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Operator Theory and Harmonic Analysis

OTHA 2020, Part I – New General Trends and Advances of the Theory

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Preface

This is the first volume of the two-volume series entitled

Operator Theory and Harmonic Analysis.

Vol. 1: New General Trends and Advances of the Theory

and

Vol. 2: Probability-Analytical Models, Methods, and Applications

Volume 1 is devoted to harmonic analysis and its applications in general, while Volume 2 is focused on probabilistic and mathematical (statistical) methods in applied sciences, but still in the context of general harmonic analysis and its numerous applications.

The volumes' readership is the pool of researchers interested in various aspects of harmonic analysis and operator theory: real and complex variable methods, applications to PDE's, mathematical modeling based on applied harmonic analysis and probability-analytical methods, and exploration of new themes and trends.

The contributions to both volumes are based on the matter supposed to be presented at the Annual International Scientific Conference on Modern Methods and Problems of Operator Theory and Harmonic Analysis and Their Applications (OTHA-2020, http://otha.sfedu.ru/), canceled due to Covid19 restrictions.

The Editors are very grateful to all the authors for their valuable contributions and for a strong willingness to support mathematical activities and communications in the hope of the soonest resumption of regular conferences and safe mutual visits. The Editors express an immense sorrow on the occasion of the recent loss of remarkable scientists and brilliant persons, Hrachik Hayrapetyan (Armenia), who is one of the authors of the first volume, Vladimir Pilidi (Russia), who was an active member of Program Committees of OTHA conferences, and Vladimir Nogin (Russia), who was a colleague and a teacher of quite a few participants of OTHA.

The first volume contains words in memoriam of our dear friends Hrachik Hayrapetyan, Vladimir Pilidi, and Vladimir Nogin.

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In Memory of Prof. Hrachik M. Hayrapetyan (25.10.1946–06.11.2020)

On November 6, 2020, the world mathematical community lost a brilliant mathematician and a wonderful personality Hrachik Hayrapetyan.

Professor Hayrapetyan was born in 1946 in Dilijan (Armenia). His mathematical talents were noticed since his adolescence. His mathematical inclinations were influenced by contacts with his first teacher A. Sahakyan. In 1964, he started his studies at the Faculty of Mathematics and Mechanics of the Yerevan State University from which he graduated in 1969. After two years of service in the Soviet Army, he entered the Institute of Mathematics of the National Academy of Sciences of Armenia as a junior scientific researcher. Since then, his collaboration with the academician Mkhitar Jrbashyan started, who proposed to him the study of free interpolation and basis properties of rational fractions. Hrachik Hayrapetyan succeeded in discovering a series of essential results in this research field. In particular, he proved that if the multiplicities in the interpolation problem are not bounded, then this problem may have no solution and the rational fractions may fail to be a basis in the closure of their linear span. In 1975, he completed his PhD thesis. In 1979, he entered the National Polytechnic University of Armenia as an associate professor of the Chair of Applied Mathematics. At that time, the scientific group of Prof. N.E. Tovmasyan was developing the theory of boundary value problems for partial differential equations. H.M. Hayrapetyan was actively involved in this research. As a specialist in the theory of complex variable functions, he was interested in theoretical-functional approach to these problems. He succeeded in obtaining a series of important results. Particularly, it is worth mentioning the new formulation of the classical Riemann boundary value problem, which allowed to solve this problem first in the classes of integrable functions and afterwards in the class of essentially bounded functions. Later, applying proposed method, Prof. Hayrapetyan with his students investigated boundary value problems in various functional spaces. The obtained results not only extended the theory of boundary value problems but also permitted to develop the theory of elliptic partial differential equations. Hrachik Hayrapetyan defended his Doctor of Science thesis "Riemann–Hilbert boundary value problem in the sense of mean convergence and applications in the theory of elliptic partial differential equations" in the M.V. Lomonosov Moscow State University. Specialists evaluated his results as a great success in the theory of boundary value problems. Developing his theory during the last decade, he studied boundary value problems in weighted classes of functions. He succeeded to describe the classes of functions, where the Dirichlet and Riemann–Hilbert boundary value problems in the classes of polyanalytic and polyharmonic functions are normally solvable in both bounded and unbounded domains. These results are highly evaluated by specialists in Armenia as well as abroad.

Hrachik Hayrapetyan was one of the members of the first elected Council of the Armenian Mathematical Union created in 1991 following Armenia's independence from the Soviet Union.

Prof. Hayrapetyan was an active organizer. He served two terms as the President of Mathematical Association of Armenia and he was the Head of specialized mathematical education chair in National Polytechnic University of Armenia and the Head of Mathematical Analysis and Function Theory chair in the Yerevan State University. His devotion to the science and excellent empathy skills helped him to interest young students in mathematical research. He was a scientific advisor of 15 PhD theses; his students continue the work on ideas of their teacher and mentor, Hrachik Hayrapetyan, in various universities and research institutions of Armenia.

The fond memory of our friend will forever rest in our hearts.

Doctor of Science, Professor Armenak H. Babayan
Doctor of Science, Professor Vanya A. Mirzoyan
Doctor of Science, Professor Levon Z. Gevorgyan
Chair of Specialized Mathematical Education of National Polytechnic University of Armenia

In Memory of Prof. Vladimir Pilidi (07.11.1946–19.01.2021)

Vladimir Pilidi became a student at the Rostov State University (now Southern Federal University) in 1964. All his scientific career from a talented student (diploma with honors) to distinguished Chair was related to this university during 57 years.

In 1972, under the guidance of Professor I.B. Simonenko, he defended his Ph.D. thesis "Local method for the study of linear operator equations of the type of bisingular integral equations," and in 1990 at the Dissertation Council of the Tbilisi Institute of Mathematics named after I. A. Razmadze of the Georgian Academy of Sciences, he defended doctoral dissertation (second degree) "Bisingular operators and classes of operators close to them", which became a significant scientific achievement that enriched the general theory of operators of local type. Seven students of Vladimir Pilidi became candidates of science (PhD's).

Professor Pilidi was a highly qualified expert in the field of mathematics and its applications, reviewer of scientific articles, member and chairman of the Dissertation Council, scientific consultant of several research institutions, chairman of the State Examination Commissions of universities. After he became the head of the Chair of Informatics and Computational Experiment in 2000, Vladimir Pilidi expanded his area of scientific interests towards the application of mathematical methods in cryptography, the theory of pattern recognition, and graphic information processing.

Professor Pilidi is known as the author of the bilocal method, an analogue of the classical local method of Simonenko, and in his research he successfully applied it to the study of bisingular and related operators, as well as algebras of such operators. Thanks to these achievements, the name of Vladimir Pilidi will forever remain among the names of outstanding researchers in analysis and operator theory.

Professor Pilidi actively participated in the scientific life of the Mathematics Department, in the organization of scientific seminars, conferences, and schools. He was one of the main organizers of the OTHA conference series, a regular participant, and a member of the Program Committee of these conferences. Professor Pilidi and his students made a valuable contribution to the development of this series of conferences and to the development of publication activity following the conferences.

Vladimir Pilidi is known as a brilliant lecturer of various courses in mathematics and computer science. He had remarkable achievements as a teacher in the lecture course in algebra and geometry for students of applied and mechanical engineering, which he taught for about 20 years. His textbook “Linear Algebra” (Vuzovskaya Kniga, Moscow, 2005), co-authored with A.V. Kozak, is standard for other authors. Other textbooks by Professor Pilidi are: “Mathematical Analysis” (Phoenix, 2009), “Mathematical Foundations of Information Security” (Southern Federal University, 2019), and Analytic Geometry (Southern Federal University, 2020). Vladimir Pilidi developed a deep modern course on mathematical methods of cryptography, which he taught to students of the Department of Fundamental Informatics and Information Technology and students of the Department of Applied Mathematics, specialized in the field of mathematical methods of information security.

His distinguished features were not only erudition and professionalism but also modesty, discreetness, and goodwill in relations with colleagues and students. Vladimir Pilidi was a wonderful head of his mathematical family. His wife, daughter, and son-in-law devoted themselves to mathematics, and his grandchildren are preparing to become mathematicians as well.

The bright memory of Vladimir Pilidi—of a mathematician, a teacher, and a brilliant person, will remain in our hearts.

I. M. Erusalimskiy
A. N. Karapetyants
V. S. Rabinovich
S. G. Samko

In Memory of Vladimir Nogin (20.12.1955–31.05.2021)

Dr. Nogin Vladimir Alexandrovich was born on December 20, 1955. Vladimir Nogin graduated from the Faculty of Mechanics and Mathematics of the Rostov State University (now it is Southern Federal University) in 1979, defended his Ph.D. thesis in 1982 and worked 35 years as an assistant professor, senior teacher and then associate professor of the Department (Chair) of Differential and Integral Equations at the same University. During his work at the university, V.A. Nogin taught courses in mathematical analysis, higher mathematics, and mathematical physics. He also developed and taught more than five special courses for undergraduate and postgraduate math students, which included contemporary results in the field of functional analysis and mathematical physics.

Dr. Nogin's scientific interests were in the classical area of analysis related to the study of operators of mathematical physics, the construction and study of fractional powers of these operators, their inversion, and the description of the image of such operators in the framework of Lebesgue spaces. At the same time, he dealt with questions of functional analysis—the description of function spaces that arise in analysis in the context of the above-mentioned theory of operators. He and his students obtained profound results in this theory; he successfully developed the so-called method of approximate inverse operators. He has published about 70 scientific papers and a significant number of textbooks.

Vladimir Alexandrovich always found enough time for his students, and scientific work was his main passion in life. 8 PhD theses defended under his supervisorship. One of his students, Mikhail Gurov, became the teacher of the year in the Russian Federation in 2020.

Vladimir Alexandrovich was distinguished by his modesty and delicacy in relation to colleagues. The bright memory of Dr. V.A. Nogin will always be in the hearts of his colleagues and students.

On behalf of the colleagues and students,

O. G. Avsyankin, A. P. Chegolin, M. N. Gurov, A. N. Karapetyants, D. N. Karasev, B. G. Vakulov

Contents

Weighted Hadamard–Bergman Convolution Operators

Smbat A. Aghekyan and Alexey N. Karapetyants

Abstract Following the ideas of the recent paper by Karapetiants and Samko (Hadamard–Bergman convolution operators. Complex analysis operator theory) we extend the introduced in the mentioned paper notion of Hadamard–Bergman convolution operators to a weighted settings. We treat operators of fractional integration and differentiation as important examples of operators in the above mentioned class, and study mapping properties of certain generalized potentials in generalized Hölder spaces.

Keywords Hadamard–Bergman convolutions · Holomorphic fractional integrals and derivatives · Hölder space

1 Introduction

In the recent paper by A. Karapetyants and S. Samko [16] there was introduced the notion of Hadamard–Bergman convolution. Here we extend the results of the mentioned paper to the weighted case, i.e., we study convolutions

$$g \times f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D},$$

where $\mathrm{dA}_\lambda(z) = (\lambda + 1)(1 - |z|^2)^\lambda \mathrm{dA}(z)$, and $d\mathrm{A}(z) = \frac{1}{\pi} dxdy$, $z = x + iy \in \mathbb{D}$. Here either f and g are both in $H(\mathbb{D})$, or $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$, see Sect. 3 for definitions.

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Introducing the weight parameter is not just a technical issue; it seems more natural to consider such weighted convolutions and corresponding weighted Hadamard–Bergman operators in view of the important examples such as the weighted operators of fractional integration and differentiation (see [26] and also Sect. 5), which are particular cases of the Hadamard–Bergman convolution operators.

Certainly the proofs for the weighted case in most occasions are similar to the unweighted situation, however the corresponding formulas and conclusions can provide some tricky issues in view of the weight parameter. Therefore we prefer to provide a reader with a proof or at least with a sketch of the proof.

The idea can be developed further in the direction of Bergman type operators which appear in the study of generalized holomorphic functions related to Vekua and some other equations. Such study attracts now attention of many authors, see e.g. [5, 6, 14] and references therein. We plan such investigation in another work.

The paper is organized as follows. Section 2 collects necessary preliminaries. In Sect. 3 we give definitions and discuss some properties of weighted Hadamard–Bergman convolutions, and in Sect. 4 we proceed with the corresponding operators. The important examples, the operators of weighted fractional integrodifferentiation, are discussed in Sect. 5. Some mapping properties in weighted Lebesgue spaces are discussed in Sects. 6, and 7 presents mappings by weighted fractional operators in generalized Hölder spaces. This section serves as an example of application of our results.

2 Preliminaries

Let $dA(z) = \frac{1}{\pi} dxdy$ be the normalized Lebesgue measure on the unit disc $\mathbb{D}$. Let $-1 < \lambda < +\infty$, $dA_\lambda(z) = (\lambda + 1)(1 - |z|^2)^\lambda dA(z)$. We equip weighted Lebesgue spaces $L^p_\lambda(\mathbb{D}) = L^p(\mathbb{D}, dA_\lambda)$ with the norm

$$\|f\|_{p,\lambda} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\lambda(z) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \tag{1}$$

and we treat the case $p = \infty$ as usual: $\|f\|_\infty = \text{esssup}_{z \in \mathbb{D}} |f(z)|$. By $\mathcal{A}^p_\lambda(\mathbb{D})$, as usual, we denote the subspace in $L^p_\lambda(\mathbb{D})$ consisting of holomorphic in $\mathbb{D}$ functions (see [3, 4, 9, 10]) and also [26, 27]. Let $H(\mathbb{D})$ be the set of functions f, holomorphic in $\mathbb{D}$, equipped with the topology defined by the countable set of norms

$$\|f\|_n = \sup_{|z| < 1 - \frac{1}{n+1}} |f(z)| = \sup_{|z| < 1 - \frac{1}{n+1}} \left| \sum_{m=0}^{\infty} f_m z^m \right|, \quad n = 1, 2, \ldots, \tag{2}$$

where f_m are the Taylor coefficients of f :

$$f_m = \frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi i}\int_{|\tau|=r}\frac{f(\tau)}{\tau^{m+1}}d\tau \qquad (3)$$
$$= \frac{1}{(\lambda+1)B(r^2; m+1, \lambda+1)}\int_{|w|<r} f(w)\overline{w}^m \mathrm{dA}_\lambda(w),$$

where

$$B(\tau; a, b) = \int_0^\tau t^{a-1}(1-t)^{b-1}dt, \quad a > 0, \quad b > 0, \quad \tau > 0$$

is the incomplete Beta-function, see [7, page 910]. The space $H(\mathbb{D})$ may be identified with the set of series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{such that} \quad \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leqslant 1.$$

It can be viewed as a subspace of the space S of all formal power series $f = \sum_{n=0}^{\infty} f_n z^n$ or, which is the same, with the space of sequences $\{f_n\}_{n=0}^{\infty}$. By $\mathcal{L}(H(\mathbb{D}))$ we denote the set of linear operators on $H(\mathbb{D})$. We refer to the book [25], Chapter 1.3.1, for more details.

The following lemma is intuitive.

Lemma 2.1 *Let* $f \in \mathcal{A}^p_\lambda(\mathbb{D})$, $1 \leqslant p < \infty$, *and* $0 \leqslant \alpha < 2$, *then*

$$\int_{\mathbb{D}} \frac{|f(w)|^p}{|w|^\alpha}\mathrm{dA}_\lambda(w) \leqslant C\|f\|^p_{p,\lambda}, \quad 1 \leqslant p < \infty,$$

where $C > 0$ *does not depend on* f.

The following lemma follows from part (a) of Lemma 3.10 in [27].

Lemma 2.2 *Let* $\alpha + \beta < 2 + \lambda$, $\lambda < \beta < \lambda + 1$. *Then* $\frac{1}{(1-|z|)^\alpha(1-z)^\beta} \in L^1_\lambda(\mathbb{D})$.

We will need the following asymptotic of the ratio of Gamma functions:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\sum_{m=0}^{N}\frac{C_m}{z^m} + z^{a-b}O\left(z^{-N-1}\right), \quad |z| \to \infty, \qquad (4)$$

where $|\arg(z+a)| < \pi$, $C_0 = 1$, and $C_m = \frac{(-1)^m\Gamma(b-a+m)}{m!\Gamma(b-a)}B_m^{a-b+1}(a)$, $m = 1, 2, \ldots, N$, are expressed in terms of generalized Bernoulli polynomials, see [19] and [22, page 17].

3 Weighted Hadamard–Bergman Convolution of Functions

3.1 Convolutions of Holomorphic Functions

Keeping a study of weighted holomorphic spaces in mind, we define the Hadamard product type composition of holomorphic functions in $H(\mathbb{D})$:

$$f(z) = \sum_{m=0}^{\infty} f_m z^m \quad \text{and} \quad g(z) = \sum_{m=0}^{\infty} g_m z^m,$$

as follows

$$f * g(z) = (\lambda + 1) \sum_{k=0}^{\infty} B(k+1, \lambda+1) f_k g_k z^k, \quad z \in \mathbb{D}.$$

See, e.g. [8, 22, 25]) for Hadamard product composition (Hadamard fractional integrodifferentiation) theory. Here the right hand side is well defined for all $f, g \in H(\mathbb{D})$, since $H(\mathbb{D})$ is identified with the set of series $\sum_{n=0}^{\infty} a_n z^n$ with $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leqslant 1$.

From the above formula at least formally we have for $f \in \mathcal{A}^2_\lambda(\mathbb{D})$:

$$f * \overline{f}(z) = \int_{\mathbb{D}} |f(z)|^2 \mathrm{dA}_\lambda(z) = (\lambda + 1) \sum_{m=0}^{\infty} B(m+1, \lambda+1) |f_m|^2.$$

See Theorem 3.1 below for the justification of this formula.

We extend the definition of Hadamard–Bergman convolution given in [16] for the weighted case, though in most places in this paper we skip the word “weighted”.

Definition 1 Let $f, g \in H(\mathbb{D})$. The construction

$$g \times f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}, \tag{5}$$

is called the (weighted) Hadamard–Bergman convolution of functions $g, f \in H(\mathbb{D})$. Here and everywhere in the sequel, the Hadamard–Bergman convolution of holomorphic functions is treated in the improper sense as justified in Theorem 3.1 below.

Theorem 3.1 *Let $f, g \in H(\mathbb{D})$. Then the following statements are valid*

1. For any $0 < r < 1$, $z \in \mathbb{D}$

$$\int_{|w|<r} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w) = \int_{|w|<r} f(z\overline{w})g(w)\mathrm{dA}_\lambda(w)$$
$$= (\lambda+1)\sum_{k=0}^{\infty} B(r^2; k+1, \lambda+1) f_k g_k z^k. \quad (6)$$

2. The integral

$$\int_{\mathbb{D}} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}, \quad (7)$$

exists as improper integral:

$$\int_{\mathbb{D}} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w) = \lim_{r\to 1}\int_{|w|<r} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w)$$
$$= (\lambda+1)\sum_{k=0}^{\infty} B(k+1, \lambda+1) f_k g_k z^k.$$

Proof We have

$$\int_{|w|<r} f(z\overline{w})g(w)\mathrm{dA}_\lambda(w) = \sum_{k=0}^{\infty} f_k z^k \int_{|w|<r} g(w)\overline{w}^k \mathrm{dA}_\lambda(w).$$

It remains to refer to (3); the first equality of (6) immediately follows. For the proof of the second statement we note that from (3) there follows that the integral

$$\int_{\mathbb{D}} f(w)\overline{w}^m \mathrm{dA}_\lambda(w) = \lim_{r\to 1}\int_{|w|<r} f(w)\overline{w}^m \mathrm{dA}_\lambda(w)$$

exists as improper integral for all $f \in H(\mathbb{D})$. Now it suffices to use (6). □

It is clear from the above, but it worth to underline separately that the convolution formula (5) is symmetric in the sense that:

$$g \times f(z) = \int_{\mathbb{D}} g(w)f(z\overline{w})\mathrm{dA}_\lambda(w), \quad (8)$$
$$= \int_{\mathbb{D}} g(z\overline{w})f(w)\mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}.$$

This formula holds in the more general “integration by parts” form:

$$\int_{\mathbb{D}} (\mathcal{D}^m f)(z\overline{w}) g(w) \mathrm{dA}_\lambda(w) = \int_{\mathbb{D}} f(w) (\mathcal{D}^m g)(z\overline{w}) \mathrm{dA}_\lambda(w), \tag{9}$$

where $\mathcal{D}^m f(z) = z^m \left(\frac{d}{dz}\right)^m f(z)$.

3.2 Convolutions of Holomorphic and non Holomorphic Functions

Convolution formula (5) may be in general considered for non necessarily holomorphic functions. In the formula (5) the function g maybe taken non holomorphic, while $f \in H(\mathbb{D})$.

However, in this case we are not able to define the convolution as an improper integral and have to assume $g \in L^1_\lambda(\mathbb{D})$. Also, the symmetry (8) does not hold, in general.

We define the Fourier type transform of a function $g \in L^1_\lambda(\mathbb{D})$:

$$g \in L^1_\lambda(\mathbb{D}) \longrightarrow \mu_{m,\lambda}(g) = \int_{\mathbb{D}} g(w) \overline{w}^m \mathrm{dA}_\lambda(w), \quad m = 0, 1, \dots . \tag{10}$$

In the case $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$ the convolution (5) also reduces to multiplier form, i.e.

$$g \times f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(g) f_m z^m, \quad z \in \mathbb{D}. \tag{11}$$

Therefore, it is clear, that the Hadamard–Bergman convolution (5) of $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$ may be represented as a convolution with a certain holomorphic kernel g_{hol}:

$$g \times f(z) = \int_{\mathbb{D}} g_{hol}(w) f(z\overline{w}) \mathrm{dA}_\lambda(w) \tag{12}$$

The relation between the holomorphic kernel g_{hol} and the initially given non holomorphic kernel g is as follows:

$$g_{hol}(z) = B^\lambda_{\mathbb{D}} g(z) = \int_{\mathbb{D}} \frac{g(w)}{(1 - z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D},$$

where $B^\lambda_{\mathbb{D}}$ is the weighted Bergman projection. Indeed, consider the holomorphic function $g_{hol}(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(g) z^m$ in $\mathbb{D}$, where $\mu_{m,\lambda}(g)$ are given in (10). We have

$$\begin{aligned} g_{hol}(z) &= \sum_{m=0}^{\infty} z^m \frac{1}{(\lambda+1)B(m+1,\lambda+1)} \int_{\mathbb{D}} g(w)\overline{w}^m \mathrm{dA}_\lambda(w) \\ &= \int_{\mathbb{D}} g(w) \left(\sum_{m=0}^{\infty} \frac{1}{(\lambda+1)B(m+1,\lambda+1)} z^m \overline{w}^m \right) \mathrm{dA}_\lambda(w) \\ &= \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w) = B_{\mathbb{D}}^\lambda g(z), \quad z \in \mathbb{D}. \end{aligned}$$

Here we used the known expansion formula for the weighed Bergman kernel for the unit disc:

$$K_\lambda(z,w) := \frac{1}{(1-z\overline{w})^{2+\lambda}} = \sum_{m=0}^{\infty} \frac{1}{(\lambda+1)B(m+1,\lambda+1)} z^m \overline{w}^m, \quad z, w \in \mathbb{D}.$$

The extension of the notion of Hadamard–Bergman convolution for non holomorphic function g is very important for further analysis. An immediate example of the Hadamard–Bergman convolution is the construction given by the weighted Bergman projection $B_{\mathbb{D}}^\lambda$ defined on $g \in L_\lambda^1(\mathbb{D})$ as

$$B_{\mathbb{D}}^\lambda g(z) = \int_{\mathbb{D}} K_\lambda(z,w) g(w) \mathrm{dA}_\lambda(w) = \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w), \ z \in \mathbb{D}.$$

Further development concerns the theory of Toeplitz operators; we plan to threat this issue in another paper.

4 Weighted Hadamard–Bergman Convolution Operators

Fix a function $g \in H(\mathbb{D})$ and define the the Hadamard–Bergman convolution operator as an operator

$$\begin{aligned} \mathbb{K}_g f(z) &= \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w) \\ &= \lim_{r\to 1} \int_{|w|<r} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad f \in H(\mathbb{D}). \end{aligned}$$

We note that for the Hadamard–Bergman convolution operator $\mathbb{K}_g$ with holomorphic kernel $g(z) = \sum_{m=0}^{\infty} g_m z^m$ we have

$$\mathbb{K}_g f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(\mathbb{K}_g) f_m z^m, \ \text{ for } \ f(z) = \sum_{m=0}^{\infty} f_m z^m \in H(\mathbb{D}),$$

where

$$\mu_{m,\lambda}(\mathbb{K}_g) \equiv \mu_{m,\lambda}(g) = (\lambda+1)B(m+1,\lambda+1)g_m, \quad m = 0, 1, \ldots. \tag{13}$$

The converse is also true: every operator $K \in \mathcal{L}(H(\mathbb{D}))$ of the form

$$Kf(z) = \sum_{m=0}^{\infty} \mu_m f_m z^m, \quad z \in \mathbb{D}, \tag{14}$$

(such operator is called a coefficient multiplier and it is automatically continuous, see e.g., [21]) is represented as Hadamard–Bergman convolution with the kernel

$$g(z) = \frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{\mu_m z^m}{B(m+1,\lambda+1)} = K\left(\frac{1}{(1-w)^{2+\lambda}}\right)(z), \quad z \in \mathbb{D}. \tag{15}$$

Indeed, we have

$$\begin{aligned} Kf(z) &= K\left(\sum_{m=0}^{\infty} f_m z^m\right) = \sum_{m=0}^{\infty} f_m \mu_m z^m \\ &= (\lambda+1)\sum_{m=0}^{\infty} B(m+1,\lambda+1) g_m f_m z^m, \end{aligned} \tag{16}$$

where $g_m = \frac{\mu_m}{(\lambda+1)B(m+1,\lambda+1)}$, and we arrive at the Hadamard–Bergman convolution operator with the kernel

$$\begin{aligned} g(z) &= \frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{\mu_m z^m}{B(m+1,\lambda+1)} \\ &= K\left(\frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{z^m}{B(m+1,\lambda+1)}\right) = K\frac{1}{(1-z)^{2+\lambda}}. \end{aligned}$$

The above results can be also obtained from the results of [24] on the general form of Hadamard or coefficient type multipliers (see also [20]).

Fix now a function $g \in L^1_\lambda(\mathbb{D})$. The Hadamard–Bergman convolution operator $\mathbb{K}_g$ is well defined as

$$\mathbb{K}_g f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}. \tag{17}$$

It possesses the multiplier realization

$$\mathbb{K}_g f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(\mathbb{K}_g) f_m z^m, \quad \text{for} \quad f(z) = \sum_{m=0}^{\infty} f_m z^m \in H(\mathbb{D}),$$

where $\mu_{m,\lambda}(\mathbb{K}_g) \equiv \mu_{m,\lambda}(g)$, and $\mu_{m,\lambda}(g)$ is given in (10).

We conclude this section with the following important remark. By a change of variables the convolution operator $\mathbb{K}_g$ with the kernel $g \in L^1_\lambda(\mathbb{D})$ can be written in the form

$$\mathbb{K}_g f(z) = \frac{\lambda+1}{|z|^2} \int_{|w|<|z|} g\left(\frac{\overline{w}}{\overline{z}}\right)\left(1-\left|\frac{\overline{w}}{\overline{z}}\right|^2\right)^{\lambda} f(w)\mathrm{dA}(w), \quad z \in \mathbb{D}. \tag{18}$$

In this form the operator $\mathbb{K}_g$ is well defined in the setting of measurable functions for which the integral converges. It is represented as the integral operator with homogeneous of degree (-2) kernel invariant with respect to rotations. Such operators of the form (18) belong to the class of operators with homogeneous kernels well studied in analysis (see books [11, 12], and review paper [13]). The operators (18) may be also considered as generalized Hardy operators. The algebra of operators with homogeneous kernels is well studied (see [1, 2] for recent development in this direction); these results may be used for the study of the algebra of Hadamard–Bergman convolution operators in the framework of holomorphic functions. We plan to study such questions in another paper. The multidimensional case is of a special interest as well. Hoverer in such a case we most likely need to deal with a general Banach lattices and orthogonally theory (see e.g., [17, 18]) using homogeneous complex polynomials which are substitution of spherical harmonics techniques proved to work very well in the case of operators with homogeneous kernel in real analysis.

5 Operators of Fractional Integrodifferentiation

The operator of fractional integration

$$\mathbb{I}^\alpha_\lambda f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\lambda-\alpha}} \mathrm{dA}_\lambda(w), \quad \alpha > 0, \quad \lambda > -1. \tag{19}$$

is an example of Hadamard–Bergman convolution. Direct calculation provides

$$\mathbb{I}^\alpha_\lambda f(z) = \frac{\Gamma(2+\lambda)}{\Gamma(2+\lambda-\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+2+\lambda-\alpha)}{\Gamma(2+m+\lambda)} f_m z^m, \tag{20}$$

$$\alpha > 0, \lambda \neq -2, -3, \ldots.$$

The formula (20) defines $\mathbb{I}_\lambda^\alpha$ for a wider range of parameter λ, i.e. $\lambda \in \mathbb{R}, \lambda \neq -2, -3, \ldots$.

We will also need the operator of fractional differentiation

$$\mathbb{D}_\lambda^\alpha f(z) = \frac{\Gamma(2+\lambda-\alpha)}{\Gamma(2+\lambda)} \sum_{m=0}^{\infty} \frac{\Gamma(2+m+\lambda)}{\Gamma(m+2+\lambda-\alpha)} f_m z^m, \tag{21}$$
$$\alpha > 0, \lambda - \alpha \neq -2, -3, \ldots,$$

so that

$$\mathbb{D}_\lambda^\alpha \mathbb{I}_\lambda^\alpha f(z) = \mathbb{I}_\lambda^\alpha \mathbb{D}_\lambda^\alpha f(z) = f(z), \quad z \in \mathbb{D}.$$

In order to distinguish between integration and differentiation we prefer to consider $\mathbb{I}_\lambda^\alpha$ and $\mathbb{D}_\lambda^\alpha$ for positive $\alpha > 0$. Clearly the corresponding constructions (20) and (21) make sense for $\alpha \in \mathbb{R}$, and, in particular,

$$\mathbb{D}_\lambda^\alpha = \mathbb{I}_{\lambda-\alpha}^{-\alpha}, \quad \lambda - \alpha \neq -2, -3, \ldots. \tag{22}$$

The operator $\mathbb{I}_\lambda^\alpha$ does not satisfy the semigroup property, however we can indicate the following rule for the composition:

$$\mathbb{I}_\lambda^\alpha \mathbb{I}_{\lambda-\alpha}^\beta = \mathbb{I}_\lambda^{\alpha+\beta}, \tag{23}$$

where $\alpha > 0, \beta > 0$ and neither λ nor $\lambda - \alpha$ equal to $-2, -3, \ldots$.

In the following theorem we find a representation of the operator (21) in convolution terms with the kernel expressed in the term of elementary function. Denote

$$\mathcal{D} f(z) = z \frac{d}{dz} f(z).$$

Theorem 5.2 *Let* $0 < \alpha < 1$ *and* $\lambda - \alpha \neq -2, -3, \ldots$, *then*

$$\mathbb{D}_\lambda^\alpha f(z) = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right) \mathbb{I}_{\lambda-\alpha}^{1-\alpha} f(z), \quad z \in \mathbb{D},$$

where $Ef = f$ *is the identity operator. Hence, for* $\lambda - \alpha > -1$ *we have*

$$\mathbb{D}_\lambda^\alpha f(z) = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right) \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{1+\lambda}} \mathrm{dA}_{\lambda-\alpha}(w), \quad z \in \mathbb{D}.$$

Proof We have $\left(E + \frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^1_\lambda = E$, hence

$$\mathbb{D}^\alpha_\lambda = \mathbb{I}^{-\alpha}_{\lambda-\alpha} = E\mathbb{I}^{-\alpha}_{\lambda-\alpha} = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^1_\lambda \mathbb{I}^{-\alpha}_{\lambda-\alpha}$$
$$= \left(E + \frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^{-\alpha}_{\lambda-\alpha}\mathbb{I}^1_\lambda = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^{1-\alpha}_{\lambda-\alpha},$$

according to (23). □

Lemma 5.3 *Let* $\alpha > 0$, $\beta \in \mathbb{R}$. *Then*

$$\mathbb{I}^\alpha_\lambda \frac{1}{(1-z)^\beta} = \frac{A(z)}{(1-z)^{\beta-\alpha}}, \quad \lambda \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{24}$$

$$\mathbb{D}^\alpha_\lambda \frac{1}{(1-z)^\beta} = \frac{B(z)}{(1-z)^{\beta+\alpha}}, \quad \lambda - \alpha \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{25}$$

where $A, B \in H(\mathbb{D})$, *and* $A \in C(\overline{\mathbb{D}})$, $A(1) = 1$ *when* $\beta - \alpha < 1$, *and* $B \in C(\overline{\mathbb{D}})$, $B(1) = 1$, *when* $\beta + \alpha < 1$.

Besides this for $m = 0, 1, 2, \ldots$

$$\mathbb{I}^\alpha_\lambda \frac{z^m}{(1-z)^{2+\lambda+m}} = \frac{z^m}{(1-z)^{2+\lambda-\alpha+m}}, \quad \lambda \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{26}$$

$$\mathbb{D}^\alpha_\lambda \frac{z^m}{(1-z)^{2+\lambda+m}} = \frac{z^m}{(1-z)^{2+\lambda+\alpha+m}}, \quad \lambda - \alpha \neq -2, -3, \ldots, \quad z \in \mathbb{D}. \tag{27}$$

Proof To prove (24) we calculate Taylor expansion of

$$R(z) \equiv \mathbb{I}^\alpha_\lambda \frac{1}{(1-z)^\beta} - \frac{\Gamma(\beta-\alpha)\Gamma(2+\lambda)}{\Gamma(\beta)\Gamma(2+\lambda-\alpha)}\frac{1}{(1-z)^{\beta-\alpha}}$$

and then use the asymptotic behaviour of Gamma function (4):

$$R(z) = \frac{\Gamma(2+\lambda)}{\Gamma(\beta)\Gamma(2+\lambda-\alpha)}\sum_{m=0}^\infty c_{n,\alpha,\beta,\lambda} z^n,$$
$$c_{n,\alpha,\beta,\lambda} = \frac{\Gamma(n+\beta)}{\Gamma(n+1)}\frac{\Gamma(n+2+\lambda-\alpha)}{\Gamma(n+2+\lambda)} - \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)}$$
$$= O\left(\frac{1}{n^{2+\alpha-\beta}}\right), \quad n \to \infty.$$

Now (24) is clear. The formula (25) follows by (22).

To prove the second statement observe that for $\lambda > -1$ we have

$$\begin{aligned}\mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} &= \int_{\mathbb{D}} \frac{dA_\lambda(w)}{(1-z\overline{w})^{2+\lambda-\alpha}(1-w)^{2+\lambda}} \\ &= \int_{\mathbb{D}} \frac{dA_\lambda(w)}{(1-z\overline{w})^{2+\lambda}(1-w)^{2+\lambda-\alpha}} \\ &= B_{\mathbb{D}}^\lambda \frac{1}{(1-z)^{2+\lambda-\alpha}} = \frac{1}{(1-z)^{2+\lambda-\alpha}},\end{aligned}$$

for $z \in \mathbb{D}$. Therefore,

$$\mathcal{D}^m \mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} = \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} \frac{z^m}{(1-z)^{2+\lambda-\alpha+m}}, \quad z \in \mathbb{D}.$$

From the other side, differentiating under the integral sign we have

$$\begin{aligned}\mathcal{D}^m \mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} &= \mathcal{D}^m \int_{\mathbb{D}} \frac{dA_\lambda(w)}{(1-w)^{2+\lambda}(1-z\overline{w})^{2+\lambda-\alpha}} \\ = z^m \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} &\int_{\mathbb{D}} \frac{\overline{w}^m dA_\lambda(w)}{(1-w)^{2+\lambda}(1-z\overline{w})^{2+\lambda+m-\alpha}} \\ = \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} &\int_{\mathbb{D}} \frac{w^m dA_\lambda(w)}{(1-z\overline{w})^{2+\lambda}(1-w)^{2+\lambda+m-\alpha}},\end{aligned}$$

for $z \in \mathbb{D}$. Now for $\lambda > -1$ the formula (26) follows by comparing the above formulas. The constructions in both sides of the formula (26), as functions of λ are holomorphic in $\mathbb{C}$ except for the poles $\lambda = -2, -3, \ldots$. Therefore, by arguments of analytic continuation, the formula (26) remains valid for all $\lambda \in \mathbb{R}$ except for $\lambda = -2, -3, \ldots$. Finally, the formula (27) follows by (22). □

6 Some Mapping Results in $L_\lambda^p(\mathbb{D})$

6.1 Young Type Theorem

Theorem 6.3 *Let* $f \in \mathcal{A}_\lambda^p(\mathbb{D})$, $g \in L_\lambda^q(\mathbb{D})$, $\lambda > -1$. *Let* $1 \leqslant p, q, r \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} \geqslant 1$, *then*

$$\|\mathbb{K}_g f\|_{r,\lambda} \leqslant C_\lambda \|g\|_{q,\lambda} \|f\|_{p,\lambda}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \tag{28}$$

where the constant does not depend on f *and* g.

Proof The case $r = p = \infty$ is obvious with $C_\lambda = 1$:

$$\|\mathbb{K}_g f\|_\infty \leqslant \|g\|_{1,\lambda}\|f\|_\infty. \tag{29}$$

Let now $r = p = 1$. By Fubini theorem we have

$$\begin{aligned}
\|\mathbb{K}_g f\|_{1,\lambda} &\leqslant \int_{\mathbb{D}} |g(w)|\mathrm{dA}_\lambda(w) \int_{\mathbb{D}} |f(z\overline{w})|\mathrm{dA}_\lambda(z)\\
&= (\lambda+1)\int_{\mathbb{D}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&= (\lambda+1)\int_{|w|<\frac{1}{2}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&+(\lambda+1)\int_{\frac{1}{2}<|w|<1} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z).
\end{aligned}$$

Due to known estimate $|f(z)| \leqslant \frac{\|f\|_{p,\lambda}}{(1-|z|^2)^{\frac{2+\lambda}{p}}}$, $z \in \mathbb{D}$ we have $|f(z)| \leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}$ for $|z| < \frac{1}{2}$, hence

$$\begin{aligned}
&(\lambda+1)\int_{|w|<\frac{1}{2}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}\int_{|w|<\frac{1}{2}} |g(w)|\mathrm{dA}_\lambda(w) \leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}\|g\|_{1,\lambda}.
\end{aligned}$$

For the second term similar estimate is also trivial:

$$\begin{aligned}
&(\lambda+1)\int_{\frac{1}{2}<|w|<1} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant 4(\lambda+1)\int_{\frac{1}{2}<|w|<1} |g(w)|\mathrm{dA}_\lambda(w)\int_{|z|<|w|} |f(z)|\left(1-|z|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant 4\|f\|_{1,\lambda}\|g\|_{1,\lambda}.
\end{aligned}$$

Hence, we obtain

$$\|\mathbb{K}_g f\|_{1,\lambda} \leqslant \left(4+\left(\frac{4}{3}\right)^{2+\lambda}\right)\|g\|_{1,\lambda}\|f\|_{1,\lambda}. \tag{30}$$

In view of (29) and (30), applying Riesz-Thorin-Stein-Weiss interpolation theorem [23] we obtain

$$\|\mathbb{K}_g f\|_{p,\lambda} \leqslant C_\lambda \|g\|_{1,\lambda} \|f\|_{p,\lambda}, \quad 1 \leqslant p \leqslant \infty. \tag{31}$$

Here C_λ does not depend on f and g.

By Hölder inequality

$$\|\mathbb{K}_g f\|_\infty \leqslant \|g\|_{p',\lambda} \|f\|_{p,\lambda}, \quad 1 \leqslant p \leqslant \infty. \tag{32}$$

Combining (31) with (32) and again interpolating between 1 and p' we finally obtain (28). □

As an anonymous reviewer kindly noticed, in fact Theorem 6.3 states that the bilinear operator $B(f; g) = f \times g$ is bounded (with respect to the norm $\|\cdot\|_{r,\lambda}$ on its rage) on the corresponding product of spaces.

6.2 Sobolev Type Theorem

Besides the operator $\mathbb{I}_\lambda^\alpha$, consider also

$$\mathbb{I}_\lambda^{\alpha,+} f(z) = \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2+\lambda-\alpha}} \mathrm{dA}_\lambda(w), \quad \alpha > 0, \quad \lambda > -1, \quad z \in \mathbb{D},$$

and the following two more general operators as well

$$T_{a,b}^\alpha f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2-\alpha+a+b}} \mathrm{dA}_b(w),$$

$$S_{a,b}^\alpha f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2-\alpha+a+b}} \mathrm{dA}_b(w).$$

The following Sobolev type theorem is valid.

Theorem 6.4 *Let* $0 < \alpha < 2$, $1 < p < \frac{2}{\alpha}$, $a + b \geqslant 0$, *and* $b > -\frac{1}{p}(1 - \frac{\alpha}{2})$. *Then*

$$\|T_{a,b}^\alpha f\|_{q,bq} \leqslant C \|f\|_{p,bp}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2},$$

$$\|S_{a,b}^\alpha f\|_{q,bq} \leqslant C \|f\|_{p,bp}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}.$$

Proof It suffices to prove one of the above estimates. For instance, for $T^{\alpha}_{a,b}$ we have

$$\|T^{\alpha}_{a,b}f\|^{q}_{q,bq} =$$

$$= (bq+1)(b+1)^{q}\int_{\mathbb{D}}\left|\int_{\mathbb{D}}\frac{f(w)(1-|w|^{2})^{b}(1-|z|^{2})^{a+b}}{(1-z\overline{w})^{2-\alpha+a+b}}\mathrm{dA}(w)\right|^{q} d\mathrm{A}(z)$$

$$\leq C_{a,b,q,p}\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{|f(w)|(1-|w|^{2})^{b}}{|1-z\overline{w}|^{2-\alpha}}d\mathrm{A}(w)\right)^{q}\mathrm{dA}(z)$$

$$\leq C\left(\int_{\mathbb{D}}|f(w)|^{p}(1-|w|^{2})^{bp}d\mathrm{A}(w)\right)^{\frac{q}{p}} = C_{1}\|f\|^{q}_{p,bp},$$

where we used known boundedness of the (unweighed) operator $\mathbb{I}^{\alpha,+} = \mathbb{I}^{\alpha,+}_{0}$ from $L^{p}(\mathbb{D})$ to $L^{q}(\mathbb{D})$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}$ (see [16]). □

Corollary 6.1 *Let* $0 < \alpha < 2$, $1 < p < \frac{2}{\alpha}$, *and* $\lambda \geqslant 0$. *Then*

$$\|\mathbb{I}^{\alpha}_{\lambda}f\|_{q,\lambda q} \leqslant C\|f\|_{p,\lambda p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2},$$

$$\|\mathbb{I}^{\alpha,+}_{\lambda}f\|_{q,\lambda q} \leqslant C\|f\|_{p,\lambda p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2},$$

and, consequently, $\mathbb{I}^{\alpha}_{\lambda}$ *is bounded from* $\mathcal{A}^{p}_{\lambda p}(\mathbb{D})$ *to* $\mathcal{A}^{q}_{\lambda q}(\mathbb{D})$.

From Corollary 6.1 we immediately obtain the following result.

Theorem 6.5 *Let the multiplier* $\mu_{n,\lambda}(\mathbb{K}_{g}) \equiv \mu_{n,\lambda}(g)$ *of the Hadamard–Bergman operator* $\mathbb{K}_{g}$ *with the kernel* g *satisfies the condition*

$$\mu_{n,\lambda}(\mathbb{K}_{g}) = \frac{C_{1}}{n^{\alpha+\lambda}} + \frac{C_{2}}{n^{\alpha+1+\lambda}} + O\left(\frac{1}{n^{\alpha+2+\lambda}}\right), \quad n\to\infty.$$

Then

$$\|\mathbb{K}f\|_{q,\lambda q} \leqslant C\|f\|_{p,\lambda p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}, \quad 0<\alpha<\frac{2}{p}, \quad \lambda \geqslant 0.$$

7 Fractional Operators in Generalized Hölder Spaces $A^{\omega}(\mathbb{D})$

By the symbol Ω we denote the set of functions $\omega : [0, 2] \to \mathbb{R}_+$ which are modulus of continuity in the usual sense:

1. ω is continuous in a neighborhood of the origin and $\omega(0) = 0$,
2. ω is almost increasing on $[0, 2]$,
3. $\frac{\omega(h)}{h}$ is almost decreasing on $[0, 2]$.

We will use the following Zygmund type conditions:

$$\int_0^t \frac{\omega(s)}{s} ds \le C\omega(t), \quad 0 < t < 2, \tag{33}$$

$$\int_t^2 \frac{\omega(s)}{s^2} ds \le C\frac{\omega(t)}{t}, \quad 0 < t < 2, \tag{34}$$

where C does not depend on t.

Let $\omega : [0, 2] \to \mathbb{R}_+$ be a modulus of continuity. By $L^{\omega}(\mathbb{D})$ denote the space of functions measurable in $\mathbb{D}$ such that

$$|f(z) - f(w)| \le C\omega(|z - w|), \quad z, w \in \mathbb{D}, \tag{35}$$

where C does not depend on z, w. The semi-norm and norm of a function $f \in L^{\omega}(\mathbb{D})$ are respectively given by

$$\|f\|_{\#,L^{\omega}(\mathbb{D})} = \sup_{z,w\in\mathbb{D}} \frac{|f(z) - f(w)|}{\omega(|z - w|)}, \quad \|f\|_{L^{\omega}(\mathbb{D})} = \|f\|_{\#,L^{\omega}(\mathbb{D})} + \|f\|_{L^{\infty}(\mathbb{D})}. \tag{36}$$

The generalized Hölder type space of holomorphic functions in the unit disc with prescribed modulus of continuity, denoted by $A^{\omega}(\mathbb{D})$, is the space of functions f from $L^{\omega}(\mathbb{D})$ holomorphic in $\mathbb{D}$, with the notation

$$\|f\|_{\#,A^{\omega}(\mathbb{D})} = \|f\|_{\#,L^{\omega}(\mathbb{D})}.$$

Under the conditions (33) and (34) the space $A^{\omega}(\mathbb{D})$ possesses equivalent description in terms of behavior of derivative of a function near the boundary $\mathbb{T}$. Indeed by $B^{\omega}(\mathbb{D})$ denote the space of functions holomorphic in $\mathbb{D}$ such that $|f'(z)| \le C\frac{\omega(1-|z|)}{1-|z|}$, $z \in \mathbb{D}$, where C does not depend on z. The semi-norm and norm of a function $f \in B^{\omega}(\mathbb{D})$ are given by

$$\|f\|_{\#,B^{\omega}(\mathbb{D})} = \sup_{z\in\mathbb{D}} |f'(z)| \frac{1 - |z|}{\omega(1 - |z|)}, \quad \|f\|_{B^{\omega}(\mathbb{D})} = \|f\|_{\#,B^{\omega}(\mathbb{D})} + \|f\|_{L^{\infty}(\mathbb{D})}.$$

We will use the following known lemmas given in [16] and in [15] respectively.

Lemma 7.4 ([16]) *Let ω_1 and ω_2 belong to Ω, such that ω_1 satisfies (34), ω_2 satisfies (33), and the product $\omega_1\omega_2$ satisfies (34). Then the following estimate holds true*

$$J_{\omega_1,\omega_2}(z) := \int_{\mathbb{D}} \frac{\omega_1(|1-z\overline{w}|)}{|1-z\overline{w}|^2}\frac{\omega_2(1-|w|)}{1-|w|}\mathrm{dA}(w) \leqslant C\frac{\omega_1(1-|z|)\omega_2(1-|z|)}{(1-|z|)}, \quad z \in \mathbb{D},$$

where the constant C does not depend on $z \in \mathbb{D}$.

Lemma 7.5 ([15]) *Let ω satisfy (33), then $B^{\omega}(\mathbb{D}) \subseteq A^{\omega}(\mathbb{D})$. Let ω satisfy (34), then $A^{\omega}(\mathbb{D}) \subseteq B^{\omega}(\mathbb{D})$. In particular, if ω satisfies both conditions* (33) *and* (34)*, then the spaces $A^{\omega}(\mathbb{D})$ and $B^{\omega}(\mathbb{D})$ coincide up to equivalence of norms.*

Let $a(1-z)$ be a holomorphic function in $z \in \mathbb{D}$. We consider generalized fractional operator

$$\mathcal{I}^a_\lambda f(z) = \int_{\mathbb{D}} \frac{a(1-w)f(z\overline{w})}{(1-w)^{2+\lambda}}\mathrm{dA}_\lambda(w), \quad \lambda > -1. \tag{37}$$

Evidently, if $a(1-z) = (1-z)^{\alpha}$, $\alpha > 0$, then $\mathcal{I}^a_\lambda = \mathbb{I}^{\alpha}_\lambda$. In the sequel we assume that the function a has radial majorant $a^* \in \Omega$, i.e., $|a(z)| \leqslant a^*(|z|)$, $z \in \mathbb{D}$, and the function $a^* = a^*(h)$ possesses properties of modulus of continuity.

Theorem 7.6 *Let $\lambda > -1$. Let a^* and ω belong to Ω, and let $\omega_a = \omega a^*$. Let also ω satisfies (33), a^* satisfies (34), and ωa^* satisfies (34). Then for the operator $\mathcal{I}^a_\lambda$ the following mapping property holds:*

$$\mathcal{I}^a_\lambda : B^{\omega}(\mathbb{D}) \to B^{\omega_a}(\mathbb{D}). \tag{38}$$

Proof For all $z \in \mathbb{D}$, $z \neq 0$ we have

$$\left(\mathcal{I}^a_\lambda f\right)'(z) = \frac{\lambda+1}{z}\int_{\mathbb{D}} \frac{a(1-w)(1-|w|^2)^{\lambda} z\overline{w} f'(z\overline{w})}{(1-w)^{2+\lambda}}\mathrm{dA}(w) = \frac{\lambda+1}{z}\int_{\mathbb{D}} \frac{a(1-z\overline{w})(1-|w|^2)^{\lambda} w f'(w)}{(1-z\overline{w})^{2+\lambda}}\mathrm{dA}(w).$$

Therefore, for $\frac{1}{2} < |z| < 1$ we estimate

$$\left|\left(\mathcal{I}^a f\right)'(z)\right| \leqslant C_\lambda \|f\|_{\#,B^{\omega}(\mathbb{D})} \int_{\mathbb{D}} \frac{a^*(|1-z\overline{w}|)}{|1-z\overline{w}|^2}\frac{\omega(1-|w|)}{1-|w|}\mathrm{dA}(w).$$

It suffices to apply Lemma 7.4 with $\omega_1 = a^*$ and $\omega_2 = \omega$, and then use Lemma 7.5. □

Theorem 7.7 *Let $\lambda > -1$. Let a^* and ω belong to Ω, and let $\omega_a = \omega a^*$. Let also ω and ωa^* satisfy both (33) and (34), and a^* satisfies (34). Then for the operator $\mathcal{I}^a_\lambda$ the following mapping property holds:*

$$\mathcal{I}^a_\lambda : A^\omega(\mathbb{D}) \to A^{\omega_a}(\mathbb{D}). \tag{39}$$

Proof Follows from Theorem 7.6 and Remark 7.5. □

The following theorem is focused on the case of $\mathbb{I}^\alpha_\lambda$. In that case we show that the mapping in Theorem 7.6 is onto.

Theorem 7.8 *Let $0 < \alpha < 1$, $\lambda > -1$, and ω belongs to Ω, and let ω and $\omega_\alpha(h) = h^\alpha \omega(h)$ satisfy both (33) and (34). Then*

$$\mathbb{I}^\alpha_\lambda \left(A^\omega(\mathbb{D})\right) = A^{\omega_\alpha}(\mathbb{D}). \tag{40}$$

Proof The mapping $\mathbb{I}^\alpha_\lambda : A^\omega(\mathbb{D}) \to A^{\omega_\alpha}(\mathbb{D})$ have been proved in Theorem 7.6, since $a(1-z) = (1-z)^\alpha$ for the case of $\mathbb{I}^\alpha_\lambda$. Let us prove that the mapping is “onto”. Since $\mathbb{I}^\alpha_\lambda$ and $\mathbb{D}^\alpha_\lambda$ are Hadamard–Bergman convolutions, we have $f(z) = \mathbb{I}^\alpha_\lambda \mathbb{D}^\alpha_\lambda f(z)$, $z \in \mathbb{D}$. Hence, it suffices to prove that $f \in A^{\omega_\alpha}(\mathbb{D})$ implies $\mathbb{D}^\alpha_\lambda f \in A^\omega(\mathbb{D})$. We have $\mathbb{D}^\alpha_\lambda f = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^{1-\alpha}_{\lambda-\alpha} f$, and $\mathcal{D}\mathbb{I}^{1-\alpha}_{\lambda-\alpha} f = \mathbb{I}^{1-\alpha}_{\lambda-\alpha}\mathcal{D} f$. It remains to show that $\mathbb{I}^{1-\alpha}_{\lambda-\alpha}\mathcal{D} f \in A^\omega(\mathbb{D})$. We have for $\lambda - \alpha > -1$

$$\begin{aligned}\partial_z \mathbb{I}^{1-\alpha}_{\lambda-\alpha}\mathcal{D} f(z) &= (1+\lambda)\int_{\mathbb{D}} \frac{\overline{w}\mathcal{D} f(w)}{(1-z\overline{w})^{2+\lambda}} \mathrm{dA}_{\lambda-\alpha}(w) \\ &= (1+\lambda)\int_{\mathbb{D}} \frac{\overline{w}\mathcal{D} f(w)\mathrm{dA}_\lambda(w)}{(1-z\overline{w})^{2+\lambda}(1-|w|^2)^\alpha}.\end{aligned} \tag{41}$$

In fact, since $f \in A^{\omega_\alpha}(\mathbb{D})$, then the integral in the right side in (41) converges absolutely for $\lambda > -1$, and therefore the formula is true for such λ. Hence, for $z \in \mathbb{D}$ and $\lambda > -1$ we estimate as follows

$$\begin{aligned}\left|\partial_z \mathbb{I}^{1-\alpha}_{\lambda-\alpha}\mathcal{D} f(z)\right| &\leqslant (1+\lambda)\int_{\mathbb{D}} \frac{|f'(w)|(1-|w|^2)^{\lambda-\alpha}}{|1-z\overline{w}|^{2+\lambda}} \mathrm{dA}(w) \\ &\leqslant C_1 \int_{\mathbb{D}} \frac{\omega(1-|w|)}{|1-z\overline{w}|^2(1-|w|)} \mathrm{dA}(w),\end{aligned}$$

where the constant C does not depend on $z \in \mathbb{D}$. Applying Lemma 7.4 we arrive at $\left|\partial_z \mathbb{I}^{1-\alpha}_{\lambda-\alpha}\mathcal{D} f(z)\right| \leqslant C_\lambda \frac{\omega(1-|z|)}{1-|z|}$, $z \in \mathbb{D}$, which in accordance with Lemma 7.5 completes the proof. □

If $\omega(h) = h^\gamma$, $0 < q < 1$ then we use standard notation $A^\gamma_\lambda(\mathbb{D})$ for the corresponding Hölder space

Corollary 7.2 *Let* $\omega(h) = h^\gamma$, $0 < \gamma < 1$, $\lambda > -1$. *Then*

$$\mathbb{I}^\alpha_\lambda(A^q(\mathbb{D})) = A^{\gamma+\alpha}_\lambda(\mathbb{D}), \quad \text{whenever} \quad 0 < \gamma + \alpha < 1.$$

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On Initial Extensions of Mappings

A. B. Antonevich and C. Dolicanin

Abstract We introduce a general concept of an extension for a mapping $f : X_0 \to Y$, where X and Y are topological spaces and $X_0 \subset X$. Three constructions of such extensions are proposed and the corresponding examples are given. In one of them, the extension coincides with the Gelfand transform. The peculiarity of these constructions is that the domain of the extended transformation does not belong to X but is a bundle over a subset of X.

