he touches on the axiomatic treatment of continuity and, as he puts it, "the dependence of the propositions of a field of knowledge on the axiom of continuity". By the "axiom of continuity", Hilbert seems to mean a number of things. In this paper I speculate on the various meanings Hilbert may have ascribed to the term. I focus in particular on interpreting the "axiom of continuity" as the central principal of Synthetic Differential Geometry that all real functions are smooth.

In Hilbert's paper "Axiomatic Thinking"—the published version[1] of his 1917 Zürich talk which the present meeting commemorates—he touches on the axiomatic treatment of continuity and, as he puts it, "the dependence of the propositions of a field of knowledge on the axiom of continuity".

By the "axiom of continuity", Hilbert seems to mean a number of things. He first assimilates it to the Archimedean axiom (which he also calls the "axiom of measurement") and observes its independence of the other axioms of the theory of real numbers. Presumably, he means the other axioms of the first-order theory of real numbers, since the Archimedean axiom is derivable in the second-order theory in which order-completeness is assumed.

Hilbert goes on to observe that the Archimedean axiom plays—implicitly at least—a role in physics.

> It seems to me that it has principal interest in physics as well; for it leads us to the following outcome. That is, the fact that we can come up with the dimensions and ranges of celestial bodies by putting together terrestrial ranges, namely measuring celestial lengths by terrestrial measure, as well as the fact that the distances inside atoms can be expressed in terms of metric measure, is by no means a merely logical consequence of propositions on the triangular congruence and the geometric configuration, but rather an investigative result of experience. The validity of the Archimedean axiom in nature, in the sense indicated above,

[1] Hilbert [6].

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> needs experimental confirmation just as much as does the proposition of the angle sums in triangle in the ordinary sense.

Hilbert asserts that the validity of the Archimedean axiom is "an investigative result of "experience". What he may mean here is that in comparing astronomical, terrestrial, and subatomic distances, none is infinitesimal, or infinitely large, with respect to the others. Thus, in principle, the radius of an electron could be used as a unit to measure terrestrial or astronomical distances.

What has this to do with continuity? Hilbert seems to imply that, so far as measurement is concerned, the empirical validity of the Archimedean axiom means that there is a kind of continuity—a smooth transition—between microcosm, mesocosm, and macrocosm. None of these realms is cut off from the others.

While the Archimedean axiom is exact, the notion of "continuity" associated with it, although suggestive, is essentially qualitative (and akin to Leibniz's principle of continuity, see below). In order to formulate an exact principle of continuity Hilbert turns to physics:

> In general, I should like to formulate the axiom of continuity in physics as follows: "If a certain arbitrary degree of exactitude is prescribed for the validity of a physical assertion, a small range shall then be specified, within which the presuppositions prepared for the assertion may freely vary so that the deviation from the assertion does notoverstep the prescribed degree of exactitude." This axiom in the main brings only that into expression which directly lies in the essence of experiments; it has always been assumed by physicists who, however, have never specifically formulated it.

(Note the little dig at physicists with which Hilbert concludes this passage—is this a foretaste of the famous, but perhaps apocryphal remark later attributed to Hilbert that "Physics is obviously much too difficult for the physicists."?).

Hilbert's formulation of the principle of continuity in physics—what I shall call the *physical continuity axiom* (**PCA**) is evidently an empirical version of the familiar (ε, δ) definition of a continuous function. More precisely, the axiom asserts that any *physical* function—that is, a function from real numbers to real numbers associated with a physical assertion—is (ε, δ)—continuous. This is an updated version of Leibniz's Principle of Continuity: *Natura non facit saltus.*

Before the nineteenth century, **PCA** would have been formulated in terms of infinitesimals, perhaps as follows:

> If the degree of exactitude is prescribed for the validity of a physical assertion, is prescribed to be within infinitesimal limits, then also within infinitesimal limits the presuppositions prepared for the assertion may freely vary so that the deviation from the assertion does not overstep the prescribed infinitesimal limits

This may be succinctly expressed as the *Principle of Infinitesimal Continuity for Physical Functions: any physical function sends infinitesimally close points to infinitesimally close points.*

These are all very strong "global" axioms which are to be contrasted with the "local" continuity axioms imposed on the system of real numbers such as the Archimedean principle or the order-completeness principle.

