# Chapter 1
# A Framework for Metamathematics

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**Abstract** First, we consider Hilbert's program, focusing on the three different aspect of mathematics called *actual mathematics*, *formal mathematics*, and *metamathematics*. Then, we investigate the relationship between *metamathematics* and *actual mathematics*, describe what shall be achieved with *metamathematics*, and propose a framework for *metamathematics*.

## 1.1 Hilbert's Program Revisited

Motivated by prior work of Frege and Russell, Hilbert describes in [3] what he calls *axiomatic thinking*. He concludes his article with the following words:

> Ich glaube: Alles, was Gegenstand des wissenschaftlichen Denkens überhaupt sein kann, verfällt, sobald es zur Bildung einer Theorie reif ist, der axiomatischen Methode und damit mittelbar der Mathematik. Durch Vordringen zu immer tieferliegenden Schichten von Axiomen [...] gewinnen wir auch in das Wesen des wissenschaftlichen Denkens selbst immer tiefere Einblicke und werden uns der Einheit unseres Wissens immer mehr bewußt. In dem Zeichen der axiomatischen Methode erscheint die Mathematik berufen zu einer führenden Rolle in der Wissenschaft überhaupt.[1]

At this early stage of Hilbert's program, the focus is on the "objects of scientific thought" which become dependent on the axiomatic method. When these "objects of

[1] I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms [...] we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science. (*Translation taken from* [2].)

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scientific thought" are mathematical objects, one can think of these objects as being part of the *real mathematical world*.

Later in 1922, Hilbert went a step further. The focus is now not on the "objects of scientific thought" which shall be axiomatized, but on the consistency of the axiomatic systems. In [5, p. 174], Hilbert summarizes his program as follows:

> Erstens: Alles, was bisher die eigentliche Mathematik ausmacht, wird nunmehr streng formalisiert, so daß die *eigentliche Mathematik* oder die Mathematik im engeren Sinne zu einem Bestande an beweisbaren Formeln wird. [. . .]
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> Zweitens: Zu dieser eigentlichen Mathematik kommt eine gewissermaßen neue Mathematik, eine *Metamathematik*, hinzu, die zur Sicherung jener dient, indem sie sie vor dem Terror der unnötigen Verbote sowie der Not der Paradoxien schützt. In dieser Metamathematik kommt—im Gegensatz zu den rein formalen Schlußweisen der eigentlichen Mathematik—das inhaltliche Schließen zur Anwendung, und zwar zum Nachweis der Widerspruchsfreiheit der Axiome.
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> Die Entwicklung der mathematischen Wissenschaft geschieht hiernach beständig wechselnd auf zweierlei Art: durch Gewinnung neuer "beweisbarer" Formeln aus den Axiomen mittels formalen Schließens und durch Hinzufügung neuer Axiome nebst dem Nachweis ihrer Widerspruchsfreiheit mittels inhaltlichen Schließens.[2]

What we see here is the beginning of a paradigm shift: In classical mathematics, axioms were a statement that was taken to be true. Axioms served as premises or starting points for further reasoning and arguments. Therefore, it would have been absurd to consider different contradicting axiom systems, since at most one of these systems can be true in an absolute sense, and all the others systems must be false or meaningless. Now, focusing on the consistency of axiom systems rather than on their inherent truth, we do not need to restrict ourselves to axiom systems which are relevant for *actual mathematics* (i.e. for the investigation of objects in the, to some extend, *real mathematical world*), but could investigate any consistent axiomatic system, no matter whether it is relevant for *actual mathematics* or not.

However, since the ultimate goal of Hilbert's program was to give a firm (i.e. provably consistent) foundation of *actual mathematics*, there are still some axiomatic systems which are more relevant for mathematics, and some which are less relevant for mathematics. This situation is similar to geometry, where one could argue that the only geometric system which is relevant, is the one which describes the space

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[2] First: everything that hitherto made up [actual mathematics] is now to be strictly formalized, so that ***actual mathematics***, or mathematics in the strict sense, becomes a stock of provable formulae. [. . .]
Secondly: in addition to actual mathematics, there appears a mathematics that is to some extent new, a ***metamathematics*** which serves to safeguard it by protecting it from the terror of unnecessary prohibitions as well as from the difficulty of paradoxes. In this metamathematics—in contrast to the purely formal modes of inference in actual mathematics—we apply contentual inference; in particular, to the proof of the consistency of the axioms.
The development of mathematical science accordingly takes place in two ways that constantly alternate: the derivation of new "provable" formulae from the axioms by means of formal inference; and the adjunctionn of new axioms together with a proof of their consistency by means of contentual inference. (*Translation taken from* [2], *except that we translated "eigentliche Mathematik" as "actual mathematics" and not as "mathematics proper"*.)

in which we live. Even though from a physical point of view this argument makes sense, from a mathematical point of view it is immaterial. For example, it is very unlikely that our space satisfies the axioms of projective geometry, but nevertheless, projective geometry is the key tool in the investigation of conic sections in Euclidian geometry.

