Evolution Equations: Present and Future

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This volume features chapters that present contemporary research focus and results in the theory of Evolution Equations and its applications in the physical and natural sciences, in a unique combination of mathematical rigor and realistic background. Evolution Equations, besides their theoretical interests, aim at describing the future time behavior of systems using time differential laws given by classes of equations that include: systems of ordinary differential equation in the form

$$du/dt = f(u, t), \quad d^2y/dt^2 = f(u, u', t), \cdots \tag{1}$$

where $u(t)$ could be regarded as the solution of the Cauchy problem; partial differential equations (e.g., of parabolic type for Heat conduction problems, and of hyperbolic type for motions of elastic continua); differential-difference equations (e.g., time-delayed finite dimensional feedback control systems in the transmission of the control signal); functional differential equations.

The goal is to be able to use the equation to predict the future of any related physical or natural system, given an "initial state" of the system and other parameters affecting the system. It is expected that one must have existence and uniqueness of forward-time solutions of all physically allowable data.

One widely studied area in the theory of Evolution Equations has been *periodicity* and its generalization in the direction of *almost periodicity*; briefly, consider for instance a Banach abstract space $\mathbb{B}$, and a $\mathbb{B}-$valued continuous function f over a

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G. M. N'Guérékata, B. Toni (eds.), *Studies in Evolution Equations and Related Topics*, STEAM-H: Science, Technology, Engineering, Agriculture, Mathematics & Health,
https://doi.org/10.1007/978-3-030-77704-3_1

time space $\mathbb{G}$, an additive semigroup. Take here $\mathbb{G} = \mathbb{R}$. The function f is time-*almost periodic* if $\forall \epsilon > 0$, $\exists F_\epsilon$ a finite subset in the algebra of $\mathbb{B}-$valued bounded continuous functions on $\mathbb{G}$, such

$$f_{\mathbb{R}} \subset B_\epsilon + F_\epsilon \tag{2}$$

where $f_{\mathbb{R}}$ is the set of all translates for f, that is, $f_{\mathbb{R}} := \{f_s : \quad s \in \mathbb{R}, \quad f_s(x) = x + s\}$, and B_ϵ is the ϵ zero neighborhood, i.e., open ball centered at 0 with radius ϵ. This is equivalent to $f_{\mathbb{G}}$ being *relatively compact* or *pre-compact* (compact closure $\bar{F_{\mathbb{R}}}$).

Almost periodicity from the earlier work by Bohr, Besicovitch, Bochner, and von Neumann, has been extensively studied, with notable contributions from many authors to include Corduneanu, Fink, and N'Guérékata, the current volume co-editor. This concept was also fittingly extended to the concept of *almost limit cycles* in the realm of continuous time dynamical systems to account for oscillations that are almost self-sustained.

A noted weakness of the theory is that the classical mathematical framework to study almost periodicity, as well as Evolution Equations, involves mostly, if not exclusively, Archimedean spaces (e.g., Euclidean, Banach, or Hilbert spaces). These are spaces that satisfy the *Archimedean Principle* in that there are all endowed with the usual/standard Euclidean norm and its induced metric which satisfied the triangle inequality; much of our visual and mental perception is based on the standard Euclidean space with its perfectly straight lines and planes; we came to see and represent the physical universe as actually Euclidean in its geometry, resulting oftentimes in biased and not so realistic mathematic model.

Indeed the physical and natural systems that Evolution Equations aim at describing are inherently non-Euclidean, with a natural geometrical ordering that is not the usual real line, but the more adequate *hierarchical generating tree.* For instance, the usual geometrical Archimedean/Euclidean distance "suitable" for measuring spatial location separation between human beings is less effective for the genetic distance measuring the hierarchical kinship relations.

Fortunately, thanks to the pioneering work by Kurt Hensel in the late nineteenth century, a new trend has been emerging in recent years studying Evolution Equations in Non-Archimedean or *p-adic* spaces, endowed with the so-called *ultrametrics;* ultrametrics could be induced by *p-adic absolute values* $|.|_p$, for p prime, that satisfy a more stringent inequality than the usual triangle inequality; specifically in a non-Archimedean space $\mathbb{K}$, the *ultrametric* distance d (or denoted $d_p(x, y)$ when induced by the *p-adic norm* $|.|_p$) satisfies the inequality

$$d(x, z) \leq max(d(x, y), d(y, z)) \quad \forall \text{ x,y,z } \in \mathbb{K}. \tag{3}$$

Recall that the set $\mathbb{R}$ of reals is the completion of the set $\mathbb{Q}$ of rationals by the usual Euclidean infinite norm denoted $|.|_\infty$, which amounts to creating new numbers as limits of Cauchy sequences that do not have rational limits. The same construction replacing the norm $|.|_\infty$ by the p-adic norm $|.|_p$ yields a new complete field, analogous to $\mathbb{R}$ denoted $\mathbb{Q}_p$, for every prime p, and called the field of *p-adic numbers*. Inside this field $\mathbb{Q}_p$ of p-adic numbers lies the ring $\mathbb{Z}_p$ of *p-adic integers* whose geometry is similar to the Cantor set. The set of rationals $\mathbb{Q}$ is densely contained in every $\mathbb{Q}_p$ as it is in the set of reals $\mathbb{R}$.

