Some Algebraic Aspects of Boolean Valued Analysis

Anatoly G. Kusraev

Abstract The article deals with a Boolean valued approach to some algebraic problems arising from functional analysis. The main results are as follows. (1) A universally complete vector lattice without locally one-dimensional bands contains an infinite direct sum of order dense sublattices each of which is a band preserving linear isomorphic (but not lattice isomorphic) copy of the whole lattice. (2) Every separated injective module over a semiprime rationally complete commutative ring admits a direct sum decomposition with homogeneous summands. (3) A semiprime rationally complete commutative ring properly embedded in a ring with projections K is a homogeneity ring of an additive mapping between appropriate K-modules.

Keywords Boolean valued analysis · Universally complete vector lattice · Injective module · Commutative ring · K-module

1 Introduction

The *Boolean valued approach* is a machinery of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras. As these models, one usually takes the classical sets in the shape of the *von Neumann universe* $\mathbb{V}$ and a properly truncated *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ in which the conventional set-theoretic concepts and constructions acquire nonstandard interpretations.

A. G. Kusraev (✉)
Southern Mathematical Institute of the Vladikavkaz Scientific Center of the RAS, Vladikavkaz, Russia

K. L. Khetagurov North-Ossetian State University, Vladikavkaz, Russia
e-mail: kusraev@smath.ru

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A general scheme of applying the Boolean valued approach is as follows, see [16, 17]. Assume that $\mathbf{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects, *external* and *internal*, respectively. Suppose we are able to prove the following

Boolean Valued Representation Result: Every external $X \in \mathbf{X}$ embeds into a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ becoming an internal object $\mathscr{X} \in \mathbb{X}$.

Boolean Valued Transfer Principle then tells us that every theorem about $\mathscr{X}$ within Zermelo–Fraenkel set theory with choice ZFC has its counterpart for the original object X interpreted as a Boolean valued object $\mathscr{X}$.

Boolean Valued Machinery enables us to perform some translation of theorems from $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ making use of appropriate general operations and the principles of Boolean valued models.

The paper deals with a Boolean-valued approach to some algebraic problems arising from functional analysis. Section 2 collects some Boolean valued requisites. The main result of Sect. 3 states that a universally complete vector lattice without locally one-dimensional bands contains an infinite direct sum of order dense sublattices each of which is a band preserving linearly isomorphic (but not lattice isomorphic) copy of the whole lattice. This problem is related to band preserving linear operators in vector lattices, see Abramovich and Kitover [2], Kusraev and Kutateladze [17, Chap. 4]. In Sect. 4 it is proved that every separated injective module over a semiprime rationally complete commutative ring admits a direct sum decomposition with homogeneous summands. Problems of this kind arise in the theory of operator algebras, see Chilin and Karimov [5], Ozawa [20]. It is proved in Sect. 5 that a semiprime rationally complete commutative ring properly embedded in a ring with projections K is a homogeneity ring of an additive mapping between appropriate K-modules. This result relates to functional equations, see Wilansky [23].

The reader can find the necessary information on the theory of vector lattices in [1, 3]; Boolean valued analysis, in [4, 15, 16]; rings and modules, in [19]. Troughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. We let := denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

2 Boolean Valued Requisites

Let $\mathbb{B}$ be a complete Boolean algebra. Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ is defined by recursion on α with α running through the class of all ordinals:

$$\mathbb{V}_\alpha^{(\mathbb{B})} = \left\{x : (\exists\, \beta \in \alpha)\ x : \operatorname{dom}(x) \to \mathbb{B},\ \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})}\right\},$$

$$\mathbb{V}^{(\mathbb{B})} := \bigcup\nolimits_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})} \quad (\mathrm{On}\ \text{ is the class of all ordinals}).$$

For making statements about $\mathbb{V}^{(\mathbb{B})}$ take a formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of *Zermelo–Fraenkel set theory* with choice ($\equiv$ ZFC) and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. There is a natural way of assigning to each such statement an element $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$ which acts as the *'Boolean truth-value'* of $\varphi(x_1, \dots, x_n)$ in the universe $\mathbb{V}^{(\mathbb{B})}$. We say that the statement $\varphi(x_1, \dots, x_n)$ *is valid within* $\mathbb{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$.

Theorem 2.1 (Transfer Principle) *All theorems of Zermelo–Fraenkel set theory with choice are true within* $\mathbb{V}^{(\mathbb{B})}$. *More precisely, if* $\varphi(u_1, \dots, u_n)$ *is a theorem of* ZFC *then*

$$(\forall x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})})\ [\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$$

is also a theorem of ZFC.

