STEAM-H: Science, Technology, Engineering, Agriculture, Mathematics & Health

Gaston M. N'Guérékata
Bourama Toni *Editors*

Studies in Evolution Equations and Related Topics

Springer

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Bourama Toni
Department of Mathematics
Howard University
Washington, DC, USA

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Gaston M. N'Guérékata • Bourama Toni
Editors

Studies in Evolution Equations and Related Topics

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Editors
Gaston M. N'Guérékata
Department of Mathematics
Morgan State University
Baltimore, MD, USA

Bourama Toni
Department of Mathematics
Howard University
Washington, DC, USA

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Preface

This book, "Studies in Evolution Equations and Related Topics" features recent developments and techniques in Evolution Equations by world-renown experts in the field; it will contribute to re-emphasize the relevance and depth of this important area of mathematics, in particular, its expanding reach into the physical, biological, social, and computational sciences. The volume provides an accessible summary of a wide range of active research topics, along with exciting new results. Topics include: Impulsive Implicit Caputo Fractional q-Difference Equations in Finite and Infinite Dimensional Banach Spaces; Optimal control of averaged state of a population dynamic model; Structural stability of nonlinear elliptic p(u)-Laplacian problem with Robin type boundary condition; Exponential dichotomy and partial neutral functional differential equations, stable and center stable manifolds of admissible class; Global attractor in alpha-norm for some Partial Functional Differential Equations of neutral and retarded type.

The volume's unique feature is to gather in a single expert book the most recent theorical developments as well as state-of-the art applications of Evolution Equations. It will certainly serve as a useful resource for both graduate students entering this research area and for more established researchers, including as a wide angle snapshot of this exciting and far-reaching research domain. It also facilitates an in-depth exchange of ideas on recent advances in the various aspects of Evolution Equations.

As such the volume is an important part of the multidisciplinary STEAM-H series (Science, Technology, Engineering, Agriculture, Mathematics and Health); the series brings together leading researchers to present their work in the perspective to advance their specific fields, and in a way to generate a genuine interdisciplinary interaction transcending disciplinary boundaries. All chapters therein were carefully edited and peer-reviewed; they are reasonably self-contained, and pedagogically exposed for a multidisciplinary readership.

Contributions are invited only, and reflect the most recent advances delivered in a high standard, self-contained in lines with the goals of the series, that is:

(1) To enhance multidisciplinary understanding between the disciplines by showing how some new advances in a particular discipline can be of interest to the other discipline, or how different disciplines contribute to a better understanding of a relevant issue at the interface of mathematics and the sciences.
(2) To promote the spirit of inquiry so characteristic of mathematics for the advances of the natural, physical and behavioral sciences by featuring leading experts.
(3) To encourage diversity in the readers' background and expertise, while at the same time structurally fostering genuine interdisciplinary interactions and networking.

Current disciplinary boundaries do not encourage effective interactions between scientists; researchers from different fields usually occupy different academic buildings, publish in journals specific to their field and attend different scientific meetings. Existing scientific meetings usually fall into either small gatherings specializing on specific questions, targeting specific and small group of scientists already aware of each other's work and potentially collaborating, or large meetings covering a wide field and targeting a diverse group of scientists but usually not allowing specific interactions to develop due to their large size and a crowded program. Here contributors focus on how to make their work intelligible, accessible to a diverse audience, which in the process enforces mastery of their own field of expertise.

This volume strongly advocates multidisciplinarity with the goal to generate new interdisciplinary approaches, instruments and models including new knowledge, transcending scientific boundaries to adopt a more holistic approach. For instance, it should be acknowledged, following Nobel laureate and president of the UK's Royal Society of Chemistry, Professor Sir Harry Kroto, "that the traditional chemistry, physics, biology departmentalised university infrastructures–which are now clearly out-of-date and a serious hindrance to progress–must be replaced by new ones which actively foster the synergy inherent in multidisciplinarity." The National Institute of Health and the Howard Hughes Medical Institute have strongly recommended that undergraduate biology education should incorporate mathematics, physics, chemistry, computer science, and engineering until "interdisciplinary thinking and work become second nature." Young physicists and chemists are encouraged to think about the opportunities waiting for them at the interface with the life sciences. Mathematics is playing an ever more important role in the physical and life sciences, engineering and technology, blurring the boundaries between scientific disciplines.

The series, through contributed volumes such as the current one, is to be a reference of choice for established interdisciplinary scientists and mathematicians, and a source of inspiration for a broad spectrum of researchers and research students, graduate and postdoctoral fellows; the sheer emphasis of these carefully selected and refereed contributed chapters is on important methods, research directions and applications of analysis including within and beyond mathematics. As such the volume implicitly promotes mathematical sciences, physical and life sciences, engineering, and technology education, as well as interdisciplinary, industrial and academic genuine cooperation.

The current book, entitled *"Studies in Evolution Equations and Related Topics*", as a whole certainly enhances the overall objective of the series, that is, to foster the readership interest and enthusiasm in the STEAM-H disciplines (Science, Technology, Engineering, Agriculture, Mathematics and Health), stimulate graduate and undergraduate research, and generate collaboration among researchers on a genuine interdisciplinary basis.

The STEAM-H series is hosted at Howard University, Washington DC, USA, an area that is socially, economically, intellectually very dynamic, and home to some of the most important research centers in the USA. This series, by now well established and published by Springer a world-renown publisher, is expected to become a national and international reference in interdisciplinary education and research.

Washington, DC, USA Bourama Toni
March 20, 2021

Acknowledgements

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Contributors

Saïd Abbas Department of Mathematics, Tahar Moulay University of Saida, Saida, Algeria

Mostafa Adimy Institut Camille Jordan, Université Claude Bernard Lyon 1, Villeurbanne Cedex, France

Badr Alqahtani Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

Catherine Beaumont INRAE, Université de Tours, UMR Biologie des oiseaux et aviculture, Nouzilly, France

Mouffak Benchohra Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, Sidi Bel-Abbes, Algeria

Younes Echarroudi Laboratory of Mathematics and Population Dynamics, Faculty of Sciences Semlalia Marrakesh, Cadi Ayyad university, Marrakesh, Morroco

Khalil Ezzinbi Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, Marrakesh, Morroco

Thi Ngoc Ha Vu School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam

Thieu Huy Nguyen School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam

Cyrille Kenne University of Buea, Department of Mathematics, Buea, Cameroon

Lahcen Maniar Cadi Ayyad University Morroco, Faculty of Sciences Semlalia Marrakesh, Marrakesh, Morroco

Catherine Marquet Université de Pau et des Pays de l'Adour, Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Pau, France

Gaston M. N’Guérékata Department of Mathematics, Morgan State University, Baltimore, MD, USA

Boniface Nkemzi University of Buea, Department of Mathematics, Buea, Cameroon

Fatemeh Norouzi Department of Mathematics, Morgan State University, Baltimore, MD, USA

Stanislas Ouaro LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences Exactes et Appliquées, Université Joseph KI ZERBO, Ouagadougou, Burkina Faso

Noufou Sawadogo LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences et Techniques, Université Nazi Boni, Bobo-Dioulasso, Burkina Faso

Bourama Toni Department of Mathematics, Howard University, Kunming, Yunnan, People’s Republic of China

Chao Wang Department of Mathematics, Yunnan University, Kunming, Yunnan, People’s Republic of China

Brice M. Yambiyo NEERLab, Department of Mathematics, Ecole Doctorale Sciences et Technologie, Université de Bangui, Bangui, Central African Republic

Xuan Yen Trinh Department of Basic Sciences, Hung Yen University of Technology and Education, Khoai Chau, Hung Yen, Vietnam

Pascal Zongo Laboratoire L3MA, DSI et IUT, Université des Antilles, Schoelcher, Martinique

Evolution Equations: Present and Future

Gaston M. N'Guérékata and Bourama Toni

This volume features chapters that present contemporary research focus and results in the theory of Evolution Equations and its applications in the physical and natural sciences, in a unique combination of mathematical rigor and realistic background. Evolution Equations, besides their theoretical interests, aim at describing the future time behavior of systems using time differential laws given by classes of equations that include: systems of ordinary differential equation in the form

$$du/dt = f(u,t), \quad d^2y/dt^2 = f(u,u',t), \cdots \tag{1}$$

where $u(t)$ could be regarded as the solution of the Cauchy problem; partial differential equations (e.g., of parabolic type for Heat conduction problems, and of hyperbolic type for motions of elastic continua); differential-difference equations (e.g., time-delayed finite dimensional feedback control systems in the transmission of the control signal); functional differential equations.

The goal is to be able to use the equation to predict the future of any related physical or natural system, given an "initial state" of the system and other parameters affecting the system. It is expected that one must have existence and uniqueness of forward-time solutions of all physically allowable data.

One widely studied area in the theory of Evolution Equations has been *periodicity* and its generalization in the direction of *almost periodicity*; briefly, consider for instance a Banach abstract space $\mathbb{B}$, and a $\mathbb{B}-$valued continuous function f over a

G. M. N'Guérékata
Department of Mathematics, Morgan State University, Baltimore, MD, USA
e-mail: Gaston.NGuerekata@morgan.edu

B. Toni (✉)
Department of Mathematics, Howard University, Washington, DC, USA
e-mail: bourama.toni@howard.edu

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time space $\mathbb{G}$, an additive semigroup. Take here $\mathbb{G} = \mathbb{R}$. The function f is time-*almost periodic* if $\forall \epsilon > 0$, $\exists F_\epsilon$ a finite subset in the algebra of $\mathbb{B}-$valued bounded continuous functions on $\mathbb{G}$, such

$$f_{\mathbb{R}} \subset B_\epsilon + F_\epsilon \tag{2}$$

where $f_{\mathbb{R}}$ is the set of all translates for f, that is, $f_{\mathbb{R}} := \{f_s : \quad s \in \mathbb{R}, \quad f_s(x) = x + s\}$, and B_ϵ is the ϵ zero neighborhood, i.e., open ball centered at 0 with radius ϵ. This is equivalent to $f_{\mathbb{G}}$ being *relatively compact* or *pre-compact* (compact closure $\bar{F_{\mathbb{R}}}$).

Almost periodicity from the earlier work by Bohr, Besicovitch, Bochner, and von Neumann, has been extensively studied, with notable contributions from many authors to include Corduneanu, Fink, and N'Guérékata, the current volume co-editor. This concept was also fittingly extended to the concept of *almost limit cycles* in the realm of continuous time dynamical systems to account for oscillations that are almost self-sustained.

A noted weakness of the theory is that the classical mathematical framework to study almost periodicity, as well as Evolution Equations, involves mostly, if not exclusively, Archimedean spaces (e.g., Euclidean, Banach, or Hilbert spaces). These are spaces that satisfy the *Archimedean Principle* in that there are all endowed with the usual/standard Euclidean norm and its induced metric which satisfied the triangle inequality; much of our visual and mental perception is based on the standard Euclidean space with its perfectly straight lines and planes; we came to see and represent the physical universe as actually Euclidean in its geometry, resulting oftentimes in biased and not so realistic mathematic model.

Indeed the physical and natural systems that Evolution Equations aim at describing are inherently non-Euclidean, with a natural geometrical ordering that is not the usual real line, but the more adequate *hierarchical generating tree.* For instance, the usual geometrical Archimedean/Euclidean distance "suitable" for measuring spatial location separation between human beings is less effective for the genetic distance measuring the hierarchical kinship relations.

Fortunately, thanks to the pioneering work by Kurt Hensel in the late nineteenth century, a new trend has been emerging in recent years studying Evolution Equations in Non-Archimedean or *p-adic* spaces, endowed with the so-called *ultrametrics;* ultrametrics could be induced by *p-adic absolute values* $|.|_p$, for p prime, that satisfy a more stringent inequality than the usual triangle inequality; specifically in a non-Archimedean space $\mathbb{K}$, the *ultrametric* distance d (or denoted $d_p(x, y)$ when induced by the *p-adic norm* $|.|_p$) satisfies the inequality

$$d(x, z) \leq max(d(x, y), d(y, z)) \quad \forall \text{ x,y,z } \in \mathbb{K}. \tag{3}$$

Recall that the set $\mathbb{R}$ of reals is the completion of the set $\mathbb{Q}$ of rationals by the usual Euclidean infinite norm denoted $|.|_\infty$, which amounts to creating new numbers as limits of Cauchy sequences that do not have rational limits. The same construction replacing the norm $|.|_\infty$ by the p-adic norm $|.|_p$ yields a new complete field, analogous to $\mathbb{R}$ denoted $\mathbb{Q}_p$, for every prime p, and called the field of *p-adic numbers*. Inside this field $\mathbb{Q}_p$ of p-adic numbers lies the ring $\mathbb{Z}_p$ of *p-adic integers* whose geometry is similar to the Cantor set. The set of rationals $\mathbb{Q}$ is densely contained in every $\mathbb{Q}_p$ as it is in the set of reals $\mathbb{R}$.

In short mathematical modelers of natural and physical have at their disposal a toolbox of various fields in the so-called *book structure* with the rationals $\mathbb{Q}$ as the book spine, and every page represents a field in which to carry mathematical modeling, to include $\mathbb{R} = \mathbb{Q}_\infty$, and $\mathbb{Q}_p$, for every p prime.

Non-Archimedean spaces, equivalently called *p-adic spaces* or *ultrametric spaces,* have some peculiar and important features described as follows:

- One immediate interesting fact about *p-adic integers* is that these integers are bounded in the norm by 1; that is, $|n|_p \leq 1$, i.e., as the number tends to infinity, its p-adic size remains less than one and tends to zero, in violation of the *Archimedean Principle*.
- The topology has a basis of *clopen* sets, i.e., sets that are simultaneously open and closed; so phrases such "open ball" and "closed balls" become meaningless.
- Every point in a p-adic ball is also its center. For instance, if *modeled p-adically* the center of the universe could be found at the nose of the nose, as once alluded to by the physicist Hawkins.
- Two balls are either disjoint or one within the other. This could hint toward the notion of parallel universes.
- The spaces are *totally disconnected*: the connected component of every point is the point itself. Consequently, the principle of *analytic continuation* is lost, as well as the *Intermediate value theorem*.
- The ultrametric geometry allows only *isosceles triangles*.
- The geometrical ordering is not along the real line but rather on a *hierarchical generating tree.* Consequently, the notion of time as we know seems to be a purely statistical construct. $\mathbb{Z}_3$ for instance is homeomorphic to the fractal-like Sierpinski Gasket.
- Convergence of p-adic series: $\sum_{n=1}^{\infty} a_n < \infty$ if and only $a_n \to 0$. (A calculus student dream!)
- The most consequential outcome and challenge is the study of Evolution Equations and their differential representations with a *p-adic time* instead of the usual real time. That is, how to model a system that evolves *p-adically.* This is still a hard open problem.
- Consider the following simple looking differential equations modeling so many natural systems:

$$\frac{dy}{dt} = \dot{y} = \lambda y, \tag{4}$$

and

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \quad \text{(Harmonic Oscillator)}. \tag{5}$$

Respectively the solutions are in the form $Ae^{\lambda x}$, and as level curves $H^{-1}(c) \quad c \in \mathbb{R}_{\geq 0}$ of a quadratic Hamiltonian H (i.e., concentric circles in the Euclidean plane $\mathbb{R}^2$).

However the *p-adic exponential function* $exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges only for $x \in B(0, r), \quad r = p^{1/(1-p)}$; that is, a very small radius convergence in contract of the standard exponential series that converges on the whole real line $\mathbb{R}$. Therefore the series $\sum_{n\geq 1} \frac{(\lambda x)^n}{n!}$ is convergent if and only if $|x| < |\lambda|^{-1} p^{1/(1-p)}$.

- Existence of *singular function* or Devil's Staircase or Cantor function: In the classic real analysis a function differentiable with vanishing derivative is a constant function. In contrast there exist in the p-adic space, functions that are *non-constant, continuous, and differentiable with vanishing derivatives*; these functions are called *pseudo-constants.* They are used to describe situation in which an object is known to move from point A to point B, but whenever it is observed it appears to be at rest. "Spooky motion" Einstein would say!
- The next most consequential notion is that of periodicity. Indeed, periodicity, an important concept in the study of Evolution Equations, is non-existent in non-Archimedean spaces; functions cannot be non-trivially periodic; however almost periodicity is allowed and seems to be the norm in the p-adic world. In other words, systems modeled do not have a time-periodic behavior in their p-adic dynamics. For instance, it has been noted (e.g., Corduneanu) that almost periodic oscillations are much more common than the periodic ones. There has been a tremendous amount of work in mathematical sciences involving periodicity, to include the famous Hilbert's 16^{th} problem, still unsolved to this day. Therefore, all these results on periodicity are made meaningless by just changing to the non-Archimedean/p-adic spaces. Such a realization is humbling and should be kept in mind when drawing conclusions based on periodicity.

In non-Archimedean/ultrametric spaces, almost periodicity is defined and understood in the following sense: consider a non-trivially valued complete algebraically closed non-Archimedean field $\mathbb{K}$, and a $\mathbb{K}-$valued function f over an additive semigroup $\mathbb{G}$.

The function f is said to be *p-adic/non-Archimedean almost periodic function* if the set $f_{\mathbb{G}}$ of all the translates of f is a *compactoid* in the $\mathbb{K}-$Banach algebra $\mathcal{B}$ of all bounded continuous functions $\mathbb{G} \longrightarrow \mathbb{K}$ with respect to pointwise operations and the supremum norm $||f||_\infty = \sup_{\mathbb{G}} |f(x)|$.

That is, $\forall \epsilon > 0$, $\exists$ a finite subset $F_\epsilon \subset \mathcal{B}$ such that

$$f_{\mathbb{G}} \subset B_\epsilon + aco(F_\epsilon) \tag{6}$$

where $aco(F_\epsilon)$ is the *smallest absolutely convex* subset containing F_ϵ. In the classic notion of almost periodicity, the set F_ϵ is only required to be *relatively compact/pre-compact.*

In the natural sciences ultrametricity is emerging as a consequence of randomness and the law of large numbers; exact in the limit $N \longrightarrow \infty$ for systems with a large number N of degrees of freedom, ultrametricity provides a more natural type of organization. It has proven effective for studying Evolution Equations describing neutral evolution of *pseudogenes*, stochastic branching processes in large space, energy landscape of *disordered frustrated systems* (spin glasses, problems in engineering and biology of combinatorial optimization), and in *taxonomy* where representation is given by dendrogram of hierarchy pictured in inverted tree. Ultrametricity serves also to better model Mental spaces, and the emerging and evolution of languages.

The richness of non-Archimedean space allows more realistic mathematical models for evolution equations, making it a state-of-the-art tool in the arsenal of researchers of Evolution Equations; however, still few researchers in Evolution Equations and its related topics have expertise in Non-Archimedean analysis. That is, the classical, Archimedean study of Evolution Equations must be redirected toward, for example, *p-adic evolution differential and pseudo-differential equations;* indeed the non-Archimedean analogs of the Heat equation and the Schrödinger equation are being considered.

All chapters in this volume present work done in the classic Archimedean setting; it could be very interesting to investigate the problems described herein in a non-Archimedean framework.

Chapter "Impulsive Implicit Caputo Fractional q-Difference Equations in Finite and Infinite Dimensional Banach Spaces" by Alqhatani et al. uses the concept of measure of noncompactness and the fixed point theory to prove some new results on the existence of solutions for a class of Caputo-fractional q-difference equations with impulses in Banach spaces.

Ezzinbi and co-authors in the chapter "Global Attractor in Alpha-Norm for Some Partial Functional Differential Equations of Neutral and Retarded Type" study the existence of a global attractor for some partial neutral functional differential equations, giving the convergence of all the solutions to the attractor in terms of a so-called alpha-norm. The global attractor is proved to exist for a compact dissipative semigroup. Illustrative applications are taken from physical systems.

In chapter "Invariant Stable Manifolds of ϵ-Class for Partial Neutral Functional Differential Equations on a Half-Line" Nguyen Thieu Huy et al. consider equations

$$\partial F u_t / \partial t = B(t) F u_t + \Phi(t, u_t), \quad t \in (0, \infty), \quad u_0 = \phi \in C([-r, 0], X)$$

and prove the existence of invariant stable and center-stable manifolds of $\epsilon-$class for solutions under some appropriate conditions; the approach to construct the manifolds of admissible classes is based on admissibility of function spaces and Lyapunov–Perron equations combined with fixed point arguments.

Chapter "Optimal Control of Averaged State of a Population Dynamics" by Cyrille Kenne and Boniface Nkemzi presents a population dynamic model with age dependence and spatial structure in a bounded domain $\Omega \subset \mathbb{R}^3$. The authors then prove that the average of the state can be brought to a desired state, using Euler–Lagrange first order optimality condition.

In chapter "Controllability of a Cascade Model in Population Dynamics", the authors Maniar and Echarroudi follow the classical track of Carleman estimates and semi-group theory to study a population dynamic cascade model with one force and two different scattering coefficients, possible null on the left hand side of the gene type domain.

Ouaro and Sawadogo study in chapter "Structural Stability of Nonlinear Elliptic $p(u)$-Laplacian Problem with Robin Type Boundary Condition" a class of nonlinear elliptic boundary value problem in a bounded open domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and prove the existence and uniqueness of weak solution and structural stability.

In chapter "C_0-Semigroup and Stepanov-Like Almost Automorphic Functions in Matched Spaces of Time Scales" Wang and N'Guérékata present the concepts of C_0-semigroup and Stepanov-like almost automorphic functions on a quantum time scale and their basic properties, generalizing the results to matched spaces of time scales.

Norouzi et al. in chapter "A Study of An Epidemic SIR Model via Homotopy Analysis Method in the Sense of Caputo-Fractional System" consider the fractional-order SIR system in the sense of Caputo-fractional differential equations to investigate the existence and local stability of equilibria; an analytic solution is also derived using the homotopy analysis method. The feasibility and validity of the results are illustrated in numerical simulations of three different cases. The results also show the asymptotic stability of the system for both similar and non-similar fractional orders at a certain limit order toward a fixed point; moreover changes in the strength of the infection do not affect the stability of the SIR model.

Finally, chapter "A Reaction-Diffusion Model for Salmonella Transmission Within an Industrial Hens House with Distributed Resistance to Salmonella Carrier-State" by Zongo and Beaumont, proposes a spatio-temporal model for Salmonella transmission within a flock of genetic heterogeneous animals with distinct levels of genetic resistance to the infection. The authors determine an explicit formula of the classic threshold predictor parameter R_0; in particular they derive the dependence of the severity of the disease transmission on the initial distribution of genetical fowls within a spatially heterogeneous environment.

Invariant Stable Manifolds of $\mathcal{E}$-Class for Partial Neutral Functional Differential Equations on a Half-Line

Thi Ngoc Ha Vu, Thieu Huy Nguyen, and Xuan Yen Trinh

2010 Mathematics Subject Classification 34K19, 35R10

1 Introduction

Consider partial neutral functional differential equation (PNFDE)

$$\frac{\partial}{\partial t} F u_t = B(t) F u_t + \Phi(t, u_t), \quad t \in [0, +\infty) \tag{1.1}$$

with the initial datum $u_0 = \phi \in \mathcal{C} := C([-r, 0], X)$ where $B(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed $t \geq 0$; $F : \mathcal{C} \to X$ is a bounded linear operator called *a difference operator*; $\Phi : \mathbb{R}_+ \times \mathcal{C} \to X$ is a continuous nonlinear operator called a *delay operator*, and u_t is the *history function* defined by $u_t(\theta) := u(t + \theta)$ for $\theta \in [-r, 0]$.

The study of invariant manifolds is one of important directions in the research for asymptotic behavior of solutions to evolution equations and has a long history. Early results come back to Hadamard [11], Perron [19, 20], Bogoliubov and Mitropolsky [2, 3] for the case of ordinary differential equations (ODE) in $\mathbb{R}^n$. Daleckii and Krein [4] proved the existence of invariant manifolds for solutions to ODE in Banach spaces. Henry [5] extended such results to the case of parabolic partial differential equations without delays. Huy [9] showed such results for general semi-linear evolution equations with nonlinear terms being φ-Lipschitz and suitable for

T. N. H. Vu · T. H. Nguyen (✉)
School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam
e-mail: ha.vuthingoc@hust.edu.vn; huy.nguyenthieu@hust.edu.vn

X. Y. Trinh
Department of Basic Sciences, Hung Yen University of Technology and Education, Khoai Chau, Hung Yen, Vietnam

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complicated diffusion processes. Moreover, in [8], Huy has proved the existence of a new type of invariant manifolds, called the *invariant stable manifolds of admissible classes*. Such manifolds have been constituted by trajectories belonging to the admissible Banach space E which can be L_p-spaces, Lorentz spaces $L_{p,q}$, or some interpolation spaces.

For the case of partial *delay* functional differential equations (i.e., the special case of (1.1) when $Fu_t = u(t)$), Minh and Wu [15] proved the existence of invariant manifolds of solutions to delay partial differential equations (see also [22] and reference therein for state of the art). Such results have then been extended by Huy and Duoc [11] to the case of φ-Lipschitz delays.

For the case of partial *neutral* functional differential equations (PNFDE) in the autonomous cases (i.e., $B(t) = B$ and $\Phi(t, \phi) = \Phi(\phi)$ do not depend on t), some results on existence of invariant manifolds have been obtained by H. Petzeltová and O.J. Staffans [18] and by R. Benkhalti, K. Ezzinbi and S. Fatajou [1]. They obtained such results under the conditions that B generates a hyperbolic analytic semigroup, and Φ is uniform Lipschitz continuous with a small Lipschitz constant.

Results on existence of invariant manifolds in the non-autonomous case for PNFDE (i.e., $B(t)$ and $\Phi(t, \phi)$ depend on time t) have been obtained by Huy and Bang under the conditions that the family $(B(t))_{t\geq 0}$ generates the dichotomic or trichotomic evolution family, and the delay term Φ is φ-Lipschitz, i.e., $\|\Phi(t, \phi) - \Phi(t, \psi)\| \leq \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$, where $\phi, \psi \in \mathcal{C}$ and $\varphi(t)$ is a real function belonging to certain admissible space.

The purpose of the present paper is to extend the results and methods in [10] combining with the methods in [8] to prove the existence of invariant stable manifolds of admissible classes (see Definition 1.2) which are constituted by trajectories of solutions belonging to certain Banach space $\mathcal{E}$ which can be an L_p-space, a Lorentz space $L_{p,q}$, or some interpolation space. We prove the existence of such manifolds for Eq. (1.1) when its linear part $(B(t))_{t\geq 0}$ generates the evolution family having an exponential dichotomy or trichotomy on the half-line, and its nonlinear term is φ-Lipschitz, i.e., $\|\Phi(t, \phi) - \Phi(t, \psi)\| \leq \varphi(t)\|\phi - \psi\|_{\mathcal{C}}$, where $\phi, \psi \in \mathcal{C}$ and $\varphi(t)$ is a real and positive function which belongs to admissible function space.

As mentioned in [10], when handling with PNFDE we face a difficult fact that the differential operators do not apply directly to $u(t)$ but to Fu_t, and hence the variation-of-constant formula is available only for Fu_t. Therefore, we write F in the form $F = \delta_0 - (\delta_0 - F)$, with Dirac distribution δ_0 concentrated at 0. Furthermore, another difficulty is lying in the fact that the admissibly inertial manifold is constituted by trajectories of the solutions belonging to (rescaledly) general admissible function spaces (see Definition 3.1 and Remark 3.2 thereafter) which are not necessarily L_∞-spaces. Therefore, the techniques and methodology used in the paper [10] cannot directly be applied here. Instead, we use the arguments together with generalized Hölder inequalities to obtain necessary estimates corresponding to the dichotomy of the evolution family. Then we apply our techniques and results in [8] (see also [12]) of using admissibility of function spaces to construct the solutions of Lyapunov–Perron's equation which will be used to derive the existence of admissibly stable manifolds of $\mathcal{E}$-class. Moreover, using these results and rescaling procedures we prove the existence of center-stable manifolds of $\mathcal{E}$-class for the mild solutions to Eq. (1.1) in the case of trichotomic linear parts under the same conditions on the nonlinear delay term Φ as in the dichotomic case. Our main results are contained in Theorems 2.7, 3.3.

We now recall some notions.

Let X be a Banach space (with a norm$\|\cdot\|$) and for a given $r > 0$ we denote by $\mathcal{C} := C([-r, 0], X)$ the Banach space of all continuous functions from $[-r, 0]$ into X, equipped with the norm $\|\phi\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} \|\phi(\theta)\|$ for $\phi \in \mathcal{C}$.

For a continuous function $w : [-r, \infty) \to X$, the *history function* $w_t \in \mathcal{C}$ is defined by $w_t(\theta) = v(t + \theta)$ for all $\theta \in [-r, 0]$.

An evolution family is now defined as follows.

Definition 1.1 A family of bounded linear operators $\{U(t, s)\}_{t\geq s\geq 0}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* if

(i) $U(t, t) = Id$ and $U(t, r)U(r, s) = U(t, s)$ for all $t \geq r \geq s \geq 0$,
(ii) The map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$,
(iii) There are constants $K, c \geq 0$ such that $\|U(t, s)x\| \leq Ke^{c(t-s)}\|x\|$ for all $t \geq s \geq 0$ and $x \in X$.

The notion of an evolution family arises naturally from the theory of well-posed, non-autonomous abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} = B(t)u(t), & t \geq s \geq 0, \\ u(s) = x \in X \end{cases} \tag{1.2}$$

Roughly speaking, the well-posedness of Problem (1.2) means that there exists an evolution family $\{U(t, s)\}_{t\geq s\geq 0}$ such that the solution of (1.2) is given by $u(t) = U(t, s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families, and applications to partial differential equations, we refer the readers to Pazy [17] (see also Nagel and Nickel [16] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$).

We then briefly recall some notions on function spaces taken from Massera and Schäffer [13], Räbiger and Schnaubelt [21], and Huy *et al.* [7–12].

Definition 1.2 Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathcal{C}) = \{f : \mathbb{R}_+ \to \mathcal{C} : f \text{ is strongly measurable and } \|f(\cdot)\|_{\mathcal{C}} \in E\}$$

(modulo λ-null functions) endowed with the norm $\|f\|_{\mathcal{E}} = \|\|f(\cdot)\|_{\mathcal{C}}\|_E$. One can easily see that $\mathcal{E}$ is a Banach space. We call it *the Banach space corresponding to the Banach function space E*.

In case $E = L^\infty(\mathbb{R}_+)$, we denote

$$\mathcal{E}_\infty := \{f : \mathbb{R}_+ \to \mathcal{C} : f \text{ is strongly measurable and } \|f(\cdot)\|_{\mathcal{C}} \in L^\infty(\mathbb{R}_+)\}$$

In order to study the invariant manifolds of $\mathcal{E}$-class for semi-linear evolution equations, we need some restrictions on the admissible Banach function spaces and assume the following hypothesis.

Standing Hypothesis 1.3 Throughout this paper we consider the admissible Banach function space E such that its associate space E' is also an admissible Banach function space. Moreover, for such an admissible Banach function space E we suppose that E' contains an *exponentially E-invariant* function, that is the function $\varphi \geq 0$ having the property that, for any fixed $\nu > 0$ the function h_ν defined by

$$h_\nu(t) := \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \text{ for } t \geq 0$$

belongs to E.

2 Stable Manifolds of $\mathcal{E}$-Class

In order to prove the existence of invariant stable manifolds, we need the following notion of exponential dichotomies of the evolution family $\{U(t,s)\}_{t\geq s\geq 0}$.

Definition 2.1 An evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on the Banach space X is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t),\ t \geq 0$, on X and positive constants $N,\ \nu$ such that

(a) $U(t,s)P(s) = P(t)U(t,s), \quad t \geq s \geq 0$,
(b) The restriction $U(t,s)_| : \mathrm{Ker}P(s) \to \mathrm{Ker}P(t),\ t \geq s \geq 0$, is an isomorphism, and we denote its inverse by $U(s,t)_| := (U(t,s)_|)^{-1}$, $0 \leq s \leq t$,
(c) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X,\ t \geq s \geq 0$,
(d) $\|U(s,t)_| x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \mathrm{Ker}P(t),\ t \geq s \geq 0$.

The projections $P(t),\ t \geq 0$, are called the *dichotomy projections*, and the constants $N,\ \nu$ the *dichotomy constants*.

Note that the exponential dichotomy of $\{U(t,s)\}_{t\geq s\geq 0}$ implies that $H := \sup_{t\geq 0}\|P(t)\| < \infty$ and the map $t \mapsto P(t)$ is strongly continuous (see [14, Lemma 4.2]). We can then define the Green's function on the half-line as follows:

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)U(t,\tau) & \text{for } t > \tau \geq 0 \\ -U(t,\tau)_|(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases} \tag{2.1}$$

It follows from the exponential dichotomy of $\{U(t,s)\}_{t\geq s\geq 0}$ that

$$\|\mathcal{G}(t,\tau)\| \leq N(1+H)e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau, \quad t,\tau \in \mathbb{R}.$$

Next, using the projections $(P(t))_{t\geq 0}$ on X, we can define the family of operators $(\widetilde{P}(t))_{t\geq 0}$ on $\mathcal{C}$ as follows:

$$\widetilde{P}(t) : \mathcal{C} \to \mathcal{C}$$
$$(\widetilde{P}(t)\phi)(\theta) = U(t-\theta, t)P(t)\phi(0) \text{ for all } \theta \in [-r, 0]. \tag{2.2}$$

Then, we have that $(\widetilde{P}(t))^2 = \widetilde{P}(t)$, and therefore the operators $\widetilde{P}(t)$, $t \geq 0$, are projections on $\mathcal{C}$. Moreover,

$$\text{Im}\widetilde{P}(t) = \{\phi \in \mathcal{C} : \phi(\theta) = U(t-\theta, t)\nu_0 \ \forall\theta \in [-r, 0] \text{ for some } \nu_0 \in \text{Im}P(t)\}. \tag{2.3}$$

To obtain the existence of invariant stable manifolds, we also need the following notion of the φ-Lipschitz of the nonlinear delay term Φ.

Definition 2.2 Let E be an admissible Banach function space and φ be a positive function belonging to E. A function $\Phi : [0, \infty) \times \mathcal{C} \to X$ is said to be *φ-Lipschitz* if Φ satisfies

(i) $\|\Phi(t, 0)\| \leq \varphi(t)$ for all $t \in \mathbb{R}_+$.
(ii) $\|\Phi(t, \phi_1) - \Phi(t, \phi_2)\| \leq \varphi(t)\|\phi_1 - \phi_2\|_{\mathcal{C}}$ for all $t \in \mathbb{R}_+$ and all $\phi_1, \ \phi_2 \in \mathcal{C}$.

Note that if $\Phi(t, \phi)$ is φ-Lipschitz, then $\|\Phi(t, \phi)\| \leq \varphi(t)(1 + \|\phi\|_{\mathcal{C}})$ for all $\phi \in \mathcal{C}$ and $t \geq 0$.

In the space of infinite dimension, instead of Eq. (1.1) we consider the following integral equation:

$$\begin{cases} Fu_t &= U(t, s)F\phi + \int_s^t U(t, \xi)\Phi(\xi, u_\xi)d\xi \text{ for } t \geq s \geq 0, \\ u_s &= \phi \in \mathcal{C}. \end{cases} \tag{2.4}$$

We note that, if the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ arises from the well-posed Cauchy problem (1.2), then the function $u : [s - r, \infty) \to X$, which satisfies (2.4) for some given function Φ, is called a *mild solution* of the semi-linear problem

$$\begin{cases} \frac{\partial}{\partial t} Fu_t = B(t)Fu_t + \Phi(t, u_t), \quad t \geq s \geq 0, \\ u_s \quad\ = \phi \in \mathcal{C}. \end{cases}$$

The reader is referred to J. Wu [22] for detailed treatments on the relations between classical and mild solutions of functional evolution equations.

We now give the notion of an invariant stable manifold for the solutions of the integral equation (2.4) the next definition.

Definition 2.3 A set $S \subset \mathbb{R}_+ \times \mathcal{C}$ is said to be an *invariant stable manifold of $\mathcal{E} - class$* for the solutions to Eq. (2.4) if for every $t \in \mathbb{R}_+$ the phase spaces $\mathcal{C}$ splits into a direct sum $\mathcal{C} = \text{Im}\widetilde{P}(t) \oplus \text{Ker}\widetilde{P}(t)$ with corresponding projections $\widetilde{P}(t)$ and there exists a family of Lipschitz continuous mappings

$$\widetilde{y}_t : \text{Im}\widetilde{P}(t) \to \text{Ker}\widetilde{P}(t), \quad t \in \mathbb{R}_+$$

with the Lipschitz constants independent of t such that

(i) $S = \{(t, \psi + \widetilde{y}_t(\psi)) \in \mathbb{R}_+ \times (\mathrm{Im}\widetilde{P}(t) \oplus \mathrm{Ker}\widetilde{P}(t)) \mid t \in \mathbb{R}_+,\ \psi \in \widetilde{X}_0(t)\}$, and we denote by

$$S_t := \{\psi + \widetilde{y}_t(\psi) : (t, \psi + \widetilde{y}_t(\psi)) \in S\},$$

(ii) S_t is homeomorphic to $\mathrm{Im}\widetilde{P}(t)$ for all $t \geq 0$,

(iii) To each $\phi \in S_s$ there corresponds one and only one solution $u(t)$ to Eq. (2.4) on $[s - r, \infty)$ satisfying the conditions that $\tilde{u}_s = \phi$, and the function $\chi_{[s,\infty)}(t)u_t, t \in \mathbb{R}$, belongs to $\mathcal{E} \cap \mathcal{E}_\infty$ where the function $\tilde{u}_s$ is defined by $\tilde{u}_s(\theta) = Fu_{s-\theta}$ for all $-r \leq \theta \leq 0$.

(iv) S is positively F-invariant under Eq. (2.4) in the sense that if $u(t)$, $t \geq s - r$, is a solution to Eq. (2.4) satisfying conditions that $\widetilde{u}_s \in S_s$ and the function $\chi_{[s,\infty)}(t)u_t,\ t \in \mathbb{R}$, belongs to $\mathcal{E}$, then we have $\tilde{u}_t \in S_t$ for all $t \geq s$, where the function $\tilde{u}_t$ is defined by

$$\tilde{u}_t(\theta) = Fu_{t-\theta} \text{ for all } -r \leq \theta \leq 0 \text{ and } t \geq 0. \tag{2.5}$$

Note that if we denoted by $\widetilde{X}_0(t) = \mathrm{Im}\widetilde{P}(t)$, $\widetilde{X}_1(t) = \mathrm{Ker}\widetilde{P}(t)$ and we identify $\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with $\widetilde{X}_0(t) \times \widetilde{X}_1(t)$, then we can write $S_t = graph(\widetilde{y}_t)$.

The following lemma gives the form of bounded solutions to Eq. (2.4). To do this, we first recall the notion of the integral translation operators $\Lambda_1, \Lambda_\nu^{'}, \Lambda_\nu^{''}$ (see [8, Def. 2.4; Pro. 2.7]) as follows: for $\varphi \in E$, $\Lambda_1\varphi$ is defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)d\tau$ belong to E for all $t \in \mathbb{R}_+$; if $\varphi \in L_{1,loc}(\mathbb{R})$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$; $\nu > 0$ then $\Lambda_\nu^{'}, \Lambda_\nu^{''}$ are defined by $\Lambda_\nu^{'}\varphi(t) = \int\limits_0^t e^{-\nu(t-s)}\varphi(s)ds$; $\Lambda_\nu^{''}\varphi(t) = \int\limits_t^\infty e^{-\nu(s-t)}\varphi(s)ds$ belong to E.

Lemma 2.4 *Let the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants N, $\nu > 0$. Assume Standing Hypothesis 1.3 and let $\varphi \in E'$ be an exponentially E-invariant function defined as in that Standing Hypothesis. Let $F : \mathcal{C} \to X$ and $\Phi : \mathbb{R}_+ \times \mathcal{C} \to X$ be respectively the difference and delay operators. Suppose that Φ is φ-Lipschitz, and that $u(t)$ is a solution to Eq. (2.4) such that, for fixed $s \geq 0$ the function $\chi_{[s,\infty)}(t)u_t,\ t \in \mathbb{R}$, belongs to $\mathcal{E} \cap \mathcal{E}_\infty$. Then, for $t \geq s$ the function $u(t)$ satisfies*

$$\begin{cases} Fu_t = U(t, s)\nu_0 + \int_s^\infty \mathcal{G}(t, \tau)\Phi(\tau, u_\tau)d\tau, \\ u_s = \phi \in \mathcal{C} \end{cases} \tag{2.6}$$

for some $\nu_0 \in X_0(s) = P(s)X$, where $\mathcal{G}(t, \tau)$ is the Green's function defined as in (2.1).

Proof Put

$$y(t) = \begin{cases} \int_s^\infty \mathcal{G}(t,\tau)\Phi(\tau,u_\tau)d\tau & \text{for } t \geq s \\ \int_s^\infty \mathcal{G}(2s-t,\tau)\Phi(\tau,u_\tau)d\tau & \text{for } s-r \leq t < s \end{cases}.$$

We have, for $t \geq s$

$$\|y(t)\| \leq \int_s^\infty N(1+H)e^{-\nu|t-\tau|}\varphi(\tau)(1+\|u_\tau\|_{\mathcal{C}})d\tau$$

$$\leq N(1+H)(1+\sup_{\xi\geq s-r}\|u(\xi)\|)\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)d\tau$$

and, for $s-r \leq t < s$

$$\|y(t)\| \leq \int_s^\infty N(1+H)e^{-\nu|2s-t-\tau|}\varphi(\tau)(1+\|u_\tau\|_{\mathcal{C}})d\tau$$

$$\leq N(1+H)(1+\sup_{\xi\geq s-r}\|u(\xi)\|)\int_0^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)d\tau.$$

Since $t+\theta \in [-r+t,t]$ for fixed $t \in [s,\infty)$ and all $\theta \in [-r,0]$, we have that

$$\|y_t\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]} y(t+\theta) \leq N(1+H)(1+\sup_{t\geq s}\|u(t)\|)e^{\nu r}\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)d\tau$$

$$\leq N(1+H)(1+\sup_{t\geq s}\|u(t)\|)e^{\nu r}\left(\Lambda_\nu'\varphi(t)+\Lambda_\nu''\varphi(t)\right) \text{ for } t \geq s.$$

Therefore, by Banach lattice properties we have that $y(\cdot) \in \mathcal{E}\cap\mathcal{E}_\infty$ and

$$\|y(\cdot)\|_{\mathcal{E}} \leq N(1+H)e^{\nu r}(1+\sup_{t\geq s}\|u(t)\|)\frac{(N_1\|\Lambda_1T_1^+\varphi\|_E + N_2\|\Lambda_1\varphi\|_E)}{1-e^{-\nu}}, \tag{2.7}$$

and

$$\|y(\cdot)\|_{\mathcal{E}_\infty} \leq N(1+H)e^{\nu r}(1+\sup_{t\geq s}\|u(t)\|)\frac{(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}$$

where T_1^+ is defined as in [8, Def. 2.4].

On the other hand,

$$\begin{aligned} U(t,s)y(s) &= -\int_s^t U(t,s)U(s,\tau)_|(I-P(\tau))\Phi(\tau,u_\tau)d\tau \\ &\quad -\int_t^\infty U(t,s)U(s,\tau)_|(I-P(\tau))\Phi(\tau,u_\tau)d\tau \\ &= -\int_s^t U(t,\tau)(I-P(\tau))\Phi(\tau,u_\tau)d\tau \\ &\quad -\int_t^\infty U(t,\tau)_|(I-P(\tau))\Phi(\tau,u_\tau)d\tau. \end{aligned}$$

Therefore,

$$y(t) = U(t,s)y(s) + \int_s^t U(t,\tau)\Phi(\tau,u_\tau)d\tau.$$

Since u_t is a solution of Eq. (2.4), we obtain that $Fu_t - y(t) = U(t,s)(Fu_s - y(s))$. Put now $\nu_0 = Fu_s - y(s)$. The boundedness of Fu_t and $y(t)$ on $[s,\infty)$ implies that $\nu_0 \in X_0(s)$ and $P(s)Fu_s = P(s)F\phi = \nu_0$. Therefore, $Fu_t = U(t,s)\nu_0 + y(t)$ for $t \geq s$. □

Remark 2.5 Equation (2.6) is called the *Lyapunov–Perron's equation*. By computing directly, we can see that the converse of Lemma 2.4 is also true. This means that, all solutions of the integral equation (2.6) satisfy Eq. (2.4) for $t \geq s$.

We come to our next result on the existence and partial stability of solutions starting from a subspace of $\mathcal{C}$.

Theorem 2.6 *Under the hypotheses of Lemma 2.4 let $\widetilde{P}(t)$, $t \geq 0$, be projections defined as in (2.2). Consider functions φ and h_ν defined as in Standing Hypothesis 1.3. Let $F : \mathcal{C} \to X$ be of the form $F = \delta_0 - \Psi$ for $\Psi \in L(\mathcal{C}, X)$ with $\|\Psi\| < 1$, and δ_0 being the Dirac function concentrated at 0. Suppose that the delay operator $\Phi : \mathbb{R}_+ \times \mathcal{C} \to X$ is φ-Lipschitz and set*

$$k := N(1+H)e^{\nu r} \times \max\left\{\|h_\nu(\cdot)\|_E, \frac{N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}\right\} \tag{2.8}$$

Then, if $\frac{k}{1-\|\Psi\|} < 1$*, there corresponds to each* $\phi \in \mathrm{Im}\widetilde{P}(s)$ *one and only one solution* $u(t)$ *of Eq. (2.6) on* $[s-r,\infty)$ *satisfying the conditions that* $\widetilde{P}(s)\tilde{u}_s = \phi$*, and function* $\chi_{[s,\infty)}(t)u_t$*,* $t \in \mathbb{R}$*, belongs to* $\mathcal{E} \cap \mathcal{E}_\infty$*, where the function* $\tilde{u}_s$ *is defined as in Definition (2.3). Moreover, the following estimate is valid for any two solutions* $u(t), v(t)$ *corresponding to different initial functions* $\phi,\ \psi \in \mathrm{Im}\widetilde{P}(s)$*:*

$$\|u_t - v_t\|_{\mathcal{C}} \le C_\mu e^{-\mu(t-s)}\|\phi(0)-\psi(0)\| \qquad \text{for all } t \ge s \ge 0, \tag{2.9}$$

where μ *is a positive constant satisfying*

$$0 < \mu < \nu + \ln\left(1 - \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)\right), \ \textit{and}$$

$$C_\mu := \frac{Ne^{\nu r}}{1-\|\Psi\| - \frac{N(1+H)e^{\nu r}}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}.$$

Proof Firstly, to prove that there corresponds to each $\phi \in \mathrm{Im}\widetilde{P}(s)$ one and only one solution $u(t)$ in $\mathcal{E} \cap \mathcal{E}_\infty$ of Eq. (2.6) on $[s-r,\infty)$, we construct a contraction mapping. To do this, we consider from (2.2) with $\phi \in \mathrm{Im}\widetilde{P}(s) = \{\phi(\theta) = U(t-\theta,t)\nu_0 : -r \le \theta \le 0;\ \nu_0 \in ImP(s)\}$. Clearly, $\nu_0 = \phi(0)$.

Denote by $C_b([s-r,\infty),X)$ the Banach space of bounded, continuous, and X-valued functions defined on $[s-r,\infty)$, which is endowed with the sup-norm $\|\cdot\|_\infty$.

We define the operator $\tilde{\Psi} : C_b([s-r,\infty),X) \to C_b([s-r,\infty),X)$ by

$$[\widetilde{\Psi}u](t) = \begin{cases} \Psi(u_t) & \text{for } s \le t \\ \Psi(u_s) & \text{for } s-r \le t \le s. \end{cases} \tag{2.10}$$

Because $\|\Psi\| < 1$ we have $\|\tilde{\Psi}\| \le \|\Psi\| < 1$. Therefore, the operator $I - \tilde{\Psi}$ is invertible. For $\nu_0 = \phi(0) \in \mathrm{Im}P(s)$ as above, we define a mapping $\widetilde{F}_\phi : C_b([s-r,\infty),X) \to C_b([s-r,\infty),X)$ by

$$(\widetilde{F}_\phi u)(t) = \begin{cases} U(t,s)\nu_0 + \int_s^\infty \mathcal{G}(t,\tau)\Phi(\tau,u_\tau)d\tau & \text{for } t \ge s \\ U(2s-t,s)\nu_0 + \int_s^\infty \mathcal{G}(2s-t,\tau)\Phi(\tau,u_\tau)d\tau & \text{for } s-r \le t \le s. \end{cases} \tag{2.11}$$

We now put $T := (I-\widetilde{\Psi})^{-1}\widetilde{F}_\phi$. We will prove the transformation T as above acts from $\mathcal{E} \cap \mathcal{E}_\infty$ into $\mathcal{E} \cap \mathcal{E}_\infty$ and is a contraction.

In fact, putting $e_\nu(t) = e^{-\nu|t|}$ and we have $\left(T_s^+ e_\nu\right)(t) = e^{-\nu(t-s)}$ (see [8, Def. 2.4; Pro. 2.7])

We have, for $t \geq s$

$$\|(\widetilde{F}_\phi u)(t)\|$$

$$\leq Ne^{-\nu(t-s)}\|\nu_0\| + \int_s^\infty N(1+H)e^{-\nu|t-\tau|}\varphi(\tau)(1+\|u_\tau\|_{\mathcal{C}})d\tau$$

$$\leq N\left(T_s^+ e_\nu\right)(t)\|\nu_0\| + N(1+H)(1+\sup_{\xi\geq s-r}\|u(\xi)\|)\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)d\tau,$$

and, for $s - r \leq t \leq s$

$$\|(\widetilde{F}_\phi u)(t)\|$$

$$\leq Ne^{-\nu(s-t)}\|\nu_0\| + \int_s^\infty N(1+H)e^{-\nu|2s-t-\tau|}\varphi(\tau)(1+\|u_\tau\|_{\mathcal{C}})d\tau$$

$$\leq N\left(T_s^+ e_\nu\right)(t)\|\nu_0\| + N(1+H)(1+\sup_{\xi\geq s-r}\|u(\xi)\|)\int_0^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)d\tau.$$

On the other hand $(Tu)(t) = \left[(I-\widetilde{\Psi})^{-1}\widetilde{F}_\phi u\right](t)$.
Since $t+\theta \in [-r+t, t]$ for fixed $t \in [s,\infty)$ and all $\theta \in [-r, 0]$, we have

$$\|(Tu)(t)\|_{\mathcal{C}} \leq \frac{1}{1-\|\Psi\|}\Big[Ne^{\nu r}\left(T_s^+ e_\nu\right)(t)\|\nu_0\| + N(1+H)(1+\sup_{t\geq s}\|u(t)\|)e^{\nu r}$$
$$\left(\Lambda_\nu^{'}\varphi(t) + \Lambda_\nu^{''}\varphi(t)\right)\Big] \text{ for } t \geq s.$$

Therefore, by Banach lattice properties we have that $Tu(\cdot) \in \mathcal{E}$ and

$$\|Tu(\cdot)\|_{\mathcal{E}} \leq \frac{1}{1-\|\Psi\|}\Big[NN_1 e^{\nu r}\|\nu_0\|\|e_\nu\|_E + N(1+H)e^{\nu r}(1+\sup_{t\geq s}\|u(t)\|)$$
$$\frac{(N_1\|\Lambda_1 T_1^+\varphi\|_E + N_2\|\Lambda_1\varphi\|_E)}{1-e^{-\nu}}\Big].$$

Similarly, we have

$$\|Tu(\cdot)\|_{\mathcal{E}_\infty} \leq \frac{1}{1-\|\Psi\|}\Big[Ne^{\nu r}\|\nu_0\| + N(1+H)e^{\nu r}(1+\sup_{t\geq s}\|u(t)\|)$$
$$\frac{(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}\Big].$$

Hence, the transformation T acts from $\mathcal{E} \cap \mathcal{E}_\infty$ into $\mathcal{E} \cap \mathcal{E}_\infty$. Next, we will prove T is a contraction mapping.

We then estimate

$$\begin{aligned}
\|(\widetilde{F}_\phi u)(t) - (\widetilde{F}_\phi v)(t)\| &\le \int_s^\infty \|\mathcal{G}(t,\tau)(\Phi(\tau,u_\tau) - \Phi(\tau,v_\tau))\| d\tau \\
&\le N(1+H)\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau, \\
&\le N(1+H)\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \quad \text{for } t \ge s,
\end{aligned}$$

and similarly

$$\begin{aligned}
\|(\widetilde{F}_\phi u)(t) - (\widetilde{F}_\phi v)(t)\| &\le \int_s^\infty \|\mathcal{G}(2s-t,\tau)(\Phi(\tau,u_\tau) - \Phi(\tau,v_\tau))\| d\tau \\
&\le N(1+H)\int_s^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \\
&\le N(1+H)\int_0^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \\
&\quad \text{for } -r+s \le t \le s.
\end{aligned}$$

Using the Neumann's series, we then have

$$\begin{aligned}
(Tu)(t) - (Tv)(t) &= \left[\left(\sum_{n=0}^\infty \widetilde{\Psi}^n\right)\widetilde{F}_\phi u\right](t) - \left[\left(\sum_{n=0}^\infty \widetilde{\Psi}^n\right)\widetilde{F}_\phi v\right](t) \\
&= \left[(\widetilde{F}_\phi u)(t) - (\widetilde{F}_\phi v)(t)\right] + \left[(\widetilde{\Psi}\widetilde{F}_\phi u)(t) - (\widetilde{\Psi}\widetilde{F}_\phi v)(t)\right] \\
&\quad + \left[(\widetilde{\Psi}^2\widetilde{F}_\phi u)(t) - (\widetilde{\Psi}^2\widetilde{F}_\phi v)(t)\right] + \dots
\end{aligned}$$

Next, by induction we can easily see that

$$\|(\widetilde{\Psi}^n\widetilde{F}_\phi u)(t) - (\widetilde{\Psi}^n\widetilde{F}_\phi v)(t)\| \le \|\Psi\|^n N(1+H)\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau$$

$$\text{for } t \ge s \text{ and } n \in \mathbb{N},$$

and

$$\|(\widetilde{\Psi}^n \widetilde{F}_\phi u)(t) - (\widetilde{\Psi}^n \widetilde{F}_\phi v)(t)\| \le \|\Psi\|^n N(1+H) \int_0^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau$$

for $-r+s \le t \le s$ and $n \in \mathbb{N}$.

From the above claim, it follows that

$$\|(Tu)(t) - (Tv)(t)\| \le \left(\sum_{n=0}^\infty \|\Psi\|^n\right) N(1+H) \int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau$$

$$\text{for } t \ge s,$$

and

$$\|(Tu)(t) - (Tv)(t)\| \le \left(\sum_{n=0}^\infty \|\Psi\|^n\right) N(1+H) \int_0^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau$$

$$\text{for } -r+s \le t \le s.$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in [s, \infty)$ and all $\theta \in [-r, 0]$ we have that,

$$\begin{aligned}
&\|(Tu)(t) - (Tv)(t)\|_{\mathcal{C}} \\
&\quad = \sup_{\theta \in [-r,0]} \|(Tu)(t+\theta) - (Tv)(t+\theta)\| \\
&\quad \le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r} \int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} \text{ for } t \ge s.
\end{aligned}$$

Since $e^{-\nu|t-\cdot|}\varphi(\cdot) \in E'$, $\|u_\tau - v_\tau\|_{\mathcal{C}} \in E$, and using the "Hölder's inequality," it follows from the above inequality that

$$\begin{aligned}
\|(Tu)(t) - (Tv)(t)\|_{\mathcal{C}} &\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r} \|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'} \| \|u_{(\cdot)} - v_{(\cdot)}\|_{\mathcal{C}}\|_E \\
&\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r} h_\nu(t)\|u(\cdot) - v(\cdot)\|_{\mathcal{E}} \text{ for } t \ge s.
\end{aligned}$$

According to the definition (1.2), we have

$$\begin{aligned}
&\|(Tu)(\cdot) - (Tv)(\cdot)\|_{\mathcal{E}} \\
&\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r} \|h_\nu(\cdot)\|_E \|u(\cdot) - v(\cdot)\|_{\mathcal{E}} \\
&\le \frac{k}{1-\|\Psi\|} \|u(\cdot) - v(\cdot)\|_{\mathcal{E}},
\end{aligned}$$

where k is defined as in (2.8). In a similar way, we have

$$\begin{aligned}
&\|(Tu)(\cdot) - (Tv)(\cdot)\|_{\mathcal{E}_\infty} \\
&\le \frac{1}{1-\|\Psi\|} N(1+H)e^{\nu r} \left(\frac{N_1\|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty}{1-e^{-\nu}} \right) \|u(\cdot) - v(\cdot)\|_{\mathcal{E}_\infty} \\
&\le \frac{k}{1-\|\Psi\|} \|u(\cdot) - v(\cdot)\|_{\mathcal{E}_\infty}.
\end{aligned}$$

Next, if $\frac{k}{1-\|\Psi\|} < 1$ the transformation $T : \mathcal{E} \cap \mathcal{E}_\infty \to \mathcal{E} \cap \mathcal{E}_\infty$ is a contraction mapping. Thus, there exists a unique $u(\cdot) \in \mathcal{E} \cap \mathcal{E}_\infty$ such that $Tu = u$. This yields that $u(t)$, $t \ge s - r$, is the unique solution of Eq. (2.6) with

$$(\widetilde{F}_\phi u_s)(\theta) = U(s-\theta, s)\nu_0 + \int_s^\infty \mathcal{G}(s-\theta, \tau)\Phi(\tau, u_\tau)d\tau \qquad \text{for all } \theta \in [-r, 0],$$

and $P(s)Fu_s = \nu_0$. Therefore, $\widetilde{P}(s)\tilde{u}_s = \phi$ by definition of $\widetilde{P}(s)$ (see Equality (2.2)).

Secondly, we now prove the inequality (2.9). Let $u(t)$, $v(t)$ be the two solutions to Eq. (2.6) corresponding to different initial functions $\phi,\ \psi \in \operatorname{Im}\widetilde{P}(s)$, respectively. We have that

$$u(t) - v(t) = (Tu)(t) - (Tv)(t) = [(I-\widetilde{\Psi})^{-1}\widetilde{F}_\phi u](t) - [(I-\widetilde{\Psi})^{-1}\widetilde{F}_\psi v](t).$$

Using Neumann's series, we arrive at

$$\begin{aligned}
u(t) - v(t) = {} & [(\widetilde{F}_\phi u)(t) - (\widetilde{F}_\psi v)(t)] + [(\widetilde{\Psi}(\widetilde{F}_\phi u)(t)) - (\widetilde{\Psi}(\widetilde{F}_\psi v)(t))] \\
& + [(\widetilde{\Psi}^2 \widetilde{F}_\phi u)(t)) - (\widetilde{\Psi}^2(\widetilde{F}_\psi v)(t))] + \dots
\end{aligned} \tag{2.12}$$

By definition of $\tilde{F}_\phi$ the norm of the first term in (2.12) can be estimated by

$$\begin{aligned}
&\|(\widetilde{F}_\phi u)(t) - (\widetilde{F}_\psi v)(t)\| \le N(T_s^+ e_\nu)(t)\|\phi(0) - \psi(0)\| + N(1+H) \\
&\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \text{ for } t \ge s.
\end{aligned}$$

Again, by induction, the norm of the n^{th} term in (2.12) can be estimated by

$$\|(\widetilde{\Psi}^n \widetilde{F}_\phi u)(t) - (\widetilde{\Psi}^n \widetilde{F}_\psi v)(t)\| \le \|\Psi\|^n \Big[N(T_s^+ e_\nu)(t)\|\phi(0) - \psi(0)\| + N(1+H)\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \Big] \quad \text{for } t \ge s$$

and

$$\|(\widetilde{\Psi}^n \widetilde{F}_\phi u)(t) - (\widetilde{\Psi}^n \widetilde{F}_\psi v)(t)\| \le \|\Psi\|^n \Big[N(T_s^+ e_\nu)(t)\|\phi(0) - \psi(0)\| + N(1+H)\int_s^\infty e^{-\nu|2s-t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \Big] \text{ for } s-r \le t \le s.$$

Again, since $t+\theta \in [-r+t, t]$ for fixed $t \in [s, \infty)$ and all $\theta \in [-r, 0]$, and using (2.12) we obtain

$$\|u_t - v_t\|_{\mathcal{C}} \le \frac{1}{1-\|\Psi\|}\Big[Ne^{\nu r}(T_s^+ e_\nu)(t)\|\phi(0) - \psi(0)\| + N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \Big], \quad t \ge s. \tag{2.13}$$

Put $h(t) = \|u_t - v_t\|_{\mathcal{C}}$. Then,

$$h(t) \le \frac{1}{1-\|\Psi\|}\Big[Ne^{\nu r}(T_s^+ e_\nu)(t)\|\phi(0) - \psi(0)\| + N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)h(\tau)\, d\tau \Big], \quad t \ge s. \tag{2.14}$$

We will use the cone inequality theorem (see [9, Theorem 2.8]) applying to Banach space E with the cone $\mathcal{K}$ being the set of all non-negative functions. We then consider the linear operator A defined for $g \in E$ by

$$(Ag)(t) = \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)g(\tau)d\tau, \quad t \ge s.$$

By "Hölder's inequality" then

$$\|(Ag)(t)\| \le \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|} h_\nu(t)\|g\|_E.$$

By the Banach laxative property of E, we have

$$\|Ag\|_E \le \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}\|h_\nu\|_E\|g\|_E.$$

Therefore $A \in \mathcal{L}(E)$ and $\|A\| \le \frac{k}{1-\|\Psi\|} < 1$. Clearly, the cone $\mathcal{K}$ is invariant under the operator A. The inequality (2.14) can now be rewritten as

$$h \le Ah + z \text{ for } z(t) = \frac{1}{1-\|\Psi\|}\Big[Ne^{\nu r}(T_s^+ e_\nu)(t)\|\phi(0)-\psi(0)\|\Big].$$

By the cone inequality theorem [9, Theorem 2.8] we obtain that $h \le g$, where g is a solution in E of the equation $g = Ag + z$ which can be rewritten as

$$\begin{aligned} g(t) = \frac{1}{1-\|\Psi\|}\Big[&Ne^{\nu r}(T_s^+ e_\nu)(t)\|\phi(0)-\psi(0)\| \\ &+ N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)g(\tau)\,d\tau\Big] \text{ for } t \ge s \ge 0. \end{aligned} \tag{2.15}$$

Next, to estimate g, we put $w(t) = e^{\mu(t-s)}g(t)$ for $t \ge s \ge 0$. Then, we obtain that

$$\begin{aligned} w(t) = \frac{1}{1-\|\Psi\|}\Big[&Ne^{\nu r}(T_s^+ e_{\nu-\mu})(t)\|\phi(0)-\psi(0)\| \\ &+N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)w(\tau)\,d\tau\Big] \text{ for } t \ge s. \end{aligned} \tag{2.16}$$

We will find w in $L_\infty[s,\infty)$ which is space of real-valued functions defined and essentially bounded on $[s,\infty)$ (endowed with the sup-norm denoted by $\|\cdot\|_\infty$). We next consider the linear operator D defined on $L_\infty[s,\infty)$ as

$$(D\phi)(t) = \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}\int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)\phi(\tau)d\tau \text{ for all } t \ge s.$$

By Proposition [10, Prop. 2.6], one can easily see that $D \in \mathcal{L}(L_\infty[s,\infty))$ and

$$\|D\| \le \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}.\left(\frac{N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-(\nu-\mu)}}\right).$$

Equation (2.16) can be rewritten as

$$w = Dw + \tilde{z} \text{ for } \tilde{z}(t) = \frac{1}{1-\|\Psi\|}Ne^{\nu r}(T_s^+ e_{\nu-\mu})(t)\|\phi(0)-\psi(0)\|.$$

We have $\|D\| < 1$ if $0 < \mu < \nu + \ln\Big(1 - \frac{N(1+H)e^{\nu r}}{1-\|\Psi\|}(N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty$ $+N_2\|\Lambda_1\varphi\|_\infty)\Big)$. Under this condition, the equation $w = Dw + \tilde{z}$ has the unique solution $w \in L_\infty[s,\infty)$ and $w = (I-D)^{-1}\tilde{z}$. Hence, we obtain that

$$\begin{aligned}\|w\|_\infty &= \|(I-D)^{-1}\tilde{z}\|_\infty \le \frac{Ne^{\nu r}}{(1-\|D\|)(1-\|\Psi\|)}\|\phi(0)-\psi(0)\| \\ &\le \frac{Ne^{\nu r}\|\phi(0)-\psi(0)\|}{1-\|\Psi\| - \frac{N(1+H)e^{\nu r}}{1-e^{-(\nu-\mu)}}.(N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)} \\ &:= C_\mu\|\phi(0)-\psi(0)\|.\end{aligned}$$

This yields that $w(t) \le C_\mu\|\phi(0)-\psi(0)\|$ for $t \ge s$. Therefore,

$$h(t) = \|u_t - v_t\|_{\mathcal{C}} \le g(t) = e^{-\mu(t-s)}w(t) \le C_\mu e^{-\mu(t-s)}\|\phi(0)-\psi(0)\| \quad \text{for } t \ge s.$$

□

We now prove our main result of this section.

Theorem 2.7 *Under the hypotheses and notations of Theorem 2.6. Put*

$$k_1 := N(1+H)e^{\nu r}\left(\frac{N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}\right).$$

Then, if $\max\{\frac{Nk_1e^{\nu r}}{1-k_1-\|\Psi\|}, \frac{k}{1-\|\Psi\|}\} < 1$ *where k is defined by (2.8), then there exists an invariant stable manifold S of $\mathcal{E}$-class for the solutions to Eq. (2.4).*

Moreover, every two solutions $u(t)$, $v(t)$ on the manifold S of Eq. (2.4) corresponding to different initial functions $\phi, \psi \in S_s$ attract each other exponentially in sense that, there exist positive constants μ and C_μ independent of $s \ge 0$ such that

$$\|u_t - v_t\|_{\mathcal{C}} \le C_\mu e^{-\mu(t-s)}\|\widetilde{P}(s)\phi - \widetilde{P}(s)\psi\|_{\mathcal{C}} \text{ for } t \ge s, \tag{2.17}$$

where $\widetilde{P}(t), t \ge 0$, are defined as in (2.2) and S_s is defined as in Definition (2.3).

Proof To prove the existence of an invariant stable manifold $S = \{(t, S_t)\}_{t\ge 0}$ of $\mathcal{E}$-class for the solution to Eq. (2.4) satisfies the conditions of Definition (2.3). We have, since $\{U(t,s)\}_{t\ge s\ge 0}$ has an exponential dichotomy, for each $t \ge 0$ the phase space $\mathcal{C}$ splits into the direct sum $\mathcal{C} = \operatorname{Im}\widetilde{P}(t) \oplus \operatorname{Ker}\widetilde{P}(t)$ where the projections $\widetilde{P}(t),\ t \ge 0$, are defined as in Equality (2.2). We determine the surface S_t for $t \ge 0$ by the formula

$$S_t := \{\phi + \widetilde{y}_t(\phi) : \phi \in \operatorname{Im}\widetilde{P}(t)\} \subset \mathcal{C},$$

where the operator $\widetilde{y}_{t_0}$ is defined for each $t_0 \ge 0$ by

$$\widetilde{y}_{t_0}(\phi)(\theta) = \int_{t_0}^{\infty} \mathcal{G}(t_0 - \theta, \tau)\Phi(\tau, u_\tau)d\tau \text{ for all } \theta \in [-r, 0],$$

here $u(\cdot)$ is the unique solution of Eq. (2.4) on $[-r + t_0, \infty)$ satisfying $\widetilde{P}(t_0)u_{t_0} = \phi$ (note that the existence and uniqueness of $u(\cdot)$ is guaranteed by Theorem 2.6). On the other hand, by the definition of the Green's function $\mathcal{G}$ (see Eq. (2.1)) we have that $\widetilde{y}_{t_0}(\phi) \in \operatorname{Ker}\widetilde{P}(t_0)$. We next show that the stable manifold S satisfies the conditions of Definition 2.3.

Firstly, we prove that $\widetilde{y}_{t_0}$ is Lipschitz continuity with Lipschitz constant independent of t_0. Indeed, for ϕ and ψ belonging to $\operatorname{Im}\widetilde{P}(t_0)$ we have

$$\begin{aligned}
&\|\widetilde{y}_{t_0}(\phi)(\theta) - \widetilde{y}_{t_0}(\psi)(\theta)\| \\
&\quad \le N(1+H)\int_{t_0}^{\infty} e^{-\nu|t_0-\theta-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}}\, d\tau \\
&\quad \le N(1+H)e^{\nu r}\sup_{\tau\ge t_0}\|u_\tau - v_\tau\|_{\mathcal{C}}\int_{t_0}^{\infty} e^{-\nu|t_0-\tau|}\varphi(\tau)\, d\tau \\
&\quad \le N(1+H)e^{\nu r}\sup_{\tau\ge t_0}\|u_\tau - v_\tau\|_{\mathcal{C}}\frac{N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}.
\end{aligned} \tag{2.18}$$

Moreover, by the Lyapunov–Perron's equation for $u(\cdot)$ and $v(\cdot)$, for $\tau \ge t_0$ we have

$$\begin{aligned}
\sup_{\tau\ge t_0}\|u_\tau - v_\tau\|_{\mathcal{C}} \le \frac{1}{1-\|\Psi\|}\Bigg[&Ne^{\nu r}\|\phi-\psi\|_{\mathcal{C}} + N(1+H)e^{\nu r} \\
&\times\left(\frac{N_1\|\Lambda_1 T_1^{+}\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}\right)\sup_{\tau\ge t_0}\|u_\tau - v_\tau\|_{\mathcal{C}}\Bigg].
\end{aligned}$$

This follows that

$$\sup_{\tau\ge t_0}\|u_\tau - v_\tau\|_{\mathcal{C}} \le \frac{Ne^{\nu r}\|\phi-\psi\|_{\mathcal{C}}}{1-k_1-\|\Psi\|}.$$

Substituting this inequality to (2.18), we obtain that

$$\|\widetilde{y}_{t_0}(\phi) - \widetilde{y}_{t_0}(\psi)\|_{\mathcal{C}} = \sup_{\theta\in[-r,0]}\|\widetilde{y}_{t_0}(\phi)(\theta) - \widetilde{y}_{t_0}(\psi)(\theta)\| \le \frac{Ne^{\nu r}k_1}{1-k_1-\|\Psi\|}\|\phi-\psi\|_{\mathcal{C}}.$$

Therefore, $\widetilde{y}_{t_0}$ is Lipschitz continuous with the Lipschitz constant $\frac{Ne^{\nu r}k_1}{1-k_1-\|\Psi\|}$ independent of t_0.

To show that S_{t_0} is homeomorphic to $\operatorname{Im}\widetilde{P}(t_0)$. We define the transformation $\mathbf{D} : \operatorname{Im}\widetilde{P}(t_0) \to S_{t_0}$ by $\mathbf{D}\phi := \phi + \widetilde{y}_{t_0}(\phi)$ for all $\phi \in \operatorname{Im}\widetilde{P}(t_0)$. Then, applying the Implicit Function Theorem for Lipschitz continuous mappings (see [15, Lemma 2.7]) we have that if the Lipschitz constant $\frac{Ne^{\nu r}k_1}{1-k_1-\|\Psi\|} < 1$ (equivalently $k_1 < (1-\|\Psi\|)/(1+Ne^{\nu r})$), then $\mathbf{D}$ is a homeomorphism. Therefore, the condition (ii) in Definition 2.3 is satisfied.

The condition (iii) in Definition 2.3 now follows from Theorem 2.6. We now prove that the condition (iv) of Definition 2.3 is satisfied. Indeed, let $u(\cdot)$ be solution of Eq. (2.4) such that the function $u_s \in S_s$. Then, by Lemma 2.4, for $t \in [s, \infty)$ the solution $u(t)$ satisfies

$$Fu_t = U(t,s)\nu_0 + \int_s^\infty \mathcal{G}(t,\tau)\Phi(\tau,u_\tau)d\tau \text{ for some } \nu_0 \in \operatorname{Im}P(s).$$

Thus, for $t \geq s$ and $\theta \in [-r, 0]$ we have

$$\begin{aligned}
&Fu_{t-\theta}\\
&= U(t-\theta,s)\nu_0 + \int_s^\infty \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau\\
&= U(t-\theta,s)\nu_0 + \int_s^t \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau + \int_t^\infty \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau\\
&= U(t-\theta,s)\nu_0 + \int_s^t U(t-\theta,\tau)P(\tau)\Phi(\tau,u_\tau)d\tau + \int_t^\infty \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau\\
&= U(t-\theta,t)[U(t,s)\nu_0 + \int_s^t U(t,\tau)P(\tau)\Phi(\tau,u_\tau)d\tau]+\\
&+ \int_t^\infty \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau \text{ for all } -r \leq \theta \leq 0.
\end{aligned}$$

Put $z_0 = U(t,s)\nu_0 + \int_s^t U(t,\tau)P(\tau)\Phi(\tau,u_\tau)d\tau$. We have $P(t)z_0 = z_0$, hence $z_0 \in \operatorname{Im}P(t)$. We thus obtain that the function $\phi(\theta) := U(t-\theta,t)z_0$, $-r \leq \theta \leq 0$ belongs to $\operatorname{Im}\widetilde{P}(t)$ and

$$Fu_{t-\theta} = U(t-\theta,t)z_0 + \int_t^\infty \mathcal{G}(t-\theta,\tau)\Phi(\tau,u_\tau)d\tau \text{ for all } -r \leq \theta \leq 0.$$

By the uniqueness of $u(\cdot)$ on $[s-r,\infty)$ as in the proof of Theorem 2.6 we have that Eq. (2.4) has a unique solution $u(\cdot)$ on $[-r+t,\infty)$ satisfying $\widetilde{P}(t)u_t = \phi$ and

$$Fu_\xi = U(2t-\xi,t)z_0 + \int_t^\infty \mathcal{G}(2\xi-t,\tau)\Phi(\tau,u_\tau)d\tau$$

for $\xi \in [-r+t,t]$. Therefore, for $t \geq s$ the function $\tilde{u}_t$ defined as in (2.5) satisfies

$$\tilde{u}_t(\theta) = Fu_{t-\theta} = U(t-\theta, t)z_0 + \int_t^{\infty} \mathcal{G}(t-\theta, \tau)\Phi(\tau, u_\tau)d\tau = \phi(\theta) + \widetilde{y}_t(\phi)(\theta)$$

$$\text{for all } -r \le \theta \le 0$$

where, as seen above, $\phi \in \operatorname{Im}\widetilde{P}(t)$.

Hence, $\tilde{u}_t \in S_t$ for $t \geq s$.

Finally, the inequality (2.17) follows from inequality (2.9) in Theorem (2.6) □

We illustrate our result by the following example.

Example 2.8 Consider the following neutral partial functional differential equation:

$$\frac{\partial w(x,t)}{\partial t} - l\frac{\partial w(x,t-1)}{\partial t} = a(t)\left[\frac{\partial^2 w(x,t)}{\partial x^2} - l\frac{\partial^2 w(x,t-1)}{\partial x^2} + \alpha w(x,t)\right] \tag{2.19}$$

$$+ b\, e^{-\alpha t}\int_{-1}^{0} \ln(1+|w(x,t+\theta)|)d\theta$$

$$\text{for} \quad 0 \le x \le \pi, t \ge 0$$

$$w(0,t) = w(\pi,t) = 0 \qquad t \ge 0$$

$$w_s(x,\theta) = w(x, s+\theta) \text{ for all } x \in [0,\pi], \theta \in [-1,0],$$

where l and α are real constants with $|l| < 1$, $\alpha > 1$ and $\alpha \neq n^2, \forall n \in \mathbb{N}$. The function $a(\cdot) \in L_{1,loc}(\mathbb{R}_+)$ and satisfies the condition $\gamma_1 \geq a(t) \geq \gamma_0 > 0$ for fixed constants γ_0, γ_1 and a.e. $t \geq 0$. We choose the Hilbert space $X := L_2[0,\pi]$, $\mathcal{C} := C([-1,0], X)$ and let $B : X \to X$ be defined by

$$B(f) = f'' + \alpha f$$

with the domain $D(B) = H_0^2[0,\pi] := \{f \in W^{2,2}[0,\pi] : f(0) = f(\pi) = 0)\}$

Also define the difference and delay operators F and Φ as

$$F : \mathcal{C} \to X, \quad F(f) := f(0) - kf(-1)$$

$$\Phi : \mathbb{R}_+ \times \mathcal{C} \to X, \quad \Phi(t,\phi) := b\, e^{-\alpha t}\int_{-1}^{0} \ln(1+|(\phi(\theta))(x)|)d\theta, x \in [0,\pi]. \tag{2.20}$$

Note that the fact that Φ takes value in $X = L_2[0,\pi]$ can be easily seen by using the Minkowski's inequality.

Putting now $B(t) := a(t)B$ Eq. (3.4) can now be rewritten as

$$\begin{cases} \frac{\partial}{\partial t} Fu_t(\cdot) = B(t)Fu_t(\cdot) + \Phi(t, u_t(\cdot,\theta)), & t \geq s \geq 0, \\ u_s(\cdot,\theta) \quad = \phi(\cdot,\theta) \in \mathcal{C} \end{cases}$$

where B is the generator of an analytic semigroup $(T(t))_{t>0}$, with $\sigma(B) = \{-1 + \alpha, -4 + \alpha, \ldots\}$. Since $\alpha \neq n^2 \forall n \in \mathbb{N}$ we have that $\sigma(B) \cap i\mathbb{R} = \emptyset$. Applying the spectral mapping theorem for analytic semigroups we get

$$\sigma(T(t)) = e^{t\sigma(B)} = \left\{ e^{t(\alpha-1)}, e^{t(\alpha-4)}, \ldots \right\}$$

and $\sigma(T(t)) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$ for all $t > 0$. Therefore, for fixed t_0, the spectrum of the operator $T(t_0)$ splits into two disjoint sets σ_0, σ_1, where $\sigma_0 \subset \{z \in \mathbb{C} : |z| < 1\}$, $\sigma_1 \subset \{z \in \mathbb{C} : |z| > 1\}$. Next, we choose $P = P(t_0)$ to be the Riesz projections corresponding to the spectral set σ_0, and $Q = I - P$. Clearly, P and Q commute with $T(t)$ for all $t > 0$. We denote by $T_Q(t) := T(t)Q$ the restriction of $T(t)$ on ImQ. As well known in Semigroup Theory, in this case, the semigroup $(T(t))_{t>0}$ is called hyperbolic (or having an exponential dichotomy) and the restriction $T_Q(t)$ is invertible. Moreover, there are positive constants N, γ such that

$$\begin{aligned} \|T(t)|_{PX}\| &\leq Ne^{-\gamma t}, \quad \forall t \geq 0 \\ \|T_Q(-t)\| = \|T_Q(t)^{-1}\| &\leq Ne^{-\gamma t}, \quad \forall t \geq 0. \end{aligned} \tag{2.21}$$

Clearly, the family $(B(t))_{t\geq 0} = (a(t)B)_{t\geq 0}$ generates the evolution family $(U(t,s))_{t\geq s\geq 0}$ defined by the formula:

$$U(t,s) = T\bigg(\int_s^t a(\tau)d\tau \bigg).$$

From the dichotomy estimates of $(T(t))_{t>0}$ in (2.21), it is straightforward to check that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with projections P and constants $N, \nu := \gamma\gamma_0$ by the following estimates:

$$\|U(t,s)|_{PX}\| = \left\| T\left(\int_s^t a(\tau)d\tau \right) |_{PX} \right\| \leq Ne^{-\nu(t-s)}$$

$$\|U(s,t)_{|}\| = \|\,(U(t,s)|_{\ker P})^{-1}\,\| = \left\| T_Q\left(-\int_s^t a(\tau)d\tau \right) \right\| \leq Ne^{-\nu(t-s)}$$

for all $t \geq s \geq 0$.