Hilbert's continuity axiom was formulated for the physical realm, but it can be extended to mathematics where it takes the form of

Brouwer's Continuity Principle:
BCP *All real functions are continuous.*

Of course, Brouwer did not regard this principle as an axiom—indeed he seems to have had a low opinion of the axiomatic method in mathematics. Rather he regarded it as a *fact* (albeit requiring demonstration) about the real numbers arising from the nature of the continuum as he conceived it.

The "infinitesimal" version of Brouwer's principle would read:

Universal Principle of Infinitesimal Continuity (**UPIC**): *any real function sends infinitesimally close points to infinitesimally close points.*

The question of the *consistency* of these extended principle of continuity arises immediately. It might seem at first glance that both **BCP** and **UPIC** are inconsistent since the "blip" function $b:\mathbb{R} \to \mathbb{R}$ defined by $b(0) = 1, b(x) = 0$ for $x \neq 0$ is obviously discontinuous, and, on any account of the notion of infinitesimal, fails to send points infinitesimally close to—but different from—0 to points infinitesimally close to 0. But the condition that b is defined on the *whole* of $\mathbb{R}$ rests on the unquestioned assumption that, for any real number x, either $x = 0$ or $x \neq 0$. This in turn rests on the *Law of Excluded Middle* (**LEM**)—the logical principle, going back to Aristotle, that, for any proposition, either it or its negation must be true. While **LEM** is a core principle of classical logic, it is not affirmed in *intuitionistic* logic, the system of logic implicit in Brouwer's conception of mathematics and later made explicit by his student Heyting.

Thus, while **BCP** and **UPIC** are inconsistent with classical mathematics, that is, mathematics based on classical logic, they can be, and in fact are, consistent with intuitionistic mathematics, that is, mathematics based on intuitionistic logic. In fact, within intuitionistic mathematics, **LEM** is *refutable* from **BCP** or **UPIC** in the sense that[2]

$$\textbf{BCP or UPIC} \Rightarrow \neg\forall x \in \mathbb{R}(x = 0 \vee x \neq 0).$$

Here, we have an example of mathematical axioms *actually refuting a logical axiom.* It is of interest to note that Cantor, in introducing his transfinite numbers, had to repudiate Euclid's 5th *axiom* that the whole is always greater than the part, and Bolyai and Lobachevsky (as well as Gauss) in their formulation of non-Euclidean geometry, were compelled to repudiate Euclid's 5th *postulate*. In both of these earlier cases the question of consistency was central, and it is equally important in the case of **BCP** or **UPIC.** In fact, just as models of non-Euclidean geometry were later constructed to establish its consistency, so models of mathematics have been constructed based on imtuitionistic logic and realizing **BCP** and **UPIC,** thus establishing the consistency of both.

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[2] Of course, in the case of **UPIC,** a precise meaning must be assigned to the term "infinitesimal". This will mot be long delayed.

An even stronger version of the continuity principle (implicitly adhered to in differential geometry) is:

SP *All real functions are smooth,* i.e. arbitrarily many times differentiable. (More generally, all functions between manifolds are smooth).

Axiom **SP** has been realized by adopting what amounts to a *synthetic* approach to differential geometry.

Traditionally, there have been two methods of deriving the theorems of (classical) geometry: the *analytic* and the *synthetic* or *axiomatic.* While the analytic method is based on the introduction of numerical coordinates, and so on the theory of real numbers, the idea behind the synthetic approach is to furnish the subject of geometry with a purely geometric foundation in which the theorems are then deduced by purely logical means from an initial body of axioms.

The most familiar examples of synthetic geometry are classical Euclidean geometry and the synthetic projective geometry introduced by Desargues in the seventeenth century and revived and developed by Carnot, Poncelet, Steiner and others during the nineteenth century.

The power of analytic geometry derives very largely from the fact that it permits the methods of the calculus, and, more generally, of mathematical analysis, to be introduced into geometry, leading in particular to *differential geometry* (a term, by the way, introduced in 1894 by the Italian geometer *Luigi Bianchi*). That being the case, the idea of a "synthetic" differential geometry seems elusive: how can differential geometry be placed on a "purely geometric" or "axiomatic" foundation when the apparatus of the calculus seems inextricably involved?