Given an arbitrary $X \in \mathbb{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ as the set $X{\downarrow} := \{x \in \mathbb{V}^{(\mathbb{B})} : [\![x \in X]\!] = \mathbb{1}\}$. Assume that $X, Y, f, P \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$ and $[\![P \subset X^2]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y and P is a binary relation on X within $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ $(x \in X{\downarrow})$ and $P{\downarrow}$ is a unique binary relation on $X{\downarrow}$ with $(x_1, x_2) \in P{\downarrow} \iff [\![(x_1, x_2) \in P]\!] = \mathbb{1}$. The *ascent* is a transformation acting in the reverse direction. i.e., sending any subset $X \subset \mathbb{V}^{(\mathbb{B})}$ into an element of $\mathbb{V}^{(\mathbb{B})}$. Along with these transformation there is the *canonical embedding* $X \mapsto X^{\wedge}$ of the class of standard sets ($\equiv$ *von Neumann universe*) $\mathbb{V}$ into a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ see [4, 15].

Let $\mathscr{R}$ stands for the field of reals within $\mathbb{V}^{(\mathbb{B})}$ i. e., $\mathscr{R} := (\mathrm{R}, \oplus, \odot, 0, 1, \leqq)$ and $[\![\varphi(\mathscr{R})]\!] = \mathbb{1}$, where $\varphi(\mathscr{R})$ is the conjunction of axioms of the reals. Consider the descent $\mathbf{R} := \mathscr{R}{\downarrow}$ of the algebraic structure $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$. In other words, $\mathbf{R} := (\mathrm{R}{\downarrow}, \oplus{\downarrow}, \odot{\downarrow}, \leqq{\downarrow}, 0, 1)$ is considered as the descent $\mathrm{R}{\downarrow}$ of the underlying set R together with the descended operations $\oplus{\downarrow}$ and $\odot{\downarrow}$ and order relation $\leqq{\downarrow}$ of the structure $\mathscr{R}$. The following fundamental result due to Gordon [7] tells us that the interpretation of reals (complexes) in a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ is a universally complete real (complex) vector lattice with the Boolean algebra of band projections isomorphic to $\mathbb{B}$.

Theorem 2.2 (Gordon) *The algebraic structure* $\mathscr{R}{\downarrow}$ *with the descended operations and order relation is a universally complete real vector lattice and a semiprime* f*-algebra with a ring and order unit* $\mathbb{1} := 1^{\wedge}$. *Moreover,* $\mathbb{R}^{\wedge}$ *is a dense subfield of* $\mathscr{R}$ *within* $\mathbb{V}^{(\mathbb{B})}$.

Proof A detailed proof can be found in [17, Sections 2.2, 2.3]. □

Gordon's theorem 2.2 raises the question of when $\mathbb{R}^{\wedge}$ and $\mathscr{R}$ coincide within $\mathbb{V}^{(\mathbb{B})}$. The answer was obtained by Gutman [10] in terms of the σ-distributivity of the Boolean algebra $\mathbb{B}$. A σ-complete Boolean algebra $\mathbb{B}$ is said to be σ*-distributive* if, for any double sequence $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$, the equality holds (cf. [22, 19.1]).

$$\bigwedge_{n\in\mathbb{N}}\bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^{\mathbb{N}}}\bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$$

Theorem 2.3 (Gutman) *Let $\mathbb{B}$ be a complete Boolean algebra and $\mathscr{R}$ the field of reals within $\mathbb{V}^{(\mathbb{B})}$. The following assertions are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*
(2) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$.
(3) *$\mathscr{R}{\downarrow}$ is locally one-dimensional.*

We also need the following result on the structure of cardinals within a Boolean-valued universe. Let $\mathrm{Card}(x)$ symbolize that x is a cardinal.