Keywords Extension of mapping · Closable mapping · Bundle space · Gelfand transform

1 Introduction

Let X and Y be topological spaces and let a map $f : X_0 \to Y_0 \subset Y$, where $X_0 \subset X$ be given. The problem under analysis in this paper is to construct in a canonical way a new map F with a wider domain. We present naturally arising different versions of such constructions which serve, in particular, not only for a single map but also for a family of maps $f_\alpha : X_0 \to Y_0$.

Usually, the following notions are considered.

Definition 1 A mapping $F : X_1 \to Y$, where $X_0 \subset X_1 \subset X$, such that $F(x) = f(x)$ for $x \in X_0$, is called *a prolongation* of the map $f : X_0 \to Y$. The prolongation F is said to be a *continuation* if it is continuous.

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The problem of constructing continuations is one of the global problems of analysis, and has been studied intensively. The results in this direction include, for example, the Tietze–Urysohn Theorem, the Hahn–Banach Theorem on the continuation of a linear continuous functional and a lot of others.

If the natural prolongation of f does not exist one arrives at a problem of its extension. The introduction of the concept we are considering here is based on the following reasoning. The initial object is the mapping f defined on a subset $X_0 \subset X$. However, X_0 can also be considered as a subset of other spaces, and this is quite a common situation, as we will see by examples. Therefore, one can waive the requirement that the domain of (an extension) F must be a subset of X and the image should belong to Y.

Definition 2 A mapping F acting from a set $\widehat{X}$ to a set $\widehat{Y}$ will be called an *extension* of the mapping $f : X_0 \to Y_0 \subset Y$ if there exist two embeddings

$$J_X : X_1 \to \widehat{X} \quad and \quad J_Y : Y_1 \to \widehat{Y},$$

where

$$X_0 \subset X_1 \subset X; \quad Y_0 \subset Y_1 \subset Y,$$

such that

$$F(J_X(x)) = J_Y(f(x)) \quad \text{for} \quad x \in X_0.$$

The extension for a family of mappings is defined in a similar way.

The main difference between an extension and a prolongation is that the domain for the extension may not be a subset of X and therefore the main step is to build new spaces $\widehat{X}$ and $\widehat{Y}$. The set $\widehat{X}$ must be constructed in such a way that it can be considered as a "natural domain" for a given mapping f, because it includes objects that can be associated with the function f and its "natural values". Moreover, such a "natural value" does not always belong to a given Y, but it may be an element of a wider set $\widehat{Y}$.

We single out particular cases of extensions in the sense of Definition 2.

If $\widehat{Y} = Y$ and we extend only the domain X, then the extension will be called *initial*.

If $\widehat{X} = X$ and only the range of values Y is extended, then the extension will be called *final*.

Of a particular interest are extensions of f acting from a space $\widehat{X}$ into itself, that is, when $\widehat{X} = \widehat{Y}$. We call such an extension *two-sided*.

We start with recalling well-known specific examples of extensions.

Example 1.1 If f is an analytic function defined on a domain $X_0 \subset \mathbb{C}$, its analytic continuation F can be constructed. It is a function defined on the Riemann surface, which is the "natural domain" of the analytic function, and may not belong to $\mathbb{C}$. Here, we have an example of an initial extension.

If an analytic function has a pole at a point z_0, then by assigning the value ∞ to the function in question at this point we get an extension that acts not in $\mathbb{C}$, but in the Riemann sphere. This is an example of a final extension.

If, at the poles of the analytic continuation F mentioned above we assign to it the values ∞, then we obtain an extension in the sense of Definition 2, where $\widehat{X}$ is a Riemann surface and $\widehat{Y}$ is the Riemann sphere.

Example 1.2 Classic example of an extension is the continuation of a uniformly continuous mapping $f : X_0 \to Y_0$ of metric spaces $X_0,\ Y_0$. In this situation F is a continuous mapping from the completion $\widehat{X}$ of X_0 to the completion $\widehat{Y}$ of Y_0 which is an extension for f.

Example 1.3 One more example is extension of differentiation. The classical differentiation $f : x \to \frac{dx}{dt}$ is a mapping in the space $X = C(\mathbb{R})$ with the domain $C^1(\mathbb{R})$. The differentiation in the sense of distribution theory, acting in the Schwartz space $\mathcal{D}'(\mathbb{R})$ is a two-sided extension of this mapping [1].

Example 1.4 A well-known example of an extension can be found in Geometry. Let X and Y be differentiable manifolds. For a smooth map $f : X \to Y$, its derivative maps the tangent bundle TX into the tangent bundle TY, and it is an extension of f. Here, with every point $x \in X$, there is associated a family points from TX consisting of tangent vectors at the point x.

The sets $\widehat{X}$ that will be constructed henceforth usually have a bundle structure, and it is convenient to use the terminology from the theory of bundle spaces to describe the arising relations.

A triple (E, B, p), where E and B are given sets, and $p : E \to B$ is a surjective map, we will call a *bundle*. Moreover, E is called the bundle space, B is the base of the bundle, and p is called the projection.

The subset of $E_b := p^{-1}(b) \subset E$ is called *the fiber over a pointb*; the space E is represented as a disjoint union of fibers, the points from the fiber E_b will be called *associated with b*. *A bundle section* is a map $S : B \to E$ such that $p(S(x)) = x$ for all $x \in B$ (i.e. S is a right inverse to the projection p). Each section defines an embedding of the base in the bundle space.

A particular important situation is the case when all the fibers are isomorphic to some space V which is called a *typical fiber*.

A bundle is called a *vector bundle* if a typical fiber V is a vector space.

Usually, *topological bundles* are considered, i.e. it is assumed that E and B are topological spaces and the projection p is continuous.

A topological bundle is called *locally trivial* if for every point x from the base, there is a neighborhood W_x such that $p^{-1}(W_x)$ is homeomorphic to the product bundle $W_x \times V$. where V is the typical fiber.

Example 1.5 A more complicated situation arises for an extension of the operation of multiplication in the space of distributions.

The Schwartz distribution space $\mathcal{D}(\mathbb{R})$ contains an everywhere dense subspace of infinitely differentiable functions C^∞ which is a differential algebra. Moreover, for $u \in \mathcal{D}'(\mathbb{R})$ and $a \in C^\infty$ the product $au \in \mathcal{D}'(\mathbb{R})$ is defined, but for arbitrary distributions of u and v the product of uv is not defined.

Formal expressions containing such products appear in different applications, in particular, when considering equations with generalized coefficients, and therefore we arrive at the problem giving a meaning to the expressions uv for arbitrary distributions, i.e. constructing an extension for the multiplication operation. This question has been investigated by different authors in various settings and extensive literature has been devoted to it [2–5].

In the initial formulation of L. Schwartz, the problem of extending the operation of multiplication was to construct a certain commutative differential algebra $\widehat{\mathcal{D}'(\mathbb{R})}$ and embeddings $R : \mathcal{D}'(\mathbb{R}) \to \widehat{\mathcal{D}'(\mathbb{R})}$, where differentiation goes into differentiation and multiplication transfers into multiplication, i.e. the equality

$$R(au) = R(a)R(u) \quad \text{for all} \quad a \in C^\infty,\ u \in \mathcal{D}'(\mathbb{R}). \tag{1}$$

holds.

However, L. Schwartz showed in 1954 that such algebra does not exist and the question of constructing of embeddings of the distribution space in algebras was not considered for some time.

Eventually the problem was considered in a weakened setting: to construct an embedding R of the distribution space in a differential algebra $\widehat{\mathcal{D}'(\mathbb{R})}$, where the differentiation transfers into differentiation, without requiring equality (1).

Under such an embedding, the product of the distributions uv is the element $R(u)R(v)$ from an algebra and the multiplication of the distribution u by the smooth function a is corrected: the 'correct' product $a \times u$ is the element $R(a)R(u) \in \widehat{\mathcal{D}'(\mathbb{R})}$ but not $R(au)$.

The greatest resonance in this direction was caused by the works of J.F. Colombeau who build the so-called algebra of new generalized functions $G(\mathcal{D}'(\mathbb{R}))$, and embedding of the space $\mathcal{D}'(\mathbb{R})$ into it, for which the equality (1) holds for infinitely differentiable functions. The Colombeau algebra $G(\mathcal{D}'(\mathbb{R}))$ is an example of a two-sided extension of the multiplication operation.

The relation between $G(\mathcal{D}'(\mathbb{R}))$ and the distribution space is the following: $G(\mathcal{D}'(\mathbb{R}))$ contains a subspace which is a vector bundle over $\mathcal{D}'(\mathbb{R})$ with infinite-dimensional typical fiber and the remaining elements are of a qualitatively different nature and are not associated with distributions.

The above discussed examples show that in a general case the construction of a desired space $\widehat{X}$ can be considered as a generalization of different known constructions: the tangent bundle to a manifold, the completion of a metric space, the spaces of distributions and algebras of new generalized functions.

In this paper, we consider three methods of constructing the spaces $\widehat{X}$ corresponding to the initial extensions. More general constructions of two-sided extensions will be given in subsequent works.

In the constructions under consideration a certain set of sequences is distinguished, and an equivalence relation is introduced in it. The desired space $\widehat{X}$ is defined then as a factor space for this equivalence relation. Since as we have mentioned the concept of naturalness can be interpreted in different ways it follows that in the problem of constructing of a “natural domain” there arise several constructions corresponding to different settings of the problem.

In some specific cases (as in the case of completions of metric spaces) a clearer implementation for the spaces constructed can be indicated, but in a general situation there is only a description for them as a factor space for a sequences space.

2 Closure of a Non-closable Mapping

Let X and Y be certain topological spaces and let a map $f : X_0 \to Y_0$, $X_0 \subset X$, $Y_0 \subset Y$ be given.

If a sequence $x_n \in X_0$ converges to $x \in X$ then the most natural candidate for the prolongation value at the point x is the element

$$F(x) = \lim_{n\to\infty} f(x_n). \tag{2}$$

Here for the correctness of this formula the two conditions are needed:

(i) existence of the limit;
(ii) independence of these limits from the choice of the sequence x_n.

The first condition can be fulfilled if one starts the extension not from the whole of X but only from the set X_1 consisting of the points x for which there exists a sequence $x_n \in X_0$ converging to x for which the sequence $f(x_n)$ also converges.

Definition 3 A mapping $f : X_0 \to Y$ is called *closable* if the condition (ii) holds; i.e.

$$\left\{\begin{matrix} x_n \to x, & x_n' \to x, & x \in X_1; \\ \lim\limits_{n\to\infty} f(x_n) = y \in Y, & \lim\limits_{n\to\infty} f(x_n') = y' \in Y \end{matrix}\right\} \quad \Longrightarrow \quad y = y'.$$

For a closable map formula (2) defines a prolongation $\overline{f} : X_1 \to Y$, which we will call the *classical closure* of f.

Note that continuity and closability properties of f are not related in the general case.

For linear operators acting in Banach spaces, the above definition coincides with the well-known definition of the closure of an operator, which is meaningful for discontinuous linear operators. Moreover, any continuous linear operator is closable and its closure is also continuous.

For a non-closable map its extension can be considered as well. The corresponding construction naturally involves consideration of a relation between X and Y which is functional with respect to x. Let us recall the necessary definitions, following, for example, [6].

By a *relation between* X *and* Y we call any subset G of the Cartesian product $X \times Y$. The domain $D(G)$ of the relation is the projection of G onto X. The relation G is called *functional with respect to* x iff

$$\{(x, y_1) \in G,\ (x, y_2) \in G\} \quad \Rightarrow \quad y_1 = y_2 .$$

A mapping from X to Y is the relation $G \subset X \times Y$ which is functional with respect to x. Namely, if we use the common definition of a mapping $f : X \to Y$ then the graph

$$Gr(f) = \{(x, f(x)) : x \in D(f)\} \subset X \times Y$$

is a relation between X and Y functional with respect to x.

Moreover, the map $J : D(f) \ni x \to (x, f(x)) \in Gr(f)$ is a bijection (the inverse is projection onto X), which allows us to identify these objects. After such identification, $Gr(f)$ can be considered as the domain of f, and then the action of mapping f is defined as the projection onto the second coordinate:

$$f : Gr(f) \ni (x, y) \to y \in Y.$$

A representation of a map in this form will be called its *normal form*.

This trivial remark shows that the initial data of the problem under consideration can be reformulated. Namely, one can assume that f is defined not on $X_0 \subset X$ but on the subset $Gr(f)$ from the Cartesian product $X \times Y$.

If $G \subset X \times Y$ is an arbitrary relation, then it is obvious that the projection on X is continuous and defines a bundle structure in G. In addition, the projection on Y is a continuous closable mapping, the domain of its closure is $\overline{G}$, and the closure also acts as projection on Y.

Bearing in mind the above presented observations we proceed to the following.

Construction 1 By the *closure of a map* f acting from a topological space X to a topological space Y we mean the map $F := \overline{f}$ defined on the closure of the graph $\overline{Gr(f)} \subset X \times Y$ and acting as a projection on the second coordinate.

In other words, the closure is defined as the classic closure for the normal form of a mapping. In the case of a closable mapping, this definition is equivalent to the classical definition of the closure, but the fundamental difference between this construction and Definition 3 is that such a non-classical closure exists for any mapping and it is an initial extension in the sense of Definition 2, where $\widehat{X} = \overline{G(f)}$.

Note that the closure of $\overline{G(f)}$ depends on the topology: when the topology on X is weakened the closure may turn to be wider and then the fibers of the constructed bundle may increase.

If a map f is not closable, then the relation $\overline{Gr(f)}$ is not functional with respect to x but has a bundle structure, where a fiber may contain several points and the fibers can have different structures. As a result a new effect arises: when passing to the extended domain $\overline{Gr(f)}$ the points from X decompose into a family of elements of a new type.

Moreover, on $\overline{Gr(f)}$ and on the basis the topologies are defined and the bundle is topological.

Thereby, the qualitative difference between the classical closure and the one described above lies in the fact that the latter one is not defined on a subset of X but on a bundle over some subset $X_1 := p_x(\overline{Gr(f)}) \subset X$.

For linear non-closable operators in Banach spaces, such construction of the closure was considered in [7, 8]. For a linear operator A, the closure of its graph $\overline{Gr(A)}$ is a vector subspace in $X \oplus Y$, the projection onto X defines the structure of a vector bundle on it where the typical fiber is the vector space

$$V := G_0 := p^{-1}(0) = \{y \in Y : (0, y) \in G\} \subset Y.$$

In special situations it is convenient to consider a sequential closure. Recall that the *sequential closure* of a subset M of a topological space is the set of limits of sequences of points from M. In the general case, the sequential closure is smaller than the topological closure but for a number of spaces, in particular, for metric ones they do coincide.

The definition of the sequential closure also makes sense for the spaces where a convergence is specified, but topology is not defined. An example is the space of measurable functions with almost everywhere convergence.

In addition elements obtained by the sequential closure may have better properties than elements obtained by the topological closure. For example, consider $C[0, 1]$ as a subset of the space $F[0, 1]$ of all the functions on the interval with the topology of point convergence. The topological closure of the set $C[0, 1]$ is the entire space $F[0, 1]$ while the sequential closure consists of Baire functions of the first class, and it is a part of the space of measurable functions.

Now we give a description of the construction of the sequential closure of a map.

Denote by $X_0(f)$ the set of sequences $x_n \in X_0$, for which the sequence of the corresponding points of the graph $Gr(f)$ converges, i.e. we have

$$\lim x_n = x \in X, \quad \lim f(x_n) = y \in Y.$$

Two such sequences $\{x_n\}$ and $\{x'_n\}$ will be considered to be equivalent if both the mentioned limits do coincide. Then, the closure $\overline{Gr(f)}$ is isomorphic to the set of classes of equivalent sequences from $X_0(f)$.

Let X be a topological vector space and let the sequences $x_n \to x$ and $z_n \to x$ define different elements from the fiber over the point x in the bundle $\overline{Gr(f)}$. Then, $z_n = x_n + v_n$, where the difference $v_n = x_n - z_n$ is infinitesimal (tends to zero in X). Therefore, the closure construction contains the introduction of infinitesimal quantities, similar to that how it is done in non-standard analysis.

We illustrate the approach described by an example of a simple non-closable linear differential operator.

Example 2.1 Consider a linear operator A defined in $C^1[0, 1] \subset L_1[0, 1]$ and acting into $Y = L_1[0, 1] \oplus \mathbb{R}^2$ by the formula

$$Au = (u', u(0), u'(0)).$$

For a comparison, we will consider in parallel the operator A_0 defined in $C^1[0, 1] \subset L_1[0, 1]$ and acting into the direct sum $Y_0 = L_1[0, 1] \oplus \mathbb{R}$ by the formula $A_0 u = (u', u(0))$. This operator is closable and its closure is defined in the space $W^1[0, 1]$, consisting of absolutely continuous functions and acts by the same formula $\overline{A_0} u = (u', u(0))$.

Unlike A_0, the operator A is not closed. The closure of its graph is the subspace

$$\widetilde{X} = \overline{Gr(A)} = \{(u, u', u(0), \xi) : u \in W^1[0, 1], \xi \in \mathbb{R}\} \subset X \times Y.$$

Since ξ is an arbitrary number, this subspace is a topological vector bundle over $W^1[0, 1]$ where the typical fiber

$$V = p^{-1}(0) = \{(0, 0, \xi)\} \subset Y$$

is a one-dimensional space. Moreover, the closure of the operator acts at $u \in \widetilde{X}$ by the formula

$$\overline{A}((u, u', u(0), \xi)) = (u', u(0), \xi) \in Y,$$

and the number ξ can be interpreted as the value of the derivative at point 0 for $u \in W^1[0, 1]$. Then, for example, for an overdetermined Cauchy problem

$$u'(t) = y(t); \quad u(0) = a, \quad u'(0) = b,$$

there exists the solution $u \in \widetilde{X}$ for any right-hand part. Note that for $u \in W^1[0, 1]$, this problem does not have sense, since the value of $u'(0)$ is not defined, and for $u \in C^1[0, 1]$ it is solvable only under the condition $b = y(0)$.

However, such a definition of the space $\widetilde{X}$ causes dissatisfaction, since here, the value of the derivative at the point 0 for a function $u \in W^1[0, 1]$ is formally assigned. At the same time the description of the space $\overline{Gr(A)}$ with the help of sequences turns out to be more meaningful.

Here the space $\overline{Gr(A)}$ is constructed by means of the sequences $u_n \in C^1$ for each of which there exist four limits: two limits of the functional sequences $u_n \to u_0$ and $u'_n \to y$ in $L_1[0, 1]$ and the limits of the two numerical sequences $u_n(0)$ and $u'_n(0)$. Two sequences are called equivalent if for them the four mentioned limits do coincide, and $\overline{Gr(A)}$ can be defined as the set of equivalence classes.

For a comparison we consider a similar construction for the operator A_0. Here the space $\overline{Gr(A_0)} \sim W^1[0, 1]$ can be constructed from the sequences $u_n \in C^1$ for each of which there exist the three limits: two limits of the functional sequences

$u_n \to u_0$ and $u_n' \to y$ and the limit for the numerical sequence $u_n(0)$. In this case, the sequences for which these three limits do coincide are considered to be equivalent.

With such a description, the effect of spreading of a point from $W^1[0, 1]$ onto a family of points from $\overline{Gr(A)}$ is clearly visible: each sequence class defining a point from $\overline{Gr(A_0)}$ contains a lot of different classes defining points from $\overline{Gr(A)}$. Infinitesimal elements are here the classes of sequences such that $u_n \to 0$, $u_n' \to 0$ and $u_n(0) \to 0$.

Here, we can say that each equivalence class from $\overline{Gr(A)}$ associated with a function $u \in W^1[0, 1]$ 'remembers' about the method of approximating of u by u_n, namely, stores information about the behavior of the values of $u_n'(0)$.

One can notice a relation of the content of the construction considered above with the of the so-called singularly perturbed problems. The simplest example is the Cauchy problem for an equation with a small parameter in the second derivative

$$\frac{1}{n}u''(x) + q_1(x)u'(x) + q_0(x)u(x) = f(x),\ u(0) = a,\ u'(0) = b_1, \tag{3}$$

where q_1 and q_2 are a continuous functions and $q_1(x) \neq 0$.

Let u_n be solutions to the problem (3) and w_n be solutions to a similar problem

$$\frac{1}{n}w''(x) + q_1(x)w'(x) + q_0(x)w(x) = f(x),\ w(0) = a,\ w'(0) = b_2.$$

Both these families converge to the same absolutely continuous function u, which is the solution to the Cauchy problem

$$q(x)u'(x) + q_0(x)u(x) = f(x),\ u(0) = a. \tag{4}$$

However, these are different objects that should not be identified, since they approach u differently, and each of them contains some additional information about the approximation method: $u_n'(0) = b_1$, and $w_n'(0) = b_2$. These sequences define different equivalence classes of the space $\overline{Gr(A)}$ and it is natural to consider elements from $\overline{Gr(A)}$, as solutions to the problem (3). Here, the difference $v_n = u_n - w_n$ is an infinitesimal quantity, which corresponds to the point zero in $W^1[0, 1]$, and in $\overline{Gr(A)}$ it is a non-zero quantity.

Example 2.2 Let $L^1_{loc}(\mathbb{R})$ be the space of locally integrable functions. The map $f : u \to f(u)$, where

$$< f(u), \varphi >= \int_{-\infty}^{\infty} u(t)\varphi(t)dt \tag{5}$$

defines an embedding $f : L^1_{loc}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$.

The question about extension of this f is the following:
for which $u \in C(\mathbb{R} \setminus 0)$ *can be constructed a corresponding distribution and what kind can be such distributions?*

For non locally integrable u integral

$$\int_{-\infty}^{\infty} u(t)\varphi(t)dt \tag{6}$$

is divergent for some $\varphi \in \mathcal{D}(\mathbb{R})$ and the problem is to give sense to such integrals for all φ.

The closure of this embedding f allows us to associate distributions with some functions $u \notin L^1_{loc}(\mathbb{R})$. Since the map f is not closable its closure F is defined not on a set consisting of functions, but on a bundle over such a set, and with a function u a family of distributions is associated.

To construct the closure of f we need to define a topological vector space X consisting of measurable functions on $\mathbb{R}$ and define also its vector subspace X_0 consisting of locally integrable functions. As a result of the closure construction we obtain the space $\widehat{X}$, where the closure F is defined. Since the map f is linear $\widehat{X}$ is a vector bundle over the vector subspace $X_1 \subset X$ consisting of these functions that can be associated with distributions.

As an example, let us consider the space $X = C(\mathbb{R}\setminus 0)$ consisting of the functions continuous at $t \neq 0$ with the topology of the uniform convergence on compact subsets of $\mathbb{R} \setminus 0$, and as X_0 we take the subspace $C(\mathbb{R})$.

Let us describe the space $\widehat{X}$ for this example.

Denote by $L = D(\mathbb{R} \setminus 0)$ the subspace of $D(\mathbb{R})$ consisting of the functions φ, each of which is equal to zero at a neighborhood of zero. If $u \in X$ and $\varphi \in L$, then linear functional u_L is defined on L by expresion

$$< u_L, \varphi >:= \int_{-\infty}^{\infty} u(t)\varphi(t)dt.$$

Let $u \in X_1$ and $\xi \in \widehat{(X)}_u$. Here, ξ is a class consisting of the sequences of $u_n \in C(\mathbb{R})$ that converge to u in X and $f(u_n)$ converge to a distribution $U = F(\xi)$ in $\mathcal{D}'(\mathbb{R})$.

If $\varphi \in L$, then

$$< U, \varphi >= \lim \int_{-\infty}^{\infty} u_n(t)\varphi(t)dt = \int_{-\infty}^{\infty} u(t)\varphi(t)dt =< u_L, \varphi > .$$

Since U is continuous, the functional u_L is continuous too and U is a continuation of u_L. For example, if $u(t) = \exp \frac{1}{t}$, then the functional u_L is discontinuous and, therefore, this function does not belong to X_1. It is easy to check that

$$X_1 = \{u \in C(\mathbb{R} \setminus 0) : u = \frac{d^m v}{dt^m}, \text{ where } v \in C(\mathbb{R})\}.$$

If $\eta \in \widehat{(X)}_u$ is another element from the fiber over u, then there exists $v_n \in X_0$ such that $v_n \to u$ and $f(v_n) \to V \in \mathcal{D}'(\mathbb{R})$.

Then $v_n - u_n \to 0$ and is infinitisimal in X. But $f(v_n) - f(u_n) \to V - U$, where the support of distribution $U - V$ is point 0. Each such distribution is a linear combination of derivatives of the δ−function, i.e.

$$V = U + \sum_{k=0}^{N} C_k \delta^{(k)}.$$

Moreover, for any distribution of this kind, it is easy to indicate explicitly a sequence that converges to it. From this, we obtain that in the corresponding vector bundle $\widehat{X}$, the typical fiber V is arranged as an infinite-dimensional vector space consisting of such linear combinations (with arbitrary N).

3 Joint Closure of a Family

The description of the closure using sequences suggests an approach to the closure of a family of mappings $f_\alpha : X_0 \to Y,\ \alpha \in \mathcal{A}$.

Construction 2 Consider the set $\widetilde{X}(\{f_\alpha\})$, consisting of the sequences of points $x_n \in X_0$, with $\lim x_n = x \in X$ and such that for any α there exists $\lim f_\alpha(x_n) \in Y$.

The sequences (x_n) and (x'_n) will be considered to be equivalent if all the indicated limits do coincide for them. Let $\widehat{X}(\{f_\alpha\})$ be the factor-space for this equivalence relation. Then, for any α the map $F_\alpha : \widehat{X}(\{f_\alpha\}) \to Y$, is defined by the formula

$$F_\alpha([(x_n)]) = \lim f_\alpha(x_n). \tag{7}$$

The constructed family of the mappings F_α defined on $\widehat{X}(\{f_\alpha\})$ will be called the *joint closure of the family of mappings* f_α.

Let $X_1 \subset X$ be the subset of limits of the sequences $(x_n) \in \widetilde{X}(\{f_\alpha\})$. Then, the map $p : \widetilde{X}(\{f_\alpha\}) \to X$ where $p([(x_n)]) = \lim x_n \in X_1$, defines the structure of the bundle over X_1. Here, the fiber over a point x consists of the functions on α with values in Y.

Example 3.1 Let $\mathcal{A}$ be the space consisting of the functions f on $X_0 := (0, 1)$ that are left continuous at each point x and have limits $f(x + 0)$ from the right. Let us construct a joint closure for these functions. Let a sequence x_n be converging to $x \in (0, 1)$. For the convergence of $f(x_n)$ (for all f) it is necessary that the sequence converges to x either from the left or from the right, and then $\lim\ f(x_n) = f(x)$ in the first case and $\lim f(x_n) = f(x + 0)$ in the second one. Denoting the class of sequences converging to x from the left by x^- and those converging to x from the right by x^+, we get that in this example the set $\widehat{X}(\{f\})$ is arranged as two copies of the interval (0, 1) with a natural projection onto (0, 1), where the fiber over each point x contains x^+ and x^-.

Therefore the extension of a function f is given by the formula $F(x^+) := f(x+0)$; $F(x^-) := f(x)$. The topology on $\widehat{X}(\{f\})$ is given in the following way: the base of neighborhoods of a point x^+ consists of the sets of the form

$$W(x^+) = (x - \varepsilon, x + \varepsilon)^+ \bigcup (x, x + \varepsilon)^-,$$

and the base of neighborhoods of a point x^- consists of the sets of the form

$$W(x^-) = (x - \varepsilon, x + \varepsilon)^- \bigcup (x - \varepsilon, x)^+,$$

where $M^\pm = \{x^\pm : x \in M\}$ for $M \subset (0, 1)$. This is the weakest topology such that all the functions F are continuous.

We emphasize that in this bundle all the fibers are arranged identically, but the bundle is not locally trivial.

4 Physical Interpretation of the Joint Closure

Let us give one of possible interpretations of the construction of a joint closure from the point of view of applications.

Let the results of observations of a given system be studied by using a set of instruments, which we denote by $\mathcal{A}$. Firstly, we assume that a mathematical model is considered, where it is assumed that the states of the system are points of a space X, and the observation results belong to a space Y. It is also assumed that observational results are known for the states from a certain subset X_0 consisting of 'simple' states, i.e. the family of mappings $f_\alpha : X_0 \to Y,\ \alpha \in \mathcal{A}$, is defined.

The problem is to find (based on the available data) the results of observations for more complex states of the system.

To clarify the practical meaning of the above setting we recall that Z is called the *set of the system states* corresponding to a given set of observations $\mathcal{A}$ if for each element of Z and $\alpha \in \mathcal{A}$ a uniquely determined observation result $< \alpha, z > \in Y$ is given i.e. for a fixed α the function $F_\alpha(z) :=< \alpha, z >$ is defined.

On the other hand it is natural to assume that observations distinguish points z and then z can be identified with the function

$$\xi_z(\alpha) =< \alpha, z > .$$

Thus, the state is a function on $\mathcal{A}$ with values in Y.

For example, in quantum mechanics some C^*-algebra $\mathcal{A}$ is considered as the set of observables and the state is a non-zero positive linear functional on it.

In the situation considered above, the constructed set $\widehat{X}(\{f_\alpha\})$ where the functions F_α are defined has the mentioned properties of the set of states.

Moreover, if among f_α there is a non-closable map then $\widetilde{X}(\{f_\alpha\})$ does not belong to X, but it is a bundle over a subset of X.

Thus, the joint closure clarifies the initial mathematical model and shows that, in a general case, the state space $\widetilde{X}(\{f_\alpha\})$ is not a subset in the originally chosen space X, but it is a bundle over $X_1 \subset X$. Moreover, the points of a given fiber can be interpreted as a hidden parameters such that the results of observations depend on them and they are not significant when considering simple states.

From this point of view, in Example 3.1, the set of simple states is the interval (0,1), and the observation of a simple state is described by a function from $\mathcal{A}$. In this example, the extended state space consists of two copies of the interval (0,1), and the hidden parameter is the sign $\pm$ at x.

5 Gelfand Extension of a Family

Firstly, we describe the joint closure in other terms. Each point of X_0 defines a mapping from $X_0 \times \mathcal{A}$ to $X \times Y$ according to the formula $\kappa_x(x, \alpha) := (x, f_\alpha(x))$. Let $X_0(X_0 \times \mathcal{A})$ is the set of such functions. Then, the domain of the functions F_α is the sequential closure of the set $X_0(X_0 \times \mathcal{A})$ in the space of all the mappings from $X_0 \times \mathcal{A}$ to $X \times Y$ in the topology of pointwise convergence. The following generalization is based on the fact that formula (7) that defines the extended map does not use the convergence of the sequence x_n.

Construction 3 Let $\mathcal{F}(\mathcal{A}, Y)$ be the space all the functions on $\mathcal{A}$ with the values in Y equipped by the topology of pointwise convergence. Each point $x \in X_0$ defines a function $\xi_x \in \mathcal{F}(\mathcal{A}, Y)$ by expression $\xi_x(\alpha) = f_\alpha(x)$. We denote by $X_0(\mathcal{A})$ the set of such functions (i.e. the set of simple states in the terminology of Sect. 4). Let $\overline{X_0(\mathcal{A})}$ be its closure in the space $\mathcal{F}(\mathcal{A}, Y)$. Then, the functions $F_\alpha(\xi) := \xi(\alpha)$ are defined on $\overline{X_0(\mathcal{A})}$.

Since $\overline{X_0(\mathcal{A})}$ is a subset of $\mathcal{F}(\mathcal{A}, Y)$, the induced topology on it is defined and the functions F_α are continuous.

By the *Gelfand extension* of a family of maps f_α we mean the family of continuous maps $F_\alpha(\xi) := \xi(\alpha)$ defined in $\overline{X_0(\mathcal{A})}$.

The *sequential Gelfand extension* of a family of mappings f_α is the family of continuous mappings $F_\alpha(\xi) := \xi(\alpha)$ defined as the sequential closure of the set $X_0(\mathcal{A})$.

The main difference between Construction 3 and the joint closure considered in Sect. 3 is that here the points of $\overline{X_0(\mathcal{A})}$ are not necessarily associated with the points of X. Such points (i.e. those that can not be associated with the points of X) arise when a sequence $x_n \in X_0$ is such that the sequences of values $f_\alpha(x_n)$ converge for all α, but the sequence x_n itself does not converge in X. Then, the class containing such a sequence defines an ideal element that cannot be associated with a point from X.

From the point of view of the concept of Sect. 4 the domain of the Gelfand extension is also the state space of the corresponding system which is wider than the one obtained when constructing the joint extension.

Let us observe that Construction 3 is a generalization of the classical Gelfand transform of a commutative Banach algebra, which clarifies the use of the name.

Let $\mathcal{A}$ be a Banach algebra with a unity which is a subalgebra of the algebra of complex-valued functions on a set X_0. For the algebra $\mathcal{A}$, there is a compact topological space $M(\mathcal{A})$ called *the spectrum of the algebra* or *the space of maximal ideals*. It can be defined as the space of all multiplicative functionals on $\mathcal{A}$ with the topology of pointwise convergence. The map $G : \mathcal{A} \to C(M(\mathcal{A}))$ that takes a function $a \in \mathcal{A}$ to the function $\widehat{a}(\xi) := \xi(a),\ \xi \in M(\mathcal{A})$ is called the *Gelfand transform* of the algebra $\mathcal{A}$.

If $\mathcal{A}$ is C^*-algebra (equipped with *sup*-norm and together with function a contains the complex conjugate function $\overline{a}$), then, according to the Gelfand–Naimark Theorem, the Gelfand transform is an isomorphism of the algebras [9].

Each point $x \in X_0$ defines a multiplicative functional $\xi_x(a) = a(x)$ and the original space X_0 is embedded in $M(\mathcal{A})$. When discussing the Gelfand transform of such algebras, it is usually interpreted that here the space $M(\mathcal{A})$ serves as the "natural domain" for the functions from $\mathcal{A}$.

Let a vector space L of functions on the set M be given. A subset of $M_0 \subset M$ is called *total* for L if once the restriction of a function $a \in L$ onto the set M_0 is zero it follows that $a = 0$. For example, for a family of analytic functions on a domain $M \subset \mathbb{C}$ any set that has a limit point in M is total.

If M is a compact space then by the Tietze–Urysohn Theorem for the space of all continuous functions on M only everywhere dense subsets are total. Therefore, only M itself is closed total subset in M.

Theorem 1 *Let $\mathcal{A}$ be a Banach algebra which is a subalgebra of functions on a set X_0 with a norm $\|\cdot\|$ such that $\|a\| \geq \sup_{X_0} |a(x)|$. Then the space obtained by using the construction of the Gelfand extension is a total closed subset of the space $M(\mathcal{A})$ of the maximal ideals and the Gelfand extension of the functions from the algebra $\mathcal{A}$ is the restriction of the Gelfand transform onto this subset.*

If $\mathcal{A}$ is a C^*-algebra then the space obtained by using the Gelfand extension construction is the space $M(\mathcal{A})$ of the maximal ideals and the Gelfand extension of the functions from the algebra $\mathcal{A}$ coincides with the Gelfand transform.

Proof We apply the Gelfand extension to $\mathcal{A}$ identifying it with a family of functions on X_0. For a fixed x the formula $\psi_x(a) = a(x)$ defines a linear multiplicative functional on $\mathcal{A}$ and the set $X_0(\mathcal{A})$ consists of such functionals, i.e. it is a part of the space $M(\mathcal{A})$. The set $\overline{X_0(\mathcal{A})}$ (i.e. the Gelfand extension) is the closure of $X_0(\mathcal{A})$ in the space of all functions on $\mathcal{A}$.

The points from this closure are also linear multiplicative functionals, and therefore $\overline{X_0(\mathcal{A})}$ is a closed subset of $M(\mathcal{A})$, and the functions constructed using the

Gelfand extension are the restrictions of the Gelfand transforms $\widehat{a}$ to $M_0 := \overline{X_0(\mathcal{A})}$. Moreover, M_0 is a total closed subset.

If $\mathcal{A}$ is a C^*-algebra then any continuous function in $M(\mathcal{A})$ is a Gelfand transform of some a and, since $\overline{X_0(\mathcal{A})}$ is a total closed subset it coincides with $M(\mathcal{A})$.

Note that the density of embedding X_0 in the space $M(\mathcal{A})$ of maximal ideals holds also for a number of algebras that do not have C^*-structure, in particular, for symmetric algebras $\mathcal{A}$ (i.e when $a \in \mathcal{A}$ implies $\overline{a} \in \mathcal{A}$).

When constructing an extension for elements of an algebra $\mathcal{A}$ it suffices to construct an extension for a system of generators. The space of functions on a countable set with the topology of pointwise convergence is metrizable and the sequential closure of it coincides with the topological closure. Therefore if an algebra $\mathcal{A}$ has a countable system of generators then the sequential extension coincides with the strong one.

The next statement highlights situations when the space of maximal ideals can be constructed quite explicitly by using the mapping closure operation.

Theorem 2 *Let $\mathcal{A}_0$ be a C^*-algebra which is a subalgebra of functions on the set X_0 and $M(\mathcal{A}_0)$ is its maximal ideals space. If the C^*-algebra $\mathcal{A}$ is obtained from $\mathcal{A}_0$ by adding a function g defined on X_0 then the space of maximal ideals $M(\mathcal{A})$ is homeomorphic to the closure of the graph of g as a function defined on a dense subset of $M(\mathcal{A}_0)$, i.e.*

$$M(\mathcal{A}) \sim \overline{Gr(g)} \subset M(\mathcal{A}_0) \times \mathbb{C}.$$

The proof can be obtained by a direct verification.

Example 5.1 Let $X = X_0 = \mathbb{R}$. We start with the algebra $\mathcal{A}_0$ consisting of continuous functions on $\mathbb{R}$ that have limits at infinity. Then the Gelfand extension of these functions is defined on the extended real line $\overline{\mathbb{R}}$ obtained by adding the point ∞ to $\mathbb{R}$ which is represented as an equivalence class consisting of all sequences diverging to ∞ and the extended map is defined at this point as

$$\widehat{a}(\infty) := \lim_{x \to \infty} a(x).$$

Here $\overline{\mathbb{R}}$ is the space of maximal ideals of the algebra $\mathcal{A}_0$.

Next, we consider the algebra $\mathcal{A}_1$ obtained by adding to $\mathcal{A}_0$ the function $g_1(x) = e^{ix}$ which is not defined at the point ∞. To construct extensions of the functions from $\mathcal{A}_1$ it is enough to construct the closure for g_1. According to the closure construction among sequences tending to infinity, we distinguish the classes

$$K_\tau = \{(x_n) : \exp(ix_n) \to \tau\}, \qquad \tau \in \mathbb{C}, \quad |\tau| = 1.$$

The set of such equivalence classes is arranged as a circle, and therefore, the set where the extensions $\widehat{f}$, $f \in \mathcal{A}_1$, are defined has a bundle structure over the extended real line where the fiber over the point ∞ is a circle, and the fibers over the points from $\mathbb{R}$ consist of one point. If we equip this space with the weakest topology where all the functions $\widehat{f}$ are continuous then this space is homeomorphic to $M(\mathcal{A}_1)$, and the map $a \to \widehat{a}$ coincides with the of Gelfand transform.

Finally, we consider a wider algebra $\mathcal{A}_2$ obtained by adding to $\mathcal{A}_1$ the function $g_2(x) := e^{i\pi x}$. Then, according to the construction of the closure, in each of the classes K_τ we define the smaller classes

$$K_{\tau\eta} = \left\{(x_n) : e^{(ix_n)} \to \tau, \quad e^{(i\pi x_n)} \to \eta\right\}, \qquad \eta \in \mathbb{C}, \quad |\eta| = 1.$$

The set of all such classes is constructed as a torus $\mathbb{T}^2$, and as a result, we get that the space $M(\mathcal{A}_2)$ has a bundle structure over $\overline{\mathbb{R}}$ where the fiber over the point ∞ is the torus.

If we consider g_2 as a function on $M(\mathcal{A}_1)$ defined on a dense subset then this subset is not closed and the closure of its graph is

$$\overline{Gr(g_2)} \subset M(\mathcal{A}_1) \times \mathbb{C}$$

homeomorphic to $M(\mathcal{A}_2)$.

6 Conclusion

In the paper we consider three constructions of initial extensions. Having their own value and applications they also serve as introductory steps to the global aim of our investigation: to give a construction of two-sided extension such that the space of distributions and the algebras of new generalized functions can be obtained as particular cases. Such a construction will be presented in subsequent works.

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On Multidimensional Integral Operators with Homogeneous Kernels in Classes with Asymptotics

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Abstract We define a special class of functions having a given asymptotic behavior in a neighborhood of zero. It is proved that this class is invariant under multidimensional integral operators with homogeneous kernels.

Keywords Integral operator · Homogeneous kernel · Asymptotics · Space of continuous functions

1 Introduction

Integral operators with homogeneous kernels play an important role in mathematics and applications. The study of multidimensional integral operators with homogeneous kernels of degree $(-n)$ was started by L. G. Mikhailov (see, e.g., [1, 2]) in research on the theory of elliptic differential equations. More precisely, such operators naturally arise when the potential method is applied to equations of the form

$$|x|^2 \Delta u + |x| \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + b(x)u = 0$$

in domain D containing the point $x = 0$. Operators with homogeneous kernels are also used in some problems of mechanics. The study of integral operators with homogeneous kernels was continued by N. K. Karapetiants. He obtained necessary and sufficient conditions of boundedness for such operators and he investigated the compactness of such operators with variable coefficients (see [3]). In recent decades the theory of integral operators with homogeneous kernels has been

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actively developed. For such operators, the criteria for invertibility and the Fredholm property were obtained, Banach algebras generated by these operators were studied, and conditions for the projection method to apply were obtained (see, e.g., [3–11] and the bibliography therein). But despite significant advances, there are still many open questions and unsolved problems in this field.

In this paper multidimensional integral operators whose kernels are homogeneous of degree $(-n)$ and invariant with respect to all rotations of the space $\mathbb{R}^n$ are considered in classes with asymptotics. More precisely, we introduce the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$, which consists of functions defined on the ball $\mathbb{B}_n$ with a given asymptotic behavior in the neighborhood of zero. It is shown that this class is invariant with respect to the considered integral operators. In the future, these results may be applied to the study of asymptotics of solutions of integral equations. In conclusion, we note that for operators with difference kernels similar results are contained in [12–14].

We use the following notation:

- $\mathbb{R}^n$ is the n-dimensional Euclidean space;
- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;
- $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$;
- $x' = x/|x|$;
- $\mathbb{Z}_+$ is the set of nonnegative integers;
- $\mathbb{B}_n(a) = \{x \in \mathbb{R}^n : \ |x| \leqslant a\}$; $\mathbb{B}_n = \mathbb{B}_n(1)$;
- $C(\mathbb{B}_n(a))$ is the space of all continuous complex-valued functions on $\mathbb{B}_n(a) \setminus \{0\}$ having finite limit as $x \to 0$;
- $C_0(\mathbb{B}_n(a)) = \{g \in C(\mathbb{B}_n(a)) \colon \lim\limits_{x\to 0} g(x) = 0\}$.

2 Statement of the Problem

Consider the operator

$$(K\varphi)(x) = \int\limits_{\mathbb{B}_n} k(x, y)\varphi(y)\, dy, \quad x \in \mathbb{B}_n, \tag{1}$$

where the function $k(x, y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ (from now on we assume that $n \geqslant 2$) and satisfies the following conditions:

1° homogeneity of degree $(-n)$, i. e.

$$k(\alpha x, \alpha y) = \alpha^{-n} k(x, y), \quad \forall\, \alpha > 0;$$

2° invariance with respect to the rotation group $SO(n)$, i. e.

$$k(\omega(x), \omega(y)) = k(x, y), \quad \forall\, \omega \in SO(n);$$

3° integrability, i. e.

$$\int\limits_{\mathbb{R}^n} |k(e_1, y)|(1 + |\ln|y||)^{\mu}\, dy < \infty, \quad e_1 = (1, 0, \dots, 0),$$

where μ is fixed nonnegative number.

For example, the function

$$k(x, y) = \frac{|x|}{|y|^{3/2}|x - y|^{n-1/2}} \left(1 + \left|\ln \frac{|y|}{|x|}\right|\right)^{-\mu}$$

satisfies the conditions 1°–3°.

It is known [1, 2] that the operator K is bounded from $C(\mathbb{B}_n)$ to $C(\mathbb{B}_n)$ and from $C_0(\mathbb{B}_n)$ to $C_0(\mathbb{B}_n)$.

In the space $C(\mathbb{B}_n)$ we define the special class of functions with a specified asymptotic behavior at zero.

Definition 1 Let $0 < \alpha \leqslant 1$, $0 < \delta < 1$ and $s \in \mathbb{Z}_+$. The class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is the set of all functions $g \in C(\mathbb{B}_n)$ such that for $|x| < \delta$ the representation

$$g(x) = b + \sum_{j=0}^{s} \frac{b_j}{(1 - \ln|x|)^{j+\alpha}} + \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v \in C_0(\mathbb{B}_n(\delta))$$

holds.

Class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is adapted for integral operators with homogeneous kernels that have a singularity at zero. This class has in the theory of operators of the form (1) the same role that the asymptotics in powers of the function $\frac{1}{t+1}$ has in the theory of one-dimensional convolution operators.

The main goal of this paper is to establish that the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is invariant with respect to the operator K, i. e. to prove that $K(\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)) \subset \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

3 Auxiliary Statements

Consider the function $(1 - t)^{-\alpha}$, where $\alpha > 0$ and $t \in (-\infty, 1)$. Applying the Taylor formula, we obtain the equality

$$\frac{1}{(1 - t)^{\alpha}} = 1 + \sum_{m=1}^{s} \frac{\alpha(\alpha + 1)\dots(\alpha + m - 1)}{m!} t^m + R_s(t), \tag{2}$$

where

$$R_s(t) = \frac{\alpha(\alpha+1)\dots(\alpha+s)}{(s+1)!}(1-\xi)^{-\alpha-s-1}t^{s+1} \tag{3}$$

and ξ between 0 and t.

Lemma 1 *For any fixed $s > 0$ the estimate*

$$|R_s(t)| \leqslant \begin{cases} C_1|t|^{s+1}, & t < 0, \\ C_2\dfrac{t^{s+1}}{(1-t)^\alpha}, & 0 < t < 1, \end{cases} \tag{4}$$

holds, where C_1 and C_2 are some constants depending on s.

Proof If $t < 0$ then $\xi < 0$ and the estimate (4) follows immediately from (3) with

$$C_1 = \frac{\alpha(\alpha+1)\dots(\alpha+s)}{(s+1)!}.$$

Let $t \in (0, 1)$. We consider two cases.

1. If $t \in (0, 1/2]$ then from (3) follows the estimate

$$|R_s(t)| \leqslant C_1 2^{\alpha+s+1} t^{s+1} \leqslant \widetilde{C}_1 \frac{t^{s+1}}{(1-t)^\alpha},$$

   where $\widetilde{C}_1 = 2^{\alpha+s+1}C_1$.
2. If $t \in (1/2, 1)$ then from formula (2) we obtain

$$|R_s(t)| \leqslant \frac{1}{(1-t)^\alpha} + \left|1 + \sum_{m=1}^{s} \frac{\alpha(\alpha+1)\dots(\alpha+m-1)}{m!} t^m\right|$$
$$\leqslant \frac{1}{(1-t)^\alpha} + C \leqslant \frac{1}{t^{s+1}} \frac{1+C}{(1-t)^\alpha} t^{s+1} \leqslant \widetilde{C} \frac{t^{s+1}}{(1-t)^\alpha},$$

   where $\widetilde{C} = 2^{s+1}(1+C)$. Put $C_2 = \max\{\widetilde{C}, \widetilde{C}_1\}$.

□

Lemma 2 *Let the function $k(x, y)$ satisfies the conditions $1°$–$3°$. If $v \in C_0(\mathbb{B}_n(a))$ then the function*

$$\psi(x) = \int\limits_{|y|\leqslant a} |k(x, y)| \left(1 + \left|\ln \frac{|y|}{|x|}\right|\right)^\mu v(y)\, dy$$

belongs to the space $C_0(\mathbb{B}_n(a))$.

Proof Let us consider the integral operator

$$(Qv)(x) = \int\limits_{\mathbb{B}_n(a)} q(x, y)v(y)\,dy, \quad x \in \mathbb{B}_n(a),$$

where

$$q(x, y) = |k(x, y)| \left(1 + \left|\ln \frac{|y|}{|x|}\right|\right)^{\mu}.$$

Its kernel $q(x, y)$ satisfies the conditions 1°, 2° and integrability condition

$$\int\limits_{\mathbb{R}^n} |q(e_1, y)|\,dy < \infty.$$

Therefore, the operator Q is bounded in the space $C_0(\mathbb{B}_n(a))$ (see [2]). Then $\psi = Qv \in C_0(\mathbb{B}_n(a))$. □

Let $0 < \alpha \leqslant 1$, $0 < a \leqslant 1$ and $s \in \mathbb{Z}_+$. Denote by $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$ the class consisting of all functions $g \in C(\mathbb{B}_n(a))$ such that

$$g(x) = b + \sum_{j=0}^{s} \frac{b_j}{(1 - \ln|x|)^{j+\alpha}} + \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}},$$

where $v \in C_0(\mathbb{B}_n(a))$. Consider the operator

$$(K_a\varphi)(x) = \int\limits_{\mathbb{B}_n(a)} k(x, y)\varphi(y)\,dy, \quad x \in \mathbb{B}_n(a),$$

where the function $k(x, y)$ satisfies the conditions 1°–3°. We wish to study the action of the operator K_a in the class $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$. First of all we consider the two simplest cases.

Lemma 3 *Let numbers s and μ be such that*

$$s \leqslant [\mu] - 1. \tag{5}$$

If $\varphi(x) = C = \text{const}$, *then*

$$(K_a\varphi)(x) = c + \frac{v_0(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v_0 \in C_0(\mathbb{B}_n(a)).$$

Proof It suffices to assume that $C = 1$. We have equality

$$\int\limits_{|y|\leqslant a} k(x,y)\,dy = \int\limits_{\mathbb{R}^n} k(x,y)\,dy - \int\limits_{|y|>a} k(x,y)\,dy.$$

We make the change of variables $y = |x|z$ in the integrals on the right and use the property 1° of the function $k(x, y)$. Then we make the rotation change of variables $z = \omega_x(t)$, where ω_x is an element of the group $SO(n)$ for which $\omega_x(e_1) = x'$ and use the property 2°. As a result, we obtain

$$\int\limits_{|y|\leqslant a} k(x,y)\,dy = \int\limits_{\mathbb{R}^n} k(e_1,t)\,dt - \int\limits_{|t|>\frac{a}{|x|}} k(e_1,t)\,dt = c + \frac{v_0(x)}{(1-\ln|x|)^{s+\alpha}},$$

where

$$c = \int\limits_{\mathbb{R}^n} k(e_1,t)\,dt, \quad v_0(x) = -\,(1-\ln|x|)^{s+\alpha} \int\limits_{|t|>\frac{a}{|x|}} k(e_1,t)\,dt.$$

Let us prove that $v_0 \in C_0(\mathbb{B}_n(a))$. It's easy to see that the function $v_0(x)$ is continuous on $\mathbb{B}_n(a) \setminus \{0\}$. Find the limit of the function $v_0(x)$ as $x \to 0$. Using the obvious inequality

$$1+\ln\frac{1}{|x|} \leqslant \left(1+\ln\frac{1}{a}\right)\left(1+\ln\frac{a}{|x|}\right) \tag{6}$$

and taking (5) into account, we get the estimate

$$\begin{aligned}
|v_0(x)| &\leqslant \left(1+\ln\frac{1}{|x|}\right)^{s+\alpha} \int\limits_{|t|>\frac{a}{|x|}} |k(e_1,t)|\,dt \\
&\leqslant \left(1+\ln\frac{1}{a}\right)^{s+\alpha}\left(1+\ln\frac{a}{|x|}\right)^{s+\alpha} \int\limits_{|t|>\frac{a}{|x|}} |k(e_1,t)|\,dt \\
&\leqslant \left(1+\ln\frac{1}{a}\right)^{s+\alpha} \int\limits_{|t|>\frac{a}{|x|}} |k(e_1,t)|\,(1+|\ln|t||)^{s+\alpha}\,dt \\
&\leqslant \left(1+\ln\frac{1}{a}\right)^{s+\alpha} \int\limits_{|t|>\frac{a}{|x|}} |k(e_1,t)|\,(1+|\ln|t||)^{\mu}\,dt.
\end{aligned}$$

Due to the condition 3° the last integral tends to zero as $x \to 0$. Therefore, $v_0(x) \to 0$ as $x \to 0$. □

Lemma 4 *Let inequality (5) hold. If*

$$\varphi(x) = \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v \in C_0(\mathbb{B}_n(a)),$$

then

$$(K_a\varphi)(x) = \frac{w(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad w \in C_0(\mathbb{B}_n(a)).$$

Proof We write the equality

$$\int\limits_{|y|\leqslant a} k(x, y)\frac{v(y)}{(1 - \ln|y|)^{s+\alpha}}\,dy = \frac{w(x)}{(1 - \ln|x|)^{s+\alpha}},$$

where

$$w(x) = \int\limits_{|y|\leqslant a} k(x, y)\left(\frac{1 - \ln|x|}{1 - \ln|y|}\right)^{s+\alpha} v(y)\,dy.$$

We will prove that $w \in C_0(\mathbb{B}_n(a))$. It is clear that the function $w(x)$ is continuous on $\mathbb{B}_n(a) \setminus \{0\}$. Moreover,

$$\begin{aligned} |w(x)| &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left|\frac{1 - \ln|x|}{1 - \ln|y|}\right|^{s+\alpha} |v(y)|\,dy \\ &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left(1 + \left|\ln\frac{|y|}{|x|}\right|\right)^{s+\alpha} |v(y)|\,dy \\ &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left(1 + \left|\ln\frac{|y|}{|x|}\right|\right)^{\mu} |v(y)|\,dy. \end{aligned}$$

The last integral tends to zero as $x \to 0$ by Lemma 2. Therefore, $w \in C_0(\mathbb{B}_n(a))$. □

Now we turn to the general case.

