Chapter 2
Simplified Cut Elimination
for Kripke-Platek Set Theory

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Abstract The purpose of this article is to present a new and simplified cut elimination procedure for KP. We start off from the basic language of set theory and add constants for all elements of the constructible hierarchy up to the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$. This enriched language is then used to set up an infinitary proof system IP whose ordinal-theoretic part is based on a specific notation system $C(\varepsilon_{\Omega+1}, 0)$ due to Buchholz and his idea of operator controlled derivations. KP is embedded into IP and complete cut elimination for IP is proved.

2.1 Introduction

Kripke-Platek set theory KP (with infinity) is a remarkable subsystems of Zermelo-Fraenkel set theory and had an enormous impact on the interaction between various fields of mathematical logic; see, for example, Barwise [1]. Its proof-theoretic analysis has been carried through in Jäger [11], and it is known that the proof-theoretic ordinal of KP is the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$ and that KP proves the same arithmetical sentences as the theory ID_1 of positive inductive definitions (cf. Feferman [8] and Buchholz, Feferman, Pohlers, and Sieg [6]). Functional interpretations of KP have been studied by Burr [7] and Ferreira [9].

The purpose of this article is to present a new and simplified cut elimination procedure for KP. We start off from the basic language of set theory and add constants for all elements of the constructible hierarchy up to the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$. This enriched language is then used to set up an infinitary proof system IP whose ordinal-theoretic part is based on a specific notation system $C(\varepsilon_{\Omega+1}, 0)$ due to Buchholz (see, for example, Buchholz [3]) and his idea of operator controlled derivations. KP is embedded into IP and complete cut elimination for IP is proved.

In the older proof-theoretic treatments of theories for admissible sets infinitary systems of ramified set theory play a central role. The build up of the set terms in

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these systems has always been complicated—requiring a lot of technical intermediate steps to deal with, for example, extensionality and equality—and is now for free.

This article is organized as follows: We begin with a very compact presentation of Kripke-Platek set theory KP (with infinity) and its Tait-style variant KP^T. Then we discuss the ordinal notation system and the derivation operators needed for our analysis. Here we can confine ourselves to a "slimmed down" version of Buchholz [3]. It follows the presentation of the new infinitary system IP with its very simple term structure. After some partial soundness and completeness results for IP we show how to embed KP^T into IP. The last two sections are dedicated to cut elimination: predicative cut elimination and collapsing. The Hauptsatz then tells us that the Bachmann-Howard ordinal is an upper bound for the cut-free embedding of the Σ fragment of KP into IP; also, the constructible hierarchy up to the Bachmann-Howard ordinal is a model of the Π_2 fragment of KP.

2.2   Kripke-Platek Set Theory

Let $\mathcal{L}$ be the standard first-order language of set theory with $\in$ as the only relation symbol, countably many set variables, and the usual connectives and quantifiers of first-order logic. With regard to the later proof-theoretic analysis we want all formulas of $\mathcal{L}$ to be in negation normal form. Thus, the *atomic formulas* of $\mathcal{L}$ are all expressions $(u \in v)$ and $(u \notin v)$. The formulas of $\mathcal{L}$ are built up from these atomic formulas by means of $\vee, \wedge, \exists, \forall$ as usual. We use as metavariables (possibly with subscripts):

- u, v, w, x, y, z for set-theoretic variables,
- A, B, C, D for formulas.

As you can see, we have no connective for negation. However, the negation $\neg A$ of A is defined via de Morgan's laws and the law of double negation. In addition, we work with the following abbreviations:

$$
\begin{aligned}
(A \to B) &:\equiv (\neg A \vee B), \\
(A \leftrightarrow B) &:\equiv ((A \to B) \wedge (B \to A)), \\
(\exists x \in u) A[x] &:\equiv \exists x (x \in u \wedge A[x]), \\
(\forall x \in u) A[x] &:\equiv \forall x (x \in u \to A[x]), \\
(u = v) &:\equiv (\forall x \in u)(x \in v) \wedge (\forall x \in v)(x \in u).
\end{aligned}
$$

To simplify the notation we often omit parentheses if there is no danger of confusion. Moreover, we shall employ the common set-theoretic terminology and the standard notational conventions. For example, A^u results from A by restricting all unbounded quantifiers to u. The $\Delta_0, \Sigma, \Pi, \Sigma_n$, and Π_n formulas of $\mathcal{L}$ are defined as usual.

The logic of Kripke-Platek set theory is classical first-order logic. The *set-theoretic axioms of* KP consist of

(Equality) $u \in w \wedge u = v \to v \in w$,
(Pair) $\exists z (u \in z \wedge v \in z)$,

(Union)	$\exists z(\forall y \in u)(\forall x \in y)(x \in z)$,
(Infinity)	$\exists z(z \neq \varnothing \land (\forall x \in z)(x \cup \{x\} \in z))$,
(Δ_0-Sep)	$\exists z(z = \{x \in u : D[x]\})$,
(Δ_0-Col)	$(\forall x \in u)\exists y D[x, y] \rightarrow \exists z(\forall x \in u)(\exists y \in z)D[x, y]$,
($\in$-Ind)	$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$.

The formulas D in the schemas (Δ_0-Sep) and (Δ_0-Col) are Δ_0 whereas the formula A in the schema ($\in$-Ind) ranges over arbitrary formulas of $\mathcal{L}$.

2.3   A Tait-Style Reformulation of KP

For the later embedding into the infinitary system IP it is technically convenient to work with a Tait-style variant KP^T of KP. In KP^T we derive finite sets of $\mathcal{L}$ formulas rather than individual formulas. In the following the Greek letters Γ, Θ, Λ (possibly with subscripts) act as metavariables for finite sets of $\mathcal{L}$ formulas. Also, we write (for example) $\Gamma, A_1, \ldots, A_n$ for $\Gamma \cup \{A_1, \ldots, A_n\}$; similarly for expressions such as Γ, Θ, A. Finite sets of formulas are to be interpreted disjunctively.

Axioms of KP^T.

(Tnd)	$\Gamma, A, \neg A$ for all $\mathcal{L}$ formulas A.
(Equality)	$\Gamma, u \in w \land u = v \rightarrow v \in w$.
(Pair)	$\Gamma, \exists z(u \in z \land v \in z)$.
(Union)	$\Gamma, \exists z(\forall y \in u)(\forall x \in y)(x \in z)$.
(Infinity)	$\Gamma, \exists z(\varnothing \neq z \land (\forall x \in z)(x \cup \{x\} \in z)$.
(Δ_0-Sep)	$\Gamma, \exists z(z = \{x \in u : D[x]\})$.
(Δ_0-Col)	$\Gamma, (\forall x \in u)\exists y D[x, y] \rightarrow \exists z(\forall x \in u)(\exists y \in z)D[x, y]$.
($\in$-Ind)	$\Gamma, \forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$.

The formulas A in the Tertium-non-datur axioms (Tnd) and $\in$-induction axioms ($\in$-Ind) range over arbitrary $\mathcal{L}$ formulas whereas the formulas D in (Δ_0-Sep) and (Δ_0-Col) are supposed to be Δ_0.

Rules of inference of KP^T.

$$(or) \quad \frac{\Gamma, A_i \quad \text{for } i \in \{0, 1\}}{\Gamma, A_0 \lor A_1} \quad (and) \quad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \land A_1}$$

$$(b\text{-}ex) \quad \frac{\Gamma, u \in v \land A[u]}{\Gamma, (\exists x \in v)A[x]} \quad (b\text{-}all) \quad \frac{\Gamma, u \in v \rightarrow A[u]}{\Gamma, (\forall x \in v)A[x]}$$

$$(ex) \quad \frac{\Gamma, A[u]}{\Gamma, \exists x A[x]} \quad (all) \quad \frac{\Gamma, A[u]}{\Gamma, \forall x A[x]}$$

$$(cut) \quad \frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}.$$

In the rules (*b-all*) and (*all*) the eigenvariable u of these rules must not occur in their conclusion.

The notions of *principal formula* and *minor formula(s)* of an inference and that of *cut formula(s)* of a cut are as usual. We say that Γ is provable in KP^T iff there exists a finite sequence of finite sets of $\mathcal{L}$ formulas

$$\Theta_0, \ldots, \Theta_n$$

such that Θ_n is the set Γ and for any $i = 0, \ldots, n$ one of the following two conditions is satisfied:

- Θ_i is an axiom of KP^T;
- Θ_i is the conclusion of an inference of KP^T whose premise(s) are among $\Theta_0, \ldots, \Theta_{i-1}$.

In this case we write $\mathsf{KP}^T \vdash \Gamma$. It is an easy exercise to show that a formula A is provable in one of the usual Hilbert-style formalizations of KP iff $\mathsf{KP}^T \vdash A$. We leave all details to the reader.