To my knowledge, there have been two attempts to develop a synthetic differential geometry. The first was initiated by Herbert Busemann in the 1940s, building on earlier work of Paul Finsler. Here, the idea was to build a differential geometry that, in its author's words, "requires no derivatives": the basic objects in Busemann's approach are not differentiable manifolds, but metric spaces of a certain type in which the notion of a geodesic can be defined in an intrinsic manner.

The second approach, that with which I shall be concerned here, was originally proposed in the 1960s by F. W. Lawvere, who was in fact striving to fashion a decisive axiomatic framework for continuum mechanics. His ideas have led to what I shall simply call *synthetic differential geometry* (**SDG**—often referred to as *smooth infinitesimal analysis* **SIA**).[3] **SDG** is formulated within *category theory,* the branch of mathematics created in 1945 by Eilenberg and Mac Lane which deals with mathematical form and structure in its most general manifestations. As in biology, the viewpoint of category theory is that mathematical structures fall naturally into species or *categories*. But a category is specified not just by identifying the species of mathematical structure which constitute its *objects*; one must also specify the transformations or *maps* linking these objects. Thus, one has, for example, the category **Set** with objects all sets and maps all functions between sets; the category **Grp** with objects all groups and maps all group homomorphisms; the category **Top** with objects all topological spaces and maps all continuous functions; and **Man**, with objects all (Hausdorff,

[3] For accounts of **SDG/SIA** see [1] and [7].

second countable) smooth manifolds and maps all smooth functions. Since differential geometry "lives" in **Man**, it might be supposed that in formulating a "synthetic differential geometry" the category-theorist's goal would be to find an axiomatic description of **Man** itself.

But in fact the category **Man** has a couple of "deficiencies" which make it unsuitable as an object of axiomatic description:

- It lacks exponentials: that is, the "space of all smooth maps" T^S from one manifold S to another T in general fails to be a manifold.
- It also lacks "infinitesimal spaces"; in particular, it contains no "infinitesimal" manifold which could serve as a *generic tangent vector* for manifolds. If we follow the nineteenth-century differential geometers in thinking of a tangent vector to a manifold as an "infinitesimal straight path" in it, then a generic tangent vector would be a an "infinitesimal" manifold $\mathbf{\Delta}$ which generates arbitrary tangent vectors in the sense that any tangent vector to a manifold M may be identified with a smooth map from $\mathbf{\Delta}$ to M. That being the case, the tangent bundle of an arbitrary manifold M becomes identifiable as the exponential "manifold" $M^{\mathbf{\Delta}}$ of all smooth maps $\mathbf{\Delta} \ \to M$. A generic tangent vector may be conceived as am "infinitesimal" straight line segment Δ of which remains *straight* and *unbroken* under any smooth map. In other words, *the effect of any smooth map on Δ is to subject it to nothing more than a Euclidean motion.*

Lawvere's idea was to enlarge **Man** to a category **S**—a category of so-called (*smooth) spaces* or a *smooth category*—which makes up for these two deficiencies, admits a simple axiomatic description, and at the same time is sufficiently similar to **Set** for mathematical construction and calculation to take place in the familiar way.

The essential features of a smooth category **S** are these:

- In enlarging **Man** to **S** no "new" maps between manifolds are added, that is, all maps in **S** between spaces of **Man** are smooth. (Notice that this is not the case when **Man** is enlarged to **Set**.)
- **S** is *Cartesian closed*, that is, for any spaces S, T in **S,** the product space $S \times T$ and the exponential space T^S of all smooth maps from S to T are both contained in **S**.
- **S** *contains a generic tangent vector.* This will be identified as a certain "infinitesimal" segment Δ of the real line R.