Theorem 2.4 *Each Boolean valued cardinal is a mixture of some set of relatively standard cardinals. More precisely, Given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \mathrm{Card}(x)$ if and only if there are nonempty set of cardinals Γ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma} \subset \mathbb{B}$ such that $x = \mathrm{mix}_{\gamma\in\Gamma} b_\gamma \gamma^{\wedge}$ and $\mathbb{V}^{(\mathbb{B}_\gamma)} \models \mathrm{Card}(\gamma^{\wedge})$ with $\mathbb{B}_\gamma := [0, b_\gamma]$ for all $\gamma \in \Gamma$.*

Proof See Bell [4, Problem 1.45 and Theorem 1.50], Kusraev and Kutataeladze and [17, Subsections 1.9.7 and 1.9.11]. □

3 Band Preserving Linear Isomorphisms

Abramovich and Kitover in [2, p. 1, Problem B] raised the question as to whether the vector lattices X and Y are lattice isomorphic whenever X and Y are d-isomorphic, that is, there exists a linear disjointness preserving operator $T : X \to Y$ such that T^{-1} is also disjointness preserving? A negative answer was given in the same work, see [2, Theorem 13.4]. The problem has a negative solution even in the class of band preserving operators, see [17, Theorem 4.6.7]. Moreover, if X is a real universally complete vector lattice without locally one-dimensional bands then $X = X_1 \oplus X_2$ for some component-wise closed and laterally complete vector sublattices $X_1 \subset X$ and $X_2 \subset X$ both d-isomorphic to X but neither X_1 nor X_2 is Dedekind complete and hence lattice isomorphic to X, see [18]. The aim of this section is to show that the latter result can be improved to infinite direct sum decomposition.

Definition 3.1 A vector lattice X is said to be *locally one-dimensional* if for every two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2 respectively such that u_1 and u_2 are proportional. Let γ be a cardinal. A vector lattice X is said to be *Hamel γ-homogeneous* whenever there exists a local Hamel basis of cardinality γ in X consisting of strongly distinct weak order units. Two elements $x, y \in X$ are said to be *strongly distinct* if $|x - y|$ is a weak order unit in X.

Lemma 3.2 *Let X be a universally complete vector lattice. There is a band X_0 in X such that $X_0^{\perp}$ is locally one-dimensional and there exists a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(X_0)$ with Γ a set of infinite cardinals such that $\pi_\gamma X_0$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$.*

Proof The proof can be found in [17, Theorem 4.6.9]. □

Lemma 3.3 *The field of reals $\mathbb{R}$ has no proper subfield of which it is a finite extension.*

Proof See, for example, Coppel [6, Lemma 17]. □

Lemma 3.4 *Let $\mathscr{R}$ be the field of reals within $\mathbb{V}^{(\mathbb{B})}$, $X := \mathscr{R}{\downarrow}$, and $b \in \mathbb{B}$. Denote by $\dim(\mathscr{R})$ the internal cardinal with $[\![\dim(\mathscr{R})$ is the algebraic dimension of the vector space $\mathscr{R}$ over $\mathbb{R}^{\wedge}]\!] = \mathbb{1}$. Then bX is strictly Hamel γ-homogeneous if and only if $b \leq [\![\dim(\mathscr{R}) = \gamma^{\wedge}]\!]$.*

Proof This can be proved as in [12, Theorem 8.3.11]. □

Lemma 3.5 *Let $\mathbb{P}$ be a proper subfield of $\mathbb{R}$. There exists an infinite cardinal $\varkappa$ and a family $(\mathscr{X}_\alpha)_{\alpha\leq\varkappa}$ of $\mathbb{P}$-linear subspace in $\mathbb{R}$ such that $\mathbb{R} = \bigoplus_{\alpha\leq\varkappa} \mathscr{X}_\alpha$ and, for every $\alpha \leq \varkappa$, the $\mathbb{P}$-vector spaces $\mathscr{X}_\alpha$ and $\mathbb{R}$ are isomorphic, whilst they are not isomorphic as ordered vector spaces over $\mathbb{P}$.*

Proof It follows from Lemma 3.3 that $\mathbb{R}$ is an infinite dimensional vector space over the field $\mathbb{P}$. Let $\mathscr{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$. Since $\varkappa := |\mathscr{E}|$ is an infinite cardinal, we have the representation $\varkappa = \sum_{\alpha\in\mathrm{A}} \varkappa_\alpha$, where $\varkappa_\alpha = \varkappa$ for all $\alpha \in \mathrm{A}$ and $|\mathrm{A}| \leq \varkappa$. It follows that there is a family of subsets $\mathscr{E}_\alpha \subset \mathscr{E}$ such that $\mathscr{E} = \bigcup_{\alpha\leq\varkappa} \mathscr{E}_\alpha$, $|\mathscr{E}_\alpha| = |\mathscr{E}|$ for all $\alpha \leq \varkappa$, and $\mathscr{E}_\alpha \cap \mathscr{E}_\beta = \varnothing$ whenever $\alpha \neq \beta$. If $\mathscr{X}_\alpha$ denotes the $\mathbb{P}$-subspace of $\mathbb{R}$ generated by $\mathscr{E}_\alpha$, then $\mathscr{X}_\alpha \subsetneq \mathbb{R}$ and $\mathscr{X}_\alpha$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$. If $\mathscr{X}_\alpha$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathscr{X}$ would be order complete and, as a consequence, we would have $\mathscr{X}_\alpha = \mathbb{R}$; a contradiction. □