2.4   An Ordinal System for the Bachmann-Howard Ordinal

Buchholz has developed several ordinal notation systems based on so called collapsing functions; see, for example Buchholz [2–4]. In the following we work with a reduced version, which is sufficient for our purposes. For that we need the following ingredients:

(1) Let *On* be the collection of all ordinals and let Ω be a sufficiently large ordinal. To simplify matters we set $\Omega := \aleph_1$, but also ω_1^{ck} or even somewhat smaller ordinals could do the job.
(2) The basic ordinal operations $\lambda\eta, \xi.(\eta + \xi)$ and $\lambda\xi.\omega^\xi$.
(3) The binary Veblen function φ, where φ_α is defined by transfinite recursion on α as the ordering function of the class

$$\{\omega^\beta : \beta \in On \ \& \ (\forall \xi < \alpha)(\varphi_\xi(\omega^\beta) = \omega^\beta)\}.$$

In the following we write $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$.
(4) An ordinal α is called *strongly critical* iff $\alpha = \varphi\alpha 0$.
(5) Every ordinal α has a *normal form*

$$\alpha =_{NF} \varphi\alpha_1\beta_1 + \ldots + \varphi\alpha_n\beta_n$$

with $\beta_i < \varphi\alpha_i\beta_i$ for $i = 1, \ldots, n$ and $\varphi\alpha_1\beta_1 \geq \ldots \geq \varphi\alpha_n\beta_n$.
(6) The collection $SC(\alpha)$ of *strongly critical components* of an ordinal α is defined by

$$SC(\alpha) := \begin{cases} \varnothing & \text{if } \alpha = 0, \\ \{\alpha\} & \text{if } \alpha \text{ is strongly critical,} \\ SC(\beta) \cup SC(\gamma) & \text{if } \alpha = \varphi\beta\gamma \text{ and } \beta, \gamma < \alpha, \\ \bigcup_{i=1}^{n} SC(\alpha_i) & \text{if } \alpha =_{NF} \alpha_1 + \ldots + \alpha_n \text{ and } n > 1. \end{cases}$$

Based on that we can now introduce, for all α and β, the ordinals $\psi(\alpha)$ and the sets of ordinals $C(\alpha, \beta)$.

Definition 2.4.1 By recursion of α we simultaneously define:

(1) $\psi(\alpha) := \min(\beta : C(\alpha, \beta) \cap \Omega = \beta)$.
(2) $C(\alpha, \beta)$ is the closure of $\beta \cup \{0, \Omega\}$ under $+$, φ, and $(\xi \mapsto \psi(\xi))_{(\xi < \alpha)}$.

Since $C(\alpha, \beta)$ is countable, it is clear that $\psi(\alpha)$ is always defined and less than Ω in case that Ω is interpreted as $\aleph_1$. If Ω is interpreted as ω^{ck} (or a smaller ordinal), then additional considerations are required.

Now we list a series of properties of the sets $C(\alpha, \beta)$. Their proofs are either standard or follow from the results in the articles of Buchholz mentioned above.

Lemma 2.4.2 *We have for all ordinals α and β:*

(1) $C(\alpha, 0) = C(\alpha, \psi(\alpha))$.
(2) $C(\alpha, \psi(\alpha)) \cap \Omega = \psi(\alpha)$.
(3) $C(\alpha, \beta) \cap \Omega$ is an ordinal.

Every set $C(\alpha, 0)$ is well-ordered by the usual less relation $<$ on the ordinals but is not an ordinal itself. For example,

$$\Omega \in C(\alpha, 0) \quad \text{and} \quad (\forall \xi < \Omega)(\psi(\alpha) \leq \xi \rightarrow \xi \notin C(\alpha, 0)).$$

If we write $ot(\alpha, \xi)$ for the order-type of an element ξ of $C(\alpha, 0)$ with respect to $C(\alpha, 0)$, then

- $ot(\alpha, \xi) = \xi$ for all $\xi \leq \psi(\alpha)$,
- $ot(\alpha, \xi) < \xi$ for all elements of $C(\alpha, 0)$ greater than $\psi(\alpha)$.

In particular, we have $ot(\alpha, \Omega) = \psi(\alpha)$ and the order types $ot(\alpha, \xi)$ of all elements ξ of $C(\alpha, 0)$ are countable.

We write $\varepsilon_{\Omega+1}$ for the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$. Its collapse $\eta := \psi(\varepsilon_{\Omega+1})$ is called the *Bachmann-Howard ordinal*. This number gained importance in proof theory since it is the proof-theoretic ordinal of the theory ID_1 of one positive inductive definition and of Kripke-Platek set theory KP; see, for example, Buchholz and Pohlers [5], Jäger [11], and Pohlers [13].

2.5 Derivation Operators

The general theory of derivation operators and operator controlled derivations has been introduced in Buchholz [3]. His main motivation was to provide a conceptually clear and flexible approach to infinitary proof theory that allows to put the finger on that part of the ordinal analysis of a sufficiently strong formal theory where the uniformity of proofs and a collapsing technique play the central role.

In this article we confine ourselves to that part of the general theory that goes along with the notation system $C(\varepsilon_{\Omega+1}, 0)$ described in the previous section.

Definition 2.5.1 Let $Pow(On)$ denote the collection of all sets of ordinals. A class function

$$\mathcal{H} : Pow(On) \to Pow(On)$$

is called a *derivation operator (d-operator for short)* iff it satisfies the following conditions for all $X, Y \in Pow(On)$:

 (i) $X \subseteq \mathcal{H}(X)$.
 (ii) $Y \subseteq \mathcal{H}(X) \implies \mathcal{H}(Y) \subseteq \mathcal{H}(X)$.
(iii) $\{0, \Omega\} \subseteq \mathcal{H}(X)$.
(iv) For all α,

$$\alpha \in \mathcal{H}(X) \iff SC(\alpha) \subseteq \mathcal{H}(X).$$

Hence every d-operator $\mathcal{H}$ is monotone, inclusive, and idempotent. Every $\mathcal{H}(X)$ is closed under $+$ and the binary Veblen function φ, and the decomposition of its members into their strongly critical components.

Let $\mathcal{H}$ be a d-operator. Then we define for all finite sets of ordinals $\mathfrak{m}$ the operators

$$\mathcal{H}[\mathfrak{m}] : Pow(On) \to Pow(On)$$

by setting for all $X \subseteq On$:

$$\mathcal{H}[\mathfrak{m}](X) := \mathcal{H}(\mathfrak{m} \cup X).$$

If $\mathcal{H}$ and $\mathcal{K}$ are d-operators, then we set

$$\mathcal{H} \subseteq \mathcal{K} := (\forall X \subseteq On)(\mathcal{H}(X) \subseteq \mathcal{K}(X)).$$

In this case $\mathcal{K}$ is called an *extension* of $\mathcal{H}$. The following observation is immediate from this definition.

Lemma 2.5.2 *If $\mathcal{H}$ is a d-operator, then we have for all finite sets of ordinals $\mathfrak{m}$, $\mathfrak{n}$:*

(1) $\mathcal{H}[\mathfrak{m}]$ is a d-operator and an extensions of $\mathcal{H}$.
(2) If $\mathfrak{m} \subseteq \mathcal{H}(\varnothing)$, then $\mathcal{H}[\mathfrak{m}] = \mathcal{H}$.
(3) $\mathfrak{n} \subseteq \mathcal{H}[\mathfrak{m}](\varnothing) \implies \mathcal{H}[\mathfrak{n}] \subseteq \mathcal{H}[\mathfrak{m}]$.

Now we turn to specific operators $\mathcal{H}_\sigma$. They will play a crucial role in connection with the embedding of KP into the infinitary proof system IP—to be introduced in the next section—and the collapsing procedure for IP.

Definition 2.5.3 We define, for all ordinals σ, the operators

$$\mathcal{H}_\sigma : Pow(On) \rightarrow Pow(On)$$

by setting for all $X \subseteq On$:

$$\mathcal{H}_\sigma(X) := \bigcap \{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \text{ and } \sigma < \alpha\}.$$

The following lemmas summarize those properties of these operators that will be needed later. For their proof we refer to [3], in particular Lemma 4.6 and Lemma 4.7. Assertion (5) is a consequence of Lemma 2.4.2(3).

Lemma 2.5.4 *We have for all ordinals σ, τ and all $X \subseteq On$:*

(1) $\mathcal{H}_\sigma$ is a derivation operator.
(2) $\mathcal{H}_\sigma(\varnothing) = C(\sigma + 1, 0))$.
(3) $\tau \leq \sigma$ and $\tau \in \mathcal{H}_\sigma(X) \implies \psi(\tau) \in \mathcal{H}_\sigma(X)$.
(4) $\sigma < \tau \implies \mathcal{H}_\sigma \subseteq \mathcal{H}_\tau$.
(5) $\mathcal{H}_\sigma(X) \cap \Omega$ is an ordinal.

Lemma 2.5.5 *Let $\mathfrak{m}$ a finite set of ordinals and σ an ordinal such that the following conditions are satisfied:*

$$\mathfrak{m} \subseteq C(\sigma + 1, \psi(\sigma + 1)) \cap \Omega \quad and \quad \sigma \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing).$$

Then we have for $\widehat{\alpha} := \sigma + \omega^{\Omega + \alpha}$ and $\widehat{\beta} := \sigma + \omega^{\Omega + \beta}$:

(1) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \implies \widehat{\alpha} \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ and $\psi(\widehat{\alpha}) \in \mathcal{H}_{\widehat{\alpha}}[\mathfrak{m}](\varnothing)$.
(2) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ and $\alpha < \beta \implies \psi(\widehat{\alpha}) < \psi(\widehat{\beta})$.
(3) $\mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \cap \Omega \subseteq \psi(\sigma + 1)$.