From the presence of the generic tangent vector Δ in **S,** it follows that curves are "infinitesimally straight" in the following sense. For consider any curve C in a space M—that is, the image of a segment of R (containing Δ) under a (smooth) map f into M. Then the image of Δ under f may considered as a short straight line segment lying within C around the point $p = f(0)$ of C coinciding locally—"infinitesimally"—with the tangent line to C at p. Thus each point of C is "encased" within an infinitesimal straight line lying entirely within C. In short, curves in **S** are "imfinitesimally straight". This is the *Principle of Infinitesimal Straightness.*

We can give an explicit description of Δ by considering the simplest curve deviating from straightness, namely, the parabola with equation $y = x^2$. Since the tangent to this curve at $x = 0$ is the x-axis, Δ may be identified with the intersection of the parabola with the x-axis. That is,

$$\Delta = \{x : x \in R \wedge x^2 = 0\}.$$

Thus, Δ consists of *real numbers whose squares vanish—"nilsquare"* infinitesimals We shall simply use the term "infinitesimal" for these, and the letter ε to denote an arbitrary infinitesimal in this sense.

A precise version of the Principle of Infinitesimal Straightness—the *Principle of Infinitesimal Linearity* (or *Kock-Lawvere axiom*)—which we now state, ensures that this is not the case in **S**. The principle states that

- in **S**, any map $f\colon \Delta \to R$ is (uniquely) *linear*, that is, for some *unique* $b \in R$, we have, for all ε,

$$f(\varepsilon) = f(0) + b\varepsilon.$$

In essence, this asserts that that the action of any real function f on Δ is a Euclidean transformation: a translation by $f(0)$ and a rotation b called the *slope of* f.

The Principle of Infinitesimal Linearity asserts also that the map $R^\Delta \to R \times R$ which assigns to each $f \in R^\Delta$ the pair ($f(0)$, slope of f) is an isomorphism:

$$R^\Delta \cong R \times R.$$

In differential geometry, $R \times R$ is the tangent bundle of R, so this isomorphism confirms that R^Δ may also be identified as that tangent bundle.

For any space M in **S**, we take the tangent bundle TM of M to be the exponential space M^Δ. Elements of M^Δ are called *tangent vectors* to M. Thus, a *tangent vector* to M at a point $p \in M$ is just a map $t\colon \Delta \to M$ with $t(0) = x$ That is, a tangent vector at p is an *infinitesimal path in M with base point p*. The *base point map* $\pi\colon TM \to M$ is defined by $\pi(t) = t(0)$. For $p \in M$, the fibre $\pi^{-1}(p) = T_pM$ is the *tangent space to M at p.*

Observe that, if we identify each tangent vector with its image in M, then *each tangent space to M may be regarded as lying in M.* In this sense, just as each curve in **S** is "infinitesimally straight", each space in **S** is "infinitesimally flat".

We check the compatibility of this definition of TM with the usual one in the case of Euclidean spaces:

$$T(R^n) = (R^n)^\Delta \cong (R^\Delta)^n \cong (R \times R)^n \cong R^n \times R^n.$$

The assignment $M \mapsto TM = M^\Delta$ can be turned into a functor in the natural way—the *tangent bundle functor*. (For $f\colon M \to N$, $Tf\colon TM \to TN$ is defined by $(Tf)t = f \circ t$ for $t \in TM$.)

The whole point of synthetic differential geometry is to render *the tangent bundle functor representable*: TM becomes identified with the space of all maps from some fixed object—in this case Δ)—to *M*. (Classically, this is impossible.) This in turn simplifies a number of fundamental definitions in differential geometry.

For instance, a *vector field* on a space *M* is an assignment of a tangent vector to *M* at each point in it, that is, a map ξ: $M \to TM = M^{\Delta}$ such that $\xi(x)(0) = x$ for all $x \in M$. This means that $\pi \circ \xi$ is the identity on *M*, so that a *vector field is a section of the base point map.*

The notions of affine connection, geodesic, and the whole apparatus of Riemannian geometry can also be developed within **SDG**.[4]

As an *axiomatic system,* **SIA** may be set up as a system of axioms for the (smooth) real line *R* involving the nilsquare infinitesimals already introduced. The core axiom in **SIA** is the aforementioned *Principle of Infinitesimal Linearity.* Again writing Δ for the set of infinitesumals, i.e.

$$\Delta = \left\{x : x \in R \wedge x^2 = 0\right\},$$

the principle can be stated:

For any f: $\Delta \to R$*, there is a* ***unique*** $b \in R$ *such that*

$$f(\varepsilon) = f(0) + b\varepsilon$$

holds for all ε.