To sum up, we can say that Hilbert's program was the beginning of a paradigm shift from "axioms as obviously true statements" towards "axioms as mutually non-contradictory statements". However, since there is still a presupposed *actual mathematics*, this paradigm shift was not carried out thoroughly. For example, let us consider the axiomatic system ZFC, which is Zermelo-Fraenkel Set Theory ZF with the Axiom of Choice AC. One of the earliest problems in set theory was the question whether the Continuum Hypothesis CH holds (which is the first of the twenty-three problems Hilbert presented at the *ICM 1900 in Paris*). On the one hand, it is known that CH is independent from ZFC (i.e. within ZFC we can neither prove nor disprove CH), and on the other hand, ZFC serves as a foundation of mathematics. Now, if one believes in a unique *actual mathematics*, then CH should be either true or false, which implies that ZFC is not strong enough to serve as the foundation of *actual mathematics*. So, we have to extend ZFC by adding new axioms in such a way that the extended systems decide CH. However, by *Gödel's Second Incompleteness Theorem*, this does not really help, since no matter how we extend ZFC, we always obtain a sentence which is undecidable within the extended system. In other words, having *Gödel's Second Incompleteness Theorem* in mind, it is not possible to axiomatize *actual mathematics* in such a way that the axiomatic system obtained fully represents *actual mathematics*, which also shows that Hilbert's program must fail.

Let us turn back to the paradigm shift from "axioms as true statements" towards "axioms as mutually non-contradictory statements", which was initiated by Hilbert's program: The above explanations show that in order to make the paradigm shift complete, we have to give up the idea of *actual mathematics* as the unique *real mathematical world*, since strictly formalizing *actual mathematics* in 1st order logic yields a formal axiomatic system of which *actual mathematics* is just one of numerous models. However, we can conceive *actual mathematics* as the collection of all models of axiomatic systems which form a foundation for mathematics. Such systems are consistent extensions of ZF, whose models are proper models for mathematics, i.e. models in which we can carry out essentially all mathematics.

In order to see how and where we build these models, we have to combine *Gödel's Completeness Theorem* for 1st order logic with Hilbert's metamathematics: *Gödel's Completeness Theorem* together with the *Soundness Theorem* states that a sentence $\phi$ is provable from an axiomatic system S, denoted $\mathsf{S} \vdash \phi$, if and only if $\phi$ is valid in each model of S. In particular, we obtain that an axiomatic system S has a model if and only if S is consistent. So, for any axiomatic system S, Hilbert's metamathematics has the task to decide whether S is consistent, or equivalently, to decide whether S has a model. By *Gödel's Incompleteness Theorems* we know that this task cannot be carried out in a formal system. In other words, Hilbert's metamathematics can not be formalized, and therefore, does not belong to actual mathematics—which is indicated by the prefix “meta”, which means “behind” (i.e. metamathematics is a kind of “background-mathematics”). Moreover, even in the case when we know that some axiomatic system $\mathsf{S}$ is consistent, and therefore has a model, in general, the construction of a model of $\mathsf{S}$ cannot be carried out in a formalized system, i.e. the construction of a model must in general be carried out in metamathematics.

Since metamathematics plays an important role in the investigation of axiomatic systems and in the construction of models, and since metamathematics cannot be formalized, it is natural to ask what kind of principles we have in metamathematics. An answer to this question is given in the next section.

## 1.2 Non-constructive Principles of Metamathematics

The previous section can be summarized as follows: In mathematics we investigate formal axiomatic systems. In particular, we investigate which sentences can be derived from a given axiomatic system $\mathsf{S}$, which sentences are consistent with $\mathsf{S}$, and which sentences are independent of $\mathsf{S}$, where the investigations themselves are based on the construction of various models of axiomatic systems. In particular, we have to construct models of variations of $\mathsf{ZF}$ (i.e. models of extensions of $\mathsf{ZF}$). The construction of models is carried out in a moderate constructive way, which we are going to circumscribe now.

### *1.2.1 What We Need*

The construction of a model for an axiomatic system is carried out by following Henkin’s proof of *Gödel’s Completeness Theorem* for 1st order logic. Now, beside the constructive parts of Henkin’s proof, which are described explicitly or by algorithms, there are also some non-constructive parts using principles which are usually tacitly assumed. The goal is now to make these principles explicit.