Clearly, the difference operator F be of form $F = \delta_0 - \Psi$ for $\Psi = l\delta_{-1}$ and $\|\Psi\| \leq |l| < 1$. We now take $E = L_p(\mathbb{R}_+)1 \leq p \leq +\infty$, the delay operator $\Phi : \mathbb{R}_+ \times \mathcal{C} \to X$ defined as in (2.20) and check that Φ is φ-Lipschitz with $\varphi(t) =$

$|b|e^{-\alpha t} \in E' = L_q(\mathbb{R}_+)$ for $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, the condition (i) in Definition 2.2 is evident. To verify the condition (ii) in that definition we use Minkowski's inequality and the fact that $\ln(1+h) \le h$ for all $h \ge 0$. Then,

$$\begin{aligned}
&\|\Phi(t,\phi_1)(x) - \Phi(t,\phi_2)(x)\|_2 \\
&= |b|\, e^{-\alpha t} \left(\int_0^\pi \left(\int_{-1}^0 \ln \frac{1 + |(\phi_1(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} d\theta \right)^2 dx \right)^{\frac{1}{2}} \\
&\le |b|\, e^{-\alpha t} \int_{-1}^0 \left(\int_0^\pi \ln^2 \frac{1 + |(\phi_1(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} dx \right)^{\frac{1}{2}} d\theta \\
&= |b|\, e^{-\alpha t} \int_{-1}^0 \left(\int_0^\pi \ln^2 \left(1 + \frac{|(\phi_1(\theta))(x)| - |(\phi_2(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} \right) dx \right)^{\frac{1}{2}} d\theta \\
&\le |b|\, e^{-\alpha t} \int_{-1}^0 \left(\int_0^\pi |(\phi_1(\theta))(x) - (\phi_2(\theta))(x)|^2 dx \right)^{\frac{1}{2}} d\theta \\
&= |b|\, e^{-\alpha t} \int_{-1}^0 \|\phi_1(\theta) - \phi_2(\theta)\|_2 \, d\theta \\
&\le |b|\, e^{-\alpha t} \sup_{\theta \in [-1,0]} \|\phi_1(\theta) - \phi_2(\theta)\|_2.
\end{aligned}$$

Hence, Φ is φ-Lipschitz. In the space $L_p(\mathbb{R}_+)$, the constants N_1, N_2 are defined by $N_1 = N_2 = 1$. We have

$$\Lambda_1 \varphi(t) = \int_t^{t+1} \varphi(\tau) d\tau \quad \text{and} \quad \Lambda_1 T_1^+ \varphi(t) = \int_{(t-1)_+}^t \varphi(\tau) d\tau$$

where $(t-1)_+ = \max\{0, t-1\}$. Hence

$$\frac{N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty}{1 - e^{-\nu}} \le \frac{|b|(e^\alpha - e^{-\alpha})}{\alpha(1 - e^{-\nu})}.$$

Also, the function $h_\nu(\cdot)$ in Theorem 2.6 can be computed by

$$h_\nu(t) = \|e^{-\nu|t-\cdot|} \varphi(\cdot)\|_{L_q} = |b| \left(\frac{e^{-\nu q t} - e^{-\alpha q t}}{(\alpha - \nu) q} + \frac{e^{-\alpha q t}}{(\alpha + \nu) q} \right)^{\frac{1}{q}} \qquad \text{for } t \ge 0.$$

Suppose that $\alpha > \nu$. Then, we can estimate

$$0 \le h_\nu(t) \le \frac{|b| e^{-\nu t}}{((\alpha - \nu) q)^{\frac{1}{q}}} \qquad \text{for } t \ge 0.$$

Therefore, $h_\nu \in L_p$ and

$$\|h_\nu\|_{L_p} \le \frac{|b|}{(\nu p)^{\frac{1}{p}}[(\alpha-\nu)q]^{\frac{1}{q}}}.$$

By Theorem 2.7, we obtain that if

$$\max\left\{\frac{N^2e^{2\nu}(1+H)|b|(e^{\alpha}-e^{-\alpha})}{(1-|l|)\alpha(1-e^{-\nu})-N(1+H)e^{\nu}|b|(e^{\alpha}-e^{-\alpha})},\right.$$
$$\left.\frac{|b|N(1+H)e^{\nu}}{(1-|l|)(\nu p)^{\frac{1}{p}}[(\alpha-\nu)q]^{\frac{1}{q}}}\right\}<1$$

then there is an invariant stable manifold S of L_p-class for the mild solutions to the problem (2.19).

3 Center-Stable Manifolds of $\mathcal{E}$-Class

In this section, we will generalize Theorem 2.7 to the case that the evolution family $\{U(t,s)\}_{t\ge s\ge 0}$ has an exponential trichotomy on $\mathbb{R}_+$ and the nonlinear forcing term Φ is φ-Lipschitz. In this case, under similar conditions as in above Section we will prove that there exists a center-invariant stable manifold of $\mathcal{E}$-class for the solutions to Eq. (2.4). We now recall the definition of an exponential trichotomy and a center-invariant stable manifold of $\mathcal{E}$-class.

Definition 3.1 A given evolution family $\{U(t,s)\}_{t\ge s\ge 0}$ is said to have an exponential trichotomy on the half-line if there are three families of projections $\{P_j(t)\}$, $t \ge 0$, $j = 1, 2, 3$, and positive constants N, α, β with $\alpha < \beta$ such that the following conditions are fulfilled:

(i) $\sup_{t\ge 0}\|P_j(t)\| < \infty$, $j = 1, 2, 3$.
(ii) $P_1(t) + P_2(t) + P_3(t) = Id$ for $t \ge 0$ and $P_j(t)P_i(t) = 0$ for all $j \ne i$.
(iii) $P_j(t)U(t,s) = U(t,s)P_j(s)$ for $t \ge s \ge 0$ and $j = 1, 2, 3$.
(iv) $U(t,s)|_{\mathrm{Im}P_j(s)}$ are homeomorphisms from $\mathrm{Im}P_j(s)$ onto $\mathrm{Im}P_j(t)$, for all $t \ge s \ge 0$ and $j = 2, 3$, respectively; we also denote the inverse of $U(t,s)|_{\mathrm{Im}P_j(s)}$ by $U(s,t)_|$, $0 \le s \le t$.
(v) For all $t \ge s \ge 0$ and $x \in X$, the following estimates hold:

$$\begin{aligned}
\|U(t,s)P_1(s)x\| &\le Ne^{-\beta(t-s)}\|P_1(s)x\|\\
\|U(s,t)_|P_2(t)x\| &\le Ne^{-\beta(t-s)}\|P_2(t)x\|\\
\|U(t,s)P_3(s)x\| &\le Ne^{\alpha(t-s)}\|P_3(s)x\|.
\end{aligned}$$

The projections $\{P_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, are called the *trichotomy projections*, and the constants N, α, β *the trichotomy constants*.

Using the trichotomy projections, we can now construct three families of projections $\{\widetilde{P}_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, on $\mathcal{C}$ as follows:

$$(\widetilde{P}_j(t)\phi)(\theta) = U(t-\theta, t)P_j(t)\phi(0) \text{ for all } \theta \in [-r, 0] \text{ and } \phi \in \mathcal{C}. \tag{3.1}$$

Definition 3.2 Let the evolution family $\{U(t, s)\}_{t\geq s\geq 0}$ have an exponential trichotomy with the trichotomy projections $\{P_j(t)\}t \geq 0$, $j = 1, 2, 3$, and constants N, α, β given as in Definition 3.1.

A set $S \subset \mathbb{R}_+ \times \mathcal{C}$ is said to be a center-invariant stable manifold of $\mathcal{E}$-class for the solutions to Eq. (2.4) if there exists a family of Lipschitz continuous mappings

$$\Phi_t : \mathrm{Im}(\widetilde{P}_1(t) + \widetilde{P}_3(t)) \to \mathrm{Im}\widetilde{P}_2(t)$$

with projections $\{\widetilde{P}_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$ defined as in Eq. (3.1), and Lipschitz constants being independent of t such that $S_t = graph(\Phi_t)$ has the following properties:

(i) S_t is homeomorphic to $\mathrm{Im}(\widetilde{P}_1(t) + \widetilde{P}_3(t))$ for all $t \geq 0$.

(ii) To each $\phi \in S_s$, there corresponds one and only one solution $u(t)$ to Eq. (2.4) on $[s-r, \infty)$ satisfying $e^{-\gamma(s+\theta)}Fu_{s-\theta} = \phi(\theta)$ for $\theta \in [-r, 0]$ and

$$z_t = \begin{cases} e^{-\gamma(t+\cdot)}u_t(\cdot) & \text{for } t \geq s \geq 0 \\ 0 & \text{for } t < s \end{cases} \quad \text{belong to } \mathcal{E}\cap\mathcal{E}_\infty, \text{ where } \gamma = \frac{\beta+\alpha}{2}.$$

Moreover, for any two solutions $u(t)$ and $v(t)$ to Eq. (2.4) corresponding to different functions $\phi, \psi \in S_s$, we have the estimate

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{(\gamma-\mu)(t-s)}\|(\widetilde{P}(s)\phi)(0) - (\widetilde{P}(s)\psi)(0)\| \text{ for } t \geq s, \tag{3.2}$$

where μ, C_μ are positive constants independent of s, $u(\cdot)$, and $v(\cdot)$ and $\widetilde{P}(t) = \widetilde{P}_1(t) + \widetilde{P}_3(t)$.

(iii) S is positively F-invariant under Eq. (2.4) in the sense that, if $u(t)$, $t \geq s-r$, is the solution to Eq. (2.4) satisfying the conditions that the function $e^{-\gamma(s+\cdot)}\tilde{u}_s(\cdot) \in S_s$ and $z_t = \begin{cases} e^{-\gamma(t+\cdot)}u_t(\cdot) & \text{for } t \geq s \geq 0 \\ 0 & \text{for } t < s \end{cases}$ belong to $\mathcal{E} \cap \mathcal{E}_\infty$, then the function $e^{-\gamma(t+\cdot)}\tilde{u}_t(\cdot) \in S_t$ for all $t \geq s$ where $\tilde{u}_t$ is defined as in (2.5).

We come to our second main result. It proves the existence of a center-stable manifold for solutions of Eq. (2.4).

Theorem 3.3 *Let the evolution family* $\{U(t,s)\}_{t\geq s\geq 0}$ *have an exponential trichotomy with the trichotomy projections* $\{P_j(t)\}t \geq 0$, $j = 1, 2, 3$, *and constants* N, α, β *given as in Definition 3.1. Assume Standing Hypothesis 1.3 and let the functions* φ, h_ν, e_ν, *and the operators* F, Φ *be determined as in Theorem 2.7. Set* $q := \sup\{\|P_j(t)\| : t \geq 0,\ j = 1,\ 3\}$, $N_0 := \max\{N,\ 2Nq\}$, $\nu = \dfrac{\beta-\alpha}{2}$ *and*

$$\widetilde{k} := (1+H)e^{\nu r}N_0\left(\frac{N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}\right). \tag{3.3}$$

Then, if $\widetilde{k} < (1-\|\Psi\|)/(1+N_0e^{\nu r})$, *for each fixed* $\beta > \alpha$ *there exists a center-invariant stable manifold of* $\mathcal{E}$*-class for the solutions to Eq. (2.4).*

Proof Set $P(t) := P_1(t) + P_3(t)$ and $Q(t) := P_2(t) = Id - P(t)$ for $t \geq 0$. We have that $P(t)$ and $Q(t)$ are projections complemented to each other on X. We then define the families of projections $\{\widetilde{P}_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, on $\mathcal{C}$ as in Equality (3.1). Setting $\widetilde{P}(t) = \widetilde{P}_1(t) + \widetilde{P}_3(t)$ and $\widetilde{Q}(t) = \widetilde{P}_2(t)$, $t \geq 0$, we obtain that $\widetilde{P}(t)$ and $\widetilde{Q}(t)$ are complemented projections on $\mathcal{C}$ for each $t \geq 0$. We consider the following rescaling evolution family:

$$\widetilde{U}(t,s) = e^{-\gamma(t-s)}U(t,s) \quad \text{for all } t \geq s \geq 0, \text{ where } \gamma := \frac{\beta+\alpha}{2}.$$

We now prove that the evolution family $\widetilde{U}(t,s)$ has an exponential dichotomy with dichotomy projections $P(t)$, $t \geq 0$. Indeed,

$$\begin{aligned} P(t)\widetilde{U}(t,s) &= e^{-\gamma(t-s)}(P_1(t) + P_3(t))U(t,s) \\ &= e^{-\gamma(t-s)}U(t,s)(P_1(s) + P_3(s)) = \widetilde{U}(t,s)P(s). \end{aligned}$$

Since $U(t,s)|_{\mathrm{Im}P_2(s)}$ is a homeomorphism from $\mathrm{Im}P_2(s)$ onto $\mathrm{Im}P_2(t)$ and $\mathrm{Im}P_2(t)$=$\mathrm{Ker}P(t)$ for all $t \geq 0$, we have that $\widetilde{U}(t,s)|_{\mathrm{Ker}P(s)}$ is also a homeomorphism from $\mathrm{Ker}P(s)$ onto $\mathrm{Ker}P(t)$, and we denote $\widetilde{U}(s,t)_| := (\widetilde{U}(t,s)|_{\mathrm{Ker}P(s)})^{-1}$ for $0 \leq s \leq t$. By the definition of exponential trichotomy, we have

$$\|\widetilde{U}(s,t)_| Q(t)x\| \leq e^{-(\beta+\gamma)(t-s)}\|Q(t)x\| \quad \text{for all } t \geq s \geq 0.$$

On the other hand,

$$\begin{aligned} \|\widetilde{U}(t,s)P(s)x\| &= e^{-\gamma(t-s)}\|U(t,s)(P_1(s)+P_3(s))x\| \\ &\leq Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)x\| + e^{\alpha(t-s)}\|P_3(s)x\|) \\ &= Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)P(s)x\| + e^{\alpha(t-s)}\|P_3(s)P(s)x\|) \end{aligned}$$

for all $t \geq s \geq 0$ and $x \in X$.

Putting $q := \sup\{\|P_j(t)\|, \quad t \geq 0, \ j = 1, 3\}$, we finally get the following estimate:

$$\|\widetilde{U}(t,s)P(s)x\| \leq 2Nqe^{-\frac{(\beta-\alpha)}{2}(t-s)}\|P(s)x\|.$$

Therefore, $\widetilde{U}(t,s)$ has an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N_0 := \max\{N, \ 2Nq\}$, $\nu := \frac{\beta-\alpha}{2}$.

Put $\widehat{u}(t) = e^{-\gamma t}u(t)$, and define the mapping $\widetilde{\Phi} : \mathbb{R}_+ \times \mathcal{C} \to X$ as follows:

$$\widetilde{\Phi}(t,\phi) = e^{-\gamma t}\Phi(t, e^{\gamma(t+\cdot)}\phi(\cdot)) \text{ for } (t,\phi) \in \mathbb{R}_+ \times \mathcal{C}.$$

Obviously, $\widetilde{\Phi}$ is also φ-Lipschitz. Thus, we can rewrite Eq. (2.4) in the new form

$$\begin{cases} F\widehat{u}_t &= \widetilde{U}(t,s)F\widehat{u}_s + \int_s^t \widetilde{U}(t,\xi)\widetilde{\Phi}(\xi,\widehat{u}_\xi)d\xi \text{ for all } t \geq s \geq 0, \\ \widehat{u}_s(\cdot) &= e^{-\gamma(s+\cdot)}\phi(\cdot) \in \mathcal{C}. \end{cases} \tag{3.4}$$

Hence, by Theorem 2.7, we obtain that, if $\widetilde{k} < (1 - \|\Psi\|)/(1 + N_0 e^{\nu r})$, then there exists an invariant stable manifold of $\mathcal{E}$-class S for the solutions to Eq. (3.4). Returning to Eq. (2.4) by using the relation $u(t) := e^{\gamma t}\widehat{u}(t)$ and Theorems 2.6, 2.7, we can easily verify the properties of S which are stated in (i), (ii), and (iii) in Definition (3.2). Thus, S is a center-invariant stable manifold of $\mathcal{E}$-class for the solutions of Eq. (2.4). □

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Global Attractor in Alpha-Norm for Some Partial Functional Differential Equations of Neutral and Retarded Type

Mostafa Adimy, Khalil Ezzinbi, and Catherine Marquet

1 Introduction

The existence of a global attractor is an interesting field of research in differential equations and dynamical systems. The attractor plays a crucial role in the asymptotic behavior of the solutions. In literature, one can find many approaches dealing with this problem. Here we use the approach based on the asymptotically smoothness of nonlinear semigroups to prove the existence of a global attractor for fully nonlinear partial neutral functional differential equations in the alpha-norm.

Let X be a Banach space, $\mathcal{L}(X)$ be the space of bounded linear operators on X, and α be a constant such that $0 < \alpha < 1$. The aim of this work is to show the existence of a global attractor in the alpha-norm for the following class of partial neutral functional differential equations (PNFDE):

$$\begin{cases} \dfrac{d}{dt} D\left(x_t\right) = -AD\left(x_t\right) + F(x_t) \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C_\alpha, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is a linear operator, $C_\alpha := C\left([-r, 0]\,; D(A^\alpha)\right)$, $r > 0$, denotes the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$, and

M. Adimy
Institut Camille Jordan, Université Claude Bernard Lyon 1, Villeurbanne Cedex, France

K. Ezzinbi (✉)
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, Marrakesh, Morroco
e-mail: ezzinbi@uca.ma

C. Marquet
Université de Pau et des Pays de l'Adour, Laboratoire de Mathématiques Appliquées CNRS UMR 5142, Pau, France

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the operator A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be described in Sect. 2. For $x \in C([-r, b]; D(A^\alpha))$, $b > 0$, and $t \in [0, b]$, x_t denotes, as usual, the element of C_α defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. F is a continuous function from C_α with values in X, and D is a bounded linear operator from $C := C([-r, 0], X)$ into X defined by $D(\varphi) = \varphi(0) - D_0(\varphi)$, for $\varphi \in C$, where the operator D_0 is given by

$$D_0(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta) \text{ for } \varphi \in C,$$

and $\eta : [-r, 0] \to \mathcal{L}(X)$ is of bounded variation and non-atomic at zero; that is

$$\underset{[-\varepsilon,0]}{var}(\eta) \to 0 \text{ as } \varepsilon \to 0.$$

It is well known, that if the phase space C_α is the space of all continuous functions from $[-r, 0]$ into X (i.e. $\alpha = 0$), then Eq. (1.1) has been studied by several authors. For more details, we refer to the book of Wu [13]. For example, Wu and Xia considered in [14] a system of partial neutral functional differential-difference equations defined on the unit circle S^1, which is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. They obtained equations of the form

$$\frac{\partial}{\partial t}[x(.,t) - qx(.,t-r)] = K\frac{\partial^2}{\partial \xi^2}[x(.,t) - qx(.,t-r)] + f(x_t), \quad t \geq 0, \tag{1.2}$$

where $\xi \in S^1$, K a positive constant, and $0 \leq q < 1$. The space of initial data was chosen to be $C\left([-r, 0]; H^1(S^1)\right)$. Motivated by this work, Hale presented, in [4, 5], the basic theory of existence and uniqueness, and properties of the solution operator, as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits for a more general class of PNFDE on the unit circle S^1. For the sake of comparison, let us briefly restate the equations considered by Hale in [4, 5]. If $\varphi \in C\left([-r, 0]; H^1(S^1)\right)$, we write it as $\varphi(\xi, \theta)$ for $\xi \in S^1$ and $\theta \in [-r, 0]$. For any function $\widetilde{f} \in C^{k+1}(C([-r, 0]; \mathbb{R}); \mathbb{R})$, $k \geq 1$, we let $f \in C^{k+1}\left(C\left([-r, 0]; H^1(S^1)\right); L^2(S^1)\right)$ be defined by $f(\varphi)(\xi) = \widetilde{f}(\varphi(\xi, .))$, $\xi \in S^1$. Let $\widetilde{D} \in \mathcal{L}(C([-r, 0]; \mathbb{R}); \mathbb{R})$ be defined by

$$\begin{cases} \widetilde{D}\psi = \psi(0) - \widetilde{g}(\psi), \\ \widetilde{g}(\psi) = \int_{-r}^{0} d\eta(\theta)\psi(\theta), \end{cases}$$

where η is of bounded variation and non-atomic at 0.

We define $D \in \mathcal{L}\left(C\left([-r,0]\,;\,H^1(S^1)\right)\;;\;H^1(S^1)\right)$ as

$$D(\varphi)(\xi) = \widetilde{D}\left(\varphi(\xi,.)\right), \quad \text{for } \xi \in S^1. \tag{1.3}$$

Hale considered, in [4, 5], PNFDE of the form

$$\frac{\partial}{\partial t} D x_t = K \frac{\partial^2}{\partial \xi^2} D x_t + f\left(x_t\right), \;\; t \geq 0, \tag{1.4}$$

with $C\left([-r,0]\,;\,H^1(S^1)\right)$ as the space of initial data. He considered the Laplace operator $A_0 = K\dfrac{\partial^2}{\partial \xi^2}$ with domain $H^2(S^1)$, which yields an operator generating an analytic semigroup.

In [3] and [11], Eq. (1.1) has been studied with respect to the α-norm, but in the particular case when $D_0 \equiv 0$. Neutral partial functional differential equations have been extensively studied in literature, for more details for the readers, we refer to [1, 3–6, 8, 9, 14].

Let $(U(t))_{t\geq 0}$ denote the semigroup solution of the partial neutral functional differential equation (1.1) and $\mathcal{A}$ be the global attractor of equation then the restriction $(U_A(t))_{t\geq 0}$ of $(U(t))_{t\geq 0}$ over $\mathcal{A}$ has interesting properties that are not satisfied by $(U(t))_{t\geq 0}$. For example if $U(t) : \mathcal{A} \to \mathcal{A}$ is one to one, then $(U_A(t))_{t\geq 0}$ is a group on $\mathcal{A}$.

This paper is organized as follows: in the first part of Sect. 2, we recall some preliminary results about analytic semigroups, fractional power associated with its generator, and the smoothness of the semigroup solution. After that, we start to prove our main results. In Sect. 3, we prove the existence of a global attractor for equation is the alpha-norm. Finally, in Sect. 4, we propose an application.

2 Well-Posedness of Eq. (1.1) in the Alpha-Norm

We firstly recall some well-known results on the existence and uniqueness of the mild solution for equation. Before that we give some essential assumptions. In the next, we assume that

($\mathbf{H_1}$) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X and satisfies $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A.

We know that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $|T(t)| \leq M e^{\omega t}$, for $t \geq 0$.

If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. Then, without loss of generality, we can assume that $0 \in \rho(A)$. This remark is valuable here only for proving existence, uniqueness, and regularity of solutions.

We consider, see Pazy [10], the fractional power $(A^{\alpha}, D(A^{\alpha}))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$. We recall the following known results.

Proposition 2.1 ([10], pp. 69–75) *Let $0 < \alpha < 1$ and assume that $(\mathbf{H_1})$ holds. Then,*

(i) $D(A^{\alpha})$ is a Banach space for the norm $|x|_{\alpha} := |A^{\alpha}x|$ for $x \in D(A^{\alpha})$,
(ii) $T(t) : X \to D(A^{\alpha})$ for every $t > 0$,
(iii) $A^{\alpha}T(t)x = T(t)A^{\alpha}x$ for every $x \in D(A^{\alpha})$ and $t \geq 0$,
(iv) For every $t > 0$, $A^{\alpha}T(t)$ is bounded on X and there exists $M_{\alpha} > 0$ such that

$$\left|A^{\alpha}T(t)\right| \leq M_{\alpha}\frac{e^{\omega t}}{t^{\alpha}} \quad \text{for every } t > 0, \tag{2.1}$$

(v) $A^{-\alpha}$ is a bounded linear operator on X with $D(A^{\alpha}) = \mathrm{Im}(A^{-\alpha})$,
(vi) There exists $N_{\alpha} > 0$ such that

$$\left|(T(t) - I)A^{-\alpha}\right| \leq N_{\alpha}t^{\alpha} \quad \text{for } t > 0 \text{ small enough.}$$

We denote by X_{α} the Banach space $(D(A^{\alpha}), |\cdot|_{\alpha})$ and by $C_{\alpha} := C([-r, 0]; X_{\alpha})$ the space of continuous functions from $[-r, 0]$ into X_{α} endowed with the norm

$$|\varphi|_{\alpha} := \sup_{\theta\in[-r,0]} |\varphi(\theta)|_{\alpha}, \quad \varphi \in C_{\alpha}.$$

Remark that $(C_{\alpha}, |\cdot|_{\alpha})$ is also a Banach space. For the existence and uniqueness of the mild solution, we need to assume that

$(\mathbf{H}_2)$ $|F(\varphi_1) - F(\varphi_2)| \leq k\,|\varphi_1 - \varphi_2|_{\alpha}$, for $\varphi_1, \varphi_2 \in C_{\alpha}$, where k is a positive constant,
$(\mathbf{H}_3)$ If $x \in X_{\alpha}$ and $\theta \in [-r, 0]$, then $\eta(\theta)x \in X_{\alpha}$ and $A^{\alpha}\eta(\theta)x = \eta(\theta)A^{\alpha}x$.

The assumption $(\mathbf{H}_3)$ implies that if $\varphi \in C_{\alpha}$, then $D_0(\varphi) \in X_{\alpha}$ and $A^{\alpha}D_0(\varphi) = D_0(A^{\alpha}\varphi)$, where the expression $A^{\alpha}\varphi$ is defined, for $\varphi \in C_{\alpha}$ and $\theta \in [-r, 0]$, by

$$(A^{\alpha}\varphi)(\theta) := A^{\alpha}(\varphi(\theta)).$$

Definition 2.2 Let $\varphi \in C_{\alpha}$. A continuous function $x : [-r, +\infty) \to X_{\alpha}$ is called a mild solution of Eq. (1.1) if

(i) $D(x_t) = T(t)D(\varphi) + \int_0^t T(t-s)F(x_s)ds$ for $t \geq 0$,
(ii) $x_0 = \varphi$.

Now, we state our first result.

Theorem 2.3 *Assume that $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, and $(\mathbf{H}_3)$ hold. Then, for $\varphi \in C_{\alpha}$, Eq. (1.1) has a unique mild solution which is defined for all $t \geq 0$.*

If instead of assuming that D_0 is given by a function of bounded variation and Condition $(\mathbf{H}_3)$, we make the assumption that

$(\mathbf{H}_3')$ $\quad D_0 \in \mathcal{L}(C_\alpha, X_\alpha)$ and $|D_0|_{\mathcal{L}(C_\alpha, X_\alpha)} < 1$,

then, we obtain the same result as in Theorem 2.3.

Proposition 2.4 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, *and* $(\mathbf{H}_3')$ *hold. Then, for* $\varphi \in C_\alpha$, *Eq. (1.1) has a unique mild solution which is defined for all* $t \geq 0$.

Define the operator $U(t)$, for $t \geq 0$, on C_α by

$$U(t)(\varphi) = x_t(., \varphi),$$

where $x(., \varphi)$ is the mild solution of Eq. (1.1) for the initial condition $\varphi \in C_\alpha$. Then, we have:

Proposition 2.5 ([1]) *The family* $(U(t))_{t\geq 0}$ *is a nonlinear strongly continuous semigroup on* C_α*; that is*

(i) $U(0) = I$,
(ii) $U(t+s) = U(t)U(s)$ *for* $t, s \geq 0$,
(iii) For all $\varphi \in C_\alpha$, $U(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* C_α,
(iv) For all $t \geq 0$, $U(t)$ *is continuous from* C_α *into* C_α,
(v) $(U(t))_{t\geq 0}$ *satisfies the following translation property, for* $t \geq 0$ *and* $\theta \in [-r, 0]$,

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0) & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases}$$

Let C be the space of continuous functions from $[-r, 0]$ into X provided with the uniform norm topology and let

$$C_D = \{\varphi \in C : D(\varphi) = 0\}.$$

Definition 2.6 ([6]) D is said to be stable if the zero solution of the difference equation

$$\begin{cases} D(y_t) = 0, & t \geq 0, \\ y_0 = \varphi \in C_D, \end{cases} \tag{2.2}$$

is exponentially stable.

Proposition 2.7 ([5]) *Let* $D(\varphi) = \sum_{k=0}^{p} a_k \varphi(-r_k)$. *Then,* D *is stable iff* $\sum_{k=0}^{p} |a_k| < 1$.

In the sequel, we assume the following:

$(\mathbf{H}_4)$ $\quad$ The operator D is stable.

($\mathbf{H}_5$) The semigroup operators $T(t)$ are compact for every $t > 0$.

We recall this important result that is the key to get the existence of a global attractor for equation.

Theorem 2.8 (Smoothness of the Semigroup $(U(t))_{t\geq 0}$**)** *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$), *and* ($\mathbf{H}_5$) *hold. Then the semigroup* $(U(t))_{t\geq 0}$ *is decomposed as follows:*

$$U(t) = U_1(t) + U_2(t), \quad \textit{for } t \geq 0,$$

where $U_1(t)$ *is an exponentially stable semigroup on* C_α *and* $U_2(t)$ *is compact on* C_α *for every* $t > 0$.

For the proof, we need the following result.

Lemma 2.9 ([1]) *If* D *is stable, then there exist positive constants* a, b, c, *and* d *such that for any* $\varepsilon \in (0, r]$ *sufficiently small and any continuous function* h *from* $[0, +\infty)$ *into* X, *the solution* v *of the equation*

$$D(v_t) = h(t), \quad t \geq 0, \tag{2.3}$$

satisfies the inequality

$$|v_t| \leq e^{-a(t-\varepsilon)} \left[b|v_0| + c \sup_{0\leq s\leq \varepsilon} |h(s)| \right] + d \sup_{\max(\varepsilon, t-r)\leq s\leq t} |h(s)|, \quad t \geq \varepsilon. \tag{2.4}$$

Proof of Theorem 2.8 Without loss of generality, we can assume that there exist positive constants M_0 and γ such that the semigroup $(T(t))_{t\geq 0}$ satisfies

$$|T(t)| \leq M_0 e^{-\gamma t} \quad \text{for } t \geq 0. \tag{2.5}$$

Let $U_1(t)$ be defined by

$$(U_1(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta) & \text{if } t+\theta \leq 0, \\ v(t+\theta) & \text{if } t+\theta \geq 0, \end{cases}$$

where v is the unique solution of the problem

$$\begin{cases} D(v_t) = T(t)D(\varphi) & \text{for } t \geq 0, \\ v(t) = \varphi & \text{for } t \in [-r, 0]. \end{cases}$$

On the other hand, the operator D is stable. We deduce after applying the operator A^α that

$$|v_t|_\alpha \leq e^{-a(t-\varepsilon)} \left[b|\varphi|_\alpha + c \sup_{0\leq s\leq \varepsilon} |T(s)D(\varphi)|_\alpha \right] + d \sup_{\max(\varepsilon, t-r)\leq s\leq t} |T(s)D(\varphi)|_\alpha .$$

So, from (2.5) we get, for some constants N and ν, that

$$|U_1(t)\varphi|_\alpha \le Ne^{-\nu t}|\varphi|_\alpha, \quad \text{for } t \ge 0.$$

Let $U_2(t)\varphi := w_t = u_t - v_t$. Then,

$$D(U_2(t)\varphi) = D(u_t) - D(v_t) = \int_0^t T(t-s)F(U(s)\varphi)\,ds.$$

Consequently,

$$\begin{cases} D(w_t) = h(t,\varphi) := \int_0^t T(t-s)F(U(s)\varphi)\,ds, & \text{for } t \ge 0, \\ w_0 = 0. \end{cases} \tag{2.6}$$

Let $(\varphi_k)_{k\ge 0}$ be a bounded sequence in C_α. We will show that the family $\{h(.,\varphi_k) : k \ge 0\}$ is equicontinuous and bounded on $C([0,\sigma]; X_\alpha)$, for any $\sigma > 0$ fixed. Let $\beta \in (\alpha, 1)$. Since $A^{-\beta} : X \to X_\alpha$ is compact, it is enough to prove that $\left\{A^\beta h(t,\varphi_k) : k \ge 0\right\}$ is bounded in X, for each $t \ge 0$. Since $(U(t))_{t\ge 0}$ is locally bounded in t and φ, it follows that there exists a positive constant λ such that

$$\left|A^\beta h(t,\varphi_k)\right| \le M_\beta \lambda \int_0^t \frac{e^{\omega s}}{s^\beta}ds \text{ for every } k \ge 0.$$

We get that $\{h(t,\varphi_k) : k \ge 0\}$ is compact in X_α, for each $t \ge 0$. It remains to prove the equicontinuity property in α-norm. Let $t > t_0$. Then,

$$A^\alpha h(t,\varphi_k) - A^\alpha h(t_0,\varphi_k) = \int_0^{t_0} A^\alpha (T(t-s) - T(t_0-s))F(U(s)\varphi_k)\,ds + \int_{t_0}^t A^\alpha T(t-s)F(U(s)\varphi_k)\,ds.$$

Consequently,

$$\left|\int_{t_0}^t A^\alpha T(t-s)F(U(s)\varphi_k)\,ds\right| \le M_\alpha \lambda \int_{t_0}^t \frac{e^{\omega s}}{s^\alpha}ds \to 0 \text{ as } t \to t_0 \text{ uniformly in } k.$$

Moreover,

$$\int_0^{t_0} A^\alpha (T(t-s) - T(t_0-s))F(U(s)\varphi_k)\,ds = (T(t-t_0) - I) \int_0^{t_0} A^\alpha T(t_0-s)F(U(s)\varphi_k)ds.$$

There is a compact set K in X such that

$$\int_0^{t_0} A^{\alpha} T(t_0 - s) F(U(s)\varphi_k) ds \in K, \quad \text{for all } k \geq 0.$$

It is well known, from Banach–Steinhaus's Theorem, that

$$\lim_{t \to t_0} \sup_{x \in K} |(T((t - t_0) - I)x| = 0.$$

This implies that

$$\lim_{t \to t_0^+} |h(t, \varphi_k) - h(t_0, \varphi_k)|_{\alpha} = 0, \quad \text{uniformly in } k \geq 0.$$

The proof is similar for $t < t_0$. Then, for any $\sigma > 0$, there exists a subsequence $(\varphi_k)_{k \geq 0}$ such that $h(t, \varphi_k)$ converges as $k \to +\infty$ uniformly on $[0, \sigma]$ to some function $h(t)$ in α-norm. Let w_t^k be the solution of Eq. (2.6) with $\varphi = \varphi_k$. Then,

$$D(w_t^j - w_t^k) = h(t, \varphi_j) - h(t, \varphi_k).$$

Consequently, there is a positive constant c such that

$$\left| w_t^j - w_t^k \right|_{\alpha} \leq c \sup_{0 \leq s \leq t} \left| h(t, \varphi_j) - h(t, \varphi_k) \right|_{\alpha}.$$

This implies that the sequence $\left(w_t^k \right)_{k \geq 0}$ is a Cauchy sequence, which proves that $U_2(t)$ is compact in C_{α}.

3 A Global Attractor for Partial Neutral Functional Differential Equations

Let Y be a Banach space and $\mathrm{S} = (S(t))_{t \geq 0}$ be a (nonlinear) strongly continuous semigroup on Y.

Definition 3.1 ([4])

(i) A set $B \subset Y$ is said to attract a set $C \subset Y$ under S if

$$dist(S(t)C, B) \to 0 \text{ as } t \to +\infty.$$

(ii) A set $B \subset Y$ is said to be invariant under S if $S(t)B = B$ for all $t \geq 0$.

(iii) S is asymptotically smooth if for any nonempty closed bounded set $B \subset Y$ for which $\mathrm{S}B \subset B$, there is a compact set $J \subset B$ such that J attracts B.

(iv) A compact invariant set $\mathcal{A}$ is said to be a maximal compact invariant set if every compact invariant set of the semigroup belongs to $\mathcal{A}$.
(v) An invariant set $\mathcal{A}$ is said to be a global attractor if $\mathcal{A}$ is maximal compact invariant set which attracts each bounded set $B \subset Y$.
(vi) The semigroup S is said to be point dissipative (compact dissipative) if there is a bounded set $B \subset Y$ that attracts each point of Y (each compact set of Y) under S.

Theorem 3.2 ([4]) *Let $S(t) : Y \to Y$ be asymptotically smooth and $S(t)$ is compact dissipative. Then there is a compact invariant set which attracts compact sets of Y.*

Theorem 3.3 ([4]) *If $S(t) : Y \to Y$ is asymptotically smooth, point dissipative and orbits of bounded sets are bounded. Then there exists a global attractor which is connected.*

Theorem 3.4 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, *and* $(\mathbf{H}_5)$ *hold. Then the semigroup $(U(t))_{t\geq 0}$ is asymptotically smooth on C_α.*

The proof is based on the following lemma.

Lemma 3.5 ([4]) *Suppose that* S *is decomposed as follows $S(t) = S_1(t) + S_2(t) : Y \to Y$, $t \geq 0$, such that $(S_1(t))_{t\geq 0}$ is compact and there is a continuous function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(t,r) \to 0$ as $t \to +\infty$ and $\|S_2(t)y\| \leq u(t,r)$ if $\|y\| \leq r$. Then, the semigroup* S *is asymptotically smooth on Y.*

Then Theorem 3.4 is a consequence for lemma 3.5.

Consequently, we deduce the following results on the existence of a global attractor for Eq. (1.1).

Theorem 3.6 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ *hold and $U(t)$ is compact dissipative. Then there is a compact invariant set of $(U(t))_{t\geq 0}$ which attracts compact sets of C_α.*

Theorem 3.7 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ *hold, $U(t)$ is point dissipative and orbits of bounded sets are bounded. Then there exists a global attractor $\mathcal{A}$ which is connected for $(U(t))_{t\geq 0}$.*

Corollary 3.8 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ *hold, $U(t)$ is point dissipative and orbits of bounded sets are bounded. Moreover, if the restriction $(U_A(t))_{t\geq 0}$ of $(U(t))_{t\geq 0}$ over $\mathcal{A}$ is one to one. Then $(U_A(t))_{t\geq 0}$ is a group on $\mathcal{A}$.*

4 Partial Functional Differential Equations with Infinite Delay

The aim of this section is to prove the existence of a global attractor for the following partial functional differential equations with infinite delay:

$$\begin{cases} \dfrac{d}{dt}x(t) = -A\left(x_t\right) + F(x_t) \text{ for } t \geq 0 \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{4.1}$$

where $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X, $\mathcal{B}$ is a Banach space consisting of functions mapping $(-\infty, 0]$ to X and satisfying some axioms that will be introduced later. For $0 < \alpha < 1$, A^α denotes the fractional power of A (see definition below), we assume that F is defined on a smaller space $\mathcal{B}_\alpha$ with values in X, where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \left\{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\right\},$$

where the function $A^\alpha\varphi$ is defined by

$$\left(A^\alpha\varphi\right)(\theta) = A^\alpha\varphi(\theta) \text{ for } \theta \leq 0.$$

We suppose that F is Lipschitz continuous with respect to the fractional power norm of A^α. For every $t \geq 0$, the history function $x_t \in \mathcal{B}_\alpha$ is defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \leq 0.$$

From now on, we use an axiomatic definition of the phase space $\mathcal{B}$ which was first introduced by Hale and Kato in [7]. We assume that $\mathcal{B}$ is the normed space of functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms:

(**A**) There exists a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$, and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,
(ii) $t \to x_t$ is continuous with respect to $|\cdot|$ on $[\sigma, a]$,
(iii) $N\left|x(t)\right| \leq |x_t| \leq K(t-\sigma) \sup\limits_{\sigma\leq s\leq t} |x(s)| + M(t-\sigma)\left|x_\sigma\right|.$

(**B**) $\mathcal{B}$ is a Banach space.

Lemma 4.1 ([8]) *Let C_{00} be the space of continuous functions mapping $(-\infty, 0]$ into X with compact supports and C_{00}^T be the subspace of functions with supports included in $[-T, 0]$ endowed with the uniform norm topology. Then,*

$$C_{00}^T \hookrightarrow \mathcal{B}.$$

In the sequel, we suppose that

(**H**$_6$) For some $0 < \alpha < 1$, one has

$$A^{-\alpha}\varphi \in \mathcal{B} \text{ for } \varphi \in \mathcal{B},$$

where the function $A^{-\alpha}\varphi$ is defined by

$$\left(A^{-\alpha}\varphi\right)(\theta) = A^{-\alpha}\varphi(\theta) \text{ for } \theta \le 0.$$

Let $\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \le 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\}$. We provide $\mathcal{B}_\alpha$ with the following norm:

$$|\varphi|_\alpha = \left|A^\alpha\varphi\right| \text{ for } \varphi \in \mathcal{B}_\alpha.$$

Lemma 4.2 ([2]) *Assume that* $(\mathbf{H}_6)$ *hold. Then,* $\mathcal{B}_\alpha$ *is a Banach space.*

Definition 4.3 A continuous function $x : (-\infty, \infty) \to X_\alpha$ is called a mild solution of Eq. (4.1) if

(i) $x(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(x_s)ds$ for $t \ge 0$
(ii) $x_0 = \varphi$.

Assume that

$(\mathbf{H}_7)$ $F : \mathcal{B}_\alpha \to X$ is a Lipschitz continuous function. Let $k > 0$ be such that

$$|F(\varphi_1) - F(\varphi_2)| \le k\,|\varphi_1 - \varphi_2|_\alpha \text{ for } \varphi_1, \varphi_2 \in \mathcal{B}_\alpha.$$

Theorem 4.4 ([2]) *Assume that* $(\mathbf{H}_6)$ *and* $(\mathbf{H}_7)$ *hold. Then, for* $\varphi \in \mathcal{B}_\alpha$*, Eq. (4.1) has a unique mild solution which is defined for all* $t \ge 0$.

For $t \ge 0$, we define the operator $U(t)$ on $\mathcal{B}_\alpha$ by

$$U(t)(\varphi) = x_t(., \varphi),$$

where $x(., \varphi)$ is the mild solution of Eq. (4.1).

Proposition 4.5 ([2]) *The family* $(U(t))_{t\ge 0}$ *is a strongly continuous semigroup on* $\mathcal{B}_\alpha$*, that is*

(i) $U(0) = I$,
(ii) $U(t+s) = U(t)U(s)$*, for* $t, s \ge 0$,
(iii) For all $\varphi \in \mathcal{B}_\alpha$*,* $U(t)(\varphi)$ *is a continuous function of* $t \ge 0$ *with values in* $\mathcal{B}_\alpha$*,*
(iv) $(U(t))_{t\ge 0}$ *satisfies the translation property, that is for* $t \ge 0$ *and* $\theta \le 0$*, one has*

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0) & \text{for } t+\theta \ge 0, \\ \varphi(t+\theta) & \text{for } t+\theta \le 0, \end{cases}$$

(v) *for all* $t \geq 0$, $U(t)$ *is Lipschitz continuous from* $\mathcal{B}_\alpha$ *to* $\mathcal{B}_\alpha$. *Moreover, for all* $a > 0$, *there exists a positive constant* $m(a)$ *such that*

$$|U(t)\phi - U(t)\psi|_\alpha \leq m(a)\,|\phi - \psi|_\alpha \quad \text{for } t \in (0, a].$$

Now, we state the following fundamental result which will play a crucial role in studying the asymptotic behavior of solutions in the linear case.

Theorem 4.6 ([2]) *Assume that* $(\mathbf{H}_5)$, $(\mathbf{H}_6)$, *and* $(\mathbf{H}_7)$ *hold. Then, the semigroup* $(U(t))_{t\geq 0}$ *is decomposed as follows* $U(t) = U_1(t) + U_2(t)$, *for* $t \geq 0$, *where* $U_2(t)$ *is compact on* $\mathcal{B}_\alpha$ *for* $t > 0$ *and* $U_1(t)$ *is the semigroup solution of the equation*

$$\begin{cases} x'(t) = -Ax(t) \text{ for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{4.2}$$

For $\varphi \in \mathcal{B}$, $t \geq 0$ and $\theta \leq 0$, we define

$$\left[\widetilde{W}(t)\varphi\right](\theta) = \begin{cases} \varphi(0) & \text{for } t + \theta \geq 0 \\ \varphi(t+\theta) & \text{for } t + \theta < 0. \end{cases}$$

Then, $(\widetilde{W}(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{B}$. We set

$$\widetilde{W}_0(t) = \widetilde{W}(t)_{/\mathcal{B}_0}, \text{ where } \mathcal{B}_0 = \{\varphi \in \mathcal{B} : \varphi(0) = 0\}.$$

Definition 4.7 We say that $\mathcal{B}$ is a uniform fading memory space if the following conditions hold:

(i) If a uniformly bounded sequence $(\varphi_n)_n$ in C_{00} converges to a function φ compactly on $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $|\varphi^n - \varphi| \to 0$ as $n \to +\infty$,
(ii) $\left|\widetilde{W}_0(t)\right| \to 0$ as $t \to +\infty$.

Let $\nu_0 = \inf\{\ \nu \in \mathbb{R}$ such that $(\mathbf{D})$ is satisfied $\}$.

Lemma 4.8 ([9]) *If* $\mathcal{B}$ *is a uniform fading memory space, then* $\nu_0 < 0$.

Lemma 4.9 ([8]) *If* $\mathcal{B}$ *is a uniform fading memory space, then* K *and* M *can be chosen such* K *is bounded on* $\mathbb{R}^+$ *and* $M(t) \to 0$ *as* $t \to 0$.

Let Z be a Banach space. We introduce the Kuratowski's measure of noncompactness $\chi(\Omega)$ of set $\Omega \subset Z$ by

$$\chi(\Omega) = \inf\{d > 0 : \Omega \text{ has a finite cover of diameter } < d\}.$$

For a bounded linear operator $\mathcal{H}$, the Kuratowski measure of noncompactness $\alpha(\mathcal{H})$ of $\mathcal{H}$ is defined by

$$\chi(\mathcal{H}) = \inf\left\{\eta \in \mathbb{R}^+ : \chi(\mathcal{H}(D)) \leq \eta\chi(D), \text{ for every bounded subset } D \text{ of } Z\right\}.$$

Lemma 4.10 ([2]) *Suppose that* $(\mathbf{H}_5)$, $(\mathbf{H}_6)$*, and* $(\mathbf{H}_7)$ *hold. Then, for all* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$\chi(U_1(t)) \leq C_\varepsilon M(t-\varepsilon), \text{ for } t > \varepsilon.$$

Let $\omega_0(U_1)$ be the growth bound of $(U_1(t))_{t\geq 0}$ which is defined by

$$\omega_0(U_1) = \inf\left\{\kappa > 0 : \sup_{t\geq 0} e^{-kt} |U_1(t)| < \infty\right\}$$

and the essential growth bound is defined also by

$$\omega_{ess}(U_1) = \inf\left\{\kappa > 0 : \sup_{t\geq 0} e^{-kt} |U_1(t)|_{ess} < \infty\right\}.$$

Then, it is well known from [12], that $\omega_0 = \max\left\{\omega_{ess}, s'(A_{U_1})\right\}$, where

$$s'(A_{U_1}) = \sup\left\{\operatorname{Re}\lambda : \lambda \in \sigma(A_{U_1}) - \sigma_{ess}(A_{U_1})\right\}.$$

where $\sigma_{ess}(A_{U_1})$ is the essential spectrum of the infinitesimal generator A_{U_1} of $(U_1(t))_{t\geq 0}$. Note that $\sigma(A_{U_1}) - \sigma_{ess}(A_{U_1})$ contains a finite number of eigenvalues of A_{U_1}.

From Lemma 4.10 we deduce that $\omega_{ess}(U_1) < 0$. Consequently, $\omega_0(U_1) < 0$ if and only if $s'(A_{U_1}) < 0$.

As an immediate consequence of that, we get the following results.

Theorem 4.11 *If* $s'(A_{U_1}) < 0$*, then* $(U(t))_{t\geq 0}$ *is asymptotically smooth on* $\mathcal{B}_\alpha$.

Theorem 4.12 *If* $(\mathbf{H}_5)$, $(\mathbf{H}_6)$*, and* $(\mathbf{H}_7)$ *hold,* $\mathcal{B}$ *is a uniform fading memory space and* $s'(A_{U_1}) < 0$*. Then each of the following conditions implies the existence of a global attractor* $\mathcal{A}$ *for Eq. (4.1).*

(a) Equation (4.1) is compact dissipative.
(b) Equation (4.1) is point dissipative and orbits of bounded sets are bounded.

5 Application

As an application of our abstract result, we consider the following partial neutral functional differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,x)-qv(t-r,x)\right]=\dfrac{\partial^2}{\partial x^2}\left[v(t,x)-qv(t-r,x)\right] \\ \quad +f\left(v(t,x),\ v(t-r,x),\ \dfrac{\partial}{\partial x}\left[v(t,x)-qv(t-r,x)\right]\right) \text{ for } x\in[0,\pi],\ t\geq 0, \\ v(t,0)=qv(t-r,0) \text{ and } \quad v(t,\pi)=qv(t-r,\pi) \ \text{ for } t\geq 0, \\ v(\theta,x)=v_0(\theta,x), \quad \text{for } \theta\in[-r,0],\ x\in[0,\pi], \end{cases} \tag{5.1}$$

where $v_0 \in C\left([-r,0]\times[0,\pi];\mathbb{R}\right)$, q is a positive constant, and $f:\mathbb{R}^3\to\mathbb{R}$ is a Lipschitz continuous function.

Let A be the operator defined on $X := L^2\left([0,\pi];\mathbb{R}\right)$ by

$$\begin{cases} D(A)=H^2(0,\pi)\cap H_0^1(0,\pi), \\ Ag=-g'', \quad g\in D(A). \end{cases}$$

Then, $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $T(t)$ is compact on X for every $t>0$. The spectrum $\sigma(-A)$ of $-A$ is equal to the point spectrum $P\sigma(-A)$ and is given by $\sigma(-A)=\left\{-n^2 : n\geq 1\right\}$ and the associated eigenfunctions $(\phi_n)_{n\geq 1}$ are given by $\phi_n(x)=\sin(nx),\ x\in[0,\pi]$. Actually, the semigroup $T(t)$ is explicitly defined by

$$T(t)y=\sum_{n=1}^{\infty} e^{-n^2 t}\left\langle y,\phi_n\right\rangle \phi_n \text{ for } \quad t\geq 0 \text{ and } \ y\in X.$$

Let $\alpha=\dfrac{1}{2}$. From [11], we have for $t\geq 0$

$$\begin{cases} A^{\frac{1}{2}}T(t)y=\sum_{n=1}^{\infty} ne^{-n^2 t}\left\langle y,\phi_n\right\rangle \phi_n \ \text{ for } y\in X, \\ A^{-\frac{1}{2}}y=\sum_{n=1}^{\infty} \dfrac{1}{n}\left\langle y,\phi_n\right\rangle \phi_n \quad \text{for } y\in X, \\ A^{\frac{1}{2}}y=\sum_{n=1}^{\infty} n\left\langle y,\phi_n\right\rangle \phi_n \quad \text{for } y\in D(A^{\frac{1}{2}}). \end{cases}$$

Lemma 5.1 ([11]) *If $\phi\in D(A^{\frac{1}{2}})$, then ϕ is absolutely continuous and $\phi'\in X$.*

Let $F : C_{\frac{1}{2}} \to X$ be the mapping defined by

$$\left(F(\varphi)\right)(x)=f\left(\varphi(0)(x);\varphi(-r)(x);\ \frac{\partial}{\partial x}\left[\varphi(0)(x)-q\varphi(-r)(x)\right]\right) \text{ for } x\in[0,\pi],$$

$D : C := C([-r,0], X) \to X$ be the bounded linear operator defined by

$$D(\varphi)(x) = \varphi(0)(x) - q\varphi(-r)(x) \quad \text{for } x \in [0,\pi],$$

$y : [-r,+\infty) \to X$ be the function defined by

$$y(t) = v(t,\cdot) \quad \text{for } t \geq 0,$$

and $\varphi(\theta) = v_0(\theta,\cdot)$, for $\theta \in [-r,0]$. Then, Eq. (5.1) takes the following abstract form:

$$\begin{cases} \dfrac{d}{dt} D(y_t) = -AD(y_t) + F(y_t) \quad \text{for } t \geq 0, \\ y_0 = \varphi \in C_{\frac{1}{2}}. \end{cases} \tag{5.2}$$

Lemma 5.2 *F is Lipschitz continuous from $C_{\frac{1}{2}}$ into X.*

Proof Let $\varphi_1, \varphi_2 \in C_{\frac{1}{2}}$. Then, for $x \in [0,\pi]$, we have

$$\begin{aligned}(F(\varphi_1) - F(\varphi_2))(x) = f\left(\varphi_1(0)(x);\ \varphi_1(-r)(x);\ \frac{\partial}{\partial x}[\varphi_1(0)(x) - q\varphi_1(-r)(x)]\right) \\ - f\left(\varphi_2(0)(x);\ \varphi_2(-r)(x);\ \frac{\partial}{\partial x}[\varphi_2(0)(x) - q\varphi_2(-r)(x)]\right).\end{aligned}$$

Since f is Lipschitz continuous, then there exists a positive constant k such that

$$\begin{aligned}|F(\varphi_1) - F(\varphi_2)(x)| \leq k\,(&|\varphi_1(0)(x) - \varphi_2(0)(x)| \\ &+ |\varphi_1(-r)(x) - \varphi_2(-r)(x)| \\ &+ \left|\frac{\partial}{\partial x}[\varphi_1(0)(x) - \varphi_2(0)(x) - q(\varphi_1(-r)(x) - \varphi_2(-r)(x))]\right|\right).\end{aligned}$$

Which implies that

$$\begin{aligned}|F(\varphi_1) - F(\varphi_2)| \leq k\left(\sqrt{\int_0^\pi |\varphi_1(0)(x) - \varphi_2(0)(x)|^2\,dx}\right. \\ + \sqrt{\int_0^\pi |\varphi_1(-r)(x) - \varphi_2(-r)(x)|^2\,dx} \\ + \sqrt{\int_0^\pi \left|\frac{\partial}{\partial x}(\varphi_1(0) - \varphi_2(0))(x)\right|^2 dx}\end{aligned}$$

$$+ q \sqrt{\int_0^{\pi} \left| \frac{\partial}{\partial x}(\varphi_1(-r) - \varphi_2(-r))(x) \right|^2 dx} \Bigg).$$

By Travis and Webb [11], page 141, we have for every $\tau \in [0, r]$

$$\sqrt{\int_0^{\pi} |\varphi_1(-\tau)(x) - \varphi_2(-\tau)(x)|^2 \, dx} \leq \|\varphi_1 - \varphi_2\|_{\frac{1}{2}}$$

and

$$\sqrt{\int_0^{\pi} \left| \frac{\partial}{\partial x}(\varphi_1(-\tau) - \varphi_2(-\tau))(x) \right|^2 dx} \leq \|\varphi_1 - \varphi_2\|_{\frac{1}{2}}.$$

Which means that F is Lipschitz continuous from $C_{\frac{1}{2}}$ into X. □

Consequently, we have the existence and uniqueness of mild solutions of Eq. (5.1).

In the sequel, we assume that

$$0 < q < 1.$$

This means that the operator D is stable.

For the next, we assume that

(H) $|F(t, \phi)| \leq N$ for $t \geq 0$ and $\phi \in C_{\frac{1}{2}}$.

Proposition 5.3 *$(U(t))_{t\geq 0}$ is point dissipative, and the orbits of bounded sets are bounded. Consequently, then there exists a global attractor $\mathcal{A}$ for Eq. (5.1) which is connected.*

Proof Let u be a mild solution of Eq. (5.1). Then

$$\begin{aligned} D(u_t) &= T(t)D(\varphi) + \int_0^t T(t-s)F(s, u_s)ds \text{ for } t \geq 0. \\ &= H(t). \end{aligned}$$

It is well known that there exists $M \geq 1$ such that

$$|T(t)| \leq Me^{\omega t} \text{ for some } -1 < \omega < 0.$$

Then it follows by taking the $\frac{1}{2}$-norm that for some positive constant $c > 0$, we have the following estimation:

$$|H(t)|_{\frac{1}{2}} \leq c \left(e^{\omega t} |\varphi|_{\frac{1}{2}} + \int_0^{\infty} \frac{e^{\omega s}}{s^{\frac{1}{2}}} ds \right).$$

Since the operator D is stable, then by Lemma 2.9, we deduce that there exists a positive constant $\widetilde{C}$ such that for any bounded set B in $C_{\frac{1}{2}}$ we have that

$$\overline{\lim_{t\to\infty}} \sup_{\varphi\in B} |u(t,\varphi)|_{\frac{1}{2}} \leq \widetilde{C}.$$

Which implies that the semigroup $(U(t))_{t\geq 0}$ is point dissipative and orbits of bounded sets are bounded. Now the existence of the global attractor is achieved by applying Theorem 3.7.

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A Study of an Epidemic SIR Model via Homotopy Analysis Method in the Sense of Caputo-Fractional System

Brice M. Yambiyo, Fatemeh Norouzi, and Gaston M. N'Guérékata

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1 Introduction

Fractional calculus is a branch of mathematics which dates back in 1695 to the end of the seventeenth century, when Newton and Leibniz developed the theoretical foundations of integral and differential calculus. Leibniz introduced the symbol $\frac{\partial^n f(x)}{\partial x^n}$ to denote the nth derivative of the function f where $n \in \mathbb{N}$. "What if $n = \frac{1}{2}$?".

At that time, there were hardly any practical applications of this theory yet. The transition from pure mathematical formulations to applications began to emerge around 1990 when fractional differential equations appeared in several fields such as physics, biology, mechanics [1, 7, 14, 16, 21, 22, 26, 27, 29, 31].

Because of this property, the fractional derivative is more suited to the problems of modeling memory-dependent phenomena, especially in most biological and physical systems. Another advantage of using a fractional derivative is broadening of the region of stability in dynamical systems.

Mathematical modeling based on a system of differential equations can provide a comprehensive mechanism for the dynamics of disease transmission. In epidemiology, numerous works involving a fractional derivative have been carried out, and most of them mainly concern SIR type models with a linear incidence rate [4, 6, 8, 9, 13, 15, 23].

B. M. Yambiyo
NEERLab, Department of Mathematics, Ecole Doctorale Sciences et Technologie, Université de Bangui, Bangui, Central African Republic
e-mail: ybricemartial@gmail.com

F. Norouzi (✉) · G. M. N'Guérékata
NEERLab, Department of Mathematics, Morgan State University, Baltimore, MD, USA
e-mail: Fatemeh.norouzi@morgan.edu; Gaston.NGuerekata@morgan.edu

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Memory effects play a crucial role in the spreading of the diseases. Including memory effects in the susceptible-infected-recovered (SIR) epidemic model which is very interesting for an investigation. In [30], Saeedian et al. studied the memory effect of an SIR epidemic model using the fractional derivative of Caputo. They have proven that this effect plays an essential role in the spread of disease. Also, they considered the SIR model on structured networks and studied the effect of topology on threshold points in a non-Markovian dynamics.

In [4, 8, 9, 20, 23, 28], several analysis methods were applied to prove the existence and the stabilities of the SIR models with nonlinear incidence rate. El-Saka [9] mainly focused on fractional-order SIRS epidemic model with constant recruitment, mass action incidence, and variable population sizes. They also investigated the stability conditions of equilibrium points of the system and presented the numerical simulation. In [18], Li and Zhang worked on a modified SIR model with nonlinear incidence and recovery rates. They tried to get the idea about the influence by any government intervention and hospitalization condition variation which affect the spread of the disease. In this way, they analyzed the existence and stability conditions of the equilibria to investigate the bifurcation conditions. In addition, Dubey, et al. [8] considered a global type of SIR dynamic model. They studied some conditions of the stability of the equilibrium points by implementing the Lyapunov function. They also investigated the Hopf bifurcation of the SIR model and presented the numerical simulation. Moreover, Khan, et al. [17] by defining a generalized incidence rate, solved the SEIR epidemic model and proved the stability and existence of the equilibria. Also, Liu and Stechlinski [20] proposed several cases of SIR epidemic model. One of the models was a new SIR model with time-varying parameters and switched nonlinear incidence rate. The other model was SIR model with pulse vaccination and pulse treatment which were applied to the model with seasonality and switched incidence rate. They determined their success in eliminating the disease by using the control strategies.

Overall from above, the authors studied epidemiological models with fractional-order differential equations, from a mathematical point of view. They mainly focused on the presentation of mathematical methods in order to solve the corresponding differential equations without touching the method of homotopy analysis which is an efficient method in solving fractional differential equations. In addition, in these various previous works, the authors rarely discuss the effect of fractional-order differential equations and memory on the different levels of the force of infection that can be the cause of an outbreak in the manner of time.

In epidemiology, the number of infected individuals per unit of time is called the incidence rate. The latter can be linear or non-linear, and depends on the different levels of infection strength of the disease. So, whether the strength of infection is high, medium, or low, the rate of infection in people may vary over time.

Our main motivation is to solve the Caputo-sense of fractional differential equations by method of homotopy analysis described in [19] and to see if, memory plays a key role on the different levels of the force of infection. In our study, we propose the analytical solution to the fractional differential equations by using homotopy analysis method which has not been used in epidemiology through the study of a SIR model with a Caputo derivation. We then prove the existence, stability, and asymptotic behavior of the SIR fractional model, and finally, we present an illustrative simulation of the results to validate our results.

The SIR epidemiological model is given by

$$\begin{cases} {}^C D_t^\alpha S(t) = \Lambda - \mu S(t) - \frac{\theta\beta S(t)I(t)}{N(t)} \\ {}^C D_t^\alpha I(t) = \frac{\theta\beta S(t)I(t)}{N(t)} - \gamma I(t) - \mu I(t) \\ {}^C D_t^\alpha R(t) = \gamma I(t) - \mu R(t) \end{cases} \tag{1}$$

Where ${}^C D_t^\alpha$ is the Caputo-fractional derivative on order $\alpha \in (0, 1)$.

In system (1), the population is divided into three compartments: $S(t)$, $I(t)$, and $R(t)$, respectively, represent the number of susceptible become infected, exposed, infectious and healed or recovered individuals at time t, and $N(t)$ is the total population size which is $N = S + I + R$.

Λ is the recruitment rate of the population, μ is the natural mortality rate, γ is the infection-related mortality rate, β is the strength of infection force, and θ is the recovery rate of infected.

Our paper is organized as follows. In Sect. 2, we recall some basic definitions of fractional calculus. In Sect. 3, we propose an approximate analytical solution of the fractional SIR system using the homotopy analysis method. The existence of equilibria and their local stability are studied in Sect. 4. In Sect. 5, we attempt to present the numerical simulation of fractional SIR model. Finally, in Section 6, concluding results and comments are given.

2 Preliminaries

In this section, we present an overview of the concept of fractional calculus, and we define some basic notions of the fractional derivative of Caputo. (See for instance: [1, 3, 16, 26, 32])

Let $I = [0, T]$ where $T > 0$. We denote by $C(I, \mathbb{R}^\mathbf{n})$ the Banach space of all continuous functions $I \longrightarrow \mathbb{R}^\mathbf{n}$ endowed with the topology of uniform convergence (the norm in this space will be denoted by $\|.\|$)

2.1 Definition and Theorem

Definition 1 The fractional integral of order $\alpha > 0$ for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{ds} \tag{2}$$

Definition 2 The fractional derivative of order α in the sense of Caputo is defined by

$$ {}_{a}^{C}D^{\alpha} f(x) = \int_{a}^{t} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)\,\mathrm{ds} \tag{3}$$

where $f^{(n)}$ is the derivative of order n of the function $f \in (C[a,b])$; with $\alpha \in (n-1,n)$ and $n \in N$ such as $n = [\alpha]+1$. If α is such that $[\alpha] = 0$, then any fraction derivative of order α in the sense of Caputo will be written in the following way:

$$ {}_{a}^{C}D^{\alpha} f(x) = \int_{a}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f^{'}(x)\,\mathrm{ds} \tag{4}$$

Definition 3 A real function $f(t)$, $(t > 0)$ is said to be in the space C_{ν}, $\nu \in (\mathbb{R})$ if there exists a real $p > \nu$ such that $f(t) = t^{p} g(t)$, where $g \in C(\mathbb{R}^{+})$.

Definition 4 A real function $f(t)$, $(t > 0)$ is said to be in the space C_{ν}^{n}, where $n \in \mathbb{N}$ if $f^{n} \in C_{\nu}$.

In the rest of our paper, ${}_{a}^{C}D^{\alpha}$ will be simply noted as D_{t}^{α}.

Theorem 1 *Consider that abstract fractional functional differential equation*

$$\begin{cases} D_t^{\alpha}(x(t) + g(t, x(t))) = Ax(t) + f(t, x(t)), \, where \;\; 0 < \alpha \leq 1 \\ and, \\ x(0) = x_0 \end{cases} \tag{5}$$

where D_t^{α} denotes Caputo's fractional derivative.

Then the system (5) is equivalent to

$$\begin{aligned} x(t) = x_0 + g(0, x_0) - g(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\mathrm{ds} \\ + \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\mathrm{ds} \end{aligned} \tag{6}$$

A being a closed linear operator. See [24] for more details of the proof.

Let $X = \mathbb{R}^{\mathbf{n}}$, with the norm $\|y\|_n = \sum_{i=1}^{n} |y_i|$ where $y = (y_1, y_2, \ldots, y_n)$ and let $C^n(I)$ be the class of all continuous column vector functions $Y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ defined on I, with norm:

$$\|Y\|_n = \sum_{i=1}^{n} \sup_{t \in I} |y_i(t)| \tag{7}$$

For $w > 0$ given, we have the following equivalent norm:

$$\|Y\|_{n,\omega} = \sum_{i=1}^{n} \sup_{t\in I} e^{-\omega t}|y_i(t)|$$

Consider the Cauchy problem with nonlocal condition:

$$\begin{cases} D_t^{\alpha}(y_i(t)) = f_i(t, y_i(t)), \quad \text{where} \quad 0 < \alpha \leq 1, \quad t \in (0, T] \\ \quad \text{and,} \\ y_i(0) = y_{0i}, \quad \text{where} \quad i = 1, 2, \ldots, n \end{cases} \tag{8}$$

Where $D_t^{\alpha}(y_i(t))$ is the fractional derivative of the function $y_i(t)$ in the sense of Caputo with order $0 < \alpha \leq 1$.

The system (8) can be reformulated as follows:

$$\begin{cases} D_t^{\alpha}(Y(t)) = F(Y(t)), \quad \text{where} \quad 0 < \alpha \leq 1, \quad t \in (0, T] \\ \quad \text{and,} \\ Y(0) = Y_0, \quad \text{where} \quad Y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \end{cases} \tag{9}$$

By Theorem (1), we obtain the equivalent equation:

$$Y(t) = Y_0 + I^{\alpha} F(Y(t)). \tag{10}$$

By making the following assumptions:

H1: $f_i : I \longrightarrow \mathbb{R}^+$ are continuous on I and $f_i \in C_{\mu}$, where $\mu \geq -1$ $i = 1, 2, \ldots, n$
H2: There exists $L > 0$ such that:

$$\|f(X(t)) - f(Y(t))\|_n < L\|X(t) - Y(t)\|_n \tag{11}$$

Theorem 2 *Under the hypotheses (***H1–H2***), the problem with initial values (9) has a unique one on I provided* $Lw^{-\alpha} < 1$.

See [25] for the proof.

3 Homotopy Analysis Method (HAM)

The homotopy method was established by He in 1998 [11]. The method was then developed and improved by himself. He has applied it to problems at the limits of the nonlinear wave equation, as well as to many subjects. The homotopy disturbance method can be considered as a universal method which is capable of solving different types of nonlinear functional equations. See [2, 12, 19] for more information.

In this section, we apply the homotopy analysis method in [19] in order to obtain an analytical solution for a fractional system defined in (1).

Using the same approach as [25], we have

$$\begin{cases} S(t) &= S_0(t) + \sum_{m=1}^{\infty} X_m(t)q^m \\ I(t) &= I_0(t) + \sum_{m=1}^{\infty} I_m(t)q^m, \\ R(t) &= R_0(t) + \sum_{m=1}^{\infty} R_m(t)q^m \end{cases} \tag{12}$$

when q increases from 0 to 1.

We then take the mth-order homotopy derivative of the zero-order and obtain the m-order deformation equation:

$$\begin{cases} L_\alpha[S_m(t) - \chi_m[S_{m-1}(t)] = h\Delta_m^S(t) \\ L_\alpha[I_m(t) - \chi_m[I_{m-1}(t)] = h\Delta_m^I(t), \\ L_\alpha[R_m(t) - \chi_m[R_{m-1}(t)] = h\Delta_m^R(t) \end{cases} \tag{13}$$

With the initial conditions: $S_m(0) = 0$, $I_m(0) = 0$, and $R_m(0) = 0$. Where,

$$\begin{cases} \Delta_m^S(t) = D_C^\alpha S_{m-1}(t) - \Lambda + \mu S_{m-1}(t) + \frac{\theta\beta S_{m-1}(t)I_{m-1}(t)}{N_{m-1}(t)} \\ \Delta_m^I(t) = D_C^\alpha I_{m-1}(t) - \frac{\theta\beta S_{m-1}(t)I_{m-1}(t)}{N_{m-1}(t)} + \gamma I_{m-1}(t) + \mu I_{m-1}(t), \\ \Delta_m^R(t) = D_C^\alpha R_{m-1}(t) - \gamma I_{m-1}(t) + \mu R_{m-1}(t) \end{cases} \tag{14}$$

In this way, it is easy to solve the linear non-homogeneous equation (13) with initial conditions mentioned above for all $m \geq 1$ and we obtain

$$\begin{cases} S_1(t) = \frac{ht^\alpha}{\Gamma(\alpha-1)}[-\Lambda + \frac{\theta\beta S_0 I_0}{N_0} - \mu S_0] \\ \\ I_1(t) = \frac{ht^\alpha}{\Gamma(\alpha-1)}[-\frac{\theta\beta S_0 I_0}{N_0} + \gamma I_0 + \mu I_0], \\ \\ R_1(t) = \frac{ht^\alpha}{\Gamma(\alpha-1)}[-\gamma I_0 + \mu R_0] \end{cases} \tag{15}$$

Proceeding similarly, the p-th term of the approximate solution is of the form:

$$\begin{cases} S(t) = S_0(t) + \sum_{m=1}^{p} X_m(t) \\ \\ I(t) = I_0(t) + \sum_{m=1}^{p} I_m(t) \\ \\ R(t) = R_0(t) + \sum_{m=1}^{p} R_m(t) \end{cases} \tag{16}$$

With $S_0(t) = S_0$, $I_0(t) = I_0$, and $R_0(t) = R_0$.

4 Equilibrium Points and Their Asymptotic Stability

In this section, we discuss the existence and the local stability of equilibria for system (1). For this, we define the basic reproduction number $\mathcal{R}_0$ of our model by

$$\mathcal{R}_0 = \frac{\theta\beta}{\gamma + \mu}$$

Thus the total population size $N = S + I + R$ may vary in time.
To evaluate the equilibrium points, let

$$^{C}D^{\alpha} S(t) = D_C^{\alpha} I(t) = D_C^{\alpha} R(t) = 0$$

So, we obtain two equilibrium points which are

$$E_{eq} = (S_{eq}, I_{eq}, R_{eq}) = \left(\frac{\Lambda}{\mu}, 0, 0\right)$$

and

$$E_* = (S_*, I_*, R_*)$$

which,

$$S_* = \frac{\Lambda(\gamma + \mu))}{\theta\beta\mu}$$

$$I_* = \frac{\Lambda(\theta\beta - (\gamma + \mu))}{\theta\beta(\gamma + \mu)}$$

$$R_* = \frac{\gamma\Lambda(\theta\beta - (\gamma + \mu))}{\theta\beta\mu(\gamma + \mu)}$$

For $E_{eq} = (S_{eq}, I_{eq}, R_{eq}) = (\frac{\Lambda}{\mu}, 0, 0)$, we find that

$$J = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & \frac{\theta\beta\Lambda}{\mu} - (\mu + \gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

and its eigenvalues are

$$\lambda_1 = \lambda_2 = -\mu < 0$$

$$\lambda_3 = -\frac{\theta\beta\Lambda}{\mu} - (\mu + \gamma) < 0$$

if

$$\frac{\theta\beta\Lambda}{\mu} < (\mu + \gamma)$$

Hence the equilibrium point $E_{eq} = (S_{eq}, I_{eq}, R_{eq}) = (\frac{\Lambda}{\mu}, 0, 0)$ is local asymptotically stable if

$$\frac{r\beta\Lambda}{\mu} < (\mu + \gamma) \tag{17}$$

We proceed in a similar way to $E_* = (S_*, I_*, R_*)$.

$$J = \begin{pmatrix} -\mu - \frac{\theta\beta I_*}{N} & -\frac{\theta\beta S_*}{N} & 0 \\ \frac{\theta\beta I_*}{N} & \frac{\theta\beta S_*}{N} - (\gamma + \mu) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}$$

A sufficient condition for the local asymptotic stability of the equilibrium point

$$E_* = (S_*, I_*, R_*)$$

$$|arg(\lambda_1)| > \frac{\alpha\pi}{2},$$

$$|arg(\lambda_2)| > \frac{\alpha\pi}{2},$$

$$|arg(\lambda_3)| > \frac{\alpha\pi}{2}$$

The characteristic polynomial of the equilibrium point $E_* = (S_*, I_*, R_*)$ is given by the expression:

$$\lambda^3 - tr(J)\lambda^2 + Z(J)\lambda - det(J)$$

with

- tr(J) the trace of the matrix J;
- $Z(J) = -\frac{1}{2}(tr(J^2) - (tr(J))^2)$
- det(J) the determinant of the matrix J

So we have:

- $If \mathcal{R}_0 < 1$, the disease-free equilibrium point is globally asymptotically stable and there is no endemic equilibrium point (the disease dies out).
- $\mathcal{R}_0 \geq 1$, the disease-free equilibrium point is unstable and a globally asymptotically stable endemic equilibrium point exists.

5 Numerical Simulation

In epidemiology, a disease is always characterized by its force of infection (weak, medium, or strong). Knowledge of this force of infection makes it possible to propose a solution to control the disease during an epidemic. In our simulation, we have the infection force β, which we are going to assign as low, medium, and high value. Of course, from the point of view of epidemiological reality, whatever the value of the force of infection β is, certain diseases can become epidemic in each country. We consider the following Caputo-fractional differential system based on the parameters in the following Table 1:

Table 1 Biological parameters in SIR fractional differential model

Parameter	Value/days	Definition
Λ	0.9	Recruitment rate of the population
β	0.01	Low strength of infection
	0.5	Medium strength of infection
	0.99	High strength of infection
μ	0.05	Natural mortality rate
θ	0.4	Recovery rate of infection
γ	0.001	Infection-related mortality rate

$$\begin{cases} {}^{C}D^{\alpha_1}S(t) = \Lambda - \mu S(t) - \frac{\theta\beta S(t)I(t)}{N(t)} \\ {}^{C}D^{\alpha_2}I(t) = \frac{\theta\beta S(t)I(t)}{N(t)} - \gamma I(t) - \mu I(t) \\ {}^{C}D^{\alpha_3}R(t) = \gamma I(t) - \mu R(t) \end{cases} \quad (18)$$

where N(t) is the total population size which is $N = S + I + R$.

Numerical simulations are conducted on fractional orders based on the predictor–corrector method of Adams-Bashforth-Moulton described in [5]. We implement this method on MATLAB by using the predictor–corrector PI rules code (FDE PI12 PC.m) by Garrappa [10] which solves the multi-order system of fractional differential equations.

The reason we choose these parameters is that based on the SIR model, we need to consider the rate of population recruitment, natural mortality rate, recovery rate of infection, infection-related mortality rate, and strength rate of infection to be between 0 and 1. We aim to investigate the behavior of the solutions of the system (18) based on different rates of strength of infection and different cases of α which is the fractional order. The time histories and phase diagrams are used to identify the dynamics of the system. Following are the results of our investigations of various cases studied:

Case 1: Commensurate order, $\alpha_1 = \alpha_2 = \alpha_3 = 0.99, 0.5, 0.8$:

This system was calculated numerically based on $\alpha_1 = \alpha_2 = \alpha_3 \in (0, 1)$ and three different values of β (0.01, 0.5, 0.99). It was found that when $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$ which is the classic case, system (18) behaves periodically. When $\beta = 0.01$, the number of susceptible people tends to increase quickly until $S(t)$ stabilizes to a fixed point. Also, the number of infected people $I(t)$ and the number of recovered people $R(t)$ move smoothly as the time increases and the disease persists. In addition, in the case of $\beta = 0.5$ and 0.99, the number of susceptible people varies quickly but it is smoother than the classic case. The behavior of $I(t)$ shows that it increases at time t, and after certain point it runs smoothly to a fixed point. Also, the behavior of $R(t)$ is similar to the classic case.

The phase plots and time series of $S(t)$, $I(t)$, $R(t)$ based on three different values of β and $\alpha = 0.99, 0.8, 0.5$ are depicted in Figs. 1, 2, and 3, respectively.

Case 2: $\alpha_1 = \alpha_2 \neq \alpha_3$

In this case, we calculate the system numerically based on the orders $\alpha_1 = \alpha_2 = 0.75, \alpha_3 = 0.05$. It was found that in this case, system (18) behaves periodically.

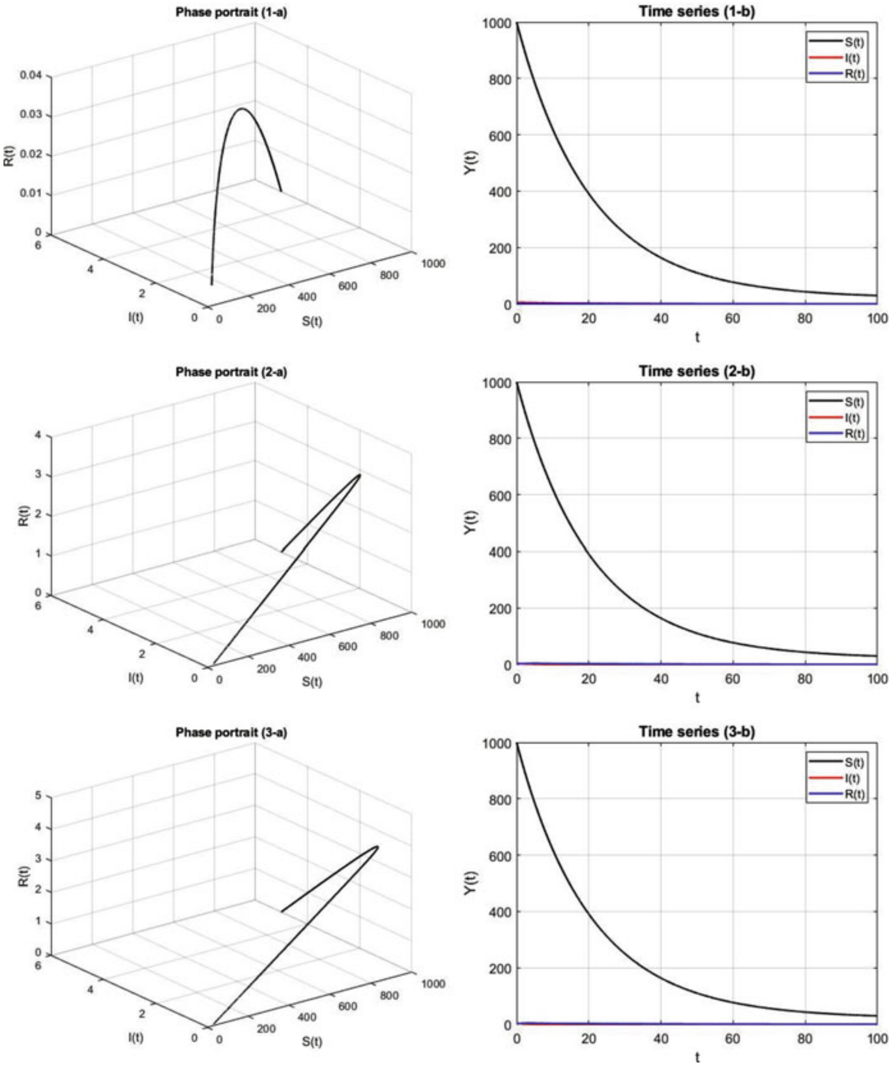

Fig. 1 Phase portraits and time histories of S(t), I(t), and R(t) for system (18) with fractional orders $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$, $(1a - 1b)\beta = 0.01$, $(2a - 2b)\beta = 0.5$, $(3a - 3b)\beta = 0.99$

When $\beta = 0.01$, the number of susceptible people tends to increase quickly until $S(t)$ stabilizes to a fixed point at time t. Also, the number of infected people $I(t)$ and the number of recovered people $R(t)$ move smoothly in a fixed behavior as the time increases. In addition, in the case of $\beta = 0.5$ and 0.99, the number of susceptible people varies quickly, but it is smoother than the classic case. The behavior of $I(t)$ shows that it increases at time t, and after certain point it runs smoothly to a fixed point. Also, the behavior of $R(t)$ is similar to the classic case.

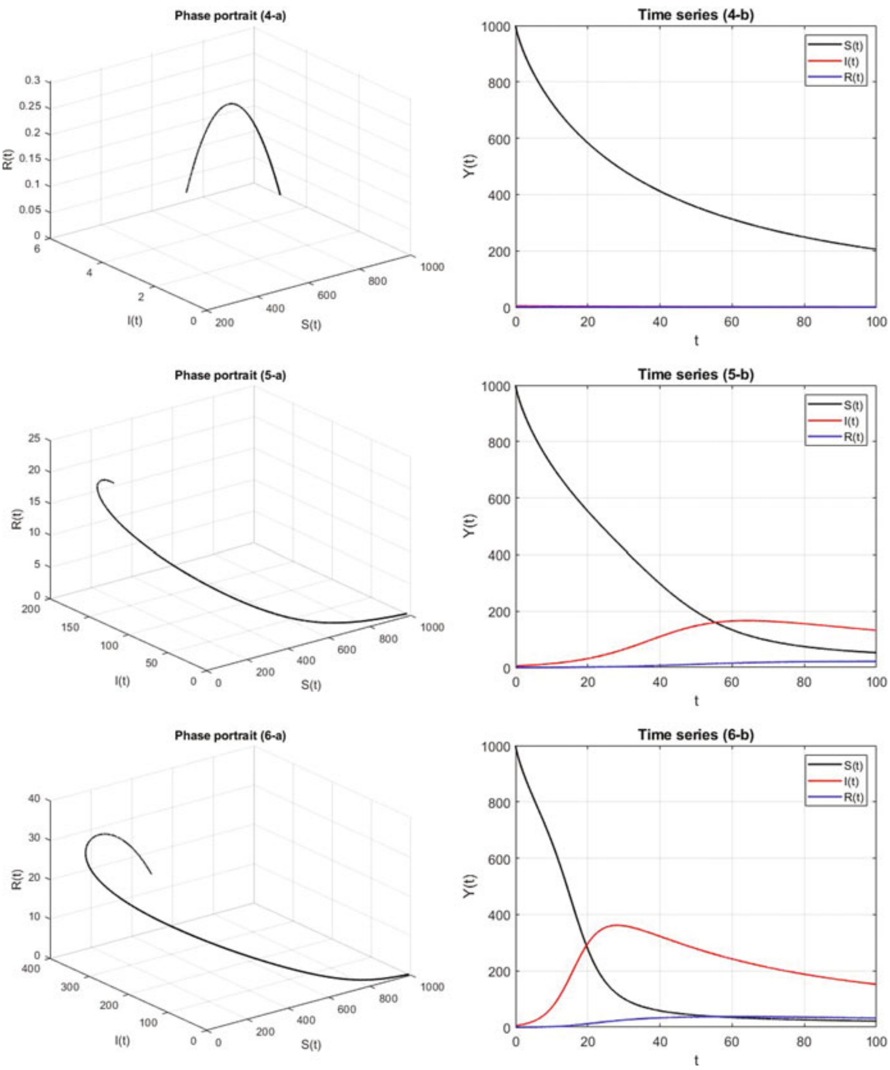

Fig. 2 Phase portraits and time histories of S(t), I(t), and R(t) for system (18) with fractional orders $\alpha_1 = \alpha_2 = \alpha_3 = 0.8$, $(4a - 4b)\beta = 0.01$, $(5a - 5b)\beta = 0.5$, $(6a - 6b)\beta = 0.99$

The phase plots and time series of $S(t)$, $I(t)$, $R(t)$ based on three different values of β and $\alpha_1 = \alpha_2 = 0.75$, $\alpha_3 = 0.05$ are depicted in Fig. 4.