This in turn gives rise to a simple definition of the derivative *f'* of *f:* given $r \in R$**,** $f'(r)$ is the unique $b \in R$ such that, for all ε, $f(r + \varepsilon) = f(r) + b\varepsilon$ (apply Infinitesimal Linearity to the function $\varepsilon \mapsto r + \varepsilon$). Then we get the equation:

$$f(r + \varepsilon) = f(r) + \varepsilon f'(r)$$

(Here, $\varepsilon f'(r)$ is the *infinitesimal increment* in the value of *f*). Similarly, we obtain all higher derivatives *f'', f''',* confirming that **SP** holds in **S**.

From the Principle of Infinitesimal Linearity we deduce the important *Principle of Infinitesimal Cancellation*, viz.

$$\textit{If}\ \varepsilon a = \varepsilon b\ \textit{for all}\ \varepsilon,\ \textit{then}\ a = b.$$

For the premise asserts that the graph of the function *g*: $\Delta \to R$ defined by $g(\varepsilon) = a\varepsilon$ has both slope *a* and slope *b*: the uniqueness condition in the Principle of Infinitesimal Linearity then gives $a = b$. The Principle of Infinitesimal Cancellation supplies the exact sense in which there are "enough" infinitesimals in **SIA**.

In **SIA**, there is a sense in which *everything is generated by the domain of infinitesmals.* Consider the set Δ^{Δ} of all maps $\Delta \to \Delta$. It follows from the Principle of Infinitesimal Linearity that *R* can be identified as the subset of Δ^{Δ} consisting of all maps vanishing at 0. In this sense, *R* is "generated" by Δ. Explicitly, Δ^{Δ} is a

[4] See [8].

monoid under composition which may be regarded as acting on Δ by composition: for $f \in \Delta^{\Delta}, f \cdot \varepsilon = f(\varepsilon)$. The subset V consisting of all maps vanishing at 0 is a submonoid naturally identified as the set of *ratios of infinitesimals.* The identification of R and V made possible by the Principle of Infinitesimal Linearity thus leads to the characterization of R itself as the set of ratios of infinitesimals. This was essentially the view of Euler, who regarded infinitesimals as formal zeros and real numbers as representing the possible values of 0/0. For this reason, Lawvere[5] has suggested that R in **SIA** should be called the space of *Euler reals.*

Once one has R, Euclidean spaces of all dimensions may be obtained as powers of R, and arbitrary manifolds may be obtained by patching together subspaces of these.

In **SIA**, the following are now easily deduced:

- $\mathbf{\Delta}$ is *nondegenerate,* i.e. $\mathbf{\Delta} \neq \{0\}$.[6]
- Call $x, y \in R$ infinitesimally close *and write* $x \approx y$ if $x - y \in \mathbf{\Delta}$. *If I is a closed interval in R, any* $f{:}I \to R$ is infinitesimally continuous *in the sense that, for* $x, y \in J$, $x \approx y$ *implies* $fx \approx fy$.

Accordingly, **UPIC** holds in **SIA**. That being the case, the postulates of **SIA** guarantee that the "blip" function is not defined on the whole of R, so that **SIA** is *incompatible with the Law of Excluded Middle of classical logic*. This fact is not "noticed" in developing mathematics in **SIA** because in basic mathematical analysis the Law of Excluded Middle is never actually invoked (through proofs by *reductio ad absurdum*, for example).

I have pointed out that, in a certain sense, **SIA** embodies the idea that *all manifolds, in particular all (smooth) spaces arising in mathematical physics, are generated by the domain of infinitesimals.* Thus a "structural" link between the infinitesimal world and the real world in its mathematical representation is "built into" **SIA**. This link between the infinitesimal and the real within **SIA** nay be considered a precise, but abstract, realization of the intuitive and somewhat nebulous—but sensationally productive—use of infinitesimals in the differential calculus of the seventeenth and eighteenth centuries. The link between the infinitesimal and the real in **SIA** can be made concrete in the following way**.** Suppose that we are investigating the behaviour of some variable quantity represented by a function F. The approach taken in **SIA**, as (implicitly) in the differential calculus, is to begin the investigation by confining it initially to the infinitesimal world. Life in the infinitesimal world is beautifully simple: curves are just straight lines and the squares of incremental changes vanish. This makes the determination of infinitesimal increments equally simple, enabling the increment $\varepsilon F'(x)$ in $F(x)$ to be presented in the form $\varepsilon k(x)$, where $k(x)$ is some explicit function whose form has been obtained by "infinitesimal" analysis. Thus, we obtain an "infinitesimal" equation of the form $\varepsilon F'(x) = \varepsilon k(x)$. Applying the Infinitesimal Cancellation Principle in turn yields the "differential" equation, but

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[5] See [9].