Lemma 5 *Let inequality (5) hold. Then the class* $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$ *is invariant with respect to the operator* K_a.

Proof Consider the function

$$\varphi(x) = b + \sum_{j=0}^{s} \frac{b_j}{(1-\ln|x|)^{j+\alpha}} + \frac{v(x)}{(1-\ln|x|)^{s+\alpha}},$$

where $v \in C_0(\mathbb{B}_n(a))$. We wish to prove that the function

$$(K_a\varphi)(x) = b \int\limits_{|y|\leqslant a} k(x,y)\,dy + \sum_{j=0}^{s} b_j \int\limits_{|y|\leqslant a} k(x,y)\frac{dy}{(1-\ln|y|)^{j+\alpha}}$$

$$+ \int\limits_{|y|\leqslant a} k(x,y)\frac{v(y)}{(1-\ln|y|)^{s+\alpha}}\,dy \tag{7}$$

belongs to the class $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$. To prove this, it suffices to show that each term on the right side of (7) belongs to the class $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$.

The first and last terms in (7) belong to the class $\boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$ by virtue of Lemma 3 and Lemma 4. Consider the other terms. We must prove that

$$\mathcal{I}_j(x) := \int\limits_{|y|\leqslant a} k(x,y)\frac{dy}{(1-\ln|y|)^{j+\alpha}} \in \boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$$

for each $j = 0, 1, \dots, s$. We consider two cases: $j = 0$ and $j > 0$.

1. Let $j = 0$. We make the change of variables $y = |x|z$ in the integral $\mathcal{I}_0(x)$ and use the property 1°, and then we make the change of variables $z = \omega_x(t)$ and use 2°. As a result, we get

$$\mathcal{I}_0(x) = \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t)\,(1-\ln|x|-\ln|t|)^{-\alpha}\,dt$$

$$= \frac{1}{(1-\ln|x|)^{\alpha}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t)\left(1-\frac{\ln|t|}{1-\ln|x|}\right)^{-\alpha} dt.$$

Applying the Taylor formula to the function $\left(1-\frac{\ln|t|}{1-\ln|x|}\right)^{-\alpha}$, we obtain

$$\mathcal{I}_0(x) = \frac{1}{(1-\ln|x|)^{\alpha}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t)\left(\sum_{m=0}^{s} c_m \frac{\ln^m|t|}{(1-\ln|x|)^m} + R_s\left(\frac{\ln|t|}{1-\ln|x|}\right)\right) dt,$$

where

$$c_0 = 1, \quad c_m = \frac{\alpha(\alpha+1)\dots(\alpha+m-1)}{m!}, \quad m = 1, 2, \dots, s,$$

and $R_s(\tau)$ is the function of the form (3). Then

$$\begin{aligned}
\mathcal{I}_0(x) &= \sum_{m=0}^{s} \frac{c_m}{(1-\ln|x|)^{\alpha+m}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t)\ln^m|t|\,dt \\
&\quad + \frac{1}{(1-\ln|x|)^{\alpha}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t) R_s\left(\frac{\ln|t|}{1-\ln|x|}\right) dt \\
&= \sum_{m=0}^{s} \frac{c_m}{(1-\ln|x|)^{\alpha+m}} \left(\int\limits_{\mathbb{R}^n} k(e_1,t)\ln^m|t|\,dt - \int\limits_{|t|> \frac{a}{|x|}} k(e_1,t)\ln^m|t|\,dt \right) \\
&\quad + \frac{1}{(1-\ln|x|)^{\alpha}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t) R_s\left(\frac{\ln|t|}{1-\ln|x|}\right) dt.
\end{aligned}$$

Let $\gamma_m = c_m \int\limits_{\mathbb{R}^n} k(e_1,t)\ln^m|t|\,dt$, where $0 \leqslant m \leqslant s$. Then we obtain

$$\begin{aligned}
\mathcal{I}_0(x) &= \sum_{m=0}^{s} \frac{\gamma_m}{(1-\ln|x|)^{\alpha+m}} - \sum_{m=0}^{s} \frac{c_m}{(1-\ln|x|)^{\alpha+m}} \int\limits_{|t|> \frac{a}{|x|}} k(e_1,t)\ln^m|t|\,dt \\
&\quad + \frac{1}{(1-\ln|x|)^{\alpha}} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t) R_s\left(\frac{\ln|t|}{1-\ln|x|}\right) dt \\
&= \sum_{m=0}^{s} \frac{\gamma_m}{(1-\ln|x|)^{\alpha+m}} + \frac{1}{(1-\ln|x|)^{s+\alpha}} \left(\sum_{m=0}^{s} w_m(x) + w(x) \right), \qquad (8)
\end{aligned}$$

where

$$w_m(x) = -c_m(1-\ln|x|)^{s-m} \int\limits_{|t|> \frac{a}{|x|}} k(e_1,t)\ln^m|t|\,dt, \quad m = 0, 1, \dots, s;$$

$$w(x) = (1-\ln|x|)^{s} \int\limits_{|t|\leqslant \frac{a}{|x|}} k(e_1,t) R_s\left(\frac{\ln|t|}{1-\ln|x|}\right) dt.$$

We wish to show that $w_m, w \in C_0(\mathbb{B}_n(a))$. Since the functions $w(x)$ and $w_m(x)$ are continuous on $B_n(a) \setminus \{0\}$, we will find the limits of these functions as $x \to 0$. Using the inequality (6), we have

$$\begin{aligned}|w_m(x)| &\leqslant c_m \left(1 + \ln \frac{1}{|x|}\right)^{s-m} \int\limits_{|t| > \frac{a}{|x|}} |k(e_1, t)| |\ln |t||^m \, dt \\ &\leqslant c_m \left(1 + \ln \frac{1}{a}\right)^{s-m} \left(1 + \ln \frac{a}{|x|}\right)^{s-m} \int\limits_{|t| > \frac{a}{|x|}} |k(e_1, t)|(1 + |\ln |t||)^m \, dt \\ &\leqslant c_m \left(1 + \ln \frac{1}{a}\right)^{s-m} \int\limits_{|t| > \frac{a}{|x|}} |k(e_1, t)|(1 + |\ln |t||)^s \, dt.\end{aligned}$$

Last integral tends to zero as $x \to 0$ due to the property 3°. Hence $w_m \in C_0(\mathbb{B}_n(a))$. Further,

$$\begin{aligned}|w(x)| &\leqslant (1 - \ln |x|)^s \int\limits_{|t| \leqslant \frac{a}{|x|}} |k(e_1, t)| \left| R_s \left(\frac{\ln |t|}{1 - \ln |x|} \right) \right| dt \\ &= (1 - \ln |x|)^s \int\limits_{|t| \leqslant 1} |k(e_1, t)| \left| R_s \left(\frac{\ln |t|}{1 - \ln |x|} \right) \right| dt \\ &\quad + (1 - \ln |x|)^s \int\limits_{1 < |t| \leqslant \frac{a}{|x|}} |k(e_1, t)| \left| R_s \left(\frac{\ln |t|}{1 - \ln |x|} \right) \right| dt.\end{aligned}$$

Applying the estimate (4), we get

$$\begin{aligned}|w(x)| &\leqslant \frac{C_1}{1 - \ln |x|} \int\limits_{|t| \leqslant 1} |k(e_1, t)| |\ln |t||^{s+1} \, dt \\ &\quad + \frac{C_2}{(1 - \ln |x|)^{1-\alpha}} \int\limits_{1 \leqslant |t| \leqslant \frac{a}{|x|}} |k(e_1, t)| \frac{|\ln |t||^{s+1}}{(1 - \ln(|x||t|))^{\alpha}} \, dt. \qquad (9)\end{aligned}$$

Put

$$M_1 = C_1 \int\limits_{|t| \leqslant 1} |k_1(e_1, t)| |\ln |t||^{s+1} dt, \quad M_2 = C_2 \max_{|x| \leqslant 1} \frac{1}{(1 - \ln |x|)^{1-\alpha}}.$$

Condition 3° and inequality (5) imply that $M_1 < \infty$. Then from (9) it follows that

$$|w(x)| \leqslant \frac{M_1}{1-\ln|x|} + M_2 \int\limits_{1\leqslant|t|\leqslant\frac{a}{|x|}} |k(e_1,t)|\,|\ln|t||^{s+1}\,\frac{dt}{(1-\ln(|x||t|))^{\alpha}}$$

$$\leqslant \frac{M_1}{1-\ln|x|} + M_2 \int\limits_{|t|\leqslant\frac{a}{|x|}} |k(e_1,t)|\,(1+|\ln|t||)^{s+1}\,\frac{dt}{(1-\ln(|x||t|))^{\alpha}}$$

$$= \frac{M_1}{1-\ln|x|} + M_2 \int\limits_{|y|\leqslant a} |k(x,y)| \left(1+\left|\ln\frac{|y|}{|x|}\right|\right)^{s+1} \frac{dy}{(1-\ln|y|)^{\alpha}}. \tag{10}$$

Since $(1-\ln|y|)^{-\alpha} \in C_0(\mathbb{B}_n(a))$ then by Lemma 2 the last term in (10) tends to zero as $x \to 0$. Hence $w \in C_0(\mathbb{B}_n(a))$. Returning to the formula (8), we conclude that $\mathcal{I}_0(x) \in \boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$.

2. Let $j > 0$. We introduce the auxiliary operator

$$(K_a^{(\ell)}\varphi)(x) = \int\limits_{|y|\leqslant a} k(x,y)\left(\ln\frac{|y|}{|x|}\right)^{\ell}\varphi(y)\,dy, \quad x \in \mathbb{B}_n(a),$$

where $\ell = 0, 1, 2, \dots, s$. We note that $K_a^{(0)} = K_a$. Then using the easily verified inequality

$$1 \equiv \left(\frac{\ln\frac{|y|}{|x|}}{1-\ln|x|}\right)^{s} + \frac{1-\ln|y|}{1-\ln|x|}\sum_{m=1}^{s}\left(\frac{\ln\frac{|y|}{|x|}}{1-\ln|x|}\right)^{m-1},$$

we obtain

$$\mathcal{I}_j(x) = \frac{1}{(1-\ln|x|)^{s}}\int\limits_{|y|\leqslant a} k(x,y)\left(\ln\frac{|y|}{|x|}\right)^{s}\frac{dy}{(1-\ln|y|)^{\alpha+j}}$$

$$+\sum_{\ell_1=1}^{s}\frac{1}{(1-\ln|x|)^{\ell_1}}\int\limits_{|y|\leqslant a} k(x,y)\left(\ln\frac{|y|}{|x|}\right)^{\ell_1-1}\frac{dy}{(1-\ln|y|)^{\alpha+j-1}}$$

$$= \frac{u_0(x)}{(1-\ln|x|)^{s+\alpha}} + \sum_{\ell_1=1}^{s}\frac{1}{(1-\ln|x|)^{\ell_1}}K_a^{(\ell_1-1)}\left(\frac{1}{(1-\ln|y|)^{\alpha+j-1}}\right), \tag{11}$$

where

$$u_0(x) = \int\limits_{|y|\leqslant a} k(x, y) \left(\ln \frac{|y|}{|x|} \right)^s \left(\frac{1 - \ln|x|}{1 - \ln|y|} \right)^\alpha \frac{dy}{(1 - \ln|y|)^j}.$$

We will prove that $u_0 \in C_0(\mathbb{B}_n(a))$. Indeed, the function $u_0(x)$ is continuous on $\mathbb{B}_n(a) \setminus \{0\}$. Moreover,

$$\begin{aligned} |u_0(x)| &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left| \ln \frac{|y|}{|x|} \right|^s \left| \frac{1 - \ln|x|}{1 - \ln|y|} \right|^\alpha \frac{dy}{(1 - \ln|y|)^j} \\ &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left| \ln \frac{|y|}{|x|} \right|^s \left(1 + \left| \ln \frac{|y|}{|x|} \right| \right)^\alpha \frac{dy}{(1 - \ln|y|)^j} \\ &\leqslant \int\limits_{|y|\leqslant a} |k(x, y)| \left(1 + \left| \ln \frac{|y|}{|x|} \right| \right)^\mu \frac{dy}{(1 - \ln|y|)^j}. \end{aligned}$$

Since $(1 - \ln|y|)^{-j} \in C_0(\mathbb{B}_n(a))$ then the last integral tends to zero as $x \to 0$ by Lemma 2. Hence $u_0(x) \to 0$ as $x \to 0$.

Further, through similar arguments we obtain that

$$\begin{aligned} K_a^{(\ell_1 - 1)} \left(\frac{1}{(1 - \ln|y|)^{\alpha + j - 1}} \right) &= \frac{u_1(x)}{(1 - \ln|x|)^{\alpha + s - \ell_1}} \\ &+ \sum_{\ell_2 = 1}^{s - \ell_1} \frac{1}{(1 - \ln|x|)^{\ell_2}} K_a^{(\ell_1 + \ell_2 - 2)} \left(\frac{1}{(1 - \ln|y|)^{\alpha + j - 2}} \right), \end{aligned} \tag{12}$$

where $u_1 \in C_0(\mathbb{B}_n(a))$. Proceeding the above mentioned process, we denote $|\widetilde{\ell}| = \ell_1 + \ell_2 + \cdots + \ell_{j-1}$ and get

$$\begin{aligned} K_a^{(|\widetilde{\ell}| - j + 1)} \left(\frac{1}{(1 - \ln|y|)^{\alpha + 1}} \right) &= \frac{u_{j-1}(x)}{(1 - \ln|x|)^{\alpha + s - |\widetilde{\ell}|}} \\ &+ \sum_{\ell_j = 1}^{s - |\widetilde{\ell}|} \frac{1}{(1 - \ln|x|)^{\ell_j}} K_a^{(|\ell| - j)} \left(\frac{1}{(1 - \ln|y|)^{\alpha}} \right), \end{aligned} \tag{13}$$

where $u_{j-1} \in C_0(\mathbb{B}_n(a))$ and $|\ell| = |\widetilde{\ell}| + \ell_j$. Using the formulas (11)–(13), we get the equality

$$\mathcal{I}_j(x) = \frac{u(x)}{(1-\ln|x|)^{\alpha+s}} + \sum_{\ell_1=1}^{s} \frac{1}{(1-\ln|x|)^{\ell_1}}$$

$$\times \sum_{\ell_2=1}^{s-\ell_1} \frac{1}{(1-\ln|x|)^{\ell_2}} \cdots \sum_{\ell_j=1}^{s-|\widetilde{\ell}|} \frac{1}{(1-\ln|x|)^{\ell_j}} K_a^{(|\ell|-j)}\left(\frac{1}{(1-\ln|y|)^{\alpha}}\right), \qquad (14)$$

where $u \in C_0(\mathbb{B}_n(a))$. Applying the result of the part 1) to the operator $K_a^{(|\ell|-j)}$, we obtain

$$K_a^{(|\ell|-j)}\left(\frac{1}{(1-\ln|y|)^{\alpha}}\right) = \frac{u_j(x)}{(1-\ln|x|)^{s-|\ell|+\alpha}} + \sum_{m=0}^{s-|\ell|} \frac{\gamma_m}{(1-\ln|x|)^{m+\alpha}}, \qquad (15)$$

where $u_j \in C_0(\mathbb{B}_n(a))$ and γ_m are some constants. We will substitute (15) for (14). Then through simple transformations, we get

$$\mathcal{I}_j(x) = \frac{w(x)}{(1-\ln|x|)^{\alpha+s}} + \sum_{\ell=j}^{s} \frac{\beta_\ell}{(1-\ln|x|)^{\ell+\alpha}},$$

where $w \in C_0(\mathbb{B}_n(a))$ and β_ℓ are some constants. Thus $\mathcal{I}_j(x) \in \boldsymbol{A}_s^{\alpha}(\mathbb{B}_n(a))$.

□

4 Main Result

The main result of this paper is the following theorem.

Theorem 1 *Let inequality (5) hold. Then the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is invariant with respect to the operator K.*

Proof Let $\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. We have to prove that $K\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. Let us consider two operators

$$(K_1\varphi)(x) = \int\limits_{|y|\leqslant\delta} k(x,y)\varphi(y)\,dy, \qquad x \in \mathbb{B}_n,$$

$$(K_2\varphi)(x) = \int\limits_{\delta\leqslant|y|\leqslant 1} k(x,y)\varphi(y)\,dy, \qquad x \in \mathbb{B}_n.$$

Since $K\varphi = K_1\varphi + K_2\varphi$, it suffices to show that the functions $K_1\varphi$ and $K_2\varphi$ belong to the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

From [1, 2] it follows that $K_1\varphi \in C(\mathbb{B}_n)$. Denote by $\varphi_\delta(x)$ the restriction of the function $\varphi(x)$ to $\mathbb{B}_n(\delta)$. Since $\varphi_\delta \in \boldsymbol{A}^{\alpha}_{s}(\mathbb{B}_n(\delta))$ then $(K_1\varphi)_\delta \in \boldsymbol{A}^{\alpha}_{s}(\mathbb{B}_n(\delta))$ by Lemma 5. Hence $K_1\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

Now consider the function $K_2\varphi$. It is obvious that $K_2\varphi \in C(\mathbb{B}_n)$. Moreover, for $|x| < \delta$ we have the equality

$$(K_2\varphi)(x) = \frac{w(x)}{(1 - \ln|x|)^{s+\alpha}},$$

where $w(x) = (1 - \ln|x|)^{s+\alpha}(K_2\varphi)(x)$. It is clear that the function $w(x)$ is continuous on $\mathbb{B}_n \setminus \{0\}$. We will find the limit of this function as $x \to 0$. Put $M = \max\limits_{\delta \leqslant |x| \leqslant 1} |\varphi(x)|$. Then applying the inequality (6), we obtain

$$\begin{aligned}
|w(x)| &\leqslant \left(1 + \ln\frac{1}{|x|}\right)^{s+\alpha} \int\limits_{\delta \leqslant |y| \leqslant 1} |k(x,y)|\,|\varphi(y)|\,dy \\
&\leqslant M\left(1 + \ln\frac{1}{\delta}\right)^{s+\alpha}\left(1 + \ln\frac{\delta}{|x|}\right)^{s+\alpha} \int\limits_{\delta \leqslant |y| \leqslant 1} |k(x,y)|\,dy \\
&\leqslant M\left(1 + \ln\frac{1}{\delta}\right)^{s+\alpha}\left(1 + \ln\frac{\delta}{|x|}\right)^{s+\alpha} \int\limits_{\frac{\delta}{|x|} \leqslant |t| \leqslant \frac{1}{|x|}} |k(e_1,t)|\;dt \\
&\leqslant M\left(1 + \ln\frac{1}{\delta}\right)^{s+\alpha} \int\limits_{|t| \geqslant \frac{\delta}{|x|}} |k(e_1,t)|\,(1 + |\ln|t||)^{s+\alpha}\,dt.
\end{aligned}$$

The last integral tends to zero as $x \to 0$ by virtue of 3°. Hence $w \in C_0(\mathbb{B}_n(\delta))$ and consequently $K_2\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. □

Corollary 1 *Let inequality (5) hold and let a function* $\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ *such that*

$$\varphi(x) = \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v \in C_0(\mathbb{B}_n(\delta))$$

for $|x| < \delta$. *Then* $K\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ *and for* $|x| < \delta$ *the representation*

$$(K\varphi)(x) = \frac{w(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad w \in C_0(\mathbb{B}_n(\delta))$$

is satisfied.

Proof The proof follows directly from the analysis of proofs of Theorem 1 and Lemma 4. □

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A Dirichlet Problem for Non-elliptic Equations and Chebyshev Polynomials

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Abstract We consider the Dirichlet problem for the linear non-elliptic fourth order partial differential equation in the unit disk. It supposed that in the equation only fourth order terms and coefficients are constant. The solvability conditions of in-homogeneous problem and the solutions of the corresponding homogeneous problem are determined in explicit form. The solutions are obtained in the form of expansions by Chebyshev polynomials.

Keywords Chebyshev polynomials · Dirichlet problem · Improperly elliptic equation · Fourth order non-elliptic equation · Correct boundary value problem

1 Formulation of the Problem and Historical Remarks

Let D be a unit disk of the complex plane and $\Gamma = \partial D$. We consider the higher order differential equation

$$\sum_{k=0}^{2N} A_k \frac{\partial^{2N} V}{\partial x^k \partial y^{2N-k}} = 0, \quad (x, y) \in D, \tag{1}$$

where A_k are constants ($A_0 \neq 0$). We denote λ_j, $j = 1, \ldots, 2N$ the roots of the characteristic equation

$$\sum_{k=0}^{2N} A_k \lambda^{2N-k} = 0. \tag{2}$$

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We are looking for a solution V of Eq. (1) in the class $C^{2N}(D) \cap C^{(N-1,\alpha)}(D \cup \Gamma)$ which satisfies the Dirichlet conditions on the boundary Γ:

$$\left.\frac{\partial^k V}{\partial r^k}\right|_\Gamma = f_k(x, y), \quad (x, y) \in \Gamma, \ \ k = \overline{0, N-1}. \tag{3}$$

Here f_j, $j = \overline{0, N-1}$ are given functions defined on Γ from the class $C^{(N-1-k,\alpha)}(\Gamma)$, while $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ stand for the derivatives with respect to radius and argument of the point of the Γ respectively ($z = x + iy = re^{i\theta}$, $(x, y) \in \Gamma$).

There are different cases, connected with position of the roots λ_j, $j = 1, \ldots, 2N$. If all roots are not real and half of them belong to the upper half-plane and another half—to the lower half-plane Eq. (1) is properly elliptic and the problem (1), (3) is correct, that is homogeneous problem (when $f_k \equiv 0$) has finite number of linearly independent solutions and for the solvability of the in-homogeneous problem it is necessary and sufficient that there is a finite number linearly independent conditions to the boundary functions f_k [1, 2]. If all roots are not real and the numbers of the roots in the upper and lower half-planes are different, then Eq. (1) is improperly elliptic. Investigation of the Dirichlet problem for such equations started in the famous paper of A.V. Bicadze [3]. Further it was shown that all classical boundary value problems are not correct (see, for example [2]) for improperly elliptic equation (1). Correct formulation of the Dirichlet problem for second order improperly equation was presented in [4]. The main problem—exact determination of the class of boundary functions. It was shown in [4] that the boundary functions must be analytic in the domain inside the unit disc. Then the problem (1), (3) for properly elliptic equation (1) was considered in [5, 6] and [7] and for improperly elliptic equation (1) in [8–10]. In [5] a method was introduced applicable not only to elliptic equation (1), but to non-elliptic (when Eq. (2) has real roots) equations also.

In the paper, we consider the case, when roots of Eq. (2) may be real. It was shown in [11] that in this case the problem (1), (3) is not correct. Then the second order hyperbolic equation (1) was considered in [12–14]. The Dirichlet problem for the first order hyperbolic system was considered in [15]. In all these works the solvability of the problem was connected with the geometry of the boundary Γ. In the [14] the homogeneous Dirichlet problem for the vibrating string equation was investigated in the unit disk. It was shown that the solution may be represented by the Chebyshev polynomials. We will show in the paper, that these polynomials are useful for the investigation of the Dirichlet problem for arbitrary higher order equation. We want to introduce the unified approach to investigation of the problem (1), (3) in the unit disk for non-elliptic and elliptic higher order equation (1). The main idea is the usage of the expansions of the solution by the Chebyshev polynomials. We consider the case of fourth order equation (1), when roots of Eq. (2) are real. In the final part of the paper we consider particular case of fourth order equation with real and complex roots.

Let's pass to exact formulation of the results. We consider fourth order equation (1), and in the first two sections we suppose that λ_j, $j = 1, 2, 3, 4$ four roots of the characteristic equation (2), are real.

First we consider the case, when $\lambda_1 = \lambda_3 \neq \lambda_2 = \lambda_4$. In this case Eq. (1) may be represented in the form:

$$\left(\frac{\partial}{\partial y} - \lambda_1 \frac{\partial}{\partial x}\right)^2 \left(\frac{\partial}{\partial y} - \lambda_2 \frac{\partial}{\partial x}\right)^2 V = 0. \quad (4)$$

Taking into account identities $\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$, we replace boundary conditions (3) by the equivalent conditions

$$V_x|_\Gamma = \cos\theta f_1(\theta) - \sin\theta f_0'(\theta) \equiv F(\theta),$$

$$V_y\big|_\Gamma = \sin\theta f_1(\theta) + \cos\theta f_0'(\theta) \equiv G(\theta); \quad (5)$$

$$V(1, 0) = f_0(0), \quad z = x + iy = e^{i\theta}, \quad (x, y) \in \Gamma.$$

Here $F, G \in C^{(\alpha)}(\Gamma)$ are uniquely determined by the boundary functions f_j, $j = 0, 1$. We will prove the following theorem

Theorem 1 *If the roots of Eq. (2) are real and satisfy the condition $\lambda_1 = \lambda_3 \neq \lambda_2 = \lambda_4$ then the problem (4), (5) is uniquely solvable.*

In Sect. 3 we consider the case, when Eq. (1) may be represented in the form

$$\left(\frac{\partial}{\partial y} - \lambda_1 \frac{\partial}{\partial x}\right)^3 \left(\frac{\partial}{\partial y} - \lambda_2 \frac{\partial}{\partial x}\right) V = 0 \quad (6)$$

that is the roots of Eq. (2) satisfy the condition $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2$ (λ_j are real). Let's denote angles α_j by the formulas

$$\cos\alpha_j = \frac{1}{\sqrt{1+\lambda_j^2}}, \quad \sin\alpha_j = \frac{\lambda_j}{\sqrt{1+\lambda_j^2}}, \quad j = 1, 2. \quad (7)$$

Then, the obtained result is the following.

Theorem 2 *We suppose that the roots of Eq. (2) satisfy the condition: $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2$. We denote $\delta = \alpha_2 - \alpha_1$, where the angles α_j are found from (7). In this case the homogeneous problem (6), (5) (when $F \equiv 0$, $G \equiv 0$ and $f_0(0) = 0$) has a unique solution if and only if the conditions*

$$\Theta_k(\delta) = (k-1)\sin(k+1)\delta - (k+1)\sin(k-1)\delta \neq 0, \quad (8)$$

for $k = 3, 4, \ldots$ hold. If the conditions (8) fail for some q (that is $\Theta_q(\delta) = 0$), then the homogeneous problem (6), (5) has one non-zero solution—polynomial of order $q + 1$. Hence, the dimension of the kernel of the problem (6), (5) is equal to the quantity of the numbers l for which (8) fail, that is $\Theta_l(\delta) = 0$.

Theorem 3 *If the angle $\delta = 0.5\pi$ then homogeneous problem (6), (5) has an infinite number of linearly independent solutions, which are polynomials of order $2m$ ($m = 2, 3, \ldots$) and the corresponding in-homogeneous problem has a solution if and only if the boundary functions F and G satisfy infinite number of linearly independent orthogonality conditions.*

The linearly independent solutions of the homogeneous problem and linearly independent solvability conditions of the in-homogeneous problem (6), (5) are determined in explicit form.

2 The Case of Double Roots of Eq. (2)

We consider the problem (4), (5). The general solution of Eq. (4) may be represented in the following form [5]:

$$V(x, y) = \Phi_0(x + \lambda_1 y) + \frac{\partial}{\partial\theta}\Phi_1(x + \lambda_1 y) + \Psi_0(x + \lambda_2 y)+$$

$$+ \frac{\partial}{\partial\theta}\Psi_1(x + \lambda_2 y), \quad x = r\cos\theta, \ y = r\sin\theta. \tag{9}$$

Here Φ_j, Ψ_j are the functions to be determined. Let's substitute the solution (9) in the boundary equations (5). Using operator identities $\frac{\partial}{\partial x}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $\frac{\partial}{\partial y}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial}{\partial y} - \frac{\partial}{\partial x}$, we get two equations for the determination of the functions $\Phi_j^{'}$, $\Psi_j^{'}$.

$$\Phi_0^{'}(S_1) + \left(\frac{\partial}{\partial\theta} + \lambda_1 I\right)\Phi_1^{'}(S_1) + \Psi_0^{'}(S_2) + \left(\frac{\partial}{\partial\theta} + \lambda_2 I\right)\Psi_1^{'}(S_2) = F(\theta);$$

$$\lambda_1\Phi_0^{'}(S_1) + \left(\lambda_1\frac{\partial}{\partial\theta} - I\right)\Phi_1^{'}(S_1) + \lambda_2\Psi_0^{'}(S_2) + \left(\lambda_2\frac{\partial}{\partial\theta} - I\right)\Psi_1^{'}(S_2) = G(\theta). \tag{10}$$

Here

$$S_j = \cos\theta + \lambda_j\sin\theta = \sqrt{1 + \lambda_j^2}\cos(\theta - \alpha_j), \ j = 1, 2 \tag{11}$$

where the angles α_j determined in (7). Thus, unknown functions on the boundary depend on $\cos(\theta - \alpha_j)$, therefore, 2π-periodic functions, symmetric relative to the angle $(\theta - \alpha_j)$. Therefore, these functions may be represented by the Fourier series, and in the corresponding Fourier expansions only $\cos k(\theta - \alpha_j)$ terms will remain. We get the representation of unknown functions on the boundary Γ:

$$\sqrt{1+\lambda_1^2}\Phi_j'(S_1) = \sum_{k=0}^{\infty} A_{kj} \cos k(\theta - \alpha_1), \quad j = 0, 1$$

$$\sqrt{1+\lambda_2^2}\Psi_j'(S_2) = \sum_{k=0}^{\infty} B_{kj} \cos k(\theta - \alpha_2), \quad j = 0, 1.$$

Recalling the definition of the first kind Chebyshev polynomials [16], we see, that

$$\sqrt{1+\lambda_1^2}\Phi_j'(S_1) = \sum_{k=0}^{\infty} A_{kj} T_k(\cos(\theta - \alpha_1)), \quad j = 0, 1$$

$$\sqrt{1+\lambda_2^2}\Psi_j'(S_2) = \sum_{k=0}^{\infty} B_{kj} T_k(\cos(\theta - \alpha_2)), \quad j = 0, 1 \tag{12}$$

Thus, we seek unknown functions by the expansions (12) for arbitrary $x + \lambda_j y$ for $(x, y) \in D$. Substituting these expansions in (10) we have:

$$\sum_{k=0}^{\infty} A_{k0} \cos\alpha_1 \cos k(\theta - \alpha_1) - \sum_{k=0}^{\infty} A_{k1} k \cos\alpha_1 \sin k(\theta - \alpha_1) + \sum_{k=0}^{\infty} A_{k1} \sin\alpha_1 \cdot$$

$$\cos k(\theta - \alpha_1) + \sum_{k=0}^{\infty} B_{k0} \cos\alpha_2 \cos k(\theta - \alpha_2) - \sum_{k=0}^{\infty} B_{k1} k \cos\alpha_2 \sin k(\theta - \alpha_2) +$$

$$\sum_{k=0}^{\infty} B_{k1} \sin\alpha_2 \cos k(\theta - \alpha_2) = F(\theta); \quad \sum_{k=0}^{\infty} A_{k0} \sin\alpha_1 \cos k(\theta - \alpha_1) -$$

$$\sum_{k=0}^{\infty} A_{k1} k \sin\alpha_1 \sin k(\theta - \alpha_1) - \sum_{k=0}^{\infty} A_{k1} \cos\alpha_1 \cos k(\theta - \alpha_1) + \sum_{k=0}^{\infty} B_{k0} \sin\alpha_2 \cdot$$

$$\cos k(\theta - \alpha_2) - \sum_{k=0}^{\infty} B_{k1} k \sin\alpha_2 \sin k(\theta - \alpha_2) - \sum_{k=0}^{\infty} B_{k1} \cos\alpha_2 \cos k(\theta - \alpha_2) = G(\theta).$$

Let's group the coefficients of $\cos k\theta$ and $\sin k\theta$ and use the Fourier expansions of the boundary functions F and G:

$$F(\theta) = \frac{F_0}{2} + \sum_{k=1}^{\infty}(F_k \cos k\theta + E_k \sin k\theta),\ G(\theta) = \frac{G_0}{2} + \sum_{k=1}^{\infty}(G_k \cos k\theta + H_k \sin k\theta). \tag{13}$$

We get

$$\sum_{k=0}^{\infty}((A_{k0}\cos\alpha_1\cos k\alpha_1 + A_{k1}(k\cos\alpha_1\sin k\alpha_1 + \sin\alpha_1\cos k\alpha_1) + B_{k0}\cos\alpha_2\cdot$$

$$\cos k\alpha_2 + B_{k1}(k\cos\alpha_2\sin k\alpha_2 + \sin\alpha_2\cos k\alpha_2))\cos k\theta + (A_{k0}\cos\alpha_1\sin k\alpha_1 +$$

$$B_{k0}\cos\alpha_2\sin k\alpha_2 + A_{k1}(-k\cos\alpha_1\cos k\alpha_1 + \sin\alpha_1\sin k\alpha_1) + B_{k1}(-k\cos\alpha_2\cdot$$

$$\cos k\alpha_2 + \sin\alpha_2\sin k\alpha_2))\sin k\theta) = \frac{F_0}{2} + \sum_{k=1}^{\infty}(F_k\cos k\theta + E_k\sin k\theta),$$

$$\sum_{k=0}^{\infty}((A_{k0}\sin\alpha_1\cos k\alpha_1 + A_{k1}(k\sin\alpha_1\sin k\alpha_1 - \cos\alpha_1\cos k\alpha_1) + B_{k0}\sin\alpha_2\cdot$$

$$\cos k\alpha_2 + B_{k1}(k\sin\alpha_2\sin k\alpha_2 - \cos\alpha_2\cos k\alpha_2))\cos k\theta + (A_{k0}\sin\alpha_1\sin k\alpha_1 +$$

$$A_{k1}(-k\sin\alpha_1\cos k\alpha_1 - \cos\alpha_1\sin k\alpha_1) + B_{k0}\sin\alpha_2\sin k\alpha_2 + B_{k1}(-k\sin\alpha_2\cdot$$

$$\cos k\alpha_2 - \cos\alpha_2\sin k\alpha_2))\sin k\theta) = \frac{G_0}{2} + \sum_{k=1}^{\infty}(G_k\cos k\theta + H_k\sin k\theta).$$

Using uniqueness of the Fourier expansions, we can equate the corresponding coefficients of $\cos k\theta$ and $\sin k\theta$ and get the system of linear equations for the determination of unknown A_{kj}, B_{kj} for $j = 0, 1$. If $k = 0$ we get

$$A_{00}\cos\alpha_1 + A_{01}\sin\alpha_1 + B_{00}\cos\alpha_2 + B_{01}\sin\alpha_2 = 0.5F_0$$

$$A_{00}\sin\alpha_1 - A_{01}\cos\alpha_1 + B_{00}\sin\alpha_2 - B_{01}\cos\alpha_2 = 0.5G_0. \tag{14}$$

For $k \geq 1$ we obtain fourth order linear system. If we introduce the following notions

$$X = (A_{k0}\ A_{k1}\ B_{k0}\ B_{k1})^T; \quad b = (F_k\ E_k\ G_k\ H_k)^T \tag{15}$$

then we get the system for determination of unknown vector X

$$\Omega_k X = b, \quad \Omega_k = (L^1_{k1}\ L^1_{k2}\ L^2_{k1}\ L^2_{k2}). \tag{16}$$

Here Ω_k is a matrix with four-dimensional columns L^j_{ki}, $i, j = 1, 2$. These columns are determined by the formulas:

$$L^j_{k1} = \begin{pmatrix} L^j_{1k1} \\ L^j_{2k1} \\ L^j_{3k1} \\ L^j_{4k1} \end{pmatrix} = \begin{pmatrix} \cos\alpha_j \cos k\alpha_j \\ \cos\alpha_j \sin k\alpha_j \\ \sin\alpha_j \cos k\alpha_j \\ \sin\alpha_j \sin k\alpha_j \end{pmatrix}$$

$$L^j_{k2} = \begin{pmatrix} L^j_{1k2} \\ L^j_{2k2} \\ L^j_{3k2} \\ L^j_{4k2} \end{pmatrix} = \begin{pmatrix} k\cos\alpha_j \sin k\alpha_j + \sin\alpha_j \cos k\alpha_j \\ -k\cos\alpha_j \cos k\alpha_j + \sin\alpha_j \sin k\alpha_j \\ k\sin\alpha_j \sin k\alpha_j - \cos\alpha_j \cos k\alpha_j \\ -k\sin\alpha_j \cos k\alpha_j - \cos\alpha_j \sin k\alpha_j \end{pmatrix}. \quad j = 1, 2. \tag{17}$$

Calculating the determinant of the system (16), we get

$$\det \Omega_k = k^2 \sin^2\delta - \sin^2 k\delta = \sin^2\delta (k^2 - U^2_{k-1}(\cos\delta)), k = 1, 2, \dots, \tag{18}$$

where U_{k-1}—second kind Chebyshev polynomial of order $k - 1$, $\delta = \alpha_2 - \alpha_1$ (α_j determined in (7)). Taking into account, that $\cos\alpha_j > 0$ (see (7)) and $\alpha_1 \neq \alpha_2$ we see that $\delta \neq 0,\ \pi$ and, therefore, $|U_{k-1}(\cos\delta)| < k$ (see [16], point 2.2) and, therefore, the $\det\Omega_k \neq 0$ for $k = 2, 3, \dots$.

Let's consider homogeneous problem (4), (5). In this case $F(\theta) \equiv 0$ and $G(\theta) \equiv 0$, hence, from unique solvability of the system (16) for $k > 1$, we see, that the solution of the homogeneous problem (4), (5) may only be the polynomial of order two (because in the expansions (12) only A_{0j}, B_{0j} and A_{1j}, B_{1j} differ from zero). But an arbitrary non-zero polynomial, which satisfies the homogeneous conditions (5) on Γ is divisible by $(1 - z\bar{z})^2$ (see [17], theorem 5.1), which means that the homogeneous problem (4), (5) has only trivial solution.

Now, we return to in-homogeneous problem (4), (5). We must consider the in-homogeneous systems (14), (16). First, taking into account, that $\det\Omega_k \neq 0$ for $k = 2, 3, \dots$, we see that in this case coefficients A_{kj} and B_{kl} are determined uniquely. System (14) is solvable for arbitrary right part, so A_{0j} and B_{0j} may also be found (not uniquely). In the system (16) for $k = 1$ left parts of second and third equations are the same. Right parts of these equations coincide also because (5) imply that the functions $V_x \sin\theta$ and $V_y \cos\theta$ differ by f_0' only, therefore, their integrals from $-\pi$ to π coincide, so $E_1 = G_1$. Summing up, we can say that all coefficients A_{kl} and B_l may be determined from the systems (14), (16), therefore, the in-homogeneous problem (4), (5) has a solution.

Now, we must show, that this solution belongs to the class $C^{(1,\alpha)}(D \cup \Gamma)$. The determinant $\det \Omega_k$ for $k \to \infty$ is equivalent to k^2 (more precise $\det \Omega_k \sim \sin^2 \delta k^2$), therefore, the coefficients A_{k1} and B_{k1} are equivalent to $\|b\|k^{-1}$ and the coefficients A_{k0} and B_{k0} are equivalent to $\|b\|$ in infinity, where b is a vector of the Fourier coefficients of boundary functions from (15). These relations imply (see [18], page 210), that all components of the solution are from the class $C^{(1,\alpha)}(\overline{D})$. Thus, the inhomogeneous problem (4), (5) has a solution in the class $C^{(1,\alpha)}(\overline{D})$ and this solution is unique. Theorem 1 is proved.

3 The Case of the Triple Root of Characteristic Equation

In this section we consider the case, when the characteristic equation has two distinct real roots, one of which has the multiplicity three, that is Eq. (1) is reduced to the form (10). In this case the solution of Eq. (6) represented in the form [5]:

$$V(x, y) = \Phi_0(x + \lambda_1 y) + \frac{\partial}{\partial\theta}\Phi_1(x + \lambda_1 y) + \left(\frac{\partial}{\partial\theta}\right)^2 \Phi_2(x + \lambda_1 y) + \Psi(x + \lambda_2 y). \tag{19}$$

Here $x = r\cos\theta$, $y = r\sin\theta$ and Φ_j, Ψ are the functions to be determined. Let's substitute the solution (19) in the boundary equations (5). Using operator identities

$$\frac{\partial}{\partial x}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial y}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial}{\partial y} - \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial\theta}\right)^2 = \left(\frac{\partial}{\partial\theta}\right)^2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial\theta}\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial\theta}\right)^2 = \left(\frac{\partial}{\partial\theta}\right)^2\frac{\partial}{\partial y} - 2\frac{\partial}{\partial\theta}\frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$

we get two equations for the determination of the functions $\Phi_j^{'}$, $\Psi^{'}$:

$$\Phi_0^{'}(S_1) + \left(\frac{\partial}{\partial\theta} + \lambda_1 I\right)\Phi_1^{'}(S_1) + \left(\left(\frac{\partial}{\partial\theta}\right)^2 + 2\lambda_1\frac{\partial}{\partial\theta} - I\right)\Phi_2^{'}(S_1) + \Psi^{'}(S_2) = F(\theta),$$

$$\lambda_1\Phi_0^{'}(S_1) + \left(\lambda_1\frac{\partial}{\partial\theta} - I\right)\Phi_1^{'}(S_1) + \left(\lambda_1\left(\frac{\partial}{\partial\theta}\right)^2 - 2\frac{\partial}{\partial\theta} - \lambda_1 I\right)\Phi_2^{'}(S_1) + \lambda_2\Psi^{'}(S_2) = G(\theta). \tag{20}$$

Here S_j were defined in (11) and unknown functions Φ'_j, Ψ may be represented in the form, analogous to (12):

$$\sqrt{1+\lambda_1^2}\Phi'_j(S_1) = \sum_{k=0}^{\infty} A_{kj} T_k(\cos(\theta-\alpha_1)), \;\; j = 0, 1, 2,$$

$$\sqrt{1+\lambda_2^2}\Psi'(S_2) = \sum_{k=0}^{\infty} B_k T_k(\cos(\theta-\alpha_2)), \tag{21}$$

We substitute the functions (21) in the boundary conditions (20):

$$\sum_{k=0}^{\infty} A_{k0}\cos\alpha_1\cos k(\theta-\alpha_1) + \sum_{k=0}^{\infty} A_{k1}\,(-k\cos\alpha_1\sin k(\theta-\alpha_1) + \sin\alpha_1\cos k(\theta-$$

$$\alpha_1)) - \sum_{k=0}^{\infty} A_{k2}\left(k^2\cos\alpha_1\cos k(\theta-\alpha_1) + 2k\sin\alpha_1\sin k(\theta-\alpha_1) + \cos\alpha_1\cos k(\theta-\right.$$

$$\alpha_1)) + \sum_{k=0}^{\infty} B_k\cos\alpha_2\cos k(\theta-\alpha_2) = F(\theta);\;\; \sum_{k=0}^{\infty} A_{k0}\sin\alpha_1\cos k(\theta-\alpha_1) + \sum_{k=0}^{\infty} A_{k1}\,(-k\cdot$$

$$\sin\alpha_1\sin k(\theta-\alpha_1) - \cos\alpha_1\cos k(\theta-\alpha_1)) - \sum_{k=0}^{\infty} A_{k2}\left(k^2\sin\alpha_1\cos k(\theta-\alpha_1)-\right.$$

$$2k\cos\alpha_1\sin k(\theta-\alpha_1) + \sin\alpha_1\cos k(\theta-\alpha_1)) + \sum_{k=0}^{\infty} B_k\sin\alpha_2\cos k(\theta-\alpha_2) = G(\theta).$$

Using Fourier expansions of the boundary functions F and G (13) and equating the coefficients of the corresponding $\cos k\theta$ and $\sin k\theta$ we get the systems of equations analogous to (14), (16) for the determination of unknown coefficients A_{kj} and B_k. For $k = 0$ we have

$$A_{00}\cos\alpha_1 + A_{01}\sin\alpha_1 - A_{02}\cos\alpha_1 + B_0\cos\alpha_2 = 0.5F_0$$

$$A_{00}\sin\alpha_1 - A_{01}\cos\alpha_1 - A_{01}\sin\alpha_1 + B_0\sin\alpha_2 = 0.5G_0. \tag{22}$$

For $k \geq 1$ if we introduce unknown vector

$$X_k = (A_{k0}\;A_{k1}\;A_{k2}\;B_k)^T \tag{23}$$

and given vector b (15) of the Fourier coefficients of the boundary functions, we get the system for determination of unknown vector X_k:

$$\Omega_k X_k = b, \quad \Omega_k = (L^1_{k1}\ L^1_{k2}\ L^1_{k3}\ L^2_{k1}). \tag{24}$$

where L^j_{k1} and L^1_{k2} are determined in (17) and

$$L^1_{k3} = \begin{pmatrix} -(k^2+1)\cos\alpha_1\cos k\alpha_1 + 2k\sin\alpha_1\sin k\alpha_1 \\ -(k^2+1)\cos\alpha_1\sin k\alpha_1 - 2k\sin\alpha_1\cos k\alpha_1 \\ -(k^2+1)\sin\alpha_1\cos k\alpha_1 - 2k\cos\alpha_1\sin k\alpha_1 \\ -(k^2+1)\sin\alpha_1\sin k\alpha_1 + 2k\cos\alpha_1\cos k\alpha_1 \end{pmatrix}. \tag{25}$$

After transformation, $\Delta_k = \det\Omega_k$, the determinant of the system (24), may be reduced to the form

$$\Delta_k = \frac{i}{16}\det\begin{pmatrix} 1 & -(k-1) & (k-1)^2 & \beta^{k-1} \\ 1 & k+1 & (k+1)^2 & \beta^{-k-1} \\ 1 & k-1 & (k-1)^2 & \beta^{-k+1} \\ 1 & -(k+1) & (k+1)^2 & \beta^{k+1} \end{pmatrix} \tag{26}$$

Here $\beta = \exp i\delta$, and $\delta = \alpha_2 - \alpha_1$ was defined in Theorem 2. Calculating the determinant (26), we get:

$$\Delta_k = k((k-1)\sin(k+1)\delta - (k+1)\sin(k-1)\delta) = k\Theta_k(\delta). \tag{27}$$

where the function Θ_k defined in (8). From definition of the angles α_j, we have $0 < |\delta| < \pi$. We see, that the solvability of the problem (6), (5) is connected to the solvability of the system (24), and, therefore, with the properties of the determinant (27).

Proof of Theorem 2 Let's consider the homogeneous problem (6), (5). This problem is reduced to the solution of homogeneous systems (22), (24) ($b = 0$, $F_0 = G_0 = 0$). If the conditions (8) hold, then the system (24) for $k \geq 3$ has the trivial solution only. If $k = 2$ the determinant (27) differs from zero for arbitrary $\delta \neq 0$. Hence, in the expansions (21) coefficients A_{kj}, B_k are equal to zero for $k > 1$. It means, that the solution of the homogeneous problem (6), (5) is at most second order polynomial. But an arbitrary non-zero polynomial, which satisfies the homogeneous conditions (5) on Γ is divisible by $(1 - z\bar{z})^2$ (see [17], theorem 5.1), therefore, has at least fourth order. Thus the homogeneous problem (6), (5) is uniquely solvable. If the condition (8) failed for some $q \geq 3$, then the homogeneous problem (6), (5) has non-trivial solution—a polynomial of order $q + 1$ (we may get in the expansions (21) $A_{kj} = B_k = 0$ for $k \neq q$ and instead of A_{qj}, B_q get the non-trivial solution of the corresponding system (24)). Theorem 2 is proved. □

Proof of Theorem 3 Now, let's suppose, that $\delta = 0.5\pi$, that is the roots λ_1, λ_2 of the characteristic equation (2) satisfy the condition $1 + \lambda_1\lambda_2 = 0$. In this case, from (8) we get, that $\Theta_{2m-1}(\delta) = 0$ for $m = 2, 3, \ldots$, therefore, the homogeneous system (24) has non-trivial solution X_{2m-1} (23). Substituting this solution in the formulas (21), and supposing $A_{kj} = B_k = 0$, for $k \neq 2m - 1$, we get a non-trivial solution of the homogeneous problem (6), (5), which is a polynomial of order $2m$. For example, $\Theta_3(0.5\pi) = 0$ and direct computation shows, that the function $(1-x^2-y^2)^2$ is a non-trivial solution of the homogeneous problem (6), (5). Let's consider in-homogeneous problem (6), (5). This problem is reduced to the in-homogeneous systems (22), (24). The system (22) always has a solution. If $k = 2m$, then the determinant (26) of the system (24) is equal to $\Delta_{2m} = 8(-1)^{m+1}m^2$, therefore, the system (24) is uniquely solvable, and the solution has following estimate for $m \to \infty$:

$$A_{2m,0} \sim \|b\|m, \;\; B_{2m} \sim \|b\|, \;\; A_{2m,1}A_{2m,2} \sim \|b\|m^{-1}, \;\; . \tag{28}$$

If $k = 2m - 1$, then third column of the matrix Ω_{2m-1} of the system is a linear combination of the first and fourth columns, and the range of Ω_{2m-1} is equal three, therefore, right parts of the system (components of the vector b) have to satisfy one solvability condition. Recalling, that the components of the vector b are the Fourier coefficients of the functions F and G, this condition may be written in the following form:

$$\int_{-\pi}^{\pi} F(\theta)S_k(\theta)d\theta = \int_{-\pi}^{\pi} G(\theta)R_k(\theta)d\theta, \quad k = 2m-1, \;\; m = 1, 2, \ldots, \tag{29}$$

where

$$S_k(\theta) = (k+1)\sin(k(\theta-\alpha_1)+\alpha_1) - (k-1)\sin(k(\theta-\alpha_1)-\alpha_1),$$

$$R_k(\theta) = (k+1)\cos(k(\theta-\alpha_1)+\alpha_1) - (k-1)\cos(k(\theta-\alpha_1)-\alpha_1). \tag{30}$$

If this condition holds, we may get $A_{k2} = 0$, and the system (24)is reduced to the system

$$\begin{pmatrix} \cos\alpha_1\cos k\alpha_1 & k\cos\alpha_1\sin k\alpha_1 + \sin\alpha_1\cos k\alpha_1 & -\sin\alpha_1\sin k\alpha_1 \\ \cos\alpha_1\sin k\alpha_1 & -k\cos\alpha_1\cos k\alpha_1 + \sin\alpha_1\sin k\alpha_1 & \sin\alpha_1\cos k\alpha_1 \\ \sin\alpha_1\cos k\alpha_1 & k\sin\alpha_1\sin k\alpha_1 - \cos\alpha_1\cos k\alpha_1 & \cos\alpha_1\sin k\alpha_1 \\ \sin\alpha_1\sin k\alpha_1 & -k\sin\alpha_1\cos k\alpha_1 - \cos\alpha_1\sin k\alpha_1 & -\cos\alpha_1\cos k\alpha_1 \end{pmatrix} Y = b, \tag{31}$$

where $Y = (A_{2m-1,0} \;\; A_{2m-1,1} \;\; (-1)^m B_{2m-1})^T$ is three-dimensional unknown vector, and b is a given vector from (15). For determination of the Y we may get arbitrary three equation of the system (31). If we get first three rows, then corresponding determinant of the system is equal $Q_k = 2(k-1)\sin(k+1)\alpha_1$, and if we get first, third and fourth rows, then corresponding determinant is equal $R_k = -2(k-1)\cos(k+1)\alpha_1$. These values ($Q_k$ and R_k) may be small for different values of α_1, when $k = 2m-1 \to \infty$, but $Q_k^2 + R_k^2 = 4(k-1)^2$, therefore, for arbitrary $k = 2m-1$ we may choose Q_k or R_k such that the modulus of the corresponding value will have $2(k-1)$ order in infinity and find Y from corresponding system. We will have the estimates

$$A_{2m-1,0} \sim \|b\|, \quad B_{2m-1} \sim \|b\|, \quad A_{2m-1,1} \sim \|b\| m^{-1}, \quad . \tag{32}$$

analogous to (28) for the obtained solution. Substituting all found coefficients A_{kj} and B_k in (21) we will get the solution of the in-homogeneous problem. Using the estimates (28), (32) and the formula (2.43) from [16] for integral of Chebyshev polynomial and the boundary conditions (20), we see that the obtained solution belongs to the prescribed class $C^{(1,\alpha)}(\overline{D})$. Theorem 3 is proved. □

4 The Case, When Eq. (2) Has Real and Complex Roots

In the final section we consider one special case of the fourth order equation (1), when the characteristic equation (2) has real and complex roots. This equation in the rectangle was investigated in [19].

Let's consider an equation

$$\frac{\partial^4 V}{\partial y^4}(x,y) - \frac{\partial^4 V}{\partial x^4}(x,y) = 0, \quad (x,y) \in D. \tag{33}$$

Unknown function V belongs to the class $C^4(D) \cap C^{(1,\sigma)}(\overline{D})$ and on the boundary $\Gamma = \partial D$ satisfies the conditions (5). The method of solution of the problem (33), (5) is analogous to the previous section consideration, hence, we explain it without details.