Case 3: $\alpha_1 \neq \alpha_2 \neq \alpha_3$s

In this case, we calculate the system numerically based on the orders $\alpha_1 = 0.5, \alpha_2 = 0.01, \alpha_3 = 0.99$. It was found that in this case, system (18) behaves periodically in all the cases for $S(t)$. When $\beta = 0.01, 0.5, 099$, we observe the number of susceptible people tends to increase quickly until $S(t)$ stabilizes to a

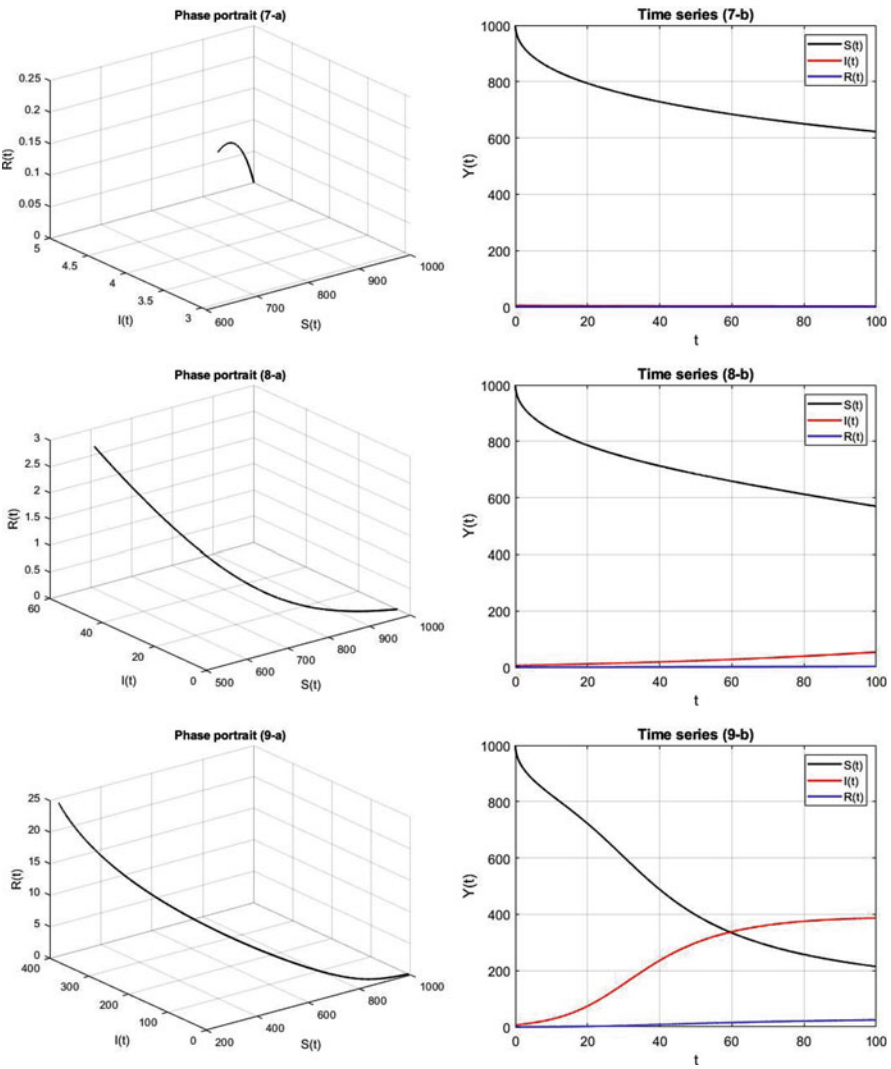

Fig. 3 Phase portraits and time histories of S(t), I(t), and R(t) for system (18) with fractional orders $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$, $(7a - 7b)\beta = 0.01$, $(8a - 8b)\beta = 0.5$, $(9a - 9b)\beta = 0.99$

fixed point at time t similarly to the case (2). Moreover, the behavior of $S(t)$, $I(t)$, and $R(t)$ are all the same for this case such that (It) and $R(t)$ run smoothly to a fixed point as time t increases and the disease persists. So, we conclude that by changing the fractional order α in the system, the stability of the model will not be affected.

The phase plots and time series of $S(t)$, $I(t)$, $R(t)$ based on three different values of β and $\alpha_1 = 0.5$, $\alpha_2 = 0.01$, $\alpha_3 = 0.99$ are shown in Fig. 5.

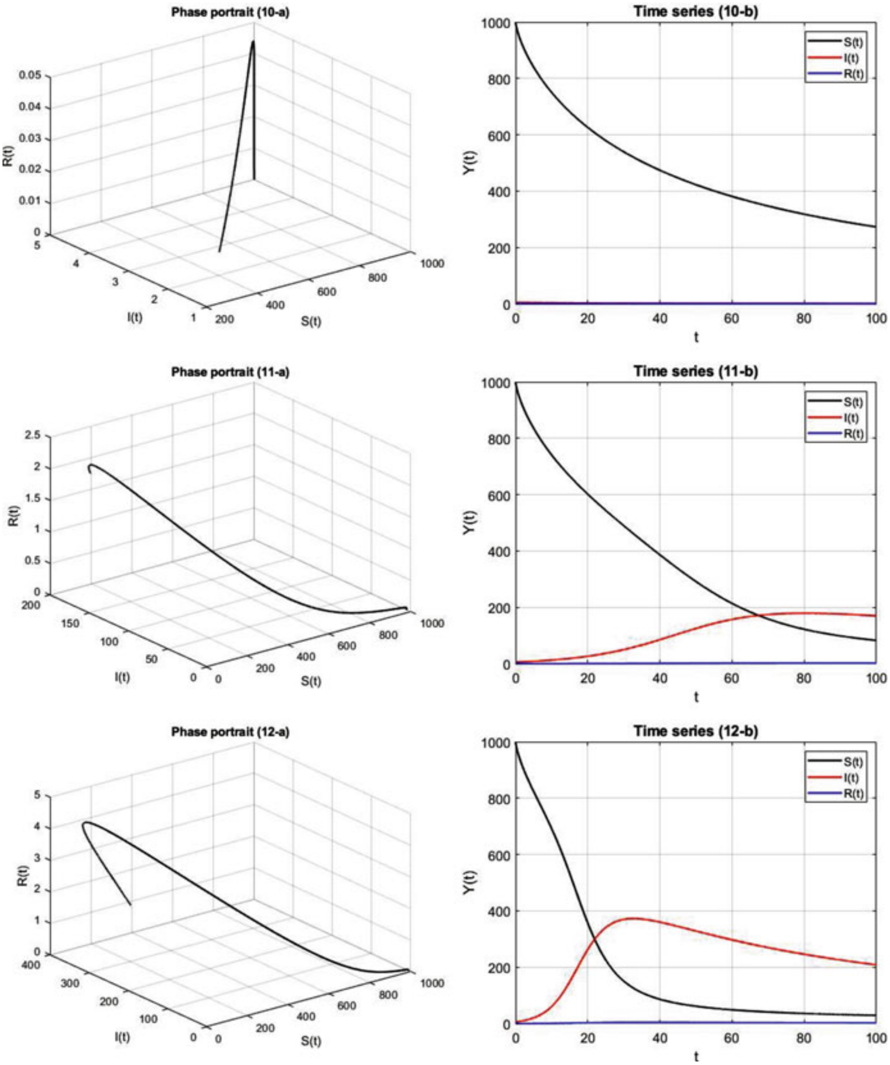

Fig. 4 Phase portraits and time histories of S(t), I(t), and R(t) for system (18) with fractional orders $\alpha_1 = \alpha_2 = 0.75$, $\alpha_3 = 0.05$, $(10a - 10b)\beta = 0.01$, $(11a - 11b)\beta = 0.5$, $(12a - 12b)\beta = 0.99$

6 Conclusion

In this paper, we proposed an approximate analytical solution to Caputo-fractional differential system of SIR epidemic model using the homotopy analysis method. We also investigated the existence of equilibria and their local stability in SIR fractional system. We have found that when $\mathcal{R}_0 = \frac{\theta\beta}{\gamma+\mu} \leq 1$, it signifies the extinction of the disease. However, when $\mathcal{R}_0 > 1$, the disease-free equilibrium becomes unstable and

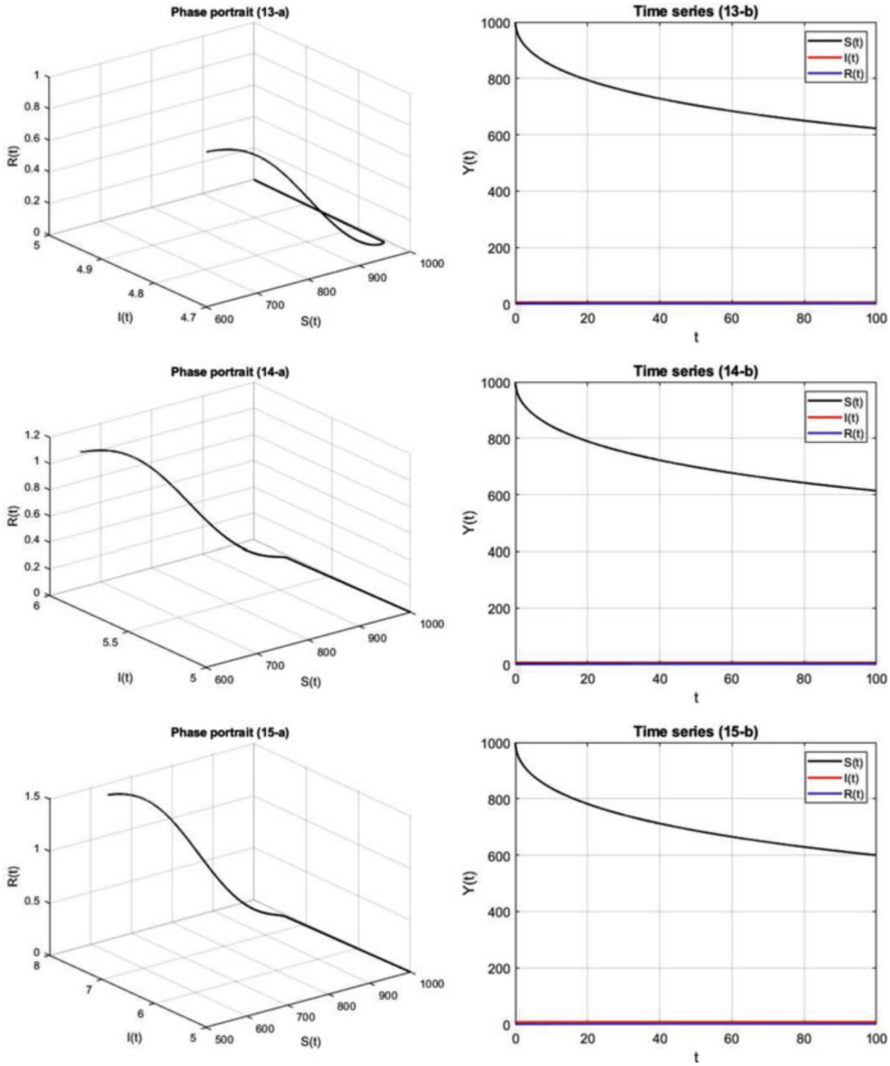

Fig. 5 Phase portraits and time histories of S(t), I(t), and R(t) for system (18) with fractional orders $\alpha_1 = 0.5$, $\alpha_2 = 0.01$, $\alpha_3 = 0.99$, $(13a-13b)\beta = 0.01$, $(14a-14b)\beta = 0.5$, $(15a-15b)\beta = 0.99$

the SIR model has an endemic equilibrium which is globally asymptotically stable. In this case, the disease persists in the population. In addition, we provided three different cases to illustrate the numerical simulation of the SIR model in the sense of Caputo. Our results satisfy similar and non-similar fractional orders. For the case of similar orders, i.e. for the same values of α (α = 0.99, 0.8, 0.05), the system showed to be asymptotically stable for $S(t)$ and $I(t)$ until it reached to a fixed point. For the cases of non-similar orders, i.e. for the values of α to be different as we showed in cases II and III, the system behaved roughly the same. We conclude that varying the fractional order α or the strength of infection β does not affect the stability of the SIR model.

Conflict of interest The authors declare that they have no conflict of interest.

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Structural Stability of Nonlinear Elliptic $p(u)$-Laplacian Problem with Robin Type Boundary Condition

Stanislas Ouaro and Noufou Sawadogo

MSC(2010) 35J60, 35D05, 76A05

1 Introduction

In this chapter, we study a nonlinear elliptic problem of the form

$$\begin{cases} b(u) - \text{div} a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u).\eta = -|u|^{r(x,u)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$ and $f \in L^1(\Omega)$.

The operator $\text{div} a(x, u, \nabla u)$ is called $p(u)$-Laplacian. It is more complicated than $p(x)$-Laplacian in terms of nonlinearity. A prototype of this operator is $\text{div}\big(|\nabla u|^{p(u)-2}.\nabla u\big)$. The variable exponent p depends both on the space variable x and on the unknown solution u.

The study of $p(u)$-Laplacian problem was first developed by Andreianov et *al.* (see [2]). The authors established the existence and uniqueness result to the weak solution and the structural stability result in the case of homogeneous Dirichlet boundary condition.

The interest of the study of this kind of problem is due to the fact that they can model various phenomena that arise in the study of elastic mechanic (see [4]), electrorheological fluids (see [16]) or image restoration (see [8]).

S. Ouaro (✉)
LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences Exactes et Appliquées, Université Joseph KI ZERBO, Ouagadougou, Burkina Faso

N. Sawadogo
LAboratoire de Mathématiques et Informatique (LA.M.I), UFR. Sciences et Techniques, Université Nazi Boni, Bobo-Dioulasso, Burkina Faso

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In [2], the authors used a structural stability theory to establish a new existence result to the problem

$$\begin{cases} b(u) - \text{div} a(x, u, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

in the Banach space $W_0^{1,p(.,u(.))}(\Omega)$, under the following assumptions:

(A_1) $f \in L^1(\Omega)$.
(A_2) b is a nondecreasing, surjective and continuous function defined on $\mathbb{R}$ such that $b(0) = 0$.

Problem (2) is adapted into a generalized Leray–Lions framework under the assumption that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with

(A_3) $a(x, z, 0) = 0$ for all $z \in \mathbb{R}$, and a.e. $x \in \Omega$,
(A_4) $\big(a(x, z, \xi) - a(x, z, \eta)\big).(\xi - \eta) > 0$ for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$,

as well as the growth and the coercivity assumptions with variable exponent

(A_5) $\big|a(x, z, \xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big)$

and

(A_6) $a(x, z, \xi).\xi \geq \dfrac{1}{C_2}|\xi|^{p(x,z)}$.

Here, C_1 and C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.

$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p_+ < \infty$ and $p'(x, z) = \dfrac{p(x, z)}{p(x, z) - 1}$ is the conjugate exponent of $p(x, z)$, with

$$p_- := ess \inf_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x, z) \text{ and } p_+ := ess \sup_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x, z).$$

In addition, we assume that

$$p_- > N \text{ and } p \text{ is log-Hölder continuous in } (x, z) \text{ uniformly on } \overline{\Omega} \times [-M, M], \text{ for all } M > 0. \tag{3}$$

The same authors in [2] established the structural stability results of weak solution u_n of the following nonlinear homogeneous Dirichlet boundary value problem:

$$\begin{cases} b(u_n) - \text{div} a_n(x, u_n, \nabla u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a_n(x, z, \xi)$ verifies $(A_3), (A_4), (A_5)$ and (A_6) with variable exponent $p_n(x, z)$ such that

$1 < p_- \leq p_n(.,.) \leq p_+ < +\infty$ and f_n is a sequence of data weakly convergent to f in $L^1(\Omega)$. In this work, instead of the Dirichlet boundary condition $u = 0$, we use the Robin boundary condition $a(x, u, \nabla u).\eta = -|u|^{r(x,u)-2}u$.

Here, $r : \partial\Omega \times \mathbb{R} \to [r_-, r_+]$ is a Carathéodory function, with $2 < r_- \leq r \leq r_+ < \infty$. Therefore, the function $t \mapsto |t|^{r(x,t)-2}t$ is continuous a.e. x on $\partial\Omega$. Furthermore, we make the following hypothesis:

$$(H) \quad t \mapsto |t|^{r(.,t)-2}t \quad \text{is increasing.}$$

Using $(A_1) - (A_6)$, we prove the existence of weak solutions when the data are bounded, thanks to the technic of pseudo-monotone operators. In this chapter, we consider the Robin boundary condition that brings some difficulties to treat the term at the boundary. In order to get our main result, we define a new space that will help us to take into account the boundary condition. This space in the context of variable exponent was for the first time introduced by Ouaro et al. (see [14]). We also establish the existence result of weak solutions for (1) and continuous dependence for weak solutions, with L^1- data, thanks to a priori estimates, the Poincaré–Wirtinger inequality with constant exponent p_- and the Young measure associated with a weakly convergence method of sequence of gradients of solution (see [9, 11]).

The remaining part of the chapter is the following: in Sect. 2, we introduce some preliminary results. In Sect. 3, we prove the existence and uniqueness result of the weak solution, when the data are in $L^1(\Omega)$. In Sect. 4, we study the continuous dependence for weak solutions.

2 Preliminary

- We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k \\ k sign_0(s) & \text{if } |s| > k \end{cases}, \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties:

$$T_k(-s) = -T_k(s), \ |T_k(s)| = \min\{|s|, k\},$$

$$\lim_{k \to +\infty} T_k(s) = s \quad \text{and} \quad \lim_{k \to 0} \frac{1}{k} T_k(s) = sign_0(s).$$

- We also need to truncate vector-valued function with the help of the mapping

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad h_m(\lambda) = \begin{cases} \lambda, & \text{if } |\lambda| \leq m \\ m\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda| > m, \end{cases} \quad \text{where } m > 0.$$

As the exponent appearing in (1) depends on x and $u(x)$, taking into account the boundary condition, we must work in Lebesgue and Sobolev spaces with variable exponent which are $L^{s(.)}(\partial\Omega)$ and $W^{1,\pi(.)}(\Omega)$, where $s(.) = r(., u(.))$ and $\pi(.) = p(., u(.))$. For the study of problem (1), we need the Sobolev spaces $W^{1,\pi(.)}(\Omega)$.

Definition 2.1 Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function.

- $L^{\pi(.)}(\Omega)$ is the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the modular

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < +\infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(.)}(\Omega)} := \inf\left\{\lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.
- $W^{1,\pi(.)}(\Omega)$ is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$. The space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,\pi(.)}(\Omega)} := ||u||_{L^{\pi(.)}(\Omega)} + ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

When $1 < p_- \leq \pi(.) \leq p_+ < +\infty$, all the above spaces are separable and reflexive Banach spaces.

We denote $\pi_n(x) := p(x, u_n(x))$ for all $x \in \Omega$ and $s_n(x) := r(x, u_n(x))$ for all $x \in \partial\Omega$.

Proposition 2.1 (See [1], Proposition 2.3) *For all measurable functions $\pi : \Omega \to [p_-, p_+]$, the following properties hold:*

(i) $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.
(ii) $L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder type inequality holds:

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \quad \left|\int_\Omega fg dx\right| \leq 2||f||_{L^{\pi(.)}(\Omega)}||g||_{L^{\pi'(.)}(\Omega)}.$$

(iii) *One has* $\rho_{\pi(.)}(f) = 1$ *if and only if* $||f||_{L^{\pi(.)}(\Omega)} = 1$*; furthermore,*
if $\rho_{\pi(.)}(f) \leq 1$*, then* $||f||^{p+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p-}_{L^{\pi(.)}(\Omega)}$*;*
if $\rho_{\pi(.)}(f) \geq 1$*, then* $||f||^{p-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p+}_{L^{\pi(.)}(\Omega)}$.

In particular, if $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^{\pi(.)}(\Omega)$*, then* $||f_n||_{L^{\pi(.)}(\Omega)}$ *tends to zero (respectively, to infinity) if and only if* $\rho_{\pi(.)}(f_n)$ *tends to zero (respectively, to infinity), as* $n \to +\infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$, we introduce the following notation:

$$\rho_{1,\pi(.)}(f) = \int_\Omega |f|^{\pi(.)}dx + \int_\Omega |\nabla f|^{\pi(.)}dx.$$

Replacing $p(x)$ by $\pi(x)$ in [7], Proposition 2.2, we get the following result that is fundamental in this chapter (see [18, 19]).

Proposition 2.2 *If* $f \in W^{1,\pi(.)}(\Omega)$*, the following properties hold:*

(i) $||f||_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow ||f||^{p-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p+}_{W^{1,\pi(.)}(\Omega)}$*;*
(ii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow ||f||^{p+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p-}_{W^{1,\pi(.)}(\Omega)}$*;*
(iii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1$*(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,\pi(.)}(f) < 1$*(respectively* $= 1; > 1$*).*

The following lemma shows that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.1 *If* $u \in W^{1,\pi(.)}(\Omega)$*, then* $T_k(u) \in W^{1,\pi(.)}(\Omega)$.

Now, we give some embedding results.

Proposition 2.3 (See [1], Proposition 2.4) *Assume that* $\pi : \Omega \to [p_-, p_+]$ *has a representative that can be extended to a continuous function up to the boundary* $\partial\Omega$ *and satisfying the log-Hölder continuity assumption:*

$$\exists L > 0, \quad \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{4}$$

(i) *Then,* $C^\infty(\overline{\Omega})$ *is dense in* $W^{1,\pi(.)}(\Omega)$.
(ii) $W^{1,\pi(.)}(\Omega)$ *is embedded into* $L^{\pi^*(.)}(\Omega)$*, where* $\pi^*(.)$ *is the Sobolev embedding exponent defined as in (5) below. If* q *is a measurable variable exponent such that ess* $\inf_{x\in\Omega}(\pi^*(.) - q(.)) > 0$*, then the embedding of* $W^{1,\pi(.)}(\Omega)$ *into* $L^{q(.)}(\Omega)$ *is compact.*

For a given $\pi(.)$*, a function taking values in* $[p_-, p_+]$*,* $\pi^*(.)$ *denotes the optimal Sobolev embedding defined for any* $x \in \Omega$ *by*

$$\pi^*(x) = \begin{cases} \dfrac{N\pi(x)}{N-\pi(x)} & \textit{if} \quad \pi(x) < N \\ \textit{any real value} & \textit{if} \quad \pi(x) = N \\ +\infty & \textit{if} \quad \pi(x) > N. \end{cases} \tag{5}$$

Put

$$\pi^\partial(x) := \big(\pi(x)\big)^\partial := \begin{cases} \dfrac{(N-1)\pi(x)}{N-\pi(x)} & \text{if} \quad \pi(x) < N \\ +\infty & \text{if} \quad \pi(x) \geq N. \end{cases} \tag{6}$$

Proposition 2.4 (See [14], Proposition 2.3) *Let $\pi(.) \in C(\overline{\Omega})$ and $p_->1$. If $q(.) \in C(\partial\Omega)$ satisfies the condition:*

$$1 \leq q(x) < \pi^\partial(x), \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular, there is a compact embedding

$$W^{1,\pi(.)}(\Omega) \hookrightarrow L^{\pi(.)}(\partial\Omega).$$

Young Measures and Nonlinear Weak-* Convergence

Throughout the chapter, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^d$.

In the following theorem, we gather the results of Ball [6], Pedregal [15] and Hungerbühler [12], which is needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us underline that the results of (ii), (iii), expressed in terms of the convergence in measure, are very convenient for the applications we have in mind.

Theorem 2.1

(i) Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$-valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$), weakly measurable in x with respect to the Lebesgue measure on Ω, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have

$$\lim_{k\to+\infty}\int_{\Omega}F(x,v_{n_k})dx=\int_{\Omega}\int_{\mathbb{R}^d}F(x,\lambda)d\nu_x(\lambda)dx, \tag{7}$$

whenever the sequence $(F(.,v_n(.)))_{n\in\mathbb{N}}$ *is equi-integrable on* Ω.

In particular,

$$v(x):=\int_{\mathbb{R}^d}\lambda d\nu_x(\lambda) \tag{8}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ *in* $L^1(\Omega)$.

The family $(\nu_x)_{x\in\Omega}$ *is called the Young measure generated by the subsequence* $(v_{n_k})_{k\in\mathbb{N}}$.

(ii) If Ω *is of finite measure, and* $(\nu_x)_{x\in\Omega}$ *is the Young measure generated by a sequence* $(v_n)_{n\in\mathbb{N}}$*, then* $\nu_x=\delta_{v(x)}$ *for a.e.* $x\in\Omega\Leftrightarrow v_n$ *converges in measure on* Ω *to* v *as* $n\to\infty$.

(iii) If Ω *is of finite measure,* $(u_n)_{n\in\mathbb{N}}$ *generates a Dirac Young measure* $(\delta_{u(x)})_{x\in\Omega}$ *on* $\mathbb{R}^{d_1}$*, and* $(v_n)_{n\in\mathbb{N}}$ *generates a Young measure* $(\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_2}$*, then the sequence* $(u_n,v_n)_{n\in\mathbb{N}}$ *generates the Young measure* $(\delta_{u(x)}\otimes\nu_x)_{x\in\Omega}$ *on* $\mathbb{R}^{d_1+d_2}$.

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [10], we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges, and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact ([1], Theorem 2.10 (i)). It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on Ω.

For the proof of the following lemma, See [1], Theorem 3.11 and [2], Step 2-proof of Theorem 2.6.

Lemma 2.2 *Assume that* $(u_n)_{n\in\mathbb{N}}$ *converges a.e. on* Ω *to some function* u*; then,*

$$\begin{gathered}|p(x,u_n(x))-p(x,u(x))| \ \text{converges in measure to } 0 \ \text{on } \Omega,\\ \text{and for all bounded subset } K \text{ of } \mathbb{R}^N,\\ \sup_{\xi\in K}|a(x,u_n(x),\xi)-a(x,u(x),\xi)| \text{ converges in measure to } 0 \text{ on } \Omega.\end{gathered} \tag{9}$$

We recall some notations.

For any $u\in W^{1,\pi(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. We will identify at the boundary u and $\tau(u)$.

3 Weak Solution

Let $f_n = T_n(f)$. Then, $(f_n)_{n\in\mathbb{N}^*}$ is bounded. Moreover, $(f_n)_{n\in\mathbb{N}^*}$ strongly converges to f in $L^1(\Omega)$ such that $||f_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}$. We consider the following problem:

$$\begin{cases} T_n(b(u_n)) - \mathrm{div} a(x, u_n, \nabla u_n) - \frac{1}{n}\triangle_{p_+} u_n + \frac{1}{n}|u_n|^{p_+-2}u_n = f_n & \text{in} \quad \Omega \\ \big(a(x, u_n, \nabla u_n) + \frac{1}{n}|\nabla u_n|^{p_+-2}\nabla u_n\big).\eta = T_n\big(-|u_n|^{s_n(.)-2}u_n\big) & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

where

$$-\triangle_{p_+} u_n := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_+-2}\frac{\partial u_n}{\partial x_i}\right).$$

In this part, we prove that the problem (10) admits at least one weak solution u_n.

We define the following reflexive space:

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u, v) \in E : \quad v = \tau(u)\}.$$

In the sequel, we will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$.

Theorem 3.1 *There exists at least one weak solution u_n for the problem (10) in the sense that $u_n \in X_0$ and for all $v \in X_0$,*

$$\begin{aligned} \int_\Omega T_n(b(u_n))v dx + \int_\Omega a(x, u_n, \nabla u_n)\nabla v dx + \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)v d\sigma \\ + \frac{1}{n}\int_\Omega \big[|u_n|^{p_+-2}u_n v + |\nabla u_n|^{p_+-2}\nabla u_n \nabla v\big]dx \\ = \int_\Omega f_n v dx. \end{aligned} \tag{11}$$

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1 (see [13], Remark 2.12**)** *Let V be a separable reflexive Banach space and A, M: $V \to V'$ such that*

(i) A is a pseudo-monotone operator;
(ii) M is a bounded hemicontinuous and monotone operator;

then, $A + M$ is pseudo-monotone.

Lemma 3.2 (See [17], Corollary 2.2) *If an operator $\mathcal{A}$ is of type (M), bounded and coercive on a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

Let

$$< A_n u, v >=< Au, v > + < G_n u, v >,$$

where

$$< Au, v >= \int_\Omega a(x, u, \nabla u).\nabla v dx, \quad < G_n u, v >= \frac{1}{n}\int_\Omega |\nabla u|^{p_+-2}\nabla u \nabla v dx$$

and

$$< B_n u, v >= \int_\Omega T_n(b(u))v dx + \int_{\partial\Omega} T_n(|u|^{s(.)-2}u)v d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+-2}uv dx,$$

with $u, v \in X_0$. Set $C_n = A_n + B_n$. The proof of Theorem 3.1 is done in three steps.

Step 1: C_n is Bounded
By using the Hölder type inequality and (A_5) with constant exponent p_+, we deduce that A is bounded. Moreover, using the same argument as in [5], Proof of Lemma 4.2, we prove that $G_n + B_n$ is bounded.Therefore, C_n is bounded.

Step 2: C_n is of Type (M)
Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u \text{ in } X_0 \\ C_n u_k \rightharpoonup \chi \text{ in } X_0' \\ \limsup_{k\to\infty} < C_n u_k, u_k >=< \chi, u > . \end{cases}$$

We will prove that $\chi = C_n u$.

As $T_n(b(u_k))u_k \geq 0$, $T_n\big(|u_k|^{s_k(.)-2}u_k\big)u_k = T_n\big(|u_k|^{r(.,u_k(.))-2}u_k\big)u_k \geq 0$ and $|u_k|^{p_+} \geq 0$, by Fatou's lemma, we deduce that

$$\liminf_{k\to\infty}\left(\int_\Omega T_n(b(u_k))u_k dx + \int_{\partial\Omega} T_n\big(|u_k|^{s_k(.)-2}u_k\big)u_k d\sigma + \frac{1}{n}\int_\Omega |u_k|^{p_+}dx\right)$$
$$\geq \int_\Omega T_n(b(u))u dx + \int_{\partial\Omega} T_n\big(|u|^{s(.)-2}u\big)u d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+}dx.$$

On the other hand, thanks to Lebesgue's dominated convergence theorem and the fact that $|u_k|^{p_+-2}u_k \rightharpoonup |u|^{p_+-2}u$ in $L^{p'_+}(\Omega)$, we have

$$\lim_{k\to\infty}\left(\int_\Omega T_n(b(u_k))v dx + \int_{\partial\Omega} T_n\big(|u_k|^{s_k(.)-2}u_k\big)v d\sigma + \frac{1}{n}\int_\Omega |u_k|^{p_+-2}u_k v dx\right)$$
$$= \int_\Omega T_n(b(u))v dx + \int_{\partial\Omega} T_n\big(|u|^{s(.)-2}u\big)v d\sigma + \frac{1}{n}\int_\Omega |u|^{p_+-2}u v dx,$$

for any $v \in X_0$. Since $t \mapsto |t|^{r(x,t)-2}t$ is continuous a.e. x on $\partial\Omega$ and $u_k \to u$ a.e. on $\partial\Omega$, $T_n(|u_k|^{s_k(.)-2}u_k) \to T_n(|u|^{s(.)-2}u)$ a.e. on $\partial\Omega$. Therefore, for k goes to ∞,

$$B_n u_k \rightharpoonup T_n(b(u)) + T_n\big(|u|^{s(.)-2}u\big) + \frac{1}{n}|u|^{p_+-2}u \text{ in } X'_0.$$

Thus, it follows that

$$A_n u_k \rightharpoonup \chi - \left(T_n(b(u)) + T_n\big(|u|^{s(.)-2}u\big) + \frac{1}{n}|u|^{p_+-2}u\right) \text{ in } X'_0, \text{ as } k \to \infty.$$

It remains to prove that A_n is of type (M). For this, we first show that G_n is bounded monotone and hemicontinuous.

From Step 1, G_n is bounded. Let us prove that G_n is monotone. For all $u, v \in W^{1,p_+}(\Omega)$, we have

$$< G_n u - G_n v, u - v >= \frac{1}{n}\int_\Omega \big(|\nabla u|^{p_+-2}\nabla u - |\nabla v|^{p_+-2}\nabla v\big)(\nabla u - \nabla v\big)dx \geq 0,$$

since $\xi \mapsto |\xi|^{p_+-2}\xi$ is increasing for $p_+ > 2$.

Moreover, G_n is hemicontinuous. Indeed, let $f : t \in \mathbb{R} \mapsto f(t) =< G_n(u + tv), v >$ and $t, t_0 \in \mathbb{R}$ such that $t \to t_0$. Let us set $w = u + tv \in W^{1,p_+}(\Omega)$ and $w_0 = u + t_0 v \in W^{1,p_+}(\Omega)$. Then,

$$\|w - w_0\|_{W^{1,p_+}(\Omega)} = \|(t - t_0)v\|_{W^{1,p_+}(\Omega)} = |t - t_0|\|v\|_{W^{1,p_+}(\Omega)} \to 0.$$

So, $w \to w_0$ in $W^{1,p_+}(\Omega)$, as $t \to t_0$, which implies that $\nabla w \to \nabla w_0$ in $L^{p_+}(\Omega)$, and we infer that

$$|\nabla w|^{p_+-2}\nabla w \to |\nabla w_0|^{p_+-2}\nabla w_0 \text{ in } L^{p'_+}(\Omega) \text{ as } t \to t_0.$$

Therefore,

$$\begin{aligned}|f(t) - f(t_0)| &= | < G_n(u + tv), v > - < G_n(u + t_0 v), v > | \\ &\leq \frac{1}{n}\int_\Omega \big||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\big||\nabla v| dx \\ &\leq \||\nabla w|^{p_+-2}\nabla w - |\nabla w_0|^{p_+-2}\nabla w_0\|_{L^{p'_+}(\Omega)}\|\nabla v\|_{L^{p_+}(\Omega)} \to 0.\end{aligned}$$

Then, we deduce that f is continuous, namely the operator G_n is hemicontinuous.

Now, we are going to prove that A is pseudo-monotone.

Let us set

$$a_1(u, v, w) = \int_\Omega a(x, u, \nabla v)\nabla w dx.$$

Then, $w \mapsto a_1(u, v, w)$ is continuous on $W^{1,p_+}(\Omega)$, thus

$$a_1(u, v, w) = \big\langle A(u, v), w\big\rangle, \quad A(u, v) \in (W^{1,p_+}(\Omega))',$$

and verify

$$A(u, u) = Au.$$

Let us Prove That A Is of Type of Calculus of Variation

- As $A(u, .)$ is bounded, we prove that $v \mapsto A(u, v)$ is hemicontinuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$.

  Since $a(x, u, \nabla(v_1 + \lambda v_2) \to a(x, u, \nabla v_1)$ in $L^{p'_+}(\Omega)$ as $\lambda \to 0$ and $u, v_1, v_2 \in W^{1,p_+}(\Omega)$, then $a_1(u, v_1 + \lambda v_2, w) \to a_1(u, v_1, w)$ as $\lambda \to 0$.

  In the same manner, we prove that $u \mapsto A(u, v)$ is hemicontinuous from $W^{1,p_+}(\Omega) \to (W^{1,p_+}(\Omega))'$.

  Moreover, for all $u, v \in W^{1,p_+}(\Omega)$, we have

$$\begin{aligned}
&< A(u, u) - A(u, v), u - v > \\
&=< A(u, u), u - v > - < A(u, v), u - v > \\
&= a_1(u, u, u - v) - a_1(u, v, u - v) \\
&= \int_\Omega a(x, u, \nabla u)\nabla(u - v)dx - \int_\Omega a(x, u, \nabla v)\nabla(u - v)dx \\
&= \int_\Omega \big(a(x, u, \nabla u) - a(x, u, \nabla v)\big)\nabla(u - v)dx \geq 0.
\end{aligned}$$

- Let us suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $< A(u_k, u_k) - A(u_k, u), u_k - u > \to 0$. We prove that

$$\forall v \in \ W^{1,p_+}(\Omega), \quad A(u_k, v) \rightharpoonup A(u, v) \text{ in } (W^{1,p_+}(\Omega))'.$$

  Let us set

$$\int_\Omega F_k dx = \big\langle A(u_k, u_k) - A(u_k, u), u_k - u\big\rangle.$$

As $u_k \rightharpoonup u$, we have

$$a(x, u_k, \nabla v) \rightharpoonup a(x, u, \nabla v) \text{ in } L^{p'_+}(\Omega)$$

(see [13], Lemma 2.2 with $m = 1$). Therefore, $A(u_k, v) \rightharpoonup A(u, v)$ in $(W^{1,p_+}(\Omega))'$.

- Now, we suppose that $u_k \rightharpoonup u$ in $W^{1,p_+}(\Omega)$ and $A(u_k, v) \rightharpoonup \Theta$ in $(W^{1,p_+}(\Omega))'$. We prove that

$$\langle A(u_k, v), u_k \rangle \to \langle \Theta, u \rangle.$$

Then, by using [13], Lemma 2.1, we obtain that $a(x, u_k, \nabla v) \to a(x, u, \nabla v)$ in $L^{p'_+}(\Omega)$ and thus $a_1(u_k, v, u_k) \to a_1(u, v, u)$.

Therefore,

$$< A(u_k, v), u_k >= a_1(u_k, v, u_k) \to < A(u, v), u > \text{ and } \Theta = A(u, v).$$

Hence, A is of Calculus of variation type. Finally, by using [13], Proposition 2.6, we deduce that A is pseudo-monotone.

Therefore, it follows from Lemma 3.1 that $A_n = A + G_n$ is pseudo-monotone. So, the operator A_n is of type (M) (see [13], Proposition 2.5), and we immediately have

$$Au + G_n u = \chi - \left(T_n(b(u)) + T_n\left(|u|^{s(.)-2}u\right) + \frac{1}{n}|u|^{p_+-2}u\right).$$

Therefore, we obtain $C_n u = \chi$.

For more understanding regarding operator of "type (M)," "calculus of variation type" and "pseudo-monotone", see [13, 17].

Step 3: C_n Is Coercive

Using (A_6) with constant exponent, we get

$$\begin{aligned} < C_n u, u > &= \int_\Omega a(x, u, \nabla u).\nabla u dx + \int_\Omega T_n(b(u))u dx \\ &+ \int_{\partial\Omega} T_n\left(|u|^{s(.)-2}u\right)u dx + \frac{1}{n}\int_\Omega \left[|u|^{p_+} + |\nabla u|^{p_+}\right]dx \\ &\geq \frac{1}{C_2}\int_\Omega |\nabla u|^{p_+}dx + \frac{1}{n}\int_\Omega |u|^{p_+}dx \\ &\geq C_3||u||^{p_+}_{W^{1,p_+}(\Omega)}, \quad \text{where } C_3 = \min\left\{\frac{1}{C_2}, \frac{1}{n}\right\}. \end{aligned}$$

We deduce that

$$\frac{< C_n u, u >}{||u||_{W^{1,p_+}(\Omega)}} \to \infty \quad \text{as} \quad ||u||_{W^{1,p_+}(\Omega)} \to \infty.$$

Hence, C_n is coercive. Then, according to Lemma 3.2, C_n is surjective.

Let $F_n = T_n(f) \in X_0'$; then, there exists at least one solution $u_n \in X_0$ of the problem

$$< C_n u_n, v >=< F_n, v >, \quad \text{for all } \; v \in X_0.$$

Therefore, u_n is a weak solution of the problem (10). This ends the proof of Theorem 3.1.

Remark 3.1 If u_n is a weak solution of the problem (10), then $u_n \in W^{1,\pi_n(.)}(\Omega)$, since $W^{1,p_+}(\Omega) \hookrightarrow W^{1,\pi_n(.)}(\Omega)$ continuously. Moreover, $a(x, u_n, \nabla u_n)$ satisfies $(A_3) - (A_6)$ with variable exponent $\pi_n(x) := p(x, u_n(x))$.

Now, we can introduce the notion of weak solution.

Definition 3.1 A measurable function $u \in W^{1,\pi(.)}(\Omega)$ for $\pi(.) = p(., u(.))$ is called a weak solution of the problem (1) if $b(u) \in L^1(\Omega)$, $|u|^{s(.)-2}u \in L^1(\partial\Omega)$ and for all $\varphi \in W^{1,\pi(.)}(\Omega)$,

$$\int_\Omega b(u)\varphi dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi dx + \int_{\partial\Omega} |u|^{s(.)-2}u\varphi d\sigma = \int_\Omega f\varphi dx. \quad (12)$$

These integrals are well defined. For the first integral and the right-hand side of the above equality, we use the fact that $\varphi \in L^\infty(\Omega)$, since $\varphi \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, for $p_- > N$. For the second integral, we use the growth assumption (A_5) to prove that $a(x, u, \nabla u)$ belongs to $L^{\pi'(.)}(\Omega)$. Moreover, as $\varphi \in C(\overline{\Omega})$, then $\varphi \in L^\infty(\partial\Omega)$, so, the third integral is well defined.

One of the main theorems of this chapter is the following.

Theorem 3.2 *Assume that* (A_1)–(A_6) *and (3) hold. Then, there exists at least one weak solution to the problem (1).*

To prove the above theorem, we need the following two lemmas.

Lemma 3.3 *Assume that* (A_2)–(A_6) *hold with variable exponent* $\pi_n(.)$. *If* u_n *is a weak solution of (10), then, we have*

$$\int_\Omega \left|T_n(b(u_n))\right| dx \le ||f||_{L^1(\Omega)}, \quad (13)$$

$$\int_{\partial\Omega} \left|T_n(|u_n|^{s_n(.)-2}u_n)\right| d\sigma \le ||f||_{L^1(\Omega)} \quad (14)$$

and

$$\|u_n\|_{W^{1,p_-}(\Omega)} \le const\big(p_-, \Omega, f\big). \quad (15)$$

Proof of Lemma 3.3 By taking $v = T_k(u_n)$, for all $k > 0$, in the weak formulation (11), we obtain

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx + \int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n)dx \\ + \int_{\partial\Omega} T_n\big(|u_n|^{s_n(.)-2}u_n\big)T_k(u_n)d\sigma \\ + \frac{1}{n}\int_\Omega \big[|u_n|^{p_+-2}u_nT_k(u_n) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n)\big]dx \\ = \int_\Omega f_nT_k(u_n)dx. \end{aligned} \tag{16}$$

Since all the terms of the left-hand side of (16) are nonnegative, we deduce that

$$\begin{aligned}\int_\Omega T_n(b(u_n))T_k(u_n)dx &\le \int_\Omega f_nT_k(u_n)dx \\ &\le k||f||_{L^1(\Omega)}\end{aligned} \tag{17}$$

and

$$\begin{aligned}\int_{\partial\Omega} T_n\big(|u_n|^{s_n(.)-2}u_n\big)T_k(u_n)d\sigma &\le \int_\Omega f_nT_k(u_n)dx \\ &\le k||f||_{L^1(\Omega)}.\end{aligned} \tag{18}$$

Dividing (17) and (18) by k and letting k goes to 0, we have

$$\int_\Omega T_n(b(u_n))sign_0(u_n)dx \le ||f||_{L^1(\Omega)}$$

and

$$\int_{\partial\Omega} T_n\big(|u_n|^{s_n(.)-2}u_n\big)sign_0(u_n)d\sigma \le ||f||_{L^1(\Omega)}.$$

Therefore,

$$\int_\Omega |T_n(b(u_n))|dx \le ||f||_{L^1(\Omega)} \tag{19}$$

and

$$\int_{\partial\Omega} \big|T_n(|u_n|^{s_n(.)-2}u_n)\big|d\sigma \le ||f||_{L^1(\Omega)}. \tag{20}$$

From the relation (19), the sequence $(T_n(b(u_n)))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. Thus, we deduce that $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. As b is continuous, nondecreasing and surjective; then, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C_4 such that

$$\int_\Omega |u_n| dx \leq C_4.$$

Hence,

$$\tilde{u}_n := \frac{1}{meas(\Omega)} \int_\Omega u_n dx \leq const(\Omega).$$

Moreover, from the Poincaré–Wirtinger inequality:

$$\int_\Omega |u_n - \tilde{u}_n|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_n|^{p_-} dx,$$

where $\tilde{u}_n$ is given by the above inequality, we deduce that

$$\int_\Omega |u_n|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_n|^{p_-} dx + \left| \int_\Omega u_n dx \right|^{p_-}.$$

Thus,

$$\int_\Omega |u_n|^{p_-} dx \leq C_5 \int_\Omega |\nabla u_n|^{p_-} dx + C_6, \tag{21}$$

where $C_5 = cons(p_-, \Omega)$ and $C_6 = const(p_-, C_4, \Omega)$ are the positive constants.

Furthermore, as $W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$, there exists a positive constant C_7 such that

$$\|u_n\|^{p_-}_{L^\infty(\Omega)} \leq C_7 \|u_n\|^{p_-}_{W^{1,p_-}(\Omega)}. \tag{22}$$

Using (A_6) with variable exponent $p(x, u_n(x))$ on $a(x, u_n, \nabla u_n)$ and Theorem 3.1, the sequence u_n satisfies

$$\begin{aligned}\int_\Omega T_n(b(u_n)) u_n dx &+ \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2} u_n) u_n d\sigma \\ &+ \frac{1}{C_2} \int_\Omega |\nabla u_n|^{\pi_n(.)} dx + \frac{1}{n}\left(\int_\Omega |\nabla u_n|^{p_+} dx + \int_\Omega |u_n|^{p_+} dx \right) \\ &\leq \int_\Omega f_n u_n dx. \end{aligned} \tag{23}$$

Applying Young's inequality on the right-hand side of (23) and using (22), we get

$$\begin{aligned}\int_\Omega f_n u_n dx &\le \int_\Omega |f||u_n|dx \\ &\le \|f\|_{L^1(\Omega)}\|u_n\|_{L^\infty(\Omega)} \\ &= \left(\frac{2C_2C_7(C_5+1)}{p_-}\right)^{\frac{1}{p_-}}\|f\|_{L^1(\Omega)}.\left(\frac{p_-}{2C_2C_7(C_5+1)}\right)^{\frac{1}{p_-}}\|u_n\|_{L^\infty(\Omega)} \\ &\le \frac{1}{p'_-}\left(\frac{2C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)}+\frac{1}{p_-}\frac{p_-}{2C_2C_7(C_5+1)}\|u_n\|^{p_-}_{L^\infty(\Omega)} \\ &\le \frac{1}{p'_-}\left(\frac{2C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)}+\frac{1}{2C_2(C_5+1)}\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)}.\end{aligned} \tag{24}$$

Moreover, as $p_- < \pi_n(.)$, we have

$$\int_\Omega |\nabla u_n|^{p_-}dx \le meas(\Omega) + \int_\Omega |\nabla u_n|^{\pi_n(.)}dx. \tag{25}$$

Combining (21) and (25), we get

$$\int_\Omega |u_n|^{p_-}dx \le C_5 meas(\Omega) + C_5\int_\Omega |\nabla u_n|^{\pi_n(.)}dx + C_6.$$

We infer from the above inequality and (25) that

$$\begin{aligned}\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_n|^{p_-}+|\nabla u_n|^{p_-}\right]dx \\ &\le (C_5+1)meas(\Omega) + C_6 + (C_5+1)\int_\Omega |\nabla u_n|^{\pi_n(.)}dx.\end{aligned} \tag{26}$$

Furthermore, using (24) and (26), we obtain

$$\begin{aligned}\int_\Omega f_n u_n dx \le \frac{1}{p'_-}\left(\frac{2C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)} + \frac{meas(\Omega)}{2C_2} \\ + \frac{C_6}{2C_2(C_5+1)} + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{\pi_n(.)}dx.\end{aligned} \tag{27}$$

Combining (23) and (27), we get

$$\int_{\Omega} T_n(b(u_n))u_n dx + \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)u_n d\sigma + \frac{1}{2C_2}\int_{\Omega} |\nabla u_n|^{\pi_n(.)} dx + \frac{1}{n}||u_n||^{p_+}_{W^{1,p_+}(\Omega)} \leq C_8, \tag{28}$$

where

$$C_8 = \frac{1}{p'_-}\left(\frac{2C_2C_7(C_5+1)}{p_-}\right)^{\frac{p'_-}{p_-}} \|f\|^{p'_-}_{L^1(\Omega)} + \frac{meas(\Omega)}{2C_2} + \frac{C_6}{2C_2(C_5+1)}.$$

Thus, we deduce from (28) that

$$\int_{\Omega} |\nabla u_n|^{\pi_n(.)} dx \leq C_9. \tag{29}$$

Now, using (26) and (29), we infer

$$\|u_n\|_{W^{1,p_-}(\Omega)} \leq const\big(p_-, \Omega, f\big). \tag{30}$$

□

Lemma 3.4 *$(u_n)_{n\in\mathbb{N}^*}$ converges a.e. in $\partial\Omega$ to v.*

Proof Since $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_n)_{n\in\mathbb{N}^*}$, it converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.

We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. Obviously, $W^{1,p_-}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ because $p_- > 1$. Therefore, $u_n \to u$ in $L^1(\partial\Omega)$ and a.e. in $\partial\Omega$. Thus, $v = u_{|\partial\Omega}$ has definite meaning. □

The following assertions are based on the Young measure and nonlinear weak-* convergence results (see [6, 12, 15]).

Assertion 1

The sequence $(\nabla u_n)_{n\in\mathbb{N}^*}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \tag{31}$$

Proof As $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_n)_{n\in\mathbb{N}^*}$, u_n converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.

Therefore, ∇u_n weakly converges to ∇u in $L^{p_-}(\Omega)$. Now, we prove that $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable.

$(|\nabla u_n|^{p_-})_{n\in\mathbb{N}}$ being bounded is equi-integrable. As $p_- > 1$, then, for all subset $E \subset \Omega$, we obtain

$$\int_E |\nabla u_n| dx \leq \int_E (1 + |\nabla u_n|^{p_-}) dx.$$

Therefore, for $meas(E)$ small enough, $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, using the representation of weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.1, property (8)), we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda).$$

□

Assertion 2
$|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$, and moreover, $u \in W^{1,\pi(.)}(\Omega)$.

Proof We know that π_n converges in measure to π. Using Theorem 2.1-[(ii),(iii)], $(\pi_n, \nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R} \times \mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi(x)} \otimes \nu_x$. Thus, we can apply the weak convergence property (7) of Theorem 2.1 to the Carathéodory function

$$F_m : (x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0},$$

with $m \in \mathbb{N}$, where h_m is defined in the preliminaries. We have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(.)} d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x(\lambda_0, \lambda)dx \\
&= \int_\Omega \int_{\mathbb{R}\times\mathbb{R}^N} F_m(x, \lambda_0, \lambda) d\mu_x(\lambda_0, \lambda)dx \\
&= \lim_{n\to\infty} \int_\Omega F_m(x, \pi_n(x), \nabla u_n(x))dx \\
&= \lim_{n\to\infty} \int_\Omega |h_m(\nabla u_n)|^{\pi_n(.)} dx \\
&\leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{\pi_n(.)} dx \\
&\leq C_9.
\end{aligned}$$

Since $h_m(\lambda) \to \lambda$, as $m \to \infty$ and as $m \mapsto h_m(\lambda)$ is increasing, using Lebesgue convergence theorem, we deduce from the above inequality that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx \le C_9.$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$.

Now, we prove that $\nabla u \in L^{\pi(.)}(\Omega)$. Using (31), Jensen's inequality and the last inequality, we get

$$\int_\Omega |\nabla u|^{\pi(.)}dx = \int_\Omega \left| \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda)\right|^{\pi(.)} dx \le \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx < \infty.$$

Thus, $\nabla u \in L^{\pi(.)}(\Omega)$. Moreover, $u \in L^{\pi(.)}(\Omega)$. Indeed, $u \in W^{1,p_-}(\Omega) \subset L^\infty(\Omega) \subset L^{\pi(.)}(\Omega)$ for $p_- > N$. Hence, $u \in W^{1,\pi(.)}(\Omega)$. □

Assertion 3

The sequence Φ_n defined by $\Phi_n := a(x, u_n, \nabla u_n)$ is equi-integrable on Ω.

Proof By using (A_5) with exponent $\pi_n(.)$, we obtain

$$|a(x, u_n, \nabla u_n)|^{\pi_n'(.)} \le C_1(|\nabla u_n|^{\pi_n(.)} + \mathcal{M}(x)).$$

The above inequality give us

$$\begin{aligned} |a(x, u_n, \nabla u_n)| &\le C\Big((1 + |\nabla u_n|^{\pi_n(.)}) + \mathcal{M}(x)\Big)^{\frac{1}{\pi_n'(.)}} \\ &\le C\Big((1 + \mathcal{M}(x))^{\frac{1}{\pi_n'(.)}} + |\nabla u_n|^{\frac{\pi_n(.)}{\pi_n'(.)}}\Big) \\ &\le C(1 + \mathcal{M}(x) + |\nabla u_n|^{\pi_n(.)-1}). \end{aligned}$$

For all set $E \subset \Omega$,

$$\begin{aligned} \int_E |a(x, u_n, \nabla u_n)|dx &\le C \int_E (1 + \mathcal{M}(x))dx \\ &\quad + C_{10}\left\| |\nabla u_n|^{\pi_n(.)-1} \right\|_{L^{\pi_n'(.)}(\Omega)} ||\chi_E||_{L^{\pi_n(.)}(\Omega)}, \end{aligned}$$

where $C_{10} = const(p_-)$. The first term on the right-hand side of the above inequality is small for $meas(E)$ small enough, since $1 + \mathcal{M} \in L^1(\Omega)$.

According to Proposition 2.1, we obtain

$$\begin{aligned} ||\chi_E||_{L^{\pi_n(.)}(\Omega)} &\le \max\left\{\rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_+}}; \rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_-}}\right\} \\ &= \max\left\{(meas(E))^{\frac{1}{p_-}}, (meas(E))^{\frac{1}{p_+}}\right\}. \end{aligned}$$

Analogously,

$$\begin{aligned}
&\left\| |\nabla u_n|^{\pi_n(.)-1} \right\|_{L^{\pi_n'(.)}(\Omega)} \\
&\quad \le \max\left\{ \left(\rho_{\pi_n'(.)}(|\nabla u_n|^{\pi_n(.)-1})^{\frac{1}{(p')_+}}\right), \left(\rho_{\pi_n'(.)}(|\nabla u_n|^{\pi_n(.)-1})^{\frac{1}{(p')_-}}\right) \right\} \\
&\quad = \max\left\{ \left(\int_\Omega |\nabla u_n|^{\pi_n(.)} dx\right)^{\frac{1}{(p')_+}}, \left(\int_\Omega |\nabla u_n|^{\pi_n(.)} dx\right)^{\frac{1}{(p')_-}} \right\}.
\end{aligned}$$

Using (29), $\int_E |a(x, u_n, \nabla u_n)| dx$ becomes small for $meas(E)$ small enough.

Hence, $(\Phi_n)_{n\in\mathbb{N}}$ is equi-integrable. □

Assertion 4

The weak limit Φ of Φ_n (or a subsequence) belongs to $L^{\pi'(.)}(\Omega)$, and we have

$$\Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x(\lambda). \tag{32}$$

Proof Set $\tilde{\Phi}_n = a(x, u(x), \nabla v_n)$ with $\nabla v_n = \nabla u_n \chi_{S_n}$, where $S_n = \{x \in \Omega, \quad |\pi(x) - \pi_n(x)| < \frac{1}{2}\}$.

We prove that $\tilde{\Phi}_n$ is equi-integrable on Ω.

We applied (A_5) with variable exponent $\pi(.)$ on $\tilde{\Phi}_n$.

Let $E \subset \Omega$, and we have

$$\begin{aligned}
\int_E |a(x, u(x), \nabla v_n)| dx &\le C \int_E (1 + \mathcal{M}(x) + |\nabla v_n|^{\pi(.)-1}) dx \\
&\le C\left(\int_E (1 + \mathcal{M}(x)) dx + \int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1} dx \right).
\end{aligned}$$

The first term on the right-hand side of the last inequality is small for $meas(E)$ small enough.

For all $x \in S_n, \pi(x) < \pi_n(x) + \frac{1}{2}$; thus,

$$\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1} dx \le \int_E \left(1 + |\nabla u_n|^{\pi_n(.)-\frac{1}{2}}\right) dx$$

and

$$\int_\Omega |\nabla u_n|^{(\pi_n(.)-\frac{1}{2})(2\pi_n(.))'} dx = \int_\Omega |\nabla u_n|^{\pi_n(.)} dx < \infty,$$

which is equivalent to saying $|\nabla u_n|^{\pi_n(.)-\frac{1}{2}} \in L^{(2\pi_n(.))'}(\Omega)$. Now, using the Hölder type inequality,

$$\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx \leq \int_E \left(1 + |\nabla u_n|^{\pi_n(.)-\frac{1}{2}}\right)dx$$
$$\leq meas(E) + 2\big|\big|\nabla u_n\big|\big|_{L^{\pi_n(.)}(\Omega)}||\chi_E||_{L^{2\pi_n(.)}(\Omega)}. \quad (33)$$

According to Proposition 2.1,

$$||\chi_E||_{L^{2\pi_n(.)}(\Omega)} \leq \max\left\{\left(\rho_{2\pi_n(.)}\left(\chi_E\right)\right)^{\frac{1}{2p-}}, \left(\rho_{2\pi_n(.)}(\chi_E)\right)^{\frac{1}{2p+}}\right\}$$
$$= \max\left\{\left(meas(E)\right)^{\frac{1}{2p-}}, \left(meas(E)\right)^{\frac{1}{2p+}}\right\}.$$

The right-hand side of (33) is uniformly small for $meas(E)$ small, and the equi-integrability of $\tilde{\Phi}_n$ follows. Therefore, up to a subsequence, $\tilde{\Phi}_n$ weakly converges in $L^1(\Omega)$ to $\tilde{\Phi}$, as $n \to \infty$.

Now, we prove that $\tilde{\Phi} = \Phi$; more precisely, we show that $\tilde{\Phi}_n - \Phi_n$ strongly converges in $L^1(\Omega)$ to 0.

From (29), $\displaystyle\int_\Omega |\nabla u_n|^{\pi_n(.)}dx$ is uniformly bounded, which implies that $\displaystyle\int_\Omega |\nabla u_n|dx$ is finite, since

$$\int_\Omega |\nabla u_n|dx \leq \int_\Omega (1 + |\nabla u_n|^{\pi_n(x)})dx.$$

By Chebyshev's inequality, we have

$$meas(\{|\nabla u_n| > L\}) \leq \frac{\int_\Omega |\nabla u_n|dx}{L}.$$

Therefore, $\sup\limits_{n\in\mathbb{N}} meas(\{|\nabla u_n| > L\})$ tends to 0 for L large enough. Since $\tilde{\Phi}_n - \Phi_n$ is equi-integrable, then for all $\beta > 0$, there exists $\delta = \delta(\beta)$ such that for all $A \subset \Omega$, $meas(A) < \delta$ and $\displaystyle\int_A |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}$.

Therefore, if we choose L large enough, we get $\dfrac{\int_\Omega |\nabla u_n|dx}{L} < \delta$, so $meas(\{|\nabla u_n| > L\}) < \delta$.

Hence,

$$\int_{\{|\nabla u_n|>L\}} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}.$$

By Lemma 2.2, we also have

$$meas\left(\left\{x \in \Omega;\quad \sup_{\lambda \in K} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\}\right) \longrightarrow 0,$$

as $n \to \infty$.

Thus, by the above equi-integrability, for all $\sigma > 0$, there exists $n_0 = n_0(\sigma, L) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{\left\{x\in\Omega;\quad \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda)-a(x,u(x),\lambda)|\geq\sigma\right\}} |\tilde{\Phi}_n - \Phi_n| dx < \frac{\beta}{4}.$$

Using the definition of Φ_n and $\tilde{\Phi}_n$, we have

$$\tilde{\Phi}_n - \Phi_n = a(x, u_n(x), \nabla u_n) - a(x, u(x), \nabla u_n) \ \text{ on } \ S_n.$$

Now, we reason on

$$S_{n,L,\sigma} := \left\{x \in \Omega;\quad \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| < \sigma, |\nabla u_n| \leq L\right\}.$$

We get

$$\begin{aligned}\int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx &\leq \int_{S_{n,L,\sigma}} \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| dx \\ &\leq \sigma meas(\Omega).\end{aligned}$$

We observe that

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n| dx = \int_{S_n\cap S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx + \int_{S_n\setminus S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx$$

and

$$S_n \setminus S_{n,L,\sigma} \subset \left\{x \in \Omega;\quad \sup_{|\lambda|\leq L} |a(x, u_n(x), \lambda) - a(x, u(x), \lambda)| \geq \sigma\right\} \cup \left\{|\nabla u_n| > L\right\}.$$

Consequently, by choosing $\sigma = \sigma(\beta) < \dfrac{\beta}{4meas(\Omega)}$, we get

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n| dx < \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = \frac{3\beta}{4},$$

for all $n \geq n_0(\sigma, L)$. By Lemma 2.2, we also have $meas(\{x \in \Omega, |\pi(x) - \pi_n(x)| \geq \frac{1}{2}\}) \to 0$ for n large enough, which means that $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Thus,

$$\int_{\Omega\setminus S_n} |\tilde{\Phi}_n - \Phi_n| dx = \int_{\Omega\setminus S_n} |\Phi_n| dx \leq \frac{\beta}{4}.$$

Therefore, for all $\beta > 0$, there exists $n_0 = n_0(\beta)$ such that for all $n \geq n_0$, $\int_\Omega |\tilde{\Phi}_n - \Phi_n| dx \leq \beta$.

Hence, $\tilde{\Phi}_n - \Phi_n$ strongly converges to 0 in $L^1(\Omega)$. We prove that

$$\Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x(\lambda) \quad a.e.\ \ x \in \Omega \ \text{ and } \ \Phi \in L^{\pi'(.)}(\Omega).$$

Notice that

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|(1 - \chi_{S_n}) dx = \lim_{n\to+\infty} \int_{\Omega\setminus S_n} |\nabla u_n| dx = 0,$$

since $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable and $meas(\Omega \setminus S_n)$ converges to 0 for n large enough.

Therefore, $(\nabla u_n)_{n\in\mathbb{N}}$ and $\nabla u_n \chi_{S_n}$ converge to the same Young measure $\nu_x(\lambda)$.

Moreover, by applying Theorem 2.1-(i) to the Carathéodory function $F(x, (\lambda_0, \lambda)) := a(x, \lambda_0, \lambda)$, we infer that

$$\tilde{\Phi}(x) = \Phi(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x(\lambda) \ \text{ a.e. } \ x \in \Omega.$$

Using (A_5), it follows that $|a(x, u(x), \lambda)|^{\pi'(.)} \leq C(\mathcal{M}(x) + |\lambda|^{\pi(.)})$. Thus, with Jensen's inequality, it follows that

$$\begin{aligned}
\int_\Omega |\Phi(x)|^{\pi'(.)} dx &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x, u(x), \lambda) d\nu_x(\lambda) \right|^{\pi'(.)} dx \\
&\leq \int_{\Omega\times\mathbb{R}^N} \left| a(x, u(x), \lambda) \right|^{\pi'(.)} d\nu_x(\lambda) dx \\
&\leq C \int_{\Omega\times\mathbb{R}^N} \left(\mathcal{M}(x) + |\lambda|^{\pi(.)} \right) d\nu_x(\lambda) dx < \infty.
\end{aligned}$$

Hence, $\Phi \in L^{\pi'(.)}(\Omega)$. □

Assertion 5

$$\int_\Omega \Phi.\nabla u dx \geq \int_{\Omega\times\mathbb{R}^N} a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx. \tag{34}$$

Proof For all $\varphi \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned}\int_\Omega T_n(b(u_n))\varphi dx + \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)\varphi d\sigma + \int_\Omega a(x,u_n,\nabla u_n).\nabla\varphi dx \\ + \frac{1}{n}\int_\Omega \left[|\nabla u_n|^{p_+-2}\nabla u_n \nabla\varphi + |u_n|^{p_+-2}u_n\varphi\right]dx \\ = \int_\Omega f_n\varphi dx.\end{aligned} \tag{35}$$

Letting n go to ∞ in (35), we obtain

$$\int_\Omega b(u)\varphi dx + \int_{\partial\Omega} |u|^{s(.)-2}u\varphi d\sigma + \int_\Omega \Phi.\nabla\varphi dx = \int_\Omega f\varphi dx. \tag{36}$$

Indeed, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in the space $W^{1,p_-}(\Omega)$ and $p_- > N$. Then, by embedding result ($W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$), $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\Omega)$. Thus, as $b(.)$ is continuous, then, $(b(u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\Omega)$. Otherwise, $T_n(b(u_n))$ converges to $b(u)$ a.e. in Ω. Therefore, thanks to Lebesgue's dominated convergence theorem, $T_n(b(u_n))$ converges to $b(u)$ in $L^1(\Omega)$. Moreover, $a(x,u_n,\nabla u_n) = \Phi_n$ weakly converges to Φ and f_n strongly converges to f in $L^1(\Omega)$. Furthermore, from (28), we deduce that

$$\frac{1}{n}\|u_n\|^{p_+}_{W^{1,p_+}(\Omega)} \leq C_8,$$

which implies that the fourth term on left-hand side of (35) goes to 0 for n large enough.

We are now interested to the second term on the left-hand side of (35).

We know that $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p_- > N$. Therefore, $(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\partial\Omega)$. We recall that $t \mapsto |t|^{r(.,t)-2}t$ is continuous a.e. in $\partial\Omega$. Thus, $(|u_n|^{r(.,u_n)-2}u_n)_{n\in\mathbb{N}^*} := (|u_n|^{s_n(.)-2}u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\partial\Omega)$. But

$$\left|T_n(|u_n|^{s_n(.)-2}u_n)\right| \leq \left||u_n|^{s_n(.)-2}u_n\right|;$$

so, the sequence $(T_n(|u_n|^{s_n(.)-2}u_n))_{n\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\partial\Omega)$. Hence, there exists a positive constant C_{11} such that

$$\left|T_n(|u_n|^{s_n(.)-2}u_n)\varphi\right| \leq C_{11}|\varphi| \text{ a.e. in } \partial\Omega.$$

Moreover, $T_n(|u_n|^{s_n(.)-2}u_n)\varphi \to |u|^{s(.)-2}u\varphi$ a.e. in $\partial\Omega$, which implies that

$$\lim_{n\to\infty}\int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)\varphi d\sigma = \int_{\partial\Omega}|u|^{s(.)-2}u\varphi d\sigma,$$

thanks to Lebesgue's dominated convergence theorem. On the other hand, by the density argument, we can replace φ with u_n in (35) to get

$$\begin{aligned}\int_\Omega T_n(b(u_n))u_n dx + \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)u_n d\sigma + \int_\Omega a(x,u_n,\nabla u_n).\nabla u_n dx \\ + \frac{1}{n}\int_\Omega \left[|\nabla u_n|^{p_+} + |u_n|^{p_+}\right]dx \\ = \int_\Omega f_n u_n dx. \qquad (37)\end{aligned}$$

$u \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ and $p(.,.)$ is locally uniformly log-Hölder continuous; then, the exponent $\pi(.)$ verifies (4). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, so we change φ by u in (36) to obtain

$$\int_\Omega b(u)u dx + \int_{\partial\Omega}|u|^{s(.)}d\sigma + \int_\Omega \Phi.\nabla u dx = \int_\Omega f u dx. \qquad (38)$$

The sequence $(T_n(b(u_n))u_n)_{n\in\mathbb{N}}$ is nonnegative. $T_n(b(u_n))u_n$ is also measurable and converges a.e. in Ω to $b(u)u$. By Fatou's lemma, we get

$$\liminf_{n\to\infty}\int_\Omega T_n(b(u_n))u_n dx \geq \int_\Omega b(u)u dx. \qquad (39)$$

In the same manner,

$$\liminf_{n\to\infty}\int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)u_n d\sigma \geq \int_{\partial\Omega}|u|^{s(.)}d\sigma. \qquad (40)$$

Moreover, the sequence $(f_n u_n)_{n\in\mathbb{N}*}$ converges a.e. in Ω to fu and

$$|f_n u_n| \leq |f|||u_n||_{L^\infty(\Omega)}.$$

$(u_n)_{n\in\mathbb{N}^*}$ is also uniformly bounded in $L^\infty(\Omega)$. Applying Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_\Omega f_n u_n dx = \int_\Omega f u dx. \qquad (41)$$

By combining (39), (40) and (41), we have

$$\liminf_{n\to\infty}\int_\Omega f_n u_n dx - \int_\Omega b(u)udx - \int_{\partial\Omega}|u|^{s(.)}d\sigma$$
$$\geq \liminf_{n\to\infty}\int_\Omega \big[f_n u_n - T_n(b(u_n))u_n\big]dx - \int_{\partial\Omega} T_n(|u_n|^{s_n(.)-2}u_n)u_n d\sigma.$$

By using (37), (38), the above inequality and the definition of Φ_n, we get

$$\int_\Omega \Phi.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega \Big(\Phi_n.\nabla u_n + \frac{1}{n}\big[|\nabla u_n|^{p_+} + |u_n|^{p_+}\big]\Big)dx$$
$$\geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx.$$

Hence,

$$\int_\Omega \Phi.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx. \tag{42}$$

By Andreianov et al. [1] Lemma 2.1, $m \mapsto a(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a(x, u_n, \nabla u_n).\nabla u_n$ for m large enough. Then,

$$a(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n) \leq a(x, u_n, \nabla u_n).\nabla u_n.$$

Therefore, by using (42) and Theorem 2.1, we have

$$\int_\Omega \Phi.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega \Phi_n.\nabla u_n dx$$
$$\geq \lim_{n\to\infty}\int_\Omega a(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n)dx$$
$$= \int_{\Omega\times\mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda)d\nu_x(\lambda)dx. \tag{43}$$

Using Lebesgue's dominated convergence theorem in (43), as m goes to ∞, we get (34). □

Assertion 6

The "div-curl" inequality holds.

$$\int_{\Omega\times\mathbb{R}^N}\big(a(x, u(x), \lambda) - a(x, u(x), \nabla u(x))\big)(\lambda - \nabla u(x))d\nu_x(\lambda)dx \leq 0.$$

Proof We have

$$\int_{\Omega\times\mathbb{R}^N}(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x)))(\lambda-\nabla u(x))d\nu_x(\lambda)dx$$
$$=\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx$$
$$-\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\nabla u(x)d\nu_x(\lambda)dx$$
$$-\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\lambda d\nu_x(\lambda)dx$$
$$+\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\nabla u(x)d\nu_x(\lambda)dx$$
$$=\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx$$
$$-\int_{\Omega}\left(\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)\right)\nabla u(x)dx$$
$$-\int_{\Omega}a(x,u(x),\nabla u(x)).\left(\int_{\mathbb{R}^N}\lambda d\nu_x\right)dx$$
$$+\int_{\Omega}a(x,u(x),\nabla u(x)).\nabla u(x)\left(\int_{\mathbb{R}^N}d\nu_x\right)dx$$
$$=\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_{\Omega}\Phi.\nabla u dx\le 0.$$

We pass from the first equality to the second equality by using Fubini's theorem and from the second inequality to the third one by using (31) and the fact that ν_x is a probability measure on $\mathbb{R}^N$. Finally, (32) and (34) give us the desired inequality. □

Assertion 7
$\Phi(x)=a(x,u(x),\nabla u(x))$ a.e. $x\in\Omega$, and ∇u_n converges to ∇u in measure on Ω, as $n\to\infty$.

Proof From Assertion 6 and relation (A_4), we deduce that

$$\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big)\big(\lambda-\nabla u(x)\big)=0\quad a.e.\quad x\in\Omega,\quad \lambda\in\mathbb{R}^N.$$

Thus, $\lambda=\nabla u(x)$ a.e. $x\in\Omega$ w.r.t. ν_x on $\mathbb{R}^N$; therefore, $\nu_x(\nabla u(x))=1$ and $\delta_{\nabla u}=\nu_x$. By using (32), we get

$$\Phi(x)=\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)=a(x,u(x),\nabla u(x))\ \text{ a.e. }\ x\in\Omega.$$

Thus, Theorem 2.1-(ii) shows that ∇u_n converges in measure to ∇u. □

Lemma 3.5 *u is a weak solution of the problem (1).*

Proof We first prove that $b(u) \in L^1(\Omega)$ and $|u|^{s(.)-2}u \in L^1(\partial\Omega)$.

Using Lemma 3.3 and Lemma 3.4, it follows from Fatou's lemma that

$$\int_\Omega |b(u)|dx \leq \liminf_{n\to\infty} \int_\Omega \left|T_n(b(u_n))\right|dx \leq ||f||_{L^1(\Omega)}$$

and

$$\int_{\partial\Omega} \left||u|^{s(.)-2}u\right|d\sigma \leq \liminf_{n\to\infty} \int_{\partial\Omega} \left|T_n(|u_n|^{s_n(.)-2}u_n)\right|d\sigma \leq ||f||_{L^1(\Omega)}.$$

Hence, $b(u) \in L^1(\Omega)$ and $|u|^{s(.)-2}u \in L^1(\partial\Omega)$.

Now, by using Assertion 4 and Assertion 7, and thanks to that the density argument ($C^\infty(\overline{\Omega})$ is dense in the space $W^{1,\pi(.)}(\Omega)$), we can take φ in $W^{1,\pi(.)}(\Omega)$ as test function in (36) to get

$$\int_\Omega b(u)\varphi dx + \int_{\partial\Omega} |u|^{s(.)-2}u\varphi d\sigma + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx = \int_\Omega f\varphi dx.$$

Hence, u is a weak solution of (1) □

Now, we state the uniqueness result of weak solution. This result uses the same arguments as in [2], Theorem 2.8.

Theorem 3.3 *Assume that b is strictly increasing. Assume that (A_3), (A_4), (A_5), (A_6), (H) and (3) hold, and $\mathcal{M}$ is taken constant in (A_5). Moreover,* ***a*** *satisfies that, for all bounded subset K of $\mathbb{R} \times \mathbb{R}^N$, there exists a constant $C(K)$ such that*

$$\begin{aligned} &a.e.\ \ x \in \Omega,\ \ \textit{for all}\ \ (z,\eta),(\tilde{z},\eta) \in K, \\ &|a(x,z,\eta) - a(x,\tilde{z},\eta)| \leq C(K)|z-\tilde{z}|. \end{aligned} \tag{44}$$

Finally, suppose the following regularity property:

$$\begin{aligned} &\textit{for all}\ \ f \in L^\infty(\Omega),\ \ \textit{there exists a weak solution of (1)}, \\ &\textit{which is Lipschitz continuous on}\ \overline{\Omega}. \end{aligned} \tag{45}$$

Then, for all $f \in L^1(\Omega)$, the problem (1) admits a unique weak solution.

Remark 3.2 As in [2], Theorem 2.8, the condition (45) goes back to idea of [3]. Moreover, in Theorem 3.3, the relation (44) is used to obtain the inequality (51) below.

Proof of Theorem 3.3 The existence has already been proved. Now, we show the uniqueness. For more details, see [2], Proof of Theorem 2.8.

Let u be a Lipschitz continuous weak solution of (1) with $f \in L^\infty(\Omega)$ and v be a weak solution in the same sense, with $\hat{f} \in L^1(\Omega)$.

The function $\phi := \dfrac{1}{k}T_k(u - v)$ is an admissible test function in the weak formulation of u and v. Indeed, as Ω is bounded open domain with smooth boundary $\partial\Omega$, the spaces of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. Moreover, ϕ belongs to $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and even $\phi \in L^\infty(\partial\Omega)$. As u is bounded, we have

$$\phi := \frac{1}{k}T_k(u - v) = \frac{1}{k}T_k(u - T_{k+||u||_{L^\infty}}(v)) \tag{46}$$

with

$$|\phi| \leq 1 \ \text{ and } \ \nabla\phi = \frac{1}{k}\nabla(u - v)\chi_{[|u-v|<k]}. \tag{47}$$

Firstly, using the fact that ∇u is bounded and assumption (A_4) of this theorem, we get

$$|a(x, u, \nabla u)| \leq C\big(|\nabla u|^{p(.,u(.))} + 1) \in L^\infty(\Omega).$$

Thus, $\phi \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ is admissible as a test function in the weak formulation for the solution u, which belongs to $W^{1,\infty}(\Omega)$. Secondly, if $v \in W^{1,p(.,v(.))}(\Omega)$, then by (46) and Lemma 2.1 and because $u \in W^{1,p_+}(\Omega) \subset W^{1,p(.,v(.))}(\Omega)$, we obtain $\phi \in W^{1,p(.,v(.))}(\Omega) \cap L^\infty(\Omega)$ as a test function in the weak formulation for the solution v.

Hence, ϕ is necessary an admissible test function for u and v. Thus, with this test function defined in (46) and (47), we have

$$\begin{aligned}\int_\Omega b(u)\frac{1}{k}T_k(u - v)dx + \int_{\partial\Omega} |u|^{r(.,u(.))-2}u\frac{1}{k}T_k(u - v)d\sigma \\ + \frac{1}{k}\int_\Omega a(x, u, \nabla u).\nabla(u - v)\chi_{[0<|u-v|<k]}dx \\ = \int_\Omega f\frac{1}{k}T_k(u - v)dx \end{aligned}\tag{48}$$

and

$$\begin{aligned}\int_\Omega b(v)\frac{1}{k}T_k(u - v)dx + \int_{\partial\Omega} |v|^{r(.,v(.))-2}v\frac{1}{k}T_k(u - v)d\sigma \\ + \frac{1}{k}\int_\Omega a(x, v, \nabla v).\nabla(u - v)\chi_{[0<|u-v|<k]}dx \\ = \int_\Omega \hat{f}\frac{1}{k}T_k(u - v)dx. \end{aligned}\tag{49}$$

Subtracting (48) from (49), we get

$$\begin{aligned}
&\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx \\
&\quad+\int_{\partial\Omega}(|u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v)\frac{1}{k}T_k(u-v)d\sigma \\
&\quad+\frac{1}{k}\int_\Omega \big(a(x,u,\nabla u)-a(x,v,\nabla v)\big).\nabla(u-v)\chi_{[0<|u-v|<k]}dx \\
&\quad=\int_\Omega \big(f-\hat{f}\big)\frac{1}{k}T_k(u-v)dx. \qquad (50)
\end{aligned}$$

Let us denote by I the third term on left-hand side of (50). We know that

$$\begin{aligned}
(a(x,u,\nabla u)-a(x,v,\nabla v))\nabla(u-v)&=(a(x,u,\nabla u)-a(x,v,\nabla u))\nabla(u-v)\\
&\quad+\underbrace{(a(x,v,\nabla u)-a(x,v,\nabla v))\nabla(u-v)}_{\geq 0}.
\end{aligned}$$

We have

$$I=I_k+\int_\Omega \big(a(x,v,\nabla u)-a(x,v,\nabla v)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx,$$

where

$$I_k=\int_\Omega \big(a(x,u,\nabla u)-a(x,v,\nabla u)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx.$$

Let us show that $I_k\to 0$ as $k\to 0$. Since u is bounded, v is also bounded on the set $[0<|u-v|<k]$. Thus,

$$\begin{aligned}
|I_k|&\leq \frac{1}{k}\int_{[0<|u-v|<k]}|a(x,u,\nabla u)-a(x,v,\nabla u)||\nabla u-\nabla v|dx\\
&\leq \frac{1}{k}\int_{[0<|u-v|<k]}C\big(||u||_{L^\infty(\Omega)},||\nabla u||_{L^\infty(\Omega)}\big)|u-v||\nabla u-\nabla v|dx \ \big(\text{by using (44)}\big)\\
&\leq C\big(||u||_{L^\infty(\Omega)},||\nabla u||_{L^\infty(\Omega)}\big)\int_{[0<|u-v|<k]}|\nabla u-\nabla v|dx\to 0,\quad \text{as } k\to 0. \qquad (51)
\end{aligned}$$

Note that $\lim_{k\to 0} meas([0<|u-v|<k])=0$ and $|\nabla u-\nabla v|\in L^1(\Omega)$.