[6] It should be noted that, while Δ does not reduce to $\{0\}$, nevertheless 0 is the sole *element* of Δ in the (weak) sense that the assertion "there exists an element of Δ which is $\neq 0$" is refutable. Figuratively speaking, Δ is the "atom" 0 encased in an infinitesimal carapace.

still an equation relating "real" quantities

$$F'(x) = k(x) \tag{5.1}$$

which holds in the real world.

The Infinitesimal Cancellation Principle thus provides a formal, astonishingly simple link between the infinitesimal world and the real world, the world "in the large". The idea of a linkage between these two worlds was, as already observed, the animating principle behind applications of the calculus throughout the seventeenth and eighteenth centuries.

In practice, of course, Eq. (5.1), while of fundamental importance, is only the first step in determining the explicit form of the function *F.* For this, it is necessary to "integrate" *k,* that is, to provide *k* with an *antiderivative,* an explicit function *G* such that *G' = k*. It will then follow that *F' = G'*, from which we will be able to conclude that *F = G.* (Strictly speaking, *F* and G may differ by a constant function but we shall ignore this here.)

Carrying out this procedure in **SIA** requires the introduction of an additional postulate linking the infinitesimal with the real. Let *I* be a closed interval. We define a *stationary point* of a function $f: I \to R$ to be a point $a \in I$ at which *f* i*s locally constant* in the sense that, for all $x \in I$, $x \approx a$ implies $fx = fa$. If *a* is a stationary point of *f,* then, for any infinitesimal ε, since $a + \varepsilon \approx a$ *so* we must have $f(a + \varepsilon) = f(a)$. This means that $f(a) + \varepsilon f'(a) = f(a)$, so that $\varepsilon f'(a) = 0$ for all ε, from which it follows by the Infinitesimal Cancellation Principle that $f'(a) = 0$. Thus, a stationary point of a function is precisely a point at which the derivative of the function vanishes.

In classical analysis, if the derivative of a function is identically zero, the function is constant. This fact is the source of the following postulate concerning stationary points adopted in **SIA**:

Constancy Principle. If every point in a closed interval *I* is a stationary point of $f: I \to R$ (that is, if f' is identically 0), then *f* is constant.

It follows from the Constancy Principle that two functions with identical derivatives differ by at most a constant.

Put succinctly, the Constancy Principle asserts that "universal local constancy implies global constancy", or "infinitesimal behaviour determines global behaviour" The Constancy Principle brings into sharp focus the difference in **SIA** between points and infinitesimals. For if in the Constancy Principle one replaces "infinitesimal constancy" by "constancy at a point" the resulting "Principle" is false because *any function whatsoever* is constant at every point. But since in **SIA** all functions on *R* are smooth, the Constancy Principle embodies the idea that for such functions local constancy is sufficient for global constancy, that a nonconstant smooth function must be somewhere nonconstant over arbitrarily small intervals.

The Constancy Principle thus provides yet another bridge between the infinitesimal world and the world "in the large". Hermann Weyl could not see such a direct linkage between the two worlds, and inferred that this absence doomed the idea of infinitesimal, leading to its inevitable replacement by the limit concept. In his *Philosophy of Mathematics and Natural Science* [13], he says:

> [In its struggle with the infinitely small] the limiting process was victorious. For the limit is an indispensable concept, whose importance is not affected by the acceptance or rejection of the infinitely small. But once the limit concept has been grasped, it is seen to render the infinitely small superfluous. Infinitesimal analysis proposes to draw conclusions by integration from the behaviour in the infinitely small, which is governed by elementary laws, to the behaviour in the large; for instance, from the universal law of attraction for two material "volume elements" to the magnitude of attraction between two arbitrarily shaped bodies with homogeneous or non-homogeneous mass distribution. If the infinitely small is not interpreted 'potentially' here, in the sense of the limiting process, then the one has nothing to do with the other, the process in infinitesimal and finite dimensions become independent of each other, the tie which binds them together is cut.