From now on the letter $\mathcal{H}$ will be used as a metavariable that ranges over d-operators.

2.6 The Infinitary Proof System IP

Henceforth, all ordinals used on the metalevel range over the set $C(\varepsilon_{\Omega+1}, 0)$ if not stated otherwise. In this section we introduce an infinitary proof system whose terms are constants for the elements of the initial segment of the constructible hierarchy L_η and whose proofs are controlled by derivation operators. Later we show that KP can be embedded into IP and that IP permits cut elimination and collapsing.

Definition 2.6.1 The language of IP is the following extension $\mathcal{L}[\eta]$ of $\mathcal{L}$:

(1) For each element a of $\boldsymbol{L_\eta}$ we fix a fresh constant $\bar{a}$. These constants are the *terms of* IP. The letters r, s, t (possibly with subscripts) act as metavariables for the terms of IP.
(2) The *level $|\bar{a}|$ of* $\bar{a}$ is the least ξ such that $a \in \boldsymbol{L_{\xi+1}}$.
(3) The *formulas of* IP are now easily obtained from the formulas of $\mathcal{L}$ by simply replacing all their free variables by terms of IP; i.e. the formulas of IP are the sentences of $\mathcal{L}[\eta]$.

Accordingly, the Δ_0, Σ, $\Pi\,\Sigma_n$, and Π_n formulas of IP are the Δ_0, Σ, $\Pi\,\Sigma_n$, and Π_n sentences of $\mathcal{L}[\eta]$, respectively.

Definition 2.6.2 Every IP formula is an expression of the form $F[\bar{a}_1, \ldots, \bar{a}_n]$ where $F[u_1, \ldots, u_n]$ is a formula of $\mathcal{L}$ with the free variables indicated and $a_1, \ldots, a_n$ are elements of $\boldsymbol{L_\eta}$. The set

$$par(F[\bar{a}_1, \ldots, \bar{a}_n]) := \{|\bar{a}_1|, \ldots, |\bar{a}_n|\}$$

is called the *parameter set* of this formula.

Observe that each Δ_0 sentence of $\mathcal{L}[\eta]$ has a non-empty parameter set. Below it will be necessary to measure the complexities of the cut formulas appearing in a derivation. To this end we assign a rank to each $\mathcal{L}[\eta]$ sentence.

Definition 2.6.3 The *rank $rk(F)$* of an $\mathcal{L}[\eta]$ sentence F is defined by induction on the number of symbols occurring in F as follows.

(1) $rk(\bar{a} \in \bar{b}) := rk(\bar{a} \notin \bar{b}) := \omega \cdot \max(|\bar{a}|, |\bar{b}|)$.
(2) $rk(F \vee G) := rk(F \wedge G) := \max(rk(F), rk(G)) + 1$.
(3) $rk((\exists x \in \bar{a}) F[x]) := rk((\forall x \in \bar{a}) F[x]) := \max(\omega \cdot |\bar{a}|, rk(F[\bar{\varnothing}]) + 1)$.
(4) $rk(\exists x\, F[x]) := rk(\forall x\, F[x]) := \max(\Omega, rk(F[\bar{\varnothing}]) + 1)$.

Finally, we define the *level $lev(F)$* of an IP formula F by

$$lev(F) := \begin{cases} \max(par(F)) & \text{if } rk(F) < \Omega, \\ \Omega & \text{if } \Omega \leq rk(F)). \end{cases}$$

Some important properties of the ranks of $\mathcal{L}[\eta]$ sentences are summarized in the following lemma. Its proof is straightforward and left to the reader.

Lemma 2.6.4 *We have for all IP formulas F, G and all $a, b \in \boldsymbol{L_\eta}$;*

(1) $rk(F) = rk(\neg F)$.
(2) $rk(F) < \omega \cdot lev(F) + \omega$.
(3) $rk(F), rk(G) < rk(F \vee G)$.
(4) $|\bar{b}| < lev(F[\bar{\varnothing}]) \implies rk(F[\bar{b}]) = rk(F[\bar{\varnothing}])$.
(5) $b \in a \implies rk(F[\bar{b}]) < rk((\exists x \in \bar{a}) F[x])$.

(6) $rk(F[\bar{b}]) < rk(\exists x F[x])$.

(7) $rk(F) \in \mathcal{H}[par(F)](\varnothing)$.

(8) $\alpha \in par(F) \implies \alpha \leq rk(F)$.

The proof system for IP will be Tait-style. From now on we let the Greek letters Γ, Θ, Λ (possibly with subscripts) also range over finite sets of $\mathcal{L}[\eta]$ sentences.

For a finite set $\mathfrak{S} = \{F, \ldots, F_m, r_1, \ldots, r_n\}$ of formulas and terms of IP we set

$$par(\mathfrak{S}) := par(F_1) \cup \ldots \cup par(F_m) \cup \{|r_1|, \ldots, |r_n|\}.$$

Accordingly,

$$\mathcal{H}[\Gamma, F_1, \ldots, F_m, r_1, \ldots, r_n] := \mathcal{H}[par(\Gamma) \cup par(\{F_1, \ldots, F_m, r_1, \ldots, r_n\})].$$

Variants of this notation may also be used. However, it will always be clear from the context what is meant.

Axioms of IP. The axioms of IP are all finite sets

$$\Gamma, (\bar{a}_1 \in \bar{b}_1) \quad \text{and} \quad \Gamma, (\bar{a}_2 \notin \bar{b}_2)$$

with $a_1, a_2, b_1, b_2 \in \boldsymbol{L_\eta}$, $a_1 \in b_1$, and $a_2 \notin b_2$.

So the axioms of IP are the finite sets that contain true atomic sentences of $\mathcal{L}[\eta]$. The next definition introduces the derivability relation, controlled by derivation operators.

Definition 2.6.5 $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma$ iff $par(\Gamma) \cup \{\alpha\} \subseteq \mathcal{H}(\varnothing)$ and one of the following cases holds:

(**Ax**) Γ is an axiom

($\vee$) $F_0 \vee F_1 \in \Gamma$ & $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, F_i$ & $\alpha_0 < \alpha$, $i \in \{0, 1\}$

($\wedge$) $F_0 \wedge F_1 \in \Gamma$ & $\mathcal{H} \left|\frac{\alpha_i}{\rho}\right. \Gamma, F_i$ & $\alpha_i < \alpha$ for $i = 0, 1$

($b\exists$) $(\exists x \in \bar{a})F[x] \in \Gamma$ & $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, F[\bar{b}]$ & $\alpha_0, |\bar{b}| < \alpha$, $b \in a$

($b\forall$) $(\forall x \in \bar{a})F[x] \in \Gamma$ & $\mathcal{H}[\bar{b}] \left|\frac{\alpha_b}{\rho}\right. \Gamma, F[\bar{b}]$ & $\alpha_b < \alpha$ for all $b \in a$

($\exists$) $\exists x F[x] \in \Gamma$ & $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, F[\bar{b}]$ & $\alpha_0, |\bar{b}| < \alpha$,

($\forall$) $\forall x F[x] \in \Gamma$ & $\mathcal{H}[\bar{b}] \left|\frac{\alpha_b}{\rho}\right. \Gamma, F[\bar{b}]$ & $\alpha_b < \alpha$ for all $b \in \boldsymbol{L_\eta}$

(**Ref**) $\exists x F^x \in \Gamma$ & $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, F$ & $\alpha_0, \Omega < \alpha$, $F \in \Sigma$

(**Cut**) $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, F$ & $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, \neg F$ & $rk(F) < \rho$, $\alpha_0 < \alpha$

We now list a series of properties of operator controlled derivations before we turn to embedding, cut elimination, and collapsing in the following sections. The first is the Weakening Lemma, and its proof is by straightforward induction on α.

Lemma 2.6.6 (Weakening) *Assume that* $\mathcal{H} \subseteq \mathcal{H}'$, $\alpha \leq \beta$, $\rho \leq \sigma$, *and that* $par(\Lambda) \cup \{\beta\} \subseteq \mathcal{H}'(\varnothing)$. *Then*

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \quad \Longrightarrow \quad \mathcal{H}' \vdash^{\beta}_{\sigma} \Gamma, \Lambda.$$

In the following, we will frequently make use of Weakening without making specific reference to it. An important such use is described in the next remark.

Remark 2.6.7 The inference rules formulated above are somewhat special in the sense that it is assumed that the principal formula of each rule (R) is an element of the premise(s) of (R). In general, however, this is not a severe limitation. Assume that we have a rule (R) with the principal formula F and the minor formulas $(F_i)_{(i:I)}$. Suppose further that F does not belong to Γ and that we have

$$\mathcal{H}_i \vdash^{\alpha_i}_{\rho} \Gamma, F_i \quad \text{and } \alpha_i < \alpha \quad \text{(for all } i \in I) \qquad (*),$$

where $\mathcal{H}_i$ is either of the form $\mathcal{H}$ or $\mathcal{H}[\bar{b}_i]$, and possibly further side conditions. If $par(F) \cup \{\alpha\} \subseteq \mathcal{H}(\varnothing)$—and this requirement will be satisfied below in all relevant cases—then we can apply Weakening and obtain

$$\mathcal{H}_i \vdash^{\alpha_i}_{\rho} \Gamma, F, F_i \quad \text{and } \alpha_i < \alpha \quad \text{(for all } i \in I).$$

Now the premises have the right form to conclude

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F. \qquad (**).$$

In the following, this intermediate step will often be omitted, and we go directly over from (*) to (**).