The general solution of Eq. (33) may be represented in the form:

$$V(x,y) = \Phi_1(x+y) + \Phi_2(x-y) + \Re\Phi_3(x+iy), \quad (x,y) \in D, \tag{34}$$

where Φ_j, $j = 1, 2$ are unknown two times differentiable functions and Φ_3 is analytic in D. We substitute these functions in the boundary conditions (5):

$$\Phi_1'(x+y) + \Phi_2'(x-y) + \Re\Phi_3'(x+iy) = F(\theta),$$

$$\Phi_1'(x+y) - \Phi_2'(x-y) + \Re i\Phi_3'(x+iy) = G(\theta), \quad x = \cos\theta, \ y = \sin\theta. \tag{35}$$

As in the formulas (12), we represent the unknown functions by the series

$$\Phi_1'(\cos\theta + \sin\theta) = \Phi_1'(\sqrt{2}(\cos(\theta - 0.25\pi)) = \sum_{k=0}^{\infty} A_{k1}\cos k(\theta - 0.25\pi),$$

or

$$\Phi_1'(\cos\theta + \sin\theta) = \sum_{k=0}^{\infty} A_{k1} T_k(\cos(\theta - 0.25\pi)),$$

and

$$\Phi_2'(\cos\theta - \sin\theta) = \sum_{k=0}^{\infty} A_{k2} T_k(\cos(\theta + 0.25\pi)). \tag{36}$$

The function Φ_3' represented in the following form:

$$\Phi_3'(\cos\theta + i\sin\theta) = \sum_{k=0}^{\infty} c_k e^{ik\theta} = \tag{37}$$

$$= \sum_{k=0}^{\infty}(B_k\cos k\theta - C_k\sin k\theta) + i\sum_{k=0}^{\infty}(B_k\sin k\theta + C_k\cos k\theta), \quad c_k = B_k + iC_k.$$

Substituting (36), (37) and expansions (13) of the boundary functions in (35), we get

$$\sum_{k=0}^{\infty}(((A_{k1} + A_{k2})\cos 0.25k\pi + B_k)\cos k\theta + (((A_{k1} - A_{k2})\sin 0.25k\pi -$$

$$-C_k)\sin k\theta) = \frac{F_0}{2} + \sum_{k=1}^{\infty}(F_k\cos k\theta + E_k\sin k\theta), \tag{38}$$

$$\sum_{k=0}^{\infty}(((A_{k1} - A_{k2})\cos 0.25k\pi - C_k)\cos k\theta + (((A_{k1} + A_{k2})\sin 0.25k\pi -$$

$$-B_k)\sin k\theta) = \frac{G_0}{2} + \sum_{k=1}^{\infty}(G_k\cos k\theta + H_k\sin k\theta). \tag{39}$$

Equating the coefficients at the same $\cos k\theta$ and $\sin k\theta$ for $k \geq 1$, we get;

$$(A_{k1} + A_{k2})\cos 0.25k\pi + B_k = F_k,$$

$$(A_{k1} + A_{k2})\sin 0.25k\pi - B_k = H_k, \tag{40}$$

$$(A_{k1} - A_{k2})\cos 0.25k\pi - C_k = G_k,$$

$$(A_{k1} - A_{k2})\sin 0.25k\pi - C_k = E_k. \tag{41}$$

Let's consider the system (40), (41). If $k = 4m$, or $k = 4m + 2$, then the system has unique solution, therefore, the coefficients $A_{4m,j}$, $A_{4m+2,j}$, B_{4m}, C_{4m}, B_{4m+2}, C_{4m+2} are uniquely determined. Let's suppose, that $k = 4m + 3$. In this case the system (40) is reduced to the form:

$$(A_{k1} + A_{k2})0.5\sqrt{2}(-1)^m - B_k = -F_k,$$

$$(A_{k1} + A_{k2})0.5\sqrt{2}(-1)^m - B_k = H_k, \quad k = 4m + 3 \tag{42}$$

therefore, for the solvability of the system the condition $F_{4m+3} = -H_{4m+3}$ has to be met, or

$$\int_{-\pi}^{\pi} F(\theta)\cos(4m+3)\theta d\theta + \int_{-\pi}^{\pi} G(\theta)\sin(4m+3)\theta d\theta = 0, \quad m = 0, 1, \ldots. \tag{43}$$

The system (41) in this case is uniquely solvable. Analogously, if $k = 4m + 1$, then the system (40) is uniquely solvable, and for the solvability of the system (41) the condition:

$$\int_{-\pi}^{\pi} F(\theta)\sin(4m+1)\theta d\theta - \int_{-\pi}^{\pi} G(\theta)\cos(4m+1)\theta d\theta = 0, \quad m = 1, 2, \ldots. \tag{44}$$

is necessary.

Let's consider homogeneous problem (33), (5). This problem is reduced to the solution of the homogeneous system (40), (41). The non-zero solution of this system will be for $k = 4m + 1$ and for $k = 4m + 3$ only. If $k = 4m + 3$, we may get

$$B_{4m+3} = 1, \quad C_{4m+3} = 0, \quad A_{4m+3,1} = A_{4m+1,2} = (-1)^m \frac{\sqrt{2}}{2}.$$

The corresponding non-trivial solution of the homogeneous problem (33), (5) is the following:

$$V_{4m+4}(x,y) = \Phi_{1,4m+3}(x+y) + \Phi_{2,4m+3}(x-y) + \Re\Phi_{3,4m+3}(x+iy),\ m=0,\dots, \tag{45}$$

where

$$\Phi'_{1,4m+3}\left(\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right) = (-1)^m\frac{\sqrt{2}}{2}\cos\left((4m+3)\left(\theta-\frac{\pi}{4}\right)\right),$$

$$\Phi'_{2,4m+3}\left(\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)\right) = (-1)^m\frac{\sqrt{2}}{2}\cos\left((4m+3)\left(\theta+\frac{\pi}{4}\right)\right),$$

$$\Phi'_{3,4m+3}\left(e^{i\theta}\right) = \cos\left((4m+3)\theta\right) + i\sin\left((4m+3)\theta\right).$$

Example 1 Let's determine the function V_4 (for $m=0$) in explicit form. We have

$$\Phi'_{1,3}\left(\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right) = \frac{\sqrt{2}}{2}\cos 3\left(\theta-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(4\left(\cos\left(\theta-\frac{\pi}{4}\right)\right)^3 - \right.$$

$$\left. -3\cos\left(\theta-\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(2\left(\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right)^3 - 3\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right),$$

and, therefore,

$$\Phi'_{1,3}(x+y) = \frac{1}{2}\left(2(x+y)^3 - 3(x+y)\right).$$

After integration, we get

$$\Phi_{1,3}(x+y) = \frac{1}{4}\left((x+y)^4 - 3(x+y)^2\right) + \widehat{c}_1,$$

and, analogously,

$$\Phi_{2,3}(x,y) = \frac{1}{4}\left((x-y)^4 - 3(x-y)^2\right) + \widehat{c}_2,\quad \Phi_{3,3}(x+iy) = \frac{1}{4}(x+iy)^4 + \widehat{c}_3.$$

Here $\widehat{c}_j$ are some constants. Finally, the solution V_4 is the following:

$$V_4(x,y) = \Phi_{1,3}(x+y) + \Phi_{2,3}(x-y) + \Re\Phi_{3,3}(x+iy) = \frac{3}{4}(1-x^2-y^2)^2 + \widehat{c} - \frac{3}{4}.$$

We determine the constant $\widehat{c}$ from the last homogeneous condition (5) $V_4(1,0) = \widehat{c} - \frac{3}{4} = 0$.

In the case $k = 4m + 1$, the system (41) is singular, therefore, as non-trivial solution of the (40), (41), we may get the following:

$$B_{4m+1} = 0, \ \ C_{4m+} = 1, \ \ A_{4m+1,1} = -A_{4m+1,2} = (-1)^m \frac{\sqrt{2}}{2}.$$

The corresponding non-trivial solution of the homogeneous problem (33), (5) is the following:

$$V_{4m+2}(x, y) = \Phi_{1,4m+1}(x + y) + \Phi_{2,4m+1}(x - y) \\ + \Re\Phi_{3,4m+1}(x + iy), \ m = 1, \dots, \tag{46}$$

where

$$\Phi'_{1,4m+1}\left(\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)\right) = (-1)^m \frac{\sqrt{2}}{2} \cos\left((4m+1)\left(\theta - \frac{\pi}{4}\right)\right),$$

$$\Phi'_{2,4m+1}\left(\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)\right) = -(-1)^m \frac{\sqrt{2}}{2} \cos\left((4m+1)\left(\theta + \frac{\pi}{4}\right)\right),$$

$$\Phi'_{3,4m+1}\left(e^{i\theta}\right) = i\left(\cos\left((4m+3)\theta\right) + i\sin\left((4m+3)\theta\right)\right).$$

Example 2 Let's determine the function V_5 (for $m = 1$) in explicit form. We have

$$\Phi'_{1,5}\left(\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)\right) = -\frac{\sqrt{2}}{2}\cos 5\left(\theta - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\left(16\left(\cos\left(\theta - \frac{\pi}{4}\right)\right)^5 - \right.$$

$$\left. -20\left(\cos\left(\theta - \frac{\pi}{4}\right)\right)^3 + 5\cos\left(\theta - \frac{\pi}{4}\right)\right),$$

and, therefore,

$$\Phi'_{1,5}(x + y) = -\frac{1}{2}\left(4(x+y)^5 - 10(x+y)^3 + 5(x+y)\right).$$

After integration, we get

$$\Phi_{1,5}(x + y) = -\frac{1}{3}(x+y)^6 + \frac{5}{4}(x+y)^4 - \frac{5}{4}(x+y)^2 + \widehat{c}_1,$$

and

$$\Phi_{2,5}(x,y)=\frac{1}{3}(x+y)^6-\frac{5}{4}(x+y)^4+\frac{5}{4}(x+y)^2+\widehat{c}_2,\quad \Phi_{3,5}(x+iy)=\frac{i}{6}(x+iy)^6+\widehat{c}_3,$$

where $\widehat{c}_j$ are some constants. Finally, the solution V_6 is the following:

$$V_6(x,y)=\Phi_{1,5}(x+y)+\Phi_{2,5}(x-y)+\Re\Phi_{3,5}(x+iy)=-5xy(1-x^2-y^2)^2.$$

The resulting constant, as in previous case, we determine from the last homogeneous condition (5) $V_4(1,0)=0$.

Summing up, we obtain the following theorem.

Theorem 4 *The homogeneous problem (33), (5) has infinitely many linearly independent solutions V_j, determined by the formulas (37), (46). The in-homogeneous problem (33), (5) has a solution in the class $C^4(D)\cap C^{(1,\sigma)}(\overline{D})$ if and only if the infinitely many linearly independent conditions (43), (44) holds. The solution is determined in explicit form by (36), (37) expansions.*

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Martingale Hardy-Amalgam Spaces: Atomic Decompositions and Duality

Justice Sam Bansah and Benoît F. Sehba

Abstract In this paper, we introduce the notion of martingale Hardy-amalgam spaces: $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$. We present two atomic decompositions for these spaces. The dual space of $H^s_{p,q}$ for $0 < p \leq q \leq 1$ is shown to be a Campanato-type space.

Keywords Martingales · Hardy-amalgam spaces · Atomic decomposition · Campanato spaces

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1 Introduction

We owe the martingale theory to J. L. Doob from his seminal work [9]. The theory was later developed by D. L. Burkholder, A. M. Garsia, R. Cairoli, B. J. Davis and their collaborators (see [4–7, 12, 24, 29] and the references therein). Martingales are particularly interesting because of their connection and applications in Fourier analysis, complex analysis and classical Hardy spaces (see for example [3, 4, 9, 10, 21, 29]). For instance in [4], we see that the methods developed for Banach valued martingales can be used to obtain sharp constants in some inequalities. The Riesz Theorem can also be proved in a probabilistic way as it is done in [3]. The martingale proof of $T(b)$ Theorem and some other martingale techniques in Harmonic analysis are discussed in [21, 22] and of course the main book by Weisz [29] contains applications where martingale techniques are used in Fourier analysis.

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The simplicity of structures in the martingale settings makes it easier when one wants to study some properties of some function spaces such as equivalences of spaces, and dualities. For instance, in 1970, one of the questions mathematicians were interested in, was the extension of R.E.A C Parley's inequality, originally proved for $1 \leq p < \infty$, to the case where $p \in (0, \infty)$. D. L. Burkholder and R. F. Gundy [5] were able to extend the result for all $p \in (0, \infty)$. Their ability to do this was essentially due to the introduction of the measurable functions $s(f)$ and $S(f)$ where f is a martingale. These measurable functions are known as conditional quadratic variation and quadratic variation respectively and were first introduced in [5]. This insight from D. L. Burkholder and R. F. Gundy gave birth to the classical martingale Hardy spaces of which many authors have contributed to the growth in the past few years [5, 7, 12, 22, 29]. These martingale techniques introduced by Burkholder and his colleagues, led to many important results in the literature. One of these results is the equivalence result named after Burkholder, Davis, and Gundy where they showed that the function space generated by the maximal function and function space generated by the quadratic variation are equivalent for $1 \leq p < \infty$ [5, 29]. Another important result obtained by applying martingale techniques is the characterization of the dual space of the spaces generated by the maximal functions when $p = 1$. This dual space was shown to be the BMO space [12].

The classical martingale Hardy spaces H_p are defined as the spaces of martingales whose maximal function, quadratic variation or conditional quadratic variation belongs to the usual Lebesgue spaces L_p with a probability measure. The atomic decompositions, martingale embeddings and dual spaces of these spaces and related spaces are discussed by F. Weisz in [29]. This type of studies has been considered by several authors for some generalizations of the classical Lebesgue spaces as Lorentz spaces, Orlicz spaces, Orlicz-Musielak spaces…(see [15, 19, 23, 26, 30–32]). Even though this paper mainly focuses on martingale Hardy-amalgam spaces, it is also worth mentioning that atomic decompositions, martingale embeddings and dual spaces have been also considered for martingale Morrey-type spaces (see for instance [8, 16, 17, 33]).

In this paper, inspired by the recent introduction of Hardy-amalgam spaces in the classical Harmonic analysis [1, 2, 34], we replace Lebesgue spaces in the definition of the classical martingale Hardy spaces by the Wiener amalgam spaces, introducing then the notion of martingale Hardy-amalgam spaces. As Wiener amalgam spaces generalize Lebesgue spaces, martingale Hardy-amalgam spaces then generalize the martingale Hardy spaces presented in [29]. We provide atomic decompositions and we characterize the dual spaces of these martingale Hardy-amalgam spaces and their associated spaces of predictive martingales and martingales with predictive quadratic variation.

We are motivated essentially by two observations. As first observation we note that in the case of classical Hardy-amalgam spaces of [1], atomic decomposition is obtained only for the range $0 < p \leq q \leq 1$. The question that came into our mind was to know if any answer could be given beyond this range. The second observation is that dyadic analogues of Hardy spaces and their dual spaces are pretty practical when comes to the study of some operators as paraproducts, Calderón–Zygmund operators and their commutators on some function spaces (see for example [20, 25, 27, 28]). It was then natural to consider dyadic analogues and more generally, martingale analogues of the Hardy-amalgam spaces of [1].

The outline of this study is as follows. In Sect. 2, we recall the definition of amalgam space and also recall the definition of the conditional quadratic variation, $s(f)$, the quadratic variation, $S(f)$ and the maximal function f^*. With these in hand, we will then be in a good position to introduce the martingale Hardy-amalgam spaces we will consider in this study. Next in Sect. 3, we present the main results of this study and in Sects. 4 and 5 we present the proofs of our main results. Finally, in Sect. 6, we conclude and introduce the reader to some unsolved problems in this area of study.

2 Preliminaries: Menagerie of Spaces

We introduce here some function spaces in relation with our concern in this paper.

2.1 Wiener Amalgam Spaces

Let Ω be an arbitrary non-empty set and let $\{\Omega_j\}_{j\in\mathbb{Z}}$ be a sequence of nonempty subsets of Ω such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$, and

$$\bigcup_{j\in\mathbb{Z}} \Omega_j = \Omega.$$

For $0 < p, q \leq \infty$, the classical amalgam of L_p and l_q, denoted $L_{p,q}$, on Ω consists of functions which are locally in L_p and have l_q behaviour, in the sense that the L_p-norms over the subsets $\Omega_j \subset \Omega$ form an l_q-sequence i.e. for $p, q \in (0, \infty)$,

$$L_{p,q} = \left\{ f : \|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} < \infty \right\}$$

where

$$\|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} := \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}, \tag{1}$$

for $0 < q < \infty$, and for $q = \infty$,

$$L_{p,\infty} = \left\{ f : \|f\|_{p,\infty} := \|f\|_{L_{p\infty}(\Omega)} := \sup_{j\in\mathbb{Z}} \left(\int_\Omega |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}} < \infty \right\}.$$

For the sake of presentation, we will use the notation

$$\|f\|_{p,q} := \left(\sum_{j\in\mathbb{Z}} \left(\int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}},$$

for $0 < p < \infty$, and $0 < q \leq \infty$, where the right handside is understood as

$$\sup_{j\in\mathbb{Z}} \left(\int_{\Omega} |f|^p \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{1}{p}}$$

when $q = \infty$. As usual, $\mathbf{1}_A$ is the indicator function of the set A. We observe the following:

- Endowed with the (quasi)-norm $\|\cdot\|_{p,q}$, the amalgam space $L_{p,q}$ is a complete space, and a Banach space for $1 \leq p, q \leq \infty$.
- $\|f\|_{p,p} = \|f\|_p$ for $f \in L_p(\Omega)$.
- $\|f\|_{p,q} \leq \|f\|_p$ if $p \leq q$ and $f \in L_p(\Omega)$.
- $\|f\|_p \leq \|f\|_{p,q}$ if $q \leq p$ and $f \in L_{p,q}(\Omega)$.

Amalgam function spaces have been essentially considered in the case $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, and in the case $d = 1$, the subsets Ω_j are just the intervals $[j, j+1)$, $j \in \mathbb{Z}$. For more on amalgam spaces and their properties and applications, we refer the reader to [11, 14, 18].

2.2 Martingale Hardy Spaces via Amalgams

In the remaining of this text, all the spaces are defined with respect to the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_n)_{n\geq 0} := (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a non-decreasing sequence of σ-algebra with respect to the complete ordering on $\mathbb{Z}_+ = \{0, 1, 2\cdots\}$ such that

$$\sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\right) = \mathcal{F}.$$

For $n \in \mathbb{Z}_+$, the expectation operator and the conditional expectation operator relatively to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$ respectively. We denote by $\mathcal{M}$ the set of all martingales $f = (f_n)_{n\geq 0}$ relatively to the filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $f_0 = 0$. We recall that for $f \in \mathcal{M}$, its martingale difference is denoted $d_n f = f_n - f_{n-1}$, $n \geq 0$ with the convention that $d_0 f = 0$.

We recall that the stochastic basis, $(\mathcal{F}_n)_{n\geq 0}$ is said to be regular if there exists $R > 0$ such that $f_n \leq R f_{n-1}$ for all non-negative martingale $(f_n)_{n\geq 0}$.

A martingale $f = (f_n)_{n\geq 0}$ is said to be L_p bounded $(0 < p \leq \infty)$ if $f_n \in L_p$ for all $n \in \mathbb{Z}_+$ and we define

$$\|f\|_p := \sup_{n\in\mathbb{N}} \|f_n\|_p < \infty.$$

We recall that

$$\|f\|_p = (\mathbb{E}(|f|^p))^{\frac{1}{p}} = \left(\int_\Omega |f|^p d\mathbb{P}\right)^{\frac{1}{p}}.$$

We denote by $\mathcal{T}$ the set of all stopping times on Ω. For $\nu \in \mathcal{T}$, and $(f_n)_{n\geq 0}$ an integrable sequence, we recall that the associated stopped sequence $f^\nu = (f_n^\nu)_{n\geq 0}$ is defined by

$$f_n^\nu = f_{n\wedge\nu}, \quad n \in \mathbb{Z}_+.$$

For a martingale $f = (f_n)_{n\geq 0}$, the quadratic variation, $S(f)$, and the conditional quadratic variation, $s(f)$, of f are defined by

$$S(f) = \left(\sum_{n\in\mathbb{N}} |d_n f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s(f) = \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}}$$

respectively. We shall agree on the notation

$$S_n(f) = \left(\sum_{i=1}^n |d_i f|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s_n(f) = \left(\sum_{i=1}^n \mathbb{E}_{i-1}|d_i f|^2\right)^{\frac{1}{2}}.$$

The maximal function f^* or $M(f)$ of the martingale f is defined by

$$M(f) = f^* := \sup_{n\in\mathbb{N}} |f_n|.$$

It is also understood in the sequel that $\mathbb{E}_n f = f_n$.

We now introduce the martingale Hardy-amalgam spaces: $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$. Let $0 < p < \infty$ and $0 < q \leq \infty$. The first space is defined as follows.

(i) $H^s_{p,q}(\Omega) = \{f \in \mathcal{M} : \|f\|_{H^s_{p,q}(\Omega)} := \|s(f)\|_{p,q} < \infty\}$.

Let Γ be the set of all sequences $\beta = (\beta_n)_{n\geq 0}$ of adapted (i.e. β_n is $\mathcal{F}_n$-measurable for any $n \in \mathbb{Z}_+$) non-decreasing, non-negative functions and define

$$\beta_\infty := \lim_{n\to\infty} \beta_n.$$

(ii) The space $\mathcal{Q}_{p,q}(\Omega)$ consists of all martingales f for which there is a sequence of functions $\beta = (\beta_n)_{n\geq 0} \in \Gamma$ such that $S_n(f) \leq \beta_{n-1}$ and $\beta_\infty \in L_{p,q}(\Omega)$.
We endow $\mathcal{Q}_{p,q}(\Omega)$ with

$$\|f\|_{\mathcal{Q}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma} \|\beta_\infty\|_{p,q}.$$

(iii) The space $\mathcal{P}_{p,q}(\Omega)$ consists of all martingales f for which there is a sequence of functions $\beta = (\beta_n)_{n\geq 0} \in \Gamma$ such that $|f_n| \leq \beta_{n-1}$ and $\beta_\infty \in L_{p,q}(\Omega)$.
We endow $\mathcal{P}_{p,q}(\Omega)$ with

$$\|f\|_{\mathcal{P}_{p,q}(\Omega)} := \inf_{\beta\in\Gamma} \|\beta_\infty\|_{p,q}.$$

A martingale $f \in \mathcal{P}_{p,q}(\Omega)$ is called predictive martingale and a martingale $f \in \mathcal{Q}_{p,q}(\Omega)$ is a martingale with predictive quadratic variation.

In the sequel, when there is no ambiguity, the spaces $H^s_{p,q}(\Omega)$, $\mathcal{Q}_{p,q}(\Omega)$ and $\mathcal{P}_{p,q}(\Omega)$ will be just denoted $H^s_{p,q}$, $\mathcal{Q}_{p,q}$ and $\mathcal{P}_{p,q}$ respectively. The same will be done for the associted (quasi)-norms.

Remark 2.3

- Observe that when $0 < p = q < \infty$, the above spaces are just the spaces H^s_p, $\mathcal{Q}_p$ and $\mathcal{P}_p$ defined and studied in [29].
- Hardy-amalgam spaces of classical functions $H_{p,q}$ of $\mathbb{R}^d$ $(d \geq 1)$ were introduced recently by V. P. Ablé and J. Feuto in [1] where they provided an atomic decomposition for these spaces for $0 < p, q \leq 1$. In [2], they also characterized the corresponding dual spaces, for $0 < p \leq q \leq 1$. A generalization of their definition and their work was pretty recently obtained in [34].
- Our definitions here are inspired from the work [1] and the usual definition of martingale Hardy spaces.

3 Presentation of the Results

We start by defining the notion of atoms.

Definition 3.1 Let $0 < p < \infty$, and $\max(p, 1) < r \leq \infty$. A measurable function a is a $(p,r)^s$-atom (resp. $(p,r)^S$-atom, $(p,r)^*$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that

(a1) $a_n := \mathbb{E}_n a = 0$ if $\nu \geq n$;

(a2) $\|s(a)\|_{r,r} := \|s(a)\|_r$ (resp. $\|S(f)\|_r$, $\|a^*\|_r$) $\leq \mathbb{P}(B_\nu)^{\frac{1}{r}-\frac{1}{p}}$

where $B_\nu = \{\nu \neq \infty\}$.

We also have the following other definition of an atom.

Definition 3.2 Let $0 < p < \infty$, $0 < q \leq \infty$ and $\max(p, 1) < r \leq \infty$. A measurable function a is a $(p, q, r)^s$-atom (resp. $(p, q, r)^S$-atom, $(p, q, r)^*$-atom) if there exists a stopping time $\nu \in \mathcal{T}$ such that condition (a1) in Definition 3.1 is satisfied and

(a3) $\|s(a)\|_r$ (resp. $\|S(a)\|_r$, $\|a^*\|_r$) $\leq \mathbb{P}(B_\nu)^{\frac{1}{r}} \|1_{B_\nu}\|_{p,q}^{-1}$.

We denote by $\mathcal{A}(p, q, r)^s$ (resp. $\mathcal{A}(p, q, r)^S$, $\mathcal{A}(p, q, r)^*$) the set of all sequences of triplets (λ_k, a^k, ν^k), where λ_k are nonnegative numbers, a^k are $(p, r)^s$-atoms (resp. $(p, r)^S$-atoms, $(p, r)^*$-atoms) and $\nu^k \in \mathcal{T}$ satisfying conditions $(a1)$ and $(a2)$ in Definition 3.1 and such that for any $0 < \eta \leq 1$,

$$\sum_k \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \in L_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

We denote by $\mathcal{B}(p, q, r)^s$ (resp. $\mathcal{B}(p, q, r)^S$, $\mathcal{B}(p, q, r)^*$) the set of all sequences of triplets (λ_k, a^k, ν^k), where λ_k are nonnegative numbers, a^k are $(p, q, r)^s$-atoms (resp. $(p, q, r)^S$-atoms, $(p, q, r)^*$-atoms) and $\nu^k \in \mathcal{T}$ satisfying conditions $(a1)$ and $(a3)$ in Definition 3.2 and such that for any $0 < \eta \leq 1$,

$$\sum_k \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \in L_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

We observe that $\mathcal{A}(p, q, r)^s \subseteq \mathcal{B}(p, q, r)^s$ if $p \leq q$ and $\mathcal{B}(p, q, r)^s \subseteq \mathcal{A}(p, q, r)^s$ if $q \leq p$. The same relation holds between the other sets of triplets.

Our first atomic decomposition of the spaces $H_{p,q}^s$ is as follows.

Theorem 3.3 *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and let* $\max(p, 1) < r \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $H_{p,q}^s$, *then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, r)^s$ *such that for all* $n \in \mathbb{N}$,

$$\sum_{k \in \mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{2}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k \geq 0} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta}, \frac{q}{\eta}}^{\frac{1}{\eta}} \leq C \|f\|_{H_{p,q}^s}. \tag{3}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely if $f \in \mathcal{M}$ *has a decomposition as in (2), then for any* $0 < \eta \leq 1$,

$$\|f\|_{H^s_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

Using our second definition of atoms, we also obtain the following atomic decomposition.

Theorem 3.4 *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and let* $\max(p, 1) < r \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $H^s_{p,q}$ *then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{B}(p, q, r)^s$ *such that for all* $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{4}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k\geq 0} \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta}, \frac{q}{\eta}} \leq C\|f\|_{H^s_{p,q}}. \tag{5}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ *as* $m \to \infty,\ l \to -\infty$.

Conversely if $f \in \mathcal{M}$ *has a decomposition as in (4), then for any* $0 < \eta \leq 1$,

$$\|f\|_{H^s_{p,q}} \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta}, \frac{q}{\eta}}.$$

For the last two spaces, we obtain the following two atomic decompositions.

Theorem 3.5 *Let* $0 < p < \infty$ *and* $0 < q \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $\mathcal{Q}_{p,q}$ *(resp.* $\mathcal{P}_{p,q}$*), then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{A}(p, q, \infty)^S$ *(resp.* $\mathcal{A}(p, q, \infty)^*$*) such that for any* $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{6}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C\|f\|_{\mathcal{Q}_{p,q}} (\text{resp. } \|f\|_{\mathcal{P}_{p,q}}). \tag{7}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ *(resp.* $\mathcal{P}_{p,q}$*) as* $m \to \infty,\ l \to -\infty$.

Conversely, if $f \in \mathcal{M}$ *has a decomposition as in (6), then for any* $0 < \eta \leq 1$,

$$\|f\|_{\mathcal{Q}_{p,q}} (\text{resp.} \|f\|_{\mathcal{P}_{p,q}}) \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Theorem 3.6 *Let* $0 < p < \infty$ *and* $0 < q \leq \infty$. *If the martingale* $f \in \mathcal{M}$ *is in* $\mathcal{Q}_{p,q}$ *(resp.* $\mathcal{P}_{p,q}$*), then there exists a sequence of triplets* $(\lambda_k, a^k, \nu^k) \in \mathcal{B}(p, q, \infty)^S$ *(resp.* $\mathcal{B}(p, q, \infty)^*$*) such that for amy* $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n \tag{8}$$

and for any $0 < \eta \leq 1$,

$$\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|1_{B_{\nu^k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq C\|f\|_{\mathcal{Q}_{p,q}} (\text{resp. } \|f\|_{\mathcal{P}_{p,q}}). \tag{9}$$

Moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ *(resp.* $\mathcal{P}_{p,q}$*) as* $m \to \infty,\ l \to -\infty$.

Conversely if $f \in \mathcal{M}$ *has a decomposition as in (8), then for any* $0 < \eta \leq 1$,

$$\|f\|_{\mathcal{Q}_{p,q}}(\text{resp.}\|f\|_{\mathcal{P}_{p,q}}) \leq C \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\|1_{B_{\nu k}}\|_{p,q}} \right)^{\eta} \mathbf{1}_{B_{\nu k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Denote by L_2^0 the set of all $f \in L_2$ such that $\mathbb{E}_0 f = 0$. For $f \in L_2^0$, put $f_n = \mathbb{E}_n f$. We recall that $(f_n)_{n\geq 0}$ is in $\mathcal{M}$ and L_2-bounded. Moreover, $(f_n)_{n\geq 0}$ converges to f in L_2 (see [24]).

Define the function $\varphi : \mathcal{F} \longrightarrow (0, \infty)$ by

$$\varphi(A) = \frac{\|\mathbf{1}_A\|_{p,q}}{\mathbb{P}(A)}$$

for all $A \in \mathcal{F}$, $P(A) \neq 0$. We then define the Campanato space $\mathcal{L}_{2,\varphi}$ as

$$\mathcal{L}_{2,\varphi} := \left\{ f \in L_2^0 : \|f\|_{\mathcal{L}_{2,\varphi}} := \sup_{\nu\in\mathcal{T}} \frac{1}{\varphi(B_\nu)} \left(\frac{1}{\mathbb{P}(B_\nu)} \int_{B_\nu} |f - f^\nu|^2 \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}} < \infty \right\}.$$

Our characterization of the dual space of $H_{p,q}^s$ spaces for $0 < p \leq q \leq 1$ is as follows.

Theorem 3.7 *Let* $0 < p \leq q \leq 1$. *For* $\kappa \in (H_{p,q}^s)^*$, *the dual space of* $H_{p,q}^s$, *there exists* $g \in \mathcal{L}_{2,\varphi}$ *such that*

$$\kappa(f) = \mathbb{E}[fg] \quad \textit{for all} \quad f \in H_{p,q}^s$$

and

$$\|g\|_{\mathcal{L}_{2,\varphi}} \leq c\|\kappa\|.$$

Conversely, let $g \in \mathcal{L}_{2,\varphi}$. *Then the mapping*

$$\kappa_g(f) = \mathbb{E}[fg] = \int_\Omega fg\, d\mathbb{P}, \quad \forall f \in L_2(\Omega)$$

can be extended to a continuously linear functional on $H_{p,q}^s$ *such that*

$$\|\kappa\| \leq c\|g\|_{\mathcal{L}_{2,\varphi}}.$$

4 Proof of Atomic Decompositions

We prove here the atomic decompositions of the spaces $H_{p,q}^s$, $\mathcal{Q}_{p,q}$, and $\mathcal{P}_{p,q}$.

Proof of Theorem 3.3 We only present here the case of $H_{p,q}^s$. The proof of the atomic decomposition of the other spaces is obtained mutatis mutandis.

Let f be in $H_{p,q}^s$ and define the stopping time as

$$\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}. \tag{10}$$

It is clear that $(\nu^k)_{k\in\mathbb{Z}}$ is nonnegative and nondecreasing. Take $\lambda_k = 2^{k+1}\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}$, and $a^k = (a_n^k)_{n\geq 0}$ where

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k}, \text{ if } \lambda_k \neq 0 \quad \text{and} \quad a_n^k = 0, \text{ if } \lambda_k = 0. \tag{11}$$

Recalling that

$$d_n f^{\nu^k} = f_n^{\nu^k} - f_{n-1}^{\nu^k} = \sum_{m=0}^{n} \mathbf{1}_{\{\nu^k \geq m\}} d_m f - \sum_{m=0}^{n-1} \mathbf{1}_{\{\nu^k \geq m\}} d_m f = \mathbf{1}_{\{\nu^k \geq n\}} d_n f,$$

we obtain that

$$\begin{aligned} s(f^{\nu^k}) &= \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n f^{\nu^k}|^2\right)^{\frac{1}{2}} = \left(\sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|\mathbf{1}_{\{\nu^k\geq n\}} d_n f|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n\in\mathbb{N}} \mathbf{1}_{\{\nu^k\geq n\}} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\nu^k} \mathbb{E}_{n-1}|d_n f|^2\right)^{\frac{1}{2}} \\ &= s_{\nu^k}(f). \end{aligned}$$

Thus by the definition of our stopping time, $s(f^{\nu^k}) = s_{\nu^k}(f) \leq 2^k$. Moreover,

$$\sum_{k\in\mathbb{Z}} (f_n^{\nu^{k+1}} - f_n^{\nu^k}) = f_n \text{ a.e.}$$

It follows that $(f_n^{\nu^k})_{n\geq 0}$ is an L_2-bounded martingale and so is $(a_n^{\nu^k})_{n\geq 0}$. Consequently, the limit

$$\lim_{n\to\infty} a_n^k$$

exists a.e. in L_2. Hence (2) is satisfied.

Let us check that a^k is an $(p,r)^s$-atom. We start by noting that on the set $\{n \leq \nu^k\}$ we have

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} = \frac{f_n - f_n}{\lambda_k} = 0 \tag{12}$$

by definition of stopped martingales and thus $\mathbb{E}_n a^k = 0$ when $\nu^k \geq n$. Hence a^k satisfies condition (a1) in Definition 3.1.

Also we note that Eq. (12) also implies that

$$\mathbf{1}_{\{\nu^k=\infty\}}[s(a^k)]^2 \leq \sum_{n\in\mathbb{N}} \mathbf{1}_{\{\nu^k \geq n\}} \mathbb{E}_{n-1}|d_n a^k|^2 = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{\nu^k \geq n\}} \mathbb{E}_{n-1}|\mathbf{1}_{\{\nu^k \geq n\}} d_n a^k|^2 = 0.$$

That is the support of $s(a^k)$ is contained in $B_{\nu^k} = \{\nu^k \neq \infty\}$.

Observing that

$$d_n a^k = \frac{d_n(f^{\nu^{k+1}} - f^{\nu^k})}{\lambda_k} = \frac{(d_n f)\mathbf{1}_{\{\nu^k < n \leq \nu^{k+1}\}}}{\lambda_k},$$

we obtain the following

$$[s(a^k)]^2 = \sum_{n\in\mathbb{N}} \mathbb{E}_{n-1}|d_n a^k|^2 \leq \left(\frac{s_{\nu^{k+1}}(f)}{\lambda_k}\right)^2 \leq \left(\frac{2^{k+1}}{\lambda_k}\right)^2 = \left(\mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}\right)^2.$$

Therefore $s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ and as $s(a^k) = 0$ outside B_{ν^k}, we easily obtain that

$$\|s(a^k)\|_r \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{r}-\frac{1}{p}}.$$

Thus condition (a2) in the definition of an $(p,r)^s$-atom also holds.

We next check that

$$\sum_k \lambda_k a^k \longrightarrow f \text{ in } H^s_{p,q}.$$

As

$$\lambda_k a^k = f^{\nu^{k+1}} - f^{\nu^k},$$

we obtain that

$$\sum_{k=l}^m \lambda_k a^k = \sum_{k=l}^m (f^{\nu^{k+1}} - f^{\nu^k}) = f^{\nu^{m+1}} - f^{\nu^l}.$$

Hence we have that

$$f - \sum_{k=l}^{m} \lambda_k a^k = (f - f^{\nu^{m+1}}) + f^{\nu^l}. \tag{13}$$

Now by definition,

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,q}} = \|s(f - f^{\nu^{m+1}})\|_{p,q}.$$

Thus for Ω_j as in the definition of amalgam spaces,

$$\begin{aligned} \|s(f - f^{\nu^{m+1}})\|_{p,q} &= \left[\sum_{j\in\mathbb{Z}} \left(\int_\Omega s^p(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \\ &= \left[\sum_{j\in\mathbb{Z}} \|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p^q\right]^{\frac{1}{q}} \end{aligned}$$

As $s^p(f - f^{\nu^{m+1}}) \le s^p(f)$ and

$$\int_{\Omega_j} s^p(f)\mathrm{d}\mathbb{P} < \infty,$$

it follows from the Dominated Convergence Theorem that

$$\|s(f - f^{\nu^{m+1}})\mathbf{1}_{\Omega_j}\|_p \longrightarrow 0$$

as $m \longrightarrow \infty$. Hence as $\sum_{j\in\mathbb{Z}} \|s(f)\mathbf{1}_{\Omega_j}\|_p^q = \|f\|^q_{H^s_{p,q}} < \infty$, applying the Dominated Convergence Theorem for the sequence space ℓ_q, we conclude that

$$\|f - f^{\nu^{m+1}}\|_{H^s_{p,q}} = \|s(f - f^{\nu^{m+1}})\|_{p,q} \longrightarrow 0$$

as $m \longrightarrow \infty$.

Also since $s(f^{\nu^k}) \le 2^k$, we have that

$$\|f^{\nu^l}\|_{H^s_{p,q}} = \|s(f^{\nu^l})\|_{p,q} \le 2^l.$$

Hence $\|f^{\nu^l}\|_{H^s_{p,q}} \longrightarrow 0$ as $l \longrightarrow -\infty$. Thus (13) implies that

$$\left\| f - \sum_{k=l}^{m} \lambda_k a^k \right\|_{H^s_{p,q}} = \left\| (f - f^{\nu^{m+1}}) + f^{\nu^l} \right\|_{H^s_{p,q}} \\ \leq \|(f - f^{\nu^{m+1}})\|_{H^s_{p,q}} + \|f^{\nu^l}\|_{H^s_{p,q}} \to 0$$

as $m \longrightarrow \infty$ and $l \longrightarrow -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ as $m \to \infty,\ l \to -\infty$.

Let us now establish (3). Let $\Omega_j \subset \Omega$ be as in the definition of amalgam space. Then by definition,

$$\left\| \sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|^{\frac{1}{\eta}}_{\frac{p}{\eta}, \frac{q}{\eta}} = \left[\sum_{j \in \mathbb{Z}} \left(\int_{\Omega} \left(\sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right)^{\frac{p}{\eta}} \mathbf{1}_{\Omega_j} \mathrm{d}\mathbb{P} \right)^{\frac{q}{p}} \right]^{\frac{\eta}{q} \cdot \frac{1}{\eta}}.$$

Considering the inner sum, we see that

$$\sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} = \sum_{k \in \mathbb{Z}} \left(\frac{2^{k+1} \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} = \sum_{k \in \mathbb{Z}} \left(2^{k+1} \right)^{\eta} \mathbf{1}_{B_{\nu^k}}.$$

We shall borrow an idea from [30, pp 21–22]. Let $G_k = B_{\nu^k} \setminus B_{\nu^{k+1}}$ where $B_{\nu^k} = \{\nu^k \neq \infty\}$. Then G_k are disjoint such that $B_{\nu^k} = \bigcup_{r=k}^{\infty} G_r$ and

$$\mathbf{1}_{B_{\nu^k}} = \sum_{r=k}^{\infty} \mathbf{1}_{G_r}. \tag{14}$$

Hence

$$\sum_{k \in \mathbb{Z}} \left(2^{k+1} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} = \sum_{k \in \mathbb{Z}} 2^{(k+1)\eta} \cdot \sum_{r=k}^{\infty} \mathbf{1}_{G_r} \\ = \sum_{r \in \mathbb{Z}} \sum_{k \leq r} 2^{(k+1)\eta} \mathbf{1}_{G_r} \\ \leq \frac{2^{\eta}}{2^{\eta} - 1} \sum_{k \in \mathbb{Z}} 2^{(k+1)\eta} \mathbf{1}_{G_k}.$$

Thus

$$\sum_{k\in\mathbb{Z}}\left(2^{k+1}\right)^{\eta}\mathbf{1}_{B_{\nu^k}} \leq \frac{4^{\eta}}{2^{\eta}-1}\left(\sum_{k\in\mathbb{Z}}s(f)\mathbf{1}_{G_k}\right)^{\eta} = \frac{(4)^{\eta}s(f)^{\eta}}{2^{\eta}-1}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}.$$

It follows that

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}} &= \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}}\\
&\leq \left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}\frac{4^p s(f)^p}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\mathbf{1}_{G_k}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}}\\
&= \left[\sum_{j\in\mathbb{Z}}\left(\frac{4^p}{(2^{\eta}-1)^{\frac{p}{\eta}}}\sum_{k\in\mathbb{Z}}\int_{G_k}s^p(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}}\\
&\leq \left(\frac{4^{\eta}}{2^{\eta}-1}\right)\left[\sum_{j\in\mathbb{Z}}\left(\int_{\Omega}s^p(f)\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{\eta}{q}}.
\end{aligned}$$

That is

$$\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_j}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \leq \left(\frac{4^{\eta}}{2^{\eta}-1}\right)^{\frac{1}{\eta}}\|s(f)\|_{p,q} = \left(\frac{4^{\eta}}{2^{\eta}-1}\right)^{\frac{1}{\eta}}\|f\|_{H^s_{p,q}}.$$

The first part of the theorem is then established.

Conversely, let the martingale f have a representation as in (2). Then as $s(a^k) \leq \mathbb{P}(\nu^k \neq \infty)^{\frac{1}{r}-\frac{1}{p}} \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ with support in B_{ν^k}, we obtain that

$$\begin{aligned}
\|f\|_{H^s_{p,q}} = \|s(f)\|_{p,q} &\leq \left\|\sum_{k\in\mathbb{Z}}\lambda_k s(a^k)\right\|_{p,q}\\
&\leq \left\|\sum_{k\in\mathbb{Z}}\lambda_k\mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\\
&= \left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}.
\end{aligned}$$

Let us quickly check that

$$\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\leq\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

for $0<\eta<1$. Indeed by definition,

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}&=\left[\sum_{j\in\mathbb{Z}}\left(\int_\Omega\left(\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right)^{p}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\&=\left[\sum_{j\in\mathbb{Z}}\left(\int_\Omega\left(\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right)^{p\frac{\eta}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{1}{q}\cdot\frac{\eta}{\eta}}\\&\leq\left[\sum_{j\in\mathbb{Z}}\left(\int_\Omega\left(\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right)^{\frac{p}{\eta}}\mathbf{1}_{\Omega_j}\mathrm{d}\mathbb{P}\right)^{\frac{q/\eta}{p/\eta}}\right]^{\frac{\eta}{q}\cdot\frac{1}{\eta}}\\&=\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.\end{aligned}$$

Thus

$$\left\|\sum_{k\in\mathbb{Z}}\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\mathbf{1}_{B_{\nu^k}}\right\|_{p,q}\leq\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}\tag{15}$$

Hence

$$\|f\|_{H^s_{p,q}}\leq\left\|\sum_{k\in\mathbb{Z}}\left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}}\right)^{\eta}\mathbf{1}_{B_{\nu^k}}\right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}$$

establishing the converse. The Theorem is proved.

□

Proof of Theorem 3.4 The proof of Theorem 3.4 follows similarly. The main changes are as follows. Let f be in $H^s_{p,q}$ and define the stopping time

$$\nu^k:=\inf\{n\in\mathbb{N}:s_{n+1}(f)>2^k\}.\tag{16}$$

The sequence here is taken as $\lambda_k = 2^{k+1}\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}$, and again $a^k = (a_n^k)_{n\geq 0}$ with

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k}, \text{ if } \lambda_k \neq 0 \quad \text{and} \quad a_n^k = 0, \text{ if } \lambda_k = 0. \tag{17}$$

As above one obtains that

$$s(a^k) \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}$$

and $s(a^k) = 0$ on $\{\nu^k = \infty\}$. It follows that

$$\|s(a^k)\|_r \leq \|\mathbf{1}_{B_{\nu^k}}\|_{p,q}^{-1}\mathbb{P}(B_{\nu^k})^{\frac{1}{r}}.$$

The remaining of the proof follows as above. □

Proof of Theorem 3.5 Let $f \in \mathcal{Q}_{p,q}$ (resp. $f \in \mathcal{P}_{p,q}$) Then there exists an adapted non-decreasing, non-negative sequence $(\beta_n)_{n\in\mathbb{N}}$ such that

$$S_n(f) \leq \beta_{n-1} \text{ (resp. } |f_n| \leq \beta_{n-1})$$

and

$$\|\beta_\infty\|_{p,q} \leq 2\|f\|_{\mathcal{Q}_{p,q}} (\text{resp. } \|f\|_{\mathcal{P}_{p,q}}).$$

As stopping time, we take

$$\nu^k := \inf\{n \in \mathbb{N} : \beta_n > 2^k\} \tag{18}$$

and define $\lambda_k = 2^{k+2}\mathbb{P}(\nu^k \neq \infty)^{\frac{1}{p}}$, and $a^k = (a_n^k)_{n\geq 0}$ with

$$a_n^k = \frac{f_n^{\nu^{k+1}} - f_n^{\nu^k}}{\lambda_k} \text{ if } \lambda_k \neq 0, \text{ and } a_n^k = 0 \text{ otherwise.} \tag{19}$$

As in Theorem 3.3, we obtain that a^k satisfies condition (a1) in the definition of $(p,\infty)^S$-atom (resp. $(p,\infty)^*$-atom). Also, $a_n^k = 0$ on $\{\nu^k = \infty\}$ for all $n \geq 0$, and the support of $S(a^k)$ (resp. $(a^k)^*$) is contained in B_{ν^k}.

We have that

$$S(f^{\nu^k}) = S_{\nu^k}(f) \leq \beta_{\nu^k - 1} \leq 2^k \text{ (resp. } (f^{\nu^k})^* \leq \beta_{\nu^k - 1} \leq 2^k).$$

We also obtain that

$$[S(a^k)]^2 = \sum_{n\geq 0} |d_n a^{\nu^k}|^2 \leq \sum_{n\geq 0} \left| \frac{(d_n f)\mathbf{1}_{\nu^k<n\leq \nu^{k+1}}}{\lambda_k} \right|^2$$
$$\leq \left(\frac{S_{\nu^{k+1}}(f)}{\lambda_k}\right)^2 \leq \left(\frac{2^{k+1}}{\lambda_k}\right)^2 \leq \mathbb{P}(B_{\nu^k})^{-\frac{2}{p}}$$

Also

$$\left(\text{resp. } (a^k)^* \leq \frac{(f^{\nu^{k+1}})^* + (f^{\nu^k})^*}{\lambda_k} \leq \frac{2^{k+2}}{\lambda_k} = \mathbb{P}(B_{\nu^k})^{-\frac{1}{p}}\right).$$

Thus $\|S(a^k)\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$ (resp. $\|(a^k)^*\|_\infty \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$). Hence condition (a2) in the definition of an $(p,\infty)^S$-atom (resp. $(p,\infty)^*$-atom) is satisfied.

We next prove that $\sum_{k=l}^m \lambda_k a^k$ converges to f in $\mathcal{Q}_{p,q}$ (resp. $\mathcal{P}_{p,q}$) as $l \to -\infty$ and $m \to \infty$. As usual, define

$$\zeta_{n-1}^j = \mathbf{1}_{\{\nu^k \leq n-1\}} \|S(a^k)\|_\infty \quad \text{and} \quad (\zeta_{n-1})^2 = \sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2$$

$$(\text{resp. } \zeta_{n-1}^j = \mathbf{1}_{\{\nu^k \leq n-1\}} \|(a^k)^*\|_\infty \quad \text{and} \quad \zeta_{n-1} = \sum_{k=m+1}^{\infty} \lambda_k \zeta_{n-1}^k).$$

Then we have (see [29, p. 17])

$$S_n(f - f^{\nu^{m+1}}) \leq \left(\sum_{k=m+1}^{\infty} \lambda_k^2 (\zeta_{n-1}^k)^2\right)^{\frac{1}{2}} = \zeta_{n-1} \text{ (resp. } |f_n - f_n^{\nu^m}| \leq \zeta_{n-1}).$$

Putting $T(a^k) = S(a^k)$, $(a^k)^*$, we obtain

$$\zeta_{n-1} \leq \sum_{k=m+1}^{\infty} \lambda_k \zeta_{n-1}^k \leq \sum_{k=m+1}^{\infty} \lambda_k \|T(a^k)\|_\infty \mathbf{1}_{\{\nu^k \leq n-1\}} \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{P(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}}.$$

It follows that

$$S(f - f^{\nu^{m+1}}) \text{ (resp. } (f - f^{\nu^{m+1}})^*) \leq \lim_{n\to\infty} \zeta_n \leq \sum_{k=m+1}^{\infty} \frac{\lambda_k}{P(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} = \sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}}.$$

Hence

$$\|f - f^{\nu^k m+1}\|^q_{\mathcal{Q}_{p,q}} \,(\text{resp. } \|f - f^{\nu^k m+1}\|^q_{\mathcal{P}_{p,q}}) \le \|\zeta_\infty\|^q_{p,q} \le \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q .$$

Proceeding as in the proof of Theorem 3.3, we obtain that

$$\left(\sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \le C\beta_\infty \mathbf{1}_{\Omega_j}.$$

Hence as $\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p < \infty$, it follows from the Dominated Convergence Theorem that

$$\left\| \left(\sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \to 0 \text{ as } m \to \infty.$$

As

$$\left\| \left(\sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p \le C\|\beta_\infty \mathbf{1}_{\Omega_j}\|_p$$

and as

$$\sum_{j\in\mathbb{Z}} \|\beta_\infty \mathbf{1}_{\Omega_j}\|_p^q = \|\beta_\infty\|^q_{p,q} < \infty,$$

an application of the Dominated Convergence Theorem for sequence spaces leads to

$$\sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k=m+1}^{\infty} 2^{k+2} \mathbf{1}_{B_{\nu^k}} \right) \mathbf{1}_{\Omega_j} \right\|_p^q \to 0 \text{ as } m \to \infty.$$

Thus $\|f - f^{\nu^k m+1}\|_{\mathcal{Q}_{p,q}} \left(\text{resp. } \|f - f^{\nu^k m+1}\|_{\mathcal{P}_{p,q}}\right) \to 0$ as $m \to \infty$.

Similarly, we obtain that $\|f^{\nu^l}\|_{\mathcal{Q}_{p,q}}$ (resp. $\|f^{\nu^l}\|_{\mathcal{P}_{p,q}}$) $\longrightarrow 0$ as $l \longrightarrow -\infty$. Therefore

$$\|f - \sum_{k=l}^{m} \lambda_k a^k\|_{\mathcal{Q}_{p,q}} \,(\text{resp. } \|f - \sum_{k=l}^{m} \lambda_k a^k\|_{\mathcal{P}_{p,q}}) \to 0$$

as $m \to \infty$ and $l \to -\infty$. Hence

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $\mathcal{Q}_{p,q}$ (resp. $\mathcal{P}_{p,q}$) as $m \to \infty$, $l \to -\infty$ and thus for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \lambda_k \mathbb{E}_n a^k = f_n.$$

Now, as in Theorem 3.3, we obtain

$$\begin{aligned}\left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} &\leq \left\| \sum_{k\in\mathbb{Z}} (2^{k+2})^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\ &\leq C \left\|\beta_\infty\right\|_{p,q} \\ &\leq 2C\|f\|_{\mathcal{Q}_{p,q}} \text{ (resp. } \|f\|_{\mathcal{P}_{p,q}}).\end{aligned}$$

Conversely, assume that $f \in \mathcal{M}$ has the decomposition (6). Define β_n by

$$\beta_n := \sum_{k\in\mathbb{Z}} \lambda_k \|S(a^k)\|_\infty \mathbf{1}_{\{\nu^k \leq n\}} \left(\text{resp. } \beta_n := \sum_{k\in\mathbb{Z}} \lambda_k \|(a^k)^*\|_\infty \mathbf{1}_{\{\nu^k \leq n\}} \right).$$

Then $(\beta_n)_{n\geq}$ is a nondecreasing nonnegative adapted sequence also, for $n \geq 0$,

$$S_n(f) \leq \beta_{n-1} \text{ (resp. } |f_n| \leq \beta_{n-1}).$$

As $\|S(a^k)\|_\infty$ (resp. $\|(a^k)^*\|_\infty) \leq \mathbb{P}(\nu^k \neq \infty)^{-\frac{1}{p}}$, it follows that

$$\|\beta_\infty\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

Thus

$$\|f\|_{\mathcal{Q}_{p,q}} \text{ (resp. } \|f\|_{\mathcal{P}_{p,q}}) \leq \|\beta_\infty\|_{p,q} \leq \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}}.$$

The proof is complete. □

The proof of Theorem 3.6 follows similarly. We leave it to the interested reader.

5 Proof of the Duality Result

We start this section by introducing the following result which is essentially [2, Proposition 2.1]. We provide a proof for the sake of completeness.

Proposition 5.1 *Let* $0 < p < 1$ *and* $0 < q \leq 1$. *For all finite sequence* $\{f_n\}_{n=-m}^{m}$ *of elements in* $L_{p,q}(\Omega)$, *we have*

$$\sum_{n=-m}^{m} \|f_n\|_{p,q} \leq \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,q}.$$

Proof Let $0 < p < 1$, $0 < q \leq 1$ and let $\{f_n\}_{n=0}^{m}$ be a finite sequence of elements of $L_{p,q}(\Omega)$. For $q = 1$, using the reverse Minkowski's inequality in L_p (see [13, p. 11–12]), we obtain

$$\sum_{n=-m}^{m} \|f_n\|_{p,1} = \sum_{j\in\mathbb{Z}} \left\| \sum_{n=-m}^{m} |f_n \mathbf{1}_{\Omega_j}| \right\|_p \leq \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,1}.$$

Now assume that $0 < q < 1$ and set

$$x_n := \left\{ \|f_n \mathbf{1}_{\Omega_j}\|_p \right\}_{j\in\mathbb{Z}} \quad \forall n = -m, \ldots, m.$$

Applying the reverse Minkowski's inequality in ℓ^q and L_p, we obtain

$$\begin{aligned} \sum_{n=-m}^{m} \|f_n\|_{p,q} &= \sum_{n=-m}^{m} \|x_n\|_{\ell^q} \leq \left\| \left\{ \sum_{n=-m}^{m} \|f_n \mathbf{1}_{\Omega_j}\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} \\ &\leq \left\| \left\{ \left\| \sum_{n=-m}^{m} f_n \mathbf{1}_{\Omega_j} \right\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} = \left\| \sum_{n=-m}^{m} |f_n| \right\|_{p,q}. \end{aligned}$$

□

We next prove our first duality result.

Proof of Theorem 3.7 Let us start by defining some spaces. For ν a stopping time, we define

$$L_2^{\nu}(\Omega) := \{f \in L_2(\Omega) : \ \mathbb{E}_n(f) = 0, \ \text{for } \nu \geq n, \ n \in \mathbb{N}\}$$

and

$$L_2^{\nu}(B_\nu) := \{f \in L_2^{\nu}(\Omega) : \text{supp}(f) \subseteq B_\nu\}.$$

We endow $L_2^\nu(B_\nu)$ with

$$\|f\|_{L_2^\nu(B_\nu)} := \left(\int_{B_\nu} |f|^2 d\mathbb{P}\right)^{\frac{1}{2}} < \infty.$$

We will first prove that any continuous linear functional on $H_{p,q}^s(\Omega)$ is also continuous on $L_2^\nu(B_\nu)$.