Definition 3.6 Let X be a vector latice and $u \in X$. An element $v \in X$ is called a *component* of v if $|v| \wedge |u - v| = 0$. The set of all components of u is denoted by $\mathbb{C}(u)$. A subset X_0 is said to be *component-wise closed* if for every $u \in X_0$ the set $\mathbb{C}(u)$ is contained in X_0. A sublattice $X_0 \subset X$ is said to be *laterally complete* if every disjoint family in X_0 has a supremum.

Lemma 3.7 *Let $\mathscr{R}$ — be the field of reals within $\mathbb{V}^{(\mathbb{B})}$ and let us consider a universally complete vector lattice $X := \mathscr{R}{\downarrow}$. For the sublattice $X_0 \subset X$ the following conditions:*

(1) *X_0 is laterally complete, component-wise closed, and $X_0^{\perp\perp} = X$.*
(2) *$X_0 = \mathscr{X}_0{\downarrow}$ for some nonzero vector sublattice $\mathscr{X}_0$ of the field $\mathscr{R}$ considered as vector lattice over the subfield $\mathbb{R}^{\wedge}$.*

Proof It directly follows from [17, Theorem 2.5.1]. □

Denote by $[A]_\sigma$ the set of all elements $x \in X$ representable in the form $x = \sum_{n=1}^{\infty} \pi_k a_k$, where (a_k) is an arbitrary sequence in A and (π_k) is a countable partition of unity in $\mathbb{P}(X)$.

Theorem 3.8 *Assume that a real universally complete vector lattice X is strictly Hamel $\varkappa$-homogeneous for some infinite cardinal $\varkappa$. Then there exists a family $(X_\alpha)_{\alpha\le\varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ satisfying the conditions:*

(1) $X = \big[\bigoplus_{\alpha\le\varkappa} X_\alpha\big]_\sigma$ *and* $X = X_\alpha^{\perp\perp}$ *for all* $\alpha \le \varkappa$.
(2) *The canonical projection* $\pi_\alpha : X \to X_\alpha$ *are all band preserving.*
(3) X_α *is d-isomorphic to* X *for all* $\alpha \le \varkappa$.
(4) X_α *is not Dedekind complete and hence not lattice isomorphic to* X *for all* $\alpha \le \varkappa$.

Proof We can assume without loss of generality that $X = \mathscr{R}{\downarrow}$. Since there is no loally one-dimensional band in X, we have $[\![\mathscr{R} \ne \mathbb{R}^\wedge]\!] = \mathbb{1}$ by Gutman's theorem 2.2. Working within $\mathbb{V}^{(\mathbb{B})}$, we can apply Lemma 3.5 by Transfer Principle and find an infinite cardinal $\varkappa$ and a family $(\mathscr{X}_\alpha)_{\alpha\le\varkappa}$ of $\mathbb{R}^\wedge$-linear subspaces of $\mathscr{R}$ such that $\mathscr{R} = \bigoplus_{\alpha\le\varkappa} \mathscr{X}_\alpha$ and, for every $\alpha \le \varkappa$, there is an $\mathbb{R}^\wedge$-isomorphism τ_α from $\mathscr{X}_\alpha$ onto $\mathscr{R}$, while $\mathscr{X}_\alpha$ and $\mathscr{R}$ are not isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$. Let $p_\alpha : \mathscr{R} \to \mathscr{X}_\alpha$ stand for the projection corresponding to the direct sum decomposition $\mathscr{R} = \bigoplus_{\alpha\le\varkappa} \mathscr{X}_\alpha$. To externalize, put $X_\alpha := \mathscr{X}_\alpha{\downarrow}$, $T_\alpha := \tau_\alpha{\downarrow}$, $S_\alpha := \tau_\alpha^{-1}{\downarrow}$, and $P_\alpha := p_\alpha{\downarrow}$. The maps $S_\alpha : X \to X_\alpha$, $T_\alpha : X_\alpha \to X$, and $P_\alpha : X \to X_\alpha$ are band preserving and $\mathbb{R}$-linear by [17, Theorem 4.3.4]. Moreover, S_α and T_α are injective and $S_\alpha = (\tau_\alpha{\downarrow})^{-1} = T_\alpha^{-1}$, see [17, 1.5.3(2)]. Since $\mathscr{X}_\alpha$ is linearly isomorphic to $\mathscr{R}$, $[\![\mathscr{X}_\alpha \ne \{0\}]\!] = \mathbb{1}$ and hence $X_\alpha^{\perp\perp} = X$. It follows from Lemma 3.6 that X_α is laterally complete and component-wise closed. It remains to observe that a) X_α and X are lattice isomorphic if and only if $\mathscr{X}$ and $\mathscr{R}$ are isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$; b) P_α is order bounded if and only if so is p_α within $\mathbb{V}^{(\mathbb{B})}$. □