In short mathematical modelers of natural and physical have at their disposal a toolbox of various fields in the so-called *book structure* with the rationals $\mathbb{Q}$ as the book spine, and every page represents a field in which to carry mathematical modeling, to include $\mathbb{R} = \mathbb{Q}_\infty$, and $\mathbb{Q}_p$, for every p prime.

Non-Archimedean spaces, equivalently called *p-adic spaces* or *ultrametric spaces,* have some peculiar and important features described as follows:

- One immediate interesting fact about *p-adic integers* is that these integers are bounded in the norm by 1; that is, $|n|_p \leq 1$, i.e., as the number tends to infinity, its p-adic size remains less than one and tends to zero, in violation of the *Archimedean Principle*.
- The topology has a basis of *clopen* sets, i.e., sets that are simultaneously open and closed; so phrases such "open ball" and "closed balls" become meaningless.
- Every point in a p-adic ball is also its center. For instance, if *modeled p-adically* the center of the universe could be found at the nose of the nose, as once alluded to by the physicist Hawkins.
- Two balls are either disjoint or one within the other. This could hint toward the notion of parallel universes.
- The spaces are *totally disconnected*: the connected component of every point is the point itself. Consequently, the principle of *analytic continuation* is lost, as well as the *Intermediate value theorem*.
- The ultrametric geometry allows only *isosceles triangles*.
- The geometrical ordering is not along the real line but rather on a *hierarchical generating tree.* Consequently, the notion of time as we know seems to be a purely statistical construct. $\mathbb{Z}_3$ for instance is homeomorphic to the fractal-like Sierpinski Gasket.
- Convergence of p-adic series: $\sum_{n=1}^{\infty} a_n < \infty$ if and only $a_n \to 0$. (A calculus student dream!)
- The most consequential outcome and challenge is the study of Evolution Equations and their differential representations with a *p-adic time* instead of the usual real time. That is, how to model a system that evolves *p-adically.* This is still a hard open problem.
- Consider the following simple looking differential equations modeling so many natural systems:

$$\frac{dy}{dt} = \dot{y} = \lambda y, \tag{4}$$

and

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \quad \text{(Harmonic Oscillator)}. \tag{5}$$

Respectively the solutions are in the form $Ae^{\lambda x}$, and as level curves $H^{-1}(c) \quad c \in \mathbb{R}_{\geq 0}$ of a quadratic Hamiltonian H (i.e., concentric circles in the Euclidean plane $\mathbb{R}^2$).

However the *p-adic exponential function* $exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges only for $x \in B(0,r), \quad r = p^{1/(1-p)}$; that is, a very small radius convergence in contract of the standard exponential series that converges on the whole real line $\mathbb{R}$. Therefore the series $\sum_{n\geq 1} \frac{(\lambda x)^n}{n!}$ is convergent if and only if $|x| < |\lambda|^{-1} p^{1/(1-p)}$.

- Existence of *singular function* or Devil's Staircase or Cantor function: In the classic real analysis a function differentiable with vanishing derivative is a constant function. In contrast there exist in the p-adic space, functions that are *non-constant, continuous, and differentiable with vanishing derivatives*; these functions are called *pseudo-constants.* They are used to describe situation in which an object is known to move from point A to point B, but whenever it is observed it appears to be at rest. "Spooky motion" Einstein would say!
- The next most consequential notion is that of periodicity. Indeed, periodicity, an important concept in the study of Evolution Equations, is non-existent in non-Archimedean spaces; functions cannot be non-trivially periodic; however almost periodicity is allowed and seems to be the norm in the p-adic world. In other words, systems modeled do not have a time-periodic behavior in their p-adic dynamics. For instance, it has been noted (e.g., Corduneanu) that almost periodic oscillations are much more common than the periodic ones. There has been a tremendous amount of work in mathematical sciences involving periodicity, to include the famous Hilbert's 16^{th} problem, still unsolved to this day. Therefore, all these results on periodicity are made meaningless by just changing to the non-Archimedean/p-adic spaces. Such a realization is humbling and should be kept in mind when drawing conclusions based on periodicity.

In non-Archimedean/ultrametric spaces, almost periodicity is defined and understood in the following sense: consider a non-trivially valued complete algebraically closed non-Archimedean field $\mathbb{K}$, and a $\mathbb{K}-$valued function f over an additive semigroup $\mathbb{G}$.

The function f is said to be *p-adic/non-Archimedean almost periodic function* if the set $f_{\mathbb{G}}$ of all the translates of f is a *compactoid* in the $\mathbb{K}-$Banach algebra $\mathcal{B}$ of all bounded continuous functions $\mathbb{G} \longrightarrow \mathbb{K}$ with respect to pointwise operations and the supremum norm $||f||_\infty = \sup_{\mathbb{G}} |f(x)|$.