For the first term on the left-hand side of (50), one has

$$\begin{aligned}
\lim_{k\to 0}\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx&=\int_\Omega (b(u)-b(v))sign_0(u-v)dx\\
&=\int_\Omega |b(u)-b(v)|dx. \qquad (52)
\end{aligned}$$

In the same manner,

$$\lim_{k\to 0}\int_{\partial\Omega}(|u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v)\frac{1}{k}T_k(u-v)d\sigma$$
$$=\int_{\partial\Omega}\big(|u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v\big)sign_0(u-v)d\sigma$$
$$=\int_{\partial\Omega}\big||u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v\big|d\sigma \tag{53}$$

and

$$\lim_{k\to 0}\int_{\Omega}\big(f-\hat{f}\big)\frac{1}{k}T_k(u-v)dx=\int_{\Omega}|f-\hat{f}|dx. \tag{54}$$

Finally, one makes k go to 0 in (50), and taking into account inequalities (51), (52), (53) and (54), we get

$$\int_{\Omega}|b(u)-b(v)|dx+\int_{\partial\Omega}\big||u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v\big|d\sigma$$
$$+\lim_{k\to 0}\int_{\Omega}\big(a(x,v,\nabla u)-a(x,v,\nabla v)\big)$$
$$\times\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx$$
$$=\int_{\Omega}|f-\hat{f}|dx. \tag{55}$$

Since the three integrals on the left-hand side of the above equality are nonnegative, we deduce that

$$\int_{\Omega}|b(u)-b(v)|dx+\int_{\partial\Omega}\big||u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v\big|d\sigma\le\int_{\Omega}|f-\hat{f}\big|dx. \tag{56}$$

Let us take a sequence $(f_i)_{i\in\mathbb{N}}\subset L^{\infty}(\Omega)$ and $(u_i)_{i\in\mathbb{N}}$ the corresponding sequence of Lipschitz continuous weak solutions. By (56), we have

$$\int_{\Omega}|b(u)-b(v)|dx+\int_{\partial\Omega}\big||u|^{r(.,u(.))-2}u-|v|^{r(.,v(.))-2}v\big|d\sigma$$
$$\le\int_{\Omega}\big[|b(u)-b(u_i)|+|b(v)-b(u_i)|\big]dx$$
$$+\int_{\partial\Omega}\Big[\big||u|^{r(.,u(.))-2}u-|u_i|^{r(.,u_i(.))-2}u_i\big|+\big||v|^{r(.,v(.))-2}v-|u_i|^{r(.,u_i(.))-2}u_i\big|\Big]d\sigma$$
$$\le\int_{\Omega}\big[|f-f_i|+|\hat{f}-f_i|\big]dx, \tag{57}$$

so that at the limit as $i \to \infty$ in (57), using the density argument between $L^\infty(\Omega)$ and $L^1(\Omega)$, we infer that

$$b(u) = b(v) \text{ a.e. in } \Omega \text{ and } |u|^{r(.,u(.))-2}u = |v|^{r(.,v(.))-2}v \text{ a.e. on } \partial\Omega.$$

Thus, using the assumption (H) and the fact that b is strictly increasing, we get

$$u = v \text{ a.e. in } \Omega \text{ and } u = v \text{ a.e. on } \partial\Omega. \qquad \square$$

4 Continuous Dependence for the Weak Solution

Here, we are interested to the stability result of weak solutions to the problems

$$(P_n)\begin{cases} b(u_n) - \text{div} a_n(x, u_n, \nabla u_n) = f_n & \text{in } \Omega \\ a_n(x, u_n, \nabla u_n).\eta = -|u_n|^{r_n(x,u_n)-2}u_n & \text{on } \partial\Omega. \end{cases}$$

$(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion flux functions such that $a_n(x, z, \xi)$ satisfies $(A_3) - (A_6)$ with variable exponent $p_n : \Omega \times \mathbb{R} \to [p_-, p_+]$, and $r_n : \partial\Omega \times \mathbb{R} \to [r_-, r_+]$ is a Carathéodory function and $(\mathcal{M}_n)_{n\in\mathbb{N}}$ equi-integrable on Ω. In the sequel, we make the log-Hölder continuous and convergence hypothesis.

$$\left[\begin{array}{l} p_n : \overline{\Omega} \times \mathbb{R} \longrightarrow [p_-, p_+], \text{ with } \quad p_- > N \text{ and } \forall M > 0, \\ p_n \text{ is log-Hölder continuous in } (x, z) \text{ uniformly on } \overline{\Omega} \times [-M, M]. \end{array}\right. \tag{58}$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R} \times \mathbb{R}^N, \\ \sup_{(z,\xi)\in K} |a_n(., z, \xi) - a(., z, \xi)| \to 0 \text{ in measure on } \Omega. \end{array}\right. \tag{59}$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R}, \\ \sup_{z\in K} |p_n(., z) - p(., z)| \text{ converges to zero in measure on } \Omega. \end{array}\right. \tag{60}$$

$$r_n(., z) \text{ converges to } r(., z) \text{ a.e. on } \partial\Omega, \quad \text{for all } z \in \mathbb{R}. \tag{61}$$

Finally, assume that

$$(f_n)_{n\in\mathbb{N}} \text{ is a sequence of data weakly convergent to } f \text{ in } L^1(\Omega). \tag{62}$$

The following structural stability result for weak solutions holds.

Theorem 4.1 *Let u_n be a weak solution of (P_n).*

Assume that $a(x, z, \xi)$ satisfies (A_3)–(A_6) with a variable exponent $p(x, z)$. Assume that (A_2) and (3) hold. Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion *flux functions of the form $a_n(x, z, \xi)$ such that (A_3)–(A_6) hold with $p_n(x, z)$, C independent of n and with a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ equi-integrable on Ω. Furthermore, let us consider the assumptions (58)–(62).*

Then, there exists a measurable function $u \in C(\overline{\Omega})$ such that u_n converges to u a.e. in Ω and a.e. on $\partial\Omega$ and ∇u_n converges to ∇u a.e. in Ω, as $n \to \infty$ (up to extraction of subsequence). The function u is a weak solution of the problem (1) associated with the diffusion flux $a(., ., .)$ and the source term f.

The proof is organized in several steps, and we reason up to an extracted subsequence of $(u_n)_{n\in\mathbb{N}}$ still denoted $(u_n)_{n\in\mathbb{N}}$.

Claim 1 Let u_n be a weak solution of (P_n). Then,

(i) the sequence $(b(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$;
(ii) there exists a positive constant $C(f, p_-)$ such that

$$\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq C(f, p_-).$$

Moreover, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p_-}(\Omega)$.

Proof (i) Taking $\varphi = T_k(u_n)$ as a test function in a weak formulation of the problem (P_n), where a_n replaces a, we get

$$\int_\Omega b(u_n)T_k(u_n)dx + \int_{\partial\Omega} |u_n|^{r_n(.,u_n)-2}u_n T_k(u_n)d\sigma + \int_\Omega a_n(x, u_n, \nabla u_n)\nabla T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx.$$

Since all the terms on the left-hand side of the above equality are nonnegative, we deduce that

$$\int_\Omega b(u_n)T_k(u_n)dx \leq \int_\Omega |f_n T_k(u_n)|dx \leq k\int_\Omega |f_n|dx.$$

As $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^1(\Omega)$, it is bounded. So, there exists a positive constant C such that

$$\int_\Omega b(u_n)T_k(u_n)dx \leq kC.$$

Dividing the above inequality by k and letting k go to 0, we get

$$\int_\Omega |b(u_n)|dx \leq C. \tag{63}$$

(ii) From above inequality, we deduce that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$, since b is continuous and onto. Thus, from the Poincaré-Wirtinger inequality, we obtain

$$\int_\Omega |u_n|^{p_-}dx \leq const(p_-, \Omega)\int_\Omega |\nabla u_n|^{p_-}dx + C_0, \tag{64}$$

where C_0 depends on $meas(\Omega)$ and p_-.

Moreover, using (A_6) with variable exponent $p_n(., u_n(.))$ on $a_n(x, u_n, \nabla u_n)$ (see the reasoning that leads to (28) in the proof of Assertion 1.), the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the standard estimation

$$\int_\Omega b(u_n)u_n dx + \int_{\partial\Omega} |u_n|^{r_n(.,u_n)}d\sigma + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq C, \tag{65}$$

with C that depends on f, $meas(\Omega)$ and p_- but not on n. From the above inequality, we deduce that

$$\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq C(f, \Omega, p_-). \tag{66}$$

Moreover, using (64), (66) and the fact that

$$\int_\Omega |\nabla u_n|^{p_-}dx \leq meas(\Omega) + \int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx,$$

we obtain

$$\begin{aligned}\|u_n\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_n|^{p_-} + |\nabla u_n|^{p_-}\right]dx \\ &\leq \left(const\left(p_-, \Omega\right) + 1\right)\left[meas(\Omega) + \int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx\right] + C_0 \\ &\leq const(p_-, \Omega, f). \end{aligned} \tag{67}$$

Therefore, $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ □

From (ii), up to a subsequence, u_n converges a.e in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to u and a.e on $\partial\Omega$. For this, see Lemma 3.3 and Lemma 3.4.

Claim 2 The sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and one has

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \tag{68}$$

Moreover, $|\lambda|^{\pi(x)}$ is integrable with respect to the measure $d\nu_x dx$ on $\mathbb{R}^N \times \Omega$ and $\nabla u \in L^{\pi(.)}(\Omega)$.

Proof Writing

$$|p_n(x, u_n(x)) - p(x, u(x))| \leq |p_n(x, u_n(x)) - p(x, u_n(x))| + |p(x, u_n(x)) - p(x, u(x))|,$$

from a.e. convergence of u_n to u, from assumption (60) and from the Lusin theorem applied to the map

$$p : \Omega \mapsto p(x, .) \in C(\mathbb{R}),$$

we deduce that $p_n(., u_n(.)) \to p(., u(.)) = \pi(.)$ in measure on Ω. Now, using Theorem 2.1—[(ii),(iii)], $(p_n(., u_n(.)), \nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R} \times \mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi}(x) \otimes \nu_x$.

Thus, we can apply the weak convergence properties of Theorem 2.1-(i) to the Carathéodory function

$F_m(x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0}$ with $m \in \mathbb{N}$, where h_m is defined in the preliminaries. Then, we obtain

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(x)} d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x(\lambda_0, \lambda)dx \\
&= \int_{\Omega}\int_{\mathbb{R}\times\mathbb{R}^N} F_m(x, \lambda_0, \lambda) d\mu_x(\lambda_0, \lambda)dx \\
&= \lim_{n\to\infty}\int_{\Omega} F_m(x, p_n(x, u_n(x)), \nabla u_n(x))dx \\
&= \lim_{n\to\infty}\int_{\Omega} |h_m(\nabla u_n)|^{p_n(x,u_n(x))}dx \\
&\leq \lim_{n\to\infty}\int_{\Omega} |\nabla u_n|^{p_n(x,u_n(x))}dx \\
&\leq C(f, \Omega, p_-) \quad \text{(using (66))}.
\end{aligned}$$

Since $h_m(\lambda) \to \lambda$, as $m \to \infty$, using Lebesgue's convergence theorem, as $m \mapsto h_m(\lambda)$ is increasing, we deduce from the last inequality that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(x)} d\nu_x(\lambda)dx \leq C(f, \Omega, p_-).$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$.

Moreover, from the last inequality and Jensen's inequality, we get

$$\int_\Omega |\nabla u|^{\pi(x)}dx = \int_\Omega \left| \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda) \right|^{\pi(x)} dx \le \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(x)} d\nu_x(\lambda)dx < \infty.$$

Thus, $\nabla u \in L^{\pi(.)}(\Omega)$. □

Claim 3

(a) The sequence $(A_n)_{n\in\mathbb{N}}$ defined by $A_n(x) := a_n(x, u_n(x), \nabla u_n(x))$ is equi-integrable.
(b) The weak limit A of (A_n) belongs to $L^{\pi'(.)}(\Omega)$, and one has

$$A(x) = \int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x(\lambda) \quad \text{a.e.} \quad x \in \Omega. \tag{69}$$

Proof (a) Using (A_5), we have

$$|a_n(x, u_n, \nabla u_n)|^{p'_n(.,u_n(.))} \le C\big(\mathcal{M}_n + |\nabla u_n|^{p_n(.,u_n(.))}\big).$$

The sequence $\big(|\nabla u_n|^{p_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$ and $\mathcal{M}_n$ is also equi-integrable on Ω; hence, $\big(|a_n(x, u_n, \nabla u_n)|^{p'_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is equi-integrable.

Otherwise, as $p'_n(., u_n(.)) > 1$, we have

$$|a_n(x, u_n, \nabla u_n)| \le 1 + |a_n(x, u_n, \nabla u_n))|^{p'_n(.,u_n(.))}.$$

Thus, for all subset $E \subset \Omega$, we have

$$\int_E |a_n(x, u_n, \nabla u_n)|dx \le meas(E) + \int_E |a_n(x, u_n, \nabla u_n)|^{p'_n(.,u_n(.))}dx.$$

Thus, for $meas(E)$ small enough, we deduce that $(A_n)_{n\in\mathbb{N}}$ is equi-integrable.

(b) Set $\nabla v_n := \nabla u_n \chi_{S'_n}$, and consider auxiliary functions $\tilde{A}_n := a(x, u, \nabla v_n)$, where

$$S'_n := \left\{ x \in \Omega, \quad |p(x, u(x)) - p_n(x, u_n(x))| < \frac{1}{2} \right\}.$$

Let us prove that for all $\sigma > 0$,

$$meas\big(\{x \in \Omega, \sup_{\lambda\in K} |a_n(x, u_n, \lambda) - a(x, u, \lambda)| \ge \sigma\}\big) \to 0, \quad \text{as} \quad n \to \infty,$$

where K is a bounded subset of $\mathbb{R}^N$. We know that

$$|a_n(x,u_n,\lambda) - a(x,u,\lambda)| \le |a_n(x,u_n,\lambda) - a(x,u_n,\lambda)| + |a(x,u_n,\lambda) - a(x,u,\lambda)|.$$

Thus, it follows from Lemma 2.2 and (59) that

$$\sup_{\lambda\in K} |a_n(x,u_n,\lambda) - a(x,u,\lambda)| \to 0 \text{ in measure, as } n\to\infty.$$

Now, using the same argument as in the proof of Assertion 4 to A_n and $\tilde{A}_n$ instead of Φ_n and $\tilde{\Phi}_n$, we get a wished result.

□

Claim 4

(a)

$$\int_\Omega A.\nabla u dx \ge \int_\Omega a(x,u(x),\lambda).\lambda d\nu_x(\lambda).$$

(b) The "div-curl" inequality holds:

$$\int_{\Omega\times\mathbb{R}^N} \big(a(x,u(x),\lambda) - a(x,u(x),\nabla u(x))\big)(\lambda - \nabla u(x)) d\nu_x(\lambda) dx \le 0. \tag{70}$$

(c)

$$A(x) = a(x,u(x),\nabla u(x)) \text{ for a.e. } x\in\Omega,$$

and ∇u_n converges to ∇u in measure on Ω, as $n\to+\infty$.

Proof We only give the proof of (a). The proofs of (b) and (c) are exactly the same as the proofs of Assertion 6 and Assertion 7.

Let $\varphi\in C^\infty(\overline{\Omega})$. For n large enough, φ is an admissible test function in the weak formulation of u_n, and we have

$$\int_\Omega b(u_n)\varphi dx + \int_{\partial\Omega} |u_n|^{r_n(.,u_n(.))-2} u_n\varphi d\sigma + \int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx = \int_\Omega f_n\varphi dx. \tag{71}$$

$(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $W^{1,p_-}(\Omega)$, so it is uniformly bounded in $L^\infty(\Omega)$. Then, $(b(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, $b(u_n)$ converges a.e. in Ω to $b(u)$. Therefore, thanks to Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_\Omega b(u_n)\varphi dx = \int_\Omega b(u)\varphi dx. \tag{72}$$

On the other hand, $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p_- > N$, so $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\partial\Omega)$. Therefore, $(|u_n|^{r_n(.,u_n)-2}u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\partial\Omega)$, for $2 < r_- \le r_n(.,u_n) \le r_+ < \infty$, since $t \mapsto |t|^{r_n(.,t)-2}t$ is continuous a.e. on $\partial\Omega$. Moreover, $|u_n|^{r_n(.,u_n)-2}u_n$ converges to $|u|^{r(.,u)-2}u$ a.e. on $\partial\Omega$. Indeed,

$$|r_n(.,u_n) - r(.,u)| \le |r_n(.,u_n) - r(.,u_n)| + |r(.,u_n) - r(.,u)|.$$

From (61), from a.e. convergence of u_n to u on $\partial\Omega$ and the fact that r is a Carathéodory function on $\partial\Omega \times \mathbb{R}$, we deduce from the above inequality that

$$r_n(.,u_n) \to r(.,u) \text{ a.e. on } \partial\Omega,$$

which implies that $|u_n|^{r_n(.,u_n)-2}u_n$ converges to $|u|^{r(.,u)-2}u$ a.e. on $\partial\Omega$. Then, by using Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_{\partial\Omega} |u_n|^{r_n(.,u_n)-2}u_n\varphi d\sigma = \int_{\partial\Omega} |u|^{r(.,u)-2}u\varphi d\sigma. \tag{73}$$

We also have

$$\lim_{n\to\infty}\int_\Omega f_n\varphi dx = \int_\Omega f\varphi dx, \tag{74}$$

since $f_n \rightharpoonup f$ in $L^1(\Omega)$, as n tends to ∞ and $\varphi \in L^\infty(\Omega)$. It remains to prove that

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\varphi dx = \int_\Omega A\nabla\varphi dx.$$

Let us prove that $(a_n(x,u_n,\nabla u_n)\nabla\varphi)_{n\in\mathbb{N}}$ is equi-integrable. Let E be a subset of Ω and by using Young's inequality, we have

$$\begin{aligned}
&\int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx \\
&\quad \le \int_E |a_n(x,u_n,\nabla u_n)||\nabla\varphi| dx \\
&\quad \le \int_E \frac{1}{p_n'(.,u_n(.))}|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}dx + \int_E \frac{1}{p_n(.,u_n(.))}|\nabla\varphi|^{p_n(.,u_n(.))}dx
\end{aligned}$$

$$\leq C\int_E \big(\mathcal{M}_n(x)+|\nabla u_n|^{p_n(.,u_n(.))}\big)dx+\int_E |\nabla\varphi|^{p_n(.,u_n(.))}dx$$
$$\leq C\int_E \big(\mathcal{M}_n(x)+|\nabla u_n|^{p_n(,u_n(.))}\big)dx$$
$$+\int_{E\cap\{|\nabla\varphi|\leq 1\}} |\nabla\varphi|^{\pi_n(.)}dx+\int_{E\cap\{|\nabla\varphi|>1\}} |\nabla\varphi|^{p_n(.,u_n(.))}dx$$
$$\leq C\int_E \big(\mathcal{M}_n(x)+|\nabla u_n|^{p_n(,u_n(.))}\big)dx+meas(E)+\int_E |\nabla\varphi|^{p+}dx \tag{75}$$

By using Claim 1-(ii) and the fact that $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is equi-integrable, $\big(\mathcal{M}_n+|\nabla u_n|^{p_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is equi-integrable. Moreover, $|\nabla\varphi|^{p+}\in L^1(\Omega)$, since $\nabla\varphi$ is bounded. Thus, we deduce from (75) that

$$\lim_{meas(E)\to 0}\int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx=0.$$

Furthermore,

$$a_n(x,u_n,\nabla u_n).\nabla\varphi\to A.\nabla\varphi \quad \text{a.e.} \quad \text{in} \ \ \Omega.$$

By applying Vitali's theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx=\int_\Omega A.\nabla\varphi dx. \tag{76}$$

We can pass to the limit as n go to ∞ in (71) and taking into account the inequalities (72), (73), (74) and (76) to obtain

$$\int_\Omega b(u)\varphi dx+\int_\Omega |u|^{s(.,u)-2}u\varphi d\sigma+\int_\Omega A.\nabla\varphi dx=\int_\Omega f\varphi dx. \tag{77}$$

By density argument, we can replace φ with u_n in (71) to get

$$\int_\Omega b(u_n)u_n dx+\int_{\partial\Omega} |u_n|^{r_n(.,u_n)}d\sigma+\int_\Omega a_n(x,u_n,\nabla u_n).\nabla u_n dx=\int_\Omega f_n u_n dx \tag{78}$$

Since $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, we replace φ by u in (77), and we have

$$\int_\Omega b(u)udx+\int_{\partial\Omega} |u|^{r(.,u)}d\sigma+\int_\Omega A.\nabla udx=\int_\Omega fudx. \tag{79}$$

By Fatou's lemma, we deduce

$$\liminf_{n\to\infty}\int_{\Omega} b(u_n)u_n dx \geq \int_{\Omega} b(u)u dx \tag{80}$$

and

$$\liminf_{n\to\infty}\int_{\partial\Omega} |u_n|^{r_n(.,u_n)} d\sigma \geq \int_{\partial\Omega} |u|^{r(.,u)} d\sigma. \tag{81}$$

Moreover,

$$\lim_{n\to\infty}\int_{\Omega} f_n u_n dx = \int_{\Omega} f u dx. \tag{82}$$

Indeed,

$$\int_{\Omega} f_n u_n dx = \int_{\Omega} f_n u dx + \int_{\Omega} f_n (u_n - u) dx. \tag{83}$$

For the first term on the right-hand side of the above equality, we have

$$\lim_{n\to\infty}\int_{\Omega} f_n u dx = \int_{\Omega} f u dx,$$

since $f_n \rightharpoonup f$ in $L^1(\Omega)$ and $u \in L^\infty(\Omega)$. For the second term on the right-hand side of (83), we have, for all $R > 0$,

$$\begin{aligned}\int_{\Omega} |f_n(u_n - u)| dx &\leq \int_{\{|f_n|>R\}} |f_n| \|u_n - u\|_{L^\infty(\Omega)} dx + R\int_{\{|f_n|\leq R\}} |u_n - u| dx \\ &\leq C\int_{\{|f_n|>R\}} |f_n| dx + R\int_{\Omega} |u_n - u| dx. \end{aligned} \tag{84}$$

For all R fixed, the second term on the right-hand side of above inequality tends to zero as $n \to \infty$. Since by Chebyshev's inequality,

$$\sup_n meas\big(\{|f_n| > R\}\big) \leq \frac{\sup_n \|f_n\|_{L^1(\Omega)}}{R} \leq \frac{C}{R} \to 0, \quad \text{as} \quad R \to \infty$$

and because a weakly convergent in $L^1(\Omega)$ sequence is equi-integrable on Ω, by the choice of R, the first term on the right-hand side of (84) can be made small as desired. Hence, we deduce that $f_n(u_n - u)$ goes to zero in $L^1(\Omega)$. Thus, (82) is justified.

Combining (80), (81) and (82), we have

$$\liminf_{n\to\infty}\int_\Omega f_n u_n dx - \int_\Omega b(u)udx - \int_{\partial\Omega}|u|^{r(.,u)}d\sigma$$
$$\geq \liminf_{n\to\infty}\left(\int_\Omega \big(f_n u_n - b(u_n)u_n\big)dx - \int_{\partial\Omega}|u_n|^{r_n(.,u_n)}d\sigma\right) \tag{85}$$

By using (78), (79), (85) and the definition of A_n, we get

$$\int_\Omega A.\nabla u dx \geq \liminf_{n\to\infty}\int_\Omega A_n.\nabla u_n dx. \tag{86}$$

By Andreianov et al. [1], Lemma 2.1, $m \mapsto a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a_n(x, u_n, \nabla u_n).\nabla u_n$ for m large enough. Then,

$$a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n) \leq a_n(x, u_n, \nabla u_n).\nabla u_n.$$

Therefore, by using (86) and the property (7) of Theorem 2.1, we have

$$\begin{aligned}\int_\Omega A.\nabla u dx &\geq \liminf_{n\to\infty}\int_\Omega A_n.\nabla u_n dx\\ &\geq \lim_{n\to\infty}\int_\Omega a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n)dx\\ &= \int_{\Omega\times\mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda)d\nu_x(\lambda)dx.\end{aligned} \tag{87}$$

Using Lebesgue's convergence theorem in (87), as m goes to ∞, we get (a). □

Lemma 4.1 *u is a weak solution of the problem (1).*

Proof Firstly, we can prove that $u \in W^{1,p(.,u(.))}(\Omega)$, thanks to Claim 2 and the fact that $u \in W^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^{p(.,u(.))}(\Omega)$. Now, we will prove the equality (12).

Let u_n be a weak solution of the problem (P_n). By using (58) and the fact that $u_n \in W^{1,p_n(.,u_n(.))}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_n(.,u_n(.))}(\Omega)$. Therefore, for n large enough, we can choose φ in $C^\infty(\overline{\Omega})$ as a test function in the weak formulation of the problem (P_n). Then, we obtain

$$\int_\Omega b(u_n)\varphi dx + \int_{\partial\Omega}|u_n|^{r_n(.,u_n(.))-2}u_n\varphi d\sigma + \int_\Omega a_n(x, u_n, \nabla u_n).\nabla\varphi dx$$
$$= \int_\Omega f_n\varphi dx. \tag{88}$$

Currently, we are looking at the third term on the left-hand side of the above equality. We know that $(a_n(x, u_n, \nabla u_n)\nabla\varphi)_{n\in\mathbb{N}}$ is equi-integrable, thanks to (75). In addition, it follows from Claim 3-(b) and Claim 4-(c) that

$$a_n(x, u_n, \nabla u_n)\nabla\varphi \to a(x, u, \nabla u)\nabla\varphi \text{ a.e. in } \Omega.$$

Therefore, by using Vitali's theorem, we get

$$\lim_{n\to\infty}\int_\Omega a_n(x, u_n, \nabla u_n)\nabla\varphi dx = \int_\Omega a(x, u, \nabla u)\nabla\varphi dx.$$

Finally using (72), (73), (74), (88) and the above equality, we get (12) for u with all test functions in $C^\infty(\overline{\Omega})$. Moreover, $b(u) \in L^1(\Omega)$ and $|u|^{r(.,u)-2}u \in L^1(\partial\Omega)$, since $b(u_n)$ strongly converges to $b(u)$ in $L^1(\Omega)$ and $|u_n|^{r_n(.,u_n)-2}u_n$ strongly converges to $|u|^{r(.,u)-2}u$ in $L^1(\partial\Omega)$ (see the reasoning that leads to (72) and (73)). This is the end of the proof of our lemma. □

Remark 4.1 Under the assumptions (44), (45) and (H), the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. in Ω and a.e. on $\partial\Omega$ and the whole sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, as $n \to \infty$.

Indeed, by Claim 1, we deduce that $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. in Ω and a.e. on $\partial\Omega$, up to extraction of subsequence. From Claim 4-(c), $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, up to extraction of subsequence.

Now, by Lemma 4.1 and the uniqueness of the weak solution to (1) done in Theorem 3.3, we conclude that all convergent subsequences of $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$ converge to the same limits u and ∇u, respectively.

The proof of Theorem 4.1 ends here.

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Optimal Control of Averaged State of a Population Dynamics Model

Cyrille Kenne and Boniface Nkemzi

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1 Introduction

The invasive species have a real impact on communities and ecosystems. They are viewed as a significant component of global change, and they have severe negative consequences for biodiversity. We consider a model describing the dynamics of an invasive species with age dependence and spatial structure, and the invasive species are diffusing in the habitat with a diffusion coefficient depending on the susceptibility of the habitat to invasion (ecological factors) and/or genetics factors of the species. We then consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^3$. We denote by Γ the boundary of the domain, and we assume that it is of class C^2. For the time $T > 0$, the life expectancy of an individual $A > 0$ and $\theta_{min}, \theta_{max} > 0$, we set $I = (\theta_{min}, \theta_{max})$, $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $\Sigma = U \times \Gamma$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$ and $Q_\omega = U \times \omega$, where ω is a non-empty open subset of Ω. For $\theta \in I$, the system reads as follows:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - d(\theta)\Delta y + \mu y = f + v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot, \cdot, \theta) = y^0 & \text{in } Q_A, \\ y(\cdot, 0, \cdot, \theta) = \displaystyle\int_0^A \beta(t, a, x) y(t, a, x, \theta)\, da & \text{in } Q_T, \end{cases} \tag{1}$$

C. Kenne (✉) · B. Nkemzi
University of Buea, Department of Mathematics, Buea, Cameroon

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where

- $y = y(t, a, x, \theta)$ is the distribution of individuals of age $a \geq 0$, at time $t \geq 0$ and location $x \in \Omega$.
- The recruitment $f \in L^2(Q)$ is a positive periodic function.
- The control $v \in L^2(Q_\omega)$, which corresponds to the removal of the individuals in a sub-domain ω of Ω, and χ_ω denote the characteristic function of the control set ω.
- The mortality rate $\mu = \mu(a) \geq 0$ is a known increasing positive function which is continuous on $[0, A]$, whereas the fertility rate $\beta = \beta(t, a, x) \in L^\infty(Q)$ is known and positive.
- $d(\theta) > 0$ is the diffusion coefficient of species dispersal in the environment and is assumed depending on susceptibility $\theta \in I$ and $d \in \mathcal{C}(I)$.

Model (1) is a system with varying parameter and our question is as follows: Let z_d be a given age-dependent distribution of species, can the average of the solution to (1) at time $t = T$ be steered to z_d upon selecting a suitable control v corresponding to a removal (eradication) of species on the sub-domain ω?

$$\lim_{a \to A} \int_0^a \mu(s) ds = +\infty,$$

Remark 1 Set

$$W(T, A) = \left\{ \rho \in L^2(U; H_0^1(\Omega)); \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2\left(U; H^{-1}(\Omega)\right) \right\}. \tag{2}$$

Then, we have (see [4]) that

$$W(T, A) \subset \mathcal{C}([0, T], L^2(Q_A)) \text{ and } W(T, A) \subset \mathcal{C}([0, A], L^2(Q_T)). \tag{3}$$

Under the assumptions on the data, and for $\theta \in I$ fixed, (1) has a unique solution $y(\theta; v) = y(t, a, x, \theta; v)$ in $W(T, A)$. More precisely, we are concerned in this chapter by the following optimization problem:

$$\inf_{v \in L^2(Q_\omega)} J(v), \tag{4}$$

where the cost function is given by

$$J(v) = \left\| \int_I y(\theta; v)(T) \, d\theta - z_d \right\|^2_{L^2(Q_A)} + N \|v\|^2_{L^2(Q_\omega)}, \tag{5}$$

with $z_d \in L^2(Q_A)$ and $N > 0$ given, $\displaystyle\int_I y(\theta; v)(T) \, d\theta \in L^2(Q_A)$, for all $v \in L^2(Q_\omega)$.

Optimal control for age-structured population was studied later by some authors like A. Ouedrogo et al. [9]. In this paper, the authors considered a nonlinear age-structured population dynamics model, and they study the existence of an optimal control making the density of the population as close as possible of some given density. In [1], B. Ainseba et al. investigated the optimal harvesting problem for a nonlinear age-dependent and spatially structured population dynamics model with a constant diffusion coefficient, where the birth process is described by a nonlocal and nonlinear boundary condition. The notion of averaged control was introduced by Zuazua [10] to analyse the problem of controlling parameter-dependent systems. In this notion, the aim is to find a control, independent of the unknown parameter, so that the average of the state is controlled. For more literature on the topic, we refer, for instance, to Lohéac and Zuazua [6], Lazar and Zuazua [5], Hafdallah and Ayadi [2] and LU and Zuazua [7], G. Mophou et al. [8] and the references therein. In this chapter, we are concerned with the control of a parameter-dependent age-structured population dynamics system. The rest of this chapter is structured as follows. In Sect. 2, we give some regularity results. In Sect. 3, we prove the existence and the uniqueness of the control and characterize an optimality system. A conclusion is given in Sect. 4.

2 Preliminary Results

In order to solve the optimization problem (4), we need some preliminary results.

In what follows, we will sometime adopt the following notation:

$$\begin{cases} L = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - d(\theta)\Delta + \mu I, \\ L^* = -\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial a} - d(\theta)\Delta + \mu I, \end{cases} \tag{6}$$

where I is the identity operator.

Remark 2 From now on, we use $C(X)$ to denote a positive constant whose value varies from a line to another but depends on X; the positive constant $d_0 = \inf_{\theta \in I} d(\theta)$ and we will denote by $(\cdot, \cdot)_H$ the scalar product in H.

Lemma 2.1 *Let $v \in L^2(Q_\omega)$ and $y \in L^2(U; H_0^1(\Omega))$ be a solution of (1); then, we have the following estimations:*

$$\|y\|_{L^2(U;H_0^1(\Omega))} \le C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right),$$

$$\|y(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)} \le C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right),$$

$$\|y(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)} \le C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right).$$

Proof We proceed as in [3]. We recall that $y = y(t, a, x, \theta; v)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - d(\theta)\Delta y + \mu y = f + v\chi_{Q_\omega} & \text{in } \ Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,\cdot,\cdot,\theta) = y^0 & \text{in } \ Q_A, \\ y(\cdot,0,\cdot,\theta) = \displaystyle\int_0^A \beta(t,a,x)y(t,a,x,\theta)\,da & \text{in } \ Q_T. \end{cases}$$

By defining $z = e^{-rt}y$ with $r > 0$, we obtain that z is the solution of the problem

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} - d(\theta)\Delta z + (\mu + r)z = f + v\chi_{Q_\omega} & \text{in } \ Q, \\ z = 0 & \text{on } \Sigma, \\ z(0,\cdot,\cdot,\theta) = y^0 & \text{in } \ Q_A, \\ z(\cdot,0,\cdot,\theta) = \displaystyle\int_0^A \beta(t,a,x)z(t,a,x,\theta)da & \text{in } \ Q_T. \end{cases} \tag{7}$$

Multiplying the first equation of system (7) by z and integrating by parts over Q, we get

$$\begin{aligned} &\frac{1}{2}\|z(T,\cdot,\cdot,\theta)\|^2_{L^2(Q_A)} - \frac{1}{2}\|z(0,\cdot,\cdot,\theta)\|^2_{L^2(Q_A)} + \frac{1}{2}\|z(\cdot,A,\cdot,\theta)\|^2_{L^2(Q_T)} \\ &\quad - \frac{1}{2}\|z(\cdot,0,\cdot,\theta)\|^2_{L^2(Q_T)} + d(\theta)\|\nabla z\|^2_{L^2(Q)} + \int_Q (r+\mu)z^2\,dxdtda \\ &\quad = \int_Q (f + v\chi_{Q_\omega})z\,dxdtda. \end{aligned}$$

Then, using the fact that $\mu \geq 0$, it follows the inequality

$$\begin{aligned} &\frac{1}{2}\|z(T,\cdot,\cdot,\theta)\|^2_{L^2(Q_A)} + \frac{1}{2}\|z(\cdot,A,\cdot,\theta)\|^2_{L^2(Q_T)} + d_0\|\nabla z\|^2_{L^2(Q)} + r\|z\|^2_{L^2(Q)} \\ &\quad \leq \frac{1}{2}\|y^0\|^2_{L^2(Q_A)} + \frac{1}{2}\|z(\cdot,0,\cdot,\theta)\|^2_{L^2(Q_T)} + \frac{1}{2}\|f + v\chi_{Q_\omega}\|^2_{L^2(Q)} \\ &\quad + \frac{1}{2}\|z\|^2_{L^2(Q)}. \end{aligned} \tag{8}$$

On the other hand, one can write for $(t,x) \in (0,T) \times \Omega$

$$z(t,0,x,\theta) = \int_0^A \beta(t,a,x)z(t,a,x,\theta)\,da,$$

then

$$\|z(\cdot, 0, \cdot, \theta)\|^2_{L^2(Q_T)} \leq \|\beta\|^2_{L^2(Q)} \|z\|^2_{L^2(Q)}.$$

Thus, (8) gives

$$\frac{1}{2}\|z(T, \cdot, \cdot, \theta)\|^2_{L^2(Q_A)} + \frac{1}{2}\|z(\cdot, A, \cdot, \theta)\|^2_{L^2(Q_T)} + d_0\|\nabla z\|^2_{L^2(Q)}$$
$$+ \left(r - \frac{1}{2}\|\beta\|^2_{L^2(Q)} - \frac{1}{2}\right)\|z\|^2_{L^2(Q)} \leq \frac{1}{2}\|f + v\chi_{Q_\omega}\|^2_{L^2(Q)}$$
$$+ \frac{1}{2}\|y^0\|^2_{L^2(Q_A)}.$$

By choosing r such that $r = \frac{1}{2}\|\beta\|^2_{L^2(Q)} + \frac{1}{2}$, one has

$$\|z(T, \cdot, \cdot, \theta)\|^2_{L^2(Q_A)} + \|z(\cdot, A, \cdot, \theta)\|^2_{L^2(Q_T)} + \|z\|^2_{L^2(U; H^1_0(\Omega))}$$
$$\leq C\left(\|y^0\|^2_{L^2(Q_A)} + \|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q_\omega)}\right).$$

This implies

$$\|y(T, \cdot, \cdot, \theta)\|^2_{L^2(Q_A)} + \|y(\cdot, A, \cdot, \theta)\|^2_{L^2(Q_T)} + \|y\|^2_{L^2(U; H^1_0(\Omega))}$$
$$\leq C(T, \|\beta\|_{L^2(Q)})\left(\|y^0\|^2_{L^2(Q_A)} + \|f\|^2_{L^2(Q)} + \|v\|^2_{L^2(Q_\omega)}\right).$$

So that,

$$\|y\|_{L^2(U; H^1_0(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)})\left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}\right),$$
$$\|y(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)} \leq C(T, \|\beta\|_{L^2(Q)})\left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}\right),$$
$$\|y(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)} \leq C(T, \|\beta\|_{L^2(Q)})\left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}\right).$$

■

Proposition 2.1 *Let $\theta \in I$, and then the map $v \mapsto y(\theta; v)$ is a continuous function from $L^2(Q_\omega)$ onto $L^2(U, H^1_0(\Omega))$.*

Proof Let $\theta \in I$ and $v_0 \in L^2(Q_\omega)$. We show that $v \mapsto y(\theta; v)$ is continuous at v_0. Set $\overline{y} = y(\theta; v) - y(\theta; v_0)$; then, $\overline{y}$ is solution to the problem

$$\begin{cases} L\overline{y} = v\chi_{Q_\omega} - v_0\chi_{Q_\omega} & \text{in } Q, \\ \overline{y} = 0 & \text{on } \Sigma, \\ \overline{y}(0,\cdot,\cdot,\theta) = 0 & \text{in } Q_A, \\ \overline{y}(\cdot,0,\cdot,\theta) = \int_0^A \beta(t,a,x)\overline{y}da & \text{in } Q_T, \end{cases} \tag{9}$$

In view of Lemma 2.1, we have that

$$\|\overline{y}\|_{L^2(U;H_0^1(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)})\|v - v_0\|_{L^2(Q_\omega)}.$$

As $v \to v_0$, we have $\overline{y} \to 0$ strongly in $L^2(U; H_0^1(\Omega))$. Hence, $y(\theta; v) \to y(\theta; v_0)$ strongly in $L^2(U; H_0^1(\Omega))$ as $v \to v_0$. ■

Proposition 2.2 *Let $\lambda > 0$. Let $v, w \in L^2(Q_\omega)$. Let also $y = y(\theta; v)$ be a solution of system (1). Set $\overline{y}_\lambda = \dfrac{y(\theta; v + \lambda w) - y(\theta; v)}{\lambda}$, and then $(\overline{y}_\lambda)$ converges strongly in $L^2(U; H_0^1(\Omega))$ as $\lambda \to 0$ to a function $\bar{y}$, which is the solution of*

$$\begin{cases} L\overline{y} = w\chi_{Q_\omega} & \textit{in } Q, \\ \overline{y} = 0 & \textit{on } \Sigma, \\ \overline{y}(0,\cdot,\cdot,\theta) = 0 & \textit{in } Q_A, \\ \overline{y}(\cdot,0,\cdot,\theta) = \int_0^A \beta(t,a,x)\overline{y}\,da & \textit{in } Q_T. \end{cases} \tag{10}$$

Proof $\overline{y}_\lambda$ is a solution to the problem

$$\begin{cases} L\overline{y}_\lambda = w\chi_{Q_\omega} & \text{in } Q, \\ \overline{y}_\lambda = 0 & \text{on } \Sigma, \\ \overline{y}_\lambda(0,\cdot,\cdot,\theta) = 0 & \text{in } Q_A, \\ \overline{y}_\lambda(\cdot,0,\cdot,\theta) = \int_0^A \beta(t,a,x)\overline{y}_\lambda\,da & \text{in } Q_T. \end{cases}$$

Define $y_\lambda = \overline{y}_\lambda - \overline{y}$, where $\overline{y}$ is a solution to (10). Then, y_λ is a solution to

$$\begin{cases} Ly_\lambda = 0 & \text{in } Q, \\ y_\lambda = 0 & \text{on } \Sigma, \\ y_\lambda(0,\cdot,\cdot,\theta) = 0 & \text{in } Q_A, \\ y_\lambda(\cdot,0,\cdot,\theta) = \int_0^A \beta(t,a,x)y_\lambda\,da & \text{in } Q_T. \end{cases} \tag{11}$$

From Lemma 2.1, we obtain that

$$\|y_\lambda\|_{L^2(U;H_0^1(\Omega))} \leq 0. \tag{12}$$

Passing to the limit in this latter identity when $\lambda \to 0$, it follows that $y_\lambda \to 0$ strongly in $L^2(U; H_0^1(\Omega))$. This means that $(\overline{y}_\lambda)$ converges to $\overline{y}$ strongly in $L^2(U; H_0^1(\Omega))$ as $\lambda \to 0$. ■

3 Existence and Characterization of the Control

In this section, we will show that the optimization problem (4) has a unique solution. Moreover, we will give the equations that characterize the control.

Proposition 3.1 *There exists a unique control $u \in L^2(Q_\omega)$ solution of (4).*

Proof Observing that we have $J(0) \geq 0$, we have that the set $\{J(v) : J(v) \geq 0, v \in L^2(Q_\omega)\}$ is a non-empty lower bounded subset of $\mathbb{R}$, and consequently $\alpha = \inf\limits_{v \in L^2(Q_\omega)} J(v)$ exists. Let $(v_n)_n$ be a minimizing sequence such that

$$J(v_n) \to \alpha, \quad \text{when} \;\; n \to +\infty. \tag{13}$$

Then, we have that there exists $C > 0$ independent of n such that for all $n \in \mathbb{N}$, $J(v_n) \leq C$; i.e.,

$$\left\| \int_I y(\theta; v_n)(T)\, d\theta - z_d \right\|^2_{L^2(Q_A)} + N \|v_n\|^2_{L^2(Q_\omega)} \leq C,$$

so

$$\|v_n\|_{L^2(Q_\omega)} \leq C, \tag{14}$$

$$\left\| \int_I y(\theta; v_n)(T)\, d\theta \right\|_{L^2(Q_A)} \leq C. \tag{15}$$

Now, $y_n = y(t, a, x, \theta; v_n)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial y_n}{\partial t} + \dfrac{\partial y_n}{\partial a} - d(\theta)\Delta y_n + \mu y_n = f + v_n \chi_{Q_\omega} & \text{in } \; Q, \\ y_n = 0 & \text{on } \; \Sigma, \\ y_n(0, \cdot, \cdot, \theta) = y^0 & \text{in } \; Q_A, \\ y_n(\cdot, 0, \cdot, \theta) = \displaystyle\int_0^A \beta(t, a, x) y_n(t, a, x, \theta)\, da & \text{in } \; Q_T. \end{cases} \tag{16}$$

In view of (14), we obtain from Lemma 2.1 that

$$\|y_n\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}), \tag{17}$$

$$\|y_n(T,\cdot,\cdot,\theta)\|_{L^2(Q_A)} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}), \tag{18}$$

$$\|y_n(\cdot,A,\cdot,\theta)\|_{L^2(Q_T)} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}). \tag{19}$$

Using (14), (17), (18) and (19), there exist $u \in L^2(Q_\omega)$, $y \in L^2(U; H_0^1(\Omega))$, $y_T \in L^2(Q_A)$, $y_A \in L^2(Q_T)$ and extracted sequences from the sequences $(v_n)_n$, $(y_n)_n$, $(y_n(T,\cdot,\cdot,\theta))_n$, $(y_n(\cdot,A,\cdot,\theta))_n$ still denoted $(v_n)_n$, $(y_n)_n$, $(y_n(T,\cdot,\cdot,\theta))_n$, $(y_n(\cdot,A,\cdot,\theta))_n$ such that the following convergences hold:

$$v_n \rightharpoonup u \ \text{ weakly in } \ L^2(Q_\omega), \tag{20}$$

$$y_n \rightharpoonup y \ \text{ weakly in } \ L^2(U; H_0^1(\Omega)), \tag{21}$$

$$y_n(T,\cdot,\cdot,\theta) \rightharpoonup y_T \ \text{ weakly in } \ L^2(Q_A), \tag{22}$$

$$y_n(\cdot,A,\cdot,\theta) \rightharpoonup y_A \ \text{ weakly in } \ L^2(Q_T). \tag{23}$$

Now, let us prove that (u, y) satisfies (1). Let $\phi \in \mathcal{D}(Q)$ be a test function. Multiplying the first equation in (16) by ϕ and integrating by parts over Q, we obtain

$$\left\langle y_n, L^*\phi \right\rangle = \langle f, \phi\rangle + \langle v_n\chi_\omega, \phi\rangle, \qquad \forall \phi \in \mathcal{D}(Q). \tag{24}$$

Taking the limit as $n \longrightarrow +\infty$ in (24) and using (20) and (21) yield

$$\left\langle y, L^*\phi \right\rangle = \langle f, \phi\rangle + \langle u\chi_\omega, \phi\rangle, \qquad \forall \phi \in \mathcal{D}(Q);$$

that is

$$\langle Ly, \phi \rangle = \langle f, \phi\rangle + \langle u\chi_\omega, \phi\rangle, \qquad \forall \phi \in \mathcal{D}(Q).$$

Thus,

$$Ly = f + u\chi_\omega \qquad \text{in } Q. \tag{25}$$

Now, on the one hand, as $y \in L^2(U; H_0^1(\Omega))$, using (25), we have that $\dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} \in L^2(U; H^{-1}(\Omega))$. This implies that $y \in W(T, A)$, and using Remark 1, it follows that $y(T,\cdot,\cdot,\theta)$ and $y(0,\cdot,\cdot,\theta)$ exist and belong to $L^2(Q_A)$ and $y(\cdot,A,\cdot)$ and $y(\cdot,0,\cdot,\theta)$ exist and belong to $L^2(Q_T)$. On the other hand, $y \in L^2(Q)$ and $d(\theta)\Delta y \in H^{-1}(U; L^2(\Omega))$; consequently, $y|_\Sigma$ and $\frac{\partial y}{\partial \nu}|_\Sigma$ exist and belong, respectively, to $H^{-1}(U; H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}(U; H^{-\frac{3}{2}}(\Gamma))$. Multiplying the first equation in (16) by $\phi \in \mathcal{C}^\infty(\overline{Q})$ such that $\phi = 0$ on Σ, $\phi(\cdot, A, \cdot) = 0$ in Q_T and integrating by parts over Q and using initials and boundary conditions, we obtain

$$
\begin{aligned}
&(y_n(T,\cdot,\cdot,\theta),\phi(0,\cdot,\cdot))_{L^2(Q_A)} - (y^0,\phi(0,\cdot,\cdot))_{L^2(Q_A)} \\
&\quad - (y_n(\cdot,0,\cdot,\theta),\phi(\cdot,0,\cdot))_{L^2(Q_T)} + \big(y_n, L^*\phi\big)_{L^2(Q)} \\
&\quad = (f,\phi)_{L^2(Q)} + (v_n,\phi)_{L^2(Q_\omega)} \\
&\quad \forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot,A,\cdot) = 0 \text{ in } Q_T.
\end{aligned}
\tag{26}
$$

Note that

$$
y_n(\cdot,0,\cdot,\theta) \rightharpoonup y^1 = \int_0^A \beta(t,a,x)y(t,a,x,\theta)\,da \qquad \text{weakly in} \quad L^2(Q_T). \tag{27}
$$

Indeed, let $\phi \in L^2(Q_T)$, and then

$$
\begin{aligned}
\int_{Q_T} y_n(t,0,x,\theta)\phi(x,t)\,dxdt &= \int_{Q_T}\left(\int_A \beta(t,a,x)y_n(t,a,x,\theta)da\right)\phi(x,t)\,dxdt \\
&= \int_Q y_n(t,a,x,\theta)\psi(t,a,x)\,dxdtda,
\end{aligned}
\tag{28}
$$

where $\psi(t,a,x) = \beta(t,a,x)\phi(x,t) \in L^2(Q)$. By letting $n \to +\infty$ in (28) while using (21), we obtain

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_{Q_T} y_n(t,0,x,\theta)\phi(x,t)\,dxdt \\
&\quad = \int_Q y(t,a,x,\theta)\psi(t,a,x)\,dxdtda \\
&\quad = \int_Q \beta(t,a,x)y(t,a,x,\theta)\phi(x,t)\,dxdtda \\
&\quad = \int_{Q_T}\left(\int_A \beta(t,a,x)y(t,a,x,\theta)da\right)\phi(x,t)\,dxdt.
\end{aligned}
$$

So that (27) holds. Moreover, by taking the limit as $n \to +\infty$ in (26) and by using (20)–(22) and (27), we are lead to

$$
\begin{aligned}
&(y_T,\phi(0,\cdot,\cdot))_{L^2(Q_A)} - (y^0,\phi(0,\cdot,\cdot))_{L^2(Q_A)} \\
&\quad - (y^1,\phi(\cdot,0,\cdot))_{L^2(Q_T)} + \big(y, L^*\phi\big)_{L^2(Q)} = (f,\phi)_{L^2(Q)} + (u,\phi)_{L^2(Q_\omega)}, \\
&\quad \forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot,A,\cdot) = 0 \text{ in } Q_T.
\end{aligned}
$$

An integration by parts gives

$$
\begin{aligned}
&(y_T,\phi(T,\cdot,\cdot))_{L^2(Q_A)} - (y^0,\phi(0,\cdot,\cdot))_{L^2(Q_A)} - (y^1,\phi(\cdot,0,\cdot))_{L^2(Q_T)} \\
&\quad - (y(T,\cdot,\cdot,\theta),\phi(T,\cdot,\cdot))_{L^2(Q_A)} + (y(0,\cdot,\cdot,\theta),\phi(0,\cdot,\cdot))_{L^2(Q_A)}
\end{aligned}
$$

$$
\begin{aligned}
&- (y(\cdot,0,\cdot,\theta), \phi(\cdot,0,\cdot))_{L^2(Q_T)} + (Ly,\phi)_{L^2(Q)} \\
&- \left\langle y, \frac{\partial \phi}{\partial \nu} \right\rangle_{H^{-1}(U;H^{-\frac{1}{2}}),H_0^1(U;H^{\frac{1}{2}})} = (f,\phi)_{L^2(Q)} + (u,\phi)_{L^2(Q_\omega)}, \\
&\forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T.
\end{aligned}
$$

That is, in view of (25),

$$
\begin{aligned}
&(y_T, \phi(T,\cdot,\cdot))_{L^2(Q_A)} - (y^0, \phi(0,\cdot,\cdot))_{L^2(Q_A)} - (y^1, \phi(\cdot,0,\cdot))_{L^2(Q_T)} \\
&\quad - (y(T,\cdot,\cdot,\theta), \phi(T,\cdot,\cdot))_{L^2(Q_A)} + (y(0,\cdot,\cdot,\theta), \phi(0,\cdot,\cdot))_{L^2(Q_A)} \\
&\quad - (y(\cdot,0,\cdot,\theta), \phi(\cdot,0,\cdot))_{L^2(Q_T)} - \left\langle y, \frac{\partial \phi}{\partial \nu} \right\rangle_{H^{-1}(U;H^{-\frac{1}{2}}),H_0^1(U;H^{\frac{1}{2}})} = 0, \\
&\forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T.
\end{aligned} \tag{29}
$$

Choosing, respectively, in (29), ϕ such that $\phi(T,\cdot,\cdot) = 0$, $\phi(\cdot,0,\cdot) = 0$, $\frac{\partial \phi}{\partial \nu} = 0$ and $\phi(\cdot,0,\cdot) = 0$, $\frac{\partial \phi}{\partial \nu} = 0$ and $\frac{\partial \phi}{\partial \nu} = 0$, we successively obtain

$$
y(0,\cdot,\cdot,\theta) = y^0 \quad \text{in } Q_A. \tag{30}
$$

$$
y_T = y(T,\cdot,\cdot,\theta) \quad \text{in } Q_A. \tag{31}
$$

$$
y(\cdot,0,\cdot,\theta) = \int_0^A \beta(t,a,x) y(t,a,x,\theta) da \quad \text{in } Q_T, \tag{32}
$$

and finally from (29),

$$
y = 0 \quad \text{on } \Sigma. \tag{33}
$$

By (25) and (30)-(33), it follows that (u, y) solves (1). Moreover, if we set

$$
V_n = \int_I y_n(T,\cdot,\cdot,\theta)\, d\theta,
$$

then in view of (15), there exist a subsequence of the sequence $(V_n)_n$ still denoted $(V_n)_n$ and $V \in L^2(Q_A)$ such that as $n \to +\infty$, $\forall \phi \in L^2(Q_A)$,

$$
\begin{aligned}
\int_{Q_A} V_n(a,x)\phi(a,x)\, dadx &= \int_I \left(\int_{Q_A} y_n(T,\cdot,\cdot,\theta)\phi(a,x) dadx \right) d\theta \\
&\to \int_{Q_A} V(a,x)\phi(a,x)\, dadx.
\end{aligned} \tag{34}
$$

Now, using (18) we deduce that the sequence $(y_n(T,\cdot,\cdot,\theta))_n$ is bounded independently of θ. Moreover, using (22) and ((31)), it follows that

$$\lim_{n\to+\infty}\int_{Q_A} y_n(T,a,x,\theta)\phi(a,x)\,dadx = \int_{Q_A} y(T,a,x,\theta)\phi(a,x)\,dadx,$$

for all $\phi \in L^2(Q_A)$. If we set $z_n = \int_{Q_A} y_n(T,a,x,\theta)\phi(a,x)\,dadx$, then using (18), we get, for all $n \in \mathbb{N}$,

$$|z_n| \le C\|\phi\|_{L^2(Q_A)}.$$

It follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}\lim_{n\to+\infty}\int_I z_n\,d\theta &= \int_I \lim_{n\to+\infty} z_n d\theta\\ &= \int_I\left(\int_{Q_A} y(T,a,x,\theta)\phi(a,x)\,dadx\right)d\theta.\end{aligned}$$

So, $\int_{Q_A} V_n(a,x)\phi(a,x)\,dadx$ converges towards

$$\int_I\left(\int_{Q_A} y_T(a,x,\theta)\phi(a,x)dadx\right)d\theta = \int_{Q_A}\left(\int_I y(T,a,x,\theta)d\theta\right)\phi(a,x)dadx,$$
$$\forall\phi\in L^2(Q_A).$$

Using (34) and the uniqueness of the limit, we have that for $(a,x)\in Q_A$

$$V(a,x) = \int_I y(T,a,x,\theta)\,d\theta,$$

so we can write

$$\int_I y_n(T,\cdot,\cdot,\theta)\,d\theta \rightharpoonup \int_I y(T,\cdot,\cdot,\theta)d\theta \qquad \text{weakly in} \quad Q_A. \tag{35}$$

According to (13), from the weak lower semi-continuity of the function $v\mapsto J(v)$, (20) and (35), we obtain $J(u) \le \liminf_{n\longrightarrow+\infty} J(v_n)$, which implies that $J(u)\le\alpha$. But since α is the lower bound, we then have $\alpha = J(u)$. In addition, the function J is strictly convex. Therefore, u is unique. ■

We can now characterize the control u.

Proposition 3.2 *Let u be the solution of (1). Then, there exists $q \in L^2(U; H_0^1(\Omega))$ such that $\int_I q(\theta)\, d\theta \in L^2(Q)$ and $\{y, q\}$ is a solution to*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - d(\theta)\Delta y + \mu y = f + u\chi_{Q_\omega} & in \ \ Q, \\ y = 0 & on \ \Sigma, \\ y(0, \cdot, \cdot, \theta) = y^0 & in \ \ Q_A, \\ y(\cdot, 0, \cdot, \theta) = \displaystyle\int_0^A \beta(t, a, x) y(t, a, x, \theta)\, da & in \ \ Q_T, \end{cases} \tag{36}$$

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - d(\theta)\Delta q + \mu q = \beta(t, a, x) q(\cdot, 0, \cdot, \theta) & in \ \ Q, \\ q = 0 & on \ \Sigma, \\ q(T, \cdot, \cdot, \theta) = \displaystyle\int_I y(\theta, u)(T)\, d\theta - z_d & in \ \ Q_A, \\ q(\cdot, A, \cdot, \theta) = 0 & in \ \ Q_T, \end{cases} \tag{37}$$

and

$$u = -\frac{1}{N} \int_I q(\theta)\, d\theta \qquad in \quad Q_\omega. \tag{38}$$

Proof We write the Euler–Lagrange first-order optimality condition that characterizes the optimal control u.

$$\lim_{\lambda \to 0} \frac{J(u + \lambda v) - J(u)}{\lambda} = 0, \qquad \forall v \in L^2(Q_\omega).$$

Let $v \in L^2(Q_\omega)$ and $\lambda > 0$. We have

$$J(u + \lambda v) - J(u) = \left\| \int_I [y(\theta; u + \lambda v)(T) - y(\theta; u)(T)]\ d\theta \right\|^2_{L^2(Q_A)}$$

$$+ N^2 \lambda^2 \|v\|^2_{L^2(Q_\omega)} + 2N\lambda \int_{L^2(Q_\omega)} uv\ dadxdt$$

$$+ 2 \left(\int_I [y(\theta; u + \lambda v)(T) - y(\theta; u)(T)]\ d\theta; \int_I y(\theta; u)(T)\, d\theta - z_d \right)_{L^2(Q_A)}.$$

Then,

$$\frac{J(u + \lambda v) - J(u)}{\lambda} = \lambda \left\| \int_I \left[\frac{y(\theta; u + \lambda v)(T) - y(\theta; u)(T)}{\lambda} \right] d\theta \right\|^2_{L^2(Q_A)}$$

$$+ N^2\lambda\|v\|^2_{L^2(Q_\omega)} + 2N\int_{L^2(Q_\omega)} uv\, dadxdt$$
$$+ 2\left(\int_I \left[\frac{y(\theta; u+\lambda v)(T) - y(\theta; u)(T)}{\lambda}\right] d\theta; \int_I y(\theta; u)(T)\, d\theta - z_d\right)_{L^2(Q_A)}. \tag{39}$$

Let us set $z_\lambda := z_\lambda(\theta; v) = \dfrac{y(\theta; u+\lambda v) - y(\theta; u)}{\lambda}$; then, using Proposition 2.2, we deduce that as $\lambda \to 0$, the sequence z_λ converges strongly to $z(\theta; v)$ in $L^2(U; H_0^1(\Omega))$, where $z = z(\theta; v)$ is solution to

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} - d(\theta)\Delta z + \mu z = v\chi_{Q_\omega} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ z(\cdot, 0, \cdot, \theta) = \displaystyle\int_0^A \beta(t, a, x) z(t, a, x, \theta)\, da & \text{in } Q_T. \end{cases} \tag{40}$$

Moreover, since $\int_I y(\theta; u)(T)\; d\theta \in L^2(Q_A)$, we also have $\int_I z(\theta; v)(T)\; d\theta \in L^2(Q_A)$. Taking (40) into account, passing to the limit as $\lambda \to 0$ in (39) and using the Lebesgue dominated convergence theorem, we are lead to

$$\left(\int_I z(T, \cdot, \cdot, \theta)\, d\theta; \int_I y(\theta; u)(T)\, d\theta - z_d\right)_{L^2(Q_A)} + N\int_{L^2(Q_\omega)} uv\, dadxdt = 0,$$
$$\forall v \in L^2(Q_\omega). \tag{41}$$

To interpret ((41)), we consider the following adjoint system:

$$\begin{cases} L^* q = \beta(t, a, x) q(\cdot, 0, \cdot, \theta) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T, \cdot, \cdot, \theta) = \displaystyle\int_I y(\theta; u)(T)\, d\theta - z_d & \text{in } Q_A, \\ q(\cdot, A, \cdot, \theta) = 0 & \text{in } Q_T. \end{cases} \tag{42}$$

where $q = q(\theta, u)$ is such that $\int_I q(\theta)\; d\theta \in L^2(Q)$. Since $\beta(t, a, x) \in L^\infty(Q)$ and $q(\cdot, 0, \cdot, \theta) \in L^2(Q_T)$, then $\beta(t, a, x) q(\cdot, 0, \cdot, \theta) \in L^2(Q)$, and using that $\int_I y(\theta; u)(T)\, d\theta - z_d \in L^2(Q_A)$, it follows that $q \in L^2(U; H_0^1(\Omega))$ and $\dfrac{\partial q}{\partial t} + \dfrac{\partial q}{\partial a} \in L^2(U; H^{-1}(\Omega))$. So, if we multiply the first equation in (40) by q solution of (42) and integrating by parts over Q, we obtain

$$
\begin{aligned}
&(z(T,\cdot,\cdot,\theta), q(T,\cdot,\cdot,\theta))_{L^2(Q_A)} + (z(\cdot,A,\cdot,\theta), q(\cdot,A,\cdot,\theta))_{L^2(Q_T)} \\
&\quad - (z(\cdot,0,\cdot,\theta), q(\cdot,0,\cdot,\theta))_{L^2(Q_T)} - \int_\Sigma q \frac{\partial z}{\partial \nu}\, d\sigma dx dt + \left(z, L^* q\right)_{L^2(Q)} \\
&\quad = (v, q)_{L^2(Q_\omega)};
\end{aligned}
$$

that is,

$$
\begin{aligned}
&(z(T,\cdot,\cdot,\theta), q(T,\cdot,\cdot,\theta))_{L^2(Q_A)} + (z(\cdot,A,\cdot,\theta), q(\cdot,A,\cdot,\theta))_{L^2(Q_T)} \\
&\quad - (z, \beta(t,a,x) q(\cdot,0,\cdot,\theta))_{L^2(Q)} - \int_\Sigma q \frac{\partial z}{\partial \nu}\, d\sigma dx dt + \left(z, L^* q\right)_{L^2(Q)} \\
&\quad = (v, q)_{L^2(Q_\omega)}. \qquad (43)
\end{aligned}
$$

Using (42), (43) is rewritten as

$$
\left(z(T,\cdot,\cdot,\theta); \int_I y(\theta; u)(T)\, d\theta - z_d\right)_{L^2(Q_A)} = \int_{Q_\omega} vq\, dx dt; \qquad (44)
$$

then, an integration by parts with respect to θ on J leads us to

$$
\left(\int_I z(T,\cdot,\cdot,\theta)\, d\theta; \int_I y(\theta; u)(T)\, d\theta - z_d\right)_{L^2(Q_A)} = \left(v, \int_I q(\theta)\, d\theta\right)_{L^2(Q_\omega)}. \qquad (45)
$$

Combining (41) and (45), we obtain

$$
\int_{L^2(Q_\omega)} v \left(\int_I q(\theta)\, d\theta\right) da dx dt + N \int_{L^2(Q_\omega)} uv\, da dx dt = 0 \quad \forall v \in L^2(Q_\omega);
$$

that is,

$$
\int_{L^2(Q_\omega)} \left(\int_I q(\theta)\, d\theta + Nu\right) v\, da dx dt = 0 \quad \forall v \in L^2(\omega_T),
$$

which implies that

$$
u = -\frac{1}{N} \int_I q(\theta)\, d\theta \qquad \text{in} \quad Q_\omega. \qquad (46)
$$

■

4 Conclusion

In this chapter, we proved that after averaging the cost function related to our model, the system is still controllable and gives an optimal control that does not depend on the unknown parameter.

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C_0-Semigroup and Stepanov-Like Almost Automorphic Functions in Matched Spaces of Time Scales

Chao Wang and Gaston M. N'Guérékata

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1 Introduction

In 1935, S. Bochner introduced the concept of almost automorphic functions referred to some research aspects of differential geometry (see [1]). As is well known that almost periodicity is a particular case of almost automorphy. For some main research contributions of almost automorphic functions and the applications to dynamic equations, one may refer to the literature [2–5]. Based on these works, some excellent contributions of generalizations such as weighted pseudo almost automorphic functions and Stepanov-like almost automorphic functions were achieved (see [6, 7]).

In 1988, the time scale theory was initiated by S. Hilger (see [8]) to unify the discrete and continuous analysis (see [9]). This powerful tool was also applied to study almost automorphic functions and dynamic equations by many researchers (see [10–14]). In 2017, the almost periodic problems in matched spaces of time scales were proposed and studied for the first time, which can cover the quantum case (see [15, 16]), and then the concept of matched spaces of time scales is introduced to enlarge the scope of suitable time scales on mathematical analysis (see [17]), which can include the new periodic time scale initiated by M. Adıvar

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C. Wang (✉)
Department of Mathematics, Yunnan University, Kunming, Yunnan, People's Republic of China
e-mail: chaowang@ynu.edu.cn

G. M. N'Guérékata
Department of Mathematics, Morgan State University, Baltimore, MD, USA
e-mail: Gaston.NGuerekata@morgan.edu

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as a special case (see [18]). Based on these works, some major contributions were achieved in relation to almost periodic and almost automorphic topic under matched spaces and irregular time scales for the first time (see [19–23]).

This chapter mainly focuses on C_0-semigroup and Stepanov-like almost automorphic functions on a quantum time scale and generalizations under matched spaces of time scales, and it is organized as follows. In Sect. 2, the C_0-semigroup on a quantum time scale is introduced and investigated. In Sect. 3, Stepanov-like almost automorphic functions on a quantum time scale are introduced and their basic properties are investigated. Section 4 is devoted to generalizing the quantum case to more comprehensive irregular time scales under the theory of matched spaces of time scales.

2 C_0-Semigroup on a Quantum Time Scale

The Π-semigroup for invariant under translation time scales was proposed in the literature [24]. In this section, we will introduce the concept of C_0-semigroup on a quantum time scale and study its properties.

Let a quantum time scale be $T_q = \{q^n : q > 1, n \in \mathbb{Z}\}$, and we introduce the notations $T_q^- = \{q^n : q > 1, n \in \mathbb{Z}^-\}$, $T_q^+ = \{q^n : q > 1, n \in \mathbb{Z}^+\}$, and $T_q = T_q^- \cup T_q^+ \cup \{1\}$.

Now, we denote $T_q^\tau = \{t \cdot \tau : t \in T_q\}$ and $T_q^{\tau^{-1}} = \{t \cdot \tau^{-1} : t \in T_q\}$ for $\tau \in T_q$. Then, if $\tau \in T_q \backslash \{1\}$, then $T_q = T_q^\tau$, i.e., T_q coincides exactly with T_q^τ after a shift τ, and we say T_q is invariant with respect to the operation $(\cdot)$.

The following theorem is obvious through direct calculation check.

Theorem 2.1 *Let T_q be a quantum time scale. Then,*

(i) $\forall \tau_1, \tau_2 \in T_q$, *we have* $\tau_1 \cdot \tau_2 \in T_q$.
(ii) $\forall \tau_1, \tau_2, \tau_3 \in T_q$, *we have* $(\tau_1 \cdot \tau_2) \cdot \tau_3 = \tau_1 \cdot (\tau_2 \cdot \tau_3)$.
(iii) *There exists an element* $e = 1 \in T_q$, *such that for all elements* $\tau \in T_q$, *the equation* $1 \cdot \tau = \tau \cdot 1 = \tau$ *holds.*
(iv) $\forall \tau \in T_q$, *there exists an element* $\tau^{-1} \in T_q$ *such that* $\tau \cdot \tau^{-1} = 1$, *where* 1 *is the identity element.*
(v) $\forall \tau_1, \tau_2 \in T_q$, *we have* $\tau_1 \cdot \tau_2 = \tau_2 \cdot \tau_1$.

According to Theorem 2.1, the following result follows immediately.

Theorem 2.2 *The pair* $(T_q, \cdot)$ *forms an Abelian group.*

Next, let $\mathbb{X}$ be a Banach space and $\mathrm{T}_\tau : \mathbb{X} \to \mathbb{X}$ be a transformation for $\tau \in T_q$. Obviously, $\{\mathrm{T}_\tau : \tau \in T_q\}$ is a set with a single parameter. Then, the multiplication is defined as follows:

$$\mathrm{T}_{\tau_1}\mathrm{T}_{\tau_2} = \mathrm{T}_{\tau_1 \cdot \tau_2}, \tag{2.1}$$

which yields that

$$\mathrm{T}_{\tau_1}\big(\mathrm{T}_{\tau_2}\mathrm{T}_{\tau_3}\big) = \big(\mathrm{T}_{\tau_1}\mathrm{T}_{\tau_2}\big)\mathrm{T}_{\tau_3} = \mathrm{T}_{\tau_1\cdot\tau_2\cdot\tau_3},$$

$I = \mathrm{T}_e$ is the identity, and $\mathrm{T}_{\tau^{-1}}$ is the inverse element of T_τ. By these definitions and notations, the following theorem is immediate.

Theorem 2.3 $\{\mathrm{T}_\tau : \tau \in T_q\}$ *forms an Abelian operator group with respect to the multiplication defined by* (2.1).

In view of Theorem 2.3, we will establish some basic concepts that are necessary to define a C_0-semigroup for a quantum invariant time scale.

Definition 2.1 Let $\mathbb{T}$ be a quantum invariant time scale and $\{\mathrm{T}_\tau\}$ be a collection of bounded linear operators on Banach space $\mathbb{X}$. If for all $\tau_1, \tau_2 \in T_q^+$, the following holds:

$$\mathrm{T}_{\tau_1\cdot\tau_2} = \mathrm{T}_{\tau_1}\mathrm{T}_{\tau_2}, \tag{2.2}$$

then $\{\mathrm{T}_\tau : \tau \in T_q^+\}$ is called a single-parameter operator semigroup; if (2.2) holds for all $\tau \in T_q$, we call $\{\mathrm{T}_\tau : \tau \in T_q\}$ a single-parameter operator group.

Definition 2.2 Let T_q be a quantum time scale and $\{\mathrm{T}_\tau : \tau \in T_q^+\}$ be an operator group on a Banach space $\mathbb{X}$, i.e.,

$$\mathrm{T}_{\tau_1}\mathrm{T}_{\tau_2} = \mathrm{T}_{\tau_1\cdot\tau_2}, \quad \tau_1, \tau_2 \in T_q^+, \quad \mathrm{T}_e = I.$$

Then, $\{\mathrm{T}_\tau : \tau \in T_q^+\}$ is said to be the strong continuous operator semigroup or the C_0-semigroup.

In what follows, we introduce the definition of infinitesimal generator of a C_0-semigroup on a quantum invariant time scale.

Definition 2.3 Let T_q be a quantum time scale and $\{\mathrm{T}_\tau : \tau \in T_q^+\}$ be a C_0-semigroup on a Banach space $\mathbb{X}$. Let $\mathscr{D}$ denote a subset of $\mathbb{X}$, which has the property that for each $x \in \mathscr{D}$ there exists a $y \in \mathbb{X}$ such that

$$\left\| \frac{1}{q-1}(\mathrm{T}_q - I)x - y \right\| = 0. \tag{2.3}$$

We define $A : \mathscr{D} \to \mathbb{X}$ satisfying $A = \frac{1}{q-1}(\mathrm{T}_q - I)$ and $Ax = y$, where y is fixed by (2.3) and A is called the infinitesimal generator of the C_0-semigroup.

Theorem 2.4 *Let T_q be a quantum time scale, $\{\mathrm{T}_\tau : \tau \in T_q^+\}$ be a C_0-semigroup on Banach space $\mathbb{X}$, and A be the infinitesimal generator of the C_0-semigroup. Then, A is a closed densely defined operator, and for every $x \in \mathscr{D}(A)$, the following holds:*

$$\frac{\mathrm{d_q}}{\mathrm{d}_q t}\mathrm{T}_\tau x = A(\mathrm{T}_\tau x) = \mathrm{T}_\tau Ax,$$

that is

$$(\mathrm{T}_\tau x) - x = \int_1^\tau A\mathrm{T}_s x \mathrm{d_q} s = \int_1^\tau \mathrm{T}_s Ax \mathrm{d_q} s,$$

where $\mathscr{D}(A)$ denotes the domain of the operator A.

Proof By Definition 2.3, this theorem is immediate by direct calculation. ■

3 Stepanov-Like Almost Automorphic Functions on a Quantum Time Scale

In this section, we will introduce the concept of Stepanov-like almost automorphic functions on a quantum time scale and provide some basic properties.

Definition 3.1 A function $f \in C(T_q, \mathbb{X})$, where $\mathbb{X}$ is a Banach space, is said to be almost automorphic (a.a. for short) in Bochner's sense if for every sequence of $(s_n^{'}) \subset T_q$, there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f(t \cdot s_n)$$

is well defined for each $t \in T_q$, and

$$\lim_{n\to\infty} g(t \cdot s_n^{-1}) = f(t)$$

for each $t \in T_q$.

If the convergence above is uniform in $t \in \mathbb{X}$, then f is almost periodic in Bochner's sense. We denote by $AA_q(\mathbb{X})$ the collection of all (Bochner) almost automorphic functions $T_q \to \mathbb{X}$. Then, similar to the results from the literature [2, 25], we have

Theorem 3.1 *If $f, f_1, f_2 \in AA_q(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA_q(\mathbb{X})$;
(ii) $\lambda f \in AA_q(\mathbb{X})$ *for any scalar* λ;
(iii) $f_\alpha \in AA_q(\mathbb{X})$ *where* $f_\alpha : T_q \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot\alpha)$;
(iv) *the range* $R_f := \{f(t) : t \in T_q\}$ *is relatively compact in* $\mathbb{X}$, *and thus* f *is bounded in norm; and*
(v) *if* $f_n \to f$ *uniformly on* T_q *where* $f_n \in AA_q(\mathbb{X})$, *then* $f \in AA_q(\mathbb{X})$.

Proof Similar to the proof process in the literature [25], one can easily prove this theorem through replacing the operation $+$ by $\cdot$, and it will be omitted here. ■

$AA_q(\mathbb{X})$ equipped with the sup-norm $\|f\|_{AA_q(\mathbb{X})} = \sup_{t\in T_q} \|f(t)\|$ turns out to be a Banach space. Now, we denote by $AA_q^u(\mathbb{X})$ the closed subspace of all functions $f \in AA_q(\mathbb{X})$ with $g \in C(T_q, \mathbb{X})$. Equivalently, $f \in AA_q^u(\mathbb{X})$ if and only if f is almost automorphic and all convergences in Definition 3.1 are uniform on compact intervals. Obviously, we have

$$AP_q(\mathbb{X}) \subseteq AA_q^u(\mathbb{X}) \subset AA_q(\mathbb{X}) \subset BC_q(\mathbb{X}),$$

where $BC_q(\mathbb{X})$ stands for the Banach space of bounded and continuous functions with values in $\mathbb{X}$.

Definition 3.2 The Bochner transform $f^b(t, s), t \in T_q, s \in [1, L]_{T_q}$, of a function $f(t)$ on T_q, with values in $\mathbb{X}$, is defined by

$$f^b(t, s) = f(t \cdot s).$$

Remark 3.1 A function $\varphi(t, s), t \in T_q, s \in [1, L]_{T_q}$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t, s) = f^b(t, s),$$

if and only if

$$\varphi(t \cdot \tau, s \cdot \tau^{-1}) = \varphi(s, t)$$

for all $t \in T_q, s \in [1, L]_{T_q}$ and $\tau \in [sL^{-1}, s]_{T_q}$.

Definition 3.3 Let $p \in [1, \infty)$. The space $BS_q^p(\mathbb{X})$ of Stepanov bounded functions, with the exponent p, consists of all measurable functions f on T_q with values in $\mathbb{X}$ such that $f^b \in L^\infty(T_q, L^p(1, L; \mathbb{X}))$. This is a Banach space with the norm:

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(T_q, L^p)} = \sup_{t\in T_q} \left(\int_t^{tL} \|f(\tau)\|_{\mathbb{X}}^p \mathrm{d_q}\tau \right)^{\frac{1}{p}}.$$

Definition 3.4 The space $AS_q^p(\mathbb{X})$ of S^p-almost automorphic functions consists of all $f \in BS_q^p(\mathbb{X})$ such that $f^b \in AA_q(L^p(1, L; \mathbb{X}))$.

Definition 3.4 also has the following equivalent form.

Definition 3.5 A function $f \in L_{loc}^p(T_q; \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : T_q \to L^p(1, L; \mathbb{X})$ is almost automorphic in the sense that for every sequence of numbers $(s_n^{'}) \subset T_q$, there exist a subsequence (s_n) and a function $g \in L_{loc}^p(T_q, \mathbb{X})$ such that

$$\left(\int_1^L \|f(t\cdot s_n\cdot s)-g(t\cdot s)\|^p \mathrm{d_q}s\right)^{\frac{1}{p}} \to 0$$

and

$$\left(\int_1^L \|g(t\cdot s_n^{-1}\cdot s)-f(t\cdot s)\|^p \mathrm{d_q}s\right)^{\frac{1}{p}} \to 0$$

as $n\to\infty$ pointwise on T_q.

Remark 3.2 Note that if $1 \le p < \hat{p} < \infty$ and $f \in L^{\hat{p}}_{loc}(T_q;\mathbb{X})$ is $S^{\hat{p}}$-almost automorphic, then f is S^p-almost automorphic. Also, if $f \in AA_q(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Remark 3.3 Note that $f \in AA^u_q(\mathbb{X})$ if and only if $f^b \in AA_q(L^\infty(1, L;\mathbb{X}))$. Hence, $AA^u_q(\mathbb{X})$ can be regarded as $AS^\infty_q(\mathbb{X})$.

Theorem 3.2 *We have the following equivalent statements:*

(i) $f \in AS^p_q(\mathbb{X})$*;*
(ii) $f^b \in AA^u_q(L^p(1, L;\mathbb{X}))$*; and*
(iii) for each sequence $(s_n') \subset T_q$*, there exists a subsequence* (s_n) *such that*

$$g(t) := \lim_{n\to\infty} f(t\cdot s_n) \tag{3.1}$$

exists in the space $L^p_{loc}(T_q;\mathbb{X})$ *and*

$$f(t) = \lim_{n\to\infty} g(t\cdot s_n^{-1}) \tag{3.2}$$

in the sense of $L^p_{loc}(T_q,\mathbb{X})$.

Proof $(ii) \Rightarrow (i)$: trivial.
$(iii) \Rightarrow (ii)$: Now, we prove that

$$\lim_{n\to\infty} f^b(t\cdot s_n,\tau) = g^b(t,\tau)$$

in $C(T_q; L^p(1, L;\mathbb{X}))$. In fact,

$$\sup_{t\in[h^{-1},h]} \|f^b(t\cdot s_n,\tau)-g^b(t,\tau)\|_{L^p(1,L;\mathbb{X})} \le \left(\int_{h^{-1}}^{hL} \|f(t\cdot s_n)-g(t)\|^p \mathrm{d_q}t\right)^{\frac{1}{p}} \to 0.$$

Similarly, we have

$$\lim_{n\to\infty} g^b(t\cdot s_n^{-1},\tau) = f^b(t,\tau)$$

in $C(T_q; L^p(1, L;\mathbb{X}))$.

$(i) \Rightarrow (iii)$: Let $f^b(t \cdot s_n, \tau) \to \varphi(t, \tau)$ pointwise, where φ is a measurable function with values in $L^p(1, L; \mathbb{X})$). By Remark 3.1, $\varphi(t, \tau) = g^b(t, \tau)$, here, $g^b \in L^p_{loc}(1, L; \mathbb{X})$. Let

$$\rho_n := \int_{h^{-1}}^{h} \|f(t \cdot s_n) - g(t)\|^p \mathrm{d_q} t.$$

We will prove that $\rho_n \to 0$. Indeed, assume that $\hat{n}(h^{-1})$ and $\hat{n}(h)$ are the positive integers such that $q^{\hat{n}(h^{-1})} = h^{-1}$ and $q^{\hat{n}(h)} = h$; then, we have

$$\begin{aligned}\rho_n &= \sum_{k=\hat{n}(h^{-1})}^{\hat{n}(h)} \int_{q^k}^{q^k L} \|f(t \cdot s_n) - g(t)\|^p \mathrm{d_q} t \\ &= \sum_{k=\hat{n}(h^{-1})}^{\hat{n}(h)} \|f^b(q^k \cdot s_n, \tau) - g^b(q^k, \tau)\|^p_{L^p(1,L;\mathbb{X})} \to 0,\end{aligned}$$

which implies (3.1).

Similarly, we can obtain (3.2). The proof is completed. ■

Theorem 3.3 *$AS^p_q(\mathbb{X})$ is a closed linear subspace of $BS^p_q(\mathbb{X})$.*

Proof First, we prove that $AS^p_q(\mathbb{X})$ is closed linear subspaces of $BS^p_q(\mathbb{X})$. Now, let $f_1, f_2 \in AS^p_q(\mathbb{X})$; then, by Definition 3.3, $f_1^b, f_2^b \in AA_q(L^p(1, L; \mathbb{X}))$, so by Definition 3.3 and Theorem 3.2, we have $f_1^b, f_2^b \in AA_q(\mathbb{X})$.

By Minkowski's lemma, we have

$$\begin{aligned}\|f_1 + f_2\| = \|f_1^b + f_2^b\|_{L^\infty(T_q, L^p)} &= \sup_{t \in T_q} \left(\int_t^{tL} \|f_1(\tau) + f_2(\tau)\|^p_{\mathbb{X}} \mathrm{d_q}\tau \right)^{\frac{1}{p}} \\ &\le \sup_{t \in T_q} \left(\int_t^{tL} \|f_1(\tau)\|^p_{\mathbb{X}} \mathrm{d_q}\tau \right)^{\frac{1}{p}} + \sup_{t \in T_q} \left(\int_t^{tL} \|f_2(\tau)\|^p_{\mathbb{X}} \mathrm{d_q}\tau \right)^{\frac{1}{p}} \\ &= \|f_1^b\|_{L^\infty(T_q, L^p)} + \|f_2^b\|_{L^\infty(T_q, L^p)} = \|f_1\|_{S^p} + \|f_2\|_{S^p}.\end{aligned}$$

Hence, we have $f_1 + f_2 \in AS^p_q(\mathbb{X})$.

Moreover, it is clear that $\lambda f_1 \in AS^p_q(\mathbb{X})$ for any scalar λ.

Finally, by employing again Minkowski's lemma, we can prove that if (f_n) is a sequence in $AS^p_q(\mathbb{X})$ that converges to f in S^p-norm, then $f \in AS^p_q(\mathbb{X})$. The proof is completed. ■

The following theorem is immediate.

Theorem 3.4 *Let $f \in AS^p_q(\mathbb{X})$ and $\mathcal{A} \in L(\mathbb{X})$, the Banach algebra of all bounded linear operators $\mathbb{X} \to \mathbb{X}$. Then, $\mathcal{A} f \in AS^p_q(\mathbb{X})$.*

Now, we have the following composition theorem.

Theorem 3.5 *Let $F : T_q \times \mathbb{X} \to \mathbb{X}$ be S^p-almost automorphic. Suppose that $F(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in T_q$, i.e., there exists $L > 0$ such that*

$$\|F(t,u) - F(t,v)\| \le L\|u - v\|$$

for all $t \in T_q$, $(u,v) \in \mathbb{X} \times \mathbb{X}$.