Corollary 3.9 *For every universally complete vector lattice X without locally one-dimensional bands there exist an infinite cardinal $\varkappa$ and a family $(X_\alpha)_{\alpha\le\varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ such that X_α is d-isomorphic but not lattice isomorphic to X for all $\alpha \le \varkappa$ and $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \ne \beta$.*

Proof This is immediate from Lemma 3.2 and Theorem 3.8. □

Remark 3.10 As can be seen from the proof of Theorem 3.8, the key role in this section is played by the Boolean valued interpretation of the Hamel basis. In some problems, a similar role belongs to the Boolean valued interpretation of a transcendence basis, see [13].

4 Classification of Injective Modules

Recently, Chilin and Karimov [5] obtained a classification result for laterally complete modules over universally complete f-algebras. They introduced the concept of the *passport* and proved that two such modules are isomorphic if and only if their passport coincide. In this section we will show that these results remain valid for a broader class of separated injective modules over semiprime rationally complete commutative rings.

In what follows, K stands for a commutative semiprime ring with unit and X denotes a unitary K-module. The Boolean valued approach to the above classification problem is based on the following two results (Theorems 4.2 and 4.4) due to Gordon [8, 9].

Definition 4.1 An *annihilator ideal* of K is a subset of the form $S^{\perp} := \{k \in K : (\forall s \in S)\ ks = 0\}$ with a nonempty subset $S \subset K$. A subset S of K is called *dense* provided that $S^{\perp} = \{0\}$; i.e., the equality $k \cdot S := \{k \cdot s : s \in S\} = \{0\}$ implies $k = 0$ for all $k \in K$. A ring K is said to be *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K with $h(x) = rx$ for all $x \in J$.

Observe that K is rationally complete if and only if the complete ring of quotients $Q(K)$ is isomorphic to K canonically, see Lambek [19, § 2.3].

Theorem 4.2 *If $\mathscr{K}$ is a field within $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete semiprime ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathbb{A}(\mathscr{K}{\downarrow})$ of the annihilator ideals of $\mathscr{K}{\downarrow}$ such that*

$$b \leq [\![x = 0]\!] \iff x \in \chi(b^{*}) \quad (x \in K,\ b \in \mathbb{B}).$$

Conversely, if K is a rationally complete semiprime ring and $\mathbb{B}$ stands for the Boolean algebra $\mathbb{A}(K)$ of all annihilator ideals of K, then there is an internal field $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that the ring K is isomorphic to $\mathscr{K}{\downarrow}$.

Proof See [16, Theorem 8.3.1] and [16, Theorem 8.3.2]. □

Definition 4.3 A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $xJ = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0.

The *Baer criterion* says that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$; see Lambek [19]. All modules under consideration are assumed to be *faithful*, that is, $Xk \neq \{0\}$ for any $0 \neq k \in K$, or equivalently, the canonical representation of K by endomorphisms of the additive group X is one-to-one.