The most important principle we need in metamathematics is the notion of FINITENESS. Hilbert writes in [4, p. 154]:

> Die beweisbaren Formeln [. . .] haben sämtlich den Charakter des Finiten, d.h. die Gedanken, deren Abbilder sie sind, können [. . .] mittels Betrachtung endlicher Gesamtheiten erhalten werden.[3]

The notion of FINITENESS plays a crucial role not only in the investigation of provable formulae, but also in the proof of *Gödel’s Incompleteness Theorems*. In fact, if the notion of FINITENESS could be formalized (i.e. if FINITENESS were a notion of

[3] The provable formulae [. . .] all have the character of the finite; that is, the thoughts whose images they are can also be obtained [. . .] from the examination of finite totalities. (*Translation taken from* [2].)

formal mathematics), then *Gödel's Incompleteness Theorems* would disappear and Hilbert's program would succeed.

What we also need to construct models is the notion of a POTENTIALLY INFINITE SET, like the natural numbers 0, 1, 2, ... Notice that we do not require to have the entire set $\mathbb{N}$ of natural numbers, which would be an actual infinite set. In fact, a closer look at Henkin's proof of *Gödel's Completeness Theorem* shows that in order to construct non-finite models (e.g. models of Peano Arithmetic $\mathsf{PA}$ or models of $\mathsf{ZFC}$), a POTENTIALLY INFINITE SET is sufficient—but also necessary.

Finally, we need a kind of LAW OF EXCLUDED MIDDLE. This law is crucial in the completion of axiomatic systems $\mathsf{S}$, since in each step of the completion of $\mathsf{S}$, for some $\phi$ we have to decide whether or not $\phi$ is consistent with the extension of $\mathsf{S}$ we already have constructed. In other words, for every axiomatic systems $\mathsf{S}$ and each sentence $\phi$, we must be able to decide whether $\phi$ is provable from $\mathsf{S}$ (i.e. $\mathsf{S} \vdash \phi$), and since a formal proof is just a special FINITE sequence of formulae, either there is such a sequence or there is no such sequence. The difficulty is, that we probably cannot decide in FINITELY many steps, whether or not $\mathsf{S} \vdash \phi$. Now, this non-constructive part in the proof of *Gödel's Completeness Theorem* is handled by the LAW OF EXCLUDED MIDDLE. If we would formalize this law, we would obtain what is known as the WEAK KÖNIG'S LEMMA, which is just KÖNIG'S LEMMA for infinite, binary 0-1-trees.

### *1.2.2 What We Obtain*

In the framework described above, we can construct models of all kind of axiomatic systems. For example, we can construct models of $\mathsf{ZFC}$, or models of $\mathsf{ZF}$ in which the Axiom of Choice fails, and we can carry out Forcing constructions in order to obtain models of $\mathsf{ZFC}$ (or of $\mathsf{ZF}$) in which certain statements become valid. In particular, we can construct models of $\mathsf{ZFC}$ in which $\mathsf{CH}$ holds or in which $\mathsf{CH}$ fails. This way, we obtain different models of the standard real numbers. On the other hand, we can also construct non-standard models of the real numbers, for example, the hyperreal numbers or the surreal numbers, which give us also non-standard models of Peano Arithmetic. In fact, even non-standard approaches to mathematics, like intuitionism, can be modelled. There is a lot of freedom we have, and it might be this freedom, which Cantor meant when he writes ([1, p. 564])

> ... das Wesen der Mathematik liegt [...] in ihrer Freiheit.[4]

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[4] ... the essence of mathematics lies [...] in its freedom.

## 1.3 Conclusion

The view of mathematics we proposed can be described as follows:

- In mathematics we investigate formal axiomatic systems. In particular, we investigate which sentences we can derive from a given axiomatic system $\mathsf{S}$, which sentences are consistent with $\mathsf{S}$, and which sentences are independent of $\mathsf{S}$.
- The investigations are based on the construction of various models of axiomatic systems, in particular, on the construction of models of variations of $\mathsf{ZFC}$.
- The construction of models is carried out in metamathematics, where metamathematics consists of all we can describe explicitly or by algorithms, together with the notions of FINITENESS and POTENTIALLY INFINITE SET, and the LAW OF EXCLUDED MIDDLE.

On the one hand, this view of mathematics is quite formal in the sense that there is no unique *real mathematical world* anymore, but on the other hand, we have a realm of models of various axiomatic systems, which distinguishes this view from pure formalism. Moreover, one of the features of this view is that we do not have any kind of "ideology" like constructivism, platonism, or intuitionism, which would lead us to the "right" mathematical world: No matter which approach we take, with Hilbert's axiomatic thinking—enriched by Gödel's work—we are able to create various mathematical worlds. With respect to this kind of mathematics, we would like to say:

> From the realm of mathematics, which Hilbert and Gödel created for us, no-one shall expel us.

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