If $\phi \in AS_q^p(\mathbb{X})$, then $\Upsilon : T_q \to \mathbb{X}$ defined by $\Upsilon(\cdot) := F(\cdot, \phi(\cdot))$ belongs to $AS_q^p(\mathbb{X})$.

Proof Since $\phi \in AS_q^p(\mathbb{X})$, for every sequence $(s_n^{'})$, there exist a subsequence (s_n) and a function $\psi \in L^p(T_q, \mathbb{X})$ such that

$$\left(\int_1^L \|\phi(t \cdot s_n \cdot s) - \psi(t \cdot s)\|^p \mathrm{d_q} s\right)^{\frac{1}{p}} \to 0, \tag{3.3}$$

and

$$\left(\int_1^L \|\psi(t \cdot s_n^{-1} \cdot s) - \phi(t \cdot s)\|^p \mathrm{d_q} s\right)^{\frac{1}{p}} \to 0 \tag{3.4}$$

as $n \to \infty$ on T_q pointwise.

Since $F : T_q \times \mathbb{X} \to \mathbb{X}$, $(t,u) \to F(t,u)$ is S^p-almost automorphic in $t \in T_q$ uniformly in $u \in \mathbb{X}$, for every sequence $(\sigma_n^{'})$, there exist a subsequence (σ_n) and a function $G(\cdot,u) \in L^p(T_q;\mathbb{X})$ such that

$$\left(\int_1^L \|F(t \cdot \sigma_n \cdot s, u) - G(t \cdot s, u)\|^p \mathrm{d_q} s\right)^{\frac{1}{p}} \to 0, \tag{3.5}$$

and

$$\left(\int_1^L \|G(t \cdot \sigma_n^{-1} \cdot s, u) - F(t \cdot s, u)\|^p \mathrm{d_q} s\right)^{\frac{1}{p}} \to 0 \tag{3.6}$$

as $n \to \infty$ on T_q pointwise for each $u \in \mathbb{X}$.

Now, by employing Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_1^L \|F(t \cdot s_n \cdot s, \phi(t \cdot s_n \cdot s)) - G(t \cdot s, \psi(t \cdot s))\|^p \mathrm{d_q} s\right)^{\frac{1}{p}} \\
&\quad \le \left(\int_1^L \|F(t \cdot s_n \cdot s, \phi(t \cdot s_n \cdot s)) - F(t \cdot s_n \cdot s, \psi(t \cdot s))\|^p \mathrm{d_q} s\right)^{\frac{1}{p}}
\end{aligned}$$

$$\left(\int_1^L \|F(t\cdot s_n\cdot s,\psi(t\cdot s))-G(t\cdot s,\psi(t\cdot s))\|^p \mathrm{d_q}s\right)^{\frac{1}{p}}$$

$$\leq L\left(\int_1^L \|\phi(t\cdot s_n\cdot s)-\psi(t\cdot s)\|^p \mathrm{d_q}s\right)^{\frac{1}{p}}$$

$$+\left(\int_1^L \|F(t\cdot s_n\cdot s,\psi(t\cdot s))-G(t\cdot s,\psi(t\cdot s))\|^p \mathrm{d_q}s\right)^{\frac{1}{p}},$$

and though (3.3) and (3.5), we have

$$\left(\int_1^L \|F(t\cdot s_n\cdot s,\phi(t\cdot s_n\cdot s))-G(t\cdot s,\psi(t\cdot s))\|^p \mathrm{d_q}s\right)^{\frac{1}{p}}\to 0$$

as $n\to\infty$.

Similar to the above, by employing Minkowski's inequality and both (3.4) and (3.6), we can also obtain

$$\left(\int_1^L \|G(t\cdot s_n^{-1}\cdot s,\psi(t\cdot s_n^{-1}\cdot s))-F(t\cdot s,\phi(t\cdot s))\|^p \mathrm{d_q}s\right)^{\frac{1}{p}}\to 0$$

as $n\to\infty$. This completes the proof. ■

4 Weak Almost Automorphy on a Quantum Time Scale

In this section, we will introduce the concept of weakly almost automorphic functions (in short w.a.a.) on a quantum time scale.

Definition 4.1 A weakly continuous function $f:T_q\to\mathbb{X}$ is weakly almost automorphic if for each sequence $(s_n^{'})\subset T_q$, there exists a subsequence (s_n) such that

$$g(t):=weak-\lim_{n\to\infty}f(t\cdot s_n)$$

is well defined for each $t\in T_q$, and

$$weak-\lim_{n\to\infty}g(t-s_n)=f(t)$$

for each $t\in T_q$.

Also, Definition 4.1 has the following equivalent form.

Definition 4.2 f is weakly almost automorphic, if for every $v \in \mathbb{X}^*$ (dual space), the numerical function $\langle v, f \rangle$ is almost automorphic. f is weakly S^p-almost automorphic, if $\langle v, f \rangle \in AS_q^p$ $\forall v \in \mathbb{X}^*$.

For convenience, we denote the collection of all weakly almost automorphic functions by $WAA_q(\mathbb{X})$ and the collection of all weakly S^p-almost automorphic functions by $WAS_q^p(\mathbb{X})$.

The following relations are immediate:

$$AA_q(\mathbb{X}) \subset WAA_q(\mathbb{X}),$$

$$AA_q^u(\mathbb{X}) \subset WAA_q^u(\mathbb{X}),$$

$$AS_q^p(\mathbb{X}) \subset WAS_q^p(\mathbb{X}).$$

Theorem 4.1 (i) *Assume that $f \in WAA_q(\mathbb{X})$. Then, f is bounded and its range is separable. As a consequence, $f \in L^\infty(T_q, \mathbb{X})$. (ii) If $f \in WAS_q^p(\mathbb{X})$, then $f \in BS_q^p(\mathbb{X})$.*

Proof (i). By contradiction. Suppose that $\sup_{t\in T_q} \|f(t)\| = \infty$, then there exists a sequence $(s_n^{'}) \subset T_q$ such that $\lim_{n\to\infty} \|f(s_n^{'})\| = \infty$. Because f is weakly almost periodic, one can extract a subsequence $(s_n^{'})$ such that

$$\text{weak} - \lim_{n\to\infty} f(s_n) = \alpha \text{ exists.}$$

Then, $(f(s_n))$ is a weakly convergent sequence, and thus it is weakly bounded and therefore bounded; this is a contradiction. From (i), we can obtain (ii) immediately. ■

Theorem 4.2 *Let $\mathbb{X}_0 \subset \mathbb{X}_1$ be a continuous and dense embedding of Banach spaces. (i) If $f \in L^\infty(T_q, \mathbb{X}_0)$ is a weakly continuous function and $f \in WAA_q(\mathbb{X}_1)$ (respectively, $f \in WAA_q^u(\mathbb{X}_1)$), then $f \in WAA_q(\mathbb{X}_0)$ (respectively, $f \in WAA_q^u(\mathbb{X}_0)$).*

(ii) If $f \in BS_q^p(\mathbb{X}_0)$ and $f \in WAS_q^p(\mathbb{X}_1)$, then $f \in WAS_q^p(\mathbb{X}_0)$.

Proof

(i) The dual embedding $\mathbb{X}_1^* \subset \mathbb{X}_0^*$ is dense and continuous. Therefore, for each $v \in \mathbb{X}_0^*$, there exists a sequence $v_n \in X_1^*$ such that $\lim_{n\to\infty} v_n = v$ in $\mathbb{X}_0^*$. Since

$$|\langle v, f(t)\rangle - \langle v_n, f(t)\rangle| \le \|f\|_{L^\infty(T_q;\mathbb{X}_0)} \|v - v_n\|_{\mathbb{X}_0^*},$$

and all functions $\langle v_n, f \rangle$ are almost automorphic, we can obtain the desired result.

(ii) Similar to the proof of (i), we can also obtain (ii) and do not repeat it here. ■

5 Shift-Semigroup Under Matched Spaces of Time Scales

In this section, we will generalize the results of the above sections in matched spaces of time scales. Now, we recall the concept of matched spaces of timescales, and one may consult [17, 19, 20] for more details.

Definition 5.1 ([17, 20]) Let Π^* be a subset of $\mathbb{R}$ together with an operation $\tilde{\delta}$ and a pair $(\Pi^*, \tilde{\delta})$ be an Abelian group, and $\tilde{\delta}$ is increasing with respect to its second argument, i.e., Π^* and $\tilde{\delta}$ satisfy the following conditions:

(1) Π^* is closed with respect to an operation $\tilde{\delta}$, i.e., for any $\tau_1, \tau_2 \in \Pi^*$, we have $\tilde{\delta}(\tau_1, \tau_2) \in \Pi^*$.
(2) For any $\tau \in \Pi^*$, there exists an identity element $e_{\Pi^*} \in \Pi^*$ such that $\tilde{\delta}(e_{\Pi^*}, \tau) = \tau$.
(3) For all $\tau_1, \tau_2, \tau_3 \in \Pi^*$, $\tilde{\delta}\big(\tau_1, \tilde{\delta}(\tau_2, \tau_3)\big) = \tilde{\delta}\big(\tilde{\delta}(\tau_1, \tau_2), \tau_3\big)$ and $\tilde{\delta}(\tau_1, \tau_2) = \tilde{\delta}(\tau_2, \tau_1)$.
(4) For each $\tau \in \Pi^*$, there exists an element $\tau^{-1} \in \Pi^*$ such that $\tilde{\delta}(\tau, \tau^{-1}) = \tilde{\delta}(\tau^{-1}, \tau) = e_{\Pi^*}$, where e_{Π^*} is the identity element in Π^*.
(5) If $\tau_1 > \tau_2$, then $\tilde{\delta}(\cdot, \tau_1) > \tilde{\delta}(\cdot, \tau_2)$.

A subset S of $\mathbb{R}$ is called relatively dense with respect to the pair $(\Pi^*, \tilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \tilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \emptyset$ for all $a \in \Pi^*$. The number L is called the inclusion length with respect to the group $(\Pi^*, \tilde{\delta})$.

Definition 5.2 ([17, 20]) Let $\mathbb{T}$ and Π be time scales, where $\mathbb{T} = \bigcup_{i\in I_1} A_i$, $\Pi = \bigcup_{i\in I_2} B_i$. If Π^* is the largest subset of the time scale Π, i.e., $\overline{\Pi^*} = \Pi$, where $\overline{A}$ denotes the closure of the set A, and $(\Pi^*, \tilde{\delta})$ is an Abelian group, and I_1 and I_2 are countable index sets, then we say Π is an adjoint set of $\mathbb{T}$ if there exists a bijective mapping:

$$F: \qquad \mathbb{T} \qquad \to \qquad \Pi$$
$$A \in \big\{A_i,\ i \in I_1\big\} \to B \in \big\{B_i,\ i \in I_2\big\},$$

i.e., $F(A) = B$. Now, F is called the adjoint mapping between $\mathbb{T}$ and Π.

Definition 5.3 ([17, 20]) Let the pair $(\Pi^*, \tilde{\delta})$ be an Abelian group and Π^*, $\mathbb{T}^*$ be the largest open subsets of the time scales Π and $\mathbb{T}$, respectively. Furthermore, let Π be a adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. The operator $\delta : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ satisfies the following properties:

(P_1) (Monotonicity) The function δ is strictly increasing with respect to its all arguments, i.e., if

$$(T_0, t),\ (T_0, u) \in \mathcal{D}_\delta := \big\{(s, t) \in \Pi^* \times \mathbb{T}^* : \delta(s, t) \in \mathbb{T}^*\big\},$$

then $t < u$ implies $\delta(T_0, t) < \delta(T_0, u)$; if $(T_1, u), (T_2, u) \in \mathcal{D}_\delta$ with $T_1 < T_2$, then $\delta(T_1, u) < \delta(T_2, u)$.

(P_2) (Existence of inverse elements) The operator δ has the inverse operator $\delta^{-1} : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ and $\delta^{-1}(\tau, t) = \delta(\tau^{-1}, t)$, where $\tau^{-1} \in \Pi^*$ is the inverse element of τ.

(P_3) (Existence of identity element) $e_{\Pi^*} \in \Pi^*$ and $\delta(e_{\Pi^*}, t) = t$ for any $t \in \mathbb{T}^*$, where e_{Π^*} is the identity element in Π^*.

(P_4) (Bridge condition) For any $\tau_1, \tau_2 \in \Pi^*$ and $t \in \mathbb{T}^*$, $\delta\big(\tilde{\delta}(\tau_1, \tau_2), t\big) = \delta\big(\tau_1, \delta(\tau_2, t)\big) = \delta\big(\tau_2, \delta(\tau_1, t)\big)$.

Then, the operator δ associated with $e_{\Pi^*} \in \mathbb{T}^*$ is said to be shift operator on the set $\mathbb{T}^*$. The variable $s \in \Pi^*$ in δ is called the shift size. The value $\delta(s, t)$ in $\mathbb{T}^*$ indicates s units shift of the term $t \in \mathbb{T}^*$. The set $\mathcal{D}_\delta$ is the domain of the shift operator δ.

Definition 5.4 ([17, 20]) Let the pair $(\Pi^*, \tilde{\delta})$ be an Abelian group and Π^*, $\mathbb{T}^*$ be the largest open subsets of the time scales Π and $\mathbb{T}$, respectively. Furthermore, let Π be a adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. If there exists the shift operator δ satisfying Definition 5.3, then we say the group $(\mathbb{T}, \Pi, F, \delta)$ is a matched space for $\mathbb{T}$.

Let $(\mathbb{T}, \Pi, F, \delta)$ be a matched space for $\mathbb{T}$, and we introduce the notations $\Pi^*_- = \big\{\tau \in \Pi^* : \tau < e\big\}$, $\Pi^*_+ = \big\{\tau \in \Pi^* : \tau > e\big\}$, and $\Pi^* = \Pi^*_- \cup \Pi^*_+ \cup \{e\}$, where e is the identity element in Π^*. For any $\tau \in \Pi^*$, we denote $\tilde{\delta}_\tau(\tau) := \delta(\tau, \tau)$, $\tilde{\delta}_{\tau^2}(\tau) := \tilde{\delta}(\tau, \tilde{\delta}(\tau, \tau)), \ldots$.

Now, we denote $\mathbb{T}^\tau = \{\delta(\tau, t) : t \in \mathbb{T}\}$ and $\mathbb{T}^{\tau^{-1}} = \{\delta(\tau^{-1}, t) : t \in \mathbb{T}\}$ for $\tau \in \Pi^*$. Then, if $\tau \in \Pi^*\backslash\{1\}$, then if $\mathbb{T} = \mathbb{T}^\tau \cup \mathbb{T}^{\tau^{-1}}$, i.e., $\mathbb{T}$ coincides exactly with $\mathbb{T}^\tau$, we say $\mathbb{T}$ is invariant with respect to the operation δ.

Let $\mathbb{X}$ be a Banach space and $\mathcal{T}_\tau : \mathbb{X} \to \mathbb{X}$ be a transformation for $\tau \in \Pi^*$. Obviously, $\{\mathcal{T}_\tau : \tau \in \Pi^*\}$ is a set with a single parameter. Then, the multiplication is defined as follows:

$$\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2} = \mathcal{T}_{\tilde{\delta}(\tau_1, \tau_2)}, \tag{5.1}$$

which yields that

$$\mathcal{T}_{\tau_1}\big(\mathcal{T}_{\tau_2}\mathcal{T}_{\tau_3}\big) = \mathcal{T}_{\tau_1}\mathcal{T}_{\tilde{\delta}(\tau_2, \tau_3)} = \mathcal{T}_{\tilde{\delta}(\tau_1, \tilde{\delta}(\tau_2, \tau_3))} = \mathcal{T}_{\tilde{\delta}(\tilde{\delta}(\tau_1, \tau_2), \tau_3))} = \big(\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}\big)\mathcal{T}_{\tau_3},$$

$I = \mathcal{T}_e$ is the identity, and $\mathcal{T}_{\tau^{-1}}$ is the inverse element of $\mathcal{T}_\tau$. By these definitions and notations, the following theorem is immediate.

Theorem 5.1 $\{\mathcal{T}_\tau : \tau \in \Pi^*\}$ *forms an Abelian operator group with respect to the multiplication defined by* (5.1).

According to Theorem 5.1, we will introduce some basic concepts to define a **shift-semigroup** for an invariant time scale.

Definition 5.5 Let $\mathbb{T}$ be an invariant time scale and $\{\mathcal{T}_\tau\}$ be a collection of bounded linear operators on Banach space $\mathbb{X}$. If for all $\tau_1, \tau_2 \in \Pi_+^*$, the following holds:

$$\mathcal{T}_{\tilde{\delta}(\tau_1,\tau_2)} = \mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2}, \tag{5.2}$$

then $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ is called a single-parameter operator shift-semigroup; if (5.2) holds for all $\tau \in \Pi^*$, we call $\{\mathcal{T}_\tau : \tau \in \Pi^*\}$ a single-parameter operator shift-group.

Definition 5.6 Let $\mathbb{T}$ be an invariant time scale and $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ be an operator group on a Banach space $\mathbb{X}$, i.e.,

$$\mathcal{T}_{\tau_1}\mathcal{T}_{\tau_2} = \mathcal{T}_{\tilde{\delta}(\tau_1,\tau_2)}, \quad \tau_1, \tau_2 \in \Pi_+^*, \quad \mathcal{T}_e = I.$$

Then, $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ is said to be the strong continuous operator shift-semigroup.

Theorem 5.2 *Let $\mathbb{T}$ be an invariant time scale and $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ be an operator semigroup on the Banach space $\mathbb{X}$, and for any $x \in \mathbb{X}$ and any $\varepsilon > 0$, there exists a neighborhood $U = (\tau_1 - \delta_0, \tau_1 + \delta_0) \cap \Pi_+^*$ for some $\delta_0 > 0$, such that*

$$\|\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}x - x\| \le \varepsilon \quad \text{for all } \tau_2 \in U; \tag{5.3}$$

then, $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ is a shift-semigroup.

Proof For any $L \in \Pi_+^*$, we claim that

$$\sup\{\|\mathcal{T}_\tau\| : \tau \in [e, L]_{\Pi^*}\} < +\infty. \tag{5.4}$$

In fact, for any $x \in \mathbb{X}$, we can take, $h \in \Pi_+^*,\ c > 0$ such that

$$\sup\{\|\mathcal{T}_\tau x\| : \tau \in [e, h]_{\Pi^*}\} \le c.$$

Now, for $\tau \in [e, L]_{\Pi^*}$, let $\tau = \tilde{\delta}(\tilde{\delta}_{h^{k-1}}(h), r),\ r \in \Pi^*$, where $k \le \tilde{\delta}(L, h^{-1}),\ 0 \le r < h$. Then, it follows that

$$\|\mathcal{T}_\tau x\| = \|\mathcal{T}_{\tilde{\delta}_{h^{k-1}}(h)}\mathcal{T}_r x\| \le \|\mathcal{T}_{\tilde{\delta}_{h^{k-1}}(h)}\| c.$$

Hence, (5.4) holds. In what follows, we let $M := \sup\{\|\mathcal{T}_\tau\| : \tau \in [e, L]_{\Pi^*}\}$.

For any $\varepsilon > 0$, there is δ_0, such that for $\tau_2 \in (\tau_1 - \delta_0, \tau_1 + \delta_0)_{\Pi_+^*}$, we have

(i) If $\tau_2 > \tau_1$, then $\sigma_\Pi(\tau_1) = \tau_1$, and we have

$$\|\mathcal{T}_{\tau_2}x - \mathcal{T}_{\tau_1}x\| \le \|\mathcal{T}_{\sigma_\Pi(\tau_1)}(\mathcal{T}_{\tilde{\delta}(\tau_2,\sigma_\Pi^{-1}(\tau_1))} - I)x + \mathcal{T}_{\tau_1}(\mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_1^{-1})} - I)x\| \le 2M\varepsilon.$$

In the above, $\sigma_\Pi(\tau_1) = \tau_1$. In fact, if $\sigma_\Pi(\tau_1) > \tau_1$, then τ_1 is a right-scattered point, which implies that $\tau_2 = \tau_1$, and this contradicts $\tau_2 > \tau_1$.

(ii) If $\tau_2 \le \tau_1$, then $\tau_2 \le \tau_1 \le \sigma_\Pi(\tau_1)$, which yields $e \le \tilde{\delta}(\sigma_\Pi(\tau_1), \tau_1^{-1}) \le \tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})$. Hence, we have

$$\|\mathcal{T}_{\tau_2}x - \mathcal{T}_{\tau_1}x\| \le \|\mathcal{T}_{\tau_2}(I - \mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})})x + \mathcal{T}_{\tau_1}(\mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_1^{-1})} - I)x\| \le 2M\varepsilon.$$

Hence, for $\tau_2 \in (\tau_1 - \delta, \tau_1 + \delta)_{\Pi_+^*}$, the following holds:

$$\|\mathcal{T}_{\tau_2}x - \mathcal{T}_{\tau_1}x\| \le 2M\varepsilon.$$

Hence, $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ is a shift-semigroup and (5.3) holds. This completes the proof. ■

In the following, the definition of infinitesimal generator of a shift-semigroup will be introduced.

Definition 5.7 Let $\mathbb{T}$ be an invariant time scale and $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ be a shift-semigroup on a Banach space $\mathbb{X}$. Let $\mathscr{D}$ denote a subset of $\mathbb{X}$, which has the property that for each $x \in \mathscr{D}$, there exists a $y \in \mathbb{X}$ such that for any $\varepsilon > 0$, there is a neighborhood $U = (\tau_1 - \delta_0, \tau_1 + \delta_0)_{\Pi_+^*}$ for some $\delta_0 > 0$, which satisfies

$$\|(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} - I)x - y|\sigma_\Pi(\tau_1) - \tau_2|\| < \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|, \quad \tau_2 \in U. \tag{5.5}$$

For $\tau_1 \to e$, we define $A : \mathscr{D} \to \mathbb{X}$ satisfying $Ax = y$, where y is fixed by (5.5). In what follows, we call this A the infinitesimal generator of this shift-semigroup.

Remark 5.1 From Definition 5.7, we can obtain

$$\lim_{\tau_1 \to e} \lim_{\tau_2 \to \tau_1} \frac{(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} - I)x}{\sigma_\Pi(\tau_1) - \tau_2} = A.$$

Remark 5.2 From (5.5), it follows that

$$\lim_{\tau_1 \to e} \left\| \frac{\mathcal{T}_{\tau_1}(\mathcal{T}_q - I)x}{(q-1)\tau_1} \right\| = \left\| \frac{1}{q-1}(T_q - I)x - y \right\| = 0,$$

which implies that Definition 5.7 is equivalent to Definition 2.3 when $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Theorem 5.3 *Let $\mathbb{T}$ be an invariant time scale, $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ be a shift-semigroup on Banach space $\mathbb{X}$ satisfying (5.3), and A be the infinitesimal generator of the shift-semigroup. Then, A is a closed densely defined operator, and for every $x \in \mathscr{D}(A)$, the following holds:*

$$(\mathcal{T}_\tau x)^{\Delta_\Pi} = A(\mathcal{T}_\tau x) = \mathcal{T}_\tau Ax, \tag{5.6}$$

that is

$$(\mathcal{T}_\tau x) - x = \int_e^\tau A\mathcal{T}_s x \Delta_\Pi s = \int_e^\tau \mathcal{T}_s A x \Delta_\Pi s, \tag{5.7}$$

where $\mathscr{D}(A)$ denotes the domain of the operator A and Δ_Π is the differential operator over the time scale Π.

Proof First, we show that A is a densely defined operator. Note that for any $x \in \mathbb{X}$, we have

$$\begin{aligned}
&\left\| \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \mathcal{T}_\theta x \Delta_\Pi \theta - \big(|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e\big) x \right\| \\
&= \left\| \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} (\mathcal{T}_\theta x - x) \Delta_\Pi \theta \right\| \\
&\leq \Big| |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e \Big| \sup_{e \leq \theta \leq |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \|\mathcal{T}_\theta x - x\| \\
&< \Big| |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e \Big| \varepsilon.
\end{aligned} \tag{5.8}$$

Let $y = \int_e^\tau \mathcal{T}_\theta x \Delta_\Pi \theta$; then,

$$\begin{aligned}
\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} y - y &= \int_e^\tau (\mathcal{T}_{\tilde{\delta}(\theta, |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|)} - \mathcal{T}_\theta x) \Delta_\Pi \theta \\
&= \int_{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|}^{\tilde{\delta}(\tau, |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|)} \mathcal{T}_\theta x \Delta_\Pi \theta - \int_e^\tau \mathcal{T}_\theta x \Delta_\Pi \theta \\
&= \int_\tau^{\tilde{\delta}(\tau, |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|)} \mathcal{T}_\theta x \Delta_\Pi \theta - \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \mathcal{T}_\theta x \Delta_\Pi \theta \\
&= \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \mathcal{T}_\theta (\mathcal{T}_\tau x) \Delta_\Pi \theta - \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \mathcal{T}_\theta x \Delta_\Pi \theta.
\end{aligned}$$

Since (5.8) holds for any $x \in \mathbb{X}$, it follows that

$$\begin{aligned}
&\left\| (\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} y - y) - \big(|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e\big)(\mathcal{T}_\tau x - x) \right\| \\
&= \left\| \int_e^{|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|} \mathcal{T}_\theta (\mathcal{T}_\tau x - x) \Delta_\Pi \theta - \big(|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e\big)(\mathcal{T}_\tau x - x) \right\| \\
&\leq \Big| |\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})| - e \Big| \varepsilon.
\end{aligned}$$

Therefore, $y \in \mathscr{D}(A)$, so $\overline{\mathscr{D}(A)} = \mathbb{X}$.

Next, we will show that (5.6) and (5.7) hold. Since

$$\lim_{\tau_2\to\tau_1}\frac{(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}-I)\mathcal{T}_{\tau_2}x}{|\sigma_\Pi(\tau_1)-\tau_2|}=\lim_{\tau_2\to\tau_1}\frac{\mathcal{T}_{\tau_2}(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}-I)x}{|\sigma_\Pi(\tau_1)-\tau_2|}=\mathcal{T}_{\tau_1}Ax,$$

we have

$$\begin{aligned}&\|\mathcal{T}_{\tau_1}(T_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|\mathcal{T}_{\tau_1}Ax\| \\ &\quad\leq\|\mathcal{T}_{\tau_1}\|\|(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|Ax\|\leq\|\mathcal{T}_{\tau_1}\|\varepsilon|\sigma_\Pi(\tau_1)-\tau_2|,\end{aligned}\tag{5.9}$$

and so, $T_{\tau_1}x\in\mathscr{D}(A)$. From (5.9), we also have

$$\|(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|}-I)x-|\sigma_\Pi(\tau_1)-\tau_2|Ax\|\leq\varepsilon|\sigma_\Pi(\tau_1)-\tau_2|.\tag{5.10}$$

(i) If $\tau_2>\tau_1$, then from (5.10) and Theorem 5.2, it follows that

$$\begin{aligned}&\|(\mathcal{T}_{\sigma_\Pi(\tau_1)}-\mathcal{T}_{\tau_2})x-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_1}Ax\| \\ &\quad\leq\|\mathcal{T}_{\sigma_\Pi(\tau_1)}(I-T_{\tilde{\delta}(\tau_2,\sigma_\Pi^{-1}(\tau_1))})x-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\sigma_\Pi(\tau_1)}Ax \\ &\qquad+(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\sigma_\Pi(\tau_1)}Ax-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_1}Ax\| \\ &\quad\leq\|\mathcal{T}_{\sigma_\Pi(\tau_1)}\|\|(\tau_2-\sigma_\Pi(\tau_1))Ax-(I-\mathcal{T}_{\tilde{\delta}(\tau_2,\sigma_\Pi^{-1}(\tau_1))})x\| \\ &\qquad+\|\mathcal{T}_{\tau_1}\|\|(I-\mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_1^{-1})})Ax\|(\tau_2-\sigma_\Pi(\tau_1)) \\ &\quad\leq M\varepsilon(\tau_2-\sigma_\Pi(\tau_1)),\end{aligned}$$

where $M:=\sup\{\|\mathcal{T}_\tau\|:\tau\in[e,L]_{\Pi^*}\}$ and $L\in\Pi^*$ is any fixed positive constant. In the above, it is necessary to note that $\sigma_\Pi(\tau_1)=\tau_1$, since if $\sigma_\Pi(\tau_1)>\tau_1$, then τ_1 is right scattered point, which implies that $\tau_2=\tau_1$, and this contradicts our assumption that $\tau_2>\tau_1$.

(ii) If $\tau_2\leq\tau_1$, then it follows from $\tau_2\leq\tau_1\leq\sigma_\Pi(\tau_1)$ that $0\leq\tau_1-\tau_2\leq\sigma_\Pi(\tau_1)-\tau_2$. Hence, from (5.10) and Theorem 5.2, we obtain

$$\begin{aligned}&\|(\mathcal{T}_{\sigma_\Pi(\tau_1)}-\mathcal{T}_{\tau_2})x-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_1}Ax\| \\ &\quad\leq\|\mathcal{T}_{\tau_2}(\mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})}-I)x-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_2}Ax \\ &\qquad+(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_2}Ax-(\sigma_\Pi(\tau_1)-\tau_2)\mathcal{T}_{\tau_1}Ax\| \\ &\quad\leq\|\mathcal{T}_{\tau_2}\|\|(\mathcal{T}_{\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})}-I)x-(\sigma_\Pi(\tau_1)-\tau_2)Ax)\| \\ &\qquad+\|\mathcal{T}_{\tau_2}\|\|(I-\mathcal{T}_{\tilde{\delta}(\tau_1,\tau_2^{-1})})Ax\|(\sigma_\Pi(\tau_1)-\tau_2) \\ &\quad\leq M\varepsilon(\sigma_\Pi(\tau_1)-\tau_2),\end{aligned}$$

where $M := \sup\{\|\mathcal{T}_\tau\| : \tau \in [0, L]_{\Pi^*}\}$, and $L \in \Pi^*$ is any fixed positive constant.

Therefore, $(\mathcal{T}_\tau x)^{\Delta_\Pi} = \mathcal{T}_\tau Ax = AT_\tau x$. Since (5.7) is the integral form of (5.6), we can conclude that (5.7) holds.

Finally, we show that A is a closed operator. Let $x_n \in \mathscr{D}(A)$, $x_n \to x$, $Ax_n \to y$, and then by (5.10), we have

$$\begin{aligned}\|(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|} - I)x - |\sigma_\Pi(\tau_1) - \tau_2|y\| = \lim_{n\to\infty} \|(\mathcal{T}_{|\tilde{\delta}(\sigma_\Pi(\tau_1),\tau_2^{-1})|} - I)x_n \\ -|\sigma_\Pi(\tau_1) - \tau_2|Ax_n\| \\ \le \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|.\end{aligned}$$

Hence, $x \in \mathscr{D}(A)$ and $Ax = y$, that is, A is a closed operator. This completes the proof. ■

Theorem 5.4 *Let $\mathbb{T}$ be an invariant time scale and $\mathbb{X}$ be a Banach space. Assume that $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$ is a shift-semigroup, A is the infinitesimal generator of the shift-semigroup, and $\mathscr{D}(A) = \mathbb{X}$, $e_A(\tilde{\delta}(\tau_1, \tau_2), e) = e_A(\tau_1, e)e_A(\tau_2, e)$ for all $\tau_1, \tau_2 \in \Pi_+^*$. Then,*

$$\mathcal{T}_\tau = e_A(\tau, e), \ \ \tau \in \Pi_+^*,$$

where $\mathscr{D}(A)$ denotes the domain of A.

Proof From Theorem 5.3, we have

$$\big(e_A(\tau, e)x\big)^{\Delta_\Pi} = Ae_A(\tau, e)x = e_A(\tau, e)Ax.$$

Furthermore, since $e_A(\tau, e)$ is Δ-differentiable on Π, then for any $\varepsilon > 0$, there is a $\delta > 0$ such that for $\tau_2 \in (\tau_1 - \delta, \tau_1 + \delta)_{\Pi_+^*}$, it follows that

$$\|(e_A(\sigma_\Pi(\tau_1), e) - e_A(\tau_2, e))x - (\sigma_\Pi(\tau_1) - \tau_2)Ae_A(\tau_1, e)x\| \le \varepsilon|\sigma_\Pi(\tau_1) - \tau_2|, \tag{5.11}$$

and hence

(i) If $\tau_2 > \tau_1$, then it follows from (5.11) that

$$\begin{aligned}&\|e_A(\sigma_\Pi(\tau_1), e)[I - e_A(\tilde{\delta}(\tau_2, \sigma_\Pi^{-1}(\tau_1)), e)x \\ &\quad -(\sigma_\Pi(\tau_1) - \tau_2)e_A(\tau_1, \sigma_\Pi(\tau_1))Ax]\| \\ &\le \|e_A(\sigma_\Pi(\tau_1), e)\|\|[I - e_A(\tilde{\delta}(\tau_2, \sigma_\Pi^{-1}(\tau_1)), e)x \\ &\quad -(\sigma_\Pi(\tau_1) - \tau_2)e_A(\tau_1, \sigma_\Pi(\tau_1))Ax]\| \\ &\le M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2|.\end{aligned}$$

In the above, $\sigma_\Pi(\tau_1) = \tau_1$. Indeed, if $\sigma_\Pi(\tau_1) > \tau_1$, then τ_1 is a righ-scattered point, and then $\tau_2 = \tau_1$, which is a contradiction since $\tau_2 > \tau_1$.

(ii) If $\tau_2 \leq \tau_1$, then it follows from $\tau_2 \leq \tau_1 \leq \sigma_\Pi(\tau_1)$ that $0 \leq \tau_1 - \tau_2 \leq \sigma_\Pi(\tau_1) - \tau_2$. Hence, from (5.11), we can obtain

$$\begin{aligned}
&\|e_A(\tau_2, e)[(e_A(\sigma_\Pi(\tau_1) - \tau_2, e) - I)x - (\sigma_\Pi(\tau_1) - \tau_2)Ax \\
&\quad +(\sigma_\Pi(\tau_1) - \tau_2)(I - e_A(\tau_1, \tau_2))Ax]\| \\
&\leq \|e_A(\tau_2, e)\| \|[(e_A(\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1}), e) - I)x - (\sigma_\Pi(\tau_1) - \tau_2)Ax]\| \\
&\quad +M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2| \leq 2M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2|,
\end{aligned}$$

where $M := \sup\{\|e_A(\tau, e)\| : \tau :\in [e, L]_{\Pi^*}\}$, and $L \in \Pi^*$ is any fixed positive constant.

From (i) and (ii), we obtain

$$\|(e_A(|\tilde{\delta}(\sigma_\Pi(\tau_1), \tau_2^{-1})|, e) - I)x - |\sigma_\Pi(\tau_1) - \tau_2|Ax\| \leq 2M\varepsilon|\sigma_\Pi(\tau_1) - \tau_2|.$$

Therefore, A is the infinitesimal generator of $\{\mathcal{T}_\tau : \tau \in \Pi_+^*\}$. This completes the proof. ■

Remark 5.3 Notice that if $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\mathcal{T}_\tau = e_A(\tau, 1) = \prod_{s \in [1, \tau)} \big[I + (q - 1)As\big].$$

Definition 5.8 Let A be the infinitesimal generator of the shift-semigroup. We call $\tilde{e}_A(t, t_0)$, $t_0 \in \mathbb{T}$, the exponential function generated by A on the time scale $\mathbb{T}$. We also let $\mathscr{T}_t = \tilde{e}_A(t, t_0)$ and call $\mathscr{T}_t$ the moving operator on $\mathbb{T}$.

Let $\mathbb{X}$ be a Banach space, and consider the following system:

$$x^\Delta = Ax(t), \quad x(t_0) = x_0, \ t_0 \in \mathbb{T}, \tag{5.12}$$

where A is the infinitesimal generator of a shift-semigroup satisfying all the conditions in Theorem 5.4, and $x : \mathbb{T} \to \mathbb{X}$.

Theorem 5.5 *The fundamental solution of the system* (5.12) *can be expressed as*

$$x(t) = \mathscr{T}_t x_0.$$

Proof From Definition 5.8, $\mathscr{T}_t = \tilde{e}_A(t, t_0)$, and hence

$$x^\Delta = (\mathscr{T}_t x_0)^\Delta = A\mathscr{T}_t x(t_0) = Ax(t).$$

Hence, $\mathscr{T}_t x_0$ is the fundamental solution of (5.12). This completes the proof. ■

From Theorem 5.5, the following result follows immediately.

Theorem 5.6 *Let A be the infinitesimal generator of the shift-semigroup, and let $\mathscr{T}_t$ be the moving operator on $\mathbb{T}$. Then,*

$$(\mathscr{T}_t x)^{\Delta} = A(\mathscr{T}_t x) = \mathscr{T}_t Ax,$$

that is

$$(\mathscr{T}_t x) - x = \int_{t_0}^{t} A\mathscr{T}_s x\Delta s = \int_{t_0}^{t} \mathscr{T}_s Ax\Delta s.$$

6 Stepanov-Like Almost Automorphic Functions in Matched Spaces of Time Scales

In this section, we will introduce the concept of Stepanov-like almost automorphic functions under matched spaces of time scales and provide some basic properties.

Definition 6.1 ([17, 19]) A function $f \in C(\mathbb{T}, \mathbb{X})$, where $\mathbb{X}$ is a Banach space, is said to be almost automorphic (a.a. for short) in Bochner's sense if for every sequence of $(s_n^{'}) \subset \Pi^*$, there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f\big(\delta(s_n, t)\big)$$

is well defined for each $t \in \mathbb{T}$, and

$$\lim_{n\to\infty} g\big(\delta(s_n^{-1}, t)\big) = f(t)$$

for each $t \in \mathbb{T}$.

If the convergence above is uniform in $t \in \mathbb{T}$, then f is almost periodic in Bochner's sense. We denote by $AA^{\delta}(\mathbb{X})$ the collection of all (Bochner) almost automorphic functions $\mathbb{T} \to \mathbb{X}$. Similar to the proof process in the literature [25], the following theorem is immediate through replacing the operation $+$ by the operation δ.

Theorem 6.1 *If $f, f_1, f_2 \in AA^{\delta}(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA^{\delta}(\mathbb{X})$;
(ii) $\lambda f \in AA^{\delta}(\mathbb{X})$ *for any scalar* λ;
(iii) $f_\alpha \in AA^{\delta}(\mathbb{X})$ *where* $f_\alpha : \mathbb{T} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\delta(\alpha, \cdot))$;
(iv) the range $R_f := \big\{f(t) : t \in \mathbb{T}\big\}$ *is relatively compact in* $\mathbb{X}$, *and thus* f *is bounded in norm; and*
(v) if $f_n \to f$ *uniformly on* $\mathbb{T}$, *where* $f_n \in AA^{\delta}(\mathbb{X})$, *then* $f \in AA^{\delta}(\mathbb{X})$.

$AA^\delta(\mathbb{X})$ equipped with the sup-norm $\|f\|_{AA^\delta(\mathbb{X})} = \sup_{t\in\mathbb{T}} \|f(t)\|$ turns out to be a Banach space. Now, we denote by $AA_u^\delta(\mathbb{X})$ the closed subspace of all functions $f \in AA^\delta(\mathbb{X})$ with $g \in C(\mathbb{T}, \mathbb{X})$. Equivalently, $f \in AA_u^\delta(\mathbb{X})$ if and only if f is almost automorphic and all convergences in Definition 6.1 are uniform on compact intervals. Obviously, we have

$$AP^\delta(\mathbb{X}) \subseteq AA_u^\delta(\mathbb{X}) \subset AA^\delta(\mathbb{X}) \subset BC(\mathbb{X}),$$

where $BC(\mathbb{X})$ stands for the Banach space of bounded and continuous functions with values in $\mathbb{X}$.

Definition 6.2 The Bochner transform $f^b(t,s), t \in \mathbb{T}, s \in [e, L]_{\Pi^*}$, of a function $f(t)$ on $\mathbb{T}$, with values in $\mathbb{X}$, is defined by $f^b(t,s) = f\big(\delta(s,t)\big)$.

Remark 6.1 A function $\varphi(t,s)$, $t \in \mathbb{T}$, $s \in [e, L]_{\Pi^*}$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t,s) = f^b(t,s),$$

if and only if

$$\varphi\big(\delta(\tau,t), \tilde{\delta}(s,\tau^{-1})\big) = \varphi(s,t)$$

for all $t \in \mathbb{T}$, $s \in [e, L]_{\Pi^*}$ and $\tau \in [\tilde{\delta}(s, L^{-1}), s]_{\Pi^*}$. In fact,

$$\begin{aligned}\varphi\big(\delta(\tau,t), \tilde{\delta}(s,\tau^{-1})\big) &= f\big(\delta(\tilde{\delta}(s,\tau^{-1}), \delta(\tau,t))\big) = f\big(\delta(s, \delta(\tau^{-1}, \delta(\tau,t)))\big)\\ &= f\big(\delta(s,t)\big) = \varphi(t,s).\end{aligned}$$

Definition 6.3 Let $p \in [1,\infty)$. The space $BS_\delta^p(\mathbb{X})$ of Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{T}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty(\mathbb{T}, L^p(e, L; \mathbb{X}))$. This is a Banach space with the norm:

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{T},L^p)} = \sup_{t\in\mathbb{T}} \left(\int_t^{\delta(L,t)} \|f(\tau)\|_{\mathbb{X}}^p \Delta\tau \right)^{\frac{1}{p}}.$$

Definition 6.4 The space $AS_\delta^p(\mathbb{X})$ of S^p-almost automorphic functions consists of all $f \in BS_\delta^p(\mathbb{X})$ such that $f^b \in AA^\delta(L^p(e, L; \mathbb{X}))$.

Definition 6.4 also has the following equivalent form.

Definition 6.5 A function $f \in L_{loc}^p(\mathbb{T}; \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{T} \to L^p(e, L; \mathbb{X})$ is almost automorphic in the sense that for every sequence of numbers $(s_n') \subset \Pi^*$, there exist a subsequence (s_n) and a function $g \in L_{loc}^p(\mathbb{T}, \mathbb{X})$ such that

$$\left(\int_e^L \|f\big(\delta(s_n, \delta(s,t))\big) - g\big(\delta(s,t)\big)\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0$$

and

$$\left(\int_e^L \|g\big(\delta(s_n^{-1}, \delta(s,t))\big) - f\big(\delta(s,t)\big)\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{T}$.

Remark 6.2 Note that if $1 \le p < \hat{p} < \infty$ and $f \in L^{\hat{p}}_{loc}(\mathbb{T}; \mathbb{X})$ is $S^{\hat{p}}$-almost automorphic, then f is S^p-almost automorphic. Also, if $f \in AA^\delta(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Remark 6.3 Note that $f \in AA^\delta_u(\mathbb{X})$ if and only if $f^b \in AA^\delta(L^\infty(e, L; \mathbb{X}))$. Hence, $AA^\delta_u(\mathbb{X})$ can be regarded as $AS^\infty_\delta(\mathbb{X})$.

Theorem 6.2 *We have the following equivalent statements:*

(i) $f \in AS^p_\delta(\mathbb{X})$*;*
(ii) $f^b \in AA^\delta_u(L^p(e, L; \mathbb{X}))$*; and*
(iii) for each sequence $(s_n^{'}) \subset \Pi^*$*, there exists a subsequence* (s_n) *such that*

$$g(t) := \lim_{n\to\infty} f\big(\delta(s_n, t)\big) \tag{6.1}$$

exists in the space $L^p_{loc}(\mathbb{T}; \mathbb{X})$ *and*

$$f(t) = \lim_{n\to\infty} g\big(\delta(s_n^{-1}, t)\big) \tag{6.2}$$

in the sense of $L^p_{loc}(\mathbb{T}, \mathbb{X})$.

Proof $(ii) \Rightarrow (i)$: trivial.

$(iii) \Rightarrow (ii)$: Now, we prove that

$$\lim_{n\to\infty} f^b\big(\delta(s_n, t), \tau\big) = g^b(t, \tau) = g\big(\delta(\tau, t)\big)$$

in $C(\mathbb{T}; L^p(e, L; \mathbb{X}))$. In fact, for any fixed $t_0 \in \mathbb{T}$ and $\tau_0 \in \Pi^*_+$, we have

$$\sup_{t\in[\delta(\tau_0^{-1}, t_0), \delta(\tau_0, t_0)]} \|f^b(\tau, \delta(s_n, t)) - g^b(\tau, t)\|_{L^p(e,L;\mathbb{X})}$$

$$\le \sup_{t\in[\delta(\tau_0^{-1}, t_0), \delta(\tau_0, t_0)]} \left(\int_e^L \|f(\delta(\tau, \delta(s_n, t))) - g(\delta(\tau, t))\|^p \Delta_\Pi \tau\right)^{\frac{1}{p}}$$

$$= \sup_{t\in[\delta(\tau_0^{-1},t_0),\delta(\tau_0,t_0)]} \left(\int_e^L \|f(\delta(s_n, \delta(\tau, t))) - g(\delta(\tau, t))\|^p \Delta_\Pi \tau \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\delta(\tau_0^{-1},t_0)}^{\delta(L,\delta(\tau_0,t_0))} \|f(\delta(s_n, s)) - g(s)\|^p \Delta s \right)^{\frac{1}{p}} \to 0.$$

Similarly, we have

$$\lim_{n\to\infty} g^b(\delta(s_n^{-1}, t), \tau) = f^b(t, \tau)$$

in $C(\mathbb{T}; L^p(e, L; \mathbb{X}))$.

$(i) \Rightarrow (iii)$: Let $f^b(\delta(s_n, t), \tau) \to \varphi(t, \tau)$ pointwise, where φ is a measurable function with values in $L^p(e, L; \mathbb{X})$). By Remark 6.1, $\varphi(t, \tau) = g^b(t, \tau)$, here $g^b \in L^p_{loc}(e, L; \mathbb{X})$. For any fixed $t_0 \in \mathbb{T}$ and $\tau_0 \in \Pi_+^*$, let

$$\rho_n := \int_{\delta(\tau_0^{-1},t_0)}^{\delta(\tau_0,t_0)} \|f(\delta(s_n, t)) - g(t)\|^p \Delta t.$$

We will prove that $\rho_n \to 0$. Indeed, assume that $\hat{n}(\tau_0^{-1})$ and $\hat{n}(\tau_0)$ are the positive integers such that

$$\tau_0^{-1} \leq \tau_{\hat{n}(\tau_0^{-1})} < \tau_{\hat{n}(\tau_0^{-1})-1} \cdots < \tau_{\hat{n}(\tau_0)-1} < \tau_{\hat{n}(\tau_0)} \leq \tau_0;$$

then, we have

$$\rho_n = \sum_{k=\hat{n}(\tau_0^{-1})}^{\hat{n}(\tau_0)} \int_{\tau_k}^{\tilde{\delta}(\tau_k,L)} \|f^b(\delta(s_n, \delta(\tau, t))) - g^b(t, \tau)\|^p \Delta_\Pi \tau$$

$$= \sum_{k=\hat{n}(\tau_0^{-1})}^{\hat{n}(\tau_0)} \|f^b(\delta(s_n, \delta(\tau, t))) - g^b(t, \tau)\|^p_{L^p(e,L;\mathbb{X})} \to 0,$$

which implies (6.1). Similarly, we can obtain (6.2). The proof is completed. ■

Theorem 6.3 *$AS^p_\delta(\mathbb{X})$ is a closed linear subspace of $BS^p_\delta(\mathbb{X})$.*

Proof First, we prove that $AS^p_\delta(\mathbb{X})$ is closed linear subspaces of $BS^p_\delta(\mathbb{X})$. Now, let $f_1, f_2 \in AS^p_\delta(\mathbb{X})$, and then by Definition 6.3, $f_1^b, f_2^b \in AA^\delta(L^p(e, L; \mathbb{X}))$, so by Definition 6.3 and Theorem 6.2, we have $f_1^b, f_2^b \in AA^\delta(\mathbb{X})$.

By Minkowski's lemma, we have

$$\begin{aligned}\|f_1+f_2\| &= \|f_1^b+f_2^b\|_{L^\infty(\mathbb{T},L^p)} = \sup_{t\in\mathbb{T}}\left(\int_t^{\delta(L,t)} \|f_1(\tau)+f_2(\tau)\|_{\mathbb{X}}^p\Delta\tau\right)^{\frac{1}{p}} \\ &\le \sup_{t\in\mathbb{T}}\left(\int_t^{\delta(L,t)} \|f_1(\tau)\|_{\mathbb{X}}^p\Delta\tau\right)^{\frac{1}{p}} + \sup_{t\in\mathbb{T}}\left(\int_t^{\delta(L,t)} \|f_2(\tau)\|_{\mathbb{X}}^p\Delta\tau\right)^{\frac{1}{p}} \\ &= \|f_1^b\|_{L^\infty(\mathbb{T},L^p)} + \|f_2^b\|_{L^\infty(\mathbb{T},L^p)} = \|f_1\|_{S^p} + \|f_2\|_{S^p}.\end{aligned}$$

Hence, we have $f_1+f_2 \in AS_\delta^p(\mathbb{X})$.

Moreover, it is clear that $\lambda f_1 \in AS_\delta^p(\mathbb{X})$ for any scalar λ.

Finally, by employing again Minkowski's lemma, we can prove that if (f_n) is a sequence in $AS_\delta^p(\mathbb{X})$ that converges to f in S^p-norm, then $f \in AS_\delta^p(\mathbb{X})$. The proof is completed. ■

The following theorem is immediate.

Theorem 6.4 *Let $f \in AS_\delta^p(\mathbb{X})$ and $\mathcal{A} \in L(\mathbb{X})$, the Banach algebra of all bounded linear operators $\mathbb{X} \to \mathbb{X}$. Then $\mathcal{A}f \in AS_\delta^p(\mathbb{X})$.*

Now, we have the following composition theorem.

Theorem 6.5 *Let $F : \mathbb{T}\times\mathbb{X} \to \mathbb{X}$ be S^p-almost automorphic. Suppose that $F(t,x)$ is Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{T}$, i.e., there exists $L > 0$ such that*

$$\|F(t,u) - F(t,v)\| \le L\|u-v\|$$

for all $t \in \mathbb{T}$, $(u,v) \in \mathbb{X}\times\mathbb{X}$. If $\phi \in AS_\delta^p(\mathbb{X})$, then $\Upsilon : \mathbb{T} \to \mathbb{X}$ defined by $\Upsilon(\cdot) := F(\cdot,\phi(\cdot))$ belongs to $AS_\delta^p(\mathbb{X})$.

Proof Since $\phi \in AS_\delta^p(\mathbb{X})$, for every sequence (s_n'), there exist a subsequence (s_n) and a function $\psi^b \in L^p(e,L;\mathbb{X})$ such that

$$\left(\int_e^L \|\phi(\delta(s_n,\delta(s,t))) - \psi(\delta(s,t))\|^p\Delta_\Pi s\right)^{\frac{1}{p}} \to 0, \tag{6.3}$$

and

$$\left(\int_e^L \|\psi(\delta(s_n^{-1},\delta(s,t))) - \phi(\delta(s,t))\|^p\Delta_\Pi s\right)^{\frac{1}{p}} \to 0 \tag{6.4}$$

as $n \to \infty$ on $\mathbb{T}$ pointwise.

Since $F : \mathbb{T}\times\mathbb{X} \to \mathbb{X}$, $(t,u) \to F(t,u)$ is S^p-almost automorphic in $t \in \mathbb{T}$ uniformly in $u \in \mathbb{X}$, for every sequence (σ_n'), there exist a subsequence (σ_n) and a function $G^b(\cdot,u) \in L^p(e,L;\mathbb{X})$ such that

$$\left(\int_e^L \|F(\delta(\sigma_n, \delta(s,t)), u) - G(\delta(s,t), u)\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0, \tag{6.5}$$

and

$$\left(\int_e^L \|G(\delta(\sigma_n^{-1}, \delta(s,t)), u) - F(\delta(s,t), u)\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0 \tag{6.6}$$

as $n \to \infty$ on $\mathbb{T}$ pointwise for each $u \in \mathbb{X}$.

Now, by employing Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_e^L \|F(\delta(s_n, \delta(s,t)), \phi(\delta(s_n, \delta(s,t)))) - G(\delta(s,t), \psi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \\
&\quad \le \left(\int_e^L \|F(\delta(s_n, \delta(s,t)), \phi(\delta(s_n, \delta(s,t)))) \right. \\
&\qquad\qquad \left. - F(\delta(s_n, \delta(s,t)), \psi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \\
&\left(\int_e^L \|F(\delta(s_n, \delta(s,t)), \psi(\delta(s,t))) - G(\delta(s,t), \phi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \\
&\quad \le L\left(\int_e^L \|\phi(\delta(s_n, \delta(s,t))) - \psi(\delta(s,t))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \\
&\qquad + \left(\int_e^L \|F(\delta(s_n, \delta(s,t)), \psi(\delta(s,t))) - G(\delta(s,t), \psi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}},
\end{aligned}$$

and though (6.3) and (6.5), we have

$$\left(\int_e^L \|F(\delta(s_n, \delta(s,t)), \phi(\delta(s_n, \delta(s,t)))) - G(\delta(s,t), \psi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$.

Similar to the above, by employing Minkowski's inequality and both (6.4) and (6.6), we can also obtain

$$\left(\int_e^L \|G(\delta(s_n^{-1}, \delta(s,t)), \psi(\delta(s_n^{-1}, \delta(s,t)))) - F(\delta(s,t), \phi(\delta(s,t)))\|^p \Delta_\Pi s\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$. This completes the proof. ■

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A Reaction–Diffusion Model for Salmonella Transmission Within an Industrial Hens House with Distributed Resistance to Salmonella Carrier State

Pascal Zongo and Catherine Beaumont

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1 Introduction

1.1 Biological Problem

It is often said that a lack of genetic diversity results in a deleterious effect on the risk of epizootic outbreaks. For example through a stochastic epidemiological model of a viral pig disease (transmissible gastroenteritis), the authors [17] explored the effect of genetic variation in the host susceptibility on probabilities and severity of epizooty. They showed that the former was reduced by genetic variation in susceptibility to infection but not in duration of recovery; however, it is to note that its severity was generally lower in genetically heterogeneous populations. Doeschl-Wilson et al. [7] note that the specific definition of heterogeneity varies among models, which may lead to partially different conclusions as in [7, 12, 22]. For example, Lloyd-Smith et al. [12] defined heterogeneity as variation in a continuously distributed "individual reproductive number" and predicted that the probability of stochastic disease extinction in heterogeneous populations increases with overdispersion (which also results in higher proportion of non-transmitting individuals). Springbett et al. [22] considered a few genotypes with distinct levels of resistance. They concluded that "more heterogeneous populations are not expected to suffer fewer epidemics on average, but are less likely to suffer catastrophic epidemics." Doeschl-Wilson et al. [7] therefore suggests using models as close as possible to what is known about genetic architecture of resistance. They modeled

P. Zongo (✉)
Laboratoire L3MA, DSI et IUT, Université des Antilles, Schoelcher, Martinique

C. Beaumont
INRAE, Université de Tours, UMR Biologie des oiseaux et aviculture, Nouzilly, France

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propagation of sheep footrope using an inverse gamma distribution to assess host variability in response to different steps of disease and predicted a strong impact of genetic heterogeneity on the disease risk, progression, and severity.

The goal of this model is to take profit of the numerous information on genetic architecture of hen resistance to Salmonella propagation (see for example [3, 4]) to develop a model as close as possible to reality, i.e., considering different resistance traits to infection: transmission rate, excretion rate as well as duration of recovery. It was shown that those traits had different genetic controls and that each of them was dependent on a large number of genes, most of which having a small effect [19, 22]. They were thus modeled with continuous and independent distributions.

The effect of genetic heterogeneity was investigated since [19] showed an effect of genetic heterogeneity considering only two subpopulations of resistant and susceptible fowls: they observed that the level of resistance influenced kinetics of propagation of Salmonella in comparison with what could be observed with a homogenous population of the same mean value for epidemic characteristics.

The spatial repartition of animals (within cages) was also considered since it was shown that the basic reproduction number $\mathcal{R}_0$ and the propagation speed were increased when the spatial distribution of fowls was heterogeneous [2, 30].

In this chapter, the combined effects of genetic and spatial distributions will be considered in order to study the impact of heterogeneities on the propagation of Salmonella, as an extension of the model developed in [19] where only two subpopulations were considered in a non-dependent spatial environment.

1.2 Mathematical Problem

In this chapter, the hen population is motionless since it is confined in cages, whereas the bacteria disperse via a diffusion process within the hen house. Moreover, the total number of hens within hen house is assumed to be constant in time and heterogeneous in space. We point out that the solution maps associated with the model system are not compact due to the lack of diffusion term according to the hen component in the model. In order to overcome this difficulty, we introduce the Kuratowski measure of noncompactness, κ (see, e.g., [5, 28, 29]), i.e., the existence of global connect attractor can be obtained through proving κ-contraction of the corresponding solution semiflow. Furthermore, we will prove the existence and uniqueness of principle eigenvalue, Λ, corresponding to the non-compact eigenvalue problem and discuss the propagation phenomenon with respect to the so-called basic reproduction number, $\mathcal{R}_0 := 1/\Lambda$.

In addition to the threshold result, we will investigate the special case where all the coefficients in (2.6) are independent of the spatial variable. Using a fluctuation method developed in [24], we show that when $\mathcal{R}_0 > 1$ the disease will become established and stabilize at a unique spatially homogeneous steady state. Moreover, the positive steady state is globally attractive, and under some appropriate conditions, it is explicitly obtained.

This chapter is organized as follows. The next section presents the model and well-posedness. The basic reproduction ratio and mathematical analysis are established in Sects. 3 and 4, respectively. Section 5 is devoted to discuss the special case with spatially independent parameters. In Sect. 6, we present some conditions required to control Salmonella by analyzing the impact of the spatial heterogeneity combined with repartition of heterogeneous fowls. Concluding remarks will follow in Sect. 7.

2 The Model and Well-Posedness

2.1 Model Formulation

In order to develop the model, assume that the habitat $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ (when $d > 1$), and ν is the outward unit normal vector on $\partial\nu$ and $\dfrac{\partial}{\partial\nu}$ means the normal derivative along ν on $\partial\Omega$. Then, we propose a reaction–diffusion model to describe the interaction between hens' population and living free bacteria in environment in a continuous spatial habitat.

Let θ denote the level of animal genetic resistance to Salmonella carrier state. In what follows, we assume that $\theta \in [0, \Theta)$, $\Theta > 0$. Furthermore, we assume that the rate of transmission, $\sigma(x, \theta)$, excretion rate in environment $\beta(x, \theta)$ and recovery rate $\gamma(x, \theta)$ of animals are depending on their level of resistance $\theta \in J$ and spatially dependent on their position $x \in \Omega$ in order to take into account the cage structure. Here, $1/\gamma(x, \theta)$ is the length of the infectious period.

Let $S(t, x, \theta)$ and $I(t, x, \theta)$ denote the densities of susceptible and infective individuals with level of resistance θ at time t and position $x \in \Omega$, respectively. The total size of the population of hens at time t and position x with respect to θ is denoted by $\mathcal{H}(t, x, \theta) = S(t, x, \theta) + I(t, x, \theta)$. The density of bacteria in the environment at time $t \geq 0$ located at a position x is denoted by $C(t, x)$. Together with these assumptions, the model we shall consider reads as follows:

$$\begin{cases} \dfrac{\partial C(t,x)}{\partial t} = D\Delta C(t,x) - \alpha(x)C(t,x) + L_\beta(I)(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial S(t,x,\theta)}{\partial t} = -\sigma(x,\theta)S(t,x,\theta)C(t,x) + \gamma(x,\theta)I(t,x,\theta), & t > 0,\ (x,\theta) \in \Omega \times J, \\ \dfrac{\partial I(t,x,\theta)}{\partial t} = \sigma(x,\theta)S(t,x,\theta)C(t,x) - \gamma(x,\theta)I(t,x,\theta), & t > 0,\ (x,\theta) \in \Omega \times J, \end{cases} \tag{2.1}$$

with $L_\beta(I)(t, x)$ denotes the flux of excreted bacteria at time t by the hens at position x. It is defined by

$$L_\beta(I)(t,x) = \int_J \beta(x,\theta)I(t,x,\theta)d\theta,\ t>0, \tag{2.2}$$

where the term $\beta(x,\theta)I(t,x,\theta)$ represents the density of bacteria excreted by hens infected with respect to their level of resistance θ at time t and position x.

We assume non-flux boundary conditions for the bacteria (reflecting the confinement of the bacteria in the domain):

$$\frac{\partial C(t,x)}{\partial \nu} = 0, \quad (t,x) \in (0,\infty)\times\partial\Omega. \tag{2.3}$$

This model is supplemented together with initial data

$$S(0,x,\theta) = S_0(x,\theta), \quad I(0,x,\theta) = I_0(x,\theta), \quad C(0,x) = C_0(x),\ x\in\Omega,\ \theta\in J. \tag{2.4}$$

In the above system, the term $\alpha(x)$ denotes the natural mortality rate of the bacteria at position x, D denotes the diffusion coefficient for their dispersal in the environment, and Δ denotes the Laplace operator.

Before going further, we need to reformulate the problem (2.1)–(2.4). By adding up the second and third equations in (2.1), we obtain the equation for the hens' total population as follows:

$$\partial\mathcal{H}(t,x,\theta)/\partial t = 0,\ t\geq 0,\ (x,\theta)\in\overline{\Omega}\times J.$$

It follows that

$$\mathcal{H}(t,x,\theta) \equiv S_0(x,\theta) + I_0(x,\theta) = H(x,\theta), \quad \forall t\geq 0,\ (x,\theta)\in\overline{\Omega}\times J.$$

Thus,

$$\limsup_{t\to\infty}\mathcal{H}(t,x,\theta) = H(x,\theta),\ \forall(x,\theta)\in\overline{\Omega}\times J.$$

In particular, $\limsup_{t\to\infty} S(t,x,\theta) + I(t,x,\theta) = H(x,\theta)$ and

$$\limsup_{t\to\infty} I(t,x,\theta) \leq H(x,\theta) \text{ and } \limsup_{t\to\infty} S(t,x,\theta) \leq H(x,\theta),\ (x,\theta)\in\overline{\Omega}\times J. \tag{2.5}$$

We further make the following assumption concerning the system (2.1):

Assumption 2.1 *$D>0$, $H\in C_{b+}\left(\overline{\Omega}\times J\right)$, $\alpha\in C_{b+}\left(\overline{\Omega}\right)$, $\gamma\in C_{b+}\left(\overline{\Omega}\times J\right)$, β, and σ are assumed nonnegative continuous functions on $\overline{\Omega}\times J$.*

Here, $C_b\left(\overline{\Sigma}\right)$ denotes the Banach space of bounded and continuous functions on $\overline{\Sigma}$ endowed with the usual supremum norm, while $C_{b+}\left(\overline{\Sigma}\right)$ denotes its positive cone, consisting of the everywhere positive functions.

Plugging $H(x,\theta)$ in (2.1) allows us to reduce the system (2.1)–(2.4) into the following problem:

$$\begin{cases} \dfrac{\partial C(t,x)}{\partial t} = D\Delta C - \alpha(x)C + \displaystyle\int_J \beta(x,\theta) I(t,x,\theta)d\theta, & t>0,\ x\in\Omega, \\ \dfrac{\partial I(t,x,\theta)}{\partial t} = \sigma(x,\theta)(H(x,\theta)-I)C - \gamma(x,\theta)I, & t>0,\ (x,\theta)\in\Omega\times J, \\ \dfrac{\partial C(t,x)}{\partial \nu} = 0, & t>0,\ x\in\partial\Omega, \\ C(0,x) = C_0(x),\ I(0,x,\theta) = I_0(x,\theta), & (x,\theta)\in\Omega\times J. \end{cases} \tag{2.6}$$

2.2 Well-Posedness of (2.6)

We first show the existence of solutions to (2.6) via a semigroup approach for which general treatments of linear or nonlinear operators in Banach spaces are given in [14, 18].

Let $\mathbb{X} := C(\overline{\Omega},\mathbb{R})$ be the Banach space with the supremum norm $\|.\|_{\mathbb{X}}$ and $\mathbb{Y} = C(J,\mathbb{X})$ with the norm $\|\phi\|_{\mathbb{Y}} = \sup_{\theta\in J}\|\phi(\theta)\|_{\mathbb{X}},\ \forall\phi\in\mathbb{Y}$. Define $\mathbb{X}^+ := C(\overline{\Omega},\mathbb{R}_+)$ and $\mathbb{Y}^+ := C(J,\mathbb{X}^+)$. Define $\mathbb{K} = \mathbb{X}\times\mathbb{Y}$ and $\mathbb{K}^+ = \mathbb{X}^+\times\mathbb{Y}^+$; then, $(\mathbb{K},\mathbb{K}^+)$ is a strongly ordered Banach space.

Let Γ be the Green's function associated with the parabolic equation $\dfrac{\partial v}{\partial t} = \Delta v$ in Ω subject to the Neumann boundary condition. Suppose that $\{T_1(t)\}_{t\geq 0} : \mathbb{X}\longrightarrow\mathbb{X}$ is the C_0-semigroups associated with $D\Delta - \alpha(.)$ subject to the Neumann boundary condition. It then follows that for any $\varphi\in\mathbb{X}$, $t\geq 0$,

$$(T_1(t)\varphi)(x) = e^{-\alpha(x)t}\int_\Omega \Gamma(\mathrm{D}t,x,y)\varphi(y)dy. \tag{2.7}$$

From [21, Section 7.1 and Corollary 7.2.3], it follows that $T_1(t) : \mathbb{X}\longrightarrow\mathbb{X}$ is compact and strongly positive, $\forall t\geq 0$.

We also define $\{T_2(t)\}_{t\geq 0} : \mathbb{Y}\longrightarrow\mathbb{Y}$ the C_0-semigroups generated by operators $-\gamma(.,.)I$ as follows: for any $\varphi\in\mathbb{Y},\ t\geq 0$,

$$(T_2(t)\varphi)(x,\theta) = e^{-\gamma(x,\theta)t}\varphi(x,\theta). \tag{2.8}$$

Let $\mathcal{A}_1 : D(\mathcal{A}_1)\subset\mathbb{X}\longrightarrow\mathbb{X}$ be the generator of T_1 and $\mathcal{A}_2 : D(\mathcal{A}_2)\subset\mathbb{Y}\longrightarrow\mathbb{Y}$ be the generator of T_2. Then, $T(t) = (T_1(t),T_2(t)) : \mathbb{K}\longrightarrow\mathbb{K},\ t\geq 0$, is a semigroup generated by the operator $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ defined on $D(A) = D(\mathcal{A}_1)\times D(\mathcal{A}_2)$.

Define $F = (F_1, F_2) : \mathbb{K}^+ \longrightarrow \mathbb{K}$ by

$$F_1(\phi)(x) = \int_J \beta(x,\theta)\phi_2(x,\theta)d\theta, \tag{2.9a}$$

$$F_2(\phi)(x,\theta) = \sigma(x,\theta)(H(x,\theta) - \phi_2(x,\theta))\phi_1(x), \tag{2.9b}$$

for all $x \in \Omega, \theta \in J$, and $\phi := (\phi_1, \phi_2)^T \in \mathbb{K}^+$.

In order to deal with this problem of existence, we consider a subset $\mathbb{X}_H$ in $\mathbb{K}_+$ defined as follows:

$$\mathbb{X}_H = \left\{ \phi := (\phi_1, \phi_2)^T \in \mathbb{K}^+ \;\middle|\; \begin{array}{c} \phi_1(x) \leq \dfrac{\beta_{\max} H_{\max} |J|}{\alpha_{\min}}, \ \forall x \in \overline{\Omega} \\ \phi_2(x,\theta) \leq H(x,\theta), \ \forall (x,\theta) \in \overline{\Omega} \times J \end{array} \right\},$$

where

$$\beta_{\max} = \max_{(x,\theta)\in\overline{\Omega}\times J} \beta(x,\theta), \ H_{\max} = \max_{(x,\theta)\in\overline{\Omega}\times J} H(x,\theta), \ \alpha_{\min} = \min_{x\in\overline{\Omega}} \alpha(x),$$

and $|J|$ represents the Lebesgue measure of J.

Then, System (2.6) can be written abstractly as an ordinary differential equation in the Banach space $\mathbb{X}_H$ as

$$\frac{du}{dt} = \mathcal{A}u + F(u), \quad t > 0 \tag{2.10a}$$

$$u(0) = \phi \in \mathbb{X}_H. \tag{2.10b}$$

Then, (2.6) can be rewritten as the following integral equation:

$$u(t) = T(t)\phi + \int_0^t T(t-s)F(u(.,s))\,\mathrm{d}s. \tag{2.11}$$

The following lemma states the local existence of solutions with values in $\mathbb{X}_H$.

Lemma 2.2 *Let Assumption 2.1 be satisfied; for any $\phi := (C_0, I_0) \in \mathbb{X}_H$, the system (2.6) has a unique mild solution $u(t,.,.,\phi) = (C(t,.,.), I(t,.,.))$ on $(0, \tau_\phi)$ with $u(0,.,.,\phi) = \phi$, where $\tau_\phi \leq \infty$. Furthermore, for $t \in (0, \tau_\phi)$, $u(t,.,\phi) \in \mathbb{X}_H$ and $u(t,.,.,\phi)$ is a classical solution of (2.6).*

Proof We apply [16, Corollary 4] or [21, Theorem 7.3.1]. It is clear that since $F : \mathbb{X}_H \longrightarrow \mathbb{X}$ is globally Lipschitz continuous, the Cauchy problem has at most one solution on $\mathbb{X}_H$. The local existence of solutions with values in $\mathbb{X}_H$ follows once we have checked the subtangential conditions

$$\lim_{t\to 0+} \frac{1}{h} d(\phi + hF(\phi), \mathbb{X}_H) = 0, \ \forall \phi \in \mathbb{X}_H, \tag{2.12}$$

where $d(z, \mathbb{X}_H) := \inf\{\|z - y\|_{\mathbb{K}}, y \in \mathbb{X}_H\}$ is the distance from the point z to the set $\mathbb{X}_H$. For any $\phi \in \mathbb{X}_H$ and $h \geq 0$, we have

$$\begin{pmatrix} \phi_1(x) + hF_1(\phi) \\ \phi_2(x,\theta) + hF_2(\phi) \end{pmatrix} = \begin{pmatrix} \phi_1(x) + h\int_J \beta(x,\theta)\phi_2(x,\theta)d\theta, \\ \phi_2(x,\theta) + h\sigma(x,\theta)(H(x,\theta) - \phi_2(x,\theta))\phi_1(x) \end{pmatrix} \tag{2.13a}$$

$$\geq \begin{pmatrix} \phi_1(x) \\ \phi_2(x,\theta)\,[1 - h\sigma_{\max}\phi_1(x)] \end{pmatrix}, \quad (x,\theta) \in \Omega \times J, \tag{2.13b}$$

where $\sigma_{\max} = \max_{(x,\theta)\in\overline{\Omega}\times J} \sigma(x,\theta)$.

The above inequalities imply that $\phi + hF(\phi) \in \mathbb{X}_H$ when h is sufficiently small, confirming that the subtangential condition is satisfied. □

To proceed further, we need some information on the following scalar reaction–diffusion equation:

$$\begin{cases} \dfrac{\partial w}{\partial t} - D\Delta w = A(x) - g(x)w, & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{2.14}$$

where $D > 0$ and A and g are continuous and positive functions on Ω.

Lemma 2.3 *System (2.14) admits a unique positive steady state $w^*(.)$ that is globally asymptotically stable in $\mathbb{X}$. Furthermore, if $g(.) \equiv g$ and $A(.) \equiv A$, then $w := A/g$, is globally attractive in $\mathbb{X}^+$.*

The proof is similar to that given in [9, Lemma 2.2].

We further obtain the following result that solutions of system (2.6) exist globally on $[0, \infty)$. We begin to recall that $\limsup_{t\to\infty} I(t,x,\theta) \leq H(x,\theta)$ from (2.5). Thus,

$$I(t,x,\theta) \leq H(x,\theta), \quad (x,\theta) \in \overline{\Omega} \times J, \; t \geq 0. \tag{2.15}$$

By virtue of first equation of (2.6) combined with (2.15), we have

$$\begin{cases} \dfrac{\partial C}{\partial t} - D\Delta C \leq \beta_{\max} H_{\max}|J| - \alpha_{\min} C, & t > 0, \; x \in \Omega, \\ \dfrac{\partial C}{\partial \nu} = 0, & t > 0, \; x \in \partial\Omega, \\ C(0,x) = C_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{2.16}$$

where

$$\beta_{\max} = \max_{(x,\theta)\in\overline{\Omega}\times J} \beta(x,\theta), \ H_{\max} = \max_{(x,\theta)\in\overline{\Omega}\times J} H(x,\theta), \ \alpha_{\min} = \min_{x\in\overline{\Omega}} \alpha(x).$$

Thus, $C(t,x,\theta)$ is a subsolution of the following problem:

$$\begin{cases} \dfrac{\partial M}{\partial t} - D\Delta M = \beta_{\max} H_{\max}|J| - \alpha_{\min} M, & x \in \Omega, \\ \dfrac{\partial M}{\partial \nu} = 0, & x \in \partial\Omega, \\ M(0,x) = \max\limits_{x\in\overline{\Omega}} C_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{2.17}$$

Denote by M the unique solution of the problem (2.17). We also observe that the positive constant

$$\widehat{C} := \max\left\{ \frac{\beta_{\max} H_{\max}|J|}{\alpha_{\min}}, \max_{x\in\overline{\Omega}} C_0(x) \right\}$$

is a supersolution of (2.17). Thus, from the well-known comparison principle for parabolic equations, it then follows that

$$C(t,x) \leq M(t,x) \leq \widehat{C}, \ x \in \overline{\Omega}, \ t \geq 0. \tag{2.18}$$

Now we are in a position to state that solutions of the system (2.6) exist globally for $t \in [0,\infty)$ in $\mathbb{X}_H$.

Theorem 2.4 *Let Assumption 2.1 be satisfied, for any* $\phi := (C_0, I_0) \in \mathbb{X}_H$*; the system (2.6) has a unique solution* $u(t,x,\theta,\phi) = (C(t,x), I(t,x,\theta))$ *defined on* $[0,\infty)$ *with* $u(0,.,.;\phi) = \phi$ *and a semiflow* $\Psi(t) : \mathbb{X}_H \longrightarrow \mathbb{X}_H$ *generated by system (2.6) that is defined by*

$$\Psi(t)\phi = u(t,.,.,\phi) := (C(t,.,\phi), I(t,.,.,\phi)), t \geq 0. \tag{2.19}$$

Furthermore, the semiflow $\Psi(t)$ *is point dissipative, and positive orbits of bounded subsets of* $\mathbb{X}_H$ *for* $\Psi(t)$ *are bounded.*

Proof For all $(x,\theta) \in \bar{\Omega} \times J$ and $t \geq 0$, $I(t,x,\theta)$ and $C(t,x)$ are ultimately bounded with respect to the maximum norm from (2.15) and (2.18), respectively, for all initial conditions $(C_0, I_0) \in \mathbb{X}_H$; thus, the solution exists globally, and moreover, the solution semiflow generated by (2.6) is point dissipative. Moreover, the positive orbits of bounded subsets of $\mathbb{X}_H$ for $\Psi(t)$ are bounded. □

Since the second equation of (2.6) has no diffusion term, the solution map $\Psi(t)$ is not compact. In order to overcome this problem, we introduce the Kuratowski measure of noncompactness κ (see, e.g., [5, 28, 29]), which is defined by

$$\kappa(B) := \inf\{r : \text{B has a finite cover of diameter} < r\}, \tag{2.20}$$

for any bounded set B. We set $\kappa(B) = \infty$ whenever B is unbounded. It is easy to see that B is precompact (i.e., $\overline{B}$ is compact) if and only if $\kappa(B) = 0$. A continuous mapping $f : X \longrightarrow X$ is said to be κ-condensing (κ-contraction of order ℓ, $0 \leq \ell < 1$) if f takes bounded sets to bounded sets and $k(f(B)) < \kappa(B)(\kappa(f(B)) < \ell\kappa(B))$ for any nonempty closed bounded set $B \in X$ with $\kappa(B) > 0$ (see [28, 29]). It is easy to see that a κ-contraction of order ℓ is κ-condensing and a compact map is a κ-contraction of order 0. It is well-known that κ-condensing maps are asymptotically smooth (see, e.g., [10, Lemma 2.3.5]).

Then, the solution map $\Psi(t)$ has some partial compactness in the following sense.

Lemma 2.5 *$\Psi(t) : \mathbb{X}_H \longrightarrow \mathbb{X}_H$ is κ-contraction on $\mathbb{X}_H$ in the sense that*

$$\kappa(\Psi(t)B) \leq e^{-\gamma_{\min} t}\kappa(B) \tag{2.21}$$

for any bounded set $B \subset \mathbb{X}_H$ where $\gamma_{\min} = \min_{(x,\theta)\in\overline{\Omega}\times J} \gamma(x,\theta) > 0$.

Proof For any $\phi = (\phi_1, \phi_2) \in \mathbb{X}_H$, it follows from Theorem 2.4 that the semiflow $\Psi(t) : \mathbb{X}_H \longrightarrow \mathbb{X}_H$ generated by system (2.6) with initial value condition: $C(0, x) = \phi_1(x),\ I(0, x, \theta) = \phi_2(x, \theta),\ (x, \theta) \in \Omega \times J$ is defined by

$$\Psi(t)\phi = (C(t, ., \phi), I(t, ., ., \phi)),\ \forall\phi \in \mathbb{X}_H,\ t \geq 0. \tag{2.22}$$

From the second equation in (2.6), it is easy to see that $I(t, ., ., \phi)$ satisfies the following equations:

$$\begin{cases} \dfrac{\partial I(t, x, \theta)}{\partial t} = \sigma(x, \theta)(H(x, \theta) - I(t, x, \theta, \phi))C(t, x, \phi) & \\ -\gamma(x, \theta)I(t, x, \theta, \phi) & t > 0,\ (x, \theta) \in \Omega \times J, \\ I(0, x, \theta) = \phi_2(x, \theta) & (x, \theta) \in \Omega \times J. \end{cases} \tag{2.23}$$

Then, for all $t > 0,\ x \in \Omega$

$$\begin{aligned} I(t, x, \theta, \phi) &= T_2(t)\phi_2(x, \theta) \\ &\quad + \sigma(x, \theta)\int_0^t T_2(t - s)\left[(H(x, \theta) - I(s, x, \theta, \phi))C(s, x, \phi))\right] ds, \end{aligned} \tag{2.24}$$

where $T_2(t)\phi_2 = e^{-\gamma(.,.)t}\phi_2,\ \forall\phi_2 \in \mathbb{Y}$. Define the linear operator

$$\mathcal{L}(t)\phi = (0, T_2(t)\phi_2),\ \forall\phi = (\phi_1, \phi_2) \in \mathbb{X}_H$$

and a nonlinear operator

$$\mathcal{N}(t)\phi = \Big(C(t,.,\phi), \sigma(x,\theta)\int_0^t T_2(t-s)\Big[(H(x,\theta) - I(s,x,\theta,\phi))C(s,x,\phi))\Big]ds\Big),\ \forall\phi\in\mathbb{X}_H,$$

where

$$C(t,x)=T_1(t)C_0(x)+\int_0^t T_1(t-s)\left(\int_J \beta(x,\theta)I(s,x,\theta)d\theta\right)ds,\quad t>0,\ x\in\Omega.$$

It is easy to see that

$$\Psi(t)\phi = \mathcal{L}(t)\phi + \mathcal{N}(t)\phi,\ \forall\phi\in\mathbb{X}_H,\ t\ge 0.$$

By the compactness of $T_1(t)$ and the boundedness of Ψ, it then follows from the expression of (2.24) that $\mathcal{N}(t):\mathbb{X}_H\longrightarrow\mathbb{X}_H$ is compact for each $t>0$, and hence, $\kappa(\mathcal{N}(t)B)=0$ for any bounded set $B\in\mathbb{X}_H$ and $t>0$.

Since

$$\sup_{\phi\in\mathbb{X}_H}\frac{\|\mathcal{L}(t)\phi\|}{\|\phi\|}\le\sup_{\phi\in\mathbb{X}_H}\frac{\|e^{-\gamma(.,.)t}\phi_2\|}{\|\phi\|}\le e^{-\gamma_{\min}t}\sup_{\phi\in\mathbb{X}_H}\frac{\|\phi_2\|}{\|\phi\|},$$

where $\gamma_{\min}:=\min_{(x,\theta)\in\overline{\Omega}\times J}\gamma(x,\theta)$, we obtain

$$\|\mathcal{L}(t)\|\le e^{-\gamma_{\min}t},\ t>0.$$

Therefore, for any bounded set $B\in\mathbb{X}_H$, we have

$$\kappa(\Psi(t)B)\le\kappa(\mathcal{L}(t)B)+\kappa(\mathcal{N}(t)B)\le\|\mathcal{L}(t)\|\kappa(B)+0\le e^{-\gamma_{\min}t}\kappa(B),\ \forall t>0.$$

Thus, $\Psi(t):\mathbb{X}_H\longrightarrow\mathbb{X}_H,\ t>0$, is κ-contraction of order $e^{-\gamma_{\min}t}$ on $\mathbb{X}_H$. □

Now we are ready to show that solutions of system (2.6) converge to a compact attractor in $\mathbb{X}_H$.

Theorem 2.6 *Let Assumption 2.1 be satisfied;* $\Psi(t):\mathbb{X}_H\longrightarrow\mathbb{X}_H$, $t>0$ *admits a connected global attractor on* $\mathbb{X}_H$.

Proof By Lemma 2.5 and Theorem 2.4, it follows that $\Psi(t)$ is point dissipative and κ-contracting on $\mathbb{X}_H$. From the proof of Theorem 2.4, we also know that the positive orbits of bounded subsets of $\mathbb{X}_H$ for $\Psi(t)$ are (uniformly) bounded. By Magal and Zhao [15, Theorem 2.6], $\Psi(t)$ has a global attractor that attracts every bounded set in $\mathbb{X}_H$. □

3 The Basic Reproduction Number

In epidemic models, the basic reproduction number $\mathcal{R}_0$ is the expected number of secondary cases produced by a typical infective individual introduced into a completely susceptible population, in the absence of any control measure. It can be computed using the next-generation approach (see, e.g., [6, 23, 26]).