SIA reconnects the infinitesimal and the extended. Behaviour "in the large" is completely determined by behaviour "in the infinitely small".

The Constancy Principle has another important consequence. Let us call a subset $D \subseteq R$ *discrete* if infinitesimally close elements of it are identical. The set of natural numbers and each of its subsets, all of which may be considered subsets of R are discrete.

It follows quickly from the Constancy Principle that *any map on* R (*or one of its closed intervals*) *to a discrete subset of* R *is constant.* To see this, let f be a map of R to a discrete set D. Then from $x \approx y$, we deduce $fx \approx fy$, and hence $fx = fy$, in D. So f is locally constant, and hence constant.

In ordinary analysis, R and each of its intervals is connected in the sense that they cannot be split into two nonempty subsets neither of which contains a limit point of the other. In **SIA**, these have the vastly stronger property of *cohesiveness:* they cannot be split *in any way whatsoever* into two disjoint nonempty subsets.[7] This follows quickly from the Constancy Principle: if $R = U \cup V$ with $U \cap V = \emptyset$, let 2 be the discrete subset $\{0, 1\}$ of R, and define $f\colon R \to 2$ by $f(x) = 1$ if $x \in U$, $f(x) = 0$ if $x \in V$. Then f is constant, that is, constantly 1 or 0. In the first case $V = \emptyset$, and in the second $U = \emptyset$.

One of the most widely discussed axioms in mathematics is the *Axiom of Choice.* Surprisingly, perhaps, this is *incompatible* with the various continuity axioms we have discussed. This is because, as shown in the 1970s, it implies **LEM**.[8] We shall demonstrate its refutability in **SIA** by deriving from it $\forall x \in R(x = 0 \vee x \neq 0)$, and hence that the discontinuous blip function is defined on the whole of R.

We take the Axiom of Choice in the particular form.

AC *for any family* A *of nonempty subsets of* R, *there is a function* $f\colon A \to R$ *such that* $f(X) \in X$ *for every* $X \in A$.

For each $x \in R$, define

$$A_x = \{y \in R : y = 0 \vee x = 0\},$$

[7] For more on cohesiveness see [2].

[8] See [4, 5].

$$B_x = \{y \in R : y = 1 \vee x = 0\}.$$

Clearly, $0 \in A_x$ and $1 \in B_x$, so these sets are both nonempty. By **AC**, we obtain a map f_x: $\{A_x, B_x\} \rightarrow R$ such that, for any $x \in R$, $f_x(A_x) \in A_x$ and $f_x(B_x) \in B_x$. Thus,

$$[f_x(A_x) = 0 \vee x = 0] \wedge [f_x(B_x) = 1 \vee x = 0].$$

Applying the distributive law for $\vee$ over $\wedge$ (valid in intuitionistic logic), we obtain

$$[f_x(A_x) = 0 \wedge f_x(B_x) = 1] \vee x = 0$$

whence

$$(*) \qquad f_x(A_x) \neq f_x(B_x) \vee x = 0.$$

Now clearly, $A_0 = B_0 = R$, so that $f_0(A_0) = f_0(B_0)$. Thus,

$$f_x(A_x) \neq f_x(B_x) \rightarrow x \neq 0.$$

So from (*), it follows that

$$x \neq 0 \vee x = 0$$

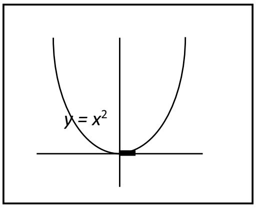
hence

$$\forall \chi \in R\, (x = 0 \vee x \neq 0).$$

The refutability of the Axiom of Choice in **SIA**, and hence its incompatibility with the Principle of Continuity which prevails there, is not surprising in view of the Axiom's well-known "paradoxical" consequences. One of these is the famous *Banach-Tarski paradox* [12] which asserts that any solid sphere can be decomposed into finitely many pieces which can themselves be reassembled to form two solid spheres each of the same size as the original, or into one solid sphere of any preassigned size. Paradoxical decompositions such as these become possible only when continuous geometric objects are, recalling Dedekind's words, "dissolved to atoms … [through a] frightful, dizzying discontinuity" into discrete sets of points which the Axion of Choice then allows to be rearranged in an arbitrary (discontinuous) manner. Such procedures violate the Principle of Continuity.