That is, $\forall \epsilon > 0$, $\exists$ a finite subset $F_\epsilon \subset \mathcal{B}$ such that

$$f_{\mathbb{G}} \subset B_\epsilon + aco(F_\epsilon) \tag{6}$$

where $aco(F_\epsilon)$ is the *smallest absolutely convex* subset containing F_ϵ. In the classic notion of almost periodicity, the set F_ϵ is only required to be *relatively compact/pre-compact.*

In the natural sciences ultrametricity is emerging as a consequence of randomness and the law of large numbers; exact in the limit $N \longrightarrow \infty$ for systems with a large number N of degrees of freedom, ultrametricity provides a more natural type of organization. It has proven effective for studying Evolution Equations describing neutral evolution of *pseudogenes*, stochastic branching processes in large space, energy landscape of *disordered frustrated systems* (spin glasses, problems in engineering and biology of combinatorial optimization), and in *taxonomy* where representation is given by dendrogram of hierarchy pictured in inverted tree. Ultrametricity serves also to better model Mental spaces, and the emerging and evolution of languages.

The richness of non-Archimedean space allows more realistic mathematical models for evolution equations, making it a state-of-the-art tool in the arsenal of researchers of Evolution Equations; however, still few researchers in Evolution Equations and its related topics have expertise in Non-Archimedean analysis. That is, the classical, Archimedean study of Evolution Equations must be redirected toward, for example, *p-adic evolution differential and pseudo-differential equations;* indeed the non-Archimedean analogs of the Heat equation and the Schrödinger equation are being considered.

All chapters in this volume present work done in the classic Archimedean setting; it could be very interesting to investigate the problems described herein in a non-Archimedean framework.

Chapter "Impulsive Implicit Caputo Fractional q-Difference Equations in Finite and Infinite Dimensional Banach Spaces" by Alqhatani et al. uses the concept of measure of noncompactness and the fixed point theory to prove some new results on the existence of solutions for a class of Caputo-fractional q-difference equations with impulses in Banach spaces.

Ezzinbi and co-authors in the chapter "Global Attractor in Alpha-Norm for Some Partial Functional Differential Equations of Neutral and Retarded Type" study the existence of a global attractor for some partial neutral functional differential equations, giving the convergence of all the solutions to the attractor in terms of a so-called alpha-norm. The global attractor is proved to exist for a compact dissipative semigroup. Illustrative applications are taken from physical systems.

In chapter "Invariant Stable Manifolds of ϵ-Class for Partial Neutral Functional Differential Equations on a Half-Line" Nguyen Thieu Huy et al. consider equations

$$\partial F u_t/\partial t = B(t)Fu_t + \Phi(t, u_t), \quad t \in (0, \infty), \quad u_0 = \phi \in C([-r, 0], X)$$

and prove the existence of invariant stable and center-stable manifolds of $\epsilon-$class for solutions under some appropriate conditions; the approach to construct the manifolds of admissible classes is based on admissibility of function spaces and Lyapunov–Perron equations combined with fixed point arguments.

Chapter "Optimal Control of Averaged State of a Population Dynamics" by Cyrille Kenne and Boniface Nkemzi presents a population dynamic model with age dependence and spatial structure in a bounded domain $\Omega \subset \mathbb{R}^3$. The authors then prove that the average of the state can be brought to a desired state, using Euler–Lagrange first order optimality condition.

In chapter "Controllability of a Cascade Model in Population Dynamics", the authors Maniar and Echarroudi follow the classical track of Carleman estimates and semi-group theory to study a population dynamic cascade model with one force and two different scattering coefficients, possible null on the left hand side of the gene type domain.

Ouaro and Sawadogo study in chapter "Structural Stability of Nonlinear Elliptic $p(u)$-Laplacian Problem with Robin Type Boundary Condition" a class of nonlinear elliptic boundary value problem in a bounded open domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and prove the existence and uniqueness of weak solution and structural stability.

In chapter "C_0-Semigroup and Stepanov-Like Almost Automorphic Functions in Matched Spaces of Time Scales" Wang and N'Guérékata present the concepts of C_0-semigroup and Stepanov-like almost automorphic functions on a quantum time scale and their basic properties, generalizing the results to matched spaces of time scales.

Norouzi et al. in chapter "A Study of An Epidemic SIR Model via Homotopy Analysis Method in the Sense of Caputo-Fractional System" consider the fractional-order SIR system in the sense of Caputo-fractional differential equations to investigate the existence and local stability of equilibria; an analytic solution is also derived using the homotopy analysis method. The feasibility and validity of the results are illustrated in numerical simulations of three different cases. The results also show the asymptotic stability of the system for both similar and non-similar fractional orders at a certain limit order toward a fixed point; moreover changes in the strength of the infection do not affect the stability of the SIR model.

Finally, chapter "A Reaction-Diffusion Model for Salmonella Transmission Within an Industrial Hens House with Distributed Resistance to Salmonella Carrier-State" by Zongo and Beaumont, proposes a spatio-temporal model for Salmonella transmission within a flock of genetic heterogeneous animals with distinct levels of genetic resistance to the infection. The authors determine an explicit formula of the classic threshold predictor parameter R_0; in particular they derive the dependence of the severity of the disease transmission on the initial distribution of genetical fowls within a spatially heterogeneous environment.