In order to define the basic reproduction ratio for model (2.6), we begin to find the disease-free equilibrium by letting the densities of the disease-related compartments C and I be zero. It is easy to see that (0, 0) is disease-free equilibrium for model (2.6). Linearizing system (2.6) at the disease-free equilibrium, we get the following system:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D\Delta u_1(t,x) - \alpha(x)u_1(t,x) + \displaystyle\int_J \beta(x,\theta)u_2(t,x,\theta)d\theta, & t>0,\ x\in\Omega, \\ \dfrac{\partial u_2(t,x,\theta)}{\partial t} = \sigma(x,\theta)H(x,\theta)u_1(t,x) - \gamma(x,\theta)u_2(t,x,\theta), & t>0,\ (x,\theta)\in\Omega\times J, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = 0, & t>0,\ x\in\partial\Omega, \\ u_1(0,x)=\phi_1(x),\ u_2(0,x,\theta)=\phi_2(x,\theta) & (x,\theta)\in\Omega\times J. \end{cases} \tag{3.25}$$

We begin to recall that $\mathbb{K} = \mathbb{X}\times\mathbb{Y}$ where $\mathbb{X} := C(\overline{\Omega},\mathbb{R})$ with the supremum norm $\|.\|_{\mathbb{X}}$ and $\mathbb{Y} = C(J,\mathbb{X})$ with the norm $\|\phi\|_{\mathbb{Y}} = \sup_{\theta\in J}\|\phi(\theta)\|_{\mathbb{X}},\ \forall\phi\in\mathbb{Y}$.

For any $\widehat{\varphi} := (\widehat{\varphi}_1, \widehat{\varphi}_2)\in\mathbb{K}$, it is easy to see that the linear system (3.25) admits a unique global solution

$$\widehat{u} = (\widehat{u}_1(t,.,\widehat{\varphi}), \widehat{u}_2(t,.,.,\widehat{\varphi})) \text{ with } \widehat{u}(0) = \widehat{\varphi}.$$

Denote by $Q(t):\mathbb{K}\longrightarrow\mathbb{K}$ the solution semiflow of (3.25) on $\mathbb{K}$, that is defined by

$$Q(t)\widehat{\varphi} = \widehat{u},\ \text{ with } \widehat{u}(0) = \widehat{\varphi}\in\mathbb{K},\ t\geq 0.$$

Since (3.25) is cooperative, $Q(t)$ is a positive C_0-semigroup on $\mathbb{K}$, in the sense that $Q(t)\mathbb{K}^+\subseteq\mathbb{K}^+,\ \forall t\geq 0$. Then, $Q(t)$ is resolvent-positive [23, Theorem 3.12], and its generator A can be written as follows:

$$A = \begin{pmatrix} D\Delta - \alpha(.) & L_\beta^0 \\ \sigma(.,.)H(.,.) & -\gamma(.,.) \end{pmatrix},$$

where the operator L_β^0 is defined as follows:

$$(L_\beta^0)(\varphi)(x) := \int_J \beta(x,y)\varphi(x,y)dy,\ \forall\varphi\in\mathbb{Y}. \tag{3.26}$$

We begin to investigate the spectral properties of the linear operator A before defining the basic reproduction ratio.

Substituting $u_1(t,x) = e^{\lambda t}\phi_1(x)$; $u_2(t,x,\theta) = e^{\lambda t}\phi_2(x,\theta)$ into (3.25), we get the following associated eigenvalue problem:

$$\lambda\phi_1(x) = D\Delta\phi_1(x) - \alpha(x)\phi_1(x) + \int_J \beta(x,y)\phi_2(x,y)dy,\ \ x \in \Omega \tag{3.27a}$$

$$\lambda\phi_2(x,\theta) = \sigma(x,\theta)H(x,\theta)\phi_1(x) - \gamma(x,\theta)\phi_2(x,\theta),\ \ (x,\theta) \in \Omega \times J \tag{3.27b}$$

$$\frac{\partial\phi_1(x)}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.27c}$$

We point out that the solution map of (3.25) is not compact due to the lack of diffusion term in the second equation in (3.25). The following lemma concerns with the existence of the principal eigenvalue of system (3.27).

Lemma 3.1 *Eigenvalue problem* (3.27) *has a principal simple eigenvalue* $\lambda(H)$ *with a positive eigenfunction.*

Proof By the same argument as Lemma 2.5, for each $t > 0$, the solution semiflow $Q(t) := \widehat{u} : \mathbb{K} \longrightarrow \mathbb{K}$ of (3.25) is κ-contraction of order $e^{-\gamma_{\min}t} \in [0,1),\ t > 0$ on $\mathbb{K}$ in the sense that

$$\kappa(Q(t)B) \leq e^{-\gamma_{\min}t}\kappa(B), \tag{3.28}$$

where $\gamma_{\min} = \min_{(x,\theta)\in\overline{\Omega}\times J} \gamma(x,\theta) > 0$. Hence, $Q(t)$ is κ-condensing, that is,

$$\kappa(Q(t)B) \leq \kappa(B) \tag{3.29}$$

for any bounded set $B \in \mathbb{K}$, with $\kappa(B) > 0$, where κ is the Kuratowski measure of noncompactness as defined in (2.20). Thus, from (3.29), it follows that the essential spectral radius $r_e(Q(t))$ of $Q(t)$ satisfies

$$r_e(Q(t)) < 1,\ \ t > 0.$$

On the other hand, the spectral radius $r(Q(t))$ of $Q(t)$ satisfies

$$r(Q(t)) = e^{s(A)t} \geq 1, t > 0$$

where $s(A)$ is the spectral bound A. This implies that $r_e(Q(t)) \leq r(Q(t))$ for any $t > 0$. Since $Q(t)$ is a strongly positive and bounded operator on $\mathbb{K}$, therefore, according to the generalized Krein–Rutman theorem (see, e.g., [25, Lemma 2.2] and [11, Chapter II.14]), it follows that system (3.27) admits a principal eigenvalue $\lambda(H)$ with a positive eigenfunction (see also [25, Lemma 3.4]). □

Lemma 3.1 means that $\lambda(H)$ is a real eigenvalue with algebraic multiplicity one, and $\mathbf{R}_e(\lambda) < \lambda(H)$ for any other eigenvalue λ of (3.27). Furthermore, $\lambda(H)$ has a corresponding eigenvector $\phi_H(x,\theta) = (\phi_{H,1}(x), \phi_{H,2}(x,\theta))$ satisfying $\phi_H(x,\theta) \gg 0$, and any other nonnegative eigenvector of (3.27) is a positive multiple of $\phi_H(x,\theta)$.

Next, we follow the framework in [13, 27] to obtain the basic reproduction number of the model (2.6).

Let $\mathcal{S}(t) : \mathbb{K} \to \mathbb{K}$ be the C_0-semigroup generated by the following reaction–diffusion system:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D\Delta u_1(t,x) - \alpha(x)u_1(t,x), & t>0,\ x\in\Omega, \\ \dfrac{\partial u_2(t,x,\theta)}{\partial t} = \sigma(x,\theta)H(x,\theta)u_1(t,x) - \gamma(x,\theta)u_2(t,x,\theta), & t>0,\ (x,\theta)\in\Omega\times J, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = 0, & x\in\partial\Omega, \\ u_1(0,x) = \phi_1(x),\ u_2(0,x,\theta) = \phi_2(x,\theta), & (x,\theta)\in\Omega\times J. \end{cases} \tag{3.30}$$

Then for all $\phi := (\phi_1, \phi_2)^T \in \mathbb{K}$, we have

$$(\mathcal{S}(t)\phi)(x,\theta) = \Big(T_1(t)\phi_1(x),\, T_2(t)\phi_2(x,\theta) + \int_0^t T_2(t-s)[(\sigma H)(x,\theta)T_1(s)\phi_1(x)]ds\Big)^T, \tag{3.31}$$

and therefore $\mathcal{S}(t)$ is a positive C_0-semigroup on $\mathbb{K}$.

Let C be the positive linear operator on $\mathbb{K}$ defined by

$$C(\phi) = \begin{pmatrix} 0 & L_\beta^0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},\ \phi := (\phi_1,\phi_2)^T \in \mathbb{K}, \tag{3.32}$$

where the operator L_β is defined in (3.26).

In order to define the basic reproduction number for the system model (2.6), we assume that the state variables are near the disease-free steady state $(0, 0)$ and one bacterium or one infectious hen is introduced at time $t = 0$ and infection occurs immediately. Then with a given initial distribution of infections described by $\phi := (\phi_1, \phi_2)^T \in \mathbb{K}$, as time evolves, those distributions reach $C(\mathcal{S}(t)\phi)$ at time t. Consequently, the distribution of the total new infections is

$$\int_0^\infty C(\mathcal{S}(t)\phi)dt.$$

Let $\mathbf{L} : \mathbb{K} \longrightarrow \mathbb{K}$ be defined by the above integral, i.e.,

$$\mathbf{L}(\phi) := \int_0^\infty C(\mathcal{S}(t)\phi)dt = C\left(\int_0^\infty \mathcal{S}(t)\phi dt\right).$$

Then, **L** is nothing but the next-generation operator of the model system (see, e.g.,[23, 26]).

By Diekmann et al. [6] and Thieme [23], the spectral radius of L is the basic reproduction number for the model (2.6), that is,

$$\mathcal{R}_0 = r(\mathrm{L}). \tag{3.33}$$

By the general results in Thieme [23] and the same arguments as in Wang and Zhao [26, Lemma 2.2], we have the following result.

Lemma 3.2 *$\mathcal{R}_0 - 1$ and $\lambda(H)$ have the same sign.*

In order to compute the basic reproduction number $\mathcal{R}_0$, we begin to characterize it in terms of the principal eigenvalue of the elliptic eigenvalue problem.

Theorem 3.3 *Let Λ be the unique positive eigenvalue of the eigenvalue problem*

$$\begin{cases} -D\Delta\phi(x) + \alpha(x)\phi(x) = \Lambda\left(\int_J \dfrac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)}dy\right)\phi(x), & x \in \Omega, \\ \dfrac{\partial\varphi(x)}{\partial\nu} = 0, & x \in \partial\Omega \end{cases} \tag{3.34}$$

with a positive eigenfunction $\varphi \in \mathbb{X}$. Then

$$\mathcal{R}_0 = \frac{1}{\Lambda}. \tag{3.35}$$

Proof Let $\mathcal{B}$ be the positive linear operator on $\mathbb{K}$ defined by

$$\mathcal{B}(\phi) = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},\ \phi := (\phi_1, \phi_2)^T \in \mathbb{K},$$

where

$$\begin{aligned} B_{11}(\phi_1) &= D\Delta\phi_1(x) - \alpha(.)\phi_1 \\ B_{22}(\phi_2) &= -\gamma(.,.)\phi_2 \\ B_{21}(\phi_1) &= \sigma(.,.)H(.,.)\phi_1. \end{aligned}$$

It is easy to see that $\mathcal{B}$ is the generator of the positive C_0-semigroup $\mathcal{S}(t)$ generated by the system (3.30) and defined in (3.31). Then, $\mathcal{B}$ is resolvent-positive [23, Theorem 3.12], and clearly, we have

$$\int_0^\infty \mathcal{S}(t)\phi dt = -\mathcal{B}^{-1}\phi,\ \phi := (\phi_1, \phi_2)^T \in \mathbb{K}. \tag{3.36}$$

A straightforward computation shows that

$$-\mathcal{B}^{-1} = \begin{pmatrix} -B_{11}^{-1} & 0 \\ B_{22}^{-1}B_{21}B_{11}^{-1} & -B_{22}^{-1} \end{pmatrix}.$$

Thus, the next-generation operator of the model (2.6) rewrites as follows: for all $\phi := (\phi_1, \phi_2)^T \in \mathbb{K}$,

$$\begin{aligned}
\mathbf{L}(\phi) =& C\left(\int_0^\infty \mathcal{S}(t)\phi dt\right) \\
=& C\left(-\mathcal{B}^{-1}\right)(\phi) \\
=& \begin{pmatrix} 0 & L_\beta^0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -B_{11}^{-1} & 0 \\ B_{22}^{-1}B_{21}B_{11}^{-1} & -B_{22}^{-1} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\
=& \begin{pmatrix} L_\beta^0[B_{22}^{-1}B_{21}B_{11}^{-1}] & L_\beta^0(-B_{22}^{-1}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.
\end{aligned}$$

Clearly, the nonzero eigenvalues of the next-generation operator $\mathbf{L}$ are equal to those of $\widehat{\mathbf{L}}$, where $\widehat{\mathbf{L}}$ is a positive linear operator on $\mathbb{X}$ defined by

$$\begin{aligned}
\widehat{\mathbf{L}}\psi =& L_\beta^0\left(B_{22}^{-1}B_{21}B_{11}^{-1}\right)\psi \\
=& L_\beta^0\left(\frac{1}{\gamma(.,.)}\sigma(.,.)H(.,.)\left[-D\Delta + \alpha(.)\right]^{-1}\right)\psi,\ \ \psi \in \mathbb{X}.
\end{aligned}$$

Thus, we have

$$\mathcal{R}_0 := r(\mathbf{L}) = r(\widehat{\mathbf{L}}).$$

Note that for all $x \in \overline{\Omega}$, we have

$$\begin{aligned}
\widehat{\mathbf{L}}\psi(x) =& \int_J \frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)}\left[-D\Delta + \alpha(x)\right]^{-1}\psi(x)dy \\
=& F(x)\left[-D\Delta + \alpha(x)\right]^{-1}\psi(x),\ \ \psi \in \mathbb{X},
\end{aligned}$$

where

$$F(x) = \int_J \frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)}dy.$$

By arguments similar to those in [26, Theorem 3.2], we have

$$\mathcal{R}_0 := r(\mathbf{L}) = r(\widehat{\mathbf{L}}) = r\left(F\,[-D\Delta + \alpha(.)]^{-1}\right) = r\left([-D\Delta + \alpha(.)]^{-1}\,F\right) = \frac{1}{\Lambda}, \tag{3.37}$$

where Λ is the unique positive eigenvalue of

$$[-D\Delta + \alpha(x)]\varphi = \Lambda F(x)\varphi, \;\; \varphi \in \mathbb{X} \tag{3.38}$$

with a positive eigenfunction. □

The following result gives the equivalent characterization of $\mathcal{R}_0$.

Corollary 3.4 *Let Assumption 2.1 be satisfied; then for system (2.6), the basic reproduction ratio $\mathcal{R}_0$ is equal to*

$$\mathcal{R}_0 = \sup_{\phi \in H^1(\Omega), \phi \neq 0} \left\{ \frac{\int_\Omega \left[\int_J \frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)} dy \right] \phi^2}{\int_\Omega [D|\nabla\phi|^2 + \alpha(x)\phi^2]} \right\}. \tag{3.39}$$

Moreover, $\mathcal{R}_0$ rewrites as follows:

$$\mathcal{R}_0 = \int_J \mathbf{r}_0(y)dy, \tag{3.40}$$

where

$$\mathbf{r}_0(y) := \sup_{\phi \in H^1(\Omega), \phi \neq 0} \left\{ \frac{\int_\Omega \left[\frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)} \right] \phi^2}{\int_\Omega [D|\nabla\phi|^2 + \alpha(x)\phi^2]} \right\} \;\forall y \in J. \tag{3.41}$$

$\mathbf{r}_0(y)$ *can be interpreted as the weight of transmission* bacteria-to-hens of type y-to-bacteria *where hens of type y represent the animals having the same level of resistance y as more thoroughly explained in the special case when all parameters are independent of the position in Sect. 5.*

Proof By virtue of the elliptic eigenvalue problem (3.34), it follows from the well-known variational characterization of the principal eigenvalue (see, e.g., [8, 28]) that the formula (3.39) holds. After changing the order of integration (3.39), we get (3.41). □

The following results give some properties of the basic reproduction number $\mathcal{R}_0$.

Corollary 3.5 *Let Assumption 2.1 be satisfied; then for system (2.6), there hold:*

(i) $\mathcal{R}_0$ *is a monotone decreasing function of D with*

$$\lim_{D\to 0} \mathcal{R}_0(D) = \max_{x\in\overline{\Omega}} \left\{ \frac{1}{\alpha(x)} \int_J \frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)} dy \right\}$$

and

$$\lim_{D\to\infty} \mathcal{R}_0(D) = \left\{ \frac{1}{\int_\Omega \alpha(x)} \int_\Omega \left[\int_J \frac{\beta(x,y)\sigma(x,y)H(x,y)}{\gamma(x,y)} dy \right] \right\}.$$

(ii)

$$\text{If } |J| \min_{y\in J} \mathbf{r}_0(y) > 1, \text{ then } \mathcal{R}_0 > 1.$$

(iii)

$$\text{If } |J| \max_{y\in J} \mathbf{r}_0(y) < 1, \text{ then } \mathcal{R}_0 < 1.$$

Proof (i) directly follows by an argument similar to the one in [1, 28, Lemma 2.3]. (ii) and (iii) directly follow from the following inequality:

$$|J| \min_{y\in J} \mathbf{r}_0(y) \le \mathcal{R}_0 \le |J| \max_{y\in J} \mathbf{r}_0(y).$$ □

4 Threshold Dynamics

First we show the strict positiveness of solutions of system (2.6). The following results will play a central role in establishing the persistence of system (2.6).

Lemma 4.1 *Let Assumption 2.1 be satisfied; suppose $u(t,x,\theta,\phi)$ is the solution of system (2.6) with initial condition $u(0,.,.;\phi) := \phi \in \mathbb{X}_H$.*

(i) *If there exits $t_1 \ge 0$ such that $u_1(t_1,.,\phi) \not\equiv 0$, then $u_i(t,.,\phi) > 0$, $\forall t > t_1$, $i = 1,2$.*
(ii) *If there exits $t_1 \ge 0$ such that $u_2(t_1,.,\phi) \not\equiv 0$, then $u_i(t,.,\phi) > 0$, $\forall t > t_1$, $i = 1,2$.*

Proof (i) Suppose that $u_1(t_1,x,\phi) \not\equiv 0$, for some $t_1 \ge 0$. According to Theorem 2.4 and the first equation of system (2.6), it is easy to see that $u_1(t,x,\phi)$ satisfies the following inequalities:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} - D\Delta u_1 \geq -\alpha(x)u_1, \ (x,\theta) \in \Omega \times J, \\ \dfrac{\partial u_1}{\partial \nu} = 0, \qquad\qquad\qquad\qquad x \in \partial\Omega. \end{cases} \tag{4.42}$$

Thus, from the strong maximum principle (see, e.g., [20, p. 172, Theorem 4] and the Hopf boundary lemma (see, e.g., [20, p. 170, Theorem 3] with the initial time at $t = t_1$ instead of $t = 0$, we get $u_1(t,x,\phi) > 0, \forall t > t_1$ and $x \in \overline{\Omega}$.

From the second equation in (2.6), $u_2(t,x,\theta)$ satisfies

$$\begin{aligned} u_2(t,x,\theta,\phi) =& T_2(t-t_1)u_2(t_1,x,\theta) + \int_0^{t-t_1} T_2(t-t_1-s)f(s,x,\theta,\phi)ds \\ \geq& \int_0^{t-t_1} T_2(t-t_1-s)f(s,x,\theta,\phi)ds, \end{aligned}$$

where $f(s,x,\theta,\phi) := \sigma(x,\theta)(H(x,\theta) - u_2(s,x,\theta,\phi))u_1(s,x,\phi) \geq 0$ and $f(s,x,\theta,\phi) \not\equiv 0$ for $s > t_1$ and $(x,\theta) \in \overline{\Omega} \times J$. According to the positivity of $T_2(t)$, it follows that $u_2(t,x,\theta) > 0, \ \forall t > t_1$. The proof of the part (i) is complete.

(ii) We consider the case where $u_2(t_1,x,\theta) \not\equiv 0$ for some $t_1 \geq 0$. From the second equation in (2.6), $u_2(t,x,\theta)$ satisfies

$$\begin{aligned} u_2(t,x,\theta,\phi) =& T_2(t-t_1)u_2(t_1,x,\theta) + \int_0^{t-t_1} T_2(t-t_1-s)f(s,x,\theta,\phi)ds \\ \geq& T_2(t-t_1)u_2(t_1,x,\theta). \end{aligned}$$

According to Theorem 2.4, we have $u_2(t,x,\theta,\phi) \geq 0$ and $u_2(t,x,\theta,\phi) \not\equiv 0, \ \forall t > t_1$. The strong positivity of $T_2(t)$ implies that $u_2(t,.,\phi) > 0, \forall t > t_1$.

On the basis of (4.42) and from the strong maximum principle and the Hopf boundary lemma as in (i), it then turns out $u_1(t,x,\phi) > 0, \forall t > t_1$ and $x \in \overline{\Omega}$. The proof of the part (ii) is complete. □

Now we are ready to prove the main result of this section, which indicates that $\mathcal{R}_0$ is a crucial index for disease persistence.

Theorem 4.2 *Let Assumption 2.1 be satisfied; suppose* $u(t,x,\theta,\phi) = (C(t,x), I(t,x,\theta))$ *is the solution of system (2.6) with initial condition* $u(0,.,.;\phi) := (C_0, I_0) \in \mathbb{X}_H$. *Then, the following two statements are valid:*

(i) *If* $\mathcal{R}_0 < 1$, *then the disease-free equilibrium* $(0,0)$ *is globally attractive in* $\mathbb{X}_H$.

(ii) *If* $\mathcal{R}_0 > 1$, *then system (2.6) admits at least one positive steady state* $\widehat{u}(x,\theta)$ *and there exists an* $\eta > 0$ *such that for any* $\phi := (\phi_1, \phi_2) \in \mathbb{X}_H$ *with* $\phi_1(0,.) \not\equiv 0$ *or* $\phi_1(0,.) \not\equiv 0$, *we have*

$$\liminf_{t\to\infty} u_1(t,x) \geq \eta, \ \text{ and } \ \liminf_{t\to\infty} u_2(t,x,\theta) \geq \eta, \tag{4.43}$$

uniformly for all $(x,\theta) \in \overline{\Omega} \times J$.

Proof (i) In the case where $\mathcal{R}_0 < 1$,, Lemma 3.2 implies that we have $\lambda(H) < 0$. Since $\lambda(m) < 0$ is continuous in m, hence there exists a sufficiently small positive number ϵ such that $\lambda(H + \epsilon) < 0$. Furthermore, from (2.6), we have

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} =D\Delta u_1(t,x) - \alpha(x)u_1(t,x) + \displaystyle\int_J \beta(x,y)u_2(t,x,y)dy, & t > t_0,\ x \in \Omega, \\ \dfrac{\partial u_2(t,x,\theta)}{\partial t} \leq \sigma(x,\theta)(H(x,\theta) + \epsilon_0)u_1(t,x) - \gamma(x,\theta)u_2(t,x,\theta), & t > 0,\ (x,\theta) \in \Omega \times J, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} =0, & t > t_0,\ x \in \partial\Omega. \end{cases} \tag{4.44}$$

Let $\psi := (\psi_1, \psi_2)^T$ be the strongly positive eigenfunction ψ corresponding to $\lambda(H + \epsilon)$ for the following system:

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} =D\Delta v_1(t,x) - \alpha(x)v_1(t,x) + \displaystyle\int_J \beta(x,y)v_2(t,x,y)dy, & t > 0,\ (x,\theta) \in \Omega \times J, \\ \dfrac{\partial v_2(t,x,\theta)}{\partial t} =\sigma(x,\theta)(H(x,\theta) + \epsilon_0)v_1(t,x) - \gamma(x,\theta)v_2(t,x,\theta), & t > 0,\ (x,\theta) \in \Omega \times J, \\ \dfrac{\partial v_1(t,x)}{\partial \nu} =0, & x \in \partial\Omega, \end{cases} \tag{4.45}$$

which has a solution $v(t,x,\theta) = (v_1(t,x), v_2(t,x,\theta)) = e^{\lambda(H+\epsilon)t}(\psi_1(x), \psi_2(x,\theta))$ for all $t \geq 0$ and $(x,\theta) \in \overline{\Omega} \times J$. Since, for any given $\phi \in \mathbb{X}_H$, there exists some $\zeta > 0$ such that $(u_1(0,x,\phi), u_2(0,x,\phi)) \leq \zeta(v_1(0,x), v_2(0,x,\theta))$; by the comparison principle, we obtain

$$(u_1(t,x,\phi), u_2(t,x,\theta,\phi)) \leq \zeta e^{\lambda(H+\epsilon)t}\psi,\ \forall x \in \overline{\Omega},\ \ t \geq 0,$$

which implies $\lim_{t\to\infty}(u_1(t,x,\phi), u_2(t,x,\theta,\phi)) = (0,0)$ uniformly for $(x,\theta) \in \overline{\Omega} \times J$. Thus, part (i) is proved.

(ii) In the case where $\mathcal{R}_0 > 1$, we have $\lambda(H) > 0$. For any $\rho \in [0, \rho^*)$, where ρ^* is a sufficient small positive number, we consider the following eigenvalue problem:

$$\lambda\phi_1(x) = D\Delta\phi_1(x) - \alpha(x)\phi_1(x) + \int_J \beta(x,y)\phi_2(x,y)dy,\ \ x \in \Omega \tag{4.46a}$$

$$\lambda u_2(x,\theta) = \sigma(x,\theta)(H(x,\theta) - \rho)\phi_1(x) - \gamma(x,\theta)\phi_2(x,\theta),\ \ (x,\theta) \in \Omega \times J \tag{4.46b}$$

$$\frac{\partial\phi_1(x)}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{4.46c}$$

By the same argument as in Lemma 3.1, eigenvalue problem (4.46) has a principal eigenvalue $\lambda(H-\rho)$ with a positive eigenfunction $\varphi_\rho(x,\theta)$. Since $\lim_{\rho\to 0}\lambda(H-\rho)=\lambda(H)$, we can fix a $\rho_0\in(0,\rho^*)$, such that $\lambda(H-\rho_0)>0$.

Let

$$\mathbb{W}_0=\{\phi\in\mathbb{X}_H:\phi_1(0,.)\not\equiv 0 \text{ and } \phi_2(0,.)\not\equiv 0\}$$

and

$$\partial\mathbb{W}_0=\mathbb{X}_H\setminus\mathbb{W}_0=\{\phi\in\mathbb{X}_H:\phi_1(0,.)\equiv 0 \text{ or } \phi_2(0,.)\equiv 0\}.$$

By Lemma 4.1, it follows that for any $\phi\in\mathbb{W}_0$, $u_1(t,x,\phi), u_2(t,x,\theta,\phi)>0$ for all $(x,\theta)\in\overline{\Omega}\times J$, namely $\Phi(t)\mathbb{W}_0\subseteq\mathbb{W}_0,\ \forall t\geq 0$. Define

$$M_\partial:=\{\phi\in\partial\mathbb{W}_0:\Phi(t)\phi\in\partial\mathbb{W}_0,\ t\geq 0\}.$$

Let $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+(\phi):=\{\Phi(t)\phi:\ \forall t\geq 0\}$.

Claim 4.3 $\omega(\psi)=\{E_0\}:=\{(0,0)^T\},\ \forall\psi\in\partial\mathbb{W}_0$.

For any given $\psi\in M_\partial$, we have $\Phi(t)\phi\in\partial\mathbb{W}_0, \forall t\geq 0$. It then follows that for each $t\geq 0$, either $u_1(t,.,\psi)\equiv 0$ or $u_2(t,.,.,\psi)\equiv 0$. In the case where $u_1(t,.,\psi)\equiv 0$, for all $t\geq 0$, in view of the u_2 equation in (2.6), we see that $\lim_{t\to 0}u_2(t,x,\theta,\psi)=0$ uniformly for $(x,\theta)\in\overline{\Omega}\times J$. In the case where $u_1(t_0,.,\psi)\not\equiv 0$ for some $t_0\geq 0$, Lemma 4.1 implies that $u_1(t_0,x,\psi)>0,\ \forall t>t_0,\ x\in\overline{\Omega}$. Thus, we have $u_2(t,.,.,\psi)\equiv 0,\ t\geq t_0$, so $\omega(\psi)=\{E_0\}, \forall\psi\in M_\partial$. This proves the Claim 4.3.

Claim 4.4 E_0 is a uniform weak repeller for $\mathbb{W}_0$ in the sense that

$$\limsup_{t\to\infty}\|\Phi(t)\phi-E_0\|\geq\rho_0, \forall\phi\in\mathbb{W}_0.$$

Suppose, by contradiction, that $\limsup_{t\to\infty}\|\Phi(t)\phi-E_0\|<\rho_0$ for some $\phi_0\in\mathbb{W}_0$. Then, there exists a $t_2>0$ such that

$$u_1(t,x,\phi_0)<\rho_0 \text{ and } u_2(t,x,\theta,\phi_0)<\rho_0,\ \forall t\geq t_2,\ (x,\theta)\in\overline{\Omega}\times J.$$

Then, $u_1(t,x,\phi_0)$ and $u_2(t,x,\theta,\phi_0)$ satisfy

$$\begin{cases}\dfrac{\partial u_1(t,x)}{\partial t}=D\Delta u_1(t,x)-\alpha(x)u_1(t,x)+\displaystyle\int_J\beta(x,\theta)u_2(t,x,\theta)d\theta, & t\geq t_2,\ x\in\Omega,\\ \dfrac{\partial u_2(t,x,\theta)}{\partial t}\geq\sigma(x,\theta)(H(x,\theta)-\rho_0)u_1(t,x)-\gamma(x,\theta)u_2(t,x,\theta), & t\geq t_2,\ (x,\theta)\in\Omega\times J,\\ \dfrac{\partial u_1(t,x)}{\partial\nu}=0, & x\in\partial\Omega.\end{cases} \tag{4.47}$$

Let $\tilde{\phi}_{\rho_0} = (\tilde{\phi}_{1\rho_0}, \tilde{\phi}_{2\rho_0})^T$ be the positive eigenfunction associated with $\lambda_{\rho_0}(H)$. Then, the linear system

$$\begin{cases} \dfrac{\partial v_1(t,x)}{\partial t} = D\Delta v_1(t,x) - \alpha(x)v_1(t,x) + \displaystyle\int_J \beta(x,\theta)v_2(t,x,\theta)d\theta, & t>0,\ x\in\Omega, \\ \dfrac{\partial v_2(t,x,\theta)}{\partial t} = \sigma(x,\theta)(H(x,\theta)-\rho_0)v_1(t,x) - \gamma(x,\theta)v_2(t,x,\theta), & t>0,\ (x,\theta)\in\Omega\times J, \\ \dfrac{\partial v_1(t,x)}{\partial \nu} = 0, & x\in\partial\Omega, \end{cases} \tag{4.48}$$

admits a solution $v(t,x,\theta) = e^{\lambda_{\rho_0}(H)t}\tilde{\phi}_{\rho_0}(x,\theta)$. Since $u_i(t,x,\phi_0) > 0, i = 1,2$ for all $t > 0$ and $(x,\theta) \in \overline{\Omega} \times J$, there exists some $\zeta_1 > 0$ such that $(u_1(t_2,x,\phi_0), u_2(t_2,x,\phi_0)) \geq \zeta_1(v_1(t_2,x), v_2(t_2,x,\theta))$; by the comparison principle, we obtain

$$(u_1(t,x,\phi_0), u_2(t,x,\theta,\phi_0)) \geq \zeta_1 e^{\lambda_{\rho_0}(H)t}\tilde{\phi}_{\rho_0}(x,\theta),\ \forall x\in\overline{\Omega},\ \ t\geq t_2.$$

Consequently, with the aid of the positivity of $\lambda_{\rho_0}(H)$, we obtain that $u_1(t,x,\phi_0)$ and $u_2(t,x,\theta,\phi_0)$ are unbounded, which is a contradiction. This proves the Claim 4.4.

To go further, define a continuous function $p : \mathbb{X}_H \longrightarrow [0,\infty)$ by

$$p(\phi) = \min\{\min_{x\in\overline{\Omega}} \phi_1(x),\ \min_{(x,\theta)\in\overline{\Omega}\times J} \phi_2(x,\theta)\},\ \forall\phi\in\mathbb{X}_H.$$

From Lemma 4.1, it follows that $p^{-1}(0,\infty) \subseteq \mathbb{W}_0$ and p has the property that if $p(\phi) = 0,\ \phi\in\mathbb{W}_0$ or $p(\phi) > 0$, then $p(\Phi(t)\phi) > 0$ for all $t > 0$. That is, p is a generalized distance function for the semiflow $\Phi(t) : \mathbb{X}_H \to \mathbb{X}_H$ (see, e.g., [21]). From Claim 4.3, it follows that any forward orbit of $\Phi(t)$ in M_∂ converges to E_0. Moreover, Claim 4.4 implies that E_0 is isolated in $\mathbb{X}_H$ and $W^s(E_0)\cap\mathbb{W}_0 = \varnothing$ where $W^s(E_0)$ is the stable set of E_0. Furthermore, there is no cycle in M_∂ from E_0 to E_0. It then follows from [21, Theorem 3] that there exists an $\widehat{\eta} > 0$ such that

$$\min_{\psi\in\omega(\phi)} p(\psi) > \widehat{\eta},\ \forall\phi\in\mathbb{W}_0.$$

Hence, $\liminf_{t\to\infty} u_i(t,.,\phi) \geq \widehat{\eta},\ \forall\phi\in\mathbb{W}_0,\ i = 1,2$, and by Lemma 4.1, we can choose an $0 < \eta \leq \widehat{\eta}$ such that $\liminf_{t\to\infty} u_i(t,.,\phi) \geq \eta,\ i = 1,2$. Thus,

$$\liminf_{t\to\infty} u_i(t,.,\phi) \geq \eta,\ i = 1,2,\ \forall\phi\in\mathbb{W}_0.$$

Therefore, the uniform persistence stated in part (ii) is valid. By Magal and Zhao [15, Theorem 3.7 and Remark 3.10], we see that $\Phi(t) : \mathbb{W}_0 \to \mathbb{W}_0$ admits a global attractor $\mathcal{A}_0$. The solution maps $\Phi(t)$ are κ-condensing. Since the set $\mathbb{W}_0$ is convex, it follows from [15, Theorem 4.7] that $\Phi(t)$ has an equilibrium $\tilde{u} \in \mathbb{W}_0$. Clearly, by Lemma 4.1, we conclude that $\tilde{u}(.)$ is a positive steady state of (2.6). □

5 A Special Case

In this section, we shall discuss a special case where all the coefficients in (2.6) are independent of the spatial variable x, that is with the following assumption:

Assumption 5.1 *For all* $(x, \theta) \in \overline{\Omega} \times J$,

$$\beta(x,\theta) \equiv \beta(\theta) > 0,\ \gamma(x,\theta) \equiv \gamma(\theta) > 0,\ H(x,\theta) \equiv H(\theta) > 0,$$
$$\alpha(x) \equiv \alpha > 0,\ \sigma(x,\theta) \equiv \sigma(\theta) > 0.$$

5.1 The Basic Reproduction Number

By Theorem 3.3, and a similar argument as [27, Theorem 2.1], we have the following formula for $\mathcal{R}_0$:

Lemma 5.2 *Let Assumption 2.1–5.1 be satisfied; then for system* (2.6)*, the basic reproduction ratio* $\mathcal{R}_0$ *is equal to*

$$\mathcal{R}_0 = \int_J \mathbf{r}_0(y)dy, \tag{5.49}$$

where

$$\mathbf{r}_0(y) = \frac{\beta(y)}{\gamma(y)} \times \frac{\sigma(y)H(y)}{\alpha},\ \forall y \in J. \tag{5.50}$$

$\mathbf{r}_0(y)$ *describes the average number of infectious hens of type* y *produced by a single bacteria introduced in the infection-free environment and is interpreted as the weight of transmission* bacteria-to-hens of type y-to-bacteria *where hens of type* y *represent the animals having the same level of resistance* y.

The quantity $\mathcal{K}_{h,b}(y) := \dfrac{\beta(y)}{\gamma(y)}$ represents the average number of bacteria produced by one infectious hen of type y during its infectious period $\dfrac{1}{\gamma(y)}$ (i.e., hens of type y-to-bacteria transmission). $\mathcal{K}_{b,h}(y) := \dfrac{\sigma(y)H(y)}{\alpha}$ depicts the average number of infectious hens of type y caused by one bacterium during its lifetime $\dfrac{1}{\alpha}$ in the environment (i.e., bacteria-to-hens of type y transmission).

5.2 Global Attractivity of the Endemic Equilibrium

In addition to the threshold result in Theorem 4.2, we are able to prove the global attractivity of the positive steady state under some appropriate conditions.

Theorem 5.3 *Let Assumption 2.1–5.1 be satisfied, and let* $u(t, x, \theta, \phi) = (C(t, x), I(t, x, \theta))$ *be the solution of system (2.6) with initial condition* $u(0, ., .; \phi) := (C_0, I_0) \in \mathbb{X}_H$*; then the following statement is valid:*

If $\mathcal{R}_0 > 1$*, then system (2.6) has a unique (spatially) constant steady state* $u^*(\theta) = (u_1^*, u_2^*(\theta))^T$ *such that for any* $\phi := (\phi_1, \phi_2)^T \in \mathbb{X}_H$ *with* $\phi_1(0, .) \not\equiv 0$ *or* $\phi_1(0, .) \not\equiv 0$,

$$\lim_{t\to\infty}(u_1(t, x), u_2(t, x, \theta)) = (u_1^*, u_2^*(\theta))^T, \tag{5.51}$$

uniformly for all $(x, \theta) \in \overline{\Omega} \times J$*; moreover, if* $\gamma(\theta) \equiv \gamma > 0$ *and* $\sigma(\theta) \equiv \sigma > 0$*, then*

$$u_1^* = \frac{\gamma}{\sigma}(\mathcal{R}_0 - 1), \text{ and } u_2^*(\theta) = \frac{H(\theta)(\mathcal{R}_0 - 1)}{\mathcal{R}_0}. \tag{5.52}$$

Proof We use a fluctuation method (see, e.g.,[13, 24]). It is easy to see from the proof of Theorem (2.4) that the set $\mathbb{X}_H$ is positively invariant for the solution semiflow $\Phi(t)$, and every forward orbit enters into $\mathbb{X}_H$ eventually. It is easy to see that a positive equilibrium $(v_1^*, v_2^*(\theta))$ of (2.6) satisfies

$$v_2^*(\theta) = \frac{\sigma(\theta)H(\theta)v_1^*}{\gamma + \sigma v_1^*}, \tag{5.53a}$$

$$0 = -\alpha + \int_J \frac{\sigma(y)\beta(y)H(y)}{\gamma(y) + \sigma(y)v_1^*}dy := f(v_1^*). \tag{5.53b}$$

Moreover, the continuous mapping $v_1^* \mapsto f(v_1^*)$ from the set $[0, \infty)$ into $\mathbb{R}$ is monotone decreasing with $f(0) = \alpha(\mathcal{R}_0 - 1)$ and $\lim_{z\to\infty} f(z) = -\alpha$. Hence, for $\mathcal{R}_0 > 1$, there exists a unique (spatially) constant endemic equilibrium $u^*(\theta) := (u_1^*, u_2^*(\theta))^T$ satisfying (5.53). In the special case where $\gamma(\theta) \equiv \gamma > 0$ and $\sigma(\theta) \equiv \sigma > 0$, u^* is explicitly given by (5.52).

To show that $\lim_{t\to\infty}(u_1(t, x), u_2(t, x, \theta)) = u^*(\theta)$, we choose a sufficiently large number $k > 0$ such as the function $ku_1 - \alpha u_1 + \int_J \beta(y)u_2(., y)dy$ is monotone increasing in u_1 for all $(u_1, u_2)^T \in \mathbb{X}_H$. It then follows that

$$u_1(t, x) = e^{-kt}\int_\Omega \Gamma(Dt, x, y)u_1(0, y)dy + \int_0^t e^{-ks}\int_\Omega \Gamma(Ds, x, y)$$
$$\times\left[ku_1(x, t - s) - \alpha u_1(t - s, x) + \int_J \beta(y)u_2(t - s, x, y)dy\right],$$

where Γ is the Green's function associated with the parabolic equation $\dfrac{\partial v}{\partial t} = \Delta v$ in Ω subject to the Neumann boundary condition.

Let

$$u_1^\infty(x) := \limsup_{t\to\infty} u_1(t,x),\ \ u_{1\infty}(x) := \liminf_{t\to\infty} u_1(t,x),$$

and for any $\theta \in J$,

$$u_2^\infty(x,\theta) := \limsup_{t\to\infty} u_2(t,x,\theta),\ \ u_{2\infty}(x,\theta) := \liminf_{t\to\infty} u_2(t,x,\theta).$$

By the uniform persistence of (2.6), there exists an $\eta > 0$ such as

$$u_1^\infty(x) \geq u_{1\infty} \geq \eta,\ \ \forall x \in \overline{\Omega}$$

and

$$u_2^\infty(x,\theta) \geq u_{2\infty}(x,\theta) \geq \eta,\ \ \forall (x,\theta) \in \overline{\Omega} \times J.$$

Using Fatou's lemma, we then get

$$u_1(x) \leq \int_0^\infty e^{-ks} \int_\Omega \Gamma(Ds,x,y) \left[ku_1^\infty - \alpha u_1^\infty + \int_J \beta(y) u_2^\infty(y)dy\right].$$

Let

$$\delta_1^\infty := \sup_{x\in\overline{\Omega}} u_1(x),\ \ \delta_{1\infty} := \inf_{x\in\overline{\Omega}} u_1(x)$$

and

$$\delta_2^\infty(\theta) := \sup_{x\in\overline{\Omega}} u_2(x,\theta),\ \ \delta_{2\infty}(\theta) := \inf_{x\in\overline{\Omega}} u_2(x,\theta).$$

Clearly,

$$\frac{\beta_{\max} H_{\max} |J|}{\alpha} \geq \delta_1^\infty \geq \delta_{1\infty} \geq \eta,\ \ \forall x \in \overline{\Omega}$$

and

$$H(\theta) \geq \delta_2^\infty(\theta) \geq \delta_{2\infty}(\theta) \geq \eta,\ \ \forall \theta \in J.$$

Since $\int_\Omega \Gamma(Dt, x, y)dy = 1$ for all $x \in \Omega,\ s > 0$, we have

$$\delta_1^\infty \leq \int_0^\infty e^{-ks}\left[k\delta_1^\infty - \alpha\delta_1^\infty + \int_J \beta(y)\delta_2^\infty(y)dy\right]$$
$$= \frac{1}{k}\left[k\delta_1^\infty - \alpha\delta_1^\infty + \int_J \beta(y)\delta_2^\infty(y)dy\right],$$

and hence,

$$0 \leq -\alpha\delta_1^\infty + \int_J \beta(y)\delta_2^\infty(y)dy. \tag{5.54}$$

Similarly, we have the following inequality:

$$0 \geq -\alpha\delta_{1\infty} + \int_J \beta(y)\delta_{2\infty}(y)dy. \tag{5.55}$$

Using the second equation in (2.6), with arguments similar to those above, we further obtain

$$0 \leq \sigma(H(\theta) - \delta_2^\infty(\theta))\delta_1^\infty - \gamma\delta_2^\infty \tag{5.56a}$$
$$0 \geq \sigma(H(\theta) - \delta_{2,\infty}(\theta))\delta_{1\infty} - \gamma\delta_{2\infty}. \tag{5.56b}$$

After reorganization of (5.56), we get

$$\delta_2^\infty(\theta) \leq \frac{\sigma(y)H(\theta)\delta_1^\infty}{\gamma + \sigma\delta_1^\infty} \tag{5.57a}$$
$$\delta_{2\infty}(\theta) \geq \frac{\sigma(y)H(\theta)\delta_{1\infty}}{\gamma + \sigma\delta_{1\infty}}. \tag{5.57b}$$

Inserting (5.57a) into (5.54), and (5.57b) into (5.55), we get

$$0 \leq -\alpha\delta_1^\infty + \int_J \frac{\sigma(y)\beta(y)H(y)\delta_1^\infty}{\gamma(y) + \sigma(y)\delta_1^\infty}dy, \tag{5.58a}$$
$$0 \geq -\alpha\delta_{1\infty} + \int_J \frac{\sigma(y)\beta(y)H(y)\delta_{1\infty}}{\gamma(y) + \sigma(y)\delta_{1\infty}}dy. \tag{5.58b}$$

Since $\delta_1^\infty \geq \eta > 0$ and $\delta_{1\infty} \geq \eta > 0$, then it follows that

$$0 \leq -\alpha + \int_J \frac{\sigma(y)\beta(y)H(y)}{\gamma(y) + \sigma(y)\delta_1^\infty}dy, \tag{5.59a}$$

$$0 \geq -\alpha + \int_J \frac{\sigma(y)\beta(y)H(y)}{\gamma(y) + \sigma(y)\delta_{1\infty}} dy. \tag{5.59b}$$

Subtracting (5.59a) from (5.59b) inequality, we get

$$(\delta_1^\infty - \delta_{1\infty}) \int_J \frac{\sigma^2(y)\beta(y)H(y)}{(\gamma(y) + \sigma(y)\delta_{1\infty})(\gamma(y) + \sigma(y)\delta_1^\infty)} dy \leq 0. \tag{5.60}$$

Therefore, we must have $\delta_1^\infty = \delta_{1\infty}$.

Subtracting (5.57a) from (5.57b) inequality and since $\delta_1^\infty = \delta_{1\infty}$ from the above equality, we get

$$-(\delta_{2\infty}(\theta) - \delta_2^\infty(\theta))(\sigma\delta_1^\infty + \gamma) \leq 0. \tag{5.61}$$

Thus, we must have $\delta_{2\infty}(\theta) = \delta_2^\infty(\theta)$ for all $\theta \in J$. It follows that

$$\lim_{t\to\infty} u(t, x, \theta, \phi) = (\delta_{1\infty}, \delta_{2\infty}(\theta))^T, \ (x, \theta) \in \overline{\Omega} \times J. \tag{5.62}$$

Let $\omega(\phi)$ be the omega limit set of ϕ for the solution semiflow $\Phi(t)$ associated with (2.6). For any $\psi \in \omega(\phi)$, there exists a sequence $t_n \to \infty$ such that $\Phi(t_n)\phi \to \psi$ in $\mathbb{X}_H$ as $n \to \infty$. Then,

$$\lim_{n\to\infty} u(t_n, x, \theta, \phi) = \psi(x, \theta), \ (x, \theta) \in \overline{\Omega} \times J \tag{5.63}$$

uniformly for $(x, \theta) \in \overline{\Omega} \times J$, and by (5.62), $\psi(.) = (\delta_{1\infty}, \delta_{2\infty}(.))^T$. Therefore, $\omega(\phi) = \{(\delta_{1\infty}, \delta_{2\infty})^T\}$ implying that $u(t, x, \theta, \phi)$ converges to $(\delta_{1\infty}, \delta_{2\infty}(\theta))^T$ in $\mathbb{X}_H$. Since $\omega(\phi)$ is invariant under $\Phi(t)$ for all $t \geq 0$, it follows that $(\delta_{1\infty}, \delta_{2\infty}(\theta))^T$ is a positive (spatially) constant steady state of (2.6) under the assumption 2.1–5.1.

□

6 Impact of Heterogeneities on $\mathcal{R}_0$

To analyze the combined effects of genetic and spatial distributions of heterogeneous fowls on $\mathcal{R}_0$, we will assume that the domain Ω is a one-dimensional interval $(0, L)$ where $L > 0$.We define the spatial average and the spatial average combined with the resistance average of a function f, respectively, by

$$\overline{f}(\theta) = \frac{1}{L}\int_0^L f(x, \theta) dx \ \text{ and } \ \widetilde{f} = \frac{1}{\Theta L}\int_0^\Theta \int_0^L f(x, \theta) dx d\theta.$$

To go further, we denote by $\overline{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_0$ the basic reproduction numbers of system (2.6), where the parameters of model are replaced by their spatial average and their spatial average combined with their resistance average, respectively. We aim to compare the reproduction number with the heterogeneities, $\mathcal{R}_0$, and the other two means. The same annotation is used for other parameters of the model.

To simplify our analyze, we shall discuss a special case where the rate of transmission, $\sigma(x,\theta)$, excretion rate in environment $\beta(x,\theta)$, and recovery rate $\gamma(x,\theta)$ of animals are only depending on their level of resistance $\theta \in [0,\Theta)$ and thus independent of the spatial variable x, except the initial distribution of hens $H(x,\theta)$ and the mortality rate of bacteria $\alpha(x)$ that depend on the position x.

Since θ represents the level of animal genetic resistance to Salmonella carrier state on $[0,\Theta)$, it is reasonable to assume that the rate of transmission, σ, the excretion rate, β, and the length of the infectious period, $1/\gamma$, are monotonically decreasing with respect to θ. Together with these hypotheses, we make the following assumption for our study:

Assumption 6.1 *For all* $(x,\theta) \in \overline{\Omega} \times J$,

$$\beta(x,\theta) \equiv \frac{\beta_0}{(\theta+1)}, \quad \gamma(x,\theta) \equiv \gamma_0(\theta+1), \quad \sigma(x,\theta) = \frac{\sigma_0}{(\theta+1)},$$

where β_0, σ_0, *and* γ_0 *are constant positive.*

Clearly,

$$\begin{aligned} \overline{\alpha} &= \widetilde{\alpha} \\ \overline{\beta}(\theta) &= \frac{\beta_0}{(\theta+1)} \text{ and } \widetilde{\beta} = \beta_0 \frac{\ln(\Theta+1)}{\Theta} \\ \overline{\sigma}(\theta) &= \frac{\sigma_0}{(\theta+1)} \text{ and } \widetilde{\sigma} = \sigma_0 \frac{\ln(\Theta+1)}{\Theta} \\ \overline{\gamma}(\theta) &= \gamma_0(\theta+1) \text{ and } \widetilde{\gamma} = \gamma_0 \frac{(\Theta+2)}{2}. \end{aligned} \tag{6.64}$$

Straightforward computations show that

$$\widetilde{\mathcal{R}}_0 := \frac{1}{\widetilde{\alpha}} \frac{\widetilde{\beta}\widetilde{\sigma}\widetilde{H}\Theta}{\widetilde{\gamma}} = \frac{1}{\overline{\alpha}} \frac{\beta_0\sigma_0}{\gamma_0} \times \frac{2\widetilde{H}\ln^2(\Theta+1)}{\Theta(\Theta+2)}. \tag{6.65}$$

Let Λ be the principal eigenvalue of system (2.6), and let $\overline{\Lambda}$ and $\widetilde{\Lambda}$ be the principal eigenvalue of system (2.6) under Assumption 6.1, where the parameters of model are replaced by their spatial average and their spatial average combined with their resistance average, respectively.

The following result implies that: (1) the spatial heterogeneity results in a basic reproduction number that is larger than the value obtained when replacing the model parameters by their spatial average. (2) The combined effects of genetic and spatial distributions result in a basic reproduction number that is larger than the product of a function of the initial distribution of fowls and the value obtained when replacing the model parameters by their spatial average.

Lemma 6.2 *Let Assumption 6.1 be satisfied, for (2.6), and there hold:*

(i)

$$\mathcal{R}_0 \geq \overline{\mathcal{R}}_0. \tag{6.66}$$

(ii)

$$\mathcal{R}_0 \geq \widetilde{\mathcal{R}}_0 F(H), \tag{6.67}$$

where

$$F(H) = \frac{\Theta(\Theta+2)}{2\widetilde{H}\ln^2(\Theta+1)} \int_0^{\Theta} \frac{\overline{H}(\theta)}{(\theta+1)^3} d\theta.$$

Proof The proof of (i) uses a similar argument to that given in [26, Lemma 4.4] with some minor modifications. Dividing (3.34) by the positive eigenfunction $L\phi^*$ associated with Λ and integrating on $(0, L)$, we have

$$-\frac{1}{L}\int_0^L \frac{D\Delta\phi^*}{\phi^*} + \overline{\alpha} = \Lambda \frac{1}{L}\int_0^L \int_0^{\Theta} \frac{\beta(\theta)\sigma(\theta)}{\gamma(\theta)} H(x,\theta) d\theta dx. \tag{6.68}$$

Using integration by parts, we further obtain

$$\overline{\alpha} \geq \Lambda \int_0^{\Theta} \frac{\beta(\theta)\sigma(\theta)}{\gamma(\theta)} \overline{H}(\theta) d\theta. \tag{6.69}$$

On the other hand, we obtain

$$\overline{\alpha} = \overline{\Lambda} \int_0^{\Theta} \frac{\beta(\theta)\sigma(\theta)}{\gamma(\theta)} \overline{H}(\theta) d\theta \tag{6.70}$$

and

$$\widetilde{\alpha} = \widetilde{\Lambda} \frac{\Theta\widetilde{\beta}\widetilde{\sigma}\widetilde{H}}{\widetilde{\gamma}}. \tag{6.71}$$

Combining (6.69) and (6.70) yields $\Lambda \leq \overline{\Lambda}$, and thus, $\mathcal{R}_0 \geq \overline{\mathcal{R}}_0$ and (i) holds.

To prove (ii), note that $\overline{\alpha} = \widetilde{\alpha}$. Thus by virtue of (6.69) combined with (6.71), we get

$$\widetilde{\alpha} = \widetilde{\Lambda} \frac{\Theta\widetilde{\beta}\widetilde{\sigma}\widetilde{H}}{\widetilde{\gamma}} \geq \Lambda \int_0^{\Theta} \frac{\beta(\theta)\sigma(\theta)}{\gamma(\theta)} \overline{H}(\theta) d\theta = \Lambda \frac{\beta_0\sigma_0}{\gamma_0} \int_0^{\Theta} \frac{\overline{H}(\theta)}{(\theta+1)^3} d\theta.$$

Thus using (6.64), we obtain

$$\widetilde{\Lambda}\frac{2\widetilde{H}\ln^2(\Theta+1)}{\Theta(\Theta+2)} \geq \Lambda\int_0^{\Theta}\frac{\overline{H}(\theta)}{(\theta+1)^3}d\theta.$$

The above inequality completes the proof of (ii). □

According to (ii) of Lemma 6.2, we conclude in this section that the severity of disease transmission is largely depending on the choice of the initial distribution H of hens: for example, when choosing H so that $F(H) \geq 1$, then $\mathcal{R}_0 \geq \widetilde{\mathcal{R}}_0$ implying that the combined effects of genetic and spatial distributions produce a larger basic reproduction number. In contrast, when $F(H) < 1$, then we cannot conclude in our study about the effects of heterogeneities on the value of $\mathcal{R}_0$. However, it should be possible to choose an initial distribution to obtain $\mathcal{R}_0 < \widetilde{\mathcal{R}}_0$ implying that the combined effects produce a smaller basic reproduction number. This choice can be achieved by solving a problem of optimization that is beyond the scope of this chapter.

7 Concluding Remarks

The objective of this chapter was to propose and study a reaction–diffusion model for Salmonella transmission within an industrial hens house when hens' distribution within the hen house varies according to their resistance to Salmonella carrier state. For the derived model (2.6), we introduced the basic reproduction number $\mathcal{R}_0$ via the next-generation operator, and we further prove that $\mathcal{R}_0$ serves as a threshold index that predicts the extinction and persistence of the disease. Furthermore, we showed the equivalent characterization of $\mathcal{R}_0$ (Theorem 3.3). Furthermore, in Corollary 3.4, $\mathcal{R}_0$ is rewritten as a function of the weight of transmission *bacteria-to-hens of type* θ*-to-bacteria*, $\mathbf{r}_0(\theta)$, where hens of type θ represent the animals having the same level of resistance θ for any $\theta \in J$. In Corollary 3.5, we give some properties linking $\mathcal{R}_0$ and $\mathbf{r}_0(\theta)$ on the control of the disease.

In Sect. 5, we investigate the special case where all the coefficients in (2.6) are independent of the spatial variable. Using a fluctuation method developed in [24], we show that when $\mathcal{R}_0 > 1$ the disease will become established and stabilize at a unique spatially homogeneous steady state. Moreover, the positive steady state is globally attractive.

The impact of heterogeneities on $\mathcal{R}_0$ was achieved in Sect. 6. We have shown that the severity of disease transmission is largely depending on the choice of the initial distribution H of hens. This was achieved by comparing the basic reproduction number of a given distribution and those when the parameters of model are replaced by their average values. The results of this model provide useful information to determine the optimum population distribution of heterogeneous fowls to minimize the basic reproduction number.

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Impulsive Implicit Caputo Fractional q-Difference Equations in Finite- and Infinite-Dimensional Banach Spaces

Badr Alqahtani, Saïd Abbas, Mouffak Benchohra, and Gaston M. N'Guérékata

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1 Introduction

Fractional differential equations have recently been applied in various areas; for some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to [1–3, 23, 25, 30, 31, 33, 34] and the references therein.

Implicit fractional differential equations have also been considered by many authors [4, 12]. Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in biological or medical domains, and in control theory, see for example the monographs of Abbas et al. [1, 2], Benchohra et al. [13], Graef et al. [19], and papers such as Abbas et al. [4], Hernández and O'Regan [21] and the references therein.

In this chapter, we first discuss the existence of solutions for the following problem of implicit fractional q-difference equations:

B. Alqahtani
Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia
e-mail: balqahtani1@ksu.edu.sa

S. Abbas
Department of Mathematics, Tahar Moulay University of Saïda, Saïda, Algeria

M. Benchohra (✉)
Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel Abbès, Sidi Bel Abbès, Algeria

G. M. N'Guérékata
Department of Mathematics, Morgan State University, Baltimore, MD, USA
e-mail: Gaston.NGuerekata@morgan.edu

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$$\begin{cases} ({}^{C}_{q}D^{r}_{t_k}u)(t) = f(t, u(t), ({}^{C}_{q}D^{r}_{t_k}u)(t)); \ t \in J_k, \ k = 0, \ldots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \ldots, m, \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $J_0 = [0, t_1]$; $J_k := (t_k, t_{k+1}]$, $k = 1, \ldots, m$; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$; $f : J_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$; $L_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ are given continuous functions, and ${}^{c}_{q}D^{r}_{t_k}$ is the Caputo fractional q-difference derivative of order $r \in (0, 1]$.

Various classes of fractional q-difference equations have been considered in the papers [5, 7, 8, 16–18]. Recently, in [3, 9–11, 14], the authors applied the measure of noncompactness to the study of some classes of functional Riemann–Liouville or Caputo fractional differential equations in Banach spaces. Motivated by the above papers, we next discuss the existence of solutions for the problem (1), when $u_0 \in E$, $f : J_k \times E \times E \to E$; $k = 1, \ldots, m$, $L_k : E \to E$; $k = 1, \ldots, m$ are given continuous functions, and E is a real (or complex) Banach space with norm $\|\cdot\|$.

This chapter initiates the study of impulsive implicit fractional q-difference equations in finite- and infinite-dimension Banach spaces.

2 Preliminaries

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from $I := [0, T]$ into E equipped with the usual norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

In the scalar case when $E = \mathbb{R}$, we replace $\|\cdot\|$ by $|\cdot|$. As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ that are Bochner integrable with the norm

$$\|v\|_1 = \int_I \|v(t)\| dt.$$

Let

$$PC = \big\{u : I \to E : u \in C(J_k); \ k = 0, \ldots, m, \ \text{and there exist } u(t_k^-) \\ \text{and } u(t_k^+); \ k = 1, \ldots, m, \ \ \text{with } u(t_k^-) = u(t_k)\big\},$$

be the Banach space with the norm

$$\|u\|_{PC} = \sup_{t \in I} \|u(t)\|.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power $(a - b)^n$ is

$$(a - b)^{(0)} = 1, \; (a - b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k); \; a, b \in \mathbb{R}, \; n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right); \; a, b, \alpha \in \mathbb{R}.$$

Definition 2.1 ([22]) The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1 - q)^{(\xi - 1)}}{(1 - q)^{\xi - 1}}; \; \xi \in \mathbb{R} - \{0, -1, -2, \ldots\}.$$

Notice that the q-gamma function satisfies $\Gamma_q(1 + \xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 2.2 ([22]) The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1 - q)t}; \; t \neq 0, \;\; (D_q u)(0) = \lim_{t \to 0} (D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t); \; t \in I, \; n \in \{1, 2, \ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 ([22]) The q-integral of a function $u : I_t \to E$ is defined by

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1 - q)q^n u(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4 ([6]) The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s;\ t \in I.$$

Lemma 2.5 ([27]) *For $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $\lambda \in (-1, \infty)$, we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(t-a)^{(\lambda+\alpha)};\ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.6 ([28]) The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t);\ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7 ([28]) The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t);\ t \in I.$$

Lemma 2.8 ([28]) *Let $\alpha \in \mathbb{R}_+$. Then, the following equality holds:*

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)}(D_q^k u)(0).$$

In particular, if $\alpha \in (0, 1)$, then

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above lemma, and in order to define the solution for our problem, we conclude the following lemma.

Lemma 2.9 *Let $f : I \times E \times E \to E$ such that $f(\cdot, u, v) \in C(I)$, for each $u, v \in E$. Then, the problem*

$$\begin{cases} ({}_q^C D_0^r u)(t) = f(t, u(t), ({}_q^C D_0^r u)(t));\ t \in [0.T], \\ u(0) = u_0, \end{cases}$$

is equivalent to the problem of obtaining the solutions of the integral equation

$$g(t) = f(t, u_0 + (I_q^{\alpha} g)(t), g(t)),$$

and if $g(\cdot) \in C(I)$ is the solution of this equation, then

$$u(t) = u_0 + (I_q^{\alpha} g)(t).$$

Lemma 2.10 *Let $h : I \to E$ be a continuous function. A function $u \in PC$ is a solution of the fractional integral equation*

$$\begin{cases} u(t) = u_0 + ({}_q I_0^r h)(t); \ if \ t \in J_0, \\ u(t) = u_0 + \sum_{i=1}^{k} L_i(u(t_i^-)) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(r-1)}}{\Gamma_q(r)} h(s) d_q s \\ + \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} h(s) d_q s; \ if \ t \in J_k, \ k = 1, \ldots, m, \end{cases} \tag{2}$$

if and only if u is a solution of the following problem:

$$\begin{cases} ({}_q^C D_{t_k}^r u)(t) = h(t); \ t \in J_k, \ k = 0, \ldots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \ldots, m, \\ u(0) = u_0. \end{cases} \tag{3}$$

Proof Assume u satisfies (3). If $t \in J_0$, then

$$({}_q^C D_0^r u)(t) = h(t).$$

Lemma 2.8 implies

$$u(t) = u_0 + ({}_q I_0^r h)(t).$$

If $t \in J_1$, then

$$({}_q^C D_{t_1}^r u)(t) = h(t).$$

Lemma 2.8 implies

$$\begin{aligned} u(t) &= u(t_1^+) + ({}_q I_{t_1}^r h)(t) \\ &= L_1(u(t_1^-)) + u(t_1^-) + ({}_q I_{t_1}^r h)(t) \\ &= L_1(u(t_1^-)) + u_0 + ({}_q I_1^r h)(t_1) + ({}_q I_{t_1}^r h)(t). \end{aligned}$$

If $t \in J_2$, then

$$({}_q^C D_2^r u)(t) = h(t).$$

We obtain

$$\begin{aligned} u(t) &= u(t_2^+) + ({}_qI_{t_2}^r h)(t) \\ &= L_2(u(t_2^-)) + u(t_2^-) + ({}_qI_{t_2}^r h)(t) \\ &= L_2(u(t_2^-)) + L_1(u(t_1^-)) \\ &\quad + u_0 + ({}_qI_1^r h)(t_1) + ({}_qI_{t_1}^r h)(t_2) + ({}_qI_{t_2}^r h)(t). \end{aligned}$$

If $t \in J_k$, then again from Lemma 2.8 we get (2).

Conversely, assume that u satisfies (2). If $t \in J_0$, then $u(t) = u_0 + ({}_qI_1^r h)(t)$. Thus, $u(0) = u_0$, and using the fact that ${}_q^C D_1^r$ is the left inverse of ${}_qI_1^r$, we get $({}_q^C D_1^r u)(t) = h(t)$.

Now, if $t \in J_k;\ k = 1, \dots, m$, we get $({}_q^C D_{t_k}^r u)(t) = h(t)$. Also, we can easily show that

$$u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)).$$

Hence, if u satisfies (2), then we get (3).

From the above Lemmas 2.10 and 2.9, we conclude with the following lemma:

Lemma 2.11 *Let $f(t, u, z) : J_k \times E \times E \to E;\ k = 0, \dots, m$, be a continuous function. Then, problem (1) is equivalent to the problem of solving the equation*

$$g(t) = f(t, u_0 + ({}_qI_{t_k}^r g)(t), g(t)),$$

and if $g(\cdot) \in C(J_k),\ k = 0, \dots, m$, is the solution of this equation, then

$$\begin{cases} u(t) = u_0 + ({}_qI_0^r g)(t);\ if\ t \in J_0, \\ u(t) = u_0 + \sum_{i=1}^{k} (L_i(u(t_i^-)) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \\ + \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s;\ if\ t \in J_k,\ k = 1, \dots, m. \end{cases}$$

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.12 Let X be a complete metric space. A map $\mu : \mathcal{M}_X \to [0, \infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$:

(a) $\mu(B) = 0$ if and only if B is precompact (regularity).
(b) $\mu(B) = \mu(\overline{B})$ (invariance under closure).
(c) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$ (semi-additivity).

Definition 2.13 ([11]) Let X be a Banach space, and let Ω_X be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0, \infty)$ defined by

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^{m} M_j, \operatorname{diam}(M_j) \le \epsilon\},$$

where $M \in \Omega_E$.

Properties

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \le \mu(M_2)$.
(4) $\mu(M_1 + M_2) \le \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M),\ c \in \mathbb{R}$.
(6) $\mu(\operatorname{conv} M) = \mu(M)$.

Definition 2.14 ([29]) A nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies one of the following conditions:

(1) For any $t > 0$, we have

$$\lim_{n\to\infty} \phi^{(n)}(t) = 0,$$

where $\phi^{(n)}$ denotes the n-th iteration of ϕ.
(2) The function ϕ is right-continuous and satisfies

$$\forall t > 0 : \phi(t) < t.$$

Remark 2.15 The choice $\phi(t) = kt$ with $0 < k < 1$ gives the classical Banach contraction mapping principle.

For our purpose, we will need the following fixed-point theorems:

Theorem 2.16 ([15, 24]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(T(x), T(y)) \le \phi(d(x, y)),$$

where ϕ is a comparison function. Then, T has a unique fixed point in X.

Theorem 2.17 (Schauder Fixed-Point Theorem [32]) *Let X be a Banach space, D be a bounded closed convex subset of X, and $T : D \to D$ be a compact and continuous map. Then, T has at least one fixed point in D.*

Theorem 2.18 (Schaefer Fixed-Point Theorem [20]) *Let X be a Banach space, and $N : X \to X$ be a completely continuous operator. If the set*

$$\mathcal{E} = \{u \in X : u = \lambda N(u);\ \textit{for some}\ \lambda \in (0, 1)\}$$

is bounded, then N has fixed points.

Theorem 2.19 (Monch's Fixed-Point Theorem [26]) *Let D be a bounded, closed, and convex subset of a Banach space such that* $0 \in D$, *and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V)\ or\ V = N(V) \cup \{0\} \Rightarrow \overline{V}\ is\ compact, \tag{4}$$

holds for every subset V of D, then N has a fixed point.

3 Existence Results in the Scalar Case

In this section, we present some results concerning the existence of solutions for the problem (1).

Definition 3.1 By a solution of the problem (1), we mean a function $u \in PC$ that satisfies the condition $u(0) = u_0$, and the equation $({}_q^C D_{t_k}^r u)(t) = f(t, u(t), ({}_q^C D_{t_k}^r u)(t))$ on $J_k;\ k = 0, \ldots, m$.

The following hypotheses will be used in the sequel:

(H_{01}) The function $f : J_k \mapsto f(t, u, v),\ k = 0, \ldots, m$, is continuous.

(H_{02}) The functions f and $L_k,\ k = 1, \ldots, m$, satisfy the generalized Lipschitz conditions:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le \phi_1(|u_1 - u_2|) + \phi_2(|v_1 - v_2|)$$

and

$$|L_k(u_1) - L_k(u_2)| \le \phi_3(|u - u_2|),$$

for $t \in I$ and $u, v \in \mathbb{R}$, where $\phi_i,\ i = 1, 2, 3$, are the comparison functions.

(H_{03}) There exists a continuous function $\ell \in C(J_k, \mathbb{R}_+),\ k = 0, \ldots, m$, such that

$$|f(t, u, v)| \le \ell(t)(1 + |u| + |v|),\ \text{for each}\ t \in J_k,\ \text{and}\ u, v \in \mathbb{R},$$

with

$$\ell^* = \sup_{t \in I} \ell(t) < 1.$$

(H_{04}) There exists a constant $l > 0$ such that

$$|L_k(u)| \leq l(1 + |u|), \ \ for\ each\ u \in \mathbb{R}.$$

Theorem 3.2 *Assume that the hypotheses* (H_{01}) *and* (H_{02}) *hold. Then, the problem (1) has a unique solution defined on* I.

Proof Consider the Banach space $C(I) := C(I, \mathbb{R})$ as a complete metric space of continuous functions from I into $\mathbb{R}$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} |u(t) - v(t)|.$$

Transform the problem (1) into a fixed-point equation. Consider the operator $N : PC \to PC$ defined by

$$\begin{cases} (Nu)(t) = u_0 + ({}_qI_0^r g)(t); \ if\ t \in J_0, \\ (Nu)(t) = u_0 + \sum_{i=1}^{k} L_i(u(t_i^-)) \\ + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \\ + \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s; \ if\ t \in J_k,\ k = 1, \ldots, m, \end{cases} \tag{1}$$

where $g(\cdot) \in C(J_k),\ k = 0, \ldots, m$, with

$$g(t) = f(t, u_0 + ({}_qI_{t_k}^r g)(t), g(t)).$$

Clearly, the fixed points of the operator N are solutions of the problem (1).

Let $u \in PC$ and $t \in J_0$. Then,

$$|(Nu)(t) - (Nv)(t)| = |({}_qI_0^r (g - h))(t)|,$$

where $g, h \in C(J_k),\ k = 0, \ldots, m$, with

$$g(t) = f(t, u_0 + ({}_qI_0^r g)(t), g(t)),\ and\ h(t) = f(t, u_0 + ({}_qI_0^r h)(t), h(t)).$$

Thus, for each $u, v \in C(I)$ and $t \in J_0$, we have

$$|(Nu)(t) - (Nv)(t)| = \int_0^t \frac{|t - qs|^{(r-1)}}{\Gamma_q(r)} |g(s) - h(s)| d_q s.$$

From (H_{02}), we have

$$|g(t) - h(t)| \leq \phi_1(|u(t) - v(t)|) + \phi_2(|g(t) - h(t)|).$$

Thus,

$$|g(t) - h(t)| \leq (Id - \phi_2)^{-1}\phi_1(|u(t) - v(t)|).$$

Hence,

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| &\leq \int_0^t \frac{|t-qs|^{(r-1)}}{\Gamma_q(r)}(Id - \phi_2)^{-1}\phi_1(|u(s) - v(s)|)d_q s \\ &\leq \frac{T^r}{\Gamma_q(1+r)}(Id - \phi_2)^{-1}\phi_1(d(u, v)) \\ &= \phi(d(u, v)), \end{aligned}.$$

where Id is the identity function, and ϕ is the comparison function defined by

$$\phi(t) = \frac{T^r}{\Gamma_q(1+r)}(Id - \phi_2)^{-1}\phi_1(t);\ t \in J_0.$$

So, we get

$$d(N(u), N(v)) \leq \phi(d(u, v)).$$

Next, for each $u, v \in C(I)$ $t \in J_k,\ k = 1, \ldots, m$, we get

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| &\leq \frac{T^r}{\Gamma_q(1+r)}(Id - \phi_2)^{-1}\phi_1(d(u, v)) + m\phi_3(d(u, v)) \\ &\leq (\frac{T^r}{\Gamma_q(1+r)}(Id - \phi_2)^{-1}\phi_1 + m\phi_3)(d(u, v)) \\ &= \phi(d(u, v)), \end{aligned}.$$

where ϕ is the comparison function defined by

$$\phi(t) = \frac{T^r}{\Gamma_q(1+r)}(Id - \phi_2)^{-1}\phi_1(t) + m\phi_3(t);\ t \in J_k :\ k = 1, \ldots, m.$$

So, we get

$$d(N(u), N(v)) \leq \phi(d(u, v)).$$

Consequently, from Theorem 2.16, the operator N has a unique fixed point, which is the unique solution of our problem (1) on I.

Theorem 3.3 *Assume that the hypotheses* (H_{01}), (H_{03}), *and* (H_{04}) *hold. If*

$$ml + \frac{2T^r\ell^*}{(1-\ell^*)\Gamma_q(1+r)} < 1,$$

then the problem (1) has at least one solution defined on I.

Proof Consider the operator $N : PC \to PC$ defined in (1). Let $R > 0$, such that

$$R \geq \frac{|u_0| + ml + \frac{2T^r \ell^*}{(1-\ell^*)\Gamma_q(1+r)}}{1 - ml - \frac{2T^r \ell^*}{(1-\ell^*)\Gamma_q(1+r)}},$$

and consider the ball $B_R := B(0, R) = \{w \in \|w\|_{PC} \leq R\}$. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.17. The proof will be given in several steps:

Step 1. $N : B_R \to B_R$ *is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in J_0$, we have

$$|(Nu_n)(t) - (Nu)(t)| \leq \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g_n(s) - g(s)| d_q s, \tag{2}$$

where $g, g_n \in C(J_0)$ with

$$g(t) = f(t, u_0 + ({}_qI_0^r g)(t), g(t)),$$

and

$$g_n(t) = f(t, u_0 + ({}_qI_0^r g_n)(t), g_n(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, (2) implies

$$\|N(u_n) - N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Also, for each $t \in J_k,\ k = 1, \ldots, m$, we have

$$|(Nu_n)(t) - (Nu)(t)|$$

$$\leq \sum_{i=1}^{k} \|L_i(u_n(t_i^-)) - L_i(u(t_i^-))\|$$
$$+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(r-1)}}{\Gamma_q(r)} |g_n(s) - g(s)| d_q s$$
$$+ \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g_n(s) - g(s)| d_q s. \tag{3}$$

Again, by the Lebesgue dominated convergence theorem, (3) implies the continuity of our operator N.

Step 2. *$N(B_R)$ is bounded.*
Let $u \in B_R$ and $t \in J_0$. Then,

$$|(Nu)(t)| = \left| u_0 + \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right|,$$

where $g(\cdot) \in C(I)$ with

$$g(t) = f(t, u_0 + ({}_qI_0^r g)(t), g(t)).$$

Thus,

$$\begin{aligned}
|(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\
&\leq |u_0| + \frac{T^r \ell^*(1+R)}{(1-\ell^*)\Gamma_q(1+r)} \\
&\leq R.
\end{aligned}$$

Next, if $u \in B_R$, and $t \in J_k : \ k = 1, \ldots, m$, we have

$$\begin{aligned}
|(Nu)(t)| &\leq |u_0| + \sum_{i=1}^{k} \|L_i(u(t_i^-))\| \\
&+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\
&+ \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\
&\leq |u_0| + ml(1+R) + \frac{2T^r \ell^*(1+R)}{(1-\ell^*)\Gamma_q(1+r)} \\
&\leq R.
\end{aligned}$$

Hence, for any $u \in B_R$, and each $t \in J$, we get

$$\|N(u)\|_{PC} \leq R.$$

This proves that N transforms the ball $B_R := B(0, R) = \{w \in \|w\|_{PC} \leq R\}$ into itself.

Step 3. *$N(B_R)$ is equicontinuous.*
Let $x_1, x_2 \in J_0$ such that $0 \leq x_1 < x_2 \leq t_1$ and let $u \in B_R$. Then,

$$
\begin{aligned}
&|(Nu)(x_2) - (Nu)(x_1)| \\
&\leq \left| \int_0^{x_2} \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s - \int_0^{x_1} \frac{(x_1 - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right|,
\end{aligned}
$$

where $g \in C(J_0)$ with

$$
g(t) = f(t, u_0 + ({}_q I_0^r g)(t), g(t)).
$$

Thus,

$$
\begin{aligned}
&|(Nu)(x_2) - (Nu)(x_1)| \\
&\leq \int_{x_1}^{x_2} \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\
&\quad + \int_0^{x_1} \frac{|(x_2 - qs)^{(r-1)} - (x_1 - qs)^{(r-1)}|}{\Gamma_q(r)} |g(s)| d_q s \\
&\leq \frac{\ell^*(1+R)(x_2 - x_1)^r}{(1-\ell^*)\Gamma_q(1+r)} \\
&\quad + \frac{\ell^*(1+R)}{1-\ell^*} \int_0^{x_1} \frac{|(x_2 - qs)^{(r-1)} - (x_1 - qs)^{(r-1)}|}{\Gamma_q(r)} d_q s.
\end{aligned}
$$

As $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. Also, if we let $x_1, x_2 \in J_k,\ k = 1, \dots, m$, such that $t_k \leq x_1 < x_2 \leq t_{k+1}$ and let $u \in B_R$, we obtain

$$
\begin{aligned}
&|(Nu)(x_2) - (Nu)(x_1)| \\
&\leq \frac{2\ell^*(1+R)}{(1-\ell^*)\Gamma_q(1+r)} |x_2 - x_1|^r \\
&\quad + \frac{2\ell^*(1+R)}{1-\ell^*} \int_0^{x_1} \left| \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} - \frac{(x_1 - qs)^{(r-1)}}{\Gamma_q(r)} \right| d_q s.
\end{aligned}
$$

Again, as $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. Hence, $N(B_R)$ is equicontinuous.

As a consequence of the above three steps, together with the Arzelá–Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 2.17, we deduce that N has a fixed point u that is a solution of problem (1).

Now, we use Schaefer's fixed-point theorem to prove the following result:

Theorem 3.4 *Assume that the hypotheses* (H_{01}), (H_{03}), (H_{04}) *and the conditions* $ml < 1$, *and*

$$ml + \frac{2T^r \ell^*}{(1-\ell^*)\Gamma_q(1+r)} < 1,$$

hold. Then, the problem (1) has at least one solution defined on I.

Proof We consider the operator $N : PC \to PC$ defined in (1). As in the proof of Theorem 3.3, we can show that $N : PC \to PC$ is continuous and compact. Now it remains to show that the set

$$\mathcal{E} = \{u \in X : u = \lambda N(u);\ for\ some\ \lambda \in (0,1)\}$$

is bounded.

Let $u \in \mathcal{E}$; then, $u = \lambda N(u)$, for some $\lambda \in (0,1)$. Thus, for each $t \in J_0$, we have

$$|u(t)| \leq \left| u_0 + \int_1^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right|,$$

where $g(\cdot) \in C(I)$ with

$$g(t) = f(t, u_0 + ({}_qI_1^r g)(t), g(t)).$$

Thus,

$$\begin{aligned} |u(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \\ &\leq |u_0| + \int_0^t \frac{(t-qs)^{(r-1)}}{(1-\ell^*)\Gamma_q(r)} \ell^*(1+|u(s)|) d_q s \\ &\leq |u_0| + \frac{T^r \ell^*}{(1-\ell^*)\Gamma_q(1+r)} + \int_0^t \frac{(t-qs)^{(r-1)}}{(1-\ell^*)\Gamma_q(r)} \ell^* |u(s)| d_q s. \end{aligned}$$

We can apply a version of Gronwall's lemma to obtain that $|u(t)| \leq M_1$, with $M_1 > 0$.

Next, for each $t \in J_k :\ k = 1, \ldots, m$, we have

$$\begin{aligned} |u(t)| \leq |u_0| &+ \sum_{i=1}^{k} \|L_i(u(t_i^-))\| \\ &+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(r-1)}}{\Gamma_q(r)} |g(s)| d_q s \end{aligned}$$

$$+\int_{t_k}^{t}\frac{(t-qs)^{(r-1)}}{\Gamma_q(r)}|g(s)|d_qs$$

$$\leq |u_0|+ml(1+|u(t)|)+\frac{2T^r\ell^*}{(1-\ell^*)\Gamma_q(1+r)}$$

$$+\int_0^t\frac{(t-qs)^{(r-1)}}{(1-\ell^*)\Gamma_q(r)}\ell^*2|u(s)|d_qs.$$

This implies that, for each $t \in J_k : \ k = 1, \ldots, m$, we get

$$|u(t)| \leq \frac{|u_0|+ml}{1-ml}+\frac{2T^r\ell^*}{(1-ml)(1-\ell^*)\Gamma_q(1+r)}$$

$$+\frac{2\ell^*}{1-ml}\int_0^t\frac{(t-qs)^{(r-1)}}{(1-\ell^*)\Gamma_q(r)}|u(s)|d_qs.$$

Also, by applying a version of Gronwall's lemma, we can obtain $|u(t)| \leq M_2$, with $M_2 > 0$. Hence, the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed-point theorem (Theorem 2.18), we deduce that N has a fixed point that is a solution of our problem (1).

4 Existence Results in Banach Spaces

In this section, we present some results concerning the existence of solutions for the problem (1) in Banach spaces.

The following hypotheses will be used in the sequel.

(H_1) The function f is continuous.

(H_2) There exists a continuous function $p \in C(J_k, \mathbb{R}_+)$, $k = 0, \ldots, m$, such that

$$\|f(t,u,v)\| \leq p(t)(1+|u|+|v|);\ \ for\ t\in J_k,\ and\ u,v\in E,$$

with $p^* = \sup_{t\in J} p(t) < 1$.

(H_3) For each bounded and measurable set $B \subset E$ and for each $t \in J_k$, $k = 0, \ldots, m$, we have

$$\mu(f(t,B,{}^C_qD^r_{t_k}B)) \leq p(t)\mu(B);\ \ t\in J_k,\ k=0,\ldots,m,$$

where ${}^C_qD^r_{t_k}B = \{{}^C_qD^r_{t_k}w : w \in B\}$.