Theorem 4.4 *Let $\mathscr{X}$ be a vector space over a field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathbb{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism in Theorem 4.2. Then $\mathscr{X}{\downarrow}$ is a separated unital injective module over $\mathscr{K}{\downarrow}$ such that $b \leq [\![x = 0]\!]$ and $\chi(b)x = 0$ are equivalent for all $x \in \mathscr{X}{\downarrow}$ and $b \in \mathbb{B}$. Conversely, if $K = \mathscr{K}{\downarrow}$ and X is a unital separated injective K-module then there exists an internal vector space $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ over $\mathscr{K}$ such that the K-module X is isomorphic to $\mathscr{X}{\downarrow}$. Moreover if $\jmath : K \to \mathscr{K}{\downarrow}$ is an isomorphism in Theorem 4.2, then one can choose an isomorphism $\imath : X \to \mathscr{X}{\downarrow}$ such that $\imath(ax) = \jmath(a)\imath(x)$ $(a \in K,\ x \in X)$.*

Proof See [16, Theorems 8.3.12] and [16, and 8.3.13]. □

Thus, Theorem 4.4 enables us to apply Boolean valued approach to unital separated injective modules over semiprime rationally complete commutative rings.

Definition 4.5 A family $\mathscr{E}$ in a K-module X is called *K-linearly independent* or symply *linearly independent* whenever, for all $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in K$, and $e_1, \dots, e_n \in \mathscr{E}$, the equality $\sum_{k=1}^{n} \alpha_k e_k = 0$ implies $\alpha_1 = \dots = \alpha_n = 0$. An inclusion maximal K-linearly independent subset of X is called a *Hamel K-basis* for X.

Every unital separated injective K-module X has a Hamel K-basis. A K-linearly independent set $\mathscr{E}$ in X is a Hamel K-basis if and only if for every $x \in X$ there exist a partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(K)$ and a family $(\lambda_{k,e})_{k\in\mathbb{N},e\in\mathscr{E}}$ in K such that

$$\pi_k x = \sum\nolimits_{e\in\mathscr{E}} \lambda_{k,e}\pi_k e \quad (k \in \mathbb{N})$$

and for every $k \in \mathbb{N}$ the set $\{e \in \mathscr{E} : \lambda_{k,e} \neq 0\}$ is finite.

Definition 4.6 Let γ be a cardinal. A K-module X is said to be *Hamel γ-homogeneous* whenever there exists a Hamel K-basis of cardinality γ in X. For $\pi \in \mathbb{P}(X)$ denote by $\varkappa(\pi)$ the least cardinal γ for which πX is Hamel γ-homogeneous. Say that X is *strictly Hamel γ-homogeneous* whenever X is Hamel γ-homogeneous and $\varkappa(\pi) = \gamma$ for all nonzero $\pi \in \mathbb{P}(X)$.

Theorem 4.7 *Let K be a semiprime rationally complete commutative ring and let X be a separated injective module over K. There exists a partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(K)$ with Γ a set of cardinals such that $e_\gamma X$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$. Moreover, X is isomorphic to $\prod_{\gamma\in\Gamma} e_\gamma X$ and the partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ is unique up to permutation.*

Proof According to Theorems 4.2 and 4.4 we may assume that $K = \mathscr{K}{\downarrow}$ and $X = \mathscr{X}{\downarrow}$, where $\mathscr{X}$ is a vector space over the field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\dim(\mathscr{X}) \in \mathbb{V}^{(\mathbb{B})}$, the algebraic dimension of $\mathscr{X}$, is an internal cardinal and, since each Boolean valued cardinal is a mixture of some set of relatively standard cardinals [17, 1.9.11], we have $\dim(\mathscr{X}) = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ where Γ is a set of cardinals and $(b_\gamma)_{\gamma\in\Gamma}$ is a partition of unity. Thus, for all $\gamma \in \Gamma$ we have $e_\gamma \leq [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]$, whence $e_\gamma X$ is strictly Hamel γ-homogeneous. The remaining details are elementary. □

Definition 4.8 The partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(K)$ in Theorem 4.7 is called the passport of K-module X is denoted by $\Gamma(X)$.

Theorem 4.9 *Faithful separated injective K-modules X and Y are isomorphic if and only if* $\Gamma(X) = \Gamma(Y)$.