Also the next lemma presents some basic properties of IP; its proof is again by induction on α.

Lemma 2.6.8 (Inversion)

(1) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F_0 \vee F_1 \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F_0, F_1$.
(2) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F_0 \wedge F_1 \ and \ i \in \{0, 1\} \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F_i$.
(3) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in \bar{a})F[x] \ and \ b \in a \quad \Longrightarrow \quad \mathcal{H}[\bar{b}] \vdash^{\alpha}_{\rho} \Gamma, F[\bar{b}]$.
(4) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F[x] \ and \ |\bar{b}| \in \mathcal{H}(\varnothing) \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F[\bar{b}]$.
(5) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F[x] \ and \ x \ does \ not \ occur \ in \ F[x] \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F$.
(6) $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F[x] \ and \ |\bar{a}| \in \mathcal{H}(\varnothing) \quad \Longrightarrow \quad \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in \bar{a})F[x]$.

We end this section with a third structural lemma. It will be essential in connection with the Collapsing Theorem of Sect. 2.10.

Lemma 2.6.9 (Boundedness) *Let F be a Σ sentence of $\mathcal{L}[\eta]$ and assume that $\alpha \leq \beta \in \mathcal{H}(\varnothing) \cap \Omega$ and that r is the constant $\bar{L}_\beta$. Then we have*

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F \implies \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F^r.$$

Proof We show this assertion by induction on α and assume $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F$. First observe that this derivation contains no instances of (Ref) since $\alpha < \Omega$. If Γ, F is an axiom, then Γ, F^r is an axiom as well, and we are done. Otherwise we distinguish cases according to the last inference of $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, F$.

(i) The last inference was ($\exists$) with principal formula F. Then F is of the form $\exists x\, G[x]$ and there exist an $\alpha_0 < \alpha$ and a set $b \in L_\alpha$ such that

$$\mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, F, G[\bar{b}],$$

where $\Lambda := \Gamma \setminus \{F\}$. By two applications of the induction hypothesis we obtain

$$\mathcal{H} \vdash^{\alpha_0}_{\rho} \Lambda, F^r, G^r[\bar{b}].$$

Since $b \in L_\alpha \subseteq L_\beta$ and $(\exists x \in r)G^r[x]$ is the formula F^r, an inference $(b\exists)$ yields

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Lambda, F^r.$$

(ii) In all other cases we apply the induction hypothesis to the premise(s) of this inference (once or twice depending on whether F was the principal formula of this inference), and then apply it again.

2.7 Partial Soundness and Completeness of IP

The design of the infinitary proof system IP is so that all its axioms and rules of inference—with the exception of (Ref)—are correct with respect to L_η. This gives us the following result.

Theorem 2.7.1 (Partial soundness of IP) *For any Γ, any d-operator $\mathcal{H}$ and all α, ρ we have that*

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \text{ and } \alpha < \Omega \implies L_\eta \models \Gamma.$$

Proof By straightforward induction on α. Observe that because of $\alpha < \Omega$, the inference rule (Ref) has not been used in this derivation.

It is clear that (Ref) is not correct in L_η. For proofs of depths greater than Ω, which use the rule (Ref), more subtle considerations are needed. We will see what to do with such derivations in Sect. 2.10.

Let us now turn to (partial) completeness. We will show that all Π sentences that are true in $\boldsymbol{L_\eta}$ can be derived in IP. To prove that we need the following auxiliary consideration. Recall the introduction of the derivation operator $\mathcal{H}_0$ in Definition 2.5.3.

Lemma 2.7.2 *If a and b are elements of $\boldsymbol{L_\eta}$ with $b \in a$, then we have for all $\mathcal{L}[\boldsymbol{\eta}]$ sentences $G[\bar{\varnothing}]$ that*

$$\mathcal{H}_0[G[\bar{b}]] \subseteq \mathcal{H}_0[(\exists x \in \bar{a})G[x]].$$

Proof Let $\mathcal{K}$ be the d-operator $\mathcal{H}_0[(\forall x \in \bar{a})G[x]]$. Then we clearly have

$$par(G[\bar{\varnothing}]) \cup \{|\bar{a}|\} \subseteq \mathcal{K}(\varnothing).$$

By Lemma 2.5.4, the set $\mathcal{K}(\varnothing) \cap \Omega$ is an ordinal. Therefore, $|\bar{b}| \in \mathcal{K}(\varnothing)$. Consequently,

$$par(G[\bar{b}]) \subseteq \mathcal{K}[\varnothing].$$

Thus Lemma 2.5.2 yields our assertion.

Theorem 2.7.3 (Partial completeness of IP) *If F is a Π sentence of $\mathcal{L}[\boldsymbol{\eta}]$, then*

$$\boldsymbol{L_\eta} \models F \quad \Longrightarrow \quad \mathcal{H}_0[F] \left|\frac{rk(F)}{0}\right. F.$$

Proof First observe that $par(F) \cup \{rk(F)\} \subseteq \mathcal{H}_0[F](\varnothing)$; see Lemma 2.6.4(7). The proof is by induction on $rk(F)$.

(i) F is atomic. Then F is an axiom and, therefore, $\mathcal{H}_0[F] \left|\frac{rk(F)}{0}\right. F$.

(ii) F is of the form $(\exists x \in \bar{a})G[x]$. Then there exists a $b \in a$ such that $\boldsymbol{L_\eta} \models G[\bar{b}]$. By the induction hypothesis we thus have

$$\mathcal{H}_0[G[\bar{b}]] \left|\frac{\alpha_b}{0}\right. G[\bar{b}]$$

for $\alpha_b := rk(G[\bar{b}])$. From the previous lemma (and Weakening) we see that

$$\mathcal{H}_0[(\exists x \in \bar{a})G[x]] \left|\frac{\alpha_b}{0}\right. (\exists x \in \bar{a})G[x], \ G[\bar{b}].$$

An application of $(b\exists)$ implies the assertion; that $|\bar{b}| < rk((\exists x \in \bar{a})G[x])$ follows from Lemma 2.6.4(8).

(iii) (ii) F is of the form $\forall x G[x]$. Then

$$\boldsymbol{L_\eta} \models G[\bar{b}]$$

for all $b \in \boldsymbol{L_\eta}$ and the induction hypothesis implies

$$\mathcal{H}_0[G[\bar{b}]] \left|\frac{\alpha_b}{0}\right. G[\bar{b}]$$

for $\alpha_b := rk(G[\bar{b}])$. Since $par(G[\bar{b}]) \subseteq par(F, |\bar{b}|)$ we obtain (Weakening)

$$\mathcal{H}_0[F, \bar{b}] \,\big|\frac{\alpha_b}{0}\, F, G[\bar{b}]$$

for all $b \in L_\eta$. It only remains to apply an ($\forall$) inference.

(iv) All other cases are similar or even simpler.

It is worth pointing out that the above approach does not work for Σ sentences true in L_η: Assume that $L_\eta \models \exists x G[x]$ for a, say, Δ_0 sentence $G[\bar{\varnothing}]$. Then there exists a $b \in L_\eta$ for which $L_\eta \models G[\bar{b}]$. With the induction hypothesis we would get

$$\mathcal{H}_0[G[\bar{b}|] \,\big|\frac{rk(G[\bar{b}])}{0}\, G[\bar{b}],$$

leading then, by an application of ($\exists$), to

$$\mathcal{H}_0[G[\bar{b}]] \,\big|\frac{rk(\exists x G[x])}{0}\, \exists x G[x].$$

But $\mathcal{H}_0[G[\bar{b}|]$ may depend on $\bar{b}$, and can this dependency be avoided?

2.8 Embedding of KP^T into IP

In order to show that KP^T can be embedded into IP we first deal with the axioms of KP^T. We show—step by step—how they can be proved in IP.

Lemma 2.8.1 (Tertium-non-datur) *For all Γ, all F, and all d-operators $\mathcal{H}$,*

$$\mathcal{H}[\Gamma, F] \,\big|\frac{\alpha}{0}\, \Gamma, F, \neg F,$$

where $\alpha := \omega^{rk(F)} \# \omega^{rk(F)}$.

Proof In view of Lemma 2.6.4(7) the ordinal condition

$$par(\Gamma, F) \cup \{\alpha\} \subseteq \mathcal{H}[\Gamma, F](\varnothing)$$

is clearly satisfied. We prove our assertion by induction on $rk(F)$.

(i) If F is atomic, the $\Gamma, F, \neg F$ is an axiom since either $L_\eta \models F$ or $L_\eta \models \neg F$.