(H_4) There exists a constant $L > 0$ such that

$$|L_k(u)| \leq L(1 + |u|), \ \ for\ each\ u \in E.$$

(H_5) There exists a constant $l > 0$ such that for each bounded set $B \subset E$ and for each $t \in J_k,\ k = 0, \ldots, m$, we have

$$\mu(L_k(B)) \leq l\mu(B).$$

Theorem 4.1 *Assume that the hypotheses* $(H_1) - (H_5)$ *hold. If*

$$\rho := mL + \frac{2p^* T^r}{\Gamma_q(1+r)} < 1, \tag{1}$$

then the problem (1) has at least one solution defined on I.

Proof Consider the operator $N : PC \to PC$ defined in (1). Let $R > 0$, such that

$$R \geq \frac{|u_0| + mL + \frac{2T^r p^*}{(1-p^*)\Gamma_q(1+r)}}{1 - mL - \frac{2T^r p^*}{(1-p^*)\Gamma_q(1+r)}}.$$

Let $u \in PC$ and $t \in J_0$. Then,

$$\|((Nu)(t)\| = \left\| u_0 + \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right\|,$$

where $g(\cdot) \in C(I)$ with

$$g(t) = f(t, u_0 + ({}_qI_0^r g)(t), g(t)).$$

Thus,

$$\begin{aligned}\|(Nu)(t)\| &\leq \|u_0\| + \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)} \|g(s)\| d_q s \\ &\leq \|u_0\| + \frac{T^r p^*(1+R)}{(1-p^*)\Gamma_q(1+r)}.\end{aligned}$$

On the other hand, if $u \in PC$ and $t \in J_k,\ k = 1, \ldots, m$, we have

$$\begin{aligned}\|(Nu)(t)\| \leq \|u_0\| &+ \sum_{i=1}^{k} \|L_i(u(t_i^-))\| \\ &+ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(r-1)}}{\Gamma_q(r)} \|g(s)\| d_q s\end{aligned}$$

$$+\int_{t_k}^{t}\frac{(t-qs)^{(r-1)}}{\Gamma_q(r)}\|g(s)\|d_qs$$
$$\leq \|u_0\|+mL+\frac{2T^r p^*(1+R)}{(1-p^*)\Gamma_q(1+r)}.$$

Hence, for any $u \in PC$ and each $t \in J$, we get

$$\|N(u)\|_{PC} \leq \|u_0\|+mL+\frac{2T^r p^*(1+R)}{(1-p^*)\Gamma_q(1+r)} \leq R.$$

This proves that N transforms the ball $B_R := B(0,R) = \{w \in \|w\|_{PC} \leq R\}$ into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.19. The proof will be given in three steps.

Step 1. *$N : B_R \to B_R$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in J_0$, we have

$$\|(Nu_n)(t)-(Nu)(t)\| \leq \int_0^t \frac{(t-qs)^{(r-1)}}{\Gamma_q(r)}\|g_n(s)-g(s)\|d_qs, \tag{2}$$

where $g, g_n \in C(J_0)$ with

$$g(t) = f(t, u_0 + ({}_qI_0^r g)(t), g(t)),$$

and

$$g_n(t) = f(t, u_0 + ({}_qI_0^r g_n)(t), g_n(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, then by the Lebesgue dominated convergence theorem, (2) implies

$$\|N(u_n)-N(u)\|_{PC} \to 0 \quad \text{as } n \to \infty.$$

Also, for each $t \in J_k,\ k = 1,\ldots,m$, we have

$$\|(Nu_n)(t)-(Nu)(t)\| \leq \sum_{i=1}^{k}\|L_i(u_n(t_i^-))-L_i(u(t_i^-))\|$$
$$+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(r-1)}}{\Gamma_q(r)}\|g_n(s)-g(s)\|d_qs$$
$$+\int_{t_k}^{t}\frac{(t-qs)^{(r-1)}}{\Gamma_q(r)}\|g_n(s)-g(s)\|d_qs. \tag{3}$$

Again, by the Lebesgue dominated convergence theorem, (3) implies the continuity of our operator N.

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded. Next, let $x_1, x_2 \in J_0$ such that $0 \leq x_1 < x_2 \leq t_1$ and let $u \in B_R$. Then,

$$\begin{aligned}&\|(Nu)(x_2) - (Nu)(x_1)\| \\ &\quad \leq \left\| \int_0^{x_2} \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s - \int_0^{x_1} \frac{(x_1 - qs)^{(r-1)}}{\Gamma_q(r)} g(s) d_q s \right\|,\end{aligned}$$

where $g \in C(J_0)$ with

$$g(t) = f(t, u_0 + ({}_q I_0^r g)(t), g(t)).$$

Thus,

$$\begin{aligned}&\|(Nu)(x_2) - (Nu)(x_1)\| \\ &\quad \leq \int_{x_1}^{x_2} \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} \|g(s)\| d_q s \\ &\qquad + \int_0^{x_1} \frac{|(x_2 - qs)^{(r-1)} - (x_1 - qs)^{(r-1)}|}{\Gamma_q(r)} \|g(s)\| d_q s \\ &\quad \leq \frac{p^*(1+R)(x_2 - x_1)^r}{(1-p^*)\Gamma_q(1+r)} \\ &\qquad + \frac{p^*(1+R)}{1-p^*} \int_0^{x_1} \frac{|(x_2 - qs)^{(r-1)} - (x_1 - qs)^{(r-1)}|}{\Gamma_q(r)} d_q s.\end{aligned}$$

As $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. Also, if we let $x_1, x_2 \in J_k,\ k = 1, \ldots, m$, such that $t_k \leq x_1 < x_2 \leq t_{k+1}$ and let $u \in B_R$, we obtain

$$\begin{aligned}&\|(Nu)(x_2) - (Nu)(x_1)\| \\ &\quad \leq \frac{2p^*(1+R)}{(1-p^*)\Gamma_q(1+r)} |x_2 - x_1|^r \\ &\qquad + \frac{2p^*(1+R)}{1-p^*} \int_0^{x_1} \left| \frac{(x_2 - qs)^{(r-1)}}{\Gamma_q(r)} - \frac{(x_1 - qs)^{(r-1)}}{\Gamma_q(r)} \right| d_q s.\end{aligned}$$

Again, as $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. Hence, $N(B_R)$ is bounded and equicontinuous.

Step 3. *The implication (4) holds.*
Now let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$, V is bounded and equicontinuous, and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on J. By (H_3) and the properties of the measure μ, for each $t \in J_0$, we have

$$
\begin{aligned}
v(t) &\leq \mu((NV)(t) \cup \{0\}) \\
&\leq \mu((NV)(t)) \\
&\leq \int_0^t \frac{(t-qs)^{(r-1)}p(s)}{\Gamma_q(r)} v(s) d_q s \\
&\leq \int_0^t \frac{(t-qs)^{(r-1)}p(s)}{\Gamma_q(r)} \mu(V(s)) d_q s \\
&\leq \frac{p^* T^r}{\Gamma_q(1+r)} \|v\|_{PC}.
\end{aligned}
$$

Thus,

$$
\|v\|_{PC} \leq \rho \|v\|_{PC}.
$$

Also, for each $t \in J_k,\ k = 1, \ldots, m$, we get

$$
\begin{aligned}
v(t) &\leq \mu((NV)(t) \cup \{0\}) \\
&\leq \mu((NV)(t)) \\
&\leq \sum_{i=1}^{k} l^* \mu(V(s)) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(r-1)}p(s)}{\Gamma_q(r)} \mu(V(s)) d_q s \\
&\quad + \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}p(s)}{\Gamma_q(r)} \mu(V(s)) d_q s \\
&\leq L \sum_{i=1}^{k} v(t) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-qs)^{(r-1)}p(s)}{\Gamma_q(r)} v(s) d_q s \\
&\quad + \int_{t_k}^{t} \frac{(t-qs)^{(r-1)}p(s)}{\Gamma_q(r)} v(s) d_q s \\
&\leq \left(mL + \frac{2p^* T^r}{\Gamma_q(1+r)} \right) \|v\|_{PC}.
\end{aligned}
$$

Hence,

$$
\|v\|_{PC} \leq \rho \|v\|_{PC}.
$$

From (1), we get $\|v\|_{PC} = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in PC. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 2.19, we conclude that N has a fixed point that is a solution of the problem (1).

5 Examples

Example 1 Consider the problem of implicit impulsive q-fractional differential equation of the form

$$\begin{cases} ({}^C_q D^r_{t_k} u)(t) = f(t, u(t), ({}^C_q D^r_{t_k} u)(t)); \ t \in J_k, \ k = 0, \dots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \dots, m, \\ u(0) = 0, \end{cases} \tag{1}$$

where $I = [0, 1], \ r \in (0, 1]$,

$$f(t, u(t), ({}^C_q D^r_{t_k} u)(t)) = \frac{\Gamma_q(1+r)t^2}{1 + |u(t)| + |{}^C_q D^r_{t_k} u(t)|} \times \left(e^{-7} + \frac{1}{e^{t+5}}\right)(2^{-n} + u_n(t)); \ t \in [0, 1],$$

$$L_k(u(t_k^-)) = \frac{1}{(3e^{45})(1 + |u(t_k^-)|)}; \ k = 1, \dots, m.$$

Clearly, the function f is continuous.

For each $t \in [0, 1]$, we have

$$|f(t, u(t), ({}^C q D^r_{t_k} u)(t))| \le \Gamma_q(1+r)t^2 \left(e^{-7} + \frac{1}{e^{t+5}}\right),$$

and

$$|L_k(u)| \le \frac{1}{3e^5}.$$

Hence, the hypothesis (H_{02}) is satisfied with $\ell^* = 2e^{-5}\Gamma_q(1+r)$, and (H_4) is satisfied with $l = \frac{1}{3e^4}$.

Simple computations show that all conditions of Theorem 3.3 are satisfied. It follows that the problem (1) has at least one solution on [0, 1].

Example 2 Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the problem of implicit impulsive q-fractional differential equation of the form

$$\begin{cases} ({}^C_q D^r_{t_k} u)(t) = f(t, u(t), ({}^C_q D^r_{t_k} u)(t)); \ t \in J_k, \ k = 0, \dots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); \ k = 1, \dots, m, \\ u(0) = 0, \end{cases} \tag{2}$$

where $I = [0, 1], \ r \in (0, 1], \ u = (u_1, u_2, \dots, u_n, \dots),$

$$f = (f_1, f_2, \dots, f_n, \dots),$$

$${}^C_q D^r_{t_k} u = ({}^C_q D^r_{t_k} u_1, {}^C_q D^r_{t_k} u_2, \dots, {}^C_q D^r_{t_k} u_n, \dots); \ k = 0, \dots, m,$$

$$f_n(t, u(t), ({}^C_q D^r_{t_k} u)(t)) = \frac{\Gamma_q(1+r)t^2}{1 + \|u(t)\|_E + \|{}^C_q D^r_{t_k} u(t)\|_E} \times \left(e^{-7} + \frac{1}{e^{t+5}} \right) (2^{-n} + u_n(t)); \ t \in [0, 1],$$

$$L_k(u(t_k^-)) = \frac{1}{(3e^{45})(1 + \|u(t_k^-)\|_E)}; \ k = 1, \dots, m.$$

For each $u \in E$ and $t \in [0, 1]$, we have

$$\|f(t, u(t), ({}^C q D^r_{t_k} u)(t))\|_E \le \Gamma_q(1+r)t^2 \left(e^{-7} + \frac{1}{e^{t+5}} \right),$$

and

$$\|L_k(u)\|_E \le \frac{1}{3e^5}.$$

Hence, the hypothesis (H_2) is satisfied with $p^* = 2e^{-5}\Gamma_q(1+r)$, and (H_4) is satisfied with $L = \frac{1}{3e^4}$.

We shall show that condition (1) holds with $T = 1$. Indeed, if we assume, for instance, that the number of impulses $m = 3$, and $r = \frac{1}{2}$, then we have

$$L := mL + \frac{2p^*T^r}{\Gamma_q(1+r)} = e^{-5} + \frac{2e^{-5}\Gamma_q(1+r)}{\Gamma_q(\frac{3}{2})} = 3e^{-5} < 1.$$

Simple computations show that all conditions of Theorem 4.1 are satisfied. It follows that the problem (2) has at least one solution on [0, 1].

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Null Controllability of a Degenerate Cascade Model in Population Dynamics

Younes Echarroudi and Lahcen Maniar

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1 Introduction

We consider the following coupled population cascade system:

$$
\begin{aligned}
\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k_1(x)y_x)_x + \mu_1(t,a,x)y &= \vartheta\chi_\omega \quad \text{in} \quad Q, \\
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k_2(x)p_x)_x + \mu_2(t,a,x)p + \mu_3(t,a,x)y &= 0 \quad \text{in} \quad Q, \\
y(t,a,1) = y(t,a,0) = p(t,a,1) = p(t,a,0) &= 0 \quad \text{on} \quad (0,T)\times(0,A), \\
y(0,a,x) = y_0(a,x);\ p(0,a,x) &= p_0(a,x) \quad \text{in} \quad Q_A, \\
y(t,0,x) &= \int_0^A \beta_1(t,a,x)y(t,a,x)da \quad \text{in} \quad Q_T, \\
p(t,0,x) &= \int_0^A \beta_2(t,a,x)p(t,a,x)da \quad \text{in} \quad Q_T,
\end{aligned}
\tag{1.1}
$$

where $Q = (0,T)\times(0,A)\times(0,1)$, $Q_A = (0,A)\times(0,1)$, $Q_T = (0,T)\times(0,1)$, $\omega \subset\subset (0,1)$, and we will denote $q = (0,T)\times(0,A)\times\omega$. The quantities $y(t,a,x)$ and $p(t,a,x)$ that are in interaction are the densities of populations of time t, age a, and gene type x. Recall that the system (1.1) above models the dispersion

Y. Echarroudi
Laboratory of Mathematics and Population Dynamics, Faculty of Sciences Semlalia Marrakesh, Cadi Ayyad university, Marrakesh, Morroco

L. Maniar (✉)
Cadi Ayyad University Morroco, Faculty of Sciences Semlalia Marrakesh, Marrakesh, Morroco
e-mail: maniar@uca.ma

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of gene in the two populations. The parameters $\beta_1(t,a,x)$ and $\beta_2(t,a,x)$ can be interpreted as the natural fertility rates, while $\mu_1(t,a,x)$ and $\mu_2(t,a,x)$ are the natural mortality rates and μ_3 stands for an interaction parameter. On the other hand, the parameters k_1 and k_2 are the coefficients of dispersion and depend on the gene-type variable x; ϑ and ω are, respectively, the control that we are looking for and the region of gene type where it acts. Such a control can be viewed in our situation as the capture strategy and corresponds in general to an external supply or to removal of individuals on the subdomain $\omega \subset\subset (0,1)$. Besides, y_0 and p_0 are the initial distributions of the two populations, whereas $\int_0^A \beta_1(t,a,x)y(t,a,x)da$ and $\int_0^A \beta_2(t,a,x)p(t,a,x)da$ are the distributions of the newborns. Finally, the two positive fixed constants T and A are, respectively, the time of control and the maximal age of expectancy that we suppose here is the same of both populations. A suitable and powerful condition will be required later on T.

The population dynamics models in their different aspects attracted many authors and were investigated from many sides (see, for example, [4, 10, 24, 26, 27, 30, 32–34, 36]). Among those questions, we find the null controllability or in general the controllability problems for age- and space-structured population dynamics models that were studied in an intensive literature like [1, 2, 5, 7]. In [1, 2], the author tried to prove both the exact and approximate controllabilities for a population dynamics model where the coefficient is a positive constant. More precisely, to prove the first type of controllability, Ainseba used the mean of observability inequality that is a consequence of Carleman estimates based on the computations done on [31] for non-degenerate heat equation. The second result of the same paper is reached by using an argument of density of the reachable set of states at time T on L^2-space for an age class $(0, a_1)$ where $a_1 < A$. Notice that the exact controllability is equivalent to the null controllability of a linear model. Based on this rule, using again the Carleman-type inequalities and with the help of the characteristics method, the workers in [5] proved under the assumptions of that the L^∞-norm of the initial data is small and the fact that the coefficient of dispersion is a positive function for all points of space domain that their population model is exact controllable. Earlier in [7], a result similar to the one in [1, 2] was shown but without calling Carleman estimate. In fact, the method used here is a combination between a contradiction process and the so-called Mizohata uniqueness theorem (see the reference for further details).

Nevertheless, the previous works were established with either a space independent or a non-degenerate dispersion coefficient contrary to our paper and the works realized in [6, 22] whose calculus are based on the papers investigating the degenerate heat equation (see for instance [16–21, 25]). In this context, [6, 22] were the first to be concerned with such a problem; each of them used a different technique and also imposed different conditions on time control T. Indeed, in [6], the authors allowed the dispersion coefficient to depend on the variable x and verify $k(0) = 0$, i.e., the coefficient of dispersion k degenerates at 0, and they tried to obtain the null controllability in such a situation with $\beta \in L^\infty(Q)$ following [9] via a new Carleman estimate for a suitable full adjoint system and afterward his observability inequality. However, the main controllability result of [6] was shown under the condition $T \geq A$ as in [11], and this constitutes a restrictiveness on the "optimality" of the control time T since it means, for example, that for a pest population whose maximal age A may equal to many days (may be many months or years), we need much time to bring the population to the zero equilibrium. In the same vocation and to overcome the condition $T \geq A$, L. Maniar et al. in [22] suggested the fixed-point technique implemented in [37] that requires that the fertility rate must belong to $C^2(Q)$ and consists briefly to demonstrate in a first time the null controllability for an intermediate system with a fertility function $f \in L^2(Q_T)$ instead of $\int_0^A \beta(t,a,x)y(t,a,x)da$ and to achieve the task via the Leray–Schauder theorem.

On the other hand, a huge amount of works are interested on the control problems of coupled systems among which we find [3, 8, 39] and the references therein. In [3], a coupled model is taken under a reaction–diffusion system describing interaction between prey and predator populations. The goal is to look for a suitable control supported on small spatial subdomain that guarantees the stabilization of the predator population to zero. The objective of [39] was different. Actually, an age-dependent pre–predator system was considered, and the authors proved the existence and uniqueness for an optimal control (also called "optimal effort") that gives the maximal harvest via the study of the optimal harvesting problem associated to their coupled model. Similarly to the case of one equation in the papers [1, 2, 5, 7], [3, 39] assumed that their coefficients of diffusion are constants. We open parentheses here to emphasize that the references [3, 39] are cited as examples of non-degenerate coupled systems, and this does not mean that (1.1) models a prey–predator model. The last two papers motivate Ait Ben Hassi et al. in [8] to generalize these works specially [3] and examined semilinear parabolic cascade systems with two different diffusion coefficients allowed to depend on the space variable and degenerate at the left boundary of the space domain. Moreover, the purpose of that paper was to bring out the null controllability via a Carleman-type inequality of the adjoint problem of the associated linearized system using the results of [9] or [15] and with the help the Schauder fixed-point theorem.

But up to now and to our best knowledge, little is known about the global null controllability question of the age-structured population dynamics cascade systems in both degenerate and non-degenerate cases, and the only item that deals with such a paradigm is the one of Boutaayamou et al. in [13]. In fact, the authors assessed (1.1) in a non-divergence form and proved its null controllability like the one in (1.4) using the classical procedure based on the observability inequality deduced from the weighted estimates of Carleman kind. To reach their purpose, Fragnelli et al. require some smooth regularity on the fertility, mortality, and interaction rates. More accurately, they assumed that such rates are continuous besides the fact that both the natural rates of fertility satisfy the following hypothesis:

$$\beta_i(a,x) = 0, \qquad \text{for all } (a,x) \in [0,\overline{a}_i] \times [0,1], \quad i = 1; 2, \tag{1.2}$$

where $\overline{a}_i, \quad i = 1; 2$, are positive constants.

As mentioned in the recent paper [13], the assumption (1.2) is natural from the biological point of view, but mathematically some interesting computations will be omitted and our work will avoid such a kind of impositions. Hence, we will address to such a control problem described by (1.1) using a different trend from the one of [13], and it will be a generalization of the results obtained in [6] and [22]. More precisely, following the global techniques of [8], we expect in this contribution to prove the global null controllability of the structured age and space system (1.1) with one control force and when

$$T \in (0, \delta), \tag{1.3}$$

where $\delta \in (0, A)$ fixed small enough. That is, we show that for all $y_0, p_0 \in L^2(Q_A)$, there exists a control $\vartheta \in L^2(Q)$ such that the associated solution of (1.1) verifies

$$\begin{cases} y(T, a, x) = 0, & a.e \text{ in } (\delta, A) \times (0, 1), \\ p(T, a, x) = 0, & a.e \text{ in } (\delta, A) \times (0, 1). \end{cases} \tag{1.4}$$

In all domains related to the population dynamics, the conditions like (1.4) are equivalent to say that we look for a control or strategy (suitable one) to steer the studied population(s) from its (their) initial distribution(s) to extinction in a finite control time T. Such a property is needed to deal with the pest populations not against the non-harmful ones or, for example, in the context of human population. It deserves to mention that the researched control ϑ depends on δ and the two initial distributions y_0 and p_0. Returning back to the condition (1.3) imposed on the fixed time of control T. This assumption is required not only for a technical cause but also is meaningful in the cost of controllability in the sense that is we will be able to drive a very wide age class of both populations to extinct fastly and quickly instead to wait for months or years like in [6] (see also [12, 23] for a similar explanation), and this will be an advantage on the optimality of the control ϑ. Note that in [13], the goal (1.4) is established for any positive time control T, and in our point of view, this implies that if T can verify (1.3), it could also be greater than age A, and this, as explained before (and also in [13]), can involve a restrictiveness on the optimality for our control ϑ. By the way, the null controllability property (1.4) does not allow to control the age class of non-fertile individuals of both populations, and this can be justified in the mathematical standpoint (see the farther proofs).

Theoretically, the result (1.4) is gotten under the conditions that all natural rates possess an L^∞-regularity (more general than the continuous one as assumed in [13]), and this will avoid us the use of the fixed-point technique needed in [22, 37], which imposed the $\mathcal{C}^2$-regularity of fertility rate β. Another striking difference with the cited references (except [13]) is that our model is a coupled dynamics system combining in the same time age and space structures and likewise the degeneracy occurring for the two different dispersion coefficients k_1 and k_2 in the left-hand side of the gene-type domain, that is $k_i(0) = 0; \quad i = 1, 2$, e.g., $k_i(x) = x^\alpha$, where α can be taken in $[0, 1)$ if we impose the Dirichlet boundary conditions or in $[1, 2)$ if we consider the Neumann boundary conditions (see the assumptions (2.5) beneath). In this case, we say that (1.1) is a degenerate population dynamics cascade system. Genetically speaking, such a property is natural since it means that if each population is not of a gene type, it cannot be transmitted to its offspring. Finally, we highlight that this work can be generalized in the case of interior degeneracy, i.e., $k_1(x_0) = 0$ and $k_2(z_0) = 0$ (e.g., $k_1(x) = |x - x_0|^\alpha$ and $k_2(x) = |x - z_0|^\alpha, \quad \alpha \in (0, 1)$), where $x_0, z_0 \in (0, 1)$ using the results proved in [12] that are based essentially on the method applied for controllability problem of interior degenerate parabolic equations [28] and can also be extended to the non-smooth case in the light of the item [29].

The remainder of this chapter is organized as follows: Sect. 2 is devoted to discussion about the well-posedness of (1.1) and establishing a new Carleman estimate of an intermediate adjoint system that helps us to provide an evidence of the main Carleman-type inequality of the associated full adjoint system. As an outcome of this latter, in Sect. 3, an observability inequality is proved with the help of the semigroups theory that allows us to obtain non-classical implicit formulas of the adjoint system solution (see [12, 23] for a similar procedure). The obtained observability inequality will play a crucial role to show the main controllability result stated in (1.4). We close this chapter in Sect. 4 that takes the form of an appendix wherein the proofs of some basic tools are provided.

2 Well-Posedness and Carleman Estimates

2.1 Well-Posedness Result

For this section and for the sequel, we assume that the dispersion coefficients $k_i, \quad i = 1, 2$, satisfy the hypotheses

$$\begin{cases} k_i \in C([0, 1]) \cap C^1((0, 1]), \quad k_i > 0 \text{ in } (0, 1] \text{ and } k_i(0) = 0, \\ \exists \gamma \in [0, 1) : x k_i'(x) \leq \gamma k_i(x), \ \ x \in (0, 1]. \end{cases} \tag{2.5}$$

The last condition on k_i means in the case of $k_i(x) = x^{\alpha_i}$ that $0 \leq \alpha_i < 1$. Similarly, all results of this chapter can also be obtained in the case of $1 \leq \alpha_i < 2$ taking, instead of Dirichlet condition on $x = 0$, the Neumann condition $(k_i(x)u_x)_x(0) = 0$. On the other hand, we assume that the rates $\mu_1, \mu_2, \mu_3, \beta_1$ and β_2 verify

$$\begin{cases} \mu_1, \mu_2, \mu_3, \beta_1, \beta_2 \in L^\infty(\overline{Q}), \\ \mu_1, \mu_2, \beta_1, \beta_2 \geq 0 \text{ and } \mu_3 > 0 \text{ a.e in } Q, \end{cases} \tag{2.6}$$

Here, we open parentheses to say that contrary to some references like [1, 2, 6], we do not require the mortality rates to satisfy $\int_0^A \mu_i(t - s, A - s, x)ds = +\infty, \quad i = 1, 2$, since these conditions do not play any role on the well-posedness result or the null controllability computations.

In summary, to justify that our model (1.1) is well-posed, we will rewrite it under an abstract Cauchy problem, and then we will combine some references namely [9, 14, 15, 23, 35, 38] to get our result. This result needs the introduction of a pertinent framework represented by weighted Sobolev spaces defined for $i = 1, 2$ by

$$\begin{cases} H^1_{k_i}(0,1) = \{u \in L^2(0,1) : u \text{ is abs. cont. in } [0,1] : \sqrt{k_i}u_x \in L^2(0,1), u(1) = u(0) = 0\}, \\ H^2_{k_i}(0,1) = \{u \in H^1_{k_i}(0,1) : k_i u_x \in H^1(0,1)\}, \end{cases} \tag{2.7}$$

endowed, respectively, with the norms,

$$\begin{cases} \|u\|^2_{H^1_{k_i}(0,1)} = \|u\|^2_{L^2(0,1)} + \|\sqrt{k_i}u_x\|^2_{L^2(0,1)}, & u \in H^1_{k_i}(0,1), \\ \|u\|^2_{H^2_{k_i}(0,1)} = \|u\|^2_{H^1_{k_i}(0,1)} + \|(k_i u_x)_x\|^2_{L^2(0,1)}, & u \in H^2_{k_i}(0,1), \end{cases}$$

with $i = 1, 2$ (see [9, 14, 15] or the references therein for the properties of such spaces).

Now, put for $i = 1, 2$, $\quad \mathrm{A}_i\theta = (k_i(x)\theta_x)_x$, with k_i, verify (2.5).

The domains of the operators A_i, $\quad i = 1, 2$, are exactly $H^2_{k_i}(0,1)$, $\quad i = 1, 2$, given in (2.7), and it is well known that such operators are closed, self-adjoint, and negative with dense domains in $L^2(0,1)$, which implies that they generate $\mathcal{C}_0$-semigroups in $L^2(0,1)$ (see [9, 14, 15] for precise proofs).

On the other hand, consider the following operators $\mathcal{A}_i$, $\quad i = 1, 2$, defined by

$$\begin{cases} \mathcal{A}_i\theta = -\dfrac{\partial\theta}{\partial a} + \mathrm{A}_i\theta, \quad \forall\theta \in D(\mathcal{A}_i), \\ D(\mathcal{A}_i) = \{u \in L^2(0,A; D(\mathrm{A}_i)); \dfrac{\partial u}{\partial a} \in L^2(0,A; H^1_{k_i}(0,1)); u(0,x) = \displaystyle\int_0^A \beta_i(a,x)u(a,x)da\}. \end{cases} \tag{2.8}$$

From [38, Theorem 4, page 23] or [38, Theorem 5, page 26] and since $(\mathrm{A}_i, D(\mathrm{A}_i))$, $\quad i = 1, 2$, are infinitesimal generators of $\mathcal{C}_0$-semigroups as mentioned before, one can conclude that $(\mathcal{A}_i, D(\mathcal{A}_i))$, $\quad i = 1, 2$, generate $\mathcal{C}_0$-semigroups in $L^2(Q_A)$. In this context, we advise the reader to take a glance for a similar discussion of the well-posedness result [23, Theorem 2.1].

Adapting these notations, the abstract Cauchy problem associated to (1.1) is formulated as

$$\begin{cases} X'(t) = (\mathbb{A} + B(t))X(t) + f(t), \\ X(0) = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}, \end{cases} \tag{2.9}$$

where $X(t) = \begin{pmatrix} y(t) \\ p(t) \end{pmatrix}$, $\mathbb{A} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}$, $D(\mathbb{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_2)$, $B(t) = \begin{pmatrix} M_{\mu_1} & 0 \\ M_{\mu_3} & M_{\mu_2} \end{pmatrix}$,

$f(t) = \begin{pmatrix} \vartheta \chi_\omega \\ 0 \end{pmatrix}$; with the generators $\mathcal{A}_i$, $\quad i = 1, 2$, are defined by (2.8),
$M_{\mu_i} w = -\mu_i w, \quad i = 1, 2, 3.$

As we can see, the operator $(\mathbb{A}, D(\mathbb{A}))$ is a diagonal matrix of generators of $\mathcal{C}_0$-semigroups; as a consequence, $(\mathbb{A}, D(\mathbb{A}))$ is also a generator of a $\mathcal{C}_0$-semigroup in $L^2(Q)$. On the other hand, the operator $B(t)$ can be viewed as a bounded perturbation of $\mathbb{A}$, so that one has $D(\mathbb{A} + B(t)) = D(\mathbb{A})$.

Gathering all these gadgets with the result [35, Theorem 2.1], we somehow justify our theorem of well-posedness.

Theorem 2.1 *The following points hold:*

1. *The operator* $(\mathbb{A} + B(t), D(\mathbb{A}))$ *generates a* $\mathcal{C}_0$*-semigroup in* $L^2(Q)$.
2. *Under the assumptions* (2.5) *and* (2.6) *and for all* $\vartheta \in L^2(Q)$ *and* $(y_0, p_0) \in D(\mathcal{A}_1) \times D(\mathcal{A}_2)$*, the system* (2.9) *admits a unique mild solution* X *belonging to* $C([0, T]; D(\mathcal{A}_1) \times D(\mathcal{A}_2))$ *and verifies the integral equation*

$$\forall t \in [0, T], \quad X(t) = e^{(\mathbb{A}+B(t))t} X_0 + \int_0^t e^{(\mathbb{A}+B(t))(t-s)} f(s) ds. \quad (2.10)$$

Before continuing, we shall make the following remark:

Remark 2.2 Since $D(\mathcal{A}_i)$, $\quad i = 1, 2$, are dense in $L^2(Q_A)$, then Theorem 2.1 can be extended to the space $L^2(Q_A)$ for the initial data (y_0, p_0) as well as our null controllability result (1.4).

2.2 Carleman Inequality Results

In this paragraph, we will focus on the so-called Carleman estimates. Generally speaking, Carleman estimate is a priori estimates for the solutions of the adjoint systems and their derivatives. The first result of this section concerns the adjoint system of the system (1.1). Classically, the adjoint system is derived by multiplying the governing equations of the direct problem by Lagrange multipliers, which means that the adjoint state is the Lagrange multiplier for the studied PDE. To obtain this model, we afterward integrate over the domains of the existing variables (herein, the time, the gene type, and the age variables). Note that it is not necessary to multiply the boundary and initial conditions of the direct problem by Lagrange multipliers because they become identically null.

In our case, the associated adjoint model of (1.1) is stated in the following proposition:

Proposition 2.3 *The adjoint system of* (1.1) *is given by*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1(x) u_x)_x - \mu_1(t, a, x) u - \mu_3 v = -\beta_1 u(t, 0, x) \quad in \quad Q, \quad (2.11)$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2(x)v_x)_x - \mu_2(t,a,x)v = -\beta_2 v(t,0,x) \quad & in \quad Q, \\
u(t,a,1) = u(t,a,0) = v(t,a,1) = v(t,a,0) = 0 \quad & on \quad (0,T)\times(0,A), \\
u(T,a,x) = u_T(a,x);\ v(T,a,x) = v_T(a,x) \quad & in \quad Q_A, \\
u(t,A,x) = v(t,A,x) = 0 \quad & in \quad Q_T,
\end{aligned}
$$

where u and v stand, respectively, for the adjoint variables of y and p.

Proof First, we define the Lagrangian L related to (1.1) by

$$
\begin{aligned}
L(y,p,u,v,\vartheta,u_0,v_0) = J(y,p,\vartheta) &+ \int_Q u\bigg(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k_1(x)y_x)_x \\
&+ \mu_1(t,a,x)y - \vartheta\chi_\omega\bigg)dtdadx \qquad (2.12)\\
&+ \int_Q v\bigg(\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k_2(x)p_x)_x \\
&+ \mu_2(t,a,x)p + \mu_3(t,a,x)y\bigg)dtdadx \\
&+ \int_{Q_A} u_0(y(0) - y_0)dadx + \int_{Q_A} v_0(p(0) - p_0)dadx,
\end{aligned}
$$

where the functional J is given by $J(y,p,\vartheta) = \frac{1}{2}\int_0^1\int_\delta^A (y^2(T,a,x) + p^2(T,a,x))dadx + \frac{1}{2}\int_Q \vartheta^2\chi_\omega dtdadx$.

Now, put

$$
I_1 = \int_Q u\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k_1(x)y_x)_x + \mu_1(t,a,x)y\right)dtdadx
$$

and

$$
I_2 = \int_Q v\left(\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k_2(x)p_x)_x + \mu_2(t,a,x)p + \mu_3(t,a,x)y\right)dtdadx.
$$

With the aid of the integration by parts technique, taking into account the first newborns equation of (1.1) and assuming that

$$
\begin{cases}
u(t,A,x) = v(t,A,x) = 0, & \text{in } (0,T)\times(0,1), \\
u(t,a,0) = u(t,a,1) = v(t,a,0) = v(t,a,1) = 0, & \text{on } (0,T)\times(0,A),
\end{cases}
\qquad (2.13)
$$

we obtain

$$I_1 = \int_{Q_A} u(T,a,x)y(T,a,x)dadx - \int_{Q_A} u(0,a,x)y(0,a,x)dadx$$
$$-\int_Q y\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u + \beta_1 u(t,0,x)\right)dtdadx \quad (2.14)$$

and

$$I_2 = \int_{Q_A} v(T,a,x)p(T,a,x)dadx - \int_{Q_A} v(0,a,x)p(0,a,x)dadx$$
$$-\int_Q p\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v + \beta_2 v(t,0,x)\right)dtdadx$$
$$+\int_Q \mu_3 v y dtdadx. \quad (2.15)$$

Combining (2.13) with (2.14) and (2.15), we get the following formula of L:

$$\begin{aligned}
L(y,p,u,v,\vartheta,u_0,v_0) = {} & \frac{1}{2}\int_{Q_A}(y^2(T,a,x)+p^2(T,a,x))dadx \\
& +\frac{1}{2}\int_Q \vartheta^2\chi_\omega dtdadx - \int_Q \vartheta u \chi_\omega dtdadx \\
& +\int_{Q_A}[u_0(y(0)-y_0)+v_0(p(0)-p_0)]dadx \\
& +\int_{Q_A} u(T,a,x)y(T,a,x)dadx \\
& -\int_{Q_A} u(0,a,x)y(0,a,x)dadx \\
& -\int_Q y\bigg(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u \\
& \qquad +\beta_1 u(t,0,x) - \mu_3 v\bigg)dtdadx \\
& +\int_{Q_A} v(T,a,x)p(T,a,x)dadx \\
& -\int_{Q_A} v(0,a,x)p(0,a,x)dadx \\
& -\int_Q p\bigg(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v \\
& \qquad +\beta_2 v(t,0,x)\bigg)dtdadx.
\end{aligned}$$

The above expression of L can be rewritten as follows:

$$
\begin{aligned}
L(y,p,u,v,\vartheta,u_0,v_0) = & \int_{Q_A}\left(\frac{1}{2}y^2(T,a,x)\chi_{(\delta,A)} + u(T,a,x)y(T,a,x)\right)dadx \\
& + \int_{Q_A}\left(\frac{1}{2}p^2(T,a,x)\chi_{(\delta,A)}\right. \\
& \qquad \left. + v(T,a,x)p(T,a,x)\right)dadx \\
& + \int_{Q}\left(\frac{1}{2}\vartheta^2\chi_\omega - \vartheta u\chi_\omega\right)dtdadx \\
& + \int_{Q_A} y(0)(u_0 - u(0,a,x))dadx - \int_{Q_A} u_0 y_0 dadx \\
& + \int_{Q_A} p(0)(v_0 - v(0,a,x))dadx - \int_{Q_A} v_0 p_0 dadx \\
& - \int_{Q} y\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u\right. \\
& \qquad \left. + \beta_1 u(t,0,x) - \mu_3 v\right)dtdadx \\
& - \int_{Q} p\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v\right. \\
& \qquad \left. + \beta_2 v(t,0,x)\right)dtdadx.
\end{aligned}
$$

Thus, for any $h \in L^2(Q)$, one has

$$dL.h = \frac{\partial L}{\partial y}.h + \frac{\partial L}{\partial p}.h + \frac{\partial L}{\partial u}.h + \frac{\partial L}{\partial v}.h + \frac{\partial L}{\partial u_0}.h + \frac{\partial L}{\partial v_0}.h + \frac{\partial L}{\partial \vartheta}.h.$$

Keep in the mind that $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial u_0}$ are, respectively, the main equation and the initial condition satisfied by y and $\frac{\partial L}{\partial v}$ and $\frac{\partial L}{\partial v_0}$ are, respectively, the main equation and the initial condition satisfied by p.

Therefore, $dL.h = \frac{\partial L}{\partial y}.h + \frac{\partial L}{\partial p}.h + \frac{\partial L}{\partial \vartheta}.h$.

Now, to reach an optimum of L, one must resolve the equation $dL.h = 0, \quad \forall h \in L^2(Q)$. Generally, in our situation, we will impose a sufficient condition like $\frac{\partial L}{\partial y}.h = \frac{\partial L}{\partial p}.h = \frac{\partial L}{\partial \vartheta}.h = 0$.

Actually, to attempt the formula of the adjoint system (2.11), we just need to have $\frac{\partial L}{\partial y}.h = \frac{\partial L}{\partial p}.h = 0$. The third equation will be used later to express the control ϑ.

On the other hand, recall that $\forall h \in L^2(Q)$, we have

$$\begin{aligned}\frac{\partial L}{\partial y}.h = &\int_{Q_A} (y(T,a,x)\chi_{(\delta,A)} + u(T,a,x))h(T,a,x)dadx \\ &+ \int_{Q_A} (u_0 - u(0,a,x))h(0,a,x)dadx \\ &- \int_Q h\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u + \beta_1 u(t,0,x) - \mu_3 v\right) dtdadx\end{aligned}$$

and

$$\begin{aligned}\frac{\partial L}{\partial p}.h = &\int_{Q_A} (p(T,a,x)\chi_{(\delta,A)} + v(T,a,x))h(T,a,x)dadx \\ &+ \int_{Q_A} (v_0 - v(0,a,x))h(0,a,x)dadx \\ &- \int_Q h\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v + \beta_2 v(t,0,x)\right) dtdadx.\end{aligned}$$

Sufficient conditions that can be applied to get both $\frac{\partial L}{\partial y}.h = \frac{\partial L}{\partial p}.h = 0$ are, respectively,

$$\begin{cases} u(T,a,x) = -y(T,a,x)\chi_{(\delta,A)}, & (0,A)\times(0,1), \\ u(0,a,x) = u_0(a,x), \text{ in } & (0,A)\times(0,1), \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1 u_x)_x - \mu_1 u - \mu_3 v = -\beta_1 u(t,0,x), & \text{in } Q, \end{cases} \tag{2.17}$$

and

$$\begin{cases} v(T,a,x) = -p(T,a,x)\chi_{(\delta,A)}, & \text{in } (0,A)\times(0,1), \\ v(0,a,x) = v_0(a,x), & \text{in } (0,A)\times(0,1), \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2 v_x)_x - \mu_2 v = -\beta_2 v(t,0,x), & \text{in } Q. \end{cases} \tag{2.18}$$

Finally, the thesis follows gathering (2.13), (2.17), and (2.18) with $u_T(a,x) = -y(T,a,x)\chi_{(\delta,A)}(a)$, in $(0,A)\times(0,1)$ and $v_T(a,x) = -p(T,a,x)\chi_{(\delta,A)}(a)$, in $(0,A)\times(0,1)$. □

Traditionally, the proof of the Carleman estimates of the full adjoint system (2.11) is based tightly on the choice of the so-called weight functions. In our case, these functions are set in the following way $\forall (t,a,x) \in Q$

$$\begin{cases} \varphi_i := \Theta(t,a)\psi_i(x), \quad i = 1, 2, \\ \Theta(t,a) := \frac{1}{(t(T-t))^4 a^4}, \\ \psi_i(x) := \lambda_i(\int_0^x \frac{r}{k_i(r)}dr - d_i) \quad i = 1, 2, \\ \phi(t,a,x) := \Theta(t,a)e^{\kappa\sigma(x)}, \\ \Phi(t,a,x) := \Theta(t,a)\Psi(x), \\ \Psi(x) := e^{\kappa\sigma(x)} - e^{2\kappa\|\sigma\|_\infty}, \end{cases} \tag{2.19}$$

where σ is the function given by

$$\begin{cases} \sigma \in C^2([0,1]), \\ \sigma(x) > 0 \quad \text{in } (0,1), \sigma(0) = \sigma(1) = 0, \\ \sigma_x(x) \neq 0 \quad \text{in } [0,1] \setminus \omega_0, \end{cases} \tag{2.20}$$

where $\omega_0 \subset\subset \omega$ an open subset. The existence of σ is proved in [31, Lemma 1.1] using a device of differential geometry. λ_i, d_i and κ are supposed to verify the following assumptions:

$$\begin{cases} d_1 > \frac{1}{k_1(1)(2-\gamma)}, \\ \frac{\lambda_1}{\lambda_2} \geq \frac{d_2}{d_1 - \int_0^1 \frac{r}{k_1(r)}dr}, \\ \kappa \geq \frac{4\ln(2)}{\|\sigma\|_\infty}, \\ d_2 \geq \frac{5}{k_2(1)(2-\gamma)}, \end{cases} \tag{2.21}$$

with $\lambda_2 \in I = [\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{d_2k_2(1)(2-\gamma)-1}, \frac{4(e^{2\kappa\|\sigma\|_\infty}-e^{\kappa\|\sigma\|_\infty})}{3d_2})$, which can be shown non-empty (see the proof of Lemma 4.3 in appendix). On the other hand, in the light of the first and fourth conditions in (2.21) on d_1 and d_2, one can observe that $\psi_i(x) < 0$, $i = 1, 2$, for all $x \in [0,1]$ and $\Theta(t,a) \longrightarrow +\infty$ as $t \longrightarrow 0^+, T^-$ and $a \longrightarrow 0^+$.

The first step to show our full ω-Carleman estimate is to show an intermediate Carleman-type inequality stated in Theorem 2.7 beneath. To this end, one needs two basic propositions concerned with Carleman-type inequalities in both degenerate and non-degenerate cases for one equation model. The first one is:

Proposition 2.4 *Consider the following system with $h \in L^2(Q)$, $\mu \in L^\infty(Q)$, and k verifies* (2.5)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k(x)u_x)_x - \mu(t,a,x)u = h \quad in \quad Q, \tag{2.22}$$

$$u(t,a,1) = u(t,a,0) = 0 \quad on \quad (0,T)\times(0,A),$$

$$u(T,a,x) = u_T(a,x) \quad in \quad Q_A,$$

$$u(t,A,x) = 0 \quad in \quad Q_T.$$

Then, there exist two positive constants C and s_0, such that every solution u of (2.22) satisfies, for all $s \geq s_0$, the following inequality:

$$\int_Q s^3\Theta^3 \frac{x^2}{k(x)} u^2 e^{2s\varphi} dtdadx + \int_Q s\Theta k(x) u_x^2 e^{2s\varphi} dtdadx$$
$$\leq C \left(\int_Q h^2 e^{2s\varphi} dtdadx + sk(1) \int_0^A \int_0^T \Theta u_x(t,a,1)^2 e^{2s\varphi(t,a,1)} dtda \right), \quad (2.23)$$

where φ and Θ are the weight functions defined by

$$\begin{cases} \varphi := \Theta(t,a)\psi(x), \\ \Theta(t,a) := \frac{1}{(t(T-t))^4 a^4}, \\ \psi(x) := c_1(\int_0^x \frac{r}{k(r)} dr - c_2), \end{cases} \quad (2.24)$$

with $c_2 > \frac{1}{k(1)(2-\gamma)}$, $c_1 > 0$, and γ is the parameter defined by (2.5).

For the proof of Proposition 2.4, we refer the reader to [22, Proposition 3.1].

Proposition 2.5 *Let us consider the following system:*

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - \mu(t,a,x)z = h \quad in \quad Q_1, \quad (2.25)$$
$$z(t,a,b_1) = z(t,a,b_2) = 0 \quad on \quad (0,T)\times(0,A),$$

where $Q_1 = (0,T)\times(0,A)\times(b_1,b_2)$, $(b_1,b_2) \subset (0,1)$, $h \in L^2(Q)$, $k \in C^1([0,1])$ is a positive function and $\mu \in L^\infty(Q_1)$. Then, there exist two positive constants C and s_0 such that for any $s \geq s_0$, z the solution of (2.25) verifies the following estimate:

$$\int_{Q_1} (s^3\phi^3 z^2 + s\phi z_x^2) e^{2s\Phi} dtdadx$$
$$\leq C \left(\int_{Q_1} h^2 e^{2s\Phi} dtdadx + \int_\omega \int_0^A \int_0^T s^3\phi^3 z^2 e^{2s\Phi} dtdadx \right), \quad (2.26)$$

where the weight functions ϕ, Θ, and Φ are defined by (2.19) and σ by (2.20).

For the proof of Proposition 2.5, careful computations allow us to adapt the same procedure of [2, Lemma 2.1] to show (2.26) in the case where k is a positive general non-degenerate coefficient, with our weight function Θ given by (2.19) and the source term h.

Besides the last two Propositions 2.4 and 2.5, we must bring out another important result:

Lemma 2.6 *Under the assumptions (2.21), the functions φ_1, φ_2, and Φ stated in (2.19) satisfy the following inequalities:*

$$\begin{cases} \varphi_1 \le \varphi_2, \\ \frac{4}{3}\Phi < \varphi_2 \le \Phi. \end{cases} \tag{2.27}$$

Proof By the definitions of $\varphi_i, \quad i = 1, 2,$ and Φ and taking into account that Θ is positive, showing the results of (2.27) is equivalent to show

$$\begin{cases} \psi_1 \le \psi_2, \\ \frac{4}{3}\Psi < \psi_2 \le \Psi. \end{cases} \tag{2.28}$$

The first inequality in (2.28) is assured by the second assumption in (2.21), while the second one is deduced from $\lambda_2 \in I$ and this achieves the proof. □

Now, we are ready to provide the proof of the following theorem:

Theorem 2.7 *Consider the following system:*

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+(k_1(x)u_x)_x-\mu_1(t,a,x)u-\mu_3(t,a,x)v&=h_1 \quad in \quad Q, \\ \frac{\partial v}{\partial t}+\frac{\partial v}{\partial a}+(k_2(x)v_x)_x-\mu_2(t,a,x)v&=h_2 \quad in \quad Q, \\ u(t,a,1)=u(t,a,0)=v(t,a,1)=v(t,a,0)&=0 \quad on \quad (0,T)\times(0,A), \\ u(T,a,x)=u_T(a,x);\ v(T,a,x)&=v_T(a,x) \quad in \quad Q_A, \\ u(t,A,x)=v(t,A,x)&=0 \quad in \quad Q_T, \end{aligned} \tag{2.29}$$

where h_1 and h_2 are $L^2(Q)$-functions. Assume that the dispersion coefficients $k_i, \quad i = 1, 2,$ satisfy (2.5) *and let $A, T > 0$ be fixed. Then, there exist two positive constants C and s_0 such that every solution (u, v) of* (2.29) *verifies for, all $s \ge s_0$, the following inequality:*

$$\begin{aligned} &\int_Q \left(s^3\Theta^3\frac{x^2}{k_1(x)}u^2+s\Theta k_1(x)u_x^2\right)e^{2s\varphi_1}dtdadx \\ &+\int_Q \left(s^3\Theta^3\frac{x^2}{k_2(x)}v^2+s\Theta k_2(x)v_x^2\right)e^{2s\varphi_2}dtdadx \\ &\le C\left(\int_Q (h_1^2+h_2^2)e^{2s\Phi}dtdadx+\int_q s^3\Theta^3(u^2+v^2)e^{2s\Phi}dtdadx\right), \end{aligned} \tag{2.30}$$

where all the weight functions are defined by (2.19).

Proof Let us introduce the smooth cut-off function $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{cases} 0 \le \xi(x) \le 1, \quad x \in \mathbb{R}, \\ \xi(x) = 1, \quad x \in [0, \frac{2x_1+x_2}{3}], \\ \xi(x) = 0, \quad x \in [\frac{2x_2+x_1}{3}, 1], \end{cases} \tag{2.31}$$

where $(x_1, x_2) \subset \omega$. Let (u, v) be the solution of (2.29) and set $w := \xi u$ and $z := \xi v$ and put $\omega^{'} := (\frac{2x_1+x_2}{3}, \frac{2x_2+x_1}{3})$. Then (w, z) satisfies the following system:

$$\begin{aligned}
&\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+(k_1(x)w_x)_x-\mu_1(t,a,x)w=\xi h_1+\mu_3(t,a,x)z+\xi_x k_1 u_x+(k_1\xi_x u)_x \quad \text{in} \quad Q, \qquad (2.32)\\
&\frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}+(k_2(x)z_x)_x-\mu_2(t,a,x)z=\xi h_2+\xi_x k_2 v_x+(k_2\xi_x v)_x \quad \text{in} \quad Q,\\
&w(t,a,1)=w(t,a,0)=z(t,a,1)=z(t,a,0)=0 \quad \text{on} \quad (0,T)\times(0,A),\\
&w(T,a,x)=w_T(a,x);\, z(T,a,x)=z_T(a,x) \quad \text{in} \quad Q_A,\\
&w(t,A,x)=z(t,A,x)=0 \quad \text{in} \quad Q_T.
\end{aligned}$$

Using Proposition 2.4 for the inhomogeneous term "$\xi h_1 + \mu_3(t, a, x)z + \xi_x k_1 u_x + (k_1\xi_x u)_x$," the definition of ξ, and Young's inequality, we get the following inequality:

$$\begin{aligned}
&\int_Q (s\Theta k_1 w_x^2 + s^3\Theta^3 \frac{x^2}{k_1} w^2)e^{2s\varphi_1} dtdadx\\
&\quad \le C\bigg(\int_Q [\xi^2(h_1+\mu_3(t,a,x)v)^2 + (\xi_x k_1 u_x + (k_1\xi_x u)_x)^2]e^{2s\varphi_1} dtdadx\\
&\qquad +sk_1(1)\int_0^A\int_0^T \Theta w_x^2(t,a,1)e^{2s\varphi_1(t,a,1)} dtda\bigg)\\
&\le \overline{C}\int_Q [\xi^2 h_1^2 + \mu_3^2(t,a,x)z^2 + (\xi_x k_1 u_x + (k_1\xi_x u)_x)^2]e^{2s\varphi_1} dtdadx. \qquad (2.33)
\end{aligned}$$

Thanks again to the definition of ξ, we have

$$\begin{aligned}
&\int_0^1 (\xi_x k_1 u_x + (k_1\xi_x u)_x)^2 e^{2s\varphi_1} dx\\
&\quad \le \int_{\omega^{'}} (8(k_1\xi_x)^2 u_x^2 + 2((k_1\xi_x))_x^2 u^2)e^{2s\varphi_1} dx\\
&\quad \le C\int_{\omega^{'}} (u^2 + u_x^2)e^{2s\varphi_1} dx. \qquad (2.34)
\end{aligned}$$

On the other hand, the third assumption in (2.5) implies that the function $x \mapsto \frac{x^2}{k_2(x)}$ is nondecreasing.

Keeping in the mind the first assumption on μ_3 (2.6) and the fact that $\varphi_1 \le \varphi_2$, then with the aid of Hardy–Poincaré inequality in [9] for the function ze^{φ_2} we conclude that

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq \frac{\|\mu_3\|_\infty^2}{k_2(1)} \int_0^1 \frac{k_2(x)}{x^2} (ze^{s\varphi_2})^2 dx$$
$$\leq C_{HP} \frac{\|\mu_3\|_\infty^2}{k_2(1)} \int_0^1 k_2(x)(ze^{s\varphi_2})_x^2 dx,$$

where $C_{HP} > 0$ is the constant of Hardy–Poincaré.

Thus, from the definition of ψ_2 in (2.19), we obtain

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq C \int_0^1 k_2(x) z_x^2 e^{2s\varphi_2} dx + C \int_0^1 s^2 \Theta^2 \frac{x^2}{k_2(x)} z^2 e^{2s\varphi_2} dx.$$

Hence, for s quite large, we have

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq \frac{1}{2} \int_0^1 s\Theta k_2(x) z_x^2 e^{2s\varphi_2} dx + \frac{1}{2} \int_0^1 s^3 \Theta^3 \frac{x^2}{k_2(x)} z^2 e^{2s\varphi_2} dx. \tag{2.35}$$

Gathering inequalities (2.33), (2.34), and (2.35), for s quite large, the following inequality holds:

$$\begin{aligned}
&\int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt da dx \leq \overline{C} \int_Q h_1^2 e^{2s\varphi_1} dt da dx \\
&+C_1 \int_{\omega'} \int_0^A \int_0^T (u^2 + u_x^2) e^{2s\varphi_1} dt da dx \\
&+\frac{1}{2} \left(\int_Q s\Theta k_2(x) z_x^2 e^{2s\varphi_2} dt da dx + \int_Q s^3 \Theta^3 \frac{x^2}{k_2(x)} z^2 e^{2s\varphi_2} dt da dx \right).
\end{aligned} \tag{2.36}$$

Applying the same way with "$\xi h_2 + \xi_x k_2 v_x + (k_2 \xi_x v)_x$," we conclude

$$\begin{aligned}
&\int_Q (s\Theta k_2 z_x^2 + s^3 \Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dt da dx \leq \overline{C_1} \int_Q h_2^2 e^{2s\varphi_2} dt da dx \\
&+C_2 \int_{\omega'} \int_0^A \int_0^T (v^2 + v_x^2) e^{2s\varphi_2} dt da dx.
\end{aligned} \tag{2.37}$$

Summing side by side (2.36) and (2.37), using the fact that $\varphi_1 \leq \varphi_2$ (Lemma 2.6), we can see that for s quite large that

$$\int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt da dx$$

$$+\int_Q (s\Theta k_2 z_x^2 + s^3\Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dtdadx \le C_4 \int_Q (h_1^2 + h_2^2) e^{2s\varphi_2} dtdadx$$
$$+C_5 \int_{\omega'} \int_0^A \int_0^T (u^2 + v^2 + u_x^2 + v_x^2) e^{2s\varphi_2} dtdadx.$$

In the light of the Caccioppoli's inequality (4.148), the last inequality becomes

$$\begin{aligned}
&\int_Q (s\Theta k_1 w_x^2 + s^3\Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dtdadx \\
&\quad + \int_Q (s\Theta k_2 z_x^2 + s^3\Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dtdadx \\
&\le C_6 \left(\int_Q (h_1^2 + h_2^2) e^{2s\varphi_2} dtdadx + \int_q s^2\Theta^2 (u^2 + v^2) e^{2s\varphi_2} dtdadx \right). \qquad (2.38)
\end{aligned}$$

Now, let $W := \eta u$ and $Z := \eta v$ with $\eta = 1 - \xi$. Then, W and Z are supported in $(x_1, 1)$ and verify the following system:

$$\begin{cases}
\frac{\partial W}{\partial t} + \frac{\partial W}{\partial a} + (k_1(x) W_x)_x - \mu_1(t,a,x) W = \eta h_1 + \mu_3(t,a,x) Z + \eta_x k_1 u_x + (k_1 \eta_x u)_x & \text{in} \quad Q_{x_1}, \\
\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial a} + (k_2(x) Z_x)_x - \mu_2(t,a,x) Z = \eta h_2 + \eta_x k_2 v_x + (k_2 \eta_x v)_x & \text{in} \quad Q_{x_1}, \\
W(t,a,1) = W(t,a,x_1) = Z(t,a,1) = Z(t,a,x_1) = 0 & \text{on} \quad (0,T) \times (0,A), \\
W(T,a,x) = W_T(a,x);\ Z(T,a,x) = Z_T(a,x) & \text{in} \quad Q_A, \\
W(t,A,x) = Z(t,A,x) = 0 & \text{in} \quad Q_T,
\end{cases} \qquad (2.39)$$

where $Q_{x_1} = (0,T) \times (0,A) \times (x_1, 1)$. Then, the system satisfied by W and Z is non-degenerate. Hence, applying Proposition 2.5 on the first equation of (2.39) for $b_1 = x_1$ and $b_2 = 1$ and $h = \eta h_1 + \mu_3(t,a,x) Z + \eta_x k_1 u_x + (k_1 \eta_x u)_x$, the following estimate occurs

$$\begin{aligned}
&\int_Q (s^3\phi^3 W^2 + s\phi W_x^2) e^{2s\Phi} dtdadx \\
&\quad \le C \left(\int_Q (\eta h_1 + \mu_3(t,a,x) Z + \eta_x k_1 u_x + (k_1 \eta_x u)_x)^2 e^{2s\Phi} dtdadx \right. \\
&\qquad \left. + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dtdadx \right).
\end{aligned}$$

Accordingly, the Caccioppoli's inequality stated in [22, Lemma 5.1], the definition of the cut-off function η, and Young's inequality and s quite large lead to

$$
\begin{aligned}
&\int_Q (s^3\phi^3 W^2 + s\phi W_x^2)e^{2s\Phi}dtdadx \\
&\quad\le \tilde{C}\bigg(\int_Q (\eta^2 h_1^2 e^{2s\Phi} + \mu_3^2 Z^2 e^{2s\Phi} + (\eta_x k_1 u_x + (k_1\eta_x u)_x)^2 e^{2s\Phi})dtdadx \\
&\qquad + \int_\omega \int_0^A \int_0^T s^3\Theta^3 u^2 e^{2s\Phi}dtdadx\bigg) \\
&\quad\le \tilde{C}_1\bigg(\int_Q (\eta^2 h_1^2 e^{2s\Phi} + \mu_3^2 Z^2 e^{2s\Phi})dtdadx + \int_{\omega'} \int_0^A \int_0^T (8(k_1\eta_x)^2 u_x^2 \\
&\qquad +2((k_1\eta_x)_x)^2 u^2) e^{2s\Phi}dtdadx\bigg) \\
&\qquad +\tilde{C}\int_\omega \int_0^A \int_0^T s^3\Theta^3 u^2 e^{2s\Phi}dtdadx \\
&\quad\le \tilde{C}_2\bigg(\int_Q (\eta^2 h_1^2 e^{2s\Phi} + \mu_3^2 Z^2 e^{2s\Phi})dtdadx \\
&\qquad + \int_{\omega'} \int_0^A \int_0^T (u^2 + u_x^2) e^{2s\Phi}dtdadx\bigg) \\
&\qquad +\tilde{C}\int_\omega \int_0^A \int_0^T s^3\Theta^3 u^2 e^{2s\Phi}dtdadx \\
&\quad\le \tilde{C}_3\bigg(\int_Q (\eta^2 h_1^2 e^{2s\Phi} + \mu_3^2 Z^2 e^{2s\Phi})dtdadx \\
&\qquad + \int_\omega \int_0^A \int_0^T s^3\Theta^3 u^2 e^{2s\Phi}dtdadx\bigg),
\end{aligned} \tag{2.40}
$$

where Φ and ϕ are defined in (2.19) and ω' is given in the beginning of this proof.

On the other hand, using the fact that $x \mapsto \frac{x^2}{k_2(x)}$ is nondecreasing, Hardy–Poincaré inequality applied for the function $We^{s\Phi}$ and taking s quite large, the same procedure employed to obtain (2.35) steers to

$$
\begin{aligned}
&\int_Q \mu_3^2 Z^2 e^{2s\Phi}dtdadx \\
&\quad\le c\left(\int_Q k_2 Z_x^2 e^{2s\Phi}dtdadx + \int_Q s^2\Theta^2 \frac{x^2}{k_2(x)} Z^2 e^{2s\Phi}dtdadx\right) \\
&\quad\le \frac{1}{2}\int_Q (s^3\phi^3 Z^2 + s\phi Z_x^2)e^{2s\Phi}dtdadx.
\end{aligned} \tag{2.41}
$$

Therefore, injecting (2.41) in (2.40), we arrive to

$$\int_Q (s^3\phi^3 W^2 + s\phi W_x^2)e^{2s\Phi}dtdadx$$
$$\leq \tilde{C}_4\left(\int_Q h_1^2 e^{2s\Phi}dtdadx + \int_\omega\int_0^A\int_0^T s^3\Theta^3 u^2 e^{2s\Phi}dtdadx\right)$$
$$+\frac{1}{2}\int_Q (s^3\phi^3 Z^2 + s\phi Z_x^2)e^{2s\Phi}dtdadx. \tag{2.42}$$

Replying the same device for the source term $h := \eta h_2 + \eta_x k_2 v_x + (k_2\eta_x v)_x$, we infer that

$$\int_Q (s^3\phi^3 Z^2 + s\phi Z_x^2)e^{2s\Phi}dtdadx$$
$$\leq \tilde{C}_5\left(\int_Q h_2^2 e^{2s\Phi}dtdadx + \int_\omega\int_0^A\int_0^T s^3\Theta^3 v^2 e^{2s\Phi}dtdadx\right). \tag{2.43}$$

Subsequently, adding (2.42) to (2.43) side by side, we merely observe that

$$\int_Q [s^3\phi^3(W^2 + Z^2) + s\phi(W_x^2 + Z_x^2)]e^{2s\Phi}dtdadx \tag{2.44}$$
$$\leq \tilde{C}_6\left(\int_Q (h_1^2 + h_2^2)e^{2s\Phi}dtdadx + \int_\omega\int_0^A\int_0^T s^3\Theta^3(u^2 + v^2)e^{2s\Phi}dtdadx\right).$$

Using the fact that $u = w + W$ and $v = z + Z$, $\varphi_1 \leq \varphi_2 \leq \Phi$, the estimates (2.38) and (2.45) lead to estimate (2.30). □

For special functions h_1 and h_2, Theorem 2.7 will play a crucial role to demonstrate the following intermediate Carleman estimate:

Theorem 2.8 *Assume that the assumptions* (2.5) *and* (2.6) *hold. Let $T, A > 0$ fixed such that $T \in (0, \delta)$ with $\delta \in (0, A)$ fixed small enough. Then, there exist two positive constants C (independent of δ) and s_0 such that for all $s \geq s_0$, every solution of* (2.11) (u, v) *satisfies*

$$\int_Q \left(s^3\Theta^3\frac{x^2}{k_1(x)}u^2 + s\Theta k_1(x)u_x^2\right)e^{2s\varphi_1}dtdadx$$
$$+\int_Q \left(s^3\Theta^3\frac{x^2}{k_2(x)}v^2 + s\Theta k_2(x)v_x^2\right)e^{2s\varphi_2}dtdadx \tag{2.45}$$
$$\leq C\left(\int_q s^3\Theta^3(u^2 + v^2)e^{2s\Phi}dtdadx + \int_0^1\int_0^\delta (u_T^2(a, x) + v_T^2(a, x))dadx\right).$$

Proof Let $h_1 := -\beta_1 u(t,0,x)$ and $h_2 := -\beta_2 v(t,0,x)$.

Therefore, thanks to hypotheses (2.6) on β_1 and β_2 and the estimate (2.30), we have the existence of two positive constants C and s_0 such that for all $s \geq s_0$ the following inequality holds:

$$\begin{aligned}
&s^3 \int_Q \Theta^3 \left(\frac{x^2}{k_1(x)} u^2 e^{2s\varphi_1} + \frac{x^2}{k_2(x)} v^2 e^{2s\varphi_2} \right) dtdadx \\
&\quad + s \int_Q \Theta (k_1(x) u_x^2 e^{2s\varphi_1} + k_2(x) v_x^2 e^{2s\varphi_2}) dtdadx \\
&\leq C \left(\int_Q ((\beta_1 u(t,0,x))^2 + (\beta_2 v(t,0,x))^2) e^{2s\Phi} dtdadx \right. \\
&\quad \left. + \int_q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dtdadx \right) \\
&\leq \tilde{C}_7 \left(\int_0^1 \int_0^T (u^2(t,0,x) + v^2(t,0,x)) dtdx \right. \\
&\quad \left. + \int_q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dtdadx \right). \qquad (2.46)
\end{aligned}$$

Set $\forall (t,a,x) \in Q$, $U(t,a,x) = u(T-t, A-a, x)$, and $V(t,a,x) = v(T-t, A-a, x)$. Then, one has

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{\partial U}{\partial a} - (k_1(x) U_x)_x + \mu_1(T-t, A-a, x) U + \mu_3(T-t, A-a, x) V \\
&\qquad = \beta_1(T-t, A-a, x) U(t, A, x) \text{ in } \quad Q, \\
&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial a} - (k_2(x) V_x)_x + \mu_2(T-t, A-a, x) V = \beta_2(T-t, A-a, x) V(t, A, x) \text{ in } \quad Q, \\
&U(t,a,1) = U(t,a,0) = V(t,a,1) = V(t,a,0) = 0 \text{ on } \quad (0,T) \times (0,A), \\
&U(0,a,x) = U_0(a,x) = u_T(A-a,x);\ V(0,a,x) = V_0(a,x) = v_T(A-a,x) \text{ in } \quad Q_A, \\
&U(t,0,x) = V(t,0,x) = 0 \text{ in } \quad Q_T.
\end{aligned} \qquad (2.47)$$

We emphasize here that similar implicit formulas of u and v given beneath are already used in the proof of observability inequality in [23] and before in the ones of the main Carleman estimate and the observability inequality in [12].

In fact, integrating along the characteristic lines, we get

$$\begin{cases}
U(t,a,.) = \int_0^a S(a-l)[\beta_1(T-t, A-l, .) U(t, A, .) - \mu_3(T-t, A-l, .) \\
\quad V(t,l,.)] dl; \text{ if } \quad t > a, \\
U(t,a,.) = S(t) U_0(a-t, .) + \int_0^t S(t-l)[\beta_1(T-l, A-a, .) U(l, A, .) \\
\quad -\mu_3(T-t, A-l, .) V(t,l,.)] dl; \text{ if } \quad t \leq a
\end{cases} \qquad (2.48)$$

and

$$\begin{cases} V(t,a,.) = \int_0^a L(a-l)\beta_2(T-t, A-l, .)V(t, A, .);\ \text{if} \quad t > a, \\ V(t,a,.) = L(t)V_0(a-t,.) + \int_0^t L(t-l)\beta_2(T-l, A-a, .)V(l, A, .)dl;\ \text{if} \\ \quad t \le a, \end{cases} \tag{2.49}$$

where $S(t)_{t\ge 0}$ and $L(t)_{t\ge 0}$ are the bounded semigroups generated, respectively, by the operators

$A_3U := -(k_1(x)U_x)_x + \mu_1(T-t, A-a, x)U$ and $A_4V := -(k_2(x)V_x)_x + \mu_2(T-t, A-a, x)V$.

On the other hand, in the light of the transformations between U and u and also between V and v and if one replaces in the implicit formulas (2.48) and (2.49) t by $T-t$ and a by $A-a$, the functions u and v can be expressed as follows:

$$\begin{cases} u(t,a,.) = \int_0^{A-a} S(A-a-l)[\beta_1(t, A-l, .)u(t,0,.) - \mu_3(t, A-l, .) \\ \quad v(t, A-l, .)]dl;\ \text{if} \quad a > t + (A-T), \\ u(t,a,.) = S(T-t)u_T(a+T-t,.) + \int_t^T S(l-t)[\beta_1(l,a,.)u(l,0,.) \\ \quad -\mu_3(l,a,.)v(l,a,.)]dl;\ \text{if} \quad a \le t + (A-T), \end{cases} \tag{2.50}$$

and

$$\begin{cases} v(t,a,.) = \int_0^{A-a} L(A-a-l)\beta_2(t, A-l, .)v(t,0,.)dl;\ \text{if} \quad a > t + (A-T), \\ v(t,a,.) = L(T-t)v_T(a+T-t,.) + \int_t^T L(l-t)\beta_2(l,a,.)v(l,0,.)dl;\ \text{if} \\ \quad a \le t + (A-T). \end{cases} \tag{2.51}$$

Thus,

$$\begin{cases} u(t,0,.) = S(T-t)u_T(T-t,.) + \int_t^T S(l-t)[\beta_1(l,0,.)u(l,0,.) \\ \quad -\mu_3(l,0,.)v(l,0,.)]dl, \\ v(t,0,.) = L(T-t)v_T(T-t,.) + \int_t^T L(l-t)\beta_2(l,0,.)v(l,0,.)dl. \end{cases} \tag{2.52}$$

Passing to the absolute value of the first equality in (2.52), we get the following relation:

$$\begin{aligned} |u(t,0,x)| &= \Big| S(T-t)u_T(T-t,.) \\ &\qquad + \int_t^T S(l-t)[\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)]dl \Big| \\ &\le |S(T-t)u_T(T-t,x)| \\ &\quad + \int_t^T |S(l-t)[\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)]|\, dl. \end{aligned}$$

Combining the last inequality with the fact that $(S(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup, we deduce readily that

$$\begin{aligned}|u(t,0,x)| &\leq |S(T-t)u_T(T-t,x)| \\ &\quad + \int_t^T \left| Me^{\lambda_3(l-t)}\left(\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)\right)\right| dl \\ &\leq |S(T-t)u_T(T-t,x)| \\ &\quad + \int_t^T \left| Me^{\lambda_3 T}\left(\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)\right)\right| dl,\end{aligned}$$

where

$$M \geq 1 \text{ and } \lambda_3 \in \mathbb{R}. \tag{2.53}$$

Applying Young's inequality to last estimate, we obtain

$$\begin{aligned}|u(t,0,x)|^2 &\leq 2|S(T-t)u_T(T-t,x)|^2 \\ &\quad +2\left[\int_t^T \left| Me^{\lambda_3 T}\left(\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)\right)\right| dl\right]^2 \\ &\leq 2|S(T-t)u_T(T-t,x)|^2 \\ &\quad + \int_t^T 2TM^2e^{2\lambda_3 T}\,|\beta_1(l,0,.)u(l,0,.) - \mu_3(l,0,.)v(l,0,.)|^2\, dl.\end{aligned}$$

Accordingly,

$$\begin{aligned}|u(t,0,x)|^2 &\leq 2|S(T-t)u_T(T-t,x)|^2 \\ &\quad + \int_t^T 4TM^2e^{2\lambda_3 T} \\ &\quad \times \left[\beta_1^2(l,0,x)u^2(l,0,x) + (\mu_3(l,0,x))^2v^2(l,0,x)\right] dl.\end{aligned} \tag{2.54}$$

Now, we claim that

$$\begin{aligned}&\forall (t,x) \in (0,T)\times(0,1), \\ &|v(t,0,x)|^2 \leq 2(M_1)^2e^{2\lambda_4 T}|v_T(T-t,x)|^2 + \tilde{M}_5\int_t^T |v_T(T-s,x)|^2 ds,\end{aligned}$$

with $\tilde{M}_5$ is a positive constant given by (2.58) and M_1 and λ_4 satisfy (2.56) below.

In fact, arguing similarly to (2.54) and taking into account that $(L(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup, one can show via the second equality of (2.52) that

$$|v(t,0,x)|^2 \le 2|L(T-t)v_T(T-t,x)|^2 \\ + \int_t^T 4T(M_1)^2 e^{2\lambda_4 T}\beta_2^2(l,0,x)v^2(l,0,x)dl, \tag{2.55}$$

where

$$M_1 \ge 1 \text{ and } \lambda_4 \in \mathbb{R}. \tag{2.56}$$

The Gronwall–Bellman's lemma applied with respect to the time variable time t in (2.55) implies

$$|v(t,0,x)|^2 \le 2|L(T-t)v_T(T-t,x)|^2 \\ + \int_t^T 8T(M_1)^2 e^{2\lambda_4 T}\left[|L(T-s)v_T(T-s,x)|^2\right]\beta_2^2(s,0,x) \\ \times \exp\left(\int_t^s 4T(M_1)^2 e^{2\lambda_4 T}\beta_2^2(\tau,0,x)d\tau\right)ds,$$

where s is not the parameter of Carleman estimates.

Thanks to the hypotheses (2.6) on the natural rates $\beta_i, \quad i = 1, 2$, we conclude

$$|v(t,0,x)|^2 \le 2|L(T-t)v_T(T-t,x)|^2 + \tilde{M_5}\int_t^T |v_T(T-s,x)|^2 ds \\ \le 2(M_1)^2 e^{2\lambda_4 T}|v_T(T-t,x)|^2 + \tilde{M_5}\int_t^T |v_T(T-s,x)|^2 ds, \tag{2.57}$$

with

$$\tilde{M_5} = 8AT(M_1)^2 e^{2\lambda_4 T}\|\beta_2\|_\infty^2 \times \exp\left(4T^2(M_1)^2 e^{2\lambda_4 T}A\|\beta_2\|_\infty^2\right). \tag{2.58}$$

Combining inequalities (2.54) and (2.57), we deduce the following estimate:

$$|u(t,0,x)|^2 \le 2|S(T-t)u_T(T-t,x)|^2 \\ +4TM^2 e^{2\lambda_3 T}\|\mu_3\|_\infty^2\left[2(M_1)^2 e^{2\lambda_4 T}\int_t^T |v_T(T-l,x)|^2 dl \right.\\ \left. +\tilde{M_5}\int_t^T\int_l^T |v_T(T-s,x)|^2 dsdl\right] \\ + \int_t^T 4TM^2 e^{2\lambda_3 T}(\beta_1(l,0,x))^2 u^2(l,0,x)dl. \tag{2.59}$$

Subsequently, Gronwall–Bellman's lemma again involves

$$
\begin{aligned}
|u(t,0,x)|^2 &\le 2|S(T-t)u_T(T-t,x)|^2 \\
&+4TM^2e^{2\lambda_3 T}\|\mu_3\|_\infty^2\Big[2(M_1)^2e^{2\lambda_4 T}\int_t^T |v_T(T-l,x)|^2dl \\
&\quad+\tilde{M}_5\int_t^T\int_l^T |v_T(T-s,x)|^2dsdl\Big] \\
&+\int_t^T 4TM^2e^{2\lambda_3 T}(\beta_1(m,0,x))^2\times\exp\left(\int_t^m 4TM^2e^{2\lambda_3 T}(\beta_1(\tau,0,x))^2d\tau\right) \\
&\times\Big[2|S(T-m)u_T(T-m,x)|^2+4TM^2e^{2\lambda_3 T}\|\mu_3\|_\infty^2 \\
&\quad\times\Big(2(M_1)^2e^{2\lambda_4 T}\int_m^T |v_T(T-l,x)|^2dl \\
&\quad+\tilde{M}_5\int_m^T\int_l^T |v_T(T-s,x)|^2dsdl\Big)\Big]dm. \qquad (2.60)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|u(t,0,x)|^2 &\le 2|S(T-t)u_T(T-t,x)|^2 \\
&+\tilde{M}_6\left[\int_t^T |v_T(T-l,x)|^2dl+\int_t^T\int_l^T |v_T(T-s,x)|^2dsdl\right] \\
&+\tilde{M}_8\int_t^T\Big[2|S(T-m)u_T(T-m,x)|^2+\int_m^T |v_T(T-l,x)|^2dl \\
&\quad+\int_m^T\int_l^T |v_T(T-s,x)|^2dsdl\Big]dm, \qquad (2.61)
\end{aligned}
$$

where

$$
\begin{cases}
\tilde{M}_6 := 4TM^2e^{2\lambda_3 T}\|\mu_3\|_\infty^2\max(\tilde{M}_5, 2(M_1)^2e^{2\lambda_4 T}), \\
\tilde{M}_8 := \max(1, \tilde{M}_5, 2(M_1)^2e^{2\lambda_4 T})\tilde{M}_7, \\
\tilde{M}_7 := 4TM^2\|\beta_1\|_\infty^2 e^{4T^2M^2e^{2\lambda_3 T}\|\beta_1\|_\infty^2+2\lambda_3 T}\max(1, 4TM^2e^{2\lambda_3 T}\|\mu_3\|_\infty^2),
\end{cases} \qquad (2.62)
$$

with $\tilde{M}_5$ is given by (2.58). Recall that the variable s in inequalities (2.57), (2.59), (2.60), and (2.61) does not represent the parameter of Carleman estimates.

Integrating inequality (2.61) over $(0,T)\times(0,1)$, we can see that

$$
\int_0^1\int_0^T |u(t,0,x)|^2dtdx \le 2\int_0^1\int_0^T |S(T-t)u_T(T-t,x)|^2dtdx
$$

$$+\tilde{M}_6 \int_0^1 \int_0^T \left[\int_t^T |v_T(T-l,x)|^2 dl + \int_t^T \int_l^T |v_T(T-s,x)|^2 ds dl \right] dt dx$$
$$+\tilde{M}_8 \int_0^1 \int_0^T \int_t^T \Big[2|S(T-m)u_T(T-m,x)|^2$$
$$+ \int_m^T |v_T(T-l,x)|^2 dl + \int_m^T \int_l^T |v_T(T-s,x)|^2 ds dl \Big] dm dt dx. \quad (2.63)$$

Set $t_1 = T - t$, $l_1 = T - l$, $s_1 = T - s$, and $m_1 = T - m$.

Hence, (2.63) becomes

$$\int_0^1 \int_0^T |u(t,0,x)|^2 dt dx \leq 2 \int_0^1 \int_0^T |S(t_1)u_T(t_1,x)|^2 dt_1 dx$$
$$+\tilde{M}_6 \int_0^1 \int_0^T \left[\int_0^{T-t} |v_T(l_1,x)|^2 dl_1 + \int_0^{T-t} \int_0^{l_1} |v_T(s_1,x)|^2 ds_1 dl_1 \right] dt dx$$
$$+\tilde{M}_8 \int_0^1 \int_0^T \int_0^{T-t} \Big[2|S(m_1)u_T(m_1,x)|^2 + \int_0^{m_1} |v_T(l_1,x)|^2 dl_1$$
$$+ \int_0^{m_1} \int_0^{l_1} |v_T(s_1,x)|^2 ds_1 dl_1 \Big] dm_1 dt dx. \quad (2.64)$$

Using again the fact that $(S(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup and the hypothesis (2.53), we can observe via (2.64) that

$$\int_0^1 \int_0^T |u(t,0,x)|^2 dt dx$$
$$\leq \left(2M^2 e^{2\lambda_3 T} + 2M^2 T e^{2\lambda_3 T} \tilde{M}_8\right) \int_0^1 \int_0^T |u_T(a,x)|^2 da dx$$
$$+ \left((T^2+T^3)\tilde{M}_8 + (T+T^2)\tilde{M}_6\right) \int_0^1 \int_0^T |v_T(a,x)|^2 da dx, \quad (2.65)$$

with $\tilde{M}_6$ and $\tilde{M}_8$ are defined by (2.62).

Taking into account that $T \in (0,\delta)$, we have the existence of two positive constants $\tilde{C}_8$ and $\tilde{C}_9$ such that

$$\int_0^1 \int_0^T |u(t,0,x)|^2 dt dx \leq \tilde{C}_8 \int_0^1 \int_0^\delta |u_T(a,x)|^2 da dx$$
$$+\tilde{C}_9 \int_0^1 \int_0^\delta |v_T(a,x)|^2 da dx, \quad (2.66)$$

where $\tilde{C}_8 := 2M^2 e^{2\lambda_3 T} + 2M^2 T e^{2\lambda_3 T} \tilde{M}_8$ and $\tilde{C}_9 := (T^2+T^3)\tilde{M}_8 + (T+T^2)\tilde{M}_6$.

Integrating (2.57) over $(0, T) \times (0, 1)$, arguing similarly as for (2.66), we can prove, exploiting again the assumption $T \in (0, \delta)$, that

$$\int_0^1 \int_0^T |v(t, 0, x)|^2 dt dx \leq \tilde{C_{10}} \int_0^1 \int_0^T |v_T(a, x)|^2 da dx$$
$$\leq \tilde{C_{10}} \int_0^1 \int_0^\delta |v_T(a, x)|^2 da dx, \tag{2.67}$$

with $\tilde{C_{10}} := T\tilde{M_5} + 2(M_1)^2 e^{2\lambda_4 T}$.

Implementing (2.66) and (2.67) in (2.46), we reach the Carleman estimate (2.46). □

Before continuing, we point out the following remark:

Remark 2.9 In general, if we want to express the implicit formula of a population dynamics model's solution, the characteristic method is the pertinent candidate. The principal of this method is to write the solution of the studied model covering all the whole $(0, T) \times (0, A)$ by deleting one of the two variables, time or age in two main sub-parts of $(0, T) \times (0, A)$. Classically, these sub-parts are separated via a given line whose the equation is $a = t + c$, where $c > 0$, which is in our case equal to $A - T$.

Following this, we will obtain the formula of our solution in both the two parts $a > t + c$ and $a \leq t + c$, and this is exactly what happened in the implicit formulas of u and v defined, respectively, by relations (2.50) and (2.51).