Let $f \in L_2^\nu(B_\nu) \setminus \{0\}$ and consider

$$a(\omega) := C\|f\|_{L_2^\nu(B_\nu)}^{-1}\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}} f(\omega), \quad \omega \in \Omega.$$

Then for an appropriate choice of the constant (for a choice of the constant, use [29, Proposition 2.6 and Theorem 2.11]), a is an $(p, 2)^s$-atom associated to $\nu \in \mathcal{T}$. Observing with Theorem 3.3 that

$$\|a\|_{H_{p,q}^s(\Omega)} \lesssim \|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}},$$

and recalling from the hypothesis that $p \leq q$ and consequently that $\|\mathbf{1}_{B_\nu}\|_{p,q} \leq \|\mathbf{1}_{B_\nu}\|_p$, we obtain

$$\begin{aligned}
\|f\|_{H_{p,q}^s(\Omega)} &= C^{-1}\|f\|_{L_2^\nu(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|a\|_{H_{p,q}^s(\Omega)} \\
&\lesssim \|f\|_{L_2^\nu(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|\mathbf{1}_{B_\nu}\|_{p,q}\mathbb{P}(B_\nu)^{-\frac{1}{p}} \\
&\lesssim \|f\|_{L_2^\nu(B_\nu)}\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}.
\end{aligned}$$

It follows that for any continuous linear functional κ on $H_{p,q}^s(\Omega)$ with operator norm $\|\kappa\|$,

$$|\kappa(f)| \leq \|\kappa\|\|f\|_{H_{p,q}^s(\Omega)} \lesssim \|\kappa\|\mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|f\|_{L_2^\nu(B_\nu)}.$$

Hence κ is continuous on $L_2^\nu(B_\nu)$ with operator norm

$$\|\kappa\|_{(L_2^\nu(B_\nu))^*} := \sup_{\substack{f\in L_2^\nu(B_\nu) \\ \|f\|_{L_2^\nu(B_\nu)}\leq 1}} \|\kappa(f)\| \lesssim \mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}}\|\kappa\|.$$

As $L_2^\nu(B_\nu)$ is a subspace of $L_2(B_\nu) = L_2(B_\nu, d\mathbb{P})$, it follows from the above observation and the Hahn-Banach Theorem that any continuous linear functional κ on $H_{p,q}^s(\Omega)$ can be extended to a continuous linear functional κ_ν on $L_2(B_\nu)$. As $L_2(B_\nu)$ is auto-dual, it follows that there exists $g \in L_2(B_\nu)$ such that

$$\kappa_\nu(f) = \int_{B_\nu} fg\, d\mathbb{P}, \quad \forall f \in L_2(B_\nu).$$

Consequently,

$$\kappa(f) = \kappa_\nu(f) = \int_{B_\nu} fg\, d\mathbb{P}, \quad \forall f \in L_2^\nu(B_\nu).$$

Next, as $L_2(\Omega)$ is a dense in $H^s_{p,q}(\Omega)$ (this follows from the fact that $p \leq q < 2$ and Theorem 3.3), we have that any element κ of the dual space of $H^s_{p,q}(\Omega)$ can be represented by

$$\kappa(f) = \int_\Omega fg\, d\mathbb{P}, \quad \forall f \in L_2(\Omega). \tag{20}$$

We are going to prove that the function g in (20) is in $\mathcal{L}_{2,\varphi}(\Omega)$.

Let $\nu \in \mathcal{T}$ and let $f \in L_2^\nu(B_\nu)$ with $\|f\|_{L_2^\nu(B_\nu)} \leq 1$. Define

$$\rho(\omega) = C\mathbb{P}(B_\nu)^{\frac{1}{2}-\frac{1}{p}} \frac{(f - f^\nu)\mathbf{1}_{B_\nu}(\omega)}{\|(f - f^\nu)\mathbf{1}_{B_\nu}\|_{L_2(\Omega)}}, \quad \omega \in \Omega.$$

Then for an appropriate choice of the constant, ρ is an $(p, 2)^s$-atom associated to the stopping time ν and $\rho \in L_2(\Omega)$. Hence

$$\kappa(\rho) = \int_\Omega \rho g\, d\mathbb{P} = \int_{B_\nu} \rho g\, d\mathbb{P}.$$

Thus

$$\begin{aligned}\left|\int_{B_\nu} \rho(g - g^\nu)\, d\mathbb{P}\right| &= \left|\int_{B_\nu} \rho g\, d\mathbb{P}\right| \\ &= |\kappa(\rho)| \\ &\leq \|\kappa\| \|\rho\|_{H^s_{p,q}(\Omega)} \\ &\lesssim \|\kappa\| \|\mathbf{1}_{B_\nu}\|_{p,q} \mathbb{P}(B_\nu)^{-\frac{1}{p}}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\int_{B_\nu} f(g - g^\nu)\, d\mathbb{P}\right| &= \left|\int_{B_\nu} (f - f^\nu)(g - g^\nu)\, d\mathbb{P}\right| \\ &\lesssim C^{-1} \|(f - f^\nu)\mathbf{1}_{B_\nu}\|_{L_2(\Omega)} \mathbb{P}(B_\nu)^{\frac{1}{p}-\frac{1}{2}} \|\kappa\| \|\mathbf{1}_{B_\nu}\|_{p,q} \mathbb{P}(B_\nu)^{-\frac{1}{p}} \\ &\lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}} \|\mathbf{1}_{B_\nu}\|_{p,q} \|\kappa\|.\end{aligned}$$

Thus

$$\left(\int_{B_\nu} |g - g^\nu|^2 \, d\mathbb{P}\right)^{\frac{1}{2}} := \sup_{\substack{f \in L_2^\nu(B_\nu) \\ \|f\|_{L_2^\nu(B_\nu)} \leq 1}} \left| \int_{B_\nu} f(g - g^\nu) \, d\mathbb{P} \right|$$

$$\lesssim \mathbb{P}(B_\nu)^{-\frac{1}{2}} \|\mathbf{1}_{B_\nu}\|_{p,q} \|\kappa\|.$$

This gives us

$$\frac{1}{\varphi(B_\nu)} \left(\frac{1}{\mathbb{P}(B_\nu)} \int_{B_\nu} |g - g^\nu|^2 \, d\mathbb{P} \right)^{\frac{1}{2}} \lesssim \|\kappa\|, \quad \forall \nu \in \mathcal{T}.$$

Hence $g \in \mathcal{L}_{2,\varphi}(\Omega)$, and the proof of the first part of the theorem is complete.

Conversely, let $g \in \mathcal{L}_{2,\varphi}(\Omega)$. Let $f \in H^s_{p,q}(\Omega)$. We know that for the stopping times

$$\nu^k := \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}, \; k \in \mathbb{Z},$$

$$\left\| \sum_{k \in \mathbb{Z}} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^\eta \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta}, \frac{q}{\eta}}^{\frac{1}{\eta}} \leq C \|f\|_{H^s_{p,q}}. \tag{21}$$

and moreover,

$$\sum_{k=l}^{m} \lambda_k a^k \longrightarrow f$$

in $H^s_{p,q}$ as $m \to \infty, \; l \to -\infty$, where $(\lambda_k, a^k, \nu_k) \in \mathcal{A}(p, q, 2)^s$. Also since a^k is L_2-bounded, for $f \in H^s_{p,q}(\Omega)$,

$$\kappa_g(f) = \mathbb{E}[fg] = \sum_{k \geq 0} \mathbb{E}[\lambda_k a^k g]$$

is well defined and linear. Using this, Schwartz's inequality, and the fact that $\|s(a^k)\|_2 \le \mathbb{P}(B_{\nu^k})^{\frac{1}{2}-\frac{1}{p}}$, we obtain

$$
\begin{aligned}
|\kappa_g(f)| &\le \sum_{k\in\mathbb{Z}} \lambda_k \left| \int_\Omega a^k (g - g^{\nu^k}) \mathrm{d}\mathbb{P} \right| \le \sum_{k\in\mathbb{Z}} \lambda_k \|a^k\|_2 \left(\int_{B_{\nu^k}} |g - g^{\nu^k}|^2 \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}} \\
&\lesssim \sum_{k\in\mathbb{Z}} \lambda_k \|s(a^k)\|_2 \left(\int_{B_{\nu^k}} |g - g^{\nu^k}|^2 \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}} \\
&= \sum_{k\in\mathbb{Z}} \lambda_k \frac{\|\mathbf{1}_{B_{\nu^k}}\|_{p,q}}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \frac{1}{\varphi(B_{\nu^k})} \left(\frac{1}{\mathbb{P}(B_{\nu^k})} \int |g - g^{\nu^k}|^2 \mathrm{d}\mathbb{P} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence using Proposition 5.1, inequalities (15) and (21), we deduce that

$$
\begin{aligned}
|\kappa_g(f)| &\lesssim \|g\|_{\mathcal{L}_{2,\varphi}} \sum_{k\in\mathbb{Z}} \left\| \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \\
&\le \|g\|_{\mathcal{L}_{2,\varphi}} \left\| \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \mathbf{1}_{B_{\nu^k}} \right\|_{p,q} \\
&\le \|g\|_{2,\varphi} \left\| \sum_{k\ge 0} \left(\frac{\lambda_k}{\mathbb{P}(B_{\nu^k})^{\frac{1}{p}}} \right)^{\eta} \mathbf{1}_{B_{\nu^k}} \right\|_{\frac{p}{\eta},\frac{q}{\eta}}^{\frac{1}{\eta}} \\
&\lesssim \|f\|_{H^s_{p,q}} \|g\|_{2,\varphi}.
\end{aligned}
$$

Thus $\kappa_g(f) = \mathbb{E}[fg]$ extends continuously on $H^s_{p,q}(\Omega)$ and the proof is complete. □

6 Concluding Comments

We start this section by introducing two additional martingale Hardy-amalgam spaces. These spaces which we denote $H^S_{p,q}$ and $H^*_{p,q}$ are defined as follows

(i) $H^S_{p,q}(\Omega) = \{f \in \mathcal{M} : \|f\|_{H^S_{p,q}(\Omega)} := \|S(f)\|_{p,q} < \infty\}$.
(ii) $H^*_{p,q}(\Omega) = \{f \in \mathcal{M} : \|f\|_{H^*_{p,q}(\Omega)} := \|f^*\|_{p,q} < \infty\}$.

We note that when $p = q$, the spaces $H^S_{p,q}$ and $H^*_{p,q}$ correspond respectively to the classical martingale Hardy spaces H^S_p and H^*_p discussed in [29]. We note that the question of atomic decompositions of the spaces H^S_p and H^*_p, is still open.

Nevertheless, embedding relations between the five classical martingale spaces are well known. Indeed we have the follows (see [29, Theorem 2.11]).

Proposition 6.1 *For any $f \in \mathcal{M}$, the following hold.*

(i) $\|f\|_{H_p^*} \leq C_p\|f\|_{H_p^s}, \quad \|f\|_{H_p^S} \leq C_p\|f\|_{H_p^s} \qquad (0 < p \leq 2)$
(ii) $\|f\|_{H_p^s} \leq C_p\|f\|_{H_p^*}, \quad \|f\|_{H_p^s} \leq C_p\|f\|_{H_p^S} \qquad (2 \leq p < \infty)$
(iii) $\|f\|_{H_p^*} \leq C_p\|f\|_{\mathcal{P}_p}, \quad \|f\|_{H_p^S} \leq C_p\|f\|_{\mathcal{Q}_p} \qquad (0 < p < \infty)$
(iv) $\|f\|_{H_p^*} \leq C_p\|f\|_{\mathcal{Q}_p}, \quad \|f\|_{H_p^S} \leq C_p\|f\|_{\mathcal{P}_p} \qquad (0 < p < \infty)$
(v) $\|f\|_{H_p^s} \leq C_p\|f\|_{\mathcal{P}_p}, \quad \|f\|_{H_p^s} \leq C_p\|f\|_{\mathcal{Q}_p} \qquad (0 < p < \infty).$

Moreover, if the $(\mathcal{F})_{n\geq 0}$ is regular, the above five spaces are equivalent. [c.f. [29, Theorem 2.11] *for the following.]*

The above result can be used to prove the Burkholder-Davis-Gundy's inequality [29, Theorem 2.12] which provides an equivalence between the spaces H_p^S and H_p^*.

We can then state our first open problem in this setting.

Question 1 Does Proposition 6.1 extend to martingale Hardy-Amalgam spaces introduced in this paper?

If one assumes that for any $j \in \mathbb{Z}$ and any $n \geq 0$, $\Omega_j \in \mathcal{F}_n$, then it is not hard to prove that the above proposition extends to our setting mutatis mutandis. Unfortunately, if this hypothesis is satisfied in the dyadic case for appropriate choice of Ω_j and $\mathcal{F}_n$, it is not sure that it works in the general case.

Our second open question about these new spaces is about the duality for large exponents.

Question 2 Is there a characterization of the dual space of the space $H_{p,q}^s$ for $\max\{p, q\} > 1$?

The dual space of H_p^s for $1 < p < \infty$ is described in [29, Theorem 2.26]. It is not clear how this can be extended to the space $H_{p,q}^s$ even under further restrictions on the family $\{\Omega_j\}_{j\in\mathbb{Z}}$.

7 Declarations

The authors declare that they have no conflict of interest regarding this work.

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Robust Estimation of European and Asian Options

G. I. Beliavsky, N. V. Danilova, and A. D. Logunov

Abstract The statistical problem of calculating upper and lower bounds for fair prices of European and Asian options within the classic Cox-Ross-Rubinstein (CRR) model with uncertain parameters is considered. The method proposed in this paper includes statistical data analysis for determining the ranges of the model parameters. Optimal portfolios are calculated simultaneously with the upper and lower bounds.

Keywords Robust methods · Cluster analysis · Quantile · Ordered sample · Median · Portfolio

1 Introduction

Consider a (B, S)-market with a risky asset S and a risk-free asset B. In the stochastic setup, the return on risky asset $\rho_n = \frac{\Delta S_n}{S_{n-1}}$ is a random variable: for example, in the CRR model the support of its distribution consists of two points a, b. For the risk-free asset, the return $r = \frac{\Delta B_n}{B_{n-1}}$ is a constant. We assume that a stochastic basis $\langle \Omega, (F_n), F, p \rangle$ is given, and consider the set of martingale measures $\tilde{P} = \{p : \langle \frac{S_n}{B_n}, F_n, p \rangle - martingale\}$ on this basis. If the set of martingale measures is non-empty, then the value $C^* = \frac{B_0}{B_N} \sup\limits_{p \in \tilde{P}} E_p f$ is called the upper price of a payoff f. The value $C_* = \frac{B_0}{B_N} \inf\limits_{p \in \tilde{P}} E_p f$ is called the lower price of this payoff; for example, see [1]. The interval $[C_*, C^*]$ is called the fair price range, because the

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price $C < C_*$ creates an arbitrage for the buyer while the price $C > C^*$ for the seller.

The robust problem of calculating the upper price for a European option with a markov payoff might look like the minimax problem $\min_{\gamma}\max_{\rho}\big(\frac{f(S_N)}{B_N} - \frac{1}{1+r}\sum_{i=1}^{N}\gamma_i\frac{S_{i-1}}{B_{i-1}}(\rho_i - r)\big)$ subject to the constraints $\rho_i \in M$, $S_n = S_{n-1}(1+\rho_n)$, $B_n = B_{n-1}(1+r)$. The initial values S_0 and B_0 are assumed to be known. The value $C^* = B_0\min_{\gamma}\max_{\rho}\left(\frac{f(S_N)}{B_N} - \frac{1}{1+r}\sum_{i=1}^{N}\gamma_i\frac{S_{i-1}}{B_{i-1}}(\rho_i - r)\right)$ is the upper price of a European option with a payoff $f(S_N)$. This price guarantees that the capital X_N^{π} of a self-financing portfolio is sufficient for paying on the payoff in the worst-case evolution of the (B, S)-market. The lower price is $C_* = B_0\max_{\gamma}\min_{\rho}\left(\frac{f(S_N)}{B_N} - \frac{1}{1+r}\sum_{i=1}^{N}\gamma_i\frac{S_{i-1}}{B_{i-1}}(\rho_i - r)\right)$ under the same constraints. The model in which the set M is an interval is presented in [1]. Robust methods are of major interest for study. As a rule, robust methods are stable against various kinds of interference [2]. In addition, robust methods reduce the effect of random spreads on the final result [3]. Robust methods are based on robust optimization. Robust convex optimization methods were described in [4–6].

Due to their high relevance, robust methods have many applications. One application is game theory. For example, in [7, 8], the models of games with incomplete (confidential) information and uncertain payoffs were considered, and the condition of an equilibrium was established. Besides game theory, robust models have numerous applications in financial mathematics. For example, in [9], the authors adopted robust optimization for estimating the prices of a wide range of options. The authors solved a robust optimization problem for determining a portfolio that minimizes the replication error of a payoff for the worst case. In [10], the problem of quantile hedging of payoff with uncertain payments from a given range was studied. In [11], the robust optimization problem for finding an optimal investment portfolio for a model with uncertain parameters was considered. In [12], a robust algorithm for finding an optimal portfolio under online trading with a fast update of portfolio data was proposed.

The problem of superhedging or the problem of calculating the upper or lower price of a payoff is studied in none of the above works. Common to all of these works is the minimax or maximin formulation of problems, which are characteristic of robust optimization. In the closest work [9] problems are reduced to linear programming problems. To describe the set of the uncertainty of model parameters, the central limit theorem for finite variance and one of the stable distributions for infinite variance are used.

The main goal of the work is the application of unsupervised learning in the problem of calculating the lower and upper prices of a payoff. To achieve this goal, the classical Cox-Ross-Rubinstein model is used, in which the parameters are replaced by non-intersecting intervals. This model is naturally related to unsupervised learning. For the new model, the seller and buyer problems are formulated as robust optimization problems, for which dynamic programming is used. The use of dynamic programming for solving the problem of robust optimization instead of linear programming significantly distinguishes our study from work [9]. Thus, the present work differs from work [9] in the formulation of the robust optimization problem, the method of its solution, and the description of the uncertainty. A significant factor in determining the calculation is the convexity of the payoff. The payoff convexity has already been applied in the mentioned model, see [1], for superhedging.

The Cox-Ross-Rubinstein model together with the continuous Black-Scholes model is the classics of financial mathematics. Let us refer to work [1] again. To describe the return behavior, using a probability distribution with two atoms, as is done in the Cox-Ross-Rubinstein model, looks unrealistic. The use of other distributions in a model with discrete time leads to a loss of market completeness, and in the case of distribution with infinite support, for example, a normal law, makes superhedging impossible. Using distributions with a finite interval as a support of the distribution can lead to a large difference between the upper and lower price, which depends on the length of the interval, see [1]. Using two disjoint intervals as distribution support can in some cases reduce the length of the fair price interval, that is, reduce uncertainty.

The structure of the work is as follows. One-step seller and buyer tasks for a two-interval model and a convex payoff are considered in Sect. 2. Problems are considered as a minimax problem for the seller and a maximin problem for the buyer. The transition to linear programming problems is made. It should be noted that this is a standard technique for solving minimax and maximin problems, see, for example, the already mentioned work [9]. The obtained computational results are based on the duality theory. These results are further used in Sect. 3 to solve multistep problems with using the dynamic programming method. Analytical formulas are obtained for the markov payoff. These formulas coincide with the Cox-Ross-Rubinstein formulas [1]. Recurrent equations are obtained for the Asian option. The main purpose of this section is to demonstrate efficient computations for a two-interval model.

The problem of estimating the parameters of the two-interval model was solved in 4 sections. Unsupervised learning was used to calculate parameter estimates. A complete overview of unsupervised learning can be found in the widely circulated monograph [13]. Unsupervised learning is used at the dividing sample into two subsets, which are then used to calculate two intervals to which a fixed proportion of the sample belongs.

A computational experiment with real data comparing the two-interval model with the CRR model is described in the fifth section.

The final section includes our conclusions and additional reference.

2 Basic Problems

In this section, we consider the problems whose solutions will be used below in the multistage models. The first problem has the form

$$\min \frac{x}{B_0}, \frac{x}{B_0} \geq \max_{\rho} \left[\frac{f(S_0(1+\rho))}{B_1} - \gamma \frac{S_0}{B_0} \frac{\rho - r}{1+r} \right], \tag{1}$$

$$\rho \in [a_1, a_2] \cup [a_3, a_4], a_2 < a_3,$$

$$\Delta B_1 = B_0 r, a_2 < r < a_3.$$

The function $f(y)$ is assumed to be convex. Since for all values γ the function $\frac{f(S_0(1+\rho))}{B_1} - \gamma \frac{S_0}{B_0} \frac{\rho-r}{1+r}$ is convex in the variable ρ,
$\max_{\rho} \left[\frac{f(S_0(1+\rho))}{B_1} - \gamma \frac{S_0}{B_0} \frac{\rho-r}{1+r} \right] = \max \left[\frac{f(S_0(1+a_1))}{B_1} - \gamma \frac{S_0}{B_0} \frac{a_1-r}{1+r}, \frac{f(S_0(1+a_4))}{B_1} - \gamma \frac{S_0}{B_0} \frac{a_4-r}{1+r} \right].$
As a result, we arrive at the linear programming problem

$$\min \frac{x}{B_0}, \frac{x}{B_0} + \gamma \frac{S_0}{B_0} \frac{a_i - r}{1+r} \geq \frac{f(S_0(1+a_i))}{B_1}, i = 1, 4 \tag{2}$$

Consider the dual linear programming problem. In the primal problem, the variables x and γ can be negative and positive; hence, the constraints in the dual problem are the equalities $y_1 + y_2 = 1$, $y_1 a_1 + y_2 a_4 = r$. The primal problem has inequality-type constraints (no matter of which sign); therefore, the dual variables are $y_i \geq 0$. Finally, the objective function of the dual problem is $F = y_1 \frac{f(S_0(1+a_1))}{B_1} + y_2 \frac{f(S_0(1+a_4))}{B_1}$. Thus, the dual problem can be written as

$$\max \left[y_1 \frac{f(S_0(1+a_1))}{B_1} + y_2 \frac{f(S_0(1+a_4))}{B_1} \right], \tag{3}$$

$$y_1 + y_2 = 1, \ y_1 a_1 + y_2 a_4 = r, \ y_i \geq 0.$$

The system of linear equations has a unique solution: $y_1 = \frac{a_4-r}{a_4-a_1}$, $y_2 = \frac{r-a_1}{a_4-a_1}$. This solution is admissible, because $a_1 < r < a_4$. Consequently, the optimal value is

$$x^* = \frac{1}{1+r} \left[\frac{a_4 - r}{a_4 - a_1} f(S_0(1+a_1)) + \frac{r - a_1}{a_4 - a_1} f(S_0(1+a_4)) \right] \tag{4}$$

The Kuhn-Tucker conditions yield the optimal value

$$\gamma^* = \frac{f(S_0(1+a_4)) - f(S_0(1+a_1))}{S_0(a_4 - a_1)} \tag{5}$$

Consider the second problem:

$$\max \frac{x}{B_0}, \frac{x}{B_0} \le \min_{\rho} \left[\frac{f(S_0(1+\rho))}{B_1} - \gamma \frac{S_0}{B_0} \frac{\rho - r}{1+r} \right]; \tag{6}$$

$$\rho \in [a_1, a_2] \cup [a_3, a_4], \ a_2 < a_3,$$

$$\Delta B_1 = B_0 r; \ a_2 < r < a_3$$

Its discrete analog obtained by the piecewise constant approximation of a continuous function on a compact set has the form

$$\frac{x}{B_0} + \gamma \frac{S_0}{B_0} \frac{\rho_i - r}{1+r} \le \frac{f(S_0(1+\rho_i))}{B_1}, \ \max \frac{x}{B_0} \tag{7}$$

Let $\rho_{i^*} = a_2$, $\rho_{i^*+1} = a_3$. Consider the dual problem

$$\sum_i y_i = 1, \ \sum_i \rho_i y_i = r, \ y_i \ge 0, \ \min F = \frac{1}{B_1} \sum_i y_i f(S_0(1+\rho_i)) \tag{8}$$

As the basic variables we choose y_{i^*}, y_{i^*+1}. The corresponding basic solution $y_{i^*} = \frac{a_3 - r}{a_3 - a_2}$, $y_{i^*+1} = \frac{r - a_2}{a_3 - a_2}$, $y_i = 0, i \ne i^*, i \ne i^* + 1$, is admissible.
The objective function can be written through the free variables as

$$B_1 F = \frac{a_3 - r}{a_3 - a_2} f(S_0(1+a_2)) + \frac{r - a_2}{a_3 - a_2} f(S_0(1+a_3)) -$$

$$- \sum_{i \ne i^*, i \ne i^*+1} \left[\frac{a_3 - \rho_i}{a_3 - a_2} f(S_0(1+a_2)) + \frac{\rho_i - a_2}{a_3 - a_2} f(S_0(1+a_3)) - f(S_0(1+\rho_i)) \right] y_i$$

Let $\rho_i > a_3$. We express $a_3 = \alpha a_2 + (1-\alpha)\rho$, and consequently $\alpha = \frac{\rho_i - a_3}{\rho_i - a_2}$. Since the function f is convex, $f(S_0(1+a_3)) \le \alpha f(S_0(1+a_2)) + (1-\alpha) f(S_0(1+\rho_i))$. This gives $\frac{1}{1-\alpha} f(S_0(1+a_3)) - \frac{\alpha}{1-\alpha} f(S_0(1+a_2)) \le f(S_0(1+\rho_i))$. Substituting α into this inequality, we obtain $\frac{a_3 - \rho_i}{a_3 - a_2} f(S_0(1+a_2)) + \frac{\rho_i - a_2}{a_3 - a_2} f(S_0(1+a_3)) \le f(S_0(1+\rho_i))$. In a similar fashion, the above inequality can be easily derived in the case $\rho_i < a_2$. Hence, the coefficients at the free variables are negative, and the basic solution is optimal. As a result,

$$x^* = \frac{1}{1+r} \left[\frac{a_3 - r}{a_3 - a_2} f(S_0(1+a_2)) + \frac{r - a_2}{a_3 - a_2} f(S_0(1+a_3)) \right], \tag{9}$$

$$\gamma^* = \frac{f(S_0(1+a_3)) - f(S_0(1+a_2))}{S_0(a_3 - a_2)} \tag{10}$$

From (9) and (10) it follows that the solution is independent of the partition points and their number. Letting the number of partition points tend to infinity (so that the diameter of partition will vanish), we establish that (9) and (10) are the solutions of the continuous problem.

The third and fourth problems have the forms

$$\min \frac{x}{B_0},\ \frac{x}{B_0} \geq \max_{\rho}\left[\frac{f(S_0+S_0(1+\rho))}{B_1} - \gamma\frac{S_0}{B_0}\frac{\rho-r}{1+r}\right],\ \rho \in [a_1,a_2]\cup[a_3,a_4],\ a_2 < a_3$$

$$\Delta B_1 = B_0 r,\ a_2 < r < a_3$$

and

$$\max \frac{x}{B_0},\ \frac{x}{B_0} \leq \min_{\rho}\left[\frac{f(S_0+S_0(1+\rho))}{B_1} - \gamma\frac{S_0}{B_0}\frac{\rho-r}{1+r}\right],\ \rho \in [a_1,a_2]\cup[a_3,a_4],\ a_2 < a_3$$

$$\Delta B_1 = B_0 r,\ a_2 < r < a_3,$$

respectively. The third and fourth problems are equivalent to the first and second ones. Therefore, we present their solutions without intermediate calculations:

$$x^* = \frac{1}{1+r}\left[\frac{a_4-r}{a_4-a_1}f(S_0+S_0(1+a_1)) + \frac{r-a_1}{a_4-a_1}f(S_0+S_0(1+a_4))\right],$$

$$\gamma^* = \frac{f(S_0+S_0(1+a_4)) - f(S_0+S_0(1+a_1))}{S_0(a_4-a_1)}$$

as the solution of the third problem;

$$x^* = \frac{1}{1+r}\left[\frac{a_3-r}{a_3-a_2}f(S_0+S_0(1+a_2)) + \frac{r-a_2}{a_3-a_2}f(S_0+S_0(1+a_3))\right],$$

$$\gamma^* = \frac{f(S_0+S_0(1+a_3)) - f(S_0+S_0(1+a_2))}{S_0(a_3-a_2)}$$

as the solution of the fourth problem.

3 Multistage Problems

Consider the multistage setup of the first problem:

$$X_0(x) = \min_{\gamma_1^N} y,\ \frac{y}{B_0} + \frac{1}{1+r}\sum_{i=1}^{N}\gamma_i\frac{S_{i-1}}{B_{i-1}}(\rho_i - r) \geq \frac{f\left(x\prod_{i=1}^{N}(1+\rho_i)\right)}{B_N},$$

$$\forall \rho_i \in [a_1,a_2]\cup[a_3,a_4],\ \Delta S_i = S_{i-1}\rho_i,\ \Delta B_i = B_{i-1}r,\ S_0 = x.$$

Here the additional notation is $\gamma_r^s = \{\gamma_r, \gamma_{r+1}, \dots, \gamma_s\}$. This multistage problem is a dynamic programming problem. For solving it, we introduce the Bellman function $X_k(x)$ for a time instant k:

$$X_k(x) = \min_{\gamma_{k+1}^N} y, \quad \frac{y}{B_k} + \frac{1}{1+r}\sum_{i=k+1}^{N} \gamma_i \frac{S_{i-1}}{B_{i-1}}(\rho_i - r) \geq \frac{f\left(x \prod_{i=1}^{N}(1+\rho_i)\right)}{B_N},$$

$$\forall \rho_i \in [a_1, a_2] \cup [a_3, a_4], \; S_i = S_{i-1}(1+\rho_i), \; B_i = B_{i-1}(1+r), \; S_k = x.$$

In other words, for calculating the value of the function $X_k(x)$ (further referred to as capital), we have to solve an optimization problem with a large number of constraints. Nevertheless, the value of $X_k(x)$ can be found using a rather simple formula; see the theorem below.

Theorem 1 (The Seller's Problem) *If $f(x)$ is a convex function, then the capital $X_k(x)$ is given by*

$$X_k(x) = \frac{1}{(1+r)^{N-k}} \sum_{j=0}^{N-k} C_{N-k}^{j} f(x(1+a_4)^j(1+a_1)^{N-k-j})(p^*)^j(q^*)^{N-k-j}$$
$$p^* = \frac{r-a_1}{a_4-a_1}, q^* = 1-p^*. \tag{11}$$

Proof This result will be established using mathematical induction. For the case $k = N-1$, see formula (4) for the one-stage problem presented above:
$X_{N-1}(x) = \frac{1}{1+r}\left[q^* f(x(1+a_1)) + p^* f(x(1+a_4))\right]$. Obviously, the desired formula holds for $N-1$. We make the inductive hypothesis, assuming that this formula is valid for $k+1$:
$X_{k+1}(x) = \frac{1}{(1+r)^{N-k-1}} \sum_{j=0}^{N-k-1} C_{N-k-1}^{j} f(x(1+a_4)^j(1+a_1)^{N-k-j-1})$ $(p^*)^j(q^*)^{N-k-j-1}$. Note that under this hypothesis, $X_{k+1}(x)$ is a convex function as a convex combination of convex function. For calculating X_k we have to solve the problem

$$X_k(x) = \min y, \quad \frac{y}{B_k} + \gamma\frac{x}{B_k}\frac{\rho - r}{1+r} \geq \frac{X_{k+1}(x(1+\rho))}{B_{k+1}}, \; \forall \rho \in [a_1, a_2] \cup [a_3, a_4].$$

Since $X_{k+1}(x(1+\rho))$ is a convex function in the variable ρ, we can use the result obtained for the one-stage model:

$$X_k(x) = \frac{1}{1+r}\left(X_{k+1}(x(1+a_4))p^* + X_{k+1}(x(1+a_1))q^*\right) \tag{12}$$

Hence,

$$X_k(x)=\frac{1}{(1+r)^{N-k}}\left(\sum_{j=0}^{N-k-1} C^j_{N-k-j} f(x(1+a_4)^{j+1}(1+a_1)^{N-k-j-1})(p^*)^j(q^*)^{N-k-j-1}+\right.$$
$$\left.+\sum_{j=0}^{N-k-1} C^j_{N-k-1} f(x(1+a_4)^{j}(1+a_1)^{N-k-j})(p^*)^j(q^*)^{N-k-j-1}\right).$$

In the first sum, we make the change of variables $j := j+1$. Then

$$X_k(x)=\frac{1}{(1+r)^{N-k}}\left(\sum_{j=1}^{N-k} C^{j-1}_{N-k-1} f(x(1+a_4)^{j}(1+a_1)^{N-k-j})(p^*)^j(q^*)^{N-k-j}+\right.$$
$$\left.+\sum_{j=0}^{N-k-1} C^j_{N-k-1} f(x(1+a_4)^{j}(1+a_1)^{N-k-j})(p^*)(q^*)^{N-k-j-1}\right)$$

In view of the relation $C^{j-1}_{N-k-1}+C^j_{N-k-1}=C^j_{N-k}$, we obtain

$$X_k(x)=\frac{1}{(1+r)^{N-k}}\left(\sum_{j=1}^{N-k-1} C^j_{N-k} f(x(1+a_4)^{j}(1+a_1)^{N-k-j})(p^*)^j(q^*)^{N-k-j}+\right.$$
$$\left.+f(x(1+a_4)^{N-k})(p^*)^{N-k}+f(x(1+a_1)^{N-k})(q^*)^{N-k}\right)=$$
$$=\frac{1}{(1+r)^{N-k}}\sum_{j=0}^{N-k} C^j_{N-k} f(x(1+a_4)^{j}(1+a_1)^{N-k-j})(p^*)^j(q^*)^{N-k-j}$$

The proof of Theorem 1 is complete. □

Obviously, the portfolio can be calculated using a formula similar to (5):

$$\gamma^*_{k+1}(x)=\frac{X_{k+1}(x(1+a_4))-X_{k+1}(x(1+a_1))}{x(a_4-a_1)} \tag{13}$$

An analogous theorem applies to the buyer's problem:

$$X_0(x)=\max_{\gamma_1^N} y,\ \frac{y}{B_0}+\frac{1}{1+r}\sum_{i=1}^{N}\gamma_i\frac{S_{i-1}}{B_{i-1}}(\rho_i-r)\le\frac{f\left(x\prod_{i=1}^{N}(1+\rho_i)\right)}{B_N},$$
$$\forall\rho_i\in[a_1,a_2]\cup[a_3,a_4],$$
$$\Delta S_i=S_{i-1}\rho_i,\ \Delta B_i=B_{i-1}r,\ S_0=x.$$

We define the capital $X_k(x)$ for the buyer's problem:

$$X_k(x) = \max_{\gamma_{k+1}^N} y,\ \frac{y}{B_k} + \frac{1}{1+r}\sum_{i=k+1}^{N} \gamma_i \frac{S_{i-1}}{B_{i-1}}(\rho_i - r) \leq \frac{f\left(x\prod_{i=1}^{N}(1+\rho_i)\right)}{B_N},$$

$$\forall \rho_i \in [a_1, a_2] \cup [a_3, a_4],\ \ \Delta S_i = S_{i-1}\rho_i,\ \ \Delta B_i = B_{i-1}r,\ \ S_k = x$$

Theorem 2 (Buyer's Problem) *If $f(x)$ is a convex function, then the capital $X_k(x)$ is given by*

$$X_k(x) = \frac{1}{(1+r)^{N-k}} \sum_{j=0}^{N-k} C_{N-k}^{j} f(x(1+a_2)^j(1+a_3)^{N-k-j})(p^*)^j(q^*)^{N-k-j},$$

$$p^* = \frac{r-a_2}{a_3-a_2}, q^* = 1-p^*. \qquad (14)$$

The portfolio can be calculated using a formula similar to (10):

$$\gamma_{k+1}^*(x) = \frac{X_{k+1}(x(1+a_3)) - X_{k+1}(x(1+a_2))}{x(a_3-a_2)} \qquad (15)$$

Consider an Asian option with the payoff function $f_N = \left(\frac{1}{N+1}\sum_{i=0}^{N} S_i - K\right)^+$. This payoff function can be interpreted as a particular case of the general function $f_N = f\left(\sum_{i=0}^{N} S_i\right)$, where $f(x)$ is a convex function. For the seller's problem, we determine a sequence of functions $X_k(u, v)$ of the two variables u and v calculated by solving the optimization problems

$$X_k(u, v) = \min y,$$

$$\frac{y}{B_k} + \frac{1}{1+r}\sum_{i=k+1}^{N} \gamma_i \frac{S_{i-1}}{B_{i-1}}(\rho_i - r) \geq \frac{f\left(v + \sum_{i=k+1}^{N} S_i\right)}{B_N},$$

$$S_i = S_{i-1}(1+\rho_i),\ \ B_i = B_{i-1}(1+r),$$

$$\forall \rho_i \in [a_1, a_2] \cup [a_3, a_4],\ \ S_k = u.$$

$X_0(u, 0)$—is the solution of the problem.

Assume that at the time instant $k + 1$ the optimal capital $X_{k+1}(u, v)$ is known. The capital $X_k(u, v)$ is given by

$$X_k(u, v) = \min y, \ \frac{y}{B_k} + \gamma \frac{S_k}{B_k} \frac{\rho - r}{1 + r} \geq \frac{X_{k+1}(u(1 + \rho), v + u(1 + \rho))}{B_{k+1}}, \quad \forall \rho \in [a_1, a_2] \cup [a_3, a_4]. \tag{16}$$

Assume that the function $X_{k+1}(u, v)$ is convex; then the function $X_{k+1}(u(1 + \rho), \ v + u(1 + \rho))$ is convex in the variable ρ. Taking into account this property and using the solution of the one-stage problem, we obtain the recursive formulas for the optimal capital and portfolio:

$$\begin{aligned} X_k(u, v) &= \\ &= \frac{1}{1+r}\left[q^* X_{k+1}(u(1 + a_1), v + u(1 + a_1)) + p^* X_{k+1}(u(1 + a_4), v + u(1 + a_4))\right], \\ X_N(u, v) &= f(v) = \left(\frac{v}{N+1} - K\right)^+, \ \forall u, \ p^* = \frac{r - a_1}{a_4 - a_1}, \ q^* = 1 - p^*. \end{aligned} \tag{17}$$

$$\begin{aligned} \gamma_{k+1}(u, v) &= \\ &= \frac{X_{k+1}(u(1 + a_4), v + u(1 + a_4)) - X_{k+1}(u(1 + a_1), v + u(1 + a_1))}{u(a_4 - a_1)} \end{aligned} \tag{18}$$

Equality (17) leads to the following (easy-to-check) fact.

Proposition The functions $X_k(u, v)$ are convex.

The convexity of the functions $X_k(u, v)$ can be established by backward induction. Similar formulas for the buyer's problem have the form

$$\begin{aligned} X_k(u, v) &= \\ &= \frac{1}{1+r}\left[q^* X_{k+1}(u(1 + a_2), v + u(1 + a_2)) + p^* X_{k+1}(u(1 + a_3), v + u(1 + a_3))\right], \\ X_N(u, v) &= f_N(v) = \left(\frac{v}{N+1} - K\right)^+, \ \forall u, \ p^* = \frac{r - a_2}{a_3 - a_2}, \ q^* = 1 - p^*. \end{aligned} \tag{19}$$

$$\begin{aligned} \gamma_{k+1}(u, v) &= \\ &= \frac{X_{k+1}(u(1 + a_3), v + u(1 + a_3)) - X_{k+1}(u(1 + a_2), v + u(1 + a_2))}{u(a_3 - a_2)} \end{aligned} \tag{20}$$

4 Statistics

In this section, we consider the estimation problem of the model parameters a_1, a_2, a_3 and a_4. Prior to this, we address parametric methods.

Parametric Methods Further presentation is based on the hypothesis that a sample of the returns $R = \{\rho_i\}_{i=1}^N$ consists of independent random variables with the same probability distribution described by a general density function $p(x) = p_1 p(x/q_1) + p_2 p(x/q_2)$, where q_1 and q_2; p_1 and p_2 are the sets of distribution parameters to be estimated.

Maximum Likelihood The maximum likelihood estimates are the solution of the optimization problem

$$\max_{p_1,p_2,q_1,q_2}\left[\sum_{i=1}^{N} ln(p_1 p(\rho_i/q_1) + p_2 p(\rho_i/q_2))\right] \tag{21}$$

Details can be found, e.g., in [13, Ch. 10]. Maximization is performed in the parameters of the distribution laws and also in the probabilities p_1 and p_2 . The algorithm proposed below is monotonic and converges to the set of stationary points, which includes the solution. For the Gaussian distributions, this algorithm converges to the maximum likelihood estimates.

Algorithm

1. Determine p_1 and p_2; the distribution parameters are q_1 and q_2.

$$\alpha_{i,1} = \frac{p(\rho_i/q_1)p_1}{p(\rho_i/q_1)p_1 + p(\rho_i/q_2)p_2},\ \alpha_{i,2} = \frac{p(\rho_i/q_2)p_2}{p(\rho_i/q_1)p_1 + p(\rho_i/q_2)p_2}$$

2.

$$p_1 = \frac{1}{N}\sum_{i=1}^{N}\alpha_{i,1},\ p_2 = \frac{1}{N}\sum_{i=1}^{N}\alpha_{i,2},$$

$$q_1 = \arg\max_q \sum_{i=1}^{N}\alpha_{i,1} ln(p(\rho_i/q)),\ q_2 = \arg\max_q \sum_{i=1}^{N}\alpha_{i,2} ln(p(\rho_i/q))$$

3. If a stopping criterion is satisfied, then finish calculations; otherwise go to Step 2.

Gaussian Law For the Gaussian laws, the parameters are $q_1 = \langle m_1, \sigma_1^2 \rangle, q_2 = \langle m_2, \sigma_2^2 \rangle$. A successive approximation of the maximum likelihood estimates is calculated by

$$\alpha_{i,1} = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(\rho_i - m_1)^2}{2\sigma_1^2}\right) p_1}{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(\rho_i - m_1)^2}{2\sigma_1^2}\right) p_1 + \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(\rho_i - m_2)^2}{2\sigma_2^2}\right) p_2}$$

$$\alpha_{i,2} = \frac{\frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(\rho_i - m_2)^2}{2\sigma_2^2}\right) p_2}{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(\rho_i - m_1)^2}{2\sigma_1^2}\right) p_1 + \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(\rho_i - m_2)^2}{2\sigma_2^2}\right) p_2}$$

$$p_1 = \frac{1}{N} \sum_{i=1}^{N} \alpha_{i,1}, \;\; p_2 = \frac{1}{N} \sum_{i=1}^{N} \alpha_{i,2}$$

$$m_1 = \frac{1}{\sum\limits_{i=1}^{N} \alpha_{i,1}} \sum_{i=1}^{N} \alpha_{i,1} \rho_i, \;\; m_2 = \frac{1}{\sum\limits_{i=1}^{N} \alpha_{i,2}} \sum_{i=1}^{N} \alpha_{i,2} \rho_i$$

$$\sigma_1^2 = \frac{1}{\sum\limits_{i=1}^{N} \alpha_{i,1}} \sum_{i=1}^{N} \alpha_{i,1} \rho_i^2 - m_1^2, \;\; \sigma_2^2 = \frac{1}{\sum\limits_{i=1}^{N} \alpha_{i,2}} \sum_{i=1}^{N} \alpha_{i,2} \rho_i^2 - m_2^2$$

As it has been noted, for two Gaussian laws the algorithm converges to the maximum likelihood estimates. For choosing the initial values of p_1, p_2 and q_1, q_2, a sample is divided into two subsets by the median: $R_1 = \{\rho \in R, \rho \leq med(R)\}$, $R_2 = R \setminus R_1$. Then the values

$$p_1 = \frac{|R_1|}{|R|}, \;\; p_2 = \frac{|R_2|}{|R|}, (p_1 = p_2 = \frac{1}{2}, \text{if} |R| \text{—even}),$$

$$m_1 = \frac{1}{|R_1|} \sum_{\rho \in R_1} \rho, \;\; m_2 = \frac{1}{|R_2|} \sum_{\rho \in R_2} \rho,$$

$$\sigma_1^2 = \frac{1}{|R_1|} \sum_{\rho \in R_1} \rho^2 - m_1^2, \;\; \sigma_2^2 = \frac{1}{|R_2|} \sum_{\rho \in R_2} \rho^2 - m_2^2$$

are calculated. After stop, the sample R is divided into two classes R_1 and R_2 as follows: an element ρ_i is attributed to R_1 if $p(\rho_i/q_1)p_1 \geq p(\rho_i/q_2)p_2$; otherwise ρ_i is attributed to R_2.

Nonparametric Methods This group includes the methods of k-means and fuzzy k-means as well as the method of hierarchical clustering. We will discuss the method of k-means. This method is based on the optimization problem

$$\min_{m,x} \sum_{i=1}^{N} (\rho_i - m_1)^2 x_i + \sum_{i=1}^{N} (\rho_i - m_2)^2 (1 - x_i), x_i \in \{0, 1\}. \tag{22}$$

The solution of this problem using coordinate descent is the main tool of clustering in the method of k-means. As is well known, being monotonic, the method does not guarantee convergence to the solution of the optimization problem. A significantly simplifying feature of the problem under study is its one-dimensionality: before clustering, one-dimensional data can be arranged in ascending order. The number of clusters is 2, and the following algorithm can be proposed.

Algorithm of the Method of k-Means

1. Set $R_1 = \{\rho_i\}$, $R_2 = \{\rho_1, \rho_2, \ldots, \rho_N\}$.
2. Calculate $m_1 = \frac{1}{|R_1|} \sum\limits_{\rho \in R_1} \rho$, $m_2 = \frac{1}{|R_2|} \sum\limits_{\rho \in R_2} \rho$.
3. Calculate a next division into classes. Let $R_1 = \{\rho_1, \rho_2, \ldots, \rho_k\}$; then the next division is

$$R_1 = \begin{cases} R_1 + \{\rho_{k+1}\}, & d(\rho_{k+1}, m_1) < d(\rho_{k+1}, m_2), \\ R_1, & otherwise \end{cases}, R_2 = R \setminus R_1$$

4. If a stopping criterion is satisfied, then finish calculations; otherwise go to Step 2.

Proposition The algorithm finds the solution of the optimization problem under study.

After clustering. Next, we use the concept of a quantile, which is important in statistics. The α-quantile of a probability distribution function $F(x)$ is the solution of the equation $F(x) = \alpha$, which may not exist if this function is discontinuous. Therefore, the following definition seems more natural: the upper quantile is $x_\alpha^+ = \min\{x : F(x) \geq \alpha\}$, and the lower quantile is $x_\alpha^- = \max\{x : F(x) \leq \alpha\}$. We will employ this concept for calculating the model parameters a_1, a_2, a_3, and a_4. The first cluster $R_1 = \{\rho_1, \rho_2, \ldots, \rho_l\}$ will be used to calculate the parameters a_1 and a_2 while the second cluster $R_2 = \{\rho_{l+1}, \rho_{l+2}, \ldots, \rho_N\}$ to calculate the parameters a_3 and a_4. Instead of the unknown probability distribution function, we use the empirical distribution function $F_\varepsilon^1(x)$ corresponding to the sample R_1, with which we find the probability that the random variable falls into a range: $P(x < \xi \leq y) = F_\varepsilon^1(y) - F_\varepsilon^1(x)$. The problem of calculating the parameters a_1 and a_2 for the sample R_1 is to find $\min(y - x)$ subject to the constraint

$F_\varepsilon^1(y) - F_\varepsilon^1(x) \geq \alpha$. For solving the problem, consider the following family of μ-problems: $max\, x$, $F_\varepsilon^1(x) \leq \mu(1-\alpha)$; $min\, y$, $F_\varepsilon^1(y) \geq \alpha(1-\mu)+\mu$. Here $\mu \in [0, 1]$ is a parameter. In accordance with the definition, the solution of the μ-problem is the interval $(x^-_{\mu(1-\alpha)}, x^+_{(1-\mu)\alpha+\mu}]$. Then, $min\left(x^+_{(1-\mu)\alpha+\mu} - x^-_{\mu(1-\alpha)}\right)$ is calculated for an approximation of the problem, namely, for $\mu \in \{\mu_0, \mu_1, \ldots, \mu_s\}$, where $(\mu_i)_{i=0}^s$ is a partition of the range [0, 1]. After solving the problem, we obtain $a_1 = x^-_{\mu^*(1-\alpha)}$, $a_2 = x^+_{(1-\mu^*)\alpha+\mu^*}$. Similarly, the estimates of a_3 and a_4 are calculated.

Sample Preprocessing As a result of the analysis of real data, especially of the data obtained with a short period, it can be concluded that the sample contains a large number of either identical or slightly different elements. Since the sample is a segment of the time series, it is precisely the neighbor values that often have almost no difference from each other. This fact has a negatively effect on further clustering. For reducing this effect, we will adopt a piecewise constant approximation to describe the time series. The idea of this approximation is as follows. Consider an arbitrary segment of the time series $R_m^n = \{\rho_m, \rho_{m+1}, \ldots, \rho_n\}$, which has to be replaced with a constant segment $\bar{R}_m^n = \{a_m^n, a_m^n, \ldots, a_m^n\}$. The resulting error can be measured in various ways, e.g., using $d_1 = \min\limits_{a_m^n} \sqrt{\frac{1}{n+1-m} \sum\limits_{i=m}^{n} (\rho_i - a_m^n)^2}$ or $d_2 = \min\limits_{a_m^n} \frac{1}{n+1-m} \sum\limits_{i=m}^{n} |\rho_i - a_m^n|$. The problem is to approximate the time series with the smallest number of constant segments, also achieving a the desired accuracy. This problem belongs to the group of problems in which a solution can be found by the exhaustive search of all alternatives. The algorithm below finds an acceptable solution, but possibly inaccurate. Let ε be a given accuracy.

Algorithm
1. Set $m = n = 1$;
2. Calculate d_1 or d_2;
3. If $d_1 \leq \varepsilon (d_2 \leq \varepsilon)$, then $n = n + 1$;
   otherwise m=n
4. If $n > N$, then STOP;
   otherwise move to Step 2.

In the algorithm the integer N is the number of time series elements.

5 Example

For calculations, we adopted Apple stock price data for 2 weeks with a period of 17 s. The relative returns were calculated. In total, about 10,000 stock price records were used. The data were preprocessed with an accuracy of ($\varepsilon = 10^{-5}$) and divided into two clusters, in accordance with the parametric and nonparametric methods described above, with the calculation of confidence intervals with a confidence level of $\alpha = 0.99$. As the result of records processing, the following clusters were obtained: $[-0.00672, -0.00501]$ and $[0.00652, 0.00702]$. The preference was given to the method of k-means, since the hypothesis of the Gaussian distribution turned out to be inconsistent. For calculations in the Cox-Ross-Rubinstein model, the medians in R_1 and R_2 were set equal to $a = -0.00542$ and $b = 0.00662$, respectively. The interest rate was set equal to $r = 0.00075$. The interest rate of the risk-free asset is chosen arbitrarily and is not associated with any asset on the market.

The calculations were performed on a binary tree. This structure was chosen to speed up the calculations using memory, since atomic computing can be easily parallelized. The optimal portfolios in each atom were calculated by formulas (11), (13) for the sellers's strategy, and (14), (15) for the buyer's strategy. 50% of the data were used for calculating the requisite ranges, and the other 50% for testing of the method.

The testing results are combined in Tables 1 and 2. More specifically, Table 1 presents the results of calculations for the European call option with a strike equal to the initial value of the stock price. Table 2 presents the corresponding results for an Asian option.

The quality of the strategy was determined by the sum $S = \sum_{i=1}^{N} (X_i^r - X_i^t)$, N-final time, the example $N = 30$. The capital X_i^t is evaluated at formulas (11), (13) for the seller's strategy, and at formulas (14), (15) for the buyer's strategy. The capital $X_i^r = \gamma_i S_i + \beta_i B_i$ is the real capital. The element of the strategy β is determined by the self-financing of the portfolio. The experiment was considered successful for the seller, if $S \geq 0$. The experiment was considered successful for the buyer, if $S \leq 0$. The number of successful experiments for the seller and the buyer is placed in the second row of the table.

For the Cox-Ross-Rubinstein model, the strategies of the seller and the buyer coincide, so a successful experiment for the seller is automatically unsuccessful for the buyer, except for the case $S = 0$. In the example, such a case was not observed. The results are shown in the third line.

The second table for the Asian option is arranged according to the same principle. The strategies and capitals of the Asian option strategies were calculated using the formulas (17)–(20).

The results presented in the first table allow us to assert that in this example the proposed method of calculating the portfolio for both the seller and the buyer has a

Table 1 European call option

Model	Seller	Buyer	Total number of experiments
Robust	1054	906	1058
Cox-Ross-Rubinstein	924	134	1058

Table 2 Asian option

Model	Seller	Buyer	Total number of experiments
Robust	1055	867	1058
Cox-Ross-Rubinstein	843	215	1058

significant advantage over the Cox-Ross-Rubinstein portfolio for the European Call option. A similar conclusion can be drawn from the results presented in the second table for the Asian Call option.

The large number of experiments carried out on the control sample gives hope that the results obtained will be extended to other examples. An important indicator for the applicability of this method is good separability of the sample into two clusters.

6 Conclusions and Some References

Summarize. The main result is associated with replacing the point estimates of the parameters of the Cox-Ross-Rubinstein model with interval estimates and performing calculations for the intervals as model parameters. To calculate the optimal strategies for European and Asian options, an essential element was the convexity of the payoff functions. Convexity and Bellman's equations made it possible to obtain the necessary formulas.

Dynamic programming is often used in discrete-time models, and the authors do not claim to be a discovery in this area. Among the related works on a discrete time, one should mention the works [14–17]. The most complete Bellman equations, as the main means of obtaining the result, are presented in the work [16]. The convexity of the payoff function is used in the work [17].

Successful application of this or any other model is impossible without sound statistics of model parameters. In our case, this is the coverage of the sample by two non-intersecting intervals. The main tool used to calculate the intervals is unsupervised learning, combined with auxiliary tools that allow you to enhance the result through preprocessing and quantile calculation of the boundaries of the intervals after training.

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On Reflexivity and Other Geometric Properties of Morrey Spaces

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Abstract We propose necessary and sufficient conditions for the local Morrey space to have one of the following properties: the absolutely continuous norm, the Fatou property, the reflexivity. We propose conditions for the global Morrey space $GM^{\tau}_{l,X}$ to have the Fatou property. We give an example of global Morrey spaces $GM^{\tau}_{l,X}$, which do have not an absolutely continuous norm, and spaces l and X have an absolutely continuous norm.

Keywords Local Morrey spaces · Approximation local Morrey spaces · Global Morrey spaces · Banach ideal spaces · Absolutely continuous norm · Fatou property · Reflexivity

AMS Mathematics Subject Classification: 46E30, 46B42, 46B70

1 Introduction

It is well known that to study specific properties of operators in analysis, for example, the Hilbert operator, the Hardy—Littlewood maximal function operator, and so on, it is very important to choose correctly spaces in which one can describe various properties of these operators. Spaces M_{λ,L^p}, introduced by Morrey [1], and their generalizations [2, 3] play an important role in harmonic analysis and in the study of partial differential equations. For the use of spaces in harmonic analysis, properties of these spaces play an important role. In this paper, we propose necessary and sufficient conditions for the local Morrey space with one of following properties: the absolutely continuous norm, the Fatou property, the reflexivity. Parameters of Morrey spaces $M^{\tau}_{l,X}$ are actually two ideal spaces: the function space X and the sequence space l. It is in terms of these spaces that the criteria of

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absolutely continuous norms, a presence of the Fatou property, and the reflexivity for local Morrey spaces will be given. Note that an important role in describing the criterion of the reflexivity is played by the theorem on the representation of the dual space to the local Morrey space, obtained in the paper of the author [4]. We propose the criterion for the presence of the Fatou property of global Morrey spaces $GM^{\tau}_{l,X}$. We demonstrate an example of global Morrey space $GM^{\tau}_{l,X}$ showing that even if both ideal spaces l, X have an absolutely continuous norm, the space $GM^{\tau}_{l,X}$ does not have this property. This example shows that the absolute continuity of the norm $GM^{\tau}_{l,X}$ depends on some other properties of X and l.