(ii) F is of the form $\exists x G[x]$. Pick an arbitrary $b \in L_\eta$. Then we define

$$\begin{aligned}
\alpha_b &:= \omega^{rk(G[\bar{b}])} \# \omega^{rk(G[\bar{b}])}, \\
\beta_b &:= |\bar{b}| + \alpha_b, \\
\gamma_b &:= \omega^{rk(\exists x G[x])} \# \omega^{rk(G[\bar{b}])}.
\end{aligned}$$

Obviously, $par(\Gamma, F, G[\bar{b}]) \subseteq par(\Gamma, F, \bar{b})$, and by some simple ordinal computatons we obtain

$$|\bar{b}| \ < \ \beta_b + 1 \ < \ \gamma_b \ < \ \alpha.$$

The induction hypothesis implies

$$\mathcal{H}[\Gamma, F, \bar{b}] \vdash_0^{\alpha_b} \Gamma, \ F, \ \neg F, \ G[\bar{b}], \ \neg G[\bar{b}],$$

and using an inference $(\exists)$ we arrive at

$$\mathcal{H}[\Gamma, F, \bar{b}] \vdash_0^{\gamma_b} \Gamma, \ F, \ \neg F, \ \neg G[\bar{b}].$$

Therefore, an application of $(\forall)$ gives the desired result.

(iii) All other cases are similar.

Lemma 2.8.2 (Equality) *For all Γ and all $a, b, c \in \mathbf{L_\eta}$,*

$$\mathcal{H}[\Gamma, \bar{a}, \bar{b}, \bar{c}] \vdash_0^{\alpha} \Gamma, \ \bar{a} = \bar{b} \wedge \bar{a} \in \bar{c} \ \rightarrow \ \bar{b} \in \bar{c},$$

where $\alpha := \max(\omega \cdot |\bar{a}| + 4, \ \omega \cdot |\bar{b}| + 4, \ \omega \cdot |\bar{c}| + 2)$.

Proof This is an immediate consequence of Theorem 2.7.3 since α is the rank of the formula $(\bar{a} = \bar{b} \wedge \bar{a} \in \bar{c} \ \rightarrow \ \bar{b} \in \bar{c})$.

Lemma 2.8.3 (Pair) *For all Γ, all $a, b \in \mathbf{L_\eta}$, and all d-operators $\mathcal{H}$,*

$$\mathcal{H}[\Gamma, \bar{a}, \bar{b}] \vdash_0^{\alpha+2} \Gamma, \ \exists x(\bar{a} \in x \wedge \bar{b} \in x),$$

where $\alpha := \max(|\bar{a}|, |\bar{b}|)$.

Proof We know that $a, b \in \mathbf{L}_{\alpha+1} \in \mathbf{L_\eta}$ and, if r is the constant for $\mathbf{L}_{\alpha+1}$,

$$par(\bar{a} \in r \wedge \bar{b} \in r) \subseteq \mathcal{H}[\bar{a}, \bar{b}](\varnothing).$$

Since $\bar{a} \in r$ and $\bar{b} \in r$ are axioms of IP we have

$$\mathcal{H}[\Gamma, \bar{a}, \bar{b}] \vdash_0^1 \Gamma, \ \bar{a} \in r \wedge \bar{b} \in r$$

and an application of $(\exists)$ yields our assertion.

Lemma 2.8.4 (Union) *For all Γ, all $a \in \mathbf{L_\eta}$, and $\alpha := |\bar{a}|$*

$$\mathcal{H}_0[\Gamma, \bar{a}] \vdash_0^{\omega \cdot \alpha + \omega} \Gamma, \ \exists z (\forall y \in \bar{a})(\forall x \in y)(x \in z).$$

Proof Let r be the constant for $\mathbf{L}_\alpha$. Then

$$\boldsymbol{L_\eta} \models (\forall y \in \bar{a})(\forall x \in y)(x \in r).$$

Now we set

$$
\begin{aligned}
F &:\equiv (\forall y \in \bar{a})(\forall x \in y)(x \in r),\\
G &:\equiv \exists z(\forall y \in \bar{a})(\forall x \in y)(x \in z).
\end{aligned}
$$

Then it is clear that

$$\mathcal{H}_0[\Gamma, F, G](\varnothing) \cup \{\omega{\cdot}\alpha + \omega\} \subseteq \mathcal{H}_0[\Gamma, \bar{a}](\varnothing).$$

Therefore, Theorem 2.7.3 and Lemma 2.6.4(2) (plus Weakening) imply that

$$\mathcal{H}_0[\Gamma, \bar{a}] \vdash^{\omega{\cdot}\alpha+n}_{0} \Gamma, \, G, \, F$$

for a suitable $n < \omega$. An application of $(\exists)$ yields our assertion.

Lemma 2.8.5 (Infinity) *For all Γ,*

$$\mathcal{H}_0[\Gamma] \vdash^{\omega^2+\omega}_{0} \Gamma, \, \exists z(x \neq \varnothing \wedge (\forall x \in z)(x \cup \{x\} \in z)).$$

Proof Clearly, $\omega \in \mathcal{H}_0(\varnothing)$ and

$$\boldsymbol{L_\eta} \models \omega \neq \varnothing \wedge (\forall y \in \omega)(y \cup \{y\} \in \omega).$$

Hence we already know, cf. Theorem 2.7.3 and Lemma 2.6.4(2), that

$$\mathcal{H}_0[\Gamma] \vdash^{\omega^2+n}_{0} \Gamma, \, \bar{\omega} \neq \varnothing \wedge (\forall y \in \bar{\omega})(y \cup \{y\} \in \bar{\omega})$$

for some $n < \omega$. Thus an application of $(\exists)$ finishes our proof.

Lemma 2.8.6 (Δ_0 Separation) *Let $A[x, y_1, \ldots, y_n]$ be a Δ_0 formula of $\mathcal{L}$ with all free variables indicated and assume $a, b_1, \ldots, b_n \in \boldsymbol{L_\eta}$. If we set*

$$\mathfrak{m} := \{|\bar{a}|, |\bar{b}_1|, \ldots, |\bar{b}_n|\} \quad and \quad \alpha := \max(\mathfrak{m}),$$

then we have that, for all Γ,

$$\mathcal{H}_0[\Gamma, \mathfrak{m}] \vdash^{\omega{\cdot}(\alpha+2)}_{0} \Gamma, \, \exists z(z = \{x \in \bar{a} : A[x, \bar{b}_1, \ldots, \bar{b}_n]\}).$$

Proof Under the above assumptions it is clear that there exists a set $c \in \boldsymbol{L}_{\alpha+2}$ such that

$$\boldsymbol{L_\eta} \models c = \{x \in a : A[x, b_1, \ldots, b_n]\}.$$

Moreover, we have for

$$F \;:\equiv\; \bar{c} = \{x \in \bar{a} : A[x, \bar{b}_1, \ldots, \bar{b}_n]\},$$
$$G \;:\equiv\; \exists y(y = \{x \in \bar{a} : A[x, \bar{b}_1, \ldots, \bar{b}_n]\}) :$$

- $rk(F) = \omega\cdot(\alpha{+}1) + n$ for some $n < \omega$,

- $\mathcal{H}_0[F, G](\varnothing) \cup \{\omega\cdot(\alpha{+}2)\} \subseteq \mathcal{H}_0[\mathfrak{m}](\varnothing)$.

Therefore, Theorem 2.7.3 (plus Weakening) gives us

$$\mathcal{H}_0[\Gamma, \mathfrak{m}] \,\big|\!\frac{\omega\cdot(\alpha+1)+n}{0}\, \Gamma, \; G, \; F$$

and after an application of ($\exists$) we are done.

Lemma 2.8.7 ($\in$-Induction) *Given the* IP *formula* $F[\bar{\varnothing}]$*, we let G be the formula* $\exists x((\forall y \in x)F[y] \wedge \neg F[x])$ *and set*

$$\gamma \;:=\; rk(\exists x\, F[x]), \quad \beta \;:=\; \omega^\gamma \,\#\, \omega^\gamma, \quad \alpha_a \;:=\; \beta + \omega\cdot|\bar{a}| + 2,$$

where $a \in \boldsymbol{L_\eta}$*. For all* Γ *and all d-operators* $\mathcal{H}$ *we then have:*

(1) $\mathcal{H}[\Gamma, G, \bar{a}] \,\big|\!\frac{\alpha_a}{0}\, \Gamma, \; G, \; F[\bar{a}]$.
(2) $\mathcal{H}[\Gamma, F[\bar{\varnothing}]] \,\big|\!\frac{\beta+\Omega+2}{0}\, \Gamma, \; \forall x((\forall y \in x)F[y] \to F[x]) \;\to\; \forall x F[x]$.

Proof We prove (1) by $\in$-induction. From Lemma 2.6.4(6) we immediately obtain that $rk(F[\bar{a}]) < \gamma$ for all $a \in \boldsymbol{L_\eta}$. Pick any $a \in \boldsymbol{L_\eta}$. Then the induction hypothesis (plus Weakening) implies

$$\mathcal{H}[\Gamma, G, \bar{a}, \bar{b}] \,\big|\!\frac{\alpha_b}{0}\, \Gamma, \; G, \; (\forall y \in \bar{a})F[y], \; F[\bar{b}]$$

for all $b \in a$. Therefore, an application of ($b\forall$) yields

$$\mathcal{H}[\Gamma, G, \bar{a}] \,\big|\!\frac{\beta+\omega\cdot|\bar{a}|}{0}\, \Gamma, \; G, \; (\forall y \in \bar{a})F[y].$$

On the other hand, Lemma 2.8.1 (together with Lemma 2.6.4(6,7) and Weakening) tells us that

$$\mathcal{H}[\Gamma, G, \bar{a}] \,\big|\!\frac{\beta}{0}\, \Gamma, \; G, \; \neg F[\bar{a}], \; F[\bar{a}].$$

Hence by ($\wedge$),

$$\mathcal{H}[\Gamma, G, \bar{a}] \,\big|\!\frac{\beta+\omega\cdot|\bar{a}|+1}{0}\, \Gamma, \; G, \; (\forall y \in \bar{a})F[y] \wedge \neg F[\bar{a}], \; F[\bar{a}].$$

From the latter we obtain our assertion (1) by an application of ($\exists$). Assertion (2) is an immediate consequence of (1).