Remark 4.10

(1) In the particular case that K is a universally complete f-algebra, we get the classification obtained by Chilin and Karimov [5, Theorems 4.2 and 4.3] without using the Boolean valued approach. In this event $\mathscr{X}$ is a vector spaces over the field of reals $\mathscr{K} = \mathscr{R}$ or complexes $\mathscr{K} = \mathscr{C}$ within $\mathbb{V}^{(\mathbb{B})}$. Another particular case of Theorem 4.7 when $\mathscr{X}$ is a vector subspace of $\mathscr{R}$ (considered as a vector space over $\mathbb{R}^\wedge$) was examined by Kusraev and Kutateladze [17, Chap. 4].
(2) The family $(e_\gamma)_{\gamma\in\Gamma}$ in Theorem 16 is called a *decomposition series* if $e_\gamma X$ is (not necessarily strict) Hamel γ-homogeneous for all $\gamma \in \Gamma$. It can be also proved that separated injective modules over $K = L^0(\mathscr{B})$ are isomorphic if and only if their decomposition series are *congruent* in the sense of Ozawa [20].
(3) An *injective Banach lattice* possesses a module structure over some ring of continuous function $C(Q)$ with Q an extremally disconnected Hausdorff topological space [17, Sections 5.10]. This enables one to apply the Boolean valued approach to the classification problem of injective Banach lattices; details can be found in Kusraev [14]. The key role here is played by the concept of the Maharam operator, see [11].

5 Homogeneity Rings of Additive Operators

In this section, a ring K is supposed to have an identity element which is distinct from zero. This implies that K is not the zero ring $\{0\}$. Accordingly, a subring F of K is required to contain the identity element of K. Let X and Y be two unitary K-modules. Then, for any additive mapping $f : X \to Y$, the subset of K defined as

$$H_f := \{k \in K : \ f(kx) = kf(x) \ \text{ for all } \ x \in X\}$$

is a subring of K, the *homogeneity ring* of f, see Rätz [21, Lemma 1]. The problem is to examine what subrings of K have the form H_f for some additive operator f from X to Y? It is proved by Rätz [21, Theorem 3] that, for $X \neq \{0\}$, $Y \neq \{0\}$ and any subring S of K for which X is a *free S-module*, there exists an additive mapping $f : X \to Y$ such that $H_f = S$. The assumption that X is a free S-module seems to be pretty restrictive. However, in a special case of vector spaces over fields this condition is fulfilled so that we have the following.

Lemma 5.1 *Let X and Y be nonzero unitary K-modules and $\{y\}$ is linearly independent for some $y \in Y$. If a subring S of K is a field then there exists an additive mapping $f : X \to Y$ such that $H_f = S$.*

Consider an f-algebra A. Given an additive operator $S : A \to A$, define the homogeneity set $H_S \subset A$ of S as $H_S := \{a \in A : S(ax) = aSx \text{ forall } x \in A\}$. Then H_S is evidently a subring of A and our problem is to examine what subrings of A have the form H_S for some additive operator S in A?

Definition 5.2 A *projection* of a ring K is an endomorphism π of K with $\pi \circ \pi = \pi$. Say that $\mathscr{B}$ is a *Boolean algebra of projections* in K if $\mathscr{B}$ consists of mutually commuting projections in K under the operations

$$\pi_1 \vee \pi_2 := \pi_1 + \pi_2 - \pi_1 \circ \pi_2, \quad \pi_1 \wedge \pi_2 := \pi_1 \circ \pi_2,$$
$$\pi^* := I_K - \pi \quad (\pi_1, \pi_2, \pi \in \mathscr{B})$$

and in which the zero and the identity operators in K serve as the top and bottom elements of $\mathscr{B}$. Given $x \in K$, the *carrier* of x is defined as the projection $[x] := \bigwedge\{\pi \in \mathscr{B} : \pi x = x\}$.

Definition 5.3 BAP-*ring* is a pair $(K, \mathscr{B})$ where K is a ring with the distinguished complete Boolean algebra of projections $\mathscr{B}$, see [15]. Say that K is a $\mathbb{B}$-complete ring if $\mathbb{B}$ is a complete Boolean algebra isomorphic to $\mathscr{B}$, $(K, \mathscr{B})$ is a BAP-ring, and for every partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathscr{B}$ the following two conditions hold:

(1) If $x \in K$ and $\pi_\xi x = 0$ for all $\xi \in \Xi$ then $x = 0$.
(2) If $(x_\xi)_{\xi \in \Xi}$ is a family in K then there exists $x \in K$ such that $\pi_\xi x = \pi_\xi x_\xi$ for all $\xi \in \Xi$.

Theorem 5.4 *Let $\mathscr{K}$ be a ring within $\mathbb{V}^{(\mathbb{B})}$ and $K := \mathscr{K}{\Downarrow}$. Then K is a $\mathbb{B}$-complete ring and there exists an isomorphism $\jmath$ from $\mathbb{B}$ onto a Boolean algebra of projections $\mathscr{B}$ in K such that*

$$b \leq [\![x = 0]\!] \iff \jmath(b)x = 0 \quad (x \in K,\ b \in \mathbb{B}).$$

Conversely, if K is a $\mathbb{B}$-complete ring then there exists $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{K}$ is a ring $]\!] = \mathbb{1}$ and the descent $\mathscr{K}{\downarrow}$ is $\mathbb{B}$-isomorphic to K.