Note that a presence of the Fatou property for classical Morrey spaces $M_{\lambda,L^p}(\mathbb{R}^n)$, ($0 < \lambda < \frac{n}{p}$), can be extracted from results of [5], and an absence of the absolutely continuous norm for of $M_{\lambda,L^p}(\mathbb{R}^n)$ can be obtained from Theorem 5 of [6]. The theorems given in this paper contain both of these results.

We note that results thus obtained are also of interest for classical Morrey spaces.

2 Preliminaries

Let μ is the Lebesgue measure in $\mathbb{R}^n$, let $S(\mu, \Omega)$ is a space of all measurable functions $x : \Omega \to \mathbb{R}$ and let $\chi(D)$ stand for characteristic function of D. Along with Lebesgue spaces L^p, $p \in [1, \infty]$ ideal and symmetric spaces X are often used in harmonic analysis. Recall their definitions (see, for example [7–9]).

Banach space X of measurable functions on Ω is said to be an ideal if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $|y(t)| \leq |x(t)|$ for almost all $t \in \Omega$ that $y \in X$ and $\|y|X\| \leq \|x|X\|$ (the symbol $\|x|X\|$ denotes the norm of the element x in the space X).

Let $v \in S(\mu)$, $v > 0$ a. e. (v is a weight). We denote by the symbol X_v a new ideal space in which the norm is given by the equation $\|x|X_v\| = \|x \cdot v|X\|$. When $X = L^p$, our definition of weighted space differs somewhat from the standard one: when $X = L^p$ the weight is usually included in the measure.

For every ideal space X, the dual ideal space X' is well defined: it consists of functionals, continuous on X and representable in the integral form, whose norm is defined by the equation

$$\|g|X'\| = \sup\{\int_{\Omega} g(t)x(t)dt : \|x|X\| \leq 1\}.$$

It is easy to verify the equality $(X_v)' = (X')_{1/v}$.

When $x : \Omega \to \mathbb{R}$ we denote by $\lambda(f, \gamma)$, ($\gamma > 0$) the distribution function of x, namely, $\lambda(x, \gamma) = \mu\{t \in \Omega : |x(t)| \leq \gamma\}$, and by x^* the rearrangement of x in nonincreasing order. An ideal space X is said to be symmetric if it follows from the condition $x \in X$, the measurability of y and the validity of the inequality $\lambda(y, \gamma) \leq \lambda(x, \gamma)$ for all $\gamma \in \mathbb{R}_+$ that $y \in X$ and $\|y|X\| \leq \|x|X\|$. Examples of symmetric spaces are Orlicz, Lorentz, and Marcinkiewicz spaces. Details can be found in [8, 9].

Along with function spaces we need ideal spaces of sequences. Let $e^i = \{\ldots, 0, 1, 0, \ldots\}$, ($i \in \mathbb{Z}$, the unit stands in the i-th place) be the standard basis in the space of two-side sequences. We denote by the symbol l an ideal space of sequences $x = \sum_{i=-\infty}^{\infty} x_i e^i$ ($x_i \in \mathbb{R}$) with the norm $\|x|l\|$.

Definitions of Lebesgue, Orlicz, Lorenz, and Marcinkiewicz symmetric sequences spaces and the weighted sequence l_ν similar to definitions of the corresponding function spaces.

All properties listed above for function spaces are preserved for sequence spaces. For details concerning the theory of sequence spaces, see [10].

Let's go to definitions of basic properties that we will explore in Morrey spaces.

Definition 2.1 (See, for example, [7, 10]). An ideal space of functions $X \subset S(\mu, \Omega)$ has an absolutely continuous norm (X has the A—property) if for every $x \in X$ following two conditions are satisfied:

$$\lim_{\delta \to 0} \sup_{\{D:\mu(D) \leq \delta\}} \|x\chi(D)|X\| = 0, \quad \lim_{R \to \infty} \|x\chi(\Omega \setminus B(R, 0)|X\| = 0.$$

A discrete ideal space l has an absolutely continuous norm if for each $x \in l$ the following two conditions are satisfied:

$$\lim_{k \to \infty} \|\Sigma_{-\infty}^{-k} e^i x_i|l\| = 0, \quad \lim_{k \to \infty} \|\Sigma_k^{\infty} e^i x_i|l\| = 0.$$

It is well known that spaces L_Ω^p (l_ω^p) for $p \in [1, \infty)$ have the A—property, and L_ω^∞ (l_ω^∞) have not the A—property.

Definition 2.2 Say (see, for example, [7–9]) that an ideal space $X \subset S(\mu, \Omega)$ has the Fatou property if from $0 \leq x_n \uparrow x$; $x_n \in X$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

The main property of spaces with the Fatou property is that the equality $X'' = X$ holds and the norms in these spaces coincide, i.e.

$$\sup\{\int_\Omega x(t)f(t)dt : \|f|X'\| \leq 1\} = \|x|X\|.$$

It is well known that Lebesgue spaces L_ω^p, (l_ω^p) for $p \in [1, \infty]$ have the Fatou property, and the space c^0 have not the Fatou property. Other examples of ideal spaces with the Fatou property are of Orlicz, Lorentz, and Marcinkiewicz spaces. Details can be found in [9].

The classical Morrey space M_{λ,L^p}, $(\lambda \in \mathbb{R})$ (see [1]), consists of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for which the following norm is finite:

$$\|f|M_{\lambda,L_p}\| = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda}\|f\chi(B(x,r))|L^p\|.$$

We note that if $\lambda = 0$, then $M_{\lambda,L_p} = L_p$, if $\lambda = \frac{n}{p}$, then $M_{\lambda,L_p} = L_\infty$, if $\lambda < 0$ or $\lambda > \frac{n}{p}$, then M_{λ,L_p} consists only of functions equivalent to zero.

If we now replace the Lebesgue space L^p in the definition of the classical Morrey space by an ideal space X, we obtain the Morrey space $M_{\lambda,X}$ constructed from the ideal space X in which the norm is defined by the equality

$$\|f|M_{\lambda,X}\| = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda}\|f\chi(B(x,r))|X\|.$$

Next step in the extension of Morrey spaces consists of the replacement of the outer sup-norm by the norm in the ideal space L and the replacement of the balls $B(0,r)$ by homothetic sets $U(0,r) \subset \mathbb{R}^n$. Below, we always assume that $0 \in U(0,1)$ and $\mu(U(0,1))) \in (0,\infty)$. Moreover, we assume that $U(0,1)$ is star-shaped with respect to the point 0, that is, if $t \in U(0,1)$, then $\gamma t \in U(0,1)$ for all $\gamma \in (0,1)$. In general, the star-shapedness assumption is not necessary, but sometimes is useful. Next, make the parameter r discrete.

We denote by Υ the set of non-negative numerical sequences $\tau = \{\tau_i\}$ each of which satisfies the conditions

$$\forall i: \quad \tau_i < \tau_{i+1}, \quad \bigcup_i (\tau_i, \tau_{i+1}] = \mathbb{R}_+.$$

When $\tau_{i+1} = \infty$, we assume that $(\tau_i,\infty] = (\tau_i,\infty)$. For every sequence $\tau = \{\tau_i\}$ we construct a family of sets $U(0,\tau_i)$ and a family of disjoint annuli $D_i = U(0,\tau_i)\backslash U(0,\tau_{i-1})$.

Definition 2.3 ([4]) Let an ideal space X on $\mathbb{R}^n$, an ideal space l of two-sided sequences with the standard basis $\{e^i\}$ and a sequence $\tau \in \Upsilon$ be given.

By the local Morrey space $M^\tau_{l,X}$ we mean the set of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for each of which the following norm is finite:

$$\|f|M^\tau_{l,X}\| = \|\sum_{i=-\infty}^{\infty} e^i\|f\chi(U(0,\tau_i))|X\||l\|.$$

By the approximation local Morrey space $\overline{M^\tau_{l,X}}$ we mean the set of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for each of which the following norm is finite:

$$\|f|\overline{M^\tau_{l,X}}\| = \|\sum_{i=-\infty}^{\infty} e^i\|f\chi(D_i))|X\||l\|.$$

By the global Morrey space $GM^{\tau}_{l,X}$ we mean the set of all functions $f \in L^{1,loc}(\mathbb{R}^n)$ for each of which the following norm is finite:

$$\|x|GM^{\tau}_{l,X}\| = \sup_{t\in\mathbb{R}^n} \| \sum_i e^i \|x(t+.)\chi(U(0,r_i))|X\| |l\|.$$

Discussion of interconnections of spaces $M^{\tau}_{l,X}$, $\overline{M^{\tau}_{l,X}}$ and their examples are given in [4].

We only note that the embedding $M^{\tau}_{l,X} \subseteq \overline{M^{\tau}_{l,X}}$ is obvious and the reverse embedding, which plays a key role in the theory of discrete Morrey spaces, is given in the following theorem.

Theorem B ([4]) *Let an ideal space X on $\mathbb{R}^n$, an ideal space L on $\mathbb{R}_+$ and a set $U(0,1) \subset \mathbb{R}^n$ for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$ and a sequence $\tau \in \Upsilon$ be given. Let spaces $M^{\tau}_{l,X}$ and $\overline{M^{\tau}_{l,X}}$ be constructed from spaces X and l, the set $U(0,1)$ and the sequence $\tau \in \Upsilon$. We introduce the operator $T : l \to l$ by the equality*

$$T(\sum_{i=-\infty}^{\infty} e^i x_i) = \sum_{k=-\infty}^{\infty} e^k y_k, \text{ where } \quad y_k = \sum_{i=-\infty}^{k} x_i. \tag{2.1}$$

When $\|T|l \to l\| = c_0 < \infty$, spaces $M^{\tau}_{l,X}$ and $\overline{M^{\tau}_{l,X}}$ have the same set of elements and the following inequalities hold:

$$\|f|\overline{M^{\tau}_{l,X}}\| \le \|f|M^{\tau}_{l,X}\| \le c_0\|f|\overline{M^{\tau}_{l,X}}\|.$$

Note that coincidence conditions do not contain restrictions on the space X and the sequence $\tau \in \Upsilon$. There is only a restriction on the sequence space l.

Everywhere below c, possibly with indices, we will denote constants whose exact value are not important.

Discrete spaces are more convenient to consider at least for following reasons. Firstly, all classical Morrey spaces can be realized as discrete Morrey spaces (see the example below), and secondly, one does not need to think about the measurability of the function $\|x(t+.)\chi(B(0,r))|X\|$.

Note that all discrete Morrey spaces are ideal.

If $l = l^p_\nu$, then it is useful to have in mind that for the norm of the operator T in space l^p_ν for $p \in (1,\infty)$ the relation is true [11]:

$$\|T|l^p_\nu \to l^p_\nu\| \approx \begin{cases} \sup_k (\sum_{-\infty}^{k} (\frac{1}{\nu_j})^{p'})^{1/p'} \cdot (\sum_k^{\infty} \nu_j{}^p)^{1/p}, & \text{for } \; p \in (1,\infty), \; \frac{1}{p'} + \frac{1}{p} = 1; \\ \sup_k \frac{1}{\nu_k} \cdot (\sum_k^{\infty} \nu_j), & \text{for } \; p = 1; \\ \sup_k \nu_k (\sum_{-\infty}^{k} \frac{1}{\nu_j}), & \text{for } \; p = \infty. \end{cases} \tag{2.2}$$

The following example shows that most recently investigated Morrey spaces can be implemented as discrete Morrey spaces.

Example 2.1 ([4].) Let $U(0, 1)$ be a star-shaped set of a positive measure, $\lambda > 0$, $p \in [1, \infty]$, and the ideal space X the space $M_{\lambda,p;X}$, the norm in which is given by the equality

$$\|x|M_{\lambda,p;X}\| = \begin{cases} \int_0^\infty (r^{-\lambda}\|x\chi(U(0,r))|X\|)^p \frac{dr}{r}\Big)^{1/p}, & \text{for } p \in [1,\infty); \\ \sup_r\{r^{-\lambda}\|x\chi(U(0,r))|X\|\}, & \text{for } p = \infty \end{cases}$$

be given.

If $p \in [1, \infty)$, then for each function $x \in M_{\lambda,p;X}$ inequalities hold:

$$2^{-\lambda}(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x\chi(U(0,2^i))|X\|)^p)^{1/p} \leq \|x|M_{\lambda,p;X}\| \leq$$

$$2^{\lambda}\cdot(ln2)^{1/p}(\sum_i (2^{-i\lambda}\|x\chi(U(0,2^i))|X\|)^p)^{1/p}.$$

Thus, for $p \in [1, \infty)$ using the equality

$$\|x|M_{\lambda,p;X}\|_b = (\sum_i (2^{-\lambda i}\|x\chi(U(0,2^i))|X\|)^p)^{1/p}$$

on the space $M_{\lambda,p;X}$ we can introduce an equivalent norm.

If $p = \infty$, then for each $x \in M_{\lambda,\infty;X}$ inequalities hold:

$$2^{-\lambda}\sup_i 2^{-i\lambda}\|x\chi(U(0,2^i))|X\| \leq \|x|M_{\lambda,\infty;X}\| \leq 2^{\lambda}\sup_i 2^{-i\lambda}\|x\chi(U(0,2^i))|X\|.$$

So using equality

$$\|x|M_{\lambda,\infty;X}\|_b = \sup_i 2^{-i\lambda}\|x\chi(U(0,2^i))|X\|$$

on the space $M_{\lambda,\infty;X}$ an equivalent norm can be introduced.

Put $\tau_i = 2^i$, $(i \in \mathbb{Z})$, according to a sequence of points $\{\tau_i\}_{-\infty}^{\infty}$ organize the partition τ for R_+ and the equality $\omega_\lambda(i) = 2^{-\lambda i}$, $(i \in \mathbb{Z})$ we define a weight sequence. Then we get that for all $p \in [1, \infty]$ up to equivalent norms are valid equality:

$$M^{\tau}_{l^p_{\omega_\lambda},X} = M_{\lambda,p;X}.$$

From the explicit form of the weight sequence ω_λ and (2.2), it follows that the operator T defined in Theorem B1 equality (2.1), is bounded from $l^p_{\omega_\lambda}$ in $l^p_{\omega_\lambda}$. It follows that up to equivalent norms, the equality holds:

$$M^\tau_{l^p_{\omega_\lambda},X} = \overline{M^\tau_{l^p_{\omega_\lambda},X}}.$$

3 Main Results

3.1 Geometric Properties of Local Morrey Spaces

We begin by characterizing local approximation Morrey spaces with the A—property.

Theorem 3.1 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

The space $\overline{M^\tau_{l,X}}$ has an absolutely continuous norm, if and only if the ideal space of functions X and the ideal space of sequences l have absolutely continuous norm.

Proof Let $x \in \overline{M^\tau_{l,X}}$ be given, for which

$$\|x|\overline{M^\tau_{l,X}}\| = \|\Sigma^\infty_{-\infty}\|x\chi(D_i)|X\|e^i|l\| = 1.$$

Let $V \subset \mathbb{R}^n$. Fix $k \in \mathbb{N}$ and define

$$s_1 = \|\Sigma^{-k}_{-\infty}\|x\chi(D_i)\chi(V)|X\|e^i|l\|, \quad s_2 = \|\Sigma^k_{-k}\|x\chi(D_i)\chi(V)|X\|e^i|l\|,$$

$$s_3 = \|\Sigma^\infty_k\|x\chi(D_i)\chi(V)|X\|e^i|l\|.$$

Then following inequalities are true

$$max\{s_1, s_2, s_3\} \le \|x\chi(V)|\overline{M^\tau_{l,X}}\| \le s_1 + s_2 + s_3. \tag{3.1}$$

Let ideal spaces X and l have absolutely continuous norm. Since the space l has absolutely continuous norm, then $\lim_{k\to\infty} s_1 + s_3 = 0$. Since the space X has absolutely continuous norm, for each fixed $k \in \mathbb{N}$ $\lim_{\mu(V)\to 0} s_2 = 0$. This implies the sufficiency of the conditions of the lemma.

Let the ideal space $\overline{M^\tau_{l,X}}$ has absolutely continuous norm. The proof that both spaces X and l have absolutely continuous norm is analogous to the proof of sufficiency, one only needs to use the left inequality in (3.1). □

From Theorem **B** and Theorem 3.1 we obtain the following theorem.

Theorem 3.2 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the operator $T : l \to l$, defined by equality (2.1), is bounded.*

The space $M^{\tau}_{l,X}$ has an absolutely continuous norm, if and only if the ideal space of functions X and the ideal space of sequences l have absolutely continuous norm.

To characterize local approximation Morrey spaces to have the Fatou property, we need one definition.

Definition 3.1 Let an ideal space X on $\mathbb{R}^n$ and a sequence $\tau \in \Upsilon$ be given.

An ideal space $X \subset S(\mu, \Omega)$ has the τ—Fatou property if for every $i \in \mathbb{Z}$ from $0 \le x_n \uparrow x$; $x_n \in X$, $supp\ x_n \in D_i$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

For each $D \subseteq \mathbb{R}^n$ we denote by $\chi(D)X$ the ideal space consisting of restrictions of functions $x \in X$ to the set D with norm $\|\chi(D)x|\chi(D)X\| = \|\chi(D)x|X\|$. Then Definition 3.1 actually means that for any $i \in \mathbb{Z}$ every ideal space $\chi(D_i)X$ from of the set of ideal spaces $\{\chi(D_i)X\}_{-\infty}^{\infty}$ have the Fatou property. Since from $0 \le x_n \uparrow x$, $x_n \in X$, it follows that $\chi(D_i)x_n \uparrow \chi(D_i)x$ for any $i \in \mathbb{Z}$, then every ideal space X with the Fatou property also has the τ—Fatou property.

The following example shows that the opposite is not true.

Example 3.1 We denote by $L_0^{\infty}[0,1]$ the subspace of $L^{\infty}[0,1]$, consisting of functions for each of which the condition is satisfied

$$\lim_{\tau\to 0} ess \sup_{t\in[0,\tau]} |x(t)| = 0.$$

Let the sequence be given $0 \ldots < \tau_{i-1} < \tau_i < \ldots$, by which the partition τ is constructed. Then the ideal space $L_0^{\infty}[0,1]$ has the τ—Fatou property. On the other hand, an example of the sequence $x_n = \chi(n^{-1}, 1)$ shows that $L_0^{\infty}[0,1]$ have not the Fatou property.

We will need another definition.

Definition 3.2 ([12]) Let an ideal space X in $S(\mu)$ and a set $D \subseteq \mathbb{R}^n$ be given. A function $e_D(.) \in X$, which is positive almost everywhere on D and outside it is equal to zero, is called the unit on D in X.

It is well known [12] that for any D units exist.

The following lemma will be necessary for us to verify the Fatou property.

Lemma 3.1 *Fix a measurable set D. Let the sequence $0 \le a_1 < a_2 < \ldots a_n < a_{n+1} < \ldots$, $lim_{n\to\infty} a_n = a_0 < \infty$ be given.*

Then there exists a sequence of $\{x_n\}$ elements of X such that: $0 \le x_1 \le x_2 \le \ldots \le x_n \le x_{n+1} \le \ldots$, for each n $supp\ x_n \subseteq D$ and the equality follows $\|x_n|X\| = a_n$.

Proof Fix the unit $e_D(.)$ on D. First, we choose a non-negative element $x_1 \in X$ with support in D for which $\|x_1|X\| = a_1$ and define the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by the equality $\varphi(x_1, \alpha) = \|x_1 + \alpha e_D(.)|X\|$. This function does not decrease with α, $\varphi(x_1, 0) = \|x_1|X\|$ and $\lim_{\alpha\to\infty} \varphi(x_1, \alpha) = \infty$. From the triangle inequality, we obtain the inequality $\varphi(\alpha+\delta)-\varphi(\alpha) = \|x_1+(\alpha+\delta)e_D(.)|X\|-\|x_1+\alpha e_D(.)|X\| \leq \delta\|e_D(.)|X\|$, which implies Lipschitz property of the function $\varphi(x_1, \alpha)$ with respect to α. We determine the number α_1 from the equality $\varphi(x_1, \alpha_1) = a_2$ and put $x_2 = x_1 + \alpha_1 e_D(.)$. Next, we define the function $\varphi_1(x_2, \alpha) = \|x_2 + \alpha e_D(.)|X\|$. This function has the same properties as the function $\varphi(x_1, \alpha) = \|x_1 + \alpha e_D(.)|X\|$. We determine the number α_2 from the equality $\varphi(x_2, \alpha_2) = a_3$ and put $x_3 = x_2 + \alpha_2 e_D(.)$, etc.

The desired sequence is built. □

Theorem 3.3 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0, 1) \subset \mathbb{R}^n$, for which $0 \in U(0, 1)$ and $\mu(U(0, 1)) \in (0, \infty)$, and a sequence $\tau \in \Upsilon$ be given.*

The space $\overline{M^{\tau}_{l,X}}$ has the Fatou property, if and only if

$$\text{the ideal sequence space } l \text{ have the Fatou property,} \tag{3.2}$$

$$\text{the ideal space } X \text{ have the } \tau-\text{Fatou property.} \tag{3.3}$$

Proof Sufficiency. Let $x_n \in \overline{M^{\tau}_{l,X}}$, $\|x_n|\overline{M^{\tau}_{l,X}}\| \leq 1$ and $x_n \uparrow x$. Then for any $i \in \mathbb{Z}$ the relation $x_n\chi(D_i) \uparrow x\chi(D_i)$ holds. Since X has the τ—Fatou property, then $\|x_n\chi(D_i)|X\| \uparrow \|x\chi(D_i)|X\|$ are fulfilled.

Therefore, for a numerical sequence, the condition holds

$$\Sigma^{\infty}_{-\infty}\|x_n\chi(D_i)|X\|e^i \uparrow \Sigma^{\infty}_{-\infty}\|x\chi(D_i)|X\|e^i.$$

Since l has the Fatou property, then

$$\lim_{n\to\infty} \|\Sigma^{\infty}_{-\infty}\|x_n\chi(D_i)|X\|e^i|l\| = \|\Sigma^{\infty}_{-\infty}\|x\chi(D_i)|X\|e^i|l\|,$$

This implies that conditions (3.2)–(3.3) are sufficient.

Necessity.

We show that X has the τ—Fatou property. Let be $\underline{x_n\chi(D_i)} \uparrow x\chi(D_i)$, $a_n = \|x_n\chi(D_i)|X\|$, $a = \|x\chi(D_i)|X\|$. Then $x_n\chi(D_i) \in \overline{M^{\tau}_{l,X}}$. Since the space $\overline{M^{\tau}_{l,X}}$ has the Fatou property, relations $\|x_n\chi(D_i)|\overline{M^{\tau}_{l,X}}\| = \|x_n\chi(D_i)|X\|\||e^i|l\| \uparrow \|x\chi(D_i)|X\|\||e^i|l\|$ are fulfilled. This means that the space X has τ—Fatou property.

We show that l has the Fatou property. Let be $b_n = \sum^{\infty}_{-\infty} b_{i,n}e^i \uparrow \overline{b} = \sum^{\infty}_{-\infty} \overline{b_i}e^i$. Then for each $i \in \mathbb{Z}$ for a numerical sequence conditions $b_{i,n} \uparrow \overline{b_i}$ are satisfied. Using Lemma 3.1, for each $i \in \mathbb{Z}$ construct a sequence of nonnegative elements $\{x_{n,i}\}^{\infty}_{-\infty}$ such that: *supp* $x_{n,i} \subseteq D_i$, $x_{n,i} \uparrow$, $\|x_{n,i}|X\| = b_{i,n}$. We defined elements y_n by the equality

$$y_n = \sum_{-\infty}^{\infty} x_{n,i}.$$

Then relations are fulfilled:

$$\|y_n|\overline{M^{\tau}_{l,X}}\| = \|\sum_{-\infty}^{\infty} \|x_{n,i}|X\|e^i|l\| = \|\sum_{-\infty}^{\infty} b_{i,n}e^i|l\|; \quad y_n \uparrow \overline{y}. \tag{3.4}$$

Since the space $\overline{M^{\tau}_{l,X}}$ has the Fatou property, relations $\overline{y} \in \overline{M^{\tau}_{l,X}}$ and $\lim_{n\to\infty} \|y_n|\overline{M^{\tau}_{l,X}}\| = \|\overline{y}|\overline{M^{\tau}_{l,X}}\|$ are fulfilled.

Put $\overline{b_i} = \lim_{n\to\infty} \|y_n\chi(D_i)|X\|$ (these limits exist) and define the vector $\overline{b}$ by the equality

$$\overline{b} = \sum_{-\infty}^{\infty} \overline{b_i}.$$

Then, by construction, the relation $b_n = \sum_{-\infty}^{\infty} b_{i,n}e^i \uparrow \overline{b}$ holds. From (3.4) we get that

$$\|b_n|l\| = \|\sum_{-\infty}^{\infty} b_{i,n}e^i|l\| = \|y_n|\overline{M^{\tau}_{l,X}}\| \uparrow \|\overline{y}|\overline{M^{\tau}_{l,X}}\| = \|\sum_{-\infty}^{\infty} \overline{b_i}e^i|l\| = \|\overline{b}|l\|.$$

□

Remark 3.1 Often, along with spaces have the Fatou property, they consider a class of ideal spaces in which an unit ball is closed with respect to convergence in measure. The closure property of an unit ball with respect to convergence in measure is often called the BC—property.

The analogue of Definition 3.1 for the BC—property looks like this.

An ideal space $X \in S(\mu, \Omega)$ has the τBC—property if for every $i \in \mathbb{Z}$ from $0 \le x_n \uparrow x$; $x_n \in X$, *supp* $x_n \in D_i$ and $\sup_n \|x_n|X\| < \infty$ it follows that $x \in X$ and $\|x|X\| = \sup_n \|x_n|X\|$.

By the scheme of the proof of Theorems 3.3, we can prove the following theorem.

Theorem 3.4 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

The space $\overline{M^{\tau}_{l,X}}$ has the BC—property, if and only if

the ideal sequence space l have the BC−property,

the ideal space X have the τ BC−property.

Now we are ready to characterize reflexive local approximation Morrey spaces.

Theorem 3.5 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

The space $\overline{M^{\tau}_{l,X}}$ will be reflexive if and only if

$$\text{the ideal sequence space } l \text{ is reflexive,} \tag{3.5}$$

$$\text{the ideal space } X \text{ has the} A-\text{property and the } \tau-\text{Fatou property,} \tag{3.6}$$

$$\text{the dual space } X' \text{ has the } A-\text{property.} \tag{3.7}$$

Proof The following criterion of reflexivity for the ideal space X is well known (see, for example, [12]): X is reflexive if and only if both spaces X and X' have the absolutely continuous norm and X has the Fatou property. We will use this criterion.

According to Theorem 3.1, the Morrey space $\overline{M^{\tau}_{l,X}}$ has the absolutely continuous norm if and only if both spaces l and X have the absolutely continuous norm.

By Theorem 3.3, the Morrey space $\overline{M^{\tau}_{l,X}}$ has the Fatou property if and only if spaces l has the Fatou property and X has the τ—Fatou property.

According to Theorem 2 [4] the dual space to $\overline{M^{\tau}_{l,X}}$ coincides with the space $\overline{M^{\tau}_{l',X'}}$.

According to Theorem 3.1 the Morrey space $\overline{M^{\tau}_{l',X'}}$ has the absolutely continuous norm if and only if both spaces l' and X' have the absolutely continuous norm.

Using the criterion of reflexivity of an ideal space again, we'll find that the reflexivity of the space $\overline{M^{\tau}_{l,X}}$ is equivalent to the fulfillment of conditions (3.5)–(3.7) □

Corollary 3.1 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

If spaces X and l are reflexive, then the space $\overline{M^{\tau}_{l,X}}$ is reflexive too.

Proof If the space X is reflexive, then, as indicated above, this is equivalent to following three conditions being satisfied: both ideal spaces X and X' have the absolutely continuous norm, the ideal space X has the Fatou property. This and the reflexivity of l imply that conditions (3.5)–(3.7) are satisfied and, therefore, the space $\overline{M^{\tau}_{l,X}}$ is reflexive too. □

From Theorem **B** and Theorem 3.5 we get the following corollary.

Corollary 3.2 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given. Let the operator $T : l \to l$, defined by equality (2.1), be bounded.*

The local Morrey space $M^{\tau}_{l,X}$ is reflexive if and only if conditions (3.5)–(3.7) are satisfied.

In particular, if spaces X and l are reflexive, then the space $M^{\tau}_{l,X}$ is reflexive too.

3.2 Geometric Properties of Global Morrey Spaces

Now we turn to a discussion of geometric properties of global Morrey spaces $GM^{\tau}_{l,X}$.

Practically verbatim repeating the proof of sufficiency in Theorem 3.3, we can prove the following theorem.

Theorem 3.6 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

If both ideal spaces l and X have the Fatou property, then the space $GM^{\tau}_{l,X}$ has the Fatou property too.

Theorem 3.6 has a complete analogue for the BC—property.

Theorem 3.7 *Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.*

If both ideal spaces l and X have the BC—property, then the space $GM^{\tau}_{l,X}$ has the BC—property too.

We turn to the study of the A—property for global Morrey spaces.

We will start with the construction of the example. This example will be important when investigating the A—property for global Morrey spaces $GM^{\tau}_{l,X}$.

Let a weight sequence $\{\omega(.)\}_{-\infty}^{\infty}$ be given. Denote by c^0_{ω} sequence space $\{a_i\}_{-\infty}^{\infty}$, each of which satisfies conditions:

$$\lim_{i\to-\infty} a_i\omega(i) = 0, \quad \lim_{i\to\infty} a_i\omega(i) = 0,$$

and a norm of $\{a_i\}_{-\infty}^{\infty}$ is given by the equality:

$$\|\sum_{-\infty}^{\infty} a_i e^i | c^0_{\omega}\| = \max_i |a_i|\omega(i).$$

It is easy to see that the space c^0_{ω} have the absolutely continuous norm.

Now we show that for any symmetric space X the Morrey space $GM^{\tau}_{c^0_{\omega},X}$ under minimal conditions on the weight sequence have not the absolutely continuous norm.

The main restriction on the weight sequence will follow from following definitions.

Definition 3.3 Let an ideal space X on $\mathbb{R}^n$, an ideal space of sequences l, a set $U(0,1) \subset \mathbb{R}^n$, for which $0 \in U(0,1)$ and $\mu(U(0,1)) \in (0,\infty)$, and a sequence $\tau \in \Upsilon$ be given.

We say that the weight $\{\omega(.)\}$ belongs to the class $B(l, X)$, ($\{\omega(.)\} \in B(l, X)$) if for each $i \in \mathbb{Z}$ the equality holds

$$\|\chi(U(0,\tau_i))|GM^{\tau}_{l_\omega,X}\| = \omega(i)\|\chi(U(0,\tau_i))|X\|. \tag{3.8}$$

If $l = c^0$, then the condition (3.8) means that relations are fulfilled

$$\omega(i-1)\|\chi(U(0,\tau_{i-1})|X\| \le \omega(i)\|\chi(U(0,\tau_i))|X\|, \quad (i \in \mathbb{Z});$$

$$\lim_{i\to-\infty} \omega(i)\|\chi(U(0,\tau_i))|X\| = 0.$$

Definition 3.4 We say that the weight $\{\omega(.)\}$ satisfies the δ_2—condition with respect to the system $\{U(0,\tau_i)\}$ if there is $m_0 \in N$ such that for any $i < j$ the inequality $\omega(i) \le \frac{1}{2}\omega(j)$ follows from the inequality $\mu(U(0,\tau_i)) \ge m_0\mu(U(0,\tau_j))$.

Let the sequence $\{\omega(.)\}$ is defined by the equality $\omega(i) = \varpi(\mu(U(0,\tau_i)))$. Then the δ_2—condition with respect to the system $\{U(0,\tau_i)\}$ is closely related to the inequality $\sup_{t\in R_+} \frac{\varpi(m_0 t)}{\varpi(t)} \le \frac{1}{2}$.

Now everything is ready for us to prove the fact that the space $GM^{\tau}_{c^0_\omega,X}$ have not absolutely continuous norm. We restrict ourselves to the one-dimensional case; moreover, we assume that all our functions outside the segment $[-1,1]$ are determined by zero. Denote by $U(0,\tau)$ a segment centered at zero of length 2τ.

Theorem 3.8 *Let a symmetric space X on $[-1,1]$ and numerical sequences $\tau \in \Upsilon$, $\{\omega(.)\}$, for which*

$$0.5 > \tau_{-1} > \tau_{-2} > \ldots > \tau_{-i} > \ldots > 0, \quad \lim_{i\to-\infty} \tau_i = 0; \quad \lim_{i\to-\infty} \omega(.) = \infty, \tag{3.9}$$

be given. Let a sequence $\{\omega(.)\}$ satisfies the δ_2—condition for the system $\{U(0,\tau_i)\}$, and $\{\omega(.)\} \in B(c^0, X)$. Let the space $GM^{\tau}_{c^0_\omega,X}$ be constructed from spaces X, c^0_ω and the sequence $\tau \in \Upsilon$.

Then the space $GM^{\tau}_{c^0_\omega,X}$ have not the absolutely continuous norm.

Remark 3.2 Note, that Theorem 3.8 holds for symmetric spaces with the A—property, for example, we can put $X = L^p$ for $p \in [1,\infty)$. It follows that even if both ideal spaces l and X in the definition of a global Morrey space $GM^{\tau}_{l,X}$ have the A—property, then the space $GM^{\tau}_{l,X}$ may have not this property. This is the fundamental difference between Theorems 3.8 and 3.1. Classical global Morrey spaces M_{λ,L^p}, $0<\lambda<\frac{n}{p}$ ($GM_{\lambda,\infty;L^p}$ in the notation of this article, see Example 2.1) have not the A –property. This fact can be obtained from the result [6]. It follows from Example 2.1 that the space M_{λ,L^p} coincides with the space $GM^{\tau}_{l^{\infty}_{\omega_\lambda},L^p}$, $\omega_\lambda(i)=2^{-i\lambda}$, $\lambda>0$, $i\in\mathbb{Z}$. Since $GM^{\tau}_{c^0_{\omega_\lambda},L^p}\subset GM^{\tau}_{l^{\infty}_{\omega_\lambda},L^p}$, our theorem is stronger than the result of Y. Savano [6].

Proof Let's choose a subsequence of negative numbers $\{n_i\}$ so that conditions were met:

$$\omega(n_j)\sum_{k=j+1}^{\infty}\frac{1}{\omega(n_k)}\leq(\frac{1}{2})^j,\quad j=1,2,\ldots..;\tag{3.10}$$

$$\sum_{i=1}^{\infty}(m_0^i+2)\mu(U(0,r_{n_i}))<1.\tag{3.11}$$

The possibility of choosing such a subsequence of numbers $\{n_i\}$ follows from (3.9).

We define the function $f:[-1,1]\to\mathbb{R}_+$ as follows. Put $s_1=\frac{1}{2}\mu(U(0,r_{n_1}))$, $t_1=\mu(U(0,r_{n_1}))$. We define the function f on $J_1=[t_1-s_1,t_1+s_1]$ by the equality:

$$f(t)=\frac{\chi(U(t_1,r_{n_1}))}{\|\chi(U(0,r_{n_1}))|X\|\omega(n_1)}.$$

Put $s_2=\frac{1}{2}\mu(U(0,r_{n_2}))$, $t_2=t_1+(2m_0+1)s_1+s_2$ and define the function f on $J_2=[t_2-s_2,t_2+s_2]$ by the equality:

$$f(t)=\frac{\chi(U(t_2,r_{n_2}))}{\|\chi(U(0,r_{n_2}))|X\|\omega(n_2)}.$$

Put $s_3=\frac{1}{2}\mu(U(0,r_{n_3}))$, $t_3=t_2+(2m_0^2+1)s_2+s_3$ and define the function f on $J_3=[t_3-s_3,t_3+s_3]$ by the equality:

$$f(t)=\frac{\chi(U(t_3,r_{n_2}))}{\|\chi(U(0,r_{n_2}))|X\|\omega(n_2)}.$$

For arbitrary $k\geq 2$ we put $s_k=\frac{1}{2}\mu(U(0,r_{n_k})$, $t_k=t_{k-1}+(2m_0^{k-1}+1)s_{k-1}+s_k$ and define the function f on $J_k=[t_k-s_k,t_k+s_k]$ by the equality:

$$f(t)=\frac{\chi(U(t_k,r_{n_k}))}{\|\chi(U(0,r_{n_k}))|X\|\omega(n_k)}.$$

Etc.

Outside of $\bigcup_k J_k$, we define the function f to zero.

Note that the distance between J_{k-1} and J_k is exactly equal to $d(J_{k-1}, J_k) = 2m_0^{k-1}s_{k-1} = m_0^{k-1}\mu(U(0, r_{n_{k-1}}))$. Therefore $U(t_l, s_{n_l}) \bigcap U(t_k, s_{n_k}) = \emptyset$, ł $\neq k$. From this relation, equalities

$$t_k = t_{k-1} + (2m_0^{k-1} + 1)s_{k-1} + s_k = t_1 + \sum_{i=1}^{k-1}(2m_0^i + 1)s_i + \sum_{i=2}^{k} s_i =$$

$$\mu(U(0, r_{n_1})) + \sum_{i=1}^{k-1}(m_0^i + \frac{1}{2})\mu(U(0, r_{n_i})) + \frac{1}{2}\sum_{i=2}^{k}\mu(U(0, r_{n_i})) =$$

$$\frac{3}{2}\mu(U(0, r_{n_1})) + \sum_{i=1}^{k-1} m_0^i\mu(U(0, r_{n_i})) + \sum_{i=2}^{k-1}\mu(U(0, r_{n_i})) + \frac{1}{2}\mu(U(0, r_{n_k}))$$

and inequalities (3.11), it follows that $f : [-1, 1] \to \mathbb{R}_+$ is defined correctly.

Let us show that $f \in GM^{\tau}_{c^0_{\omega}, X}$ and

$$\|f|GM^{\tau}_{c^0_{\omega}, X}\| \leq 1. \tag{3.12}$$

For the proof (3.12), we estimate the value:

$$S = \omega(i)\|f\chi(U(t, r_i))|X\| = \omega(i)\|\chi(U(t, r_i))\sum_{k\geq 1}\frac{\chi(U(t_k, r_{n_k}))}{\|\chi(U(0, r_{n_k}))|X\|\omega(n_k)}|X\| =$$

$$\omega(i)\|\sum_{k\geq 1}\frac{\chi(U(t, r_i))\chi(U(t_k, r_{n_k}))}{\|\chi(U(0, r_{n_k}))|X\|\omega(n_k)}|X\|.$$

When $l + 1$ summands in this sum are no equality zero, then they go in a row due to the construction (l can take the value ∞, arguments are is valid in this case as well). Let's denote the number of the first non-zero summand by n_j, and the last one by n_{j+l}. Let's first $l \geq 1$. Then

$$S \leq \omega(i)\sum_{k=j}^{j+l}\frac{\|\chi(U(t, r_i))\bigcap(U(t_2, r_{n_k}))|X\|}{\|\chi(U(0, r_{n_k}))|X\|\omega(n_k)} \leq \omega(i)\sum_{k=j}^{j+l}\frac{1}{\omega(n_k)}. \tag{3.13}$$

Since $U(t, r_i)$ intersects $U(t_j, r_{n_j})$ and $U(t_{j+1}, r_{n_{j+1}})$, then the inequality $\mu(U(t, r_i)) \geq d(J_j, J_{j+1}) = m_0^j\mu(U(0, r_{n_j}))$ holds. Therefore, the inequality $\omega(i) \leq (\frac{1}{2})^j\omega(n_j)$ follows from the δ_2—condition. From the last ratio and (3.10) we get:

$$S \leq (\frac{1}{2})^j \omega(n_j) \sum_{k=j}^{j+l} \frac{1}{\omega(n_k)} \leq 1. \tag{3.14}$$

Let's now analyze the case of $l = 0$. Then the sum will consist of a single term. Let's evaluate this term:

$$S = \frac{\omega(i)}{\omega(n_j)} \frac{\|\chi(U(t, r_i)) \bigcap (U(t_j, r_{n_j}))|X\|}{\|\chi(U(0, r_{n_j}))|X\|}.$$

If $i \geq n_j$, then the monotonicity of the weight sequence implies the inequality

$$S \leq \frac{\omega(i)}{\omega(n_j)} \frac{\|(U(t_j, r_{n_j}))|X\|}{\|\chi(U(0, r_{n_j}))|X\|} = \frac{\omega(i)}{\omega(n_j)} \leq 1. \tag{3.15}$$

If $i < n_j$, then from $\omega(.) \in B(c^0, X)$ we get:

$$S \leq \frac{\omega(i)}{\omega(n_j)} \frac{\|\chi(U(t, r_i))|X\|}{\|\chi(U(0, r_{n_j}))|X\|} \leq 1. \tag{3.16}$$

From (3.14), (3.15), and (3.16) it follows the inequality (3.12).

Checking that the condition is met:

$$\lim_{i \to -\infty} \omega(i) \|f\chi(U(t, r_i))|X\| = 0. \tag{3.17}$$

Put $M = \overline{\bigcup_{k \geq 1} J_k}$. The set $t \in \mathbb{R} \backslash M$ is open. So if $t \in \mathbb{R} \backslash M$, the condition (3.17) will be met. Let $t \in J_k$. Then for $i \to -\infty$ the analogue of the inequality (3.16) holds. Using (3.16) and $\omega(.) \in B(c^0, X)$, we obtain the equality

$$\lim_{i \to -\infty} \omega(i) \|f\chi(U(t, r_i))|X\| = \lim_{i \to -\infty} \frac{\omega(i)}{\omega(n_k)} \frac{\|\chi(U(t, r_i))|X\|}{\|\chi(U(0, r_{n_k}))} = 0.$$

It remains to consider the point $t_* = \lim_{k \to -\infty} t_k$. Fix $U(t_*, r_i)$.

Denote by $n_j(i)$ the number of the first set for which $U(t_*, r_i) \bigcap U(t_j, r_{n_j(i)}) \neq \emptyset$. Then, by analogy with the inequality (3.14), we get

$$\omega(i)\|f\chi(U(t, r_i))|X\| \leq (\frac{1}{2})^{j(i)} \omega(n_j(i)) \sum_{k=j(i)}^{\infty} \frac{1}{\omega(n_k)} \to 0 \text{ for } i \to -\infty.$$

Therefore (3.17) also holds at the point t_*.

Thus, $f \in GM^{\tau}_{c^0_\omega, X}$.

We show that for every $n \in N$ the inequality $\|\chi(t_*-n^{-1}, t_*+n^{-1})f|GM^{\tau}_{c^0_\omega,X}\| \geq 1$ holds. From this it follows that $GM^{\tau}_{c^0_\omega,X}$ have not the absolutely continuous norm.

Indeed, the interval $(t_* - n^{-1}, t_* + n^{-1})$ contains the set $U(t_k, r_{n_k})$ for a sufficiently large k. Therefore

$$\|\chi(t_* - n^{-1}, t_* + n^{-1})f|GM^{\tau}_{c^0_\omega,X}\| \geq \|\frac{\chi(U(t_k, r_{n_k}))}{\|\chi(U(0, r_{n_k}))|X\|\omega(n_k)}|GM^{\tau}_{c^0_\omega,X}\| = 1.$$

□

Corollary 3.3 *Suppose that conditions of Theorem 3.8 are satisfied.*

Then the global Morrey space $GM^{\tau}_{c^0_\omega,X}$ contains an isometric copy of the space l^∞.

Proof It is enough to define the operator $S\sum a_k e^k : l^\infty \to GM^{\tau}_{c^0_\omega,X}$, which performs the isometry by the equality:

$$S\sum a_k e^k = \sum a_k \frac{\chi(U(t_k, r_{n_k}))}{\|\chi(U(0, r_{n_k}))|X\|\omega(n_k)}.$$

□

Remark 3.3 In this article we considered Morrey spaces of functions defined on $\mathbb{R}^n$. If we consider Morrey spaces of functions on a subset $\Omega \subset \mathbb{R}^n$, $(0 \in \Omega)$, then in Definition 3.1 is necessary to replace $U(0, \tau)$ by $U(0, \tau) \cap \Omega$. All results will remain true.

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Justification of the Averaging Method for a System with Multipoint Boundary Value Conditions

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Abstract The Krylov-Bogolyubov averaging method is justified for normal systems of ordinary differential equations with multipoint boundary value conditions.

Keywords Normal system · Mutlipoint boundary value problem · Averaging method

1 Introduction

The Krylov-Bogolyubov averaging method [1–3] is one of the most important asymptotic methods. It is widely known to be developed with great completeness, especially for the systems of ordinary differential equations referred to in this paper. However, for multipoint boundary value problems, it has not been sufficiently studied. We mention three works known to us: [4, 5] where the averaging method is justified for two-point boundary value problems, and [6]: where it is justified for multi-point problems for an arbitrary number of points $m \geq 2$. The researches in [4, 5] are based on the classical implicit function theorem, which was apparently for the first time applied in the theory of the averaging method by I. B. Simonenko in [7]. The authors of the work [6], in contrast of [4, 5] , use the classic Gronwall lemma. Note that in [6], it is assumed that there exists a solution not only to the

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averaged problem (this is the traditional requirement in the theory of averaging method), but also the solvability of a much more complicated perturbed problem. In this paper, we use the approach specified above used in the works [4, 5], based on the implicit function theorem, which avoids the last hard assumption based on the solvability of perturbed problem. In fact, this approach required the strengthening of the requirements for smoothness of the problem data in comparison with [6].

2 Main Results

Let Ω denote a set in the space $\mathbb{R}^n$,[1] $T > 0$, $\Pi = \{(x, t) : x \in \Omega, t \in [0, T]\}$ and $Q = \{(x, t, \tau) : x \in \Omega, t \in [0, T], \tau \in [0, \infty)\}$.

On the set Π we consider the m-point boundary value problem governed by the following system

$$\begin{cases} \dfrac{dx}{dt} = f(x, t, \omega t) \\ \displaystyle\sum_{k=1}^{m} A_k(\omega)x(t_k) = a(\omega) \end{cases} \tag{1}$$

Here $\omega \gg 1$, $A_k(\omega)$—constant square matrices of order n with real elements, $0 = t_1 < t_2 \ldots < t_m = T$, $a(\omega)$– is a constant n-dimensional vector with real components. Let $f(x, t, \tau)$ be n-dimensional vector-function defined on the set Q with values in $\mathbb{R}^n$, satisfying the following conditions

1. Vector function $f(x, t, \tau)$ together with its first partial derivatives with respect to x, is continuous on the set Q. The corresponding Jacobi matrix will be denoted by $\frac{\partial f}{\partial x}(x, t, \tau)$;
2. Vector function $f(x, t, \tau)$ and Jacobi matrix $\frac{\partial f}{\partial x}(x, t, \tau)$ are uniformly bounded on Q.
3. For any $(x_1, t_1, \tau), (x_2, t_2, \tau) \in Q$, the following inequalities hold
   $|f(x_2, t_2, \tau) - f(x_1, t_1, \tau)| \le \gamma(|x_2 - x_1| + |t_2 - t_1|)$,
   $\left\| \frac{\partial f(x_2,t_2,\tau)}{\partial x} - \frac{\partial f(x_1,t_1,\tau)}{\partial x} \right\| \le \gamma(|x_2 - x_1| + |t_2 - t_1|)$,
   where $\gamma(r)$, $r \ge 0$ is a continuous function at zero such that $\gamma(0) = 0$.
   In this paper, the symbols $|x|$ and $\|A\|$ denote the compatible norms of the vector $x \in \mathbb{R}^n$ and the n-dimensional square matrix A , i.e. $|Ax| \le \|A\||x|$.
   For example $|x| = \max\limits_{1 \le i \le n} |x_i|$, $\|A\| = \max\limits_{1 \le i, j \le n} |a_{ij}|$,
   where x_j, a_{ij} are the components of the vector x and the elements of matrix A, respectively.

[1]We shall assume Ω any bounded set in $\mathbb{R}^n$, containing the solution $\overset{o}{y}(t)$ of the averaged problem(see condition 7).

4. On the set Π, a vector function $F(x,t)$ is continuously differentiable with respect to x. For vector–function $F(x,t)$ and the corresponding Jacobi matrix $\frac{\partial F}{\partial x}(x,t)$ uniformly with respect to $(x,t) \in \Pi$ the following limits hold:
   $\langle f(x,t,\tau)\rangle = \lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f(x,t,\tau)\,d\tau = F(x,t).$
   $\langle \frac{\partial f}{\partial x}(x,t,\tau)\rangle = \lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T \frac{\partial f}{\partial x}(x,t,\tau)\,d\tau = \frac{\partial F}{\partial x}(x,t).$
   here the symbol $\langle , \rangle$ denotes the time-averaging operator in classical meaning [1].
5. There exists a constant n-dimensional vector a_0 for which the limit equality holds $\lim\limits_{\omega\to+\infty} |a(\omega) - a_0| = 0.$
6. There exist square matrices B_k, of order n, for which the limit equality holds $\lim\limits_{\omega\to+\infty} \|A_k(\omega) - B_k\| = 0\,.$
   Along with the perturbed problem (1) consider the averaged problem described by the system as follows

$$\begin{cases} \dfrac{dy}{dt} = F(y,t) \\ \sum\limits_{k=1}^{m} B_k y(t_k) = a_0 \end{cases} \tag{2}$$

7. Assume that the system (2) has a solution $\overset{o}{y}(t)$, $t \in [0,T]$, which, together with some $\rho(>0)$– neighborhood lies in Ω, that is, for every $t \in [0,T]$ the distance from $\overset{o}{y}(t)$ to the boundary of Ω is greater than ρ.
   By the symbol $\Phi(t)$, $t \in [0,T]$, we denote the matriciant[2] of the system

$$\frac{dx}{dt} = \frac{\partial F}{\partial y}(\overset{o}{y}(t),t)x \tag{3}$$

8. Assume that the following relation holds
   $\Delta = det\left[\sum\limits_{k=1}^{m} B_k \Phi(t_k)\right] \neq 0\,.$

In the following by symbol $C_\mu([0,T])$, $\mu \in (0,1)$ we denote the well-known Hölder space, i.e, the space of continuous vector-functions $x : [0,T] \longrightarrow \mathbb{R}^n$ satisfying the condition :
$||x||_{C_\mu([0,T])} = \max\limits_{0\le t\le T}|x(t)| + \sup\limits_{0\le t_1<t_2\le T} \frac{|x(t_2)-x(t_1)|}{(t_2-t_1)^\mu} < \infty.$
We are now able to formulate the following main theorem

Theorem 1 *Let the conditions* $1-8$ *be satisfied. Then there exists* $\omega_0 > 0$ *such that for every* $\omega > \omega_0$ *there exists a unique real solution* x_ω *of problem (1),*

[2]Recall that the matriciant of system (3) is the normalized fundamental matrix of solutions $\Phi(t)$ satisfying $\Phi(0) = I$ at $t = 0$.

which belongs to a neighborhood of $\overset{o}{y}$ *in some* $C_\mu([0,T])$ *and* $\lim\limits_{\omega\to\infty} ||x_\omega(t) - \overset{o}{y}(t)||_{C_\mu([0,T])} = 0.$

Proof Substituting

$$x = u + \overset{o}{y}. \tag{4}$$

into (1) we get

$$\begin{cases} \dfrac{du}{dt} = \dfrac{\partial F}{\partial y}(\overset{o}{y}(t), t)u + K(u, t, \omega t) \\ \displaystyle\sum_{k=1}^{m} A_k(\omega)u(t_k) = C(\omega), \omega \gg 1 \end{cases} \tag{5}$$

Here
$K(u,t,\tau) = f(u + \overset{o}{y}, t, \tau) - F(\overset{o}{y}, t) - [\frac{\partial F}{\partial y}(\overset{o}{y}(t), t)]u \equiv H(u,t,\tau) + \Psi(u,t),$
$H(u,t,\tau) = f(u + \overset{o}{y}, t, \tau) - F(u + \overset{o}{y}, t),$
$\Psi(u,t) = F(u + \overset{o}{y}, t) - F(\overset{o}{y}, t) - [\frac{\partial F}{\partial y}(\overset{o}{y}(t), t)]u,$
$C(\omega) = a(\omega) - \sum\limits_{k=1}^{m} A_k(\omega)\overset{o}{y}(t_k)$
From (5), consider the problem

$$\begin{cases} \dfrac{dv}{dt} = \dfrac{\partial F}{\partial y}(\overset{o}{y}(t), t)v + \Psi(v, t) \\ \displaystyle\sum_{k=1}^{m} B_k v(t_k) = 0 \end{cases} \tag{6}$$

which obviously has a solution $v(t) = 0$.

Note that the equivalent systems of (5) and (6) are described by the integral equations as follows

$$u(t) = \Phi(t)\int_0^t \Phi^{-1}(s)K(u(s), s, \omega s)\mathrm{d}s + \\ + \Phi(t)\left[\sum_{k=1}^{m} A_k(\omega)\Phi(t_k)\right]^{-1} \\ \times \left[C(\omega) - \sum_{k=1}^{m} A_k(\omega)\Phi(t_k)\int_0^{t_k} \Phi^{-1}(s)K(u(s), s, \omega s)\mathrm{d}s\right] \tag{7}$$

$$v(t) = \Phi(t)\int_0^t \Phi^{-1}(s)\Psi(v(s),s)\mathrm{d}s -$$
$$- \Phi(t)\left[\sum_{k=1}^{m} B_k\Phi(t_k)\right]^{-1}\sum_{k=1}^{m} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)\Psi(v(s),s)\mathrm{d}s \qquad (8)$$

Proceeding from equalities (7) and (8), we define in some neighborhood of the point $(0,\infty)$ of Hölder space $C_\mu([0,T]) \times [1,\infty]$ operator $N : C_\mu([0,T]) \times [1,\infty] \to C_\mu([0,T])$, $0 < \mu \le 1$ by the following formula:

$$[N(u,\omega)](t) = \begin{cases} u(t) - \Phi(t)\int\limits_0^t \Phi^{-1}(s)K(u(s),s,\omega s)\mathrm{d}s \\ \quad -\Phi(t)\left[\sum\limits_{k=1}^{m} A_k(\omega)\Phi(t_k)\right]^{-1} \\ \left[C(\omega) - \sum\limits_{k=1}^{m} A_k(\omega)\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)K(u(s),s,\omega s)\mathrm{d}s\right], \omega \neq \infty; \\ u(t) - \Phi(t)\int\limits_0^t \Phi^{-1}(s)\Psi(u(s),s)\mathrm{d}s + \\ +\Phi(t)\left[\sum\limits_{k=1}^{m} B_k\Phi(t_k)\right]^{-1} \\ \quad \times \sum\limits_{k=1}^{m} B_k\Phi(t_k)\int_0^{t_k}\Phi^{-1}(s)\Psi(u(s),s)\mathrm{d}s, \omega = \infty, \end{cases}$$

Theorem 1 follows from the implicit function theorem and the following main statement.