I conclude with some historical observations. While **SIA** was not developed until the 1960s, the idea of treating infinitesimals as nilpotent quantities was first put forward in works of 1694–6 by the Dutch physician **Bernard Nieuwentijdt** (1654–1718). Nieuwentijdt developed his account of infinitesimals—a striking example of axiomatic thinking—in conscious opposition to Leibniz's well-known theory of differentials. Nieuwentijdt postulates a domain of quantities, or numbers, subject to an ordering relation of greater or less. This domain includes the ordinary finite quantities, but it is also presumed to contain infinitesimal and infinite quantities—a quantity being infinitesimal, or infinite, when it is smaller, or, respectively, greater, than any arbitrarily given finite quantity. The whole domain is governed by a version of the Archimedean principle to the effect that zero is the only quantity incapable of being multiplied sufficiently many times to equal any given quantity. Infinitesimal quantities may be characterized as quotients *b/m* of a finite quantity *b* by an infinite quantity *m*. In contrast with Leibniz's differentials, Nieuwentijdt's infinitesimals have the property that the product of any pair of them vanishes[9]; in particular, squares and all higher powers of infinitesimals are zero. This fact enables Nieuwentijdt to show that, for any curve given by an algebraic equation, the hypotenuse of the differential triangle generated by an infinitesimal absciss increment e coincides with the segment of the curve between x and $x + e$. That is, a curve is locally straight, or, in seventeenth-century parlance, an "infinilateral polygon".

In responding to Nieuwentijdt's assertion that squares and higher powers of infinitesimals vanish, Leibniz remarked that "it is rather strange to posit that a segment dx is different from zero and at the same time that the area of a square with side dx is equal to zero". Yet this oddity may be regarded as a consequence—apparently unremarked by Leibniz himself—of one of his own key principles, namely, that curves may be considered as infinilateral polygons. Consider the curve $y = x^2$ below. Given that the curve is an infinilateral polygon, the infinitesimal straight portion of the curve between the abscissae 0 and dx must coincide with the tangent to the curve at the origin—in this case, the axis of abscissae—between those two points. But then the point (dx, dx^2) must lie on the axis of abscissae, which means that d$x^2 = 0$.



Now Leibniz could retort that this argument depends crucially on the assumption that the portion of the curve between abscissae 0 and dx, while undoubtedly infinitesimal, is indeed *straight*. If this be denied, then of course it does not follow that d$x^2 = 0$. But still, if one grants, as Leibniz does, that there is an infinitesimal portion of the curve between abscissae 0 and e (say) which is straight and does not reduce to a single point (so that e cannot be equated to 0), then the above argument does show that $e^2 = 0$. It follows that, *if curves are infinilateral polygons, then the "lengths" of the sides of these latter must be nilsquare infinititesimals.*[10] Accordingly, to do full justice to Leibniz's conception, *two* sorts of infinitesimals are required: first, "differentials"

[9] Here, Nieuwentijdt's theory conflicts with **SIA**, for in the latter it is not hard to refute the assertion that the product of any pair of infinitesimals vanishes. For more on this, see Bell [3].

[10] This is essentially the converse of Nieuwentijdt's observation above.

obeying—as laid down by Leibniz—the same algebraic laws as finite quantities; and second, the (necessarily smaller) nilsquare infinitesimals which measure the lengths of the sides of infinilateral polygons. It may be said that Leibniz recognized the need for the first, but not the second type of infinitesimal and Nieuwentijdt, *vice versa.* It is of interest to note that Leibnizian infinitesimals (differentials) are realized in *nonstandard analysis*,[11] the other major modern account of mathematical analysis built on a theory of infinitesimals. In fact it has been shown to be possible to construct models of **SIA** which at the same time embody enough of the theory of nonstandard analysis[12] to allow for the presence of Leibnizian infinitesimals in addition to the nilsquare variety.

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[11] See [11].

[12] See [10].