Lemma 2.8.8 (Δ_0 Collection) *Let* $A[x, y, z_1, \ldots, z_n]$ *be a* Δ_0 *formula of* $\mathcal{L}$ *with the indicated free variables and assume* $a, b_1, \ldots, b_n \in \boldsymbol{L_\eta}$*. If we set*

$$\mathfrak{m} := \{|\bar{a}|, |\bar{b}_1|, \ldots, |\bar{b}_n|\}, \qquad \mathbf{b} := \bar{b}_1, \ldots, \bar{b}_n,$$
$$\alpha := rk((\forall x \in \bar{a})\exists y A[x, y, \mathbf{b}]), \quad \beta := \omega^\alpha \,\#\, \omega^\alpha,$$

then we have

$$\mathcal{H}[\Gamma, \mathfrak{m}] \,\vdash^{\beta+3}_0\, \Gamma, \; (\forall x \in \bar{a})\exists y A[x, y, \mathbf{b}] \;\rightarrow\; \exists z (\forall x \in \bar{a})(\exists y \in z) A[x, y, \mathbf{b}]$$

for all Γ and all d-operators $\mathcal{H}$.

Proof From Lemma 2.8.1 we know that

$$\mathcal{H}[\Gamma, \mathfrak{m}] \,\vdash^{\beta}_0\, \Gamma, \; \neg(\forall x \in \bar{a})\exists y A[x, y, \mathbf{b}], \; (\forall x \in \bar{a})\exists y A[x, y, \mathbf{b}].$$

Now we apply (Ref) and obtain

$$\mathcal{H}[\Gamma, \mathfrak{m}] \,\vdash^{\beta+1}_0\, \Gamma, \; \neg(\forall x \in \bar{a})\exists y A[x, y, \mathbf{b}], \; \exists z(\forall x \in \bar{a})(\exists y \in z) A[x, y, \mathbf{b}].$$

Then two applications of $(\lor)$ finish our proof.

Now we know that all axioms of KP^T can be embedded into IP. Before turning to the embedding theorem we need some further notation. If $\Gamma[x_1, \ldots, x_k]$ is the finite set of $\mathcal{L}$ formulas

$$\{A_1[x_1, \ldots, x_k], \ldots, A_n[x_1, \ldots, x_k]\}$$

whose free variables are among $x_1, \ldots, x_k$ and if $r_1, \ldots, r_k$ are terms of IP (i.e. constants for elements of $\boldsymbol{L}_\eta$), then $\Gamma[r_1, \ldots, r_k]$ stands for the set of IP formulas

$$\{A_1[r_1, \ldots, r_k], \ldots, A_n[r_1, \ldots, r_k]\}.$$

Theorem 2.8.9 (Embedding) *If KP^T proves $\Gamma[u_1, \ldots, u_k]$, then there exist $m, n < \omega$ such that*
$$\mathcal{H}_0[r_1, \ldots, r_k] \,\vdash^{\omega^{\Omega+m}}_{\Omega+n}\, \Gamma[r_1, \ldots, r_k]$$

for all IP terms $r_1, \ldots, r_k$.

Proof Note that

- $par(\Gamma[r_1, \ldots, r_k]) \cup \{\omega^{\Omega+m}\} \subseteq \mathcal{H}_0[r_1, \ldots, r_k](\varnothing)$.

- If $A[u_1, \ldots, u_k]$ is a formula of $\mathcal{L}$ with at most the indicated free variables, then there exists an $n_A < \omega$ such that

$$rk(A[r_1, \ldots, r_k]) < \Omega + n_A$$

for all terms $r_1, \ldots, r_k$ of IP; see Lemma 2.6.4. Since every derivation in KP^T contains only finitely many formulas A, we simply choose n to be the maximum of all such n_A.

Now we proceed by induction on the length of the derivation of $\Gamma[x_1, \ldots, x_k]$ in KP^T.

(i) $\Gamma[u_1, \ldots, u_k]$ is an axiom of KP^T. By Lemmas 2.8.1–2.8.8 we then have

$$\mathcal{H}_0[r_1, \ldots, r_k] \left|\frac{\omega^{\Omega+2}}{0}\right. \Gamma[r_1, \ldots, r_k]$$

for all IP terms $r_1, \ldots, r_k$.

(ii) The last inference was (ex). Then $\Gamma[u_1, \ldots, u_k]$ contains a formula $\exists x\, F[x, u_1, \ldots, u_k]$ and the premise of this inference is

$$\Gamma[u_1, \ldots, u_k], F[v, u_1, \ldots, u_k]$$

for some variable v. By the induction hypothesis there exists an $m_0 < \omega$ such that

$$\mathcal{H}_0[r_1, \ldots, r_k] \left|\frac{\omega^{\Omega+m_0}}{\Omega+n}\right. \Gamma[r_1, \ldots, r_k],\ F[s, r_1, \ldots, r_k]$$

for all IP terms $r_1, \ldots, r_k$. Here s is the term r_i if v is the variable u_i for some $i \in \{1, \ldots, k\}$ and the term $\bar{\varnothing}$ if v is different from all $u_1, \ldots, u_k$. An application of $(\exists)$ yields our assertion for $m := m_0 + 1$.

(ii) The last inference was (cut). Then the two premises of (cut) are

(1) $\Gamma[u_1, \ldots, u_k], A[u_1, \ldots, u_k, v_1, \ldots v_\ell]$,

(2) $\Gamma[u_1, \ldots, u_k], \neg A[u_1, \ldots, u_k, v_1, \ldots v_\ell]$,

with cut formula $A[u_1, \ldots, u_k, v_1, \ldots v_\ell]$, where $v_1, \ldots, v_\ell$ lists its free variables not belonging to $u_1, \ldots, u_k$. By the induction hypothesis we have $m_0, m_1 < \omega$ such that

(3) $\mathcal{H}_0[r_1, \ldots, r_k] \left|\frac{\omega^{\Omega+m_0}}{\Omega+n}\right. \Gamma[r_1, \ldots, r_k],\ A[r_1, \ldots, r_k, \bar{\varnothing}, \ldots \bar{\varnothing}]$,

(4) $\mathcal{H}_0[r_1, \ldots, r_k] \left|\frac{\omega^{\Omega+m_1}}{\Omega+n}\right. \Gamma[r_1, \ldots, r_k],\ \neg A[r_1, \ldots, r_k, \bar{\varnothing}, \ldots \bar{\varnothing}]$.

(Cut) applied to (3) and (4) yields our assertion for $m := \max(m_0, m_1) + 1$.

(iii) All other cases can be treated accordingly.

2.9 Predicative Cut Elimination

The rules of inference of IP can be divided into two classes: (i) In all rules with the exception of (Ref) the principal formula is more complex (or complicated if you want) than the respective minor formula(s). We, therefore, consider rules of this sort as *predicative* rules. (ii) On the other hand, in (Ref) we transform a Σ sentence into a (generally) less complex Σ_1 sentence. This is a sort of impredicativity and,

consequently, we consider (Ref) as an impredicative inference. This distinction is also reflected in the elimination of cuts from proofs in IP; it proceeds in two steps.

In this section we show that all cut formulas, that come from a predicative inference rule, can be eliminated by standard methods as described, for example in Schütte [16]. The principal formula of an (Ref) inference has rank Ω and so we do not touch, in this section, cut formulas of rank Ω.

Lemma 2.9.1 (Reduction) *Let F be an $\mathcal{L}[\eta]$ sentence of the form $(G_0 \vee G_1)$, $(\exists x \in \bar{a})G[x]$, or $\exists x G[x]$ and assume that $\rho := rk(F)$ is different from Ω. Then we have, for all $\Gamma, \Lambda, \alpha, \beta$ and all d-operators $\mathcal{H}$,*

$$(a) \ \mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \neg F \quad and \quad (b) \ \mathcal{H} \vdash^{\beta}_{\rho} \Lambda, F \implies \mathcal{H} \vdash^{\alpha+\beta}_{\rho} \Gamma, \Lambda.$$

Proof One easily checks that

$$par(\Gamma, \Lambda) \cup \{\alpha + \beta\}) \subseteq \mathcal{H}(\varnothing)$$

follows from (a) and (b). The proof is by induction on β, and we distinguish the following cases.

(i) Λ, F is an axiom of IP. Then Λ, F contains an atomic formula true in L_η, and this formula must be different from F. Hence Γ, Λ contains this formula as well and so is an axiom of IP.