Lemma 1 *The operator $N(u,\omega)$ is defined in neighborhood of the point $(0,\infty)$, is continuous and continuously differentiable at the same point. Furthermore, $N(0,\infty) = 0$, and the Fréchet derivative $D_u N(0,\infty) = I$ where I is the identity operator in $C_\mu([0,T])$.*

Lemma 1 is proved similarly to lemma in [5].

Note that for large ω, vector–function $x_\omega(t)$ which is expressed through $u_\omega(t)$ by formula (4) is the unique solution to the problem (1) in some $C_\mu([0,T])$ neighborhood of the vector-function $\overset{o}{y}(t)$. Moreover, from Lemma 1 follows existence of unique solution $u_\omega \underset{\omega\to\infty}{\to} 0$ of Eq. (5) on the time interval $t \in [0,T]$. Therefore, according to the formula (4), uniformly with respect to $t \in [0,T]$, the following limit holds

$$\lim_{\omega\to\infty} ||x_\omega(t) - \overset{o}{y}(t)||_{C_\mu([0,T])} = 0.$$

This completes the proof of Theorem 1. ∎

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Spanne-Guliyev Type Characterization for Fractional Integral Operator and Its Commutators in Generalized Orlicz–Morrey Spaces on Spaces of Homogeneous Type

Fatih Deringoz

Abstract This paper establishes necessary and sufficient condition for the Spanne-Guliyev type boundedness of the fractional integral operator and its commutators in generalized Orlicz–Morrey spaces over spaces of homogeneous type which satisfy the Q-homogeneous (Ahlfors regular) condition.

Keywords Generalized Orlicz–Morrey space · Fractional integral operator · Commutator · Spaces of homogeneous type

AMS Mathematics Subject Classification 42B20, 42B25, 42B35

1 Introduction

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then the fractional integral (also known as Riesz potential) $I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$ $(0 < \alpha < n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$. The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory. Later then, this result has been extended from Lebesgue spaces to various function spaces.

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, named after Morrey, seem to describe the boundedness property of the classical fractional integral operators more precisely than Lebesgue spaces. There are two remarkable results on the Morrey boundedness of I_α. The first result is due to Spanne.

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Theorem 1.1 (Spanne, But Published by Peetre [21]) *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. *Moreover, let* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $\frac{\lambda}{p} = \frac{\mu}{q}$. *Then the operator* I_α *is bounded from* $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\mu}(\mathbb{R}^n)$.

The second milestone result is due to Adams (see also Chiarenza and Frasca [3]), which turned out sharp (see [20]).

Theorem 1.2 (Adams [1]) *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. *Then the operator* I_α *is bounded from* $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\lambda}(\mathbb{R}^n)$.

In the classical case due to the fact that Morrey spaces are nested we can say that the Adams-type boundedness is stronger than the Spanne-type boundedness. However, we need to depend on the pointwise estimate of Hedberg-type [15], so the Adams-type boundedness is unavailable for local Morrey spaces.

In proving the boundedness of the fractional integral operators on various spaces, some researchers find that the translation invariance and the doubling properties of the Lebesgue measure play an important role. This is also true in studying other operators such as maximal operators and various types of singular integral operators. Thus, inspired by this fact, they studied the operators in the homogeneous setting.

In [5], the generalized Orlicz–Morrey space was introduced to unify Orlicz spaces and generalized Morrey spaces. Other definitions of generalized Orlicz–Morrey spaces can be found in [19, 22]. In words of [12], our generalized Orlicz–Morrey space is the third kind and the ones in [19] and [22] are the first kind and the second kind, respectively. Notice that the definition of the space of the third kind relies only on the fact that $L^\Phi(\mathbb{R}^n)$ is a normed linear space, which is independent of the condition that it is generated by modulars.

We refer to [8, 10, 14] and references therein for the Spanne-Guliyev type results in generalized Morrey spaces. Spanne-Guliyev type boundedness of fractional integral operator and its commutators was investigated in the generalized Orlicz–Morrey spaces on the n-dimensional Euclidean space $\mathbb{R}^n$ in [6, 11]. The purpose of this paper is to extend these Euclidean results to the spaces of homogeneous type setting. Note that, Adams type boundedness of fractional integral operator and its commutators was investigated in the generalized Orlicz–Morrey spaces on the spaces of homogeneous type in [13].

The structure of the remaining part of the present paper is as follows: In Sect. 2 we gather together the necessary definitions and a number of preliminary results about spaces of homogeneous type, fractional integral operator and its commutators, Orlicz and BMO spaces. In Sect. 3, we investigate the structure of generalized Orlicz–Morrey spaces defined on spaces of homogeneous type $\mathcal{M}^{\Phi,\varphi}(X)$. We give characterizations for the Spanne type boundedness of I_α and $[b, I_\alpha]$ in $\mathcal{M}^{\Phi,\varphi}(X)$ in Sects. 4 and 5, respectively.

At the end of this section, we make some conventions. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

In order to extend the traditional Euclidean space to build a general underlying structure for the real harmonic analysis, the notion of spaces of homogeneous type was introduced by Coifman and Weiss [4].

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x),$$

$$d(x, y) \leq K_1(d(x, z) + d(z, y)), \tag{2.1}$$

the balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x, 2r)) \leq K_2\, \mu(B(x, r)) < \infty, \tag{2.2}$$

where $K_i \geq 1$ $(i = 1, 2)$ are constants independent of $x, y, z \in X$ and $r > 0$. The dilation of a ball $B = B(x, r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$.

In the sequel, we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L^1(X, \mu)$ and that X is Q-homogeneous $(Q > 0)$, i.e.

$$K_3^{-1} r^Q \leq \mu(B(x, r)) \leq K_3 r^Q, \tag{2.3}$$

where $K_3 \geq 1$ is a constant independent of x and r. The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous.

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty.$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a measurable set $\Omega \subset X$, a measurable function f and $t > 0$, let $m(\Omega, f, t) = \mu(\{x \in \Omega : |f(x)| > t\})$. In the case $\Omega = X$, we shortly denote it by $m(f, t)$.

The Orlicz spaces and weak Orlicz spaces on spaces of homogeneous type are defined as follows.

Definition 2.2 For a Young function Φ,

$$L^{\Phi}(X) = \left\{ f \in L_1^{\mathrm{loc}}(X) : \int_X \Phi(\epsilon|f(x)|)d\mu(x) < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} \equiv \|f\|_{L^\Phi(X)} = \inf\left\{ \lambda > 0 : \int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)d\mu(x) \le 1 \right\},$$

$$WL^{\Phi}(X) = \left\{ f \in L^1_{\mathrm{loc}}(X) : \sup_{r>0} \Phi(r)m\Big(r, \epsilon f\Big) < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{WL^\Phi} \equiv \|f\|_{WL^\Phi(X)} = \inf\left\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t)m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1 \right\}.$$

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{ for } 0 \le r < \infty. \tag{2.4}$$

We also note that,

$$\|\chi_B\|_{WL^\Phi} = \|\chi_B\|_{L^\Phi} = \frac{1}{\Phi^{-1}\left(\mu(B)^{-1}\right)}, \tag{2.5}$$

where B is a μ-measurable set in X with $\mu(B) < \infty$ and χ_B is the characteristic function of B.

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$.

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = \infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.6}$$

Note that by the convexity of Φ and concavity of Φ^{-1} we have the following properties

$$\begin{cases} \Phi(\alpha t) \le \alpha\Phi(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), & \text{if } \quad \alpha > 1 \end{cases} \text{ and } \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha\Phi^{-1}(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), & \text{if } \quad \alpha > 1. \end{cases} \tag{2.7}$$

Remark 2.3 Thanks to (2.3) and (2.7) we have

$$\Phi^{-1}(\mu(B(x,r))^{-1}) \approx \Phi^{-1}(r^{-Q}).$$

The following analogue of the Hölder inequality is known,

$$\int_X |f(x)g(x)|d\mu(x) \le 2\|f\|_{L^\Phi}\|g\|_{L^{\widetilde{\Phi}}}. \tag{2.8}$$

When we prove our main estimates, we use the following lemma, which follows from (2.8), (2.5) and (2.6).

Lemma 2.4 *For a Young function Φ and $B = B(x,r)$, the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2\mu(B)\Phi^{-1}\left(\mu(B)^{-1}\right)\|f\|_{L^\Phi(B)}.$$

For a Q-homogeneous space (X, d, μ), let

$$I_\alpha f(x) = \int_X \frac{f(y)}{d(x,y)^{Q-\alpha}}d\mu(y), \qquad 0 < \alpha < Q.$$

For proving our main results, we need the following estimate.

Lemma 2.5 ([13]) *If* $B_0 := B(x_0, r_0)$*, then* $r_0^\alpha \le C I_\alpha \chi_{B_0}(x)$ *for every* $x \in B_0$.

The known boundedness statement for I_α in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 2.6 ([13, 18]) *Let* (X, d, μ) *be* $Q-$*homogeneous,* $0 < \alpha < Q$ *and* $\Phi, \Psi \in \mathcal{Y}$. *If*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\left(t^{-Q}\right) dt \lesssim r^\alpha \Phi^{-1}\left(r^{-Q}\right) \qquad \text{for } 0 < r < \infty, \tag{2.9}$$

holds, then the condition

$$r^\alpha \Phi^{-1}\left(r^{-Q}\right) \lesssim \Psi^{-1}\left(r^{-Q}\right) \qquad \text{for } 0 < r < \infty. \tag{2.10}$$

is necessary and sufficient for the boundedness of I_α *from* $L^\Phi(X)$ *to* $WL^\Psi(X)$. *Moreover, if* $\Phi \in \nabla_2$, *the condition* (2.10) *is necessary and sufficient for the boundedness of* I_α *from* $L^\Phi(X)$ *to* $L^\Psi(X)$.

We recall that the space $BMO(X) = \{b \in L^1_{\rm loc}(X) \ : \ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in X,\, r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r)}| d\mu(y) < \infty,$$

where $b_{B(x,r)} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} b(y) d\mu(y)$. We will need the following properties of BMO-functions:

Lemma 2.7 ([16, Lemma 7.1]) *Let* $b \in BMO(X)$.

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_* \ln \frac{6t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{2.11}$$

where C *does not depend on* b, x, r *and* t.

Lemma 2.8 *[13, Lemma 4.7] Let* $b \in BMO(X)$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$. *Then*

$$\|b\|_* \approx \sup_{x\in X,\, r>0} \Phi^{-1}\big(r^{-Q}\big) \left\| b(\cdot) - b_{B(x,r)} \right\|_{L^\Phi(B(x,r))}.$$

The commutators generated by $b \in L^1_{\rm loc}(X)$ and the operator I_α are defined by

$$[b, I_\alpha] f(x) = \int_X \frac{b(x) - b(y)}{d(x, y)^{Q-\alpha}} f(y) d\mu(y), \qquad 0 < \alpha < Q.$$

The operator $|b, I_\alpha|$ is defined by

$$|b, I_\alpha| f(x) = \int_X \frac{|b(x) - b(y)|}{d(x, y)^{Q-\alpha}} f(y) d\mu(y), \qquad 0 < \alpha < Q.$$

For proving our main results, we need the following estimate.

Lemma 2.9 ([13]) *If $b \in L^1_{\mathrm{loc}}(X)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{B_0}| \le C |b, I_\alpha| \chi_{B_0}(x)$$

for every $x \in B_0$.

The known boundedness statements for the commutator operator $[b, I_\alpha]$ on Orlicz spaces run as follows.

Theorem 2.10 ([13]) *Let (X, d, μ) be Q−homogeneous, $0 < \alpha < Q$, $b \in BMO(X)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-Q}\big) + \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \frac{dt}{t} \le C \Psi^{-1}\big(r^{-Q}\big) \tag{2.12}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(X)$ to $L^\Psi(X)$.
2. *If $\Psi \in \Delta_2$, then the condition (2.10) is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(X)$ to $L^\Psi(X)$.*
3. *Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$. If the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \frac{dt}{t} \le C r^\alpha \Phi^{-1}\big(r^{-Q}\big) \tag{2.13}$$

   holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (2.10) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(X)$ to $L^\Psi(X)$.

Example If $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $p, q \in [1, \infty)$ and $0 < \alpha < Q/p$, then

$$r^\alpha \Phi^{-1}\big(r^{-Q}\big) \approx \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-Q}\big) t^\alpha \frac{dt}{t} = r^{\alpha - Q/p} \text{and} \Psi^{-1}\big(r^{-Q}\big) = r^{-Q/q}.$$

In this case,

$$(2.12) \iff r^{\alpha - Q/p} \lesssim r^{-Q/q},\ r \in (0, \infty) \iff \alpha - Q/p = -Q/q.$$

Therefore, Chanillo's [2] well-known result which says $[b, I_\alpha]$ with $b \in BMO$ has the same boundedness as I_α is a corollary of Theorem 2.10.

3 Generalized Orlicz–Morrey Spaces

The generalized Orlicz–Morrey spaces and the weak generalized Orlicz–Morrey spaces on spaces of homogeneous type are defined as follows.

Definition 3.1 Let (X, d, μ) be $Q-$homogeneous, $\varphi(r)$ be a positive measurable function on $(0, \infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(X)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L^{\Phi}_{\rm loc}(X)$ with finite quasinorm

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} \equiv \|f\|_{\mathcal{M}^{\Phi,\varphi}(X)} = \sup_{x\in X, r>0} \varphi(r)^{-1}\Phi^{-1}(\mu(B(x,r))^{-1})\|f\|_{L^{\Phi}(B(x,r))},$$

where $L^{\Phi}_{\rm loc}(X)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(X)$ for all balls $B \subset X$.

Also by $W\mathcal{M}^{\Phi,\varphi}(X)$ we denote the weak generalized Orlicz–Morrey space of all functions $f \in WL^{\Phi}_{\rm loc}(X)$ for which

$$\|f\|_{W\mathcal{M}^{\Phi,\varphi}} \equiv \|f\|_{W\mathcal{M}^{\Phi,\varphi}(X)} = \sup_{x\in X, r>0} \varphi(r)^{-1}\Phi^{-1}(\mu(B(x,r))^{-1})$$
$$\|f\|_{WL^{\Phi}(B(x,r))} < \infty,$$

where $WL^{\Phi}_{\rm loc}(X)$ is defined as the set of all functions f such that $f\chi_B \in WL^{\Phi}(X)$ for all balls $B \subset X$.

According to this definition, we recover the generalized Morrey space $\mathcal{M}^{p,\varphi}(X)$ and weak generalized Morrey space $W\mathcal{M}^{p,\varphi}(X)$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$. If $\Phi(r) = r^p$, $1 \le p < \infty$ and $\varphi(r) = r^{\frac{\lambda-Q}{p}}$, $0 \le \lambda \le Q$, then $\mathcal{M}^{\Phi,\varphi}(X)$ and $W\mathcal{M}^{\Phi,\varphi}(X)$ coincide with $\mathcal{M}^{p,\lambda}(X)$ and $W\mathcal{M}^{p,\lambda}(X)$, respectively and if $\varphi(r) = \Phi^{-1}(r^{-Q})$, then $\mathcal{M}^{\Phi,\varphi}(X)$ and $W\mathcal{M}^{\Phi,\varphi}(X)$ coincide with the $L^{\Phi}(X)$ and $WL^{\Phi}(X)$, respectively.

A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C \ge 1$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp.}\varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_{\Phi}$ the set of all $\varphi : (0, \infty) \to (0, \infty)$ functions such that $t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-Q})}$ is almost increasing and $t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-Q})t^{Q}}$ is almost decreasing. Note that $\varphi \in \mathcal{G}_{\Phi}$ implies doubling condition of φ.

An observation similar to the one made by Nakai [17, p. 446] it can be assumed that $\varphi \in \mathcal{G}_{\Phi}$ in the definition of $\mathcal{M}^{\Phi,\varphi}(X)$. See [7, Section 5] for more details.

As the following lemma shows, $\mathcal{G}_\Phi$ is useful:

Lemma 3.2 *[13] Let $B_0 := B(x_0, r_0)$. If $\varphi \in \mathcal{G}_\Phi$ is almost decreasing, then there exist $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{B_0}\|_{W\mathcal{M}^{\Phi,\varphi}} \leq \|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi}} \leq \frac{C}{\varphi(r_0)}.$$

4 Fractional Integral Operator in Generalized Orlicz–Morrey Spaces

The following lemma was generalization of the Guliyev type local estimates (see [9, 10]) for Orlicz spaces.

Lemma 4.1 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $\Phi, \Psi \in \mathcal{Y}$. Assume that the conditions (2.9) and (2.10) are fulfilled. Then there exists a positive constant C such that, for all $f \in L^{\Phi}_{\text{loc}}(X)$ and $B = B(x_0, r)$,*

$$\|I_\alpha f\|_{WL^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_r^\infty \Psi^{-1}\big(t^{-Q}\big)\, \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t}. \tag{4.1}$$

Moreover if we assume $\Phi \in \nabla_2$, the following inequality is also valid:

$$\|I_\alpha f\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_r^\infty \Psi^{-1}\big(t^{-Q}\big)\, \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t}. \tag{4.2}$$

Proof Let $\Phi \in \nabla_2$. We put $f = f_1 + f_2$, where $f_1 = f\chi_{B(x_0,2kr)}$ and $f_2 = f\chi_{\complement_{B(x_0,2kr)}}$, where k is the constant from the triangle inequality (2.1). Hence

$$\|I_\alpha f\|_{L^\Psi(B)} \leq \|I_\alpha f_1\|_{L^\Psi(B)} + \|I_\alpha f_2\|_{L^\Psi(B)}\,.$$

Estimation of $I_\alpha f_1$: By Theorem 2.6 we have

$$\|I_\alpha f_1\|_{L^\Psi(B)} \leq \|I_\alpha f_1\|_{L^\Psi(X)} \lesssim \|f_1\|_{L^\Phi(X)} = \|f\|_{L^\Phi(B(x,2kr))}.$$

We note that $\Psi^{-1}\big(\tau\big)/\tau$ is decreasing, since $\Psi^{-1}\big(0\big) = 0$ and Ψ^{-1} concave. By this fact, we have

$$\begin{aligned}\Psi^{-1}\big(r^{-Q}\big) &\approx \Psi^{-1}\big(r^{-Q}\big) r^Q \int_{2kr}^\infty \frac{dt}{t^{Q+1}} \\ &\lesssim \int_{2kr}^\infty \Psi^{-1}\big(t^{-Q}\big)\frac{dt}{t}.\end{aligned}$$

Therefore using the monotonicity of the function $\|f\|_{L^\Phi(B(x_0,t))}$ with respect to t, we get

$$\|f\|_{L^\Phi(2kB)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2kr}^{\infty} \|f\|_{L^\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-Q}\big) \frac{dt}{t}. \tag{4.3}$$

Consequently we have

$$\|I_\alpha f_1\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{r}^{\infty} \Psi^{-1}\big(t^{-Q}\big) \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t}. \tag{4.4}$$

Estimation of $I_\alpha f_2$ For $x \in B$ we have

$$|I_\alpha f_2(x)| \lesssim \int_{\complement_{(2kB)}} \frac{|f(y)|}{d(x_0, y)^{Q-\alpha}} d\mu(y)$$

since $x \in B$ and $y \in {}^{\complement}(2kB)$ implies

$$\frac{1}{2k} d(x_0, y) \leq d(x, y) \leq (k + \frac{1}{2}) d(x_0, y).$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\complement_{(2kB)}} \frac{|f(y)|}{d(x_0, y)^{Q-\alpha}} d\mu(y) &\approx \int_{\complement_{(2kB)}} |f(y)| \int_{d(x_0,y)}^{\infty} \frac{dt}{t^{Q+1-\alpha}} d\mu(y) \\
&\approx \int_{2kr}^{\infty} \int_{2kr \leq d(x_0,y) \leq t} |f(y)| d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim \int_{2kr}^{\infty} \int_{B(x_0,t)} |f(y)| d\mu(y) \frac{dt}{t^{Q+1-\alpha}}.
\end{aligned}$$

Hence by Lemma 2.4 and (2.10)

$$I_\alpha f_2(x) \lesssim \int_{r}^{\infty} \Psi^{-1}\big(t^{-Q}\big) \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t}. \tag{4.5}$$

Thus the function $I_\alpha f_2(x)$, with fixed x_0 and r, is dominated by the expression not depending on x. Then we integrate the obtained estimate for $I_\alpha f_2(x)$ in x over B, we get

$$\|I_\alpha f_2\|_{L^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{r}^{\infty} \Psi^{-1}\big(t^{-Q}\big) \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t}. \tag{4.6}$$

Gathering the estimates (4.4) and (4.6) we arrive at (4.2).

Let now Φ be an arbitrary Young function. It is obvious that

$$\|I_\alpha f\|_{WL^\Psi(B)} \le \|I_\alpha f_1\|_{WL^\Psi(B)} + \|I_\alpha f_2\|_{WL^\Psi(B)}.$$

By the boundedness of the operator I_α from $L^\Phi(X)$ to $WL^\Psi(X)$, provided by Theorem 2.6, we have

$$\|I_\alpha f_1\|_{WL^\Psi(B)} \lesssim \|f\|_{L^\Phi(B(x,2kr))}.$$

By using (4.3), (4.5) and (2.5) we arrive at (4.1). □

The following theorem gives a necessary and sufficient condition for Spanne-Guliyev type boundedness of the operator I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Theorem 4.2 (Spanne-Guliyev Type Result) *Let (X, d, μ) be $Q-$homogeneous, $\Phi, \Psi \in \mathcal{Y}$ and $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$.*

1. *Assume that conditions (2.9) and (2.10) are satisfied. Then the condition*

$$\int_r^\infty \varphi_1(t) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \frac{dt}{t} \le C\, \varphi_2(r), \tag{4.7}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$. Moreover, if $\Phi \in \nabla_2$, then the condition (4.7) is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.
2. *Let φ_1 be almost decreasing. Then the condition*

$$\varphi_1(r) r^\alpha \le C \varphi_2(r), \tag{4.8}$$

   for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$ and hence $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.
3. *Let φ_1 be almost decreasing. Assume that conditions (2.9), (2.10) and*

$$\int_r^\infty \varphi_1(t) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \frac{dt}{t} \le C\, \varphi_1(r) r^\alpha,$$

   for all $r > 0$, where $C > 0$ does not depend on r, are satisfied. Then the condition (4.8) is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$. Moreover, if $\Phi \in \nabla_2$, then the condition (4.8) is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Proof

1. By (4.1) and (4.7) we have

$$\|I_\alpha f\|_{\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}} \lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \int_r^\infty \|f\|_{L^\Phi(B(x,t))}\, \Psi^{-1}\big(t^{-Q}\big)\frac{dt}{t}$$
$$\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}} \sup_{x\in X, r>0} \varphi_2(r)^{-1} \int_r^\infty \varphi_1(t)\frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)}\frac{dt}{t}$$
$$\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}.$$

Simply replace $\mathrm{W}L^\Psi(B)$ with $L^\Psi(B)$ and $\mathrm{W}\mathcal{M}^{\Psi,\varphi_2}(X)$ with $\mathcal{M}^{\Psi,\varphi_2}(X)$ for the strong estimate.

2. We will now prove the necessity. Let $B_0 = B(x_0, t_0)$ and $x \in B_0$. By Lemma 2.5 we have $t_0^\alpha \lesssim I_\alpha \chi_{B_0}(x)$. Therefore, by (2.5) and Lemma 3.2, we have

$$t_0^\alpha \lesssim \Psi^{-1}(\mu(B_0)^{-1})\|I_\alpha \chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \varphi_2(t_0)\|I_\alpha \chi_{B_0}\|_{W\mathcal{M}^{\Psi,\varphi_2}}$$
$$\lesssim \varphi_2(t_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)}.$$

Since this is true for every $t_0 > 0$, we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem.

□

Remark 4.3 Note that, in the case $X = \mathbb{R}^n$, Theorem 4.2 was proved in [6, 11].

5 Commutators of Fractional Integral Operator in Generalized Orlicz–Morrey Spaces

The following Guliyev type local estimate is valid.

Lemma 5.1 *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $b \in BMO(X)$. Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$ and the condition (2.12) holds, then the inequality*

$$\|[b, I_\alpha]f\|_{L^\Psi(B(x_0,r))} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^\infty \Big(1 + \ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-Q}\big)$$
$$\|f\|_{L^\Phi(B(x_0,t))}\frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^\Phi_{\mathrm{loc}}(X)$.

Proof For arbitrary $x_0 \in X$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2kB}$ and $f_2 = f\chi_{\complement_{(2kB)}}$, where k is the constant from the triangle inequality (2.1). Hence

$$\|[b, I_\alpha]f\|_{L^\Psi(B)} \le \|[b, I_\alpha]f_1\|_{L^\Psi(B)} + \|[b, I_\alpha]f_2\|_{L^\Psi(B)} .$$

From the boundedness of $[b, I_\alpha]$ from $L^\Phi(X)$ to $L^\Psi(X)$ (see, Theorem 2.10) it follows that

$$\begin{aligned}\|[b, I_\alpha]f_1\|_{L^\Psi(B)} &\le \|[b, I_\alpha]f_1\|_{L^\Psi(X)} \\ &\lesssim \|b\|_* \|f_1\|_{L^\Phi(X)} = \|b\|_* \|f\|_{L^\Phi(2kB)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|[b, I_\alpha]f_2(x)| &\lesssim \int_X \frac{|b(y) - b(x)|}{d(x, y)^{Q-\alpha}} |f_2(y)| d\mu(y) \\ &\approx \int_{\complement_{(2kB)}} \frac{|b(y) - b(x)|}{d(x_0, y)^{Q-\alpha}} |f(y)| d\mu(y) \\ &\lesssim \int_{\complement_{(2kB)}} \frac{|b(y) - b_B|}{d(x_0, y)^{Q-\alpha}} |f(y)| d\mu(y) \\ &\quad + \int_{\complement_{(2kB)}} \frac{|b(x) - b_B|}{d(x_0, y)^{Q-\alpha}} |f(y)| d\mu(y) = J_1 + J_2(x),\end{aligned}$$

since $x \in B$ and $y \in {}^\complement(2kB)$ implies

$$\frac{1}{2k} d(x_0, y) \le d(x, y) \le (k + \frac{1}{2}) d(x_0, y).$$

Let us estimate J_1.

$$\begin{aligned}J_1 &= \int_{\complement_{(2kB)}} \frac{|b(y) - b_B|}{d(x_0, y)^{Q-\alpha}} |f(y)| d\mu(y) \\ &\approx \int_{\complement_{(2kB)}} |b(y) - b_B||f(y)| \int_{d(x_0,y)}^\infty \frac{dt}{t^{Q+1-\alpha}} d\mu(y) \\ &\approx \int_{2kr}^\infty \int_{2kr \le d(x_0,y) \le t} |b(y) - b_B||f(y)| d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\ &\lesssim \int_{2kr}^\infty \int_{B(x_0,t)} |b(y) - b_B||f(y)| d\mu(y) \frac{dt}{t^{Q+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality, by (2.6), (2.11) and Lemmas 2.4 and 2.8 we get

$$\begin{aligned}
J_1 &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim \int_{2r}^{\infty} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L^{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L^{\Phi}(B(x_0,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\quad + \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\big(\mu(B(x_0,t))^{-1}\big) \frac{dt}{t^{1-\alpha}} \\
&\lesssim \|b\|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\big(\mu(B(x_0,t))^{-1}\big) \frac{dt}{t^{1-\alpha}}. \\
&\lesssim \|b\|_* \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(B(x_0,t))} \Psi^{-1}\big(t^{-Q}\big) \frac{dt}{t}.
\end{aligned}$$

In order to estimate J_2, by Lemmas 2.4 and 2.8 and condition (2.12), we also get

$$\begin{aligned}
\|J_2\|_{L^{\Psi}(B)} &= \left\| \int_{\complement_{(2kB)}} \frac{|b(\cdot) - b_B|}{d(x_0,y)^{Q-\alpha}} |f(y)|d\mu(y) \right\|_{L^{\Psi}(B)} \\
&\approx \|b(\cdot) - b_B\|_{L^{\Psi}(B)} \int_{\complement_{(2kB)}} \frac{|f(y)|}{d(x_0,y)^{Q-\alpha}} d\mu(y) \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{\complement_{(2kB)}} \frac{|f(y)|}{d(x_0,y)^{Q-\alpha}} d\mu(y) \\
&\approx \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{\complement_{(2kB)}} |f(y)| \int_{d(x_0,y)}^{\infty} \frac{dt}{t^{Q+1-\alpha}} d\mu(y) \\
&\approx \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2kr}^{\infty} \int_{2kr \le d(x_0,y) < t} |f(y)|d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|d\mu(y) \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Phi^{-1}\big(t^{-Q}\big) t^{\alpha-1} dt \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(B(x_0,t))} \Psi^{-1}\big(t^{-Q}\big) \frac{dt}{t}.
\end{aligned}$$

Summing up J_1 and J_2 we get

$$\|[b, I_\alpha]f_2\|_{L^\Psi(B)} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-Q}\big) \frac{dt}{t}. \tag{5.1}$$

Finally,

$$\|[b, I_\alpha]f\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi(2kB)} + \frac{\|b\|_*}{\Psi^{-1}\big(r^{-Q}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(B(x_0,t))} \Psi^{-1}\big(t^{-Q}\big) \frac{dt}{t},$$

and the statement of Lemma 5.1 follows by (4.3). □

The following theorem gives a necessary and sufficient condition for Spanne-Guliyev type boundedness of the operator $[b, I_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Theorem 5.2 (Spanne-Guliyev Type Result) *Let (X, d, μ) be $Q-$homogeneous, $0 < \alpha < Q$ and $b \in BMO(X)$, Φ, Ψ be Young functions, and let $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$.*

1. *Assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$ and the condition (2.12) is satisfied. Then the condition*

$$\int_{r}^{\infty} \varphi_1(t)\Big(1 + \ln \frac{t}{r}\Big) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \frac{dt}{t} \le C\, \varphi_2(r), \tag{5.2}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

2. *Let φ_1 be almost decreasing and $\Psi \in \Delta_2$. Then the condition (4.8) is necessary for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.*
3. *Let φ_1 be almost decreasing, $\Phi \in \Delta_2 \cap \nabla_2$ and $\Psi \in \Delta_2$. Assume that conditions (2.12) and*

$$\int_{r}^{\infty} \varphi_1(t)\Big(1 + \ln \frac{t}{r}\Big) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)} \frac{dt}{t} \le C\, \varphi_1(r) r^{\alpha},$$

for all $r > 0$, where $C > 0$ does not depend on r, are satisfied. Then the condition (4.8) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi_1}(X)$ to $\mathcal{M}^{\Psi,\varphi_2}(X)$.

Proof

1. By (5.1) and (5.2) we have

$$\|[b, I_\alpha]f\|_{\mathcal{M}^{\Psi,\varphi_2}} \lesssim \sup_{x\in X, r>0} \varphi_2(r)^{-1} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L^\Phi(B(x,t))} \Psi^{-1}\big(t^{-Q}\big)\frac{dt}{t}$$

$$\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}} \sup_{x\in X, r>0} \varphi_2(r)^{-1} \int_r^\infty \varphi_1(t)\Big(1+\ln\frac{t}{r}\Big) \frac{\Psi^{-1}\big(t^{-Q}\big)}{\Phi^{-1}\big(t^{-Q}\big)}\frac{dt}{t} \lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}.$$

2. We will now prove the necessity. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 2.9 we have $r_0^\alpha|b(x) - b_{B_0}| \lesssim |b, I_\alpha|\chi_{B_0}(x)$. Therefore, by Lemma 2.8 and Lemma 3.2

$$r_0^\alpha \lesssim \frac{\||b, I_\alpha|\chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \||b, I_\alpha|\chi_{B_0}\|_{L^\Psi(B_0)}\Psi^{-1}(\mu(B_0)^{-1})$$

$$\lesssim \varphi_2(r_0)\||b, I_\alpha|\chi_{B_0}\|_{\mathcal{M}^{\Psi,\varphi_2}} \lesssim \varphi_2(r_0)\|\chi_{B_0}\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(r_0)}{\varphi_1(r_0)}.$$

Since this is true for every $r_0 > 0$, we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem.

□

Remark 5.3 Note that, in the case $X = \mathbb{R}^n$, Theorem 5.2 was proved in [6, 11].

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Spanne Type Characterization of Parabolic Fractional Maximal Function and Its Commutators in Parabolic Generalized Orlicz–Morrey Spaces

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Abstract In this paper, we shall give necessary and sufficient conditions for the Spanne type boundedness of the parabolic fractional maximal operator and its commutators on parabolic generalized Orlicz–Morrey spaces, res-pectively. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators.

Keywords Parabolic generalized Orlicz–Morrey space · Parabolic fractional maximal operator · Commutator · BMO

AMS Mathematics Subject Classification 42B25, 42B35, 46E30

1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have

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good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

Generalized Orlicz–Morrey spaces are generalization of Orlicz spaces and generalized Morrey spaces. There are three versions of generalized Orlicz–Morrey spaces, i.e.: Nakai's [18], Gala-Sawano-Tanaka's [10] and Deringoz-Guliyev-Samko's [8] versions. We define the parabolic generalized Orlicz–Morrey space of the third kind $M^P_{\Phi,\varphi}(\mathbb{R}^n)$ (see [8, Definition 2.9 in p.6]).

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \ \ (t > 0)$, and set $\gamma = tr P$. Then, there exists a quasi-distance ρ associated with P such that

$$(a) \ \ \rho(A_t x) = t\rho(x), \ \ t > 0, \ \ \text{for every} \ \ x \in \mathbb{R}^n;$$

$$(b) \ \ \rho(0) = 0, \ \ \rho(x-y) = \rho(y-x) \geq 0$$

$$\text{and} \ \ \rho(x-y) \leq k(\rho(x-z) + \rho(y-z));$$

$$(c) \ \ dx = \rho^{\gamma-1} d\sigma(w) d\rho, \ \ \text{where} \ \ \rho = \rho(x), w = A_{\rho^{-1}}x$$

$$\text{and } d\sigma(w) \text{is a } C^\infty \text{measure on the ellipsoid } \{w : \rho(w) = 1\}.$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space [5, 6]. The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = B(x,r)$. Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$.

The parabolic fractional maximal function $M^P_\alpha f$, $0 \leq \alpha < \gamma$ of a function $f \in L_1^{\rm loc}(\mathbb{R}^n)$ is defined by

$$M^P_\alpha f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

and the parabolic fractional maximal commutator of M^P_α with a locally integrable function b is defined by

$$M^P_{b,\alpha} f(x) = \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}} |b(x) - b(y)||f(y)| dy,$$

where the supremum is taken over all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$ containing x. If $\alpha = 0$, then $M^P \equiv M_0^P$ is the parabolic maximal operator and $M_b^P \equiv M_{b,0}^P$ is the parabolic maximal commutator operator. If $P = I$, then $M_\alpha \equiv M_\alpha^I$ is the classical fractional maximal operator and $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator. It is well known that the parabolic fractional maximal operator play an important role in harmonic analysis (see [23]).

On the other hand, we can define the commutator of the operators M_α and I_α with a locally integrable function b by

$$[b, M_\alpha^P]f(x)f = b(x)M_\alpha^P f(x) - M_\alpha^P (bf)(x),$$
$$[b, I_\alpha^P]f(x)f = b(x)I_\alpha^P f(x) - I_\alpha^P (bf)(x).$$

For more details about the operators $M_{b,\alpha}^P$ and $[b, M_\alpha^P]$, where $0 \le \alpha < \gamma$, we refer to [12, 14] and references therein.

In this work we present the characterization for parabolic fractional maximal operator M_α^P (Theorems 4.6, 4.9) and its commutators $M_{b,\alpha}^P$ (Theorems 5.2, 5.5) in parabolic generalized Orlicz–Morrey spaces $M_{\Phi,\varphi}^P(\mathbb{R}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

2.1 *On Young Functions and Orlicz Spaces*

First, we recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that $0 < \Phi(r) < \infty$ for $0 < r < \infty$ will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let $\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}$. If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} \,, & r \in [0, \infty) \\ \infty \,, & r = \infty. \end{cases}$$

Note that by the convexity of Φ and concavity of Φ^{-1} we have the following properties

$$\begin{cases} \Phi(\alpha t) \le \alpha \Phi(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi(\alpha t) \ge \alpha \Phi(t), & \text{if } \alpha > 1 \end{cases} \text{ and } \begin{cases} \Phi^{-1}(\alpha t) \ge \alpha \Phi^{-1}(t), & \text{if } 0 \le \alpha \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha \Phi^{-1}(t), & \text{if } \alpha > 1. \end{cases} \tag{2.2}$$

Remark 2.2 Thanks to $|\mathcal{E}(x, r)| = v_\rho r^\gamma$ and (2.2) we have

$$\Phi^{-1}(|\mathcal{E}(x, r)|^{-1}) \approx \Phi^{-1}(r^{-\gamma}).$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if $\Phi(2r) \le C\Phi(r)$, $r > 0$ for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if $\Phi(r) \le \frac{1}{2C}\Phi(Cr)$, $r \ge 0$ for some $C > 1$.

The Orlicz space and weak Orlicz space are defined as follows.

Definition 2.3 (Orlicz Space) For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L_\Phi^{\text{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L_\Phi(\mathbb{R}^n)$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$.

$L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$. The weak Orlicz space $WL_\Phi(\mathbb{R}^n) = \{f \in L_1^{\text{loc}}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < \infty\}$ is defined by the norm $\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 \ :\ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \le 1\right\}$.

The following analogue of the Hölder's inequality is well known (see, for example, [19]).

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f, g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid $\int_\Omega |f(x)g(x)|dx \le 2\|f\|_{L_\Phi(\Omega)}\|g\|_{L_{\widetilde{\Phi}}(\Omega)}$.*

By elementary calculations we have the following property.

Lemma 2.5 *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L_\Phi} = \|\chi_{\mathcal{E}}\|_{WL_\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 2.4, Lemma 2.5 and (2.1) we get the following estimate.

Lemma 2.6 *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x, r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)|dy \le 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\|f\|_{L_\Phi(\mathcal{E})}.$$

3 Parabolic Fractional Maximal Function and Its Commutators in Orlicz Spaces

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained, see also [2].

Theorem 3.1 ([1]) *Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

In [14] the boundedness of the parabolic fractional maximal operator M_α^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained, see also [12, 20].

Theorem 3.2 ([14]) *Let $0 \le \alpha < \gamma$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$. The condition*

$$r^{-\frac{\alpha}{\gamma}}\Phi^{-1}(r) \le C\Psi^{-1}(r) \tag{3.1}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of M_α^P from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.1) is necessary and sufficient for the boundedness of M_α^P from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) = \begin{cases} 0 & \text{if } b(x) \ge 0 \\ |b(x)| & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha^P]$ and $M_{b,\alpha}^P$ are valid.

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} \big|[b, M_\alpha^P]f(x)\big| &= \big|b(x)M_\alpha^P f(x) - M_\alpha^P(bf)(x)\big| \\ &= \big|M_\alpha^P(b(x)f)(x) - M_\alpha^P(bf)(x)\big| \\ &\le M_\alpha^P(|b(x) - b|f)(x) \le M_{b,\alpha}^P(f)(x). \end{aligned}$$

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha^P]f(x)| \le M_{b,\alpha}^P(f)(x) + 2b^-(x)M_\alpha^P f(x), \qquad x \in \mathbb{R}^n \tag{3.2}$$

holds for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see, for example, [12, 24]).

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = |\mathcal{E}(x,r)|^{-1} \int_{\mathcal{E}(x,r)} b(y)dy$. We will need the following property of BMO-functions:

$$\big|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\big| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{3.3}$$

where C does not depend on b, x, r and t. We refer for instance to [15] and [16] for details on this space and properties.

Lemma 3.3 ([3]) *Let $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-\gamma}\big) \left\|b(\cdot) - b_{\mathcal{E}(x,r)}\right\|_{L_\Phi(\mathcal{E}(x,r))}.$$

Remark 3.4 Note that Lemma 3.3 in a more general case of spaces of homogeneous type was proved in [13, Lemma 4.7].

The known boundedness statements for the commutator operator $M_{b,\alpha}^P$ on Orlicz spaces run as follows, see [7, Corollary 2.3] and [12].

Theorem 3.5 *Let $0 \le \alpha < \gamma$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined by its inverse $\Psi^{-1}(r) \approx \Phi^{-1}(r)\, r^{-\frac{\alpha}{\gamma}}$. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then $M_{b,\alpha}^P$ is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$*

4 Parabolic Fractional Maximal Function in Parabolic Generalized Orlicz–Morrey Spaces

In this section, we shall give a necessary and sufficient condition for the boundedness of M_α^P on parabolic generalized Orlicz–Morrey spaces.

Various versions of generalized Orlicz–Morrey spaces were introduced in [17, 22] and [8]. We used the definition of [8] which runs as follows.

Definition 4.1 Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and Φ any Young function. We denote by $M^P_{\Phi,\varphi}(\mathbb{R}^n)$ the parabolic generalized Orlicz–Morrey space, the space of all functions $f \in L^{\text{loc}}_\Phi(\mathbb{R}^n)$ for which

$$\|f\|_{M^P_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})\|f\|_{L_\Phi(\mathcal{E}(x,r))} < \infty.$$

Lemma 4.2 ([4]) *Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$.*

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})}{\varphi(x, r)} = \infty \quad \text{for some } t > 0 \text{ and for all } x \in \mathbb{R}^n, \tag{4.1}$$

then $M^P_{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x, r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } x \in \mathbb{R}^n, \tag{4.2}$$

then $M^P_{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 4.3 Let Φ be a Young function. We denote by $\Omega_{\Phi,P}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|\mathcal{E}(x, r)|^{-1})}{\varphi(x, r)} \right\|_{L_\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x, r)^{-1} \right\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 4.2, we always assume that $\varphi \in \Omega_{\Phi,P}$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{1}{\Phi^{-1}(t^{-\gamma})}\varphi(t)$ is almost increasing.

Lemma 4.4 ([2]) *Let $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$. If $\varphi \in \mathcal{G}_\Phi$, then there exist $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{E}_0}\|_{M^P_{\Phi,\varphi}} \le \frac{C}{\varphi(r_0)}.$$

The following local estimate for the parabolic fractional maximal operator M^P_α in Orlicz space is valid.

Lemma 4.5 *Let $0 \le \alpha < \gamma$, Φ, Ψ Young functions and (Φ,Ψ) satisfy the conditions (3.1). Then*

$$\|M^P_\alpha f\|_{WL_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.3}$$

holds for any ball $\mathcal{E} = \mathcal{E}(x,r)$ and for all $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$.

If $\Phi \in \nabla_2$, then

$$\|M^P_\alpha f\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} \tag{4.4}$$

holds for any ball $\mathcal{E} = \mathcal{E}(x,r)$ and for all $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$.

Proof Let $0 \le \alpha < \gamma$, $\Phi \in \nabla_2$ and (Φ,Ψ) satisfy the conditions (3.1). We put $f = f_1 + f_2$, where $f_1 = f\chi_{\mathcal{E}(x,2kr)}$ and $f_2 = f\chi_{\complement_{\mathcal{E}(x,2kr)}}$, where k is the constant from the triangle inequality.

Estimation of $M^P_\alpha f_1$: By Theorem 3.2 we have

$$\|M^P_\alpha f_1\|_{L_\Psi(\mathcal{E})} \le \|M^P_\alpha f_1\|_{L_\Psi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using the monotonicity of the functions $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$, $\Phi^{-1}\big(t\big)$ with respect to t we get,

$$\begin{aligned}\frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} &\sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} \\ &\ge \frac{\|f\|_{L_\Phi(\mathcal{E}(x,2kr))}}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big) \gtrsim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.\end{aligned} \tag{4.5}$$

Consequently we have

$$\|M_\alpha^P f_1\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}. \tag{4.6}$$

Estimation of $M_\alpha^P f_2$] Let y be an arbitrary point from $\mathcal{E}$. If $\mathcal{E}(y,t)\cap {}^{\complement}(\mathcal{E}(x,2kr)) \neq \emptyset$, then $t > r$. Indeed, if $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then $t > \rho(y-z) \geq \frac{1}{k}\rho(x-z) - \rho(x-y) > 2r - r = r$.

On the other hand, $\mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr)) \subset \mathcal{E}(x,2kt)$. Indeed, if $z \in \mathcal{E}(y,t) \cap {}^{\complement}(\mathcal{E}(x,2kr))$, then we get $\rho(x-z) \leq k\rho(y-z) + k\rho(x-y) < kt + kr < 2kt$.

Therefore,

$$M_\alpha^P f_2(y) = \sup_{t>0} |\mathcal{E}(y,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(y,t)\cap {}^{\complement}(\mathcal{E}(x,2kr))} |f(z)|dz$$
$$\leq \sup_{t>r} |\mathcal{E}(y,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,2kt)} |f(z)|dz$$
$$\lesssim \sup_{t>r} |\mathcal{E}(y,2kt)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,2kt)} |f(z)|dz = \sup_{t>2kr} |\mathcal{E}(y,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,t)} |f(z)|dz.$$

Hence by Lemma 2.6

$$\begin{aligned} M_\alpha^P f_2(y) &\lesssim \sup_{t>2kr} \frac{|\mathcal{E}(x,t)|}{|\mathcal{E}(y,t)|^{1-\frac{\alpha}{\gamma}}} \Phi^{-1}\big(|\mathcal{E}(x,t)|^{-1}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} \\ &\lesssim \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}. \end{aligned} \tag{4.7}$$

Thus the function $M_\alpha^P f_2(y)$, with fixed x and r, is dominated by the expression not depending on y. Then we integrate the obtained estimate for $M_\alpha^P f_2(y)$ in y over $\mathcal{E}$, we get

$$\|M_\alpha^P f_2\|_{L_\Psi(\mathcal{E})} \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}. \tag{4.8}$$

Gathering the estimates (4.6) and (4.8) we arrive at (4.4).

Let now Φ be an arbitrary Young function. It is obvious that

$$\|M_\alpha^P f\|_{WL_\Phi(\mathcal{E})} \leq \|M_\alpha^P f_1\|_{WL_\Phi(\mathcal{E})} + \|M_\alpha^P f_2\|_{WL_\Phi(\mathcal{E})}.$$

By the boundedness of the operator M_α^P from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$, provided by Theorem 3.2, we have

$$\|M_\alpha^P f_1\|_{WL_\Phi(\mathcal{E})} \lesssim \|f\|_{L_\Phi(\mathcal{E}(x,2kr))}.$$

By using (4.5), (4.7) and Lemma 2.5 we arrive at (4.3). □

Theorem 4.6 *Let* $0 \le \alpha < \gamma$, (Φ, Ψ) *be Young functions, and* $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$. *Let the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the conditions* (3.1) *and*

$$\sup_{r<t<\infty} \Psi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \le C\, \varphi_2(x,r), \tag{4.9}$$

where C *does not depend on* r. *Then the operator* M_α^P *is bounded from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $WM^P_{\Psi,\varphi_2}(\mathbb{R}^n)$ *and for* $\Phi \in \nabla_2$, *the operator* M_α^P *is bounded from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof Note that $\Big(\operatorname*{ess\,inf}_{x\in A} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in A} \frac{1}{f(x)}$ is true for any real-valued nonnegative function f and measurable on A and the fact that $\|f\|_{L_\Phi(\mathcal{E}(x,t))}$ is a nondecreasing function of t

$$\begin{aligned}\frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}\limits_{0<t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\left(s^{-\gamma}\right)}} &= \operatorname*{ess\,sup}_{0<t<s<\infty} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\varphi_1(x,s)} \\ &\le \sup_{x\in\mathbb{R}^n, r>0} \frac{\Phi^{-1}\big(s^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,s))}}{\varphi_1(x,s)} = \|f\|_{M^P_{\Phi,\varphi_1}}.\end{aligned}$$

Since (φ_1, φ_2) and (Φ, Ψ) satisfy the condition (4.9),

$$\begin{aligned}&\sup_{r<t<\infty} \|f\|_{L_\Phi(\mathcal{E}(x,t))} \Psi^{-1}\big(t^{-\gamma}\big) \\ &\le \sup_{r<t<\infty} \frac{\|f\|_{L_\Phi(\mathcal{E}(x,t))}}{\operatorname*{ess\,inf}\limits_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\left(s^{-\gamma}\right)}} \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \Psi^{-1}\big(t^{-\gamma}\big) \\ &\le C\|f\|_{M^P_{\Phi,\varphi_1}} \sup_{r<t<\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-\gamma}\big)}\right) \Psi^{-1}\big(t^{-\gamma}\big) \\ &\le C\varphi_2(x,r)\|f\|_{M^P_{\Phi,\varphi_1}}.\end{aligned} \tag{4.10}$$

Then by (4.9) and (4.10) we get

$$\|M_\alpha^P f\|_{M^P_{\Psi,\varphi_2}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \sup_{t>r} \Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x,t))} = \|f\|_{M^P_{\Phi,\varphi_1}}.$$

The estimate $\|M_\alpha^P f\|_{WM^P_{\Psi,\varphi_2}} \lesssim \|f\|_{M^P_{\Phi,\varphi_1}}$ can be proved similarly by the help of local estimate (4.3). □

Remark 4.7 Note that Theorem 4.6 in the isotropic case $P = I$ were proved in [11], see also [21].

For proving our main results, we need the following estimate.

Lemma 4.8 *If $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then $r_0^{\alpha} \le C M_{\alpha}^{P} \chi_{\mathcal{E}_0}(x)$ for every $x \in \mathcal{E}_0$.*

Proof It is well-known that

$$\mathrm{M}_{\alpha}^{P} f(x) \le 2^{\gamma-\alpha} M_{\alpha}^{P} f(x), \tag{4.11}$$

where $\mathrm{M}_{\alpha}^{P}(f)(x) = \sup\limits_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}} |f(y)| dy$.

Now let $x \in \mathcal{E}_0$. By using (4.11), we get

$$\begin{aligned} M_{\alpha}^{P} \chi_{\mathcal{E}_0}(x) \ge C\mathrm{M}_{\alpha}^{P} \chi_{\mathcal{E}_0}(x) &\ge C \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} |\mathcal{E} \cap \mathcal{E}_0| \\ &\ge C|\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}} |\mathcal{E}_0 \cap \mathcal{E}_0| = C r_0^{\alpha}. \end{aligned}$$

□

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator M_{α}^{P} from $M_{\Phi,\varphi_1}^{P}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^{P}(\mathbb{R}^n)$.

Theorem 4.9 *(Spanne-type result) Let $0 \le \alpha < \gamma$, (Φ, Ψ) be Young functions, and let $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$.*

1. *If the functions (Φ, Ψ) satisfy the condition (3.1), then the condition*

$$\sup_{t<r<\infty} \Psi^{-1}\big(r^{-\gamma}\big) \operatorname*{ess\,inf}_{r<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \le C\,\varphi_2(t), \tag{4.12}$$

   for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of M_{α}^{P} from $M_{\Phi,\varphi_1}^{P}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^{P}(\mathbb{R}^n)$.
2. *If the function $\varphi_1 \in \mathcal{G}_{\Phi}$, then the condition*

$$t^{\alpha}\varphi_1(t) \le C\varphi_2(t), \tag{4.13}$$

   for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of M_{α}^{P} from $M_{\Phi,\varphi_1}^{P}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}^{P}(\mathbb{R}^n)$.
3. *Let the functions (Φ, Ψ) satisfy the condition (3.1). If $\varphi_1 \in \mathcal{G}_{\Phi}$ satisfies the condition*

$$\sup_{t<r<\infty} \frac{\Psi^{-1}\big(r^{-\gamma}\big)}{\Phi^{-1}\big(r^{-\gamma}\big)} \varphi_1(r) \le C t^{\alpha} \varphi_1(t), \tag{4.14}$$

for all $t > 0$*, where* $C > 0$ *does not depend on* t*, then the condition* (4.13) *is necessary and sufficient for the boundedness of* M_α^P *from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof The first part of the theorem was proved in Theorem 4.6.

We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, t_0)$ and $x \in \mathcal{E}_0$. By Lemma 4.8 we have $t_0^\alpha \leq C M_\alpha \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 2.5 and Lemma 4.4

$$
\begin{aligned}
t_0^\alpha &\leq C\Psi^{-1}(|\mathcal{E}_0|^{-1}) \|M_\alpha^P \chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)} \leq C\varphi_2(t_0)\|M_\alpha^P \chi_{\mathcal{E}_0}\|_{M^P_{\Psi,\varphi_2}} \\
&\leq C\varphi_2(t_0)\|\chi_{\mathcal{E}_0}\|_{M^P_{\Phi,\varphi_1}} \leq C\frac{\varphi_2(t_0)}{\varphi_1(t_0)}
\end{aligned}
$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. ◻

Remark 4.10 Note that Theorem 4.9 in the isotropic case $P = I$ were proved in [9], see also [11].

5 Parabolic Fractional Maximal Commutators in Parabolic Generalized Orlicz–Morrey Spaces

In this section, we consider the boundedness of parabolic fractional maximal commutator operator $M^P_{b,\alpha}$ and commutator of parabolic fractional maximal operator $[b, M^P_\alpha]$ on parabolic generalized Orlicz–Morrey spaces, when b belongs to the BMO space, by which some new characterizations of BMO spaces are given.

Lemma 5.1 *Let* $0 \leq \alpha < \gamma$ *and* $b \in BMO(\mathbb{R}^n)$*. Let* Φ *be a Young function and* Ψ *defined by its inverse* $\Psi^{-1}(r) \approx \Phi^{-1}(r)\, r^{-\frac{\alpha}{\gamma}}$ *and* $\Phi, \Psi \in \Delta_2 \cap \nabla_2$*, then the inequality*

$$
\|M^P_{b,\alpha} f\|_{L_\Psi(\mathcal{E}(x_0,r))} \lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-\gamma}\big)\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}
$$

holds for any ball $\mathcal{E}(x_0, r)$ *and for all* $f \in L^{\rm loc}_\Phi(\mathbb{R}^n)$.