Proof The first part is proved in [15, Theorem 4.2.8], while the second part can be deduced from [15, Theorem 4.3.3]. The reader is referred to [16, Theorems 8.1.4 and 5.1.7] for complete proofs and many pertinent results. □

Lemma 5.5 *Let $\mathscr{K}$ be a ring within $\mathbb{V}^{(\mathbb{B})}$ and $K := \mathscr{K}{\Downarrow}$. For a semiprime commutative subring $F \subset K$ the following are equivalent:*

(1) *F is rationally complete and each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$.*
(2) *F is regular, $\mathbb{B}$-complete, and $xy = 0$ implies $[x] \circ [y] = 0$ for all $x, y \in F$.*
(3) *$F = \mathscr{F}{\downarrow}$ for some field $\mathscr{F}$ which is a subring of $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$.*

Proof Clearly, $[x] \circ [y] = 0$ implies $xy = 0$ for all $x, y \in K$. It follows that $\jmath(b)F$ is an annihilator ideal. Observe that if each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$ or $xy = 0$ implies $[x] \circ [y] = 0$ for all $x, y \in F$, then $\mathbb{B}$ is isomorphic to the Boolean algebra of all annihilator ideals; the isomorphism is given by assigning $b \mapsto \jmath(b)F$ $(b \in \mathbb{B})$. Now, to ensure that the conditions (1) and (2) are equivalent it remains to observe that a semiprime ring is rationally complete if and if it is regular and $\mathbb{B}$-complete with $\mathbb{B}$ the Boolean algebra of annihilator ideals [15, Theorem 4.5.4]. The equivalence of (3) to both (1) and (2) follows from Theorem 4.2. □

Definition 5.6 A separated K-module X is said to be $\mathbb{B}$-complete if K is $\mathbb{B}$-complete and for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family$(x_\xi)_{\xi\in\Xi}$ in X there exists $x \in X$ such that $\jmath(b_\xi)x = \jmath(b_\xi)x_\xi$ for all $\xi \in \Xi$.

Theorem 5.7 *Let $\mathscr{X}$ be a modules over a ring $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Then $X := \mathscr{X}\!\downarrow$ is a $\mathbb{B}$-complete module over the $\mathbb{B}$-complete ring $K := \mathscr{K}\!\downarrow$. Conversely, if X is a $\mathbb{B}$-complete K-module then there exists $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{X}$ is a $\mathscr{K}$-module$]\!] = \mathbb{1}$ and there is an isomorphism ι_X from X onto $\mathscr{X}\!\downarrow$ such that $\iota_X(ax) = \iota_K(a)\iota_X(x)$ for all $a \in K$ and $x \in X$, where ι_K is a ring isomorphism from K onto $\mathscr{K}\!\downarrow$ in Theorem 5.4.*

Proof The proof runs along the lines of the proof of Theorems 8.3.1 and 8.3.2 in [16]. □

We are now ready to state and proof the main result of this section.

Theorem 5.8 *Let X and Y be unitary $\mathbb{B}$-complete K-modules. If a subring F of K is rationally complete and each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$ then there exists an additive mapping $f : X \to Y$ such that $H_f = F$.*

Proof According to Theorem 5.4 we can assume that $K = \mathscr{K}\!\downarrow$, where $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ and $[\![\mathscr{K}$ is a ring$]\!] = \mathbb{1}$. By Lemma 5.5 it follows that $F = \mathscr{F}\!\downarrow$ for some field $\mathscr{F}$ which is a subring of $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Using Theorem 5.7 we can find $\mathscr{K}$-modules $\mathscr{X}$ and $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$ such that $X = \mathscr{X}\!\downarrow$ and $Y = \mathscr{Y}\!\downarrow$. The Transfer Principle (Theorem 2.1) guarantees that Lemma 5.1 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exists an additive function $\varphi : \mathscr{X} \to \mathscr{Y}$ with $H_\varphi = \mathscr{F}$. Put $f := \varphi\!\downarrow$ and note that $H_f = H_\varphi\!\downarrow$. It follows that $H_f = F$. □

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