(ii) The last inference was ($\exists$) with principal formula F. Then F has the form $\exists x G[x]$ and there exist a $b \in L_\beta$ and a $\beta_0 < \beta$ such that

$$\mathcal{H} \vdash^{\beta_0}_{\rho} \Lambda, F, G[\bar{b}]. \tag{2.1}$$

By the induction hypothesis we have

$$\mathcal{H} \vdash^{\alpha+\beta_0}_{\rho} \Gamma, \Lambda, G[\bar{b}]. \tag{2.2}$$

From (2.1) we also obtain

$$|\bar{b}| \in \mathcal{H}(\varnothing) \ \text{ provided that } \bar{b} \text{ occurs in } G[\bar{b}], \tag{2.3}$$

$$par(\Lambda) \cup \{\beta_0\} \subseteq \mathcal{H}(\varnothing). \tag{2.4}$$

(a) and (2.3) yield by Inversion that

$$\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \neg G[\bar{b}]$$

and therefore, by (2.4) and Weakening,

$$\mathcal{H} \left|\frac{\alpha+\beta_0}{\rho}\right. \Gamma, \Lambda, \neg G[\bar{b}]. \tag{2.5}$$

Since $rk(G[\bar{b}]) < \rho$ it only remains to apply (Cut) to (2.2) and (2.5).

(iii) All other last inferences with principal formula F are treated analogously. F cannot be the principal formula of (Ref) because of $rk(F) \neq \Omega$.

(iv) Finally, if F was not the principal formula of the last inference, we apply the induction hypothesis the premise(s) of this inference and then carry it out again.

As mentioned above, we restrict ourselves in this section to eliminate all cuts whose cut formulas do not have rank Ω. With the help of the previous lemma the proof of the following theorem is standard.

Theorem 2.9.2 (Predicative elimination) *We have for all Γ and all derivation operators $\mathcal{H}$:*

(1) $\mathcal{H} \left|\frac{\alpha}{\Omega+n+1}\right. \Gamma \implies \mathcal{H} \left|\frac{\omega_n(\alpha)}{\Omega+1}\right. \Gamma.$

(2) $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma \quad and \quad \rho \in \mathcal{H}(\varnothing) \cap \Omega \implies \mathcal{H} \left|\frac{\varphi\rho\alpha}{0}\right. \Gamma.$

For details see Buchholz [3] and/or Schütte [16]. Recall that $\omega_0(\alpha) := \alpha$ and $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$.

2.10 Collapsing Theorem

We begin this section by showing how to eliminate cut formulas of rank Ω. More precisely, we will show that any operator controlled proof of a set Γ of Σ sentences in which all cut formulas have ranks $\leq \Omega$ can be collapsed into a proof of depth and cut rank less than Ω. This technique—called collapsing technique—is the corner stone of *impredicative proof theory*.

Together with the results of the previous sections this collapsing theorem will then lead to the Hauptsatz of this article.

Theorem 2.10.1 (Collapsing) *Let Γ be a finite set of Σ sentences of $\mathcal{L}[\eta]$ and γ an ordinal such that*

$$par(\Gamma) \subseteq C(\gamma+1, \psi(\gamma+1)) \quad and \quad \gamma \in \mathcal{H}_\gamma[\Gamma](\varnothing).$$

Then we have, for all α,

$$\mathcal{H}_\gamma[\Gamma] \left|\frac{\alpha}{\Omega+1}\right. \Gamma \implies \mathcal{H}_{\hat{\alpha}}[\Gamma] \left|\frac{\psi(\hat{\alpha})}{\psi(\hat{\alpha})}\right. \Gamma,$$

where $\hat{\alpha} := \gamma + \omega^{\Omega+\alpha}$.

Proof We show this assertion by induction on α and assume $\mathcal{H}_\gamma[\Gamma] \frac{\alpha}{\Omega+1} \Gamma$. Then we have (see Lemmas 2.5.4 and 2.5.5):

$$par(\Gamma) \cup \{\alpha\} \subseteq \mathcal{H}_\gamma[\Gamma](\varnothing) \subseteq \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing), \tag{2.1}$$

$$\widehat{\alpha} \in \mathcal{H}_\gamma[\Gamma](\varnothing) \quad \text{and} \quad \psi(\widehat{\alpha}) \in \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing). \tag{2.2}$$

Now we proceed with a distinction by cases according to the last inference of $\mathcal{H}_\gamma[\Gamma] \frac{\alpha}{\Omega+1} \Gamma$ and note that this cannot be ($\forall$). In the following we confine our attention to the interesting cases; all others can be dealt with in a similar manner.

(i) Γ is an axiom. Then $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})} \Gamma$ follows from (2.1) and (2.2).

(ii) The last inference was ($b\forall$). Then Γ contains a formula $(\forall x \in \bar{a}) F[x]$ and we have

$$\mathcal{H}_\gamma[\Gamma, \bar{b}] \frac{\alpha_b}{\Omega+1} \Gamma, F[\bar{b}] \tag{2.3}$$

with $\alpha_b < \alpha$ for all $b \in a$. We also know that $|\bar{a}| \in \mathcal{H}_\gamma[\Gamma](\varnothing)$. Since $|\bar{b}| < |\bar{a}|$ for all $b \in a$, Lemma 2.5.2, Lemma 2.5.4(5), and (2.1) yield

$$|\bar{b}| \in \mathcal{H}_\gamma[\Gamma](\varnothing) \subseteq \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing), \quad \mathcal{H}_\gamma[\Gamma, \bar{b}] = \mathcal{H}_\gamma[\Gamma], \quad \mathcal{H}_{\widehat{\alpha}}[\Gamma, \bar{b}] = \mathcal{H}_{\widehat{\alpha}}[\Gamma]. \tag{2.4}$$

Hence (2.3) gives us

$$\mathcal{H}_\gamma[\Gamma] \frac{\alpha_b}{\Omega+1} \Gamma, F[\bar{b}],$$

and by the induction hypothesis we obtain that

$$\mathcal{H}_{\widehat{\alpha}_b}[\Gamma] \frac{\psi(\widehat{\alpha}_b)}{\psi(\widehat{\alpha}_b)} \Gamma, F[\bar{b}],$$

and, therefore,

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \frac{\psi(\widehat{\alpha}_b)}{\psi(\widehat{\alpha}_b)} \Gamma, F[\bar{b}], \tag{2.5}$$

always for all $b \in a$. Furthermore, from (2.3) and (2.4) we can also deduce that $\alpha_b \in \mathcal{H}_\gamma[\Gamma](\varnothing)$ for all $b \in a$. In view of Lemma 2.5.5(2) we thus have $\psi(\widehat{\alpha}_b) < \psi(\widehat{\alpha})$ for all $b \in a$. An application of ($b\forall$) yields $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})} \Gamma$.

(iii) The last inference was ($\exists$). Then Γ contains a formula $\exists x F[x]$ such that

$$\mathcal{H}_\gamma[\Gamma] \frac{\alpha_0}{\Omega+1} \Gamma, F[\bar{b}] \tag{2.6}$$

for some $\alpha_0 < \alpha$ and some set $b \in \mathbf{L}_\alpha$. Now set

$$r \ := \ \begin{cases} \bar{b} & \text{if } \bar{b} \text{ occurs in } F[\bar{b}], \\ \bar{\varnothing} & \text{if } \bar{b} \text{ does not occur in } F[\bar{b}], \end{cases}$$

In view of (2.6) we have $\alpha_0, |r| \in \mathcal{H}_\gamma[\Gamma](\varnothing)$. The induction hypothesis yields

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F[r],$$

thus also

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F[r].$$

Now recall that from Lemma 2.5.5(3) that

$$\mathcal{H}_\gamma[\Gamma](\varnothing) \cap \Omega \subseteq \psi(\gamma + 1) \leq \psi(\widehat{\alpha})$$

and conclude that $|r| < \psi(\widehat{\alpha})$. In addition, $\psi(\widehat{\alpha}_0) < \psi(\widehat{\alpha})$ follows from Lemma 2.5.5(2). Therefore, ($\exists$) applied to (2.5) yields $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \left|\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\right. \Gamma$.

(iv) The last inference was (Ref). Then Γ contains a formula $\exists x F^x$, where F is a Σ sentence, and there exists an $\alpha_0 < \alpha$ such that

$$\mathcal{H}_\gamma[\Gamma] \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma, F$$

and $\alpha_0 \in \mathcal{H}_\gamma[\Gamma](\varnothing)$. Now the induction hypothesis yields

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F,$$

hence

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F.$$

We know $\psi(\widehat{\alpha}_0) \in \mathcal{H}_{\widehat{\alpha}}[\Gamma](\varnothing)$ and therefore, since F is a Σ sentence, the Boundedness Lemma gives us

$$\mathcal{H}_{\widehat{\alpha}}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F^r, \tag{2.7}$$

where r is the constant for the set $\boldsymbol{L}_{\psi(\widehat{\alpha}_0)}$. In view of $\alpha_0 \in \mathcal{H}_\gamma[\Gamma](\varnothing)$ and Lemma 2.5.5(2) we also have $\psi(\widehat{\alpha}_0) < \psi(\widehat{\alpha})$. Hence we can apply ($\exists$) to (2.7) and obtain $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \left|\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\right. \Gamma$.