Proof For $\mathcal{E} = \mathcal{E}(x_0, r)$, write $f = f_1 + f_2$ with $f_1 = f\chi_{2k\mathcal{E}}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality, so that

$$
\left\|M^P_{b,\alpha} f\right\|_{L_\Psi(\mathcal{E})} \leq \left\|M^P_{b,\alpha} f_1\right\|_{L_\Psi(\mathcal{E})} + \left\|M^P_{b,\alpha} f_2\right\|_{L_\Psi(\mathcal{E})}.
$$

By the boundedness of the operator $M^P_{b,\alpha}$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ provided by Theorem 3.5, we obtain

$$\|M^P_{b,\alpha} f_1\|_{L_\Psi(\mathcal{E})} \le \|M^P_{b,\alpha} f_1\|_{L_\Psi(\mathbb{R}^n)} \lesssim \|b\|_* \, \|f_1\|_{L_\Phi(\mathbb{R}^n)} = \|b\|_* \, \|f\|_{L_\Phi(2k\mathcal{E})}. \tag{5.1}$$

As we proceed in the proof of Lemma 4.5, we have for $x \in \mathcal{E}$

$$M^P_{b,\alpha}(f_2)(x) \lesssim \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(y)-b(x)||f(y)|dy.$$

Then

$$\|M^P_{b,\alpha} f_2\|_{L_\Psi(\mathcal{E})} \lesssim \left\| \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(y)-b(\cdot)||f(y)|dy \right\|_{L_\Psi(\mathcal{E})}$$

$$\lesssim J_1 + J_2 = \left\| \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}}||f(y)|dy \right\|_{L_\Psi(\mathcal{E})}$$

$$+ \left\| \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(\cdot)-b_{\mathcal{E}}||f(y)|dy \right\|_{L_\Psi(\mathcal{E})}.$$

For the term J_1 by Lemma 2.5 we obtain

$$J_1 \approx \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}}||f(y)|dy$$

and split it as follows:

$$J_1 \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |b(y)-b_{\mathcal{E}(x_0,t)}||f(y)|dy$$

$$+ \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \int_{\mathcal{E}(x_0,t)} |f(y)|dy.$$

Applying Hölder's inequality, by Lemmas 2.6 and 3.3 and (3.3) we get

$$J_1 \lesssim \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \left\|b(\cdot)-b_{\mathcal{E}(x_0,t)}\right\|_{L_{\widetilde{\Phi}}(\mathcal{E}(x_0,t))} \|f\|_{L_\Phi(\mathcal{E}(x_0,t))}$$

$$+ \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} t^{-\gamma+\alpha} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \, t^{\gamma} \, \Phi^{-1}\big(t^{-\gamma}\big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))}$$

$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \sup_{t>2kr} \Psi^{-1}\big(t^{-\gamma}\big)\Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_\Phi(\mathcal{E}(x_0,t))}.$$

For J_2 we obtain

$$J_2 \approx \|b(\cdot)-b_B\|_{L_\Psi(\mathcal{E})} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x_0,t)} |f(y)|dy$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}(r^{-\gamma})} \sup_{t>2kr} \Psi^{-1}(t^{-\gamma})\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}$$

gathering the estimates for J_1 and J_2, we get

$$\|M^P_{b,\alpha} f_2\|_{L_\Psi(\mathcal{E})} \lesssim \frac{\|b\|_*}{\Psi^{-1}(r^{-\gamma})} \sup_{t>2kr} \Psi^{-1}(t^{-\gamma})\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}. \tag{5.2}$$

By using (4.5) we unite (5.2) with (5.1), which completes the proof. □

Theorem 5.2 *Let $0 \le \alpha < \gamma$, $b \in BMO(\mathbb{R}^n)$, (Φ,Ψ) be Young functions, and $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$. Let $\Psi^{-1}(r) \approx \Phi^{-1}(r)\, r^{-\frac{\alpha}{\gamma}}$, $\Phi,\Psi \in \Delta_2 \cap \nabla_2$, and the functions (φ_1,φ_2) and (Φ,Ψ) satisfy the condition*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}(t^{-\gamma}) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(s^{-\gamma})} \le C\,\varphi_2(x,r), \tag{5.3}$$

where C does not depend on x, r. Then the operator $M^P_{b,\alpha}$ is bounded from $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof The proof is similar to the proof of Theorem 4.6 thanks to Lemma 5.1. □

Remark 5.3 Note that Theorem 5.2 in the isotropic case $P = I$ were proved in [11].

For proving our main results, we need the following estimate.

Lemma 5.4 *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E}_0 := \mathcal{E}(x_0,r_0)$, then*

$$r_0^\alpha |b(x)-b_{\mathcal{E}_0}| \le C M^P_{b,\alpha}\chi_{\mathcal{E}_0}(x) \ \ \textit{for every} \ \ x \in \mathcal{E}_0.$$

Proof It is well-known that

$$\mathrm{M}^P_{b,\alpha} f(x) \le 2^{\gamma-\alpha} M^P_{b,\alpha} f(x), \tag{5.4}$$

where $\mathrm{M}^P_{b,\alpha}(f)(x) = \sup\limits_{\mathcal{E}\ni x} |\mathcal{E}|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}} |b(x)-b(y)||f(y)|dy$.

Now let $x \in \mathcal{E}_0$. By using (5.4), we get

$$M^P_{b,\alpha}\chi_{\mathcal{E}_0}(x) \geq C\mathrm{M}^P_{b,\alpha}f(x) = C\sup_{B\ni x}|B|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}}|b(x)-b(y)|\chi_{\mathcal{E}_0}dy$$
$$= C\sup_{\mathcal{E}\ni x}|\mathcal{E}|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}\cap\mathcal{E}_0}|b(x)-b(y)|dy \geq C|\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}_0\cap\mathcal{E}_0}|b(x)-b(y)|dy$$
$$\geq \Big|C|\mathcal{E}_0|^{-1+\frac{\alpha}{\gamma}}\int_{\mathcal{E}_0}(b(x)-b(y))dy\Big| = Cr_0^\alpha|b(x)-b_{\mathcal{E}_0}|.$$

□

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator $M^P_{b,\alpha}$ from $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Theorem 5.5 *Let* $0 \leq \alpha < \gamma$, $b \in BMO(\mathbb{R}^n)$, (Φ,Ψ) *be Young functions, and let* $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$.

1. *Let* $\Psi^{-1}(t) \approx t^{-\alpha/\gamma}\,\Phi^{-1}(t)$ *and* $\Phi,\Psi \in \Delta_2 \cap \nabla_2$, *then the condition*

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Psi^{-1}\big(t^{-\gamma}\big)\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \leq C\,\varphi_2(r), \tag{5.5}$$

   for all $r > 0$, *where* $C > 0$ *does not depend on* r, *is sufficient for the boundedness of* $M^P_{b,\alpha}$ *from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.
2. *If* $\Psi \in \Delta_2$ *and* $\varphi_1 \in \mathcal{G}_\Phi$, *then the condition* (4.13) *is necessary for the boundedness of* $M^P_{b,\alpha}$ *from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.
3. *Let* $\Psi^{-1}(t) \approx t^{-\alpha/\gamma}\,\Phi^{-1}(t)$ *and* $\Phi,\Psi \in \Delta_2 \cap \nabla_2$. *If* $\varphi_1 \in \mathcal{G}_\Phi$ *satisfies the condition*

$$\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)t^\alpha\,\varphi_1(t) \leq Cr^\alpha\varphi_1(r), \tag{5.6}$$

   for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* (4.13) *is necessary and sufficient for the boundedness of* $M^P_{b,\alpha}$ *from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof The first part of the theorem is a corollary of Theorem 5.2.

We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemma 5.4 we have $r_0^\alpha|b(x)-b_{\mathcal{E}_0}| \leq CM^P_{b,\alpha}\chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 3.3 and Lemma 4.4

$$r_0^\alpha \leq C\frac{\|M^P_{b,\alpha}\chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)}}{\|b(\cdot)-b_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)}} \leq \frac{C}{\|b\|_*}\|M^P_{b,\alpha}\chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)}\Psi^{-1}(|\mathcal{E}_0|^{-1})$$
$$\leq \frac{C}{\|b\|_*}\varphi_2(r_0)\|M^P_{b,\alpha}\chi_{\mathcal{E}_0}\|_{M^P_{\Psi,\varphi_2}} \leq C\varphi_2(r_0)\|\chi_{\mathcal{E}_0}\|_{M^P_{\Phi,\varphi_1}} \leq C\frac{\varphi_2(r_0)}{\varphi_1(r_0)}.$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

Remark 5.6 Note that Theorem 5.5 in the isotropic case $P = I$ were proved in [9], see also [11].

By (3.2) and Theorems 4.6 and 5.5 we get the following corollary.

Corollary 5.7 *Let* $0 \le \alpha < \gamma$, $b \in BMO(\mathbb{R}^n)$, (Φ, Ψ) *be Young functions, and let* $\varphi_1 \in \Omega_{\Phi,P}$, $\varphi_2 \in \Omega_{\Psi,P}$. *Let also* $b^- \in L_\infty(\mathbb{R}^n)$, $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ *and* $\Psi^{-1}(r) \approx r^{-\frac{\alpha}{\gamma}}\, \Phi^{-1}(r)$. *If the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the condition* (5.3)*, then the operator* $[b, M_\alpha^P]$ *is bounded from* $M^P_{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $M^P_{\Psi,\varphi_2}(\mathbb{R}^n)$.

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Eigenvalues of Tridiagonal Hermitian Toeplitz Matrices with Perturbations in the Off-diagonal Corners

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Abstract In this paper we study the asymptotic behavior of the eigenvalues of Hermitian Toeplitz matrices with the entries $2, -1, 0, \ldots, 0, -\alpha$ in the first column. Notice that the generating symbol depends on the order n of the matrix. This matrix family is a particular case of periodic Jacobi matrices. If $|\alpha| \leq 1$, then the eigenvalues belong to $[0, 4]$ and are asymptotically distributed as the function $g(x) = 4\sin^2(x/2)$ on $[0, \pi]$. The situation changes drastically when $|\alpha| > 1$ and n tends to infinity. For this case, we prove that the two extreme eigenvalues (the minimal and the maximal one) lay out of $[0, 4]$ and converge rapidly to certain limits determined by the value of α, whilst all others belong to $[0, 4]$ and are asymptotically distributed as g. In all cases, we derive asymptotic formulas for the eigenvalues and transform the characteristic equation to a form convenient to solve by numerical methods.

Keywords Eigenvalue · Periodic Jacobi matrix · Toeplitz matrix · Tridiagonal matrix · Perturbation · Asymptotic expansion

Mathematics Subject Classification (2010) 15B05, 47B36, 15A18, 41A60, 65F15, 47A55

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1 Introduction

In this paper we analyze the eigenvalues of $n \times n$ matrices $A_{\alpha,n}$, depending on the complex parameter α, of the following form:

$$A_{\alpha,6} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Such matrices may appear in the study of one-dimensional shift-invariant models on a finite interval, with some special interactions between the extremes of the interval. This matrix family belongs simultaneously to three matrix classes:

- periodic Jacobi matrices (in this case, with constant diagonals);
- Hermitian Toeplitz matrices; notice that the generating symbol of $A_{\alpha,n}$ is the following Laurent polynomial depending on the parameters α and n:

$$-\overline{\alpha}t^{-n+1} - t^{-1} + 2 - t - \alpha t^{n-1}; \qquad (1)$$

- perturbed tridiagonal Toeplitz matrices, with values $-1, 2, -1$ on the diagonals and perturbations in the off-diagonal corners $(n, 1)$ and $(1, n)$.

Periodic Jacobi matrices were studied by Ferguson, da Fonseca, and other authors [12, 13]. In particular, they found formulas for the characteristic polynomial and analyzed the inverse eigenvalue problem.

Toeplitz matrices are naturally associated to discrete truncated shift-invariant models. The general theory of such matrices is explained in the books and reviews [6–8, 11, 17, 18]. Explicit formulas for the determinants of banded symmetric Toeplitz matrices were found in [22]. The determinants, minors, cofactors, and components of eigenvectors of banded Toeplitz matrices were recently expressed in terms of skew Schur polynomials, see [1, 19]. These formulas are not needed in the present paper, but may be useful in the study of pentadiagonal Toeplitz matrices with perturbations on the corners.

The individual behavior of the eigenvalues of Hermitian Toeplitz matrices was investigated in [2–5, 10].

Determinants of non-singular Toeplitz matrices with low-rank perturbations were studied in [6]. The eigenvalues and eigenvectors of tridiagonal Toeplitz matrices with some special perturbations on the diagonal corners are computed in [9, Section 1.1] and [14, 20]. The determinants and inverses of a family of non-symmetric tridiagonal Toeplitz matrices with perturbed corners are computed in [24].

The localization of the eigenvalues of a family of non-Hermitian Jacobi matrices (which can be viewed as a family of tridiagonal Toeplitz matrices with perturbation in the position $(n, n-1)$) was studied in [15].

Yueh and Cheng [26] considered the tridiagonal Toeplitz matrices with four perturbed corners. Using the techniques of finite differences they derived the characteristic equation in a trigonometric form and formulas for the eigenvectors, in terms of the eigenvalues. Unlike the present paper, [26] deals with arbitrary complex coefficients, but does not contain the analysis of the localization of the eigenvalues nor approximate formulas for the eigenvalues.

The generating symbol (1) of our matrices $A_{\alpha,n}$, after the change of variable $t = \exp(ix)$, results in

$$h_{\alpha,n}(x) = 4\sin^2(x/2) - 2\,\mathrm{Re}(\alpha\, e^{i(n-1)x}).$$

For $\alpha = 0$, the matrix $A_{\alpha,n}$ is the well studied tridiagonal Toeplitz matrix with the symbol $g := h_{0,n}$, i.e.,

$$g(x) := 4\sin^2(x/2). \tag{2}$$

The characteristic polynomial of $A_{0,n}$ is $\det(\lambda I_n - A_{0,n}) = U_n((\lambda - 2)/2)$, where U_n is the nth Chebyshev polynomial of the second type, and the eigenvalues of $A_{0,n}$ are $g(j\pi/(n+1))$, $1 \le j \le n$. For general α, the characteristic polynomial of $A_{\alpha,n}$ can be expressed in terms of U_n and U_{n-2}. We are able to compute the eigenvalues of $A_{\alpha,n}$ explicitly only for $|\alpha| = 1$ or $\alpha = 0$. For $|\alpha| \neq 1$, applying an appropriate trigonometric or hyperbolic change of variable, we describe the localization of the eigenvalues, transform the characteristic equation to a form that can be solved by the fixed point iteration, and get asymptotic formulas.

It turns out that the cases $|\alpha| < 1$ ("weak perturbations") and $|\alpha| > 1$ ("strong perturbations") are essentially different: if $|\alpha| > 1$ and n is large enough, then the extreme eigenvalues go outside the interval $[0, 4]$ and need a special treatment. Below we present the corresponding results separately, starting with the simpler case $|\alpha| < 1$. In Sects. 3 and 5 we give the corresponding proofs. The case $|\alpha| = 1$ can be viewed as a limit of the case $|\alpha| < 1$, but the results in this case are much simpler, see Sect. 4. Finally, in Sect. 6 we discuss some numerical tests.

2 Main Results

The matrices $A_{\alpha,n}$ are Hermitian, their eigenvalues are real, and we enumerate them in the ascending order:

$$\lambda_{\alpha,n,1} \le \lambda_{\alpha,n,2} \le \cdots \le \lambda_{\alpha,n,n}.$$

2.1 Main Results for Weak Perturbations

Recall that the function g is defined by (2). It strictly increases on $[0, \pi]$ taking values from 0 to 4.

Theorem 1 (Localization of the Eigenvalues for Weak Perturbations) *Let* $\alpha \in \mathbb{C}$, $|\alpha| \le 1$, $\alpha \notin \{-1, 1\}$, *and* $n \ge 3$. *Then the matrix* $A_{\alpha,n}$ *has* n *different eigenvalues belonging to* $(0, 4)$. *More precisely, for every* j *in* $\{1, \ldots, n\}$,

$$g\left(\frac{(j-1)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\frac{j\pi}{n}\right). \tag{3}$$

The minimal value of the generating function $h_{\alpha,n}$, for $|\alpha| \le 1$, may be strictly negative. So, even the inequality $\lambda_{\alpha,n,1} > 0$, which is a very small part of Theorem 1, is not obvious.

Theorem 1 implies that the eigenvalues of $A_{\alpha,n}$, as n tends to ∞, are asymptotically distributed as the values of the function g. This follows also from the theory of locally Toeplitz sequences [16, 21, 23] or, more specifically, from Cauchy's interlacing theorem, since the matrices $A_{\alpha,n}$ are obtained from the tridiagonal Toeplitz matrices $A_{0,n}$ by low-rank perturbations.

Our next goal is to transform the characteristic equation into a convenient form. For every α in $\mathbb{C}$ with $\alpha \notin \{-1, 1\}$ and every integer j we define the function $\eta_{\alpha,j}\colon [0, \pi] \to \mathbb{R}$ by

$$\eta_{\alpha,j}(x) := -2 \arctan\left(\left((-1)^{j+1} k_\alpha \cot(x) + \sqrt{k_\alpha^2 \cot^2(x) + l_\alpha^2}\right)^{(-1)^j}\right), \tag{4}$$

where

$$k_\alpha := \frac{1 - |\alpha|^2}{|1 + \alpha|^2}, \qquad l_\alpha := \frac{|1 - \alpha|}{|1 + \alpha|}. \tag{5}$$

In fact, $\eta_{\alpha,j}$ depends only on α and on the parity of j. Thus, for every α there are only two different functions: $\eta_{\alpha,1}$ and $\eta_{\alpha,2}$. These functions take values in $[-\pi, 0]$. See a couple of examples on Fig. 1.

Motivated by (3), we use the function g as a change of variable in the characteristic equation and put $\vartheta_{\alpha,n,j} := g^{-1}(\lambda_{\alpha,n,j})$. Inequality (3) is equivalent to

$$\frac{(j-1)\pi}{n} < \vartheta_{\alpha,n,j} < \frac{j\pi}{n}. \tag{6}$$

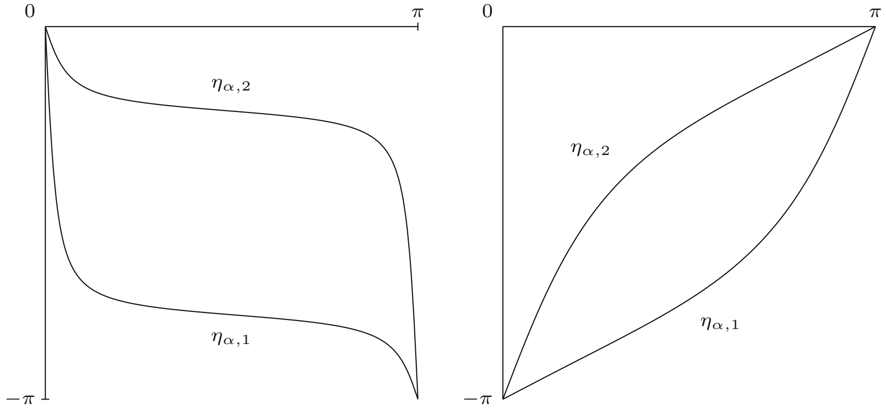

Fig. 1 Functions (4) for $\alpha = 0.7 + 0.6i$ (left) and $\alpha = 2 + i$ (right)

Theorem 2 (Characteristic Equation for Weak Perturbations) *Let* $\alpha \in \mathbb{C}$, $|\alpha| \le 1$, $\alpha \notin \{-1, 1\}$, $n \ge 3$, *and* $1 \le j \le n$. *Then the number* $\vartheta_{\alpha,n,j}$ *satisfies*

$$\vartheta_{\alpha,n,j} = \frac{j\pi + \eta_{\alpha,j}(\vartheta_{\alpha,n,j})}{n}. \tag{7}$$

For every α in $\mathbb{C}$ with $|\alpha| \ne 1$, we put

$$N_1(\alpha) := \frac{|\alpha - 1|^2}{||\alpha|^2 - 1|},\ N_2(\alpha) := \frac{|\alpha + 1|^2}{||\alpha|^2 - 1|},\ N_3(\alpha) := \max\{N_1(\alpha), N_2(\alpha)\}. \tag{8}$$

In Sect. 3 we show that for $n > N_3(\alpha)$ the function in the right-hand side of (7) is contractive. Hence, Eq. (7) can be solved by the fixed point iteration. Furthermore, we use (7) to derive asymptotic formulas for the eigenvalues.

For $|\alpha| \le 1$ and $1 \le j \le n$, define $\lambda_{\alpha,n,j}^{\text{asympt}}$ by

$$\begin{aligned}\lambda_{\alpha,n,j}^{\text{asympt}} &:= g\left(\frac{j\pi}{n}\right) + \frac{g'\left(\frac{j\pi}{n}\right)\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} \\ &+ \frac{g'\left(\frac{j\pi}{n}\right)\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)\eta'_{\alpha,j}\left(\frac{j\pi}{n}\right) + \frac{1}{2}g''\left(\frac{j\pi}{n}\right)\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)^2}{n^2}.\end{aligned} \tag{9}$$

Theorem 3 (Asymptotic Expansion of the Eigenvalues for Weak Perturbations) *Let $\alpha \in \mathbb{C}$, $|\alpha| \le 1$. Then there exists $C_1(\alpha) > 0$ not depending on n such that for n large enough and $1 \le j \le n$,*

$$|\lambda_{\alpha,n,j} - \lambda_{\alpha,n,j}^{\mathrm{asympt}}| \le \frac{C_1(\alpha)}{n^3}. \tag{10}$$

In other words, Theorem 3 claims that $\lambda_{\alpha,n,j} = \lambda_{\alpha,n,j}^{\mathrm{asympt}} + O_\alpha\left(\frac{1}{n^3}\right)$, where the constant in the upper estimate of the residue term depends on α. For simplicity, we state and justify only this asymptotic formula with three exact terms, but there are similar formulas with more terms.

Since the eigenvectors were found in [26] for a more general matrix family, we give the corresponding formulas in Propositions 8 and 16 without proofs.

2.2 Main Results for Strong Perturbations

Recall that $N_1(\alpha)$, $N_2(\alpha)$, and $N_3(\alpha)$ are defined by (8).

Theorem 4 (Localization of the Eigenvalues for Strong Perturbations) *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and $n \ge 3$.*

1. *If $n > N_1(\alpha)$, then $\lambda_{\alpha,n,1} < 0$, else $0 \le \lambda_{\alpha,n,1} < g(\pi/n)$.*
2. *If n is odd and $n > N_1(\alpha)$, or n is even and $n > N_2(\alpha)$, then $\lambda_{\alpha,n,n} > 4$. In the other case, $g((n-1)\pi/n) < \lambda_{\alpha,n,n} \le 4$.*
3. *For $2 \le j \le n-1$, the eigenvalues $\lambda_{\alpha,n,j}$ belong to $(0,4)$ and satisfy (3).*

So, if $n \ge N_3(\alpha)$, then both extreme eigenvalues go outside the segment [0, 4]:

$$\lambda_{\alpha,n,1} < 0, \qquad \lambda_{\alpha,n,n} > 4.$$

In order to solve the characteristic equation for $\lambda < 0$ and $\lambda > 4$, we use the following changes of variables, respectively:

$$g_-(x) := -4\sinh^2(x/2), \qquad g_+(x) := 4 + 4\sinh^2(x/2) \qquad (x > 0). \tag{11}$$

Theorem 5 (Characteristic Equations for Strong Perturbations) *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and $n \ge N_3(\alpha)$. Then*

$$\lambda_{\alpha,n,1} = g_-(\vartheta_{\alpha,n,1}), \qquad \lambda_{\alpha,n,n} = g_+(\vartheta_{\alpha,n,n}),$$

where $\vartheta_{\alpha,n,1}$ is the unique positive solution of the equation

$$x = \operatorname{arctanh} \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha+1|^2\tanh^2\frac{nx}{2} + |\alpha-1|^2}, \tag{12}$$

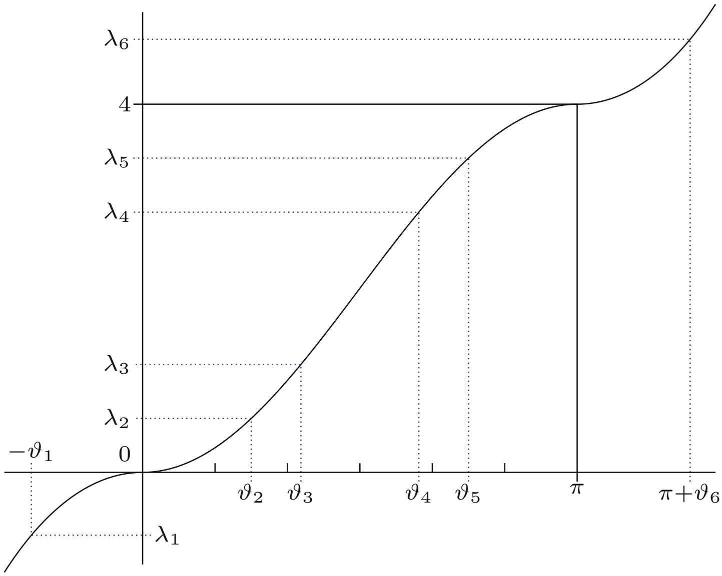

Fig. 2 The points $\vartheta_{\alpha,n,j}$ and the corresponding values of $\lambda_{\alpha,n,j}$ for $\alpha = 2 + i$, $n = 6$. Different scales are used for the axis

and $\vartheta_{\alpha,n,n}$ *is the unique positive solution of the equation*

$$x = \operatorname{arctanh} \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + (-1)^n|^2 \tanh^2 \frac{nx}{2} + |\alpha - (-1)^n|^2}. \tag{13}$$

For $2 \le j \le n - 1$*, the eigenvalues* $\lambda_{\alpha,n,j}$ *can be found as in Theorem 2.*

Figure 2 shows an example. In this figure, we glued together three changes of variables (g_-, g, and g_+) into one spline, using appropriate shifts or reflections.

There is another way to see the changes of variables (11): after extending g to an entire function, $g_-(x) = g(ix)$ and $g_+(x) = g(\pi + ix)$. So, the eigenvalues are obtained by evaluating the function g at some points belonging to a piecewise linear path on the complex plane, see Fig. 3.

In Sect. 5 we prove that for $n \ge N_4(\alpha)$, where

$$N_4(\alpha) := \frac{20\log(|\alpha| + 1) - 4\log(\log(|\alpha|))}{\log|\alpha|}, \tag{14}$$

the right-hand sides of (12) and (13) are contractive functions on the segment

$$S_\alpha := \left[\frac{\log|\alpha|}{2}, \frac{3\log|\alpha|}{2}\right]. \tag{15}$$

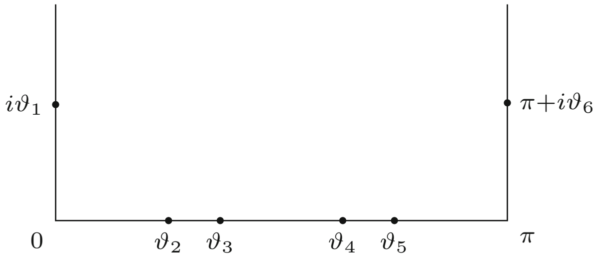

Fig. 3 Points on the complex plane that yield the eigenvalues $\lambda_{\alpha,n,j}$ after applying the function g. In this example, $\alpha = 2 + i$ and $n = 6$

In order to describe the asymptotic behavior of the extreme eigenvalues $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$, we introduce the following notation:

$$s_\alpha := |\alpha| - 2 + \frac{1}{|\alpha|}, \quad \text{i.e.} \quad s_\alpha = \frac{(|\alpha|-1)^2}{|\alpha|} = \left(\sqrt{|\alpha|} - \frac{1}{\sqrt{|\alpha|}}\right)^2. \tag{16}$$

Theorem 6 (Asymptotic Expansion of the Eigenvalues for Strong Perturbations) *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$. As n tends to infinity, the extreme eigenvalues of $A_{\alpha,n}$ converge exponentially to $-s_\alpha$ and $4 + s_\alpha$, respectively:*

$$|\lambda_{\alpha,n,1} + s_\alpha| \le \frac{C_2(\alpha)}{|\alpha|^n}, \tag{17}$$

$$|\lambda_{\alpha,n,n} - 4 - s_\alpha| \le \frac{C_2(\alpha)}{|\alpha|^n}. \tag{18}$$

Here $C_2(\alpha)$ is a positive constant depending only on α. For $2 \le j \le n-1$, the eigenvalues $\lambda_{\alpha,n,j}$ satisfy the asymptotic formulas (10).

According to Theorem 6, for $|\alpha| > 1$ we define $\lambda^{\text{asympt}}_{\alpha,n,j}$ by (9) when $2 \le j \le n-1$, and for $j = 1$ or $j = n$ we put

$$\lambda^{\text{asympt}}_{\alpha,n,1} := -s_\alpha, \qquad \lambda^{\text{asympt}}_{\alpha,n,j} := 4 + s_\alpha. \tag{19}$$

Formulas (9) and (19) yield very efficient approximations of the eigenvalues, when n is large enough.

Theorem 6 implies that for a fixed α with $|\alpha| > 1$ and $n \to \infty$, the number s_α is the "asymptotical lower spectral gap" and also the "asymptotical upper spectral gap" of the matrices $A_{\alpha,n}$, in the following sense:

$$\lim_{n\to\infty} (\lambda_{\alpha,n,2} - \lambda_{\alpha,n,1}) = 0 - (-s_\alpha) = s_\alpha,$$
$$\lim_{n\to\infty} (\lambda_{\alpha,n,n} - \lambda_{\alpha,n,n-1}) = (4 + s_\alpha) - 4 = s_\alpha.$$

Recall that for a sequence of Toeplitz matrices, generated by a bounded real-valued symbol *not depending on* n, the eigenvalues asymptotically fill the whole interval between the essential infimum and the essential supremum of the symbol, see [25, formula (3)]. In the case of continuous real-valued symbols, this also follows from the Szegő's theorem about the asymptotic distribution of the eigenvalues, see [18]. Nevertheless, a splitting phenomenon is known for the singular values of some sequences of non-Hermitian Toeplitz matrices [8, Section 4.3].

3 Proofs for the Case of Weak Perturbations

Denote by U_n the Chebyshev polynomial of the second kind of degree n. It is well known that

$$U_n(\cos(x)) = \frac{\sin((n+1)x)}{\sin(x)}, \quad U_n(\cosh(x)) = \frac{\sinh((n+1)x)}{\sinh(x)}. \tag{20}$$

In what follows, we denote by $D_{\alpha,n}(\lambda)$ the characteristic polynomial of $A_{\alpha,n}$:

$$D_{\alpha,n}(\lambda) \coloneqq \det(\lambda I_n - A_{\alpha,n}).$$

The next proposition is a particular case of [13, Corollary 2.4]; it is also easy to prove directly expanding by cofactors.

Proposition 1 *For every* α, λ *in* $\mathbb{C}$ *and* $n \geq 3$,

$$D_{\alpha,n}(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) - |\alpha|^2 U_{n-2}\left(\frac{\lambda-2}{2}\right) - 2(-1)^n \operatorname{Re}(\alpha). \tag{21}$$

Using (21) and (20), in the next proposition we compute the values of $D_{\alpha,n}$ at the points $g(j\pi/n)$. We also indicate the signs of this monic polynomial at $+\infty$ and $-\infty$.

Proposition 2 *Let* $\alpha \in \mathbb{C}$ *and* $n \geq 3$. *Then*

$$D_{\alpha,n}(0) = (-1)^n \left(n(1-|\alpha|^2) + |1-\alpha|^2\right), \tag{22}$$

$$D_{\alpha,n}(4) = n(1-|\alpha|^2) + |1-(-1)^n\alpha|^2, \tag{23}$$

$$D_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = (-1)^{n+j}\left|\alpha - (-1)^j\right|^2 \qquad (1 \leq j \leq n-1). \tag{24}$$

Moreover,

$$\lim_{\lambda\to+\infty} D_{\alpha,n}(\lambda) = +\infty, \qquad \lim_{\lambda\to-\infty} \left((-1)^n D_{\alpha,n}(\lambda)\right) = +\infty. \tag{25}$$

Proposition 2 implies that for $|\alpha| < 1$ and $n \geq 3$, the numbers $D_{\alpha,n}(g(j\pi/n))$, $0 \leq j \leq n$ are nonzero, and their signs alternate. By the intermediate value theorem, this yields the result of Theorem 1.

If $\lambda \in (0, 4)$, then we use the trigonometric change of variables $\lambda = g(x)$ in the characteristic polynomial.

Proposition 3 *Let* $\alpha \in \mathbb{C}$, $\alpha \notin \{-1, 1\}$, $n \geq 3$, $x \in (0, \pi)$. *Then*

$$D_{\alpha,n}(g(x)) = \frac{(-1)^{n+1}|\alpha+1|^2\left(\tan^2\frac{nx}{2} - 2k_\alpha\cot(x)\tan\frac{nx}{2} - l_\alpha^2\right)}{1+\tan^2\frac{nx}{2}}. \tag{26}$$

Proof We start with (21), write λ as $2 - 2\cos(x)$, use the parity or imparity of U_n and the trigonometric formula (20) for U_n:

$$U_n(-\cos(x)) = (-1)^n U_n(\cos(x)) = (-1)^n \frac{\sin((n+1)x)}{\sin(x)}.$$

Then

$$D_{\alpha,n}(g(x)) = \frac{(-1)^n}{\sin(x)}\Big(\sin((n+1)x) - |\alpha|^2\sin((n-1)x) - 2\operatorname{Re}(\alpha)\sin(x)\Big). \tag{27}$$

Applying the trigonometric identities

$$\sin((n\pm1)x) = \sin(nx)\cos(x) \pm \cos(nx)\sin(x),$$

$$\sin(nx) = \frac{2\tan\frac{nx}{2}}{1+\tan^2\frac{nx}{2}}, \qquad \cos(nx) = \frac{1-\tan^2\frac{nx}{2}}{1+\tan^2\frac{nx}{2}},$$

and regrouping the summands, we get

$$D_{\alpha,n}(g(x)) = \frac{(-1)^{n+1}}{1+\tan^2\frac{nx}{2}} \times$$
$$\times\left(|\alpha+1|^2\tan^2\frac{nx}{2} - 2(1-|\alpha|^2)\cot(x)\tan\frac{nx}{2} - |\alpha-1|^2\right), \tag{28}$$

which is equivalent to (26). □

Notice that (27), up to a nonzero factor, is a particular case of the expression that appears in [26, eq. (3.10)].

For $|\alpha| < 1$ and j in $\mathbb{Z}$, we define $u_{\alpha,j}\colon (0,\pi) \to \mathbb{R}$ by

$$u_{\alpha,j}(x) := k_\alpha\cot(x) + (-1)^{j+1}\sqrt{k_\alpha^2\cot^2(x) + l_\alpha^2}. \tag{29}$$

For x in $(0, \pi)$, different from the points $j\pi/n$, the expression $\frac{(-1)^{n+1}|\alpha+1|^2}{1+\tan^2\frac{nx}{2}}$ takes finite nonzero values. Omitting this factor, we consider the right-hand side of (26) as a quadratic polynomial in $\tan\frac{nx}{2}$, with coefficients depending on α and x. The roots of this quadratic polynomial are $u_{\alpha,1}(x)$ and $u_{\alpha,2}(x)$. So, the characteristic equation $D_{\alpha,n}(g(x)) = 0$ is equivalent to the disjunction of the equations

$$\tan\frac{nx}{2} = u_{\alpha,1}(x), \qquad \tan\frac{nx}{2} = u_{\alpha,2}(x). \tag{30}$$

For each $n \geq 3$ and $1 \leq j \leq n$, we denote by $I_{n,j}$ the open interval $\left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n}\right)$.

Proposition 4 *Let $|\alpha| < 1$, $n \geq 3$, and $1 \leq j \leq n$. Then the equation*

$$\tan\frac{nx}{2} = u_{\alpha,j}(x) \tag{31}$$

has a unique solution x in $I_{n,j}$, and the corresponding value $g(x)$ is $\lambda_{\alpha,n,j}$.

Proof Applying the intermediate value theorem, it is easy to see that the first of the Eqs. (30) has a solution on each interval $I_{n,j}$ with j odd, and the second one has a solution on each interval $I_{n,j}$ with j even. The uniqueness of the solutions follows from Theorem 1, but it can also be verified directly, since the first derivative of $u_{\alpha,j}$ is negative. Figure 4 illustrates the ideas of this proof. □

Proposition 4 implies Theorem 2 and provides an alternative proof of Theorem 1, for $|\alpha| < 1$.

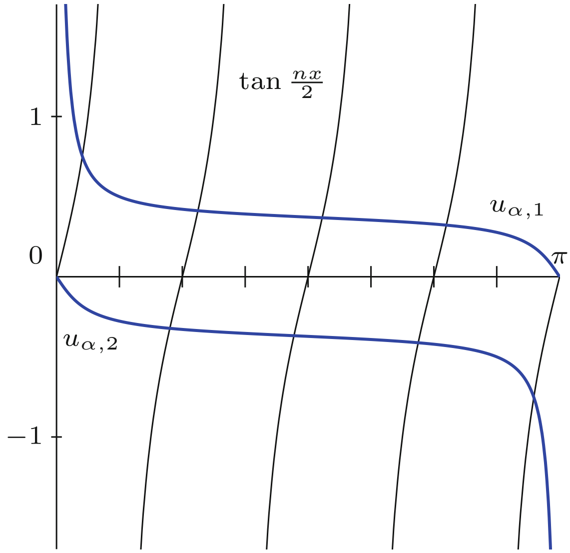

Fig. 4 Left-hand sides (black) and right-hand sides (blue) of Eqs. (31) for $\alpha = 0.7 + 0.6i$, $n = 8$, $1 \leq j \leq n$. Notice that $[0, \pi]$ is divided into 8 equal subintervals, and each subinterval corresponds to its proper equation

Recall that η is defined by (4). A straightforward computation yields

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1 + \frac{(-1)^{j+1}k_\alpha \cot(x)}{\sqrt{k_\alpha^2 \cot^2(x)+l_\alpha^2}}\right)\left(1+\cot^2(x)\right)}{1 + \left((-1)^{j+1}k_\alpha \cot(x) + \sqrt{k_\alpha^2 \cot^2(x) + l_\alpha^2}\right)^2}. \tag{32}$$

Equivalently,

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1 + \frac{(-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x))}{\sqrt{k_\alpha^2+l_\alpha^2 \tan^2(x)}}\right)\left(1+\tan^2(x)\right)}{\tan^2(x) + \left((-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x)) + \sqrt{k_\alpha^2 + l_\alpha^2 \tan^2(x)}\right)^2}. \tag{33}$$

Proposition 5 *Let $\alpha \in \mathbb{C}$, $|\alpha| \neq 1$, $j \in \mathbb{Z}$. Then each derivative of $\eta_{\alpha,j}$ is a bounded function on $(0, \pi)$. In particular,*

$$\sup_{0<x<\pi} |\eta'_{\alpha,j}(x)| = N_3(\alpha). \tag{34}$$

Proof Case I: $(-1)^{j+1}k_\alpha \tan(x) > 0$, i.e., $(-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x)) = |k_\alpha|$. Let us denote $\tan^2(x)$ by t. Then (33) simplifies to

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1 + t + \frac{|k_\alpha|\,(1+t)}{\sqrt{k_\alpha^2+l_\alpha^2 t}}\right)}{2k_\alpha^2 + (1+l_\alpha^2)t + 2|k_\alpha|\sqrt{k_\alpha^2 + l_\alpha^2 t}}. \tag{35}$$

Since $l_\alpha \geq |k_\alpha|$, $k_\alpha^2 + l_\alpha^2 t \geq k_\alpha^2(1+t)$, and $1 + l_\alpha^2 \geq 2k_\alpha^2$,

$$|\eta'_{\alpha,j}(x)| \leq \frac{2|k_\alpha|\,(1+t+\sqrt{1+t})}{2k_\alpha^2\,(1+t+\sqrt{1+t})} = \frac{1}{|k_\alpha|} = N_1(\alpha).$$

Case II: $(-1)^{j+1}k_\alpha \tan(x) < 0$, i.e., $(-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x)) = -|k_\alpha|$. Denote again $\tan^2(x)$ by t. Then, using the identity

$$\left(\sqrt{k_\alpha^2 + l_\alpha^2 t} - |k_\alpha|\right)\left(\sqrt{k_\alpha^2 + l_\alpha^2 t} + |k_\alpha|\right) = l_\alpha^2 t,$$

we transform (33) to

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha l_\alpha^2(1+t)}{\left(1 + l_\alpha^2 + \frac{(1-l_\alpha^2)|k_\alpha|}{\sqrt{k_\alpha^2+l_\alpha^2 t}}\right)\left(k_\alpha^2 + l_\alpha^2 t\right)}. \tag{36}$$

Since $k_\alpha^2 + l_\alpha^2 t \geq k_\alpha^2(1+t)$,

$$|\eta'_{\alpha,j}(x)| \leq \frac{2l_\alpha^2}{|k_\alpha|} \cdot \frac{1}{1 + l_\alpha^2 + \frac{(1-l_\alpha^2)|k_\alpha|}{\sqrt{k_\alpha^2 + l_\alpha^2 t}}}. \tag{37}$$

Denote by $R(t)$ the expression in the right-hand side of (37). If $l_\alpha \geq 1$, then R decreases, and

$$\sup_{0<t<+\infty} R(t) = \lim_{t\to 0^+} R(t) = \frac{2l_\alpha^2}{|k_\alpha|} \cdot \frac{1}{2} = N_2(\alpha).$$

If $l_\alpha < 1$, then R increases, and

$$\sup_{0<t<+\infty} R(t) = \lim_{t\to +\infty} R(t) \leq \frac{2l_\alpha^2}{|k_\alpha|} \cdot \frac{1}{1+l_\alpha^2} < \frac{1}{|k_\alpha|} = N_1(\alpha).$$

In both cases I and II, the inequality $\leq$ in (34) is proven.

The limit values of $|\eta'_{\alpha,j}|$ at the points 0 and π can be computed by applying (35) and (36), and coincide with $N_1(\alpha)$ and $N_2(\alpha)$, or vice versa, depending on the sign of $(-1)^{j+1}k_\alpha$. This implies the inverse inequality $\geq$ in (34).

For the higher derivatives of $\eta_{\alpha,j}$, the explicit estimates are too tedious, and we purpose the following argument. By (32), $\eta'_{\alpha,j}$ is analytic in a neighborhood of x, for any x in $(0,\pi)$. Moreover, formulas (35) and (36) show that $\eta'_{\alpha,j}$ has an analytic extension in some neighborhoods of the points 0 and π. Hence, $\eta'_{\alpha,j}$ has an analytic extension to a certain open set in the complex plane containing the segment $[0,\pi]$. Therefore, this function and all their derivatives are bounded on $(0,\pi)$. □

Let $f_{\alpha,n,j}$ be the function defined on $[0,\pi]$ by the right-hand side of (7):

$$f_{\alpha,n,j}(x) := \frac{j\pi + \eta_{\alpha,j}(x)}{n}. \tag{38}$$

Proposition 6 *Let* $|\alpha| < 1$, $N_3(\alpha)$ *be defined by* (8), $n > N_3(\alpha)$, *and* $1 \leq j \leq n$. *Then* $f_{\alpha,n,j}$ *is contractive in* $[0,\pi]$, *and its fixed point belongs to* $I_{n,j}$.

Proof Since the function $\eta_{\alpha,j}$ takes values in $[-\pi, 0]$ and its derivative is bounded by (34), it is easy to see that $f_{\alpha,n,j}(x) \in [0,\pi]$ for every x in $[0,\pi]$, and

$$|f'_{\alpha,n,j}(x)| \leq \frac{N_1(\alpha)}{n} < 1.$$

Moreover, the assumption $|\alpha| < 1$ implies that $\eta_{\alpha,j}(0) = 0$ and $\eta_{\alpha,j}(\pi) = -\pi$. Therefore

$$f_{\alpha,n,j}(0) > 0, \qquad f_{\alpha,n,j}(\pi) < \pi.$$

So, if x is the fixed point of $f_{\alpha,n,j}$, then $0 < x < \pi$. Thus, $-\pi < \eta_{\alpha,j}(x) < 0$ and

$$x = f_{\alpha,n,j}(x) = \frac{j\pi + \eta_{\alpha,j}(x)}{n} \in \left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n}\right) = I_{n,j}.$$

In particular, this implies that the fixed point of $f_{\alpha,n,j}$ coincides with $\vartheta_{\alpha,n,j}$. □

Proposition 7 *Let* $\alpha \in \mathbb{C}$, $|\alpha| < 1$. *Then there exists* $C_3(\alpha) > 0$ *such that for* n *large enough and* $1 \le j \le n$,

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)\eta'_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n^2} + r_{\alpha,n,j}, \tag{39}$$

where $|r_{\alpha,n,j}| \le \frac{C_3(\alpha)}{n^3}$.

Proof Theorem 1 assures the initial approximation $\vartheta_{\alpha,n,j} = j\pi/n + O(1/n)$. Substitute it into the right-hand side of (7) and expand $\eta_{\alpha,j}$ by Taylor's formula around $j\pi/n$:

$$\vartheta_{\alpha,n,j} = \frac{j\pi + \eta_{\alpha,j}\left(\frac{j\pi}{n} + O\left(\frac{1}{n}\right)\right)}{n} = \frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + O_\alpha\left(\frac{1}{n^2}\right).$$

Iterate once again in (7):

$$\vartheta_{\alpha,n,j} = \frac{j\pi + \eta_{\alpha,j}\left(\frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + O_\alpha\left(\frac{1}{n^2}\right)\right)}{n}.$$

Expanding $\eta_{\alpha,j}$ around $j\pi/n$ with two exact term and estimating the residue term with Proposition 5 we obtain the desired result. □

Theorem 3 follows from Proposition 7: we just evaluate g at the expression (39) and expand it by Taylor's formula around $j\pi/n$. See [2] for a more general scheme.

The next proposition can be seen as a particular case of [26], therefore we do not include the proof.

Proposition 8 (The Eigenvectors for Weak Perturbations) *Let* $\alpha \in \mathbb{C}$, $|\alpha| < 1$, $n \ge 3$, $1 \le j \le n$. *Then the vector* $v_{\alpha,n,j} = \left[v_{\alpha,n,j,k}\right]_{k=1}^{n}$ *with components*

$$v_{\alpha,n,j,k} := \sin(k\vartheta_{\alpha,n,j}) + \overline{\alpha}\sin((n-k)\vartheta_{\alpha,n,j}) \tag{40}$$

is an eigenvector of the matrix $A_{\alpha,n}$ *associated to the eigenvalue* $\lambda_{\alpha,n,j}$.

4 Case $|\alpha| = 1$

For $|\alpha| = 1$, the eigenvalues of $A_{\alpha,n}$ can be computed explicitly.

Proposition 9 *Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\alpha \neq \pm 1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{\alpha,n,j} = g(\vartheta_{\alpha,n,j})$, where*

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} - \frac{2}{n} \arctan\left(l_\alpha^{(-1)^j}\right). \tag{41}$$

Furthermore, $\vartheta_{\alpha,n,j} \in I_{n,j}$, and the vector with components (40) is an eigenvector of $A_{\alpha,n}$ associated to $\lambda_{\alpha,n,j}$.

Proof The condition about α implies that $k_\alpha = 0$. In this case, the functions $\eta_{\alpha,1}$ and $\eta_{\alpha,2}$ are just constants:

$$\eta_{\alpha,j}(x) = -2 \arctan\left(l_\alpha^{(-1)^j}\right),$$

and the characteristic equation (7) simplifies to the direct formula (41). □

Proposition 9 implies Theorems 1, 2, and 3, in the case $|\alpha| = 1$ and $\alpha \neq \pm 1$.

For $\alpha = \pm 1$, the situation is different: the definition of $\eta_{\alpha,j}$ does not make sense, each number $\vartheta_{\alpha,n,j}$ coincides with one of the extremes of $I_{n,j}$, and most of the eigenvalues are double.

Proposition 10 *Let $\alpha = 1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{1,n,j} = g(\vartheta_{1,n,j})$, where*

$$\vartheta_{1,n,j} = \left(j - \frac{1 - (-1)^j}{2}\right)\frac{\pi}{n} = \begin{cases} \frac{2q\pi}{n}, & j = 2q, \\ \frac{2q\pi}{n}, & j = 2q + 1. \end{cases} \tag{42}$$

The vector $v_{1,n,j} = [v_{1,n,j,k}]_{k=1}^n$ with components

$$v_{1,n,j,k} := \sin\left(k\vartheta_{1,n,j} + \frac{(1 - (-1)^j)\pi}{4}\right) = \begin{cases} \sin\frac{2kq\pi}{n}, & j = 2q, \\ \cos\frac{2kq\pi}{n}, & j = 2q + 1, \end{cases} \tag{43}$$

is an eigenvector of $A_{1,n}$ associated to $\lambda_{1,n,j}$.

Proof The numbers $\vartheta_{1,n,j}$ can be found by passing to the limit $\alpha \to 1^-$ in (7). The equalities $A_{1,n}v_{1,n,j} = \lambda_{1,n,j}v_{1,n,j}$ are easy to verify directly. □

The formulas for the eigenvalues of $A_{1,n}$ also follow from the theory of circulant matrices, since the matrix $A_{1,n}$ is circulant. Notice that

$$\lambda_{1,n,1} < \lambda_{1,n,2} = \lambda_{1,n,3} < \lambda_{1,n,4} = \lambda_{1,n,5} < \dots,$$

i.e., each of the eigenvalues $\lambda_{1,n,2}$, $\lambda_{1,n,4}$, etc. is double and has two linearly independent eigenvectors.

Proposition 11 *Let $\alpha = -1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{-1,n,j} = g(\vartheta_{-1,n,j})$, where*

$$\vartheta_{-1,n,j} = \left(j - \frac{1+(-1)^j}{2}\right)\frac{\pi}{n} = \begin{cases} \dfrac{(2q-1)\pi}{n}, & j = 2q-1, \\ \dfrac{(2q-1)\pi}{n}, & j = 2q. \end{cases} \tag{44}$$

The vector $v_{-1,n,j} = [v_{-1,n,j,k}]_{k=1}^{n}$ with components

$$v_{-1,n,j,k} := \sin\left(k\vartheta_{-1,n,j} + \frac{(1+(-1)^j)\pi}{4}\right) = \begin{cases} \sin\dfrac{k(2q-1)\pi}{n}, & j = 2q-1, \\ \cos\dfrac{k(2q-1)\pi}{n}, & j = 2q, \end{cases} \tag{45}$$

is an eigenvector of $A_{-1,n}$ associated to $\lambda_{-1,n,j}$.

Proof Similar to the proof of Proposition 10. □

By Proposition 11,

$$\lambda_{-1,n,1} = \lambda_{-1,n,2} < \lambda_{-1,n,3} = \lambda_{-1,n,4} < \dots,$$

i.e. each of the eigenvalues $\lambda_{-1,n,1}$, $\lambda_{-1,n,3}$, etc. is double and has two linearly independent eigenvectors.

5 Proofs for the Case of Strong Perturbations

Theorem 4 following directly from Proposition 2. In order to find the eigenvalues outside of $[0, 4]$, we use the change of variables $\lambda = g_-(x)$ or $\lambda = g_+(x)$, defined by (11).

Proposition 12 *For $x > 0$ and $\lambda = g_-(x)$, the equation* $\det(\lambda I_n - A_{\alpha,n}) = 0$ *is equivalent to*

$$\tanh(x) = \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + 1|^2 \tanh^2\frac{nx}{2} + |\alpha - 1|^2}. \tag{46}$$

For $x > 0$ and $\lambda = g_+(x)$, the equation $\det(\lambda I_n - A_{\alpha,n}) = 0$ *is equivalent to*

$$\tanh(x) = \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + (-1)^n|^2 \tanh^2\frac{nx}{2} + |\alpha - (-1)^n|^2}. \tag{47}$$

Proof For $\lambda = g_-(x) = 2 - 2\cosh(x)$, we apply (21) and (20). After some simple transformations,

$$\begin{aligned}\det(g_-(x)I_n - A_{\alpha,n}) = \frac{(-1)^n}{1 - \tanh^2\frac{nx}{2}}\biggl(&|\alpha+1|^2\tanh^2\frac{nx}{2} \\ &- 2(|\alpha|^2 - 1)\tanh\frac{nx}{2}\coth(x) + |\alpha - 1|^2\biggr).\end{aligned} \tag{48}$$

This expression for the characteristic polynomial yields (46). The proof of (47) is analogous. □

Remark 1 In formula (28), $\tan\frac{nx}{2}$ is rapidly oscillating and $\cot(x)$ is much slower, therefore we solve (28) for $\tan\frac{nx}{2}$. The situation in (48) is different: if x is separated from zero and n is large enough, then $\tanh\frac{nx}{2}$ is almost a constant, and we prefer to solve (48) for $\tanh(x)$.

At the moment, we have proven Theorem 5.

Let us show that for $|\alpha| > 1$, n large enough and $2 \le j \le n-1$, the situation is nearly the same as in Proposition 6. Recall that $N_3(\alpha)$ and $f_{\alpha,n,j}$ are defined by (8) and (38).

Proposition 13 *Let $|\alpha| > 1$, $n > N_3(\alpha)$, and $2 \le j \le n-1$. Then $f_{\alpha,n,j}$ is contractive on* $[0, \pi]$. *If x is the fixed point of $f_{\alpha,n,j}$, then $x \in I_{n,j}$ and $g(x) = \lambda_{\alpha,n,j}$.*

Proof Inequality (34) and the assumption $n \ge N_1(\alpha)$ imply that $f_{\alpha,n,j}$ is contractive on $[0, \pi]$. Unlike in the case of weak perturbations, in the case $|\alpha| > 1$ we have

$$\eta_{\alpha,j}(0) = -\pi, \qquad \eta_{\alpha,j}(\pi) = 0. \tag{49}$$

Now the condition $2 \le j \le n-1$ assures that 0 and π are not fixed points of $f_{\alpha,n,j}$:

$$f_{\alpha,n,j}(0) = \frac{j\pi - \pi}{n} > 0, \qquad f_{\alpha,n,j}(\pi) = \frac{j\pi + 0}{n} < \pi.$$

Let x be the fixed point of $f_{\alpha,n,j}$. Then $0 < x < \pi$. Hence $-\pi < \eta_{\alpha,j}(x) < 0$ and $x = f_{\alpha,n,j}(x) \in I_{n,j}$. □

If $|\alpha| > 1$ and $n \geq N_3(\alpha)$, then the functions $f_{\alpha,n,1}$ and $f_{\alpha,n,n}$ are contractive, but their fixed points are 0 and π. The corresponding values of g, i.e. the points 0 and 4, are not eigenvalues of $A_{\alpha,n}$. Hence, for $|\alpha| > 1$ and n large enough, the trigonometric change of variables $\lambda = g(x)$ with real x allows us to find only $n-2$ eigenvalues.

In what follows, we restrict ourselves to the analysis of Eq. (46), because (47) is similar. Define $\psi_\alpha\colon [0,1] \to [0,+\infty)$ by

$$\psi_\alpha(t) := \frac{2(|\alpha|^2-1)t}{|\alpha+1|^2t^2+|\alpha-1|^2}. \tag{50}$$

Notice that

$$\psi_\alpha(1) = \frac{2(|\alpha|^2-1)}{|\alpha+1|^2+|\alpha-1|^2} = \frac{|\alpha|^2-1}{|\alpha|^2+1} = \tanh(\log|\alpha|). \tag{51}$$

We are going to construct explicitly a left neighborhood of 1 where the values of ψ_α are close enough to $\tanh(\log|\alpha|)$.

Lemma 1 *Let $|\alpha| > 1$. Then for every t with*

$$1 - \frac{|\alpha|-1}{(|\alpha|+1)^3} \leq t \leq 1, \tag{52}$$

the following inequalities hold:

$$|\psi_\alpha'(t)| \leq 1, \tag{53}$$

$$\tanh\frac{\log|\alpha|}{2} \leq \psi_\alpha(t) \leq \tanh\frac{3\log|\alpha|}{2}, \tag{54}$$

$$1 - \psi_\alpha^2(t) \geq \frac{2}{(|\alpha|+1)^3}. \tag{55}$$

Proof The proof is