If $A = T$, i.e., $(0, T) \times (0, A)$, is a square, we will get the classical implicit formulas in the two parts $a > t$ and $a \leq t$ by dividing the pavement $(0, T) \times (0, A)$ with respect to the first bisector given by the equation $a = t$.

Since that our aim is to prove the null controllability property (1.4) for one control force problem, one must somehow "delete" the adjoint variable to the non-controlled solution that is in our case v from the right-hand side of Carleman estimate (2.46). In other words, we presume that our full ω-Carleman estimate is written as follows:

Theorem 2.10 *Let the assumptions on k_i, $i = 1, 2$, (2.5) and on the natural rates (2.6) be verified. Let $A, T > 0$ be given and fixed such that $T \in (0, \delta)$, where $\delta \in (0, A)$ fixed small enough. Assume that there exists a positive constant ν such that*

$$\mu_3 \geq \nu \quad \text{on } [0, T] \times [0, A] \times \omega_1 \quad \text{for some } \omega_1 \subset\subset \omega. \tag{2.68}$$

Then, every solution (u, v) of (2.11) satisfies

$$\int_Q \left(s^3 \Theta^3 \frac{x^2}{k_1(x)} u^2 + s\Theta k_1(x) u_x^2 \right) e^{2s\varphi_1} dt da dx$$

$$+\int_Q \left(s^3\Theta^3\frac{x^2}{k_2(x)}v^2 + s\Theta k_2(x)v_x^2\right)e^{2s\varphi_2}dtdadx$$
$$\leq C_\delta\left(\int_q u^2dtdadx + \int_0^1\int_0^\delta (u_T^2(a,x)+v_T^2(a,x))dadx\right). \quad (2.69)$$

This theorem is the novelty of this contribution and is an immediate outcome of Theorem 2.8 applied to ω_1 and the following lemma.

Lemma 2.11 *Assume that* (2.5) *and* (2.6) *hold, and let* $A, T > 0$ *be given and fixed such that* $T \in (0, \delta)$, *where* $\delta \in (0, A)$ *fixed small enough. We also suppose that* (2.68) *holds.*

Then, for all $\varepsilon > 0$, *there exist two positive constants* C *and* M_ε *such that for every solution* (u, v) *of* (2.11) *the following inequality occurs:*

$$\int_\omega\int_0^A\int_0^T s^3\Theta^3v^2e^{2s\Phi}dtdadx$$
$$\leq \varepsilon C\left(\int_Q\left(s^3\Theta^3\frac{x^2}{k_2(x)}v^2 + s\Theta k_2(x)v_x^2\right)e^{2s\varphi_2}dtdadx\right)$$
$$+M_\varepsilon\left(\int_q u^2dtdadx + \int_0^1\int_0^\delta (u_T^2(a,x)+v_T^2(a,x))dadx\right). \quad (2.70)$$

Proof Let $\chi : \mathbb{R} \to \mathbb{R}$ be the non-negative cut-off function defined as follows:

$$\begin{cases} \chi \in \mathcal{C}^\infty(0,1), \\ supp(\chi) \subset \omega, \\ \chi \equiv 1 \quad \text{on } \omega_1. \end{cases} \quad (2.71)$$

Recall that $(x_1, x_2) \subset \omega \subset\subset (0, 1)$. Multiplying the first equation of (2.11) by $\chi s^3\Theta^3ve^{2s\Phi}$ and after an integration by parts, we get

$$\int_Q \chi s^3\Theta^3ve^{2s\Phi}u_tdtdadx = -\int_Q (3+2s\Phi)\chi s^3\Theta_t\Theta^2uve^{2s\Phi}dtdadx$$
$$-\int_Q \chi s^3\Theta^3uv_te^{2s\Phi}dtdadx.$$

$$\int_Q \chi s^3\Theta^3ve^{2s\Phi}u_adtdadx = -\int_Q (3+2s\Phi)\chi s^3\Theta_a\Theta^2uve^{2s\Phi}dtdadx$$
$$-\int_Q \chi s^3\Theta^3uv_ae^{2s\Phi}dtdadx.$$

$$\begin{aligned}\int_Q \chi s^3\Theta^3 v e^{2s\Phi}(k_1u_x)_x dtdadx &= -\int_Q \chi s^3\Theta^3 k_1 e^{2s\Phi} u_x v_x dtdadx\\ &\quad + \int_Q s^3\Theta^3 k_1(\chi e^{2s\Phi})_x u v_x dtdadx\\ &\quad + \int_Q s^3\Theta^3 (k_1(\chi e^{2s\Phi})_x)_x u v dtdadx.\\ &\quad - \int_Q \chi s^3\Theta^3 v e^{2s\Phi}\mu_1 u dtdadx\\ &= -\int_Q \chi s^3\Theta^3 \mu_1 u v e^{2s\Phi} dtdadx\\ &\quad - \int_Q \chi s^3\Theta^3 v e^{2s\Phi}\mu_3 v dtdadx\\ &= -\int_Q \chi s^3\Theta^3 \mu_3 v^2 e^{2s\Phi} dtdadx.\end{aligned}$$

Then, summing all these identities side by side, using the second equation of (2.11), and integrating again by parts,

$$\int_Q \chi s^3\Theta^3\mu_3 v^2 e^{2s\Phi} dtdadx = I_1 + I_2 + I_3 + I_4 + I_5, \tag{2.72}$$

where $I_1 := \int_Q \chi s^3\Theta^3\beta_1 v u(t,0,x) e^{2s\Phi} dtdadx$,
$I_2 := -\int_Q ((3+2s\Phi)s^3\Theta_t\Theta^2 + (3+2s\Phi)s^3\Theta_a\Theta^2 + \mu_1 s^3\Theta^3 + \mu_2 s^3\Theta^3)\chi$
$e^{2s\Phi} uvdtdadx + \int_Q s^3\Theta^3(k_1(\chi e^{2s\Phi})_x)_x uvdtdadx$,
$I_3 := \int_Q \chi s^3\Theta^3\beta_2 u v(t,0,x) e^{2s\Phi} dtdadx$,
$I_4 := \int_Q s^3\Theta^3 (k_1 - k_2)(x) u v_x (\chi e^{2s\Phi})_x dtdadx$,
$I_5 := -\int_Q \chi s^3\Theta^3 (k_1 + k_2)(x) u_x v_x e^{2s\Phi} dtdadx$.

On one hand, we have by Young's inequality and definition of χ

$$\begin{aligned} I_5 &\le \varepsilon \int_Q s\Theta k_2(x) v_x^2 e^{2s\varphi_2} dtdadx\\ &\quad + \frac{1}{4\varepsilon}\int_Q \frac{\chi^2 s^5\Theta^5 (k_1+k_2)^2 u_x^2 e^{2s(2\Phi-\varphi_2)}}{k_2} dtdadx\\ &\le \varepsilon \int_Q s\Theta k_2(x) v_x^2 e^{2s\varphi_2} dtdadx\\ &\quad + \frac{\max\limits_{[0,1]}(k_1+k_2)^2}{4\varepsilon \min\limits_{\omega} k_2}\int_Q \chi s^5\Theta^5 u_x^2 e^{2s(2\Phi-\varphi_2)} dtdadx. \end{aligned} \tag{2.73}$$

Put $L := \int_Q \chi s^5 \Theta^5 u_x^2 e^{2s(2\Phi-\varphi_2)} dtdadx$. Now, we have to find an upper bound of L. To do this, we multiply the first equation of (2.11) by $\frac{\chi s^5 \Theta_5 e^{2s(2\Phi-\varphi_2)}}{k_1} u$ and after an integration by parts

$$\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1} u u_t dtdaddx$$
$$= -\frac{1}{2}\int_Q \frac{s^5\chi}{k_1}\Theta^4\Theta_t(5+2s(2\Phi-\varphi_2))e^{2s(2\Phi-\varphi_2)}u^2 dtdadx.$$

$$\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1} u u_a dtdaddx$$
$$= -\frac{1}{2}\int_Q \frac{s^5\chi}{k_1}\Theta^4\Theta_a(5+2s(2\Phi-\varphi_2))e^{2s(2\Phi-\varphi_2)}u^2 dtdadx.$$

$$\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1} u(k_1 u_x)_x dtdadx$$
$$= -\int_Q \chi s^5 \Theta^5 u_x^2 e^{2s(2\Phi-\varphi_2)} dtdadx$$
$$+\frac{1}{2}\int_Q s^5\Theta^5\left(k_1\left(\frac{\chi e^{2s(2\Phi-\varphi_2)}}{k_1}\right)_x\right)_x u^2 dtdadx.$$

$$-\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1} u\mu_1 u dtdadx = -\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1}\mu_1 u^2 dtdadx.$$

$$-\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1} u\mu_3 v dtdadx = -\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1}\mu_3 uv dtdadx.$$

Hence, adding these equalities side by side, we get

$$L = L_1 + L_2 + L_3, \tag{2.74}$$

where $L_1 := \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{k_1}\beta_1 uu(t,0,x) dtdadx$,

$L_2 := -\int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{\mu}{}_3 uv dtdadx$,

$L_3 \quad := \quad -\int_Q \Big(\frac{\chi s^5 \Theta^5}{k_1}\mu_1 + \frac{1}{2}\frac{s^5\chi}{k_1}\Theta^5\Theta_t(\frac{5}{\Theta} + 2s(2\Psi - \psi_2)) + \frac{1}{2}\frac{s^5\chi}{k_1}\Theta^5\Theta_a$
$(\frac{5}{\Theta} + 2s(2\Psi - \psi_2))\Big) e^{2s(2\Phi-\varphi_2)}u^2 dtdadx$
$+ \frac{1}{2}\int_Q s^5\Theta^5 \Big(k_1 \Big(\frac{\chi e^{2s(2\Phi-\varphi_2)}}{k_1}\Big)_x\Big)_x dtdaddx.$

The assumptions in (2.6) on β_1 and μ_3 together with Young's inequality, inequality (2.66), $T \in (0, \delta)$, the definitions of χ and Θ, the fact that the function $x \mapsto \frac{k_2}{x}$ is non-increasing, $|\Theta_t| \le C\Theta^2$ and $|\Theta_a| \le \tilde{C}\Theta^2$, and

$$\sup_{(t,a,x)\in Q} s^{r_1}\Theta^{r_1}e^{2s(2\Phi-\varphi_2)} < +\infty, \qquad \text{for } r_1 \in \mathbb{R}, \tag{2.75}$$

lead to

$$\begin{aligned}
L_1 &\le \frac{1}{4\varepsilon}\int_Q \frac{\chi s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}}{(k_1)^2}u^2 dtdadx \\
&\quad +\varepsilon \int_Q \chi s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}(\beta_1)^2 u^2(t,0,x)dtdadx \\
&\le \frac{\tilde{K}_1}{4\varepsilon}\int_Q \chi s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2 dtdadx \\
&\quad +\varepsilon K_1 \int_0^1\int_0^A\int_0^T \chi u^2(t,0,x)dtdadx \\
&\le \frac{\tilde{K}_1}{4\varepsilon}\int_Q \chi s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2 dtdadx \\
&\quad +\varepsilon K_2 \left(\int_0^1\int_0^\delta \chi u_T^2(a,x)dadx + \int_0^1\int_0^\delta \chi v_T^2(a,x)dadx\right),
\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}
L_2 &\le \varepsilon^2 \int_Q \frac{x^2}{k_2}s^3\Theta^3 e^{2s\varphi_2}v^2 dtdadx \\
&\quad + \frac{1}{4\varepsilon^2}\int_Q \chi^2 \frac{s^7\Theta^7}{(k_1)^2}e^{2s(4\Phi-3\varphi_2)}\frac{K_2}{x^2}(\mu_3)^2 u^2 dtdaddx \\
&\le \varepsilon^2 \int_Q \frac{x^2}{k_2}s^3\Theta^3 e^{2s\varphi_2}v^2 dtdadx \\
&\quad + \frac{K_4}{4\varepsilon^2}\int_\omega\int_0^A\int_0^T s^7\Theta^7 e^{2s(4\Phi-3\varphi_2)}u^2 dtdadx,
\end{aligned} \tag{2.77}$$

and

$$|L_3| \leq K_5 \int_\omega \int_0^A \int_0^T s^7\Theta^7 e^{2s(2\Phi-\varphi_2)}u^2 dtdadx, \tag{2.78}$$

where $K_4 := \frac{\|\mu_3\|_\infty^2 k_2(x_1)}{(x_1)^2 \min\limits_\omega k_1}$. On the other hand, by second inequality of (2.27) stated in Lemma 2.6, we have

$$e^{2s(2\Phi-\varphi_2)} \leq e^{2s(4\Phi-3\varphi_2)}. \tag{2.79}$$

Then, combining relations (2.74), (2.76), (2.77), and (2.78), we conclude

$$\begin{aligned} L \leq \varepsilon^2 \int_Q \frac{x^2}{k_2} s^3\Theta^3 e^{2s\varphi_2} v^2 dtdadx + K_\varepsilon \int_\omega \int_0^A \int_0^T s^7\Theta^7 e^{2s(2\Phi-\varphi_2)}u^2 dtdadx \\ +\varepsilon K_2 \left(\int_0^1 \int_0^\delta u_T^2(a,x)dadx + \int_0^1 \int_0^\delta v_T^2(a,x)dadx \right). \end{aligned} \tag{2.80}$$

Hence, by inequalities (2.73) and (2.80), we deduce that

$$\begin{aligned} I_5 \leq \varepsilon C \left(\int_Q \frac{x^2}{k_2} s^3\Theta^3 e^{2s\varphi_2} v^2 dtdadx + \int_Q s\Theta k_2(x) v_x^2 e^{2s\varphi_2} dtdadx \right) \\ +K_\varepsilon^1 \int_\omega \int_0^A \int_0^T s^7\Theta^7 e^{2s(4\Phi-3\varphi_2)}u^2 dtdadx \\ +K_2 \left(\int_0^1 \int_0^\delta u_T^2(a,x)dadx + \int_0^1 \int_0^\delta v_T^2(a,x)dadx \right), \end{aligned} \tag{2.81}$$

where K_ε^1 is a positive constant depending on ε.

Similarly, we will find upper bounds of I_1, I_2, I_3, and I_4. First, we will start by I_2. One has the following relations:

$$\begin{aligned} &\left| \int_Q \chi(3+2s\Phi)s^3\Theta_t\Theta^2 e^{2s\Phi} uvdtdadx \right| \\ &\leq \int_Q \chi|3+2s\Phi|s^3|\Theta_t|\Theta^2 e^{2s\Phi}|uv|dtdadx \\ &\leq C\int_Q \chi|3+2s\Phi|s^3\Theta^4 e^{2s\Phi}|uv|dtdadx \\ &\leq \varepsilon \int_Q s^3\Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dtdadx \\ &\quad +C_\varepsilon \int_\omega \int_0^A \int_0^T s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2 dtdadx, \end{aligned} \tag{2.82}$$

$$\left|\int_Q \chi(3+2s\Phi)s^3\Theta_a\Theta^2 e^{2s\Phi}uvdtdadx\right|$$
$$\leq \varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+C_\varepsilon^1\int_\omega\int_0^A\int_0^T s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2dtdadx, \tag{2.83}$$

$$\left|\int_Q \chi(\mu_1+\mu_2)s^3\Theta^3 e^{2s\Phi}uvdtdadx\right|$$
$$\leq \varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+C_\varepsilon^2\int_\omega\int_0^A\int_0^T s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2dtdadx, \tag{2.84}$$

$$\left|\int_Q s^3\Theta^3(k_1(\chi e^{2s\Phi})_x)_x uvdtdadx\right|$$
$$\leq \varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+\frac{1}{4\varepsilon}\int_Q s^3\Theta^3\frac{x^2}{k_2}(k_1(chie^{2s\Phi})_x^2)_x e^{-2s\varphi_2}u^2dtdadx$$
$$\leq \varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+\frac{C_2}{4\varepsilon}\int_Q s^3\Theta^3\frac{x^2}{k_2}(\chi^2+\chi_x^2+\chi_{xx}^2)e^{2s(2\Phi-\varphi_2)}u^2dtdadx$$
$$\leq \varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+C_\varepsilon^3\int_\omega\int_0^A\int_0^T s^3\Theta^3 e^{2s(2\Phi-\varphi_2)}u^2dtdadx. \tag{2.85}$$

Hence, summing inequalities (2.82)–(2.85), we obtain

$$I_2 \leq 4\varepsilon\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx$$
$$+C_\varepsilon^4\int_\omega\int_0^A\int_0^T s^5\Theta^5 e^{2s(2\Phi-\varphi_2)}u^2dtdadx. \tag{2.86}$$

For the remaining integrals

$$
\begin{aligned}
I_1 &= \int_Q \chi s^3\Theta^3\beta_1 vu(t,0,x)e^{2s\Phi}dtdadx \\
&\le \varepsilon \int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx \\
&\quad + C_\varepsilon^5\left(\int_0^1\int_0^\delta u_T^2(a,x)dadx + \int_0^1\int_0^\delta v_T^2(a,x)dadx\right).
\end{aligned} \tag{2.87}
$$

$$
\begin{aligned}
I_3 &= \int_Q \chi s^3\Theta^3\beta_2 uv(t,0,x)e^{2s\Phi}dtdadx \\
&\le \varepsilon\tilde{C_{10}}\int_0^1\int_0^\delta v_T^2(a,x)dadx \\
&\quad + \frac{1}{4\varepsilon}\int_\omega\int_0^A\int_0^T s^7\Theta^7 e^{2s(2\Phi-\varphi_2)}u^2dtdadx,
\end{aligned} \tag{2.88}
$$

with $\tilde{C_{10}}$ is the constant defined in inequality (2.67).

$$
\begin{aligned}
I_4 &= \int_Q s^3\Theta^3(k_1-k_2)(x)uv_x(\chi e^{2s\Phi})_x dtdadx \\
&= \int_Q s^3\Theta^3(k_1-k_2)(x)uv_x(\chi_x+2s\Phi_x\chi)e^{2s\Phi}dtdadx \\
&\le \varepsilon\int_Q s\Theta k_2 v_x^2 e^{2s\varphi_2}dtdadx \\
&\quad + \frac{1}{4\varepsilon}\int_Q s^5\Theta^5\frac{(k_1-k_2)^2}{k_2}(\chi_x+2s\Phi_x\chi)^2e^{2s(2\Phi-\varphi_2)}u^2dtdadx \\
&\le \varepsilon\int_Q s\Theta k_2v_x^2e^{2s\varphi_2}dtdadx + C_\varepsilon^6\int_\omega\int_0^A\int_0^T s^7\Theta^7e^{2s(2\Phi-\varphi_2)}u^2dtdadx.
\end{aligned} \tag{2.89}
$$

Subsequently, inequalities (2.81), (2.86), (2.87), (2.88), (2.90), and again (2.79) involve

$$
\begin{aligned}
&\int_Q \chi s^3\Theta^3\mu_3 v^2e^{2s\Phi}dtdadx \\
&\quad \le \varepsilon C_7\left(\int_Q s^3\Theta^3\frac{x^2}{k_2}e^{2s\varphi_2}v^2dtdadx + \int_Q s\Theta k_2v_x^2e^{2s\varphi_2}dtdadx\right)
\end{aligned}
$$

$$+C_\varepsilon^8 \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(4\Phi-3\varphi_2)} u^2 dtdadx$$
$$+C_\varepsilon^9 \left(\int_0^1 \int_0^\delta u_T^2(a,x)dadx + \int_0^1 \int_0^\delta v_T^2(a,x)dadx \right).$$

Finally, the hypothesis (2.68), the definition of χ, and the relation

$$\sup_{(t,a,x)\in Q} s^{r_1} \Theta^{r_1} e^{2s(4\Phi-3\varphi_2)} < +\infty, \quad \text{for } r_1 \in \mathbb{R}, \tag{2.90}$$

yield

$$\begin{aligned}
&\int_{\omega_1} \int_0^A \int_0^T s^3 \Theta^3 \mu_3 v^2 e^{2s\Phi} dtdadx \\
&\leq \varepsilon C_{10} \left(\int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dtdadx + \int_Q s\Theta k_2 v_x^2 e^{2s\varphi_2} dtdadx \right) \\
&\quad + C_\varepsilon^{11} \Bigg(\int_\omega \int_0^A \int_0^T u^2 dtdadx + \int_0^1 \int_0^\delta u_T^2(a,x)dadx \\
&\quad + \int_0^1 \int_0^\delta v_T^2(a,x)dadx \Bigg),
\end{aligned} \tag{2.91}$$

which achieves the task. □

The full ω-Carleman estimate (2.69) can be used in a standard way to obtain the null controllability of the cascade system with one control force (1.1). This will be reached showing a relevant observability inequality of system (2.11).

3 Observability Inequality and Null Controllability Result

3.1 Observability Inequality Result

This paragraph is devoted to the observability inequality of system (2.11).The proof is based essentially on the semigroup approach, Carleman estimate (2.69), and with the help of Hardy–Poincaré and Hölder inequalities.

Proposition 3.1 *Assume that* (2.5) *and* (2.6) *hold. Suppose also that the assumption* (2.68) *is fulfilled, and let* $A, T > 0$ *be given and fixed such that* $T < \delta$ *with* $\delta \in (0, A)$ *fixed small enough. Then, there exists a positive constant* C_δ *such that for every solution* (u, v) *of* (2.11) *the following observability inequality is satisfied:*

$$\int_0^1 \int_0^A (u^2(0,a,x) + v^2(0,a,x))dadx$$
$$\leq C_\delta \left(\int_q u^2 dtdadx + \int_0^1 \int_0^\delta (u_T^2(a,x) + v_T^2(a,x))dadx \right). \quad (3.92)$$

Proof For $\kappa_1 > 0$ to be defined later, let $\tilde{u} = e^{\kappa_1 t} u$ and $\tilde{v} = e^{\kappa_1 t} v$, where (u, v) is the solution of (2.11).

Then, $(\tilde{u}, \tilde{v})$ verifies the system

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial a} + (k_1(x)\tilde{u}_x)_x - (\mu_1 + \kappa_1)\tilde{u} = \mu_3 \tilde{v} - \beta_1 \tilde{u}(t,0,x) \quad \text{in} \quad Q, \\
&\frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{v}}{\partial a} + (k_2(x)\tilde{v}_x)_x - (\mu_2 + \kappa_1)\tilde{v} = -\beta_2 \tilde{v}(t,0,x) \quad \text{in} \quad Q, \\
&\tilde{u}(t,a,1) = \tilde{u}(t,a,0) = \tilde{v}(t,a,1) = \tilde{v}(t,a,0) = 0 \quad \text{on} \quad (0,T) \times (0,A), \\
&\tilde{u}(T,a,x) = e^{\kappa_1 T} u_T(a,x);\ \tilde{v}(T,a,x) = e^{\kappa_1 T} v_T(a,x) \quad \text{in} \quad Q_A, \\
&\tilde{u}(t,A,x) = \tilde{v}(t,A,x) = 0 \quad \text{in} \quad Q_T.
\end{aligned} \quad (3.93)$$

Multiplying the first and second equations of (3.93) by $\tilde{u}$ and $\tilde{v}$, respectively, integrating by parts the new equations over $Q_t = (0,t) \times (0,A) \times (0,1)$, and taking into account the rest of equations in (3.93), we get

$$\begin{aligned}
&-\frac{1}{2} \int_0^1 \int_0^A \tilde{u}^2(t,a,x)dadx + \frac{1}{2} \int_0^1 \int_0^A \tilde{u}^2(0,a,x)dadx \\
&+\frac{1}{2} \int_0^1 \int_0^t \tilde{u}^2(\tau,0,x)d\tau dx + \int_{Q_t} k_1 \tilde{u}_x^2 d\tau dadx \\
&+\int_{Q_t} (\kappa_1 + \mu_1)\tilde{u}^2 d\tau dadx = \int_{Q_t} \beta_1 \tilde{u}\tilde{u}(\tau,0,x)d\tau dadx - \int_{Q_t} \mu_3 \tilde{v}\tilde{u} d\tau dadx
\end{aligned} \quad (3.94)$$

and

$$\begin{aligned}
&-\frac{1}{2} \int_0^1 \int_0^A \tilde{v}^2(t,a,x)dadx + \frac{1}{2} \int_0^1 \int_0^A \tilde{v}^2(0,a,x)dadx \\
&+\frac{1}{2} \int_0^1 \int_0^t \tilde{v}^2(\tau,0,x)d\tau dx + \int_{Q_t} k_2 \tilde{v}_x^2 d\tau dadx \\
&+\int_{Q_t} (\kappa_1 + \mu_2)\tilde{v}^2 d\tau dadx = \int_{Q_t} \beta_2 \tilde{v}\tilde{v}(\tau,0,x)d\tau dadx.
\end{aligned} \quad (3.95)$$

Summing (3.94) and (3.95), we have

$$\begin{aligned}
&\int_0^1\int_0^A(\tilde{u}^2(0,a,x)+\tilde{v}^2(0,a,x))dadx\\
&\quad+\int_0^1\int_0^t(\tilde{u}^2(\tau,0,x)+\tilde{v}^2(\tau,0,x))d\tau dx+2\int_{Q_t}(k_1\tilde{u}_x^2+k_2\tilde{v}_x^2)d\tau dadx\\
&\quad+2\int_{Q_t}\mu_1\tilde{u}^2d\tau dadx+2\int_{Q_t}\mu_2\tilde{v}^2d\tau dadx+2\int_{Q_t}\kappa_1(\tilde{u}^2+\tilde{v}^2)d\tau dadx\\
&=2\left(\int_{Q_t}\beta_1\tilde{u}\tilde{u}(\tau,0,x)d\tau dadx+\int_{Q_t}\beta_2\tilde{v}\tilde{v}(\tau,0,x)d\tau dadx\right)\\
&\quad+\int_0^1\int_0^A(\tilde{u}^2(t,a,x)+\tilde{v}^2(t,a,x))dadx-2\int_{Q_t}\mu_3\tilde{v}\tilde{u}d\tau dadx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^1\int_0^A(\tilde{u}^2(0,a,x)+\tilde{v}^2(0,a,x))dadx\\
&\quad+\int_0^1\int_0^t(\tilde{u}^2(\tau,0,x)+\tilde{v}^2(\tau,0,x))d\tau dx+2\int_{Q_t}\kappa_1(\tilde{u}^2+\tilde{v}^2)d\tau dadx\\
&\le 2\left(\int_{Q_t}\beta_1\tilde{u}\tilde{u}(\tau,0,x)d\tau dadx+\int_{Q_t}\beta_2\tilde{v}\tilde{v}(\tau,0,x)d\tau dadx\right)\\
&\quad+\int_0^1\int_0^A(\tilde{u}^2(t,a,x)+\tilde{v}^2(t,a,x))dadx-2\int_{Q_t}\mu_3\tilde{v}\tilde{u}d\tau dadx. \qquad (3.96)
\end{aligned}$$

With the help of Young's inequality, one can check out the following relations:

$$\begin{aligned}
2\int_{Q_t}\beta_1\tilde{u}\tilde{u}(\tau,0,x)d\tau dadx&=2\int_{Q_t}\frac{\beta_1}{4\sqrt{\varepsilon'}}\tilde{u}4\sqrt{\varepsilon'}\tilde{u}(\tau,0,x)d\tau dadx\\
&\le\frac{\|\beta_1\|_\infty^2}{16\varepsilon'}\int_{Q_t}\tilde{u}^2d\tau dadx\\
&\quad+16\varepsilon'\int_{Q_t}\tilde{u}^2(\tau,0,x)d\tau dadx, \qquad (3.97)
\end{aligned}$$

$$\begin{aligned}
2\int_{Q_t}\beta_1\tilde{v}\tilde{v}(\tau,0,x)d\tau dadx&\le\frac{\|\beta_2\|_\infty^2}{16\varepsilon'}\int_{Q_t}\tilde{v}^2d\tau dadx\\
&\quad+16\varepsilon'\int_{Q_t}\tilde{v}^2(\tau,0,x)d\tau dadx, \qquad (3.98)
\end{aligned}$$

and

$$-2\int_{Q_t}\mu_3\tilde{v}\tilde{u}d\tau dadx \leq 16\varepsilon^{'}\|\mu_3\|_\infty^2\int_{Q_t}\tilde{v}^2d\tau dadx$$
$$+\frac{1}{16\varepsilon^{'}}\int_{Q_t}\tilde{u}^2d\tau dadx. \tag{3.99}$$

As a consequence of (3.96), (3.97), (3.98), and (3.99), one has

$$\begin{aligned}
&\int_0^1\int_0^A(\tilde{u}^2(0,a,x)+\tilde{v}^2(0,a,x))dadx+\int_0^1\int_0^t(\tilde{u}^2(\tau,0,x)\\
&\quad+\tilde{v}^2(\tau,0,x))d\tau dx+2\int_{Q_t}\kappa_1(\tilde{u}^2+\tilde{v}^2)d\tau dadx\\
&\leq\left(\frac{\|\beta_1\|_\infty^2}{16\varepsilon^{'}}+\frac{1}{16\varepsilon^{'}}\right)\int_{Q_t}\tilde{u}^2d\tau dadx+\left(\frac{\|\beta_2\|_\infty^2}{16\varepsilon^{'}}+16\varepsilon^{'}\|\mu_3\|_\infty^2\right)\\
&\quad\times\int_{Q_t}\tilde{v}^2d\tau dadx+16A\varepsilon^{'}\int_0^1\int_0^t(\tilde{u}^2(\tau,0,x)+\tilde{v}^2(\tau,0,x))d\tau dx\\
&\quad+\int_0^1\int_0^A(\tilde{u}^2(t,a,x)+\tilde{v}^2(t,a,x))dadx.
\end{aligned}\tag{3.100}$$

For $\varepsilon^{'}<\frac{1}{16A}$, we deduce from (3.100) that

$$\begin{aligned}
&\int_0^1\int_0^A(\tilde{u}^2(0,a,x)+\tilde{v}^2(0,a,x))dadx+2\int_{Q_t}\kappa_1(\tilde{u}^2+\tilde{v}^2)d\tau dadx\\
&\leq\max\left(\left(\frac{\|\beta_1\|_\infty^2}{16\varepsilon^{'}}+\frac{1}{16\varepsilon^{'}}\right),\left(\frac{\|\beta_2\|_\infty^2}{16\varepsilon^{'}}+16\varepsilon^{'}\|\mu_3\|_\infty^2\right)\right)\\
&\quad\times\int_{Q_t}(\tilde{u}^2+\tilde{v}^2)d\tau dadx+\int_0^1\int_0^A(\tilde{u}^2(t,a,x)+\tilde{v}^2(t,a,x))dadx.
\end{aligned}$$

Subsequently,

$$\begin{aligned}
&\int_0^1\int_0^A(\tilde{u}^2(0,a,x)+\tilde{v}^2(0,a,x))dadx+2\int_{Q_t}\kappa_1(\tilde{u}^2+\tilde{v}^2)d\tau dadx\\
&\leq\max\left(\left(\frac{\|\beta_1\|_\infty^2}{16\varepsilon^{'}}+\frac{1}{16\varepsilon^{'}}\right),\left(\frac{\|\beta_2\|_\infty^2}{16\varepsilon^{'}}+16\varepsilon^{'}\|\mu_3\|_\infty^2\right)\right)\\
&\quad\times\int_{Q_t}(\tilde{u}^2+\tilde{v}^2)d\tau dadx+\int_0^1\int_0^A(\tilde{u}^2(t,a,x)+\tilde{v}^2(t,a,x))dadx.
\end{aligned}\tag{3.101}$$

Taking now $\kappa_1 \geq \frac{1}{2}\max\left(\left(\frac{\|\beta_1\|_\infty^2}{16\varepsilon'} + \frac{1}{16\varepsilon'}\right), \left(\frac{\|\beta_2\|_\infty^2}{16\varepsilon'} + 16\varepsilon'\|\mu_3\|_\infty^2\right)\right)$ and thanks to the definitions of $\tilde{u}$ and $\tilde{v}$, inequality (3.101) is reduced to

$$\int_0^1\int_0^A (u^2(0,a,x)+v^2(0,a,x))dadx \leq e^{2\kappa_1 T}\int_0^1\int_0^A (u^2(t,a,x)+v^2(t,a,x))dadx. \tag{3.102}$$

Integrating (3.102) over $(\frac{T}{4}, \frac{3T}{4})$, we get

$$\int_0^1\int_0^A (u^2(0,a,x)+v^2(0,a,x))dadx \leq \frac{2e^{2\kappa_1 T}}{T}\int_0^1\int_0^A\int_{\frac{T}{4}}^{\frac{3T}{4}} (u^2(t,a,x) + v^2(t,a,x))dtdadx. \tag{3.103}$$

Henceforth, the crucial step to establish the observability inequality (3.92) is to show the existence of a positive constant $\hat{C}$ such that

$$\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{\frac{T}{4}}^{\frac{3T}{4}} (u^2(t,a,x)+v^2(t,a,x))dtdadx \leq \hat{C}\int_0^1\int_0^\delta (u_T^2(a,x)+v_T^2(a,x))dadx. \tag{3.104}$$

To this end, we will use the implicit formulas of u and v given, respectively, by (2.50) and (2.51), the formulas of the initial datum with respect to the age (2.52). Such a proof will be split on the cases when $a > t + (A - T)$ and $a \leq t + (A - T)$ (see again the two references [12] and [23] for a similar argumentation and also Remark 2.9).

In fact, if $a > t + (A - T)$, one has after a careful calculus

$$\begin{cases} u(t,a,.) = \int_0^{A-a} S(A-a-l)\beta_1(t,A-l,.)S(T-t)u_T(T-t,.)dl \\ + \int_0^{A-a} S(A-a-l)\Big[\beta_1(t,A-l,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \\ -\mu_3(m,0,.)v(m,0,.))dm\Big]dl - \int_0^{A-a} S(A-a-l)\mu_3(t,A-l,.)v(t,A-l,.)dl, \\ v(t,a,.) = \int_0^{A-a} L(A-a-l)\beta_2(t,A-l,.)L(T-t)v_T(T-t,.)dl \\ + \int_0^{A-a} L(A-a-l)\beta_2(t,A-l,.)\left(\int_t^T L(m-t)\beta_2(m,0,.)v(m,0,.)dm\right)dl, \end{cases} \tag{3.105}$$

where $(S(t))_{t\geq 0}$ and $(L(t))_{t\geq 0}$ are the semigroups defined after the relations (2.48) and (2.49). Thus, we claim from (3.105) that $\exists \tilde{M}_3, \tilde{M}_4 > 0$ such that

$$\begin{cases} \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u(t,a,x)|^2 dtdadx \leq \tilde{M_3} \int_0^1 \int_0^{\delta} |u_T(a,x)|^2 dadx \\ \qquad +\tilde{M_{13}} \int_0^1 \int_0^{\delta} |v_T(a,x)|^2 dadx, \\ \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} v^2(t,a,x) dtdadx \leq \tilde{M_4} \int_0^1 \int_0^{\delta} v_T^2(a,x) dadx. \end{cases} \tag{3.106}$$

The proofs of the two last inequalities are similar, so we will restrict ourselves to show the first one.

From the first equality of (3.105), one has

$$\begin{aligned} |u(t,a,.)| = |& \int_0^{A-a} S(A-a-l)\beta_1(t,A-l,.)S(T-t)u_T(T-t,.)dl \\ &+ \int_0^{A-a} S(A-a-l)\Big[\beta_1(t,A-l,.) \\ &\times \int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.)-\mu_3(m,0,.)v(m,0,.))dm\Big]dl \\ &- \int_0^{A-a} S(A-a-l)\mu_3(t,A-l,.)v(t,A-l,.)dl|. \end{aligned} \tag{3.107}$$

Subsequently,

$$\begin{aligned} |u(t,a,.)| \leq & \left|\int_0^{A-a} S(A-a-l)\beta_1(t,A-l,.)S(T-t)u_T(T-t,.)dl\right| \\ &+\Big|\int_0^{A-a} S(A-a-l)\Big[\beta_1(t,A-l,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \\ &-\mu_3(m,0,.)v(m,0,.))dm\Big]dl\Big| \\ &+\left|\int_0^{A-a} S(A-a-l)\mu_3(t,A-l,.)v(t,A-l,.)dl\right| \\ \leq & \int_0^{A-a} |S(A-a-l)\beta_1(t,A-l,.)S(T-t)u_T(T-t,.)|dl \\ &+\int_0^{A-a} \Big|S(A-a-l)\Big[\beta_1(t,A-l,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \\ &-\mu_3(m,0,.)v(m,0,.))dm\Big]\Big|dl \\ &+\int_0^{A-a} |S(A-a-l)\mu_3(t,A-l,.)v(t,A-l,.)|dl. \end{aligned} \tag{3.108}$$

With the variable change $r = A - l$, (3.108) becomes

$$
\begin{aligned}
|u(t,a,.)| \leq & \int_a^A |S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|dr \\
& + \int_a^A \left| S(r-a)\Big[\beta_1(t,r,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \right. \\
& \left. -\mu_3(m,0,.)v(m,0,.))dm\Big] \right| dr \\
& + \int_a^A |S(r-a)\mu_3(t,r,.)v(t,r,.)|dr.
\end{aligned} \tag{3.109}
$$

Since $(S(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup, then

$$
\|S(r-a)\| \leq Me^{\lambda_3(r-a)} \leq Me^{A\lambda_3}, \tag{3.110}
$$

where M and λ_3 are the same in (2.53).

Hence,

$$
\begin{aligned}
|u(t,a,.)| \leq & \int_a^A |S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|dr \\
& + \int_a^A Me^{A\lambda_3}\left| \Big[\beta_1(t,r,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \right. \\
& \left. -\mu_3(m,0,.)v(m,0,.))dm\Big] \right| dr + \int_a^A Me^{A\lambda_3}|\mu_3(t,r,.)v(t,r,.)|dr.
\end{aligned}
$$

Afterward,

$$
\begin{aligned}
|u(t,a,.)|^2 \leq & [\int_a^A |S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|dr \\
& + \int_a^A Me^{A\lambda_3}\left| \Big[\beta_1(t,r,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.) \right. \\
& \left. -\mu_3(m,0,.)v(m,0,.)dm)\Big] \right| dr \\
& + \int_a^A Me^{A\lambda_3}|\mu_3(t,r,.)v(t,r,.)|]^2 dr \\
\leq & \; 3\left(\int_a^A |S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|dr\right)^2
\end{aligned}
$$

$$+3\Big(\int_a^A Me^{A\lambda_3}\Big|\Big[\beta_1(t,r,.)\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.)$$
$$-\mu_3(m,0,.)v(m,0,.))dm\Big]\Big|dr\Big)^2$$
$$+3\left(\int_a^A Me^{A\lambda_3}|\mu_3(t,r,.)v(t,r,.)|dr\right)^2. \tag{3.111}$$

Applying now Hölder inequality to (3.111), we obtain

$$|u(t,a,.)|^2 \le 3\int_a^A A|S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|^2 dr$$
$$+3\int_a^A M^2e^{2A\lambda_3}\Big[\beta_1(t,r,.)\Big(\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.)$$
$$-\mu_3(m,0,.)v(m,0,.))dm\Big)\Big]^2 dr$$
$$+3\int_a^A M^2e^{2A\lambda_3}|\mu_3(t,r,.)v(t,r,.)|^2 dr. \tag{3.112}$$

On the other hand, the relation

$$\|S(m-t)\| \le Me^{\lambda_3(m-t)} \le Me^{T\lambda_3}, \tag{3.113}$$

together with hypotheses (2.6) on β_1 and again Hölder inequality, lead, respectively, to the following successive estimates:

$$3\int_a^A M^2e^{2A\lambda_3}\Big[\beta_1(t,r,.)\Big(\int_t^T S(m-t)(\beta_1(m,0,.)u(m,0,.)$$
$$-\mu_3(m,0,.)v(m,0,.))dm\Big)\Big]^2 dr$$
$$\le 3AM^4e^{2(A+T)\lambda_3}\int_a^A\Big[\Big(\int_t^T \beta_1(t,r,.)(\beta_1(m,0,.)u(m,0,.)$$
$$-\mu_3(m,0,.)v(m,0,.))dm\Big)\Big]^2 dr$$
$$\le 3AM^4e^{2(A+T)\lambda_3}\max(\|\beta_1\|_\infty^2,\|\mu_3\|_\infty^2)$$
$$\times\int_a^A\left(\int_t^T \beta_1(t,r,.)(u(m,0,.)-v(m,0,.))dm\right)^2 dr$$

$$\leq 6ATM^4e^{2(A+T)\lambda_3}\max(\|\beta_1\|_\infty^2, \|\mu_3\|_\infty^2)$$
$$\times \int_a^A \int_t^T \beta_1^2(t,r,.)(u^2(m,0,.)+v^2(m,0,.))dmdr, \qquad (3.114)$$

wherein M and λ_3 are the same as in (2.53).

The combination between (3.112) and (3.114) steers to the following inequality:

$$|u(t,a,.)|^2 \leq 3\int_a^A A|S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|^2dr$$
$$+6ATM^4e^{2(A+T)\lambda_3}\max(\|\beta_1\|_\infty^2, \|\mu_3\|_\infty^2)$$
$$\times \int_a^A \int_t^T \beta_1^2(t,r,.)(u^2(m,0,.)+v^2(m,0,.))dmdr$$
$$+3\int_a^A M^2e^{2A\lambda_3}|\mu_3(t,r,.)v(t,r,.)|^2dr. \qquad (3.115)$$

On the other hand and keeping in the mind that $(L(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup, the relation (3.113) is applied to $(L(t))_{t\geq 0}$, the formula of v in (2.51); a similar evidence as the one of (3.115) leads to

$$|v(t,r,.)|^2 \leq 2\int_r^A A|L(r_1-r)\beta_2(t,r_1,.)L(T-t)v_T(T-t,.)|^2dr_1$$
$$+2A(M_1)^2e^{2\lambda_4(A+T)}\|\beta_2\|_\infty^2 \int_r^A \int_t^T \beta_2^2(t,r_1,.)v^2(m,0,.)dmdr_1$$
$$\leq 2\int_r^A A|L(r_1-r)\beta_2(t,r_1,.)L(T-t)v_T(T-t,.)|^2dr_1$$
$$+2A^2(M_1)^2e^{2\lambda_4(A+T)}\|\beta_2\|_\infty^4 \int_t^T v^2(m,0,.)dm, \qquad (3.116)$$

where M_1 and λ_4 are defined in (2.56).

Integrating over (t, T) inequality (2.57), we get

$$\int_t^T |v(m,0,.)|^2dm \leq 2(M_1)^2e^{2\lambda_4 T}\int_t^T |v_T(T-m,.)|^2dm$$
$$+\tilde{M}_5\int_t^T \int_m^T |v_T(T-s,.)|^2dsdm$$
$$\leq 2(M_1)^2e^{2\lambda_4 T}\int_0^{T-t} |v_T(m_1,.)|^2dm_1 + \tilde{M}_5\int_t^T \int_0^{m_1} |v_T(s_1,.)|^2ds_1dm$$
$$\leq 2(M_1)^2e^{2\lambda_4 T}\int_0^{T-t} |v_T(m_1,.)|^2dm_1 + (T-t)\tilde{M}_5\int_0^{T-t} |v_T(s_1,.)|^2ds_1$$

$$\leq 2(M_1)^2 e^{2\lambda_4 T} \int_0^T |v_T(m_1,.)|^2 dm_1 + T\tilde{M}_5 \int_0^T |v_T(s_1,.)|^2 ds_1, \tag{3.117}$$

with $m_1 := T - m$ and $s_1 := T - s$, $\tilde{M}_5$ is the positive constant defined by (2.58), and s is not the parameter of Carleman estimates.

To overcome the ambiguity, we set $\int_0^T |v_T(m_1,.)|^2 dm_1 := \int_0^T |v_T(a_1,.)|^2 da_1$ and $\int_0^T |v_T(s_1,.)|^2 ds_1 := \int_0^T |v_T(a_1,.)|^2 da_1$. Thus, (3.117) becomes

$$\int_t^T |v(m,0,.)|^2 dm \leq (T\tilde{M}_5 + 2(M_1)^2 e^{2\lambda_4 T}) \int_0^T |v_T(a_1,.)|^2 da_1. \tag{3.118}$$

As a conclusion of inequalities (3.116) and (3.118), one has

$$\begin{aligned}|v(t,r,.)|^2 \leq\, & 2\int_r^A A|L(r_1-r)\beta_2(t,r_1,.)L(T-t)v_T(T-t,.)|^2 dr_1 \\ & +2A^2(M_1)^2 e^{2\lambda_4(A+T)}\|\beta_2\|_\infty^4 (T\tilde{M}_5 + 2(M_1)^2 e^{2\lambda_4 T}) \int_0^T |v_T(a_1,.)|^2 da_1.\end{aligned} \tag{3.119}$$

Thanks to the same arguments used to obtain (2.65), we can show that

$$\begin{aligned}\int_t^T |u(m,0,.)|^2 dm \leq\, & \left(2M^2 e^{2\lambda_3 T} + 2M^2 T e^{2\lambda_3 T}\tilde{M}_8\right) \int_0^T |u_T(a_1,.)|^2 da_1 \\ & + \left((T^2+T^3)\tilde{M}_8 + (T+T^2)\tilde{M}_6\right) \int_0^T |v_T(a_1,.)|^2 da_1,\end{aligned} \tag{3.120}$$

with $\tilde{M}_6$ and $\tilde{M}_8$ are defined by (2.62).

As in the above, we will choose in the sequel of this proof the variable a_1 to unify the integral variables that are in $(0,T)$ and represent the age variable.

Consequently, by inequalities (3.115), (3.118), (3.119), and (3.120), it follows that

$$\begin{aligned}|u(t,a,.)|^2 \leq\, & 3A\int_a^A |S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|^2 dr \\ & +6A^2 T M^4 e^{2(A+T)\lambda_3}\|\beta_1\|_\infty^2 \max(\|\beta_1\|_\infty^2, \|\mu_3\|_\infty^2) \\ & \times\Bigg[\left(2M^2 e^{2\lambda_3 T} + 2M^2 T e^{2\lambda_3 T}\tilde{M}_8\right)\int_0^T |u_T^2(a_1,.)|^2 da_1 \\ & +((T^2+T^3)\tilde{M}_8 + (T^2+T)\tilde{M}_6 + T\tilde{M}_5 + 2(M_1)^2 e^{2\lambda_4 T}) \int_0^T |v_T(a_1,.)|^2 da_1\Bigg] \\ & +3\|\mu_3\|_\infty^2 M^2 e^{2A\lambda_3}\end{aligned} \tag{3.121}$$

$$\times\Big[2A\int_a^A\int_r^A |L(r_1-r)\beta_2(t,r_1,.)L(T-t)v_T(T-t,.)|^2dr_1dr$$
$$+2A^{3(}M_1)^2e^{2(A+T)\lambda_4}\|\beta_2\|_\infty^4(T\tilde{M}_5+2(M_1)^2e^{2\lambda_4 T})\int_0^T|v_T(a_1,.)|^2da_1\Big].$$

Put

$$\begin{cases} \tilde{M_{11}} := 6A^2TM^4e^{2(A+T)\lambda_3}\|\beta_1\|_\infty^2\max(\|\beta_1\|_\infty^2,\|\mu_3\|_\infty^2)(2M^2e^{2\lambda_3 T} \\ \qquad +2M^2Te^{2\lambda_3 T}\tilde{M}_8), \\ \tilde{M_{12}} := 6A^2TM^4e^{2(A+T)\lambda_3}\|\beta_1\|_\infty^2\max(\|\beta_1\|_\infty^2,\|\mu_3\|_\infty^2)((T^2+T^3)\tilde{M}_8 \\ \qquad +(T^2+T)\tilde{M}_6+T\tilde{M}_5+2(M_1)^2e^{2\lambda_4 T}) \\ +6A^3M^2(M_1)^2\|\mu_3\|_\infty^2\|\beta_2\|_\infty^4e^{2(A+T)\lambda_4+2A\lambda_3}(T\tilde{M}_5+2(M_1)^2e^{2\lambda_4 T}). \end{cases} \tag{3.122}$$

Under these notations, (3.122) becomes

$$|u(t,a,.)|^2 \le 3A\int_a^A|S(r-a)\beta_1(t,r,.)S(T-t)u_T(T-t,.)|^2dr$$
$$+6A\|\mu_3\|_\infty^2M^2e^{2A\lambda_3}$$
$$\int_a^A\int_r^A|L(r_1-r)\beta_2(t,r_1,.)L(T-t)v_T(T-t,.)|^2dr_1dr$$
$$+\tilde{M_{11}}\int_0^T|u_T(a_1,.)|^2da_1+\tilde{M_{12}}\int_0^T|v_T(a_1,.)|^2da_1. \tag{3.123}$$

An integration of (3.123) over $(\frac{T}{4},\frac{3T}{4})\times(0,\delta-\frac{3T}{4})\times(0,1)$ steers to

$$\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{\frac{T}{4}}^{\frac{3T}{4}}|u(t,a,x)|^2dtdadx$$
$$\le 3A\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_a^A|S(r-a)\beta_1(t,r,x)$$
$$\times S(T-t)u_T(T-t,x)|^2drdtdadx$$
$$+6A\|\mu_3\|_\infty^2M^2e^{2A\lambda_3}\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_a^A\int_r^A|L(r_1-r)\beta_2(t,r_1,x)$$
$$\times L(T-t)v_T(T-t,x)|^2dr_1drdtdadx$$
$$+\tilde{M_{11}}\int_0^1\int_0^T|u_T(a_1,x)|^2da_1dx+\tilde{M_{12}}\int_0^1\int_0^T|v_T(a_1,x)|^2da_1dx. \tag{3.124}$$

The variable change $\tilde{s} = T - t$ in the right-hand side of (3.124) allows us to say that

$$\begin{aligned}
&\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u(t,a,x)|^2 dt da dx \\
&\quad \le 3A \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_a^A |S(r-a)\beta_1(T-\tilde{s},r,x) \\
&\qquad \times S(\tilde{s})u_T(\tilde{s},x)|^2 dr d\tilde{s} da dx \\
&\qquad +6A\|\mu_3\|_\infty^2 M^2 e^{2A\lambda_3} \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_a^A \int_r^A |L(r_1-r)\beta_2(T-\tilde{s},r_1,x) \\
&\qquad \times L(\tilde{s})v_T(\tilde{s},x)|^2 dr_1 dr d\tilde{s} da dx \\
&\quad +\tilde{M}_{11} \int_0^1 \int_0^T |u_T(a_1,x)|^2 da_1 dx + \tilde{M}_{12} \int_0^1 \int_0^T |v_T(a_1,x)|^2 da_1 dx.
\end{aligned} \tag{3.125}$$

Since $T < \delta$, exploiting again the relation (3.113) for the $\mathcal{C}_0$-semigroups $(S(t))_{t\ge0}$ and $(L(t))_{t\ge0}$ (with λ_4 and M_1 instead, respectively, λ_3 and M) and with the help of assumptions (2.6) on β_1 and β_2, one can transform (3.125) into

$$\begin{aligned}
&\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u(t,a,x)|^2 dt da dx \\
&\quad \le \left(\tilde{M}_{11} + 3A^2M^4e^{2\lambda_3(A+T)}\left(\delta - \frac{3T}{4}\right)\|\beta_1\|_\infty^2\right) \int_0^\delta |u_T(a,x)|^2 da_1 da dx \\
&\qquad + \left(\tilde{M}_{12} + 6A^3\|\beta_2\|_\infty^2\|\mu_3\|_\infty^2 M^2(M_1)^4 e^{2\lambda_4(A+T)+2\lambda_3 A}\left(\delta - \frac{3T}{4}\right)\right) \\
&\qquad \times \int_0^\delta |v_T(a,x)|^2 da dx.
\end{aligned} \tag{3.126}$$

Hence,

$$\begin{aligned}
\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u(t,a,x)|^2 dt da dx &\le \tilde{M}_3 \int_0^1 \int_0^\delta |u_T(a,x)|^2 da dx \\
&\quad +\tilde{M}_{13} \int_0^1 \int_0^\delta |v_T(a,x)|^2 da dx,
\end{aligned} \tag{3.127}$$

where

$$\begin{cases} \tilde{M_3} := \tilde{M_{11}} + 3A^2M^4e^{2\lambda_3(A+T)}\left(\delta - \frac{3T}{4}\right), \\ \tilde{M_{13}} := \tilde{M_{12}} + 6A^3\|\beta_2\|_\infty^2\|\mu_3\|_\infty^2 M^2(M_1)^4 e^{2\lambda_4(A+T)+2\lambda_3 A}\left(\delta - \frac{3T}{4}\right), \end{cases} \tag{3.128}$$

and $\tilde{M_{11}}$ and $\tilde{M_{12}}$ are given in (3.122).

Consequently, the first relation of (3.106) is achieved. Likewise, we can prove the second inequality of (3.106) using the same procedure as we mentioned previously and the inequality (3.119).

Finally, the inequality (3.104) is true in the case where $a > t + (A - T)$.

Let us now address to the case when $a \le t + (A - T)$.

Like the first case, mixing the implicit formula (2.50) and (2.51) (in the case when $a \le t + (A - T)$) and (2.52), we obtain

$$\begin{cases} u(t,a,.) = S(T-t)u_T(a+T-t,.) \\ + \int_t^T S(l-t)\beta_1(l,a,.)\Big[S(T-l)u_T(T-l,.) \\ \qquad + \int_l^T S(r-l)\left(\beta_1(r,0,.)u(r,0,.) - \mu_3(r,0,.)v(r,0.)\right)dr\Big]dl \\ - \int_t^T S(l-t)\mu_3(l,a,.)\Big[L(T-l)v_T(a+T-l,.) \\ \qquad + \int_l^T L(r-l)\beta_2(r,a,.)v(r,0,.)dr\Big]dl, \\ v(t,a,.) = L(T-t)v_T(a+T-t,.) + \int_t^T L(l-t)\beta_2(l,a,.) \\ \qquad \Big[L(T-l)v_T(T-l,.) + \int_l^T L(r-l)\beta_2(r,0,.)v(r,0,.)dr\Big]dl. \end{cases} \tag{3.129}$$

Set $\forall(t,a) \in (0,T)\times(0,A)$, $R(t,a,.) := \int_t^T S(l-t)\beta_1(l,a,.)\Big[S(T-l)u_T(T-l,.) + \int_l^T S(r-l)\left(\beta_1(r,0,.)u(r,0,.) - \mu_3(r,0,.)v(r,0.)\right)dr\Big]dl - \int_t^T S(l-t)\mu_3(l,a,.)\Big[L(T-l)v_T(a+T-l,.) + \int_l^T L(r-l)\beta_2(r,a,.) v(r,0,.)dr\Big]dl.$

Following this, the first solution u becomes

$$\forall(t,a) \in (0,T)\times(0,A), \quad u(t,a,.) = S(T-t)u_T(a+T-t,.) + R(t,a,.). \tag{3.130}$$

If one mimics the same blend of semigroup theory and Hölder inequality used in the computations in the case "$a > t + (A - T)$," we can establish the existence of a positive constant $\tilde{M_{14}}$ and $\tilde{M_{15}}$ such that

$$\int_0^1\int_0^{\delta-\frac{3T}{4}}\int_{\frac{T}{4}}^{\frac{3T}{4}} |R(t,a,x)|^2 dtdadx \le \tilde{M_{14}}\int_0^1\int_0^\delta |u_T(a,x)|^2 dadx$$
$$+\tilde{M_{15}}\int_0^1\int_0^\delta |v_T(a,x)|^2 dadx. \tag{3.131}$$

On the other hand, we can observe that

$$\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |S(T-t)u_T(a+T-t,x)|^2 dt da dx$$
$$= \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |S(\tilde{t})u_T(a+\tilde{t},x)|^2 d\tilde{t} da dx$$
$$\leq M^2 e^{\frac{3\lambda_3 T}{2}} \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u_T(a+\tilde{t},x)|^2 d\tilde{t} da dx,$$

where M and λ_3 are the same of (2.53) and $\tilde{t} := T - t$.

With the variable change $\tilde{a} = a + \tilde{t}$, it follows that

$$\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |S(T-t)u_T(a+T-t,x)|^2 dt da dx$$
$$\leq M^2 e^{\frac{3\lambda_3 T}{2}} \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{a+\frac{T}{4}}^{a+\frac{3T}{4}} |u_T(\tilde{a},x)|^2 d\tilde{a} da dx$$
$$\leq M^2 e^{\frac{3\lambda_3 T}{2}} \int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\delta} |u_T(\tilde{a},x)|^2 d\tilde{a} da dx$$
$$\leq \left(\delta - \frac{3T}{4}\right) M^2 e^{\frac{3\lambda_3 T}{2}} \int_0^1 \int_{\frac{T}{4}}^{\delta} |u_T(\tilde{a},x)|^2 d\tilde{a} dx$$
$$\leq \left(\delta - \frac{3T}{4}\right) M^2 e^{\frac{3\lambda_3 T}{2}} \int_0^1 \int_0^{\delta} |u_T(\tilde{a},x)|^2 d\tilde{a} dx, \tag{3.132}$$

and the second inequality obtained in (3.132) is an outcome of the inclusion $(a + \frac{T}{4}, a + \frac{3T}{4}) \subset (\frac{T}{4}, \delta), \quad \forall a \in (0, \delta - \frac{3T}{4})$.

It is clear now from (3.130), (3.131), and (3.132) that there exists a positive constant $\tilde{M_{16}}$ such that

$$\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |u(t,a,x)|^2 dt da dx \leq \tilde{M_{16}} \int_0^1 \int_0^{\delta} |u_T(a,x)|^2 da dx$$
$$+\tilde{M_{15}} \int_0^1 \int_0^{\delta} |u_T(a,x)|^2 da dx, \tag{3.133}$$

accurately, $\tilde{M_{16}} := (\tilde{M_{14}} + (\delta - \frac{3T}{4}) M^2 e^{\frac{3\lambda_3 T}{2}})$.

In the same manner, one can bring out a similar inequality for the solution v through the second implicit formula in (3.129), i.e., the existence of $\tilde{M_{17}} > 0$ such that

$$\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} |v(t,a,x)|^2 dtdadx \le \tilde{M_{17}} \int_0^1 \int_0^{\delta} |v_T(a,x)|^2 dadx. \quad (3.134)$$

Hence, summing up (3.133) and (3.134), the inequality (3.104) holds in the current case.

Abstractly,

$$\int_0^1 \int_0^{\delta-\frac{3T}{4}} \int_{\frac{T}{4}}^{\frac{3T}{4}} (u^2(t,a,x) + v^2(t,a,x)) dtdadx$$
$$\le \hat{C} \int_0^1 \int_0^{\delta} (u_T^2(a,x) + v_T^2(a,x)) dadx$$

is satisfied in both cases $a > t + (A - T)$ and $a \le t + (A - T)$, where $\hat{C}$ is a positive constant.

The last inequality together with (3.103) implies that

$$\int_0^1 \int_0^A (u^2(0,a,x) + v^2(0,a,x)) dadx$$
$$\le \frac{2e^{2\kappa_1 T}}{T} \int_0^1 \int_{\delta-\frac{3T}{4}}^{\delta} \int_{\frac{T}{4}}^{\frac{3T}{4}} (u^2(t,a,x) + v^2(t,a,x)) dtdadx$$
$$+ \hat{C} \frac{2e^{2\kappa_1 T}}{T} \int_0^1 \int_0^{\delta} (u_T^2(a,x) + v_T^2(a,x)) dadx.$$

Subsequently, with the help of the Hardy–Poincaré inequality and the definitions of $\varphi_i, \quad i = 1, 2$, stated in (2.19), we arrive to

$$\int_0^1 \int_0^A (u^2(0,a,x) + v^2(0,a,x)) dadx$$
$$\le \hat{C} \frac{2e^{2\kappa_1 T}}{T} \int_0^1 \int_0^{\delta} (u_T^2(a,x) + v_T^2(a,x)) dadx$$
$$+ \tilde{M_{18}} \frac{2e^{2\kappa_1 T}}{T} \Bigg(\int_0^1 \int_{\delta-\frac{3T}{4}}^{\delta} \int_{\frac{T}{4}}^{\frac{3T}{4}} s\Theta k_1(x) u_x^2(t,a,x) e^{2s\varphi_1} dtdadx$$
$$+ \int_0^1 \int_{\delta-\frac{3T}{4}}^{\delta} \int_{\frac{T}{4}}^{\frac{3T}{4}} s\Theta k_2(x) v_x^2(t,a,x) e^{2s\varphi_2} dtdadx \Bigg). \quad (3.135)$$

The proof of observability inequality (3.92) is finished thanks to (3.135) and applying the Carleman estimate (2.69) stated in Theorem 2.10 on the second term in the right-hand side of last inequality. □

With the aid of the observability inequality (3.92), we are now able to show the result of null controllability (1.4) related to the cascade model (1.1).

3.2 Null Controllability Result

This section is interested in the null controllability property (1.4) of system (1.1). It is stipulated as follows:

Theorem 3.2 *Assume that* (2.5) *and* (2.6) *hold. Let* $A, T > 0$ *be given and fixed such that* $T < \delta$ *where* $\delta \in (0, A)$ *fixed small enough. Then,* $\forall (y_0, p_0) \in L^2(Q_A) \times L^2(Q_A)$*, there exists a control* $\vartheta \in L^2(q)$ *depending on* δ *such that the associated solution* (y, p) *of* (1.1) *verifies*

$$\begin{cases} y(T, a, x) = 0, & a.e \text{ in } (\delta, A) \times (0, 1), \\ p(T, a, x) = 0, & a.e \text{ in } (\delta, A) \times (0, 1). \end{cases} \tag{3.136}$$

Recall that $q = (0, T) \times (0, A) \times \omega$*.*

Proof Let $\epsilon > 0$ and consider the following cost function:

$$J_\epsilon = \frac{1}{2\epsilon} \int_0^1 \int_\delta^A (y^2(T, a, x) + p^2(T, a, x)) dadx + \frac{1}{2} \int_q \vartheta^2(t, a, x) dtdadx.$$

We can prove that J_ϵ is continuous, convex, and coercive. Then, it admits at least one minimizer ϑ_ϵ, and we have

$$\vartheta_\epsilon = -u_\epsilon(t, a, x)\chi_\omega(x) \quad \text{in } Q, \tag{3.137}$$

with u_ϵ is the solution of the following system:

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} + \frac{\partial u_\epsilon}{\partial a} + (k_1(x)(u_\epsilon)_x)_x - \mu_1(t, a, x)u_\epsilon - \mu_3(t, a, x)v_\epsilon &= -\beta_1 u_\epsilon(t, 0, x) && \text{in} \quad Q, \\ u_\epsilon(t, a, 1) = u_\epsilon(t, a, 0) &= 0 && \text{on} \quad (0, T) \times (0, A), \\ u_\epsilon(T, a, x) &= \frac{1}{\epsilon} y_\epsilon(T, a, x)\chi_{(\delta, A)} && \text{in} \quad (0, A) \times (0, 1); \\ u_\epsilon(t, A, x) &= 0 && \text{in} \quad Q_T, \end{aligned} \tag{3.138}$$

with v_ϵ the solution of the following system:

$$\begin{aligned} \frac{\partial v_\epsilon}{\partial t} + \frac{\partial v_\epsilon}{\partial a} + (k_2(x)(v_\epsilon)_x)_x - \mu_2(t, a, x)v_\epsilon &= -\beta_2 v_\epsilon(t, 0, x) && \text{in} \quad Q, \\ v_\epsilon(t, a, 1) = v_\epsilon(t, a, 0) &= 0 && \text{on} \quad (0, T) \times (0, A), \\ v_\epsilon(T, a, x) &= \frac{1}{\epsilon} p_\epsilon(T, a, x)\chi_{(\delta, A)} && \text{in} \quad Q_A, \\ v_\epsilon(t, A, x) &= 0 && \text{in} \quad Q_T, \end{aligned} \tag{3.139}$$

where (y_ϵ, p_ϵ) is the solution of (1.1) associated to the control ϑ_ϵ.

Multiplying the first equation of (3.138) by y_ϵ and integrating over Q, we obtain

$$
\begin{aligned}
&\int_Q y_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}+\frac{\partial u_\epsilon}{\partial a}+(k_1(x)(u_\epsilon)_x)_x-\mu_1(t,a,x)u_\epsilon-\mu_3(t,a,x)v_\epsilon\right)dtdadx\\
&=-\int_Q u_\epsilon\left(\frac{\partial y_\epsilon}{\partial t}+\frac{\partial y_\epsilon}{\partial a}-(k_1(x)(y_\epsilon)_x)_x+\mu_1(t,a,x)y_\epsilon\right)dtdadx\\
&\quad-\int_Q \mu_3(t,a,x)y_\epsilon v_\epsilon dtdadx+\int_0^1\int_0^A y_\epsilon(T,a,x)u_\epsilon(T,a,x)dadx\\
&\quad-\int_0^1\int_0^A y_\epsilon(0,a,x)u_\epsilon(0,a,x)dadx-\int_0^1\int_0^T y_\epsilon(t,0,x)u_\epsilon(t,0,x)dtdx.
\end{aligned}
\tag{3.140}
$$

Similarly, multiplying the first equation of (3.139) by p_ϵ and integrating over Q

$$
\begin{aligned}
&\int_Q p_\epsilon\left(\frac{\partial v_\epsilon}{\partial t}+\frac{\partial v_\epsilon}{\partial a}+(k_2(x)(v_\epsilon)_x)_x-\mu_2(t,a,x)v_\epsilon\right)dtdadx\\
&=-\int_Q v_\epsilon\left(\frac{\partial p_\epsilon}{\partial t}+\frac{\partial p_\epsilon}{\partial a}-(k_2(x)(p_\epsilon)_x)_x+\mu_2(t,a,x)p_\epsilon\right)dtdadx\\
&\quad+\int_0^1\int_0^A p_\epsilon(T,a,x)v_\epsilon(T,a,x)dadx\\
&\quad-\int_0^1\int_0^A p_\epsilon(0,a,x)v_\epsilon(0,a,x)dadx-\int_0^1\int_0^T p_\epsilon(t,0,x)v_\epsilon(t,0,x)dtdx.
\end{aligned}
\tag{3.141}
$$

Hence, summing side by side (3.140) and (3.141), we arrive to

$$
\begin{aligned}
&\int_Q p_\epsilon\left(\frac{\partial v_\epsilon}{\partial t}+\frac{\partial v_\epsilon}{\partial a}+(k_2(x)(v_\epsilon)_x)_x-\mu_2(t,a,x)v_\epsilon\right)dtdadx\\
&\quad+\int_Q y_\epsilon\Bigg(\frac{\partial u_\epsilon}{\partial t}+\frac{\partial u_\epsilon}{\partial a}+(k_1(x)(u_\epsilon)_x)_x\\
&\quad-\mu_1(t,a,x)u_\epsilon-\mu_3(t,a,x)v_\epsilon\Bigg)dtdadx\\
&=-\int_Q u_\epsilon\left(\frac{\partial y_\epsilon}{\partial t}+\frac{\partial y_\epsilon}{\partial a}-(k_1(x)(y_\epsilon)_x)_x+\mu_1(t,a,x)y_\epsilon\right)dtdadx\\
&\quad+\int_0^1\int_0^A y_\epsilon(T,a,x)u_\epsilon(T,a,x)dadx-\int_0^1\int_0^A y_\epsilon(0,a,x)u_\epsilon(0,a,x)dadx
\end{aligned}
$$

$$-\int_0^1\int_0^T y_\epsilon(t,0,x)u_\epsilon(t,0,x)dtdx$$
$$-\int_Q v_\epsilon\left(\frac{\partial p_\epsilon}{\partial t}+\frac{\partial p_\epsilon}{\partial a}-(k_2(x)(p_\epsilon)_x)_x\right.$$
$$\left.+\mu_2(t,a,x)p_\epsilon+\mu_3(t,a,x)y_\epsilon\right)dtdadx$$
$$+\int_0^1\int_0^A p_\epsilon(T,a,x)v_\epsilon(T,a,x)dadx$$
$$-\int_0^1\int_0^A p_\epsilon(0,a,x)v_\epsilon(0,a,x)dadx$$
$$-\int_0^1\int_0^T p_\epsilon(t,0,x)v_\epsilon(t,0,x)dtdx. \tag{3.142}$$

Thus,

$$\int_Q y_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}+\frac{\partial u_\epsilon}{\partial a}+(k_1(x)(u_\epsilon)_x)_x-\mu_1(t,a,x)u_\epsilon-\mu_3(t,a,x)v_\epsilon\right)dtdadx$$
$$+\int_Q p_\epsilon\left(\frac{\partial v_\epsilon}{\partial t}+\frac{\partial v_\epsilon}{\partial a}+(k_2(x)(v_\epsilon)_x)_x-\mu_2(t,a,x)v_\epsilon\right)dtdadx$$
$$=-\int_Q u_\epsilon\vartheta_\epsilon\chi_\omega dtdadx+\int_0^1\int_\delta^A\frac{1}{\epsilon}y_\epsilon^2(T,a,x)dadx$$
$$-\int_0^1\int_0^A y_\epsilon(0,a,x)u_\epsilon(0,a,x)dadx$$
$$-\int_0^1\int_0^A\int_0^T\beta_1 y_\epsilon(t,a,x)u_\epsilon(t,0,x)dtdadx$$
$$-\int_0^1\int_0^A p_\epsilon(0,a,x)v_\epsilon(0,a,x)dadx$$
$$-\int_0^1\int_0^A\int_0^T\beta_2 p_\epsilon(t,a,x)v_\epsilon(t,0,x)dtdadx$$
$$+\int_0^1\int_\delta^A\frac{1}{\epsilon}p_\epsilon^2(T,a,x)dadx. \tag{3.143}$$

Also, one can see that

$$\int_Q y_\epsilon\left(\frac{\partial u_\epsilon}{\partial t}+\frac{\partial u_\epsilon}{\partial a}+(k_1(x)(u_\epsilon)_x)_x-\mu_1(t,a,x)u_\epsilon-\mu_3(t,a,x)v_\epsilon\right)dtdadx$$

$$= -\int_0^1\int_0^A\int_0^T \beta_1 y_\epsilon(t,a,x)u_\epsilon(t,0,x)dtdadx \tag{3.144}$$

and

$$\int_Q p_\epsilon\left(\frac{\partial v_\epsilon}{\partial t}+\frac{\partial v_\epsilon}{\partial a}+(k_2(x)(v_\epsilon)_x)_x-\mu_2(t,a,x)v_\epsilon\right)dtdadx$$
$$= -\int_0^1\int_0^A\int_0^T \beta_2 p_\epsilon(t,a,x)v_\epsilon(t,0,x)dtdadx. \tag{3.145}$$

Consequently, (3.137), (3.143), (3.144), and (3.145) lead to

$$\int_q \vartheta_\epsilon^2 dtdadx+\int_0^1\int_\delta^A \frac{1}{\epsilon}y_\epsilon^2(T,a,x)dadx+\int_0^1\int_\delta^A \frac{1}{\epsilon}p_\epsilon^2(T,a,x)dadx$$
$$=\int_0^1\int_0^A y_\epsilon(0,a,x)u_\epsilon(0,a,x)dadx+\int_0^1\int_0^A p_\epsilon(0,a,x)v_\epsilon(0,a,x)dadx. \tag{3.146}$$

Applying Young's inequality to the right-hand side of (3.146)

$$\int_q \vartheta_\epsilon^2 dtdadx+\frac{1}{\epsilon}\int_0^1\int_\delta^A (y_\epsilon^2(T,a,x)+p_\epsilon^2(T,a,x))dadx$$
$$\leq \frac{1}{4C_{obs,\delta}}\int_0^1\int_0^A (u_\epsilon^2(0,a,x)+v_\epsilon^2(0,a,x))dadx$$
$$+C_{obs,\delta}\int_0^1\int_0^A (y_0^2(a,x)+p_0^2(a,x))dadx$$

with $C_{obs,\delta}$ is the constant of observability inequality (3.92).

This together with observability inequality (3.92) allows us to say that

$$\int_q \vartheta_\epsilon^2 dtdadx+\frac{1}{\epsilon}\int_0^1\int_\delta^A (y_\epsilon^2(T,a,x)+p_\epsilon^2(T,a,x))dadx$$
$$\leq \frac{1}{4}\left(\int_q u_\epsilon^2 dtdadx+\int_0^1\int_0^\delta (u_\epsilon^2(T,a,x)+v_\epsilon^2(T,a,x))dadx\right)$$
$$+C_{obs,\delta}\int_0^1\int_0^A (y_0^2(a,x)+p_0^2(a,x))dadx.$$

Since $\forall(a,x)\in(0,\delta)\times(0,1),\quad u_\epsilon(T,a,x)=v_\epsilon(T,a,x)=0$, and keeping in the mind the relation (3.137), then the last inequality reads as

$$\frac{3}{4}\int_q \vartheta_\epsilon^2 dtdadx + \frac{1}{\epsilon}\int_0^1\int_\delta^A (y_\epsilon^2(T,a,x) + p_\epsilon^2(T,a,x))dadx$$
$$\leq C_{obs,\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))dadx.$$

Hence, it follows that

$$\begin{cases} \int_q \vartheta_\epsilon^2 dtdadx \leq \frac{4C_{obs,\delta}}{3}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))dadx, \\ \int_0^1\int_\delta^A y_\epsilon^2(T,a,x)dadx \leq \epsilon C_{obs,\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))dadx, \\ \int_0^1\int_\delta^A p_\epsilon^2(T,a,x)dadx \leq \epsilon C_{obs,\delta}\int_0^1\int_0^A (y_0^2(a,x) + p_0^2(a,x))dadx. \end{cases} \tag{3.147}$$

Then, we can extract two subsequences of (y_ϵ, p_ϵ) and ϑ_ϵ also denoted by (y_ϵ, p_ϵ) and ϑ_ϵ that converge weakly toward (y, p) and ϑ in $L^2((0,T)\times(0,A), H^1_{k_1}(0,1)\times H^1_{k_2}(0,1))$ and $L^2(q)$, respectively. Now, by a variational technique, we prove that (y, p) is a solution of (1.1) corresponding to the control ϑ, and, by the second and third estimates of (3.147), (y, p) satisfies (1.4). Another deduction from (3.147) specially the first inequality is that the researched control ϑ depends on δ. □

Remark 3.3 The result of Theorem 3.2 is important since it is equivalent to say that we can control with one control force a very wide age class of the two coupled populations in a minimum time of control and then with a minimum cost control $C_{obs,\delta}$.

4 Appendix

As is mentioned in the introduction, this section is devoted to the proofs of some intermediate results useful to show the full ω-Carleman estimate associated to the system (2.11). First, we begin by the Caccioppoli's inequality stated in the following lemma:

Lemma 4.1 *Let ω' be a subset of ω such that $\omega' \subset\subset \omega$. Let (u, v) be a solution of* (2.29)*. Then, there exists a positive constant C such that*

$$\int_{\omega'}\int_0^A\int_0^T (u_x^2 + v_x^2)e^{2s\varphi_i}dtdadx$$
$$\leq C\left(\int_q s^2\Theta^2(u^2 + v^2)e^{2s\varphi_i}dtdadx + \int_q (h_1^2 + h_2^2)e^{2s\varphi_i}dtdadx\right), \tag{4.148}$$

with φ_i, $i = 1, 2$ are defined by (2.19).

Proof The proof of this result is similar to the one of [22, Lemma 5.1]. Indeed, consider the smooth cut-off function ζ defined by

$$\begin{cases} 0 \le \zeta(x) \le 1, & x \in \mathbb{R}, \\ \zeta(x) = 0, & x < x_1 \text{ and } x > x_2, \\ \zeta(x) = 1, & x \in \omega^{'}. \end{cases} \tag{4.149}$$

Put $(.)_l = \frac{\partial .}{\partial l}$, where $l = t, a, x$.

For (u, v) solution of (2.29), one has

$$\begin{aligned}
0 &= \int_0^T \frac{d}{dt}\left[\int_0^1 \int_0^A \zeta^2 e^{2s\varphi_i}(u^2+v^2)dadx\right]dt \\
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2(\varphi_i)_t(u^2+v^2)e^{2s\varphi_i}dtdadx \\
&\quad +2\int_0^1\int_0^A\int_0^T \zeta^2 uu_t e^{2s\varphi_i}dtdadx + 2\int_0^1\int_0^A\int_0^T \zeta^2 vv_t e^{2s\varphi_i}dtdadx \\
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2(\varphi_i)_t(u^2+v^2)e^{2s\varphi_i}dtdadx \\
&\quad +2\int_0^1\int_0^A\int_0^T \zeta^2 u(h_1 - u_a - (k_1u_x)_x + \mu_1 u + \mu_3 v)e^{2s\varphi_i}dtdadx \\
&\quad +2\int_0^1\int_0^A\int_0^T \zeta^2 v(h_2 - v_a - (k_2v_x)_x + \mu_2 v)e^{2s\varphi_i}dtdadx.
\end{aligned}$$

Then, integrating by parts, we obtain

$$\begin{aligned}
&2\int_Q \zeta^2(k_1u_x^2 + k_2v_x^2)e^{2s\varphi_i}dtdadx \\
&= -2s\int_Q \zeta^2(u^2+v^2)\psi_i(\Theta_a + \Theta_t)e^{2s\varphi_i}dtdadx \\
&\quad -2\int_Q \zeta^2(uh_1 + vh_2)e^{2s\varphi_i}dtdadx - 2\int_Q \zeta^2(\mu_1u^2 + \mu_2v^2)e^{2s\varphi_i}dtdadx \\
&\quad +\int_Q (k_1(\zeta e^{2s\varphi_i})_x)_x u^2 dtdadx + \int_Q (k_2(\zeta e^{2s\varphi_i})_x)_x v^2 dtdadx \\
&\quad -2\int_Q \zeta^2\mu_3 uve^{2s\varphi_i}dtdadx.
\end{aligned}$$

On the other hand, by the definitions of ζ given in (4.149), ψ_i, $i = 1, 2$, and Θ given in (2.19), using Young's inequality, and taking s quite large, one can prove the existence of a positive constant c such that

$$2\int_Q \zeta^2(k_1u_x^2+k_2v_x^2)e^{2s\varphi_i}dtdadx$$

$$\geq 2\min(\min_{x\in\omega'}k_1(x),\min_{x\in\omega'}k_2(x))\int_{\omega'}\int_0^A\int_0^T(u_x^2+v_x^2)e^{2s\varphi_i}dtdadx,$$

$$\int_Q(k_1(\zeta e^{2s\varphi_i})_x)_xu^2dtdadx\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2u^2e^{2s\varphi_i}dtdadx,$$

$$\int_Q(k_2(\zeta e^{2s\varphi_i})_x)_xv^2dtdadx\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2v^2e^{2s\varphi_i}dtdadx,$$

$$-2s\int_Q\zeta^2(u^2+v^2)\psi_i(\Theta_a+\Theta_t)e^{2s\varphi_i}dtdadx$$

$$\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)e^{2s\varphi_i}dtdadx,$$

$$-2\int_Q\zeta^2(uh_1+vh_2)e^{2s\varphi_i}dtdadx\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)e^{2s\varphi_i}dtdadx$$

$$+c\int_\omega\int_0^A\int_0^T(h_1^2+h_2^2)e^{2s\varphi_i}dtdadx,$$

$$-2\int_Q\zeta^2(\mu_1u^2+\mu_2v^2)e^{2s\varphi_i}dtdadx$$

$$\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)e^{2s\varphi_i}dtdadx,$$

$$-2\int_Q\zeta^2\mu_3uve^{2s\varphi_i}dtdadx\leq c\int_\omega\int_0^A\int_0^T s^2\Theta^2(u^2+v^2)e^{2s\varphi_i}dtdadx.$$

Combining all the previous inequalities, we reach finally the estimate (4.148). □

Remark 4.2 In Lemma 4.1, the set ω' is chosen so that 0 that is exactly the point of degeneracy of the dispersion coefficients k_i, $i=1,2$, does not belong to $\overline{\omega'}$. More generally, if the degeneracy occurs at a point $x_0\in(0,1)$, one must take x_0 out of $\overline{\omega'}$ in the case of interior degeneracy to establish a Caccioppoli's type inequality (see [28] for more details in this context).

We close this section by the following result:

Lemma 4.3 *Assume that the conditions* (2.21) *hold. Then, the interval* $I=[\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{d_2k_2(1)(2-\gamma)-1},\frac{4(e^{2\kappa\|\sigma\|_\infty}-e^{\kappa\|\sigma\|_\infty})}{3d_2})$ *is not empty.*

Proof Indeed, one has

$$\begin{aligned}
&\frac{4(e^{2\kappa\|\sigma\|_\infty}-e^{\kappa\|\sigma\|_\infty})}{3d_2}-\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{d_2k_2(1)(2-\gamma)-1}\\
&=\frac{4(e^{2\kappa\|\sigma\|_\infty}-e^{\kappa\|\sigma\|_\infty})(d_2k_2(1)(2-\gamma)-1)-3d_2k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&=\frac{e^{2\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-4)-4e^{\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-1)}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&\quad+\frac{k_2(1)(2-\gamma)}{d_2k_2(1)(2-\gamma)-1}\\
&=\frac{e^{\kappa\|\sigma\|_\infty}[e^{\kappa\|\sigma\|_\infty}(d_2k_2(1)(2-\gamma)-4)-4(d_2k_2(1)(2-\gamma)-1)]}{3d_2(d_2k_2(1)(2-\gamma)-1)}\\
&\quad+\frac{k_2(1)(2-\gamma)}{d_2k_2(1)(2-\gamma)-1}.
\end{aligned}$$

Using the fact that $d_2 \geq \frac{5}{k_2(1)(2-\gamma)}$, we can conclude that $\frac{4(d_2k_2(1)(2-\gamma)-1)}{d_2k_2(1)(2-\gamma)-4} \leq 16$. Since $\kappa \geq \frac{4\ln(2)}{\|\sigma\|_\infty}$, then we have $e^{\kappa\|\sigma\|_\infty} \geq 16$. Therefore, the previous difference is positive and subsequently $I \neq \emptyset$. □

Dedication

The first author would like to dedicate this work to his parents Mohamed Echarroudi and Fatima Fakhri, his brothers and sister, to his step parents El Youfi Mohamed and Fatima Rim as well as to his wife and daughter.

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