(v) The last inference was (Cut). Then there exist an ordinal $\alpha_0 < \alpha$ and a sentence F with $rk(F) \leq \Omega$ such that

$$\mathcal{H}_\gamma[\Gamma] \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma, F \quad \text{and} \quad \mathcal{H}_\gamma[\Gamma] \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma, \neg F \tag{2.8}$$

and $par(F) \cup \{\alpha_0\} \subseteq \mathcal{H}_\gamma[\Gamma](\varnothing)$. We distinguish two cases:

(v.1) $rk(F) < \Omega$. Then from Lemma 2.6.4(7) and Lemma 2.5.5(3) we conclude that $rk(F) < \psi(\gamma + 1) \leq \psi(\widehat{\alpha})$. Also, F and $\neg F$ are Σ sentences. Therefore the induction hypothesis applied to (2.8) yields

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, F \quad \text{and} \quad \mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \left|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\right. \Gamma, \neg F$$

As before, $\psi(\widehat{\alpha}_0) < \psi(\widehat{\alpha})$. Hence (Cut) gives us $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \big|\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\ \Gamma$.

(v.2) $rk(F) = \Omega$. Then F or $\neg F$ is of the form $\exists x G[x]$ with $G[\bar{\varnothing}]$ being a Δ_0 sentence. We assume that F is $\exists x G[x]$. The induction hypothesis applied to the left hand side of (2.8) yields

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \big|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\ \Gamma, F$$

Since $\psi(\widehat{\alpha}_0) \in \mathcal{H}_{\widehat{\alpha}_0}[\Gamma](\varnothing)$ we can apply the Boundedness Lemma and obtain

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \big|\frac{\psi(\widehat{\alpha}_0)}{\psi(\widehat{\alpha}_0)}\ \Gamma, F^r, \tag{2.9}$$

where r is the constant for the set $\boldsymbol{L}_{\psi(\widehat{\alpha}_0)}$. By applying Lemma 2.6.8 to the right hand side of (2.8), we get

$$\mathcal{H}_{\widehat{\alpha}_0}[\Gamma] \big|\frac{\alpha_0}{\Omega+1}\ \Gamma, \neg F^r.$$

We also have $\widehat{\alpha}_0 \in \mathcal{H}_\gamma[\Gamma](\varnothing) \subseteq \mathcal{H}_{\widehat{\alpha}_0}[\Gamma](\varnothing)$ and

$$par(\Gamma) \subseteq C(\gamma + 1, \psi(\gamma + 1)) \subseteq C(\widehat{\alpha}_0 + 1, \psi(\widehat{\alpha}_0 + 1)).$$

Hence, since $\neg F^r$ is a Σ sentence, we are in the position to apply the induction hypothesis, yielding

$$\mathcal{H}_{\alpha_1}[\Gamma] \big|\frac{\psi(\alpha_1)}{\psi(\alpha_1)}\ \Gamma, \neg F^r, \tag{2.10}$$

where $\alpha_1 := \widehat{\alpha}_0 + \omega^{\Omega+\alpha_0} = \gamma + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \gamma + \omega^{\Omega+\alpha} = \widehat{\alpha}$ and, in addition, $\alpha_1 \in \mathcal{H}_\gamma[\Gamma](\varnothing)$; see Lemma 2.5.5(1).

Clearly, $rk(F^r) < \Omega$ and, according to (2.9), we have $par(F^r) \subseteq \mathcal{H}_{\widehat{\alpha}_0}[\Gamma](\varnothing)$. Hence $rk(F^r) < \psi(\widehat{\alpha}_0 + 1)$ by Lemma 2.6.4(7) and Lemma 2.5.5(3). Furthermore, $\psi(\widehat{\alpha}_0) < \psi(\alpha_1) < \psi(\widehat{\alpha})$. Therefore, we can apply (Cut) to (2.9) and (2.10) and obtain $\mathcal{H}_{\widehat{\alpha}}[\Gamma] \big|\frac{\psi(\widehat{\alpha})}{\psi(\widehat{\alpha})}\ \Gamma$.

Theorem 2.10.2 (Hauptsatz) *Let $A[u_1, \ldots, u_k]$ be a Σ formula of $\mathcal{L}$ with at most the indicated free variables and suppose that it is provable in KP^T. For all $a_1, \ldots, a_k \in \boldsymbol{L}_\eta$ there exist a d-operator $\mathcal{H}$ and an ordinal $\alpha < \boldsymbol{\eta}$ such that*

(1) $\mathcal{H}[\bar{a}_1, \ldots, \bar{a}_k] \big|\frac{\alpha}{\alpha}\ A[\bar{a}_1, \ldots, \bar{a}_n].$

(2) $\mathcal{H}[\bar{a}_1, \ldots, \bar{a}_k] \big|\frac{\varphi\alpha\alpha}{0}\ A[\bar{a}_1, \ldots, \bar{a}_n].$

Proof By Theorem 2.8.9 there are $m, n < \omega$ with

$$\mathcal{H}_0[r_1, \ldots, r_k] \big|\frac{\omega^{\Omega+m}}{\Omega+n+1}\ A[r_1, \ldots, r_k]$$

for all IP terms $r_1, \ldots, r_k$. By a first application of predicative cut elimination, cf. Theorem 2.9.2(1), we obtain

$$\mathcal{H}_0[r_1, \ldots, r_k] \big|\frac{\beta}{\Omega+1}\ A[r_1, \ldots, r_k] \tag{2.1}$$

for $\beta := \omega_{n+1}(\Omega + m)$ and all IP terms $r_1, \ldots, r_k$.

Now we pick specific $a_1, \ldots, a_n \in \boldsymbol{L_\eta}$. Then there exits an $\ell < \omega$ such that

$$a_1, \ldots, a_n \in \boldsymbol{L}_{\psi(\gamma)} \quad \text{for} \quad \gamma := \omega_\ell(\Omega + 1).$$

It follows that

$$\{|\bar{a}_1|, \ldots, |\bar{a}_k|\} \subseteq C(\gamma + 1, \psi(\gamma + 1)) \quad \text{and} \quad \gamma \in \mathcal{H}_\gamma[\bar{a}_1, \ldots, \bar{a}_k](\varnothing) \quad (2.2)$$

and from (2.1) we obtain

$$\mathcal{H}_\gamma[\bar{a}_1, \ldots, \bar{a}_k] \big|\frac{\beta}{\Omega+1} \ A[\bar{a}_1, \ldots, \bar{a}_k]. \tag{2.3}$$

Because of (2.2) and (2.3) we can apply Theorem 2.10.1, yielding

$$\mathcal{H}_{\widehat{\beta}}[\bar{a}_1, \ldots, \bar{a}_k] \big|\frac{\psi(\widehat{\beta})}{\psi(\widehat{\beta})} \ A[\bar{a}_1, \ldots, \bar{a}_k] \tag{2.4}$$

for $\widehat{\beta} := \gamma + \omega^{\Omega+\beta}$. A simple calculation shows that $\alpha := \psi(\widehat{\beta})$ is smaller than η and thus the required ordinal for (1).

By making use of Theorem 2.9.2, the second assertion is an immediate consequence of (1). It only has to be verified with the help of Lemma 2.5.5 that $\psi(\widehat{\beta}) \in \mathcal{H}_{\widehat{\beta}}[\bar{a}_1, \ldots, \bar{a}_k](\varnothing)$.

The notion of Π_2 *ordinal of a theory T* has been introduced in Jäger [12]. It is defined there as the least ordinal α such that $\boldsymbol{L}_\alpha \models A$ for all Π_2 consequences of T. One of the most prominent results along these lines and a direct consequence of Jäger [11] states that η is the Π_2 ordinal of KP. Besides that [12] contains a series of results about Π_2 ordinals of theories for iterated admissible sets.

Π_2 ordinals have also been considered in Rathjen [15] and Pohlers [14]. Some remarks have been made in these articles and in [12] about the relationship between proof-theoretic ordinals and Π_2 ordinals. It is planned to come back to this topic from a more general perspective in Jäger [10].

It is not surprising that our Hauptsatz directly implies that η is the Π_2 ordinal of KP.

Corollary 2.10.3 (Π_2 ordinal of KP) *If A is a Π_2 sentence of $\mathcal{L}$, the we have*

$$\text{KP} \vdash A \quad \Longrightarrow \quad \boldsymbol{L_\eta} \models A.$$

Proof A has the form $\forall x \exists y B[x, y]$ such that $B[x, y]$ is a Δ_0 formula of $\mathcal{L}$ with at most x, y free. Since KP proves A, it is clear that KP^T proves $\exists y B[u, y]$ for any u. Now pick an arbitrary $a \in \boldsymbol{L_\eta}$. Then the Hauptsatz tells us that

$$\mathcal{H}[\bar{a}] \big|\frac{\alpha}{\alpha} \ \exists y B[\bar{a}, y]$$

for some d-operator $\mathcal{H}$ and some $\alpha < \eta$. Therefore,

$$\boldsymbol{L_\eta} \models \exists y B[\bar{a}, y]$$

according to Theorem 2.7.1. Since this is so for any element a of $\boldsymbol{L_\eta}$, we have $\boldsymbol{L_\eta} \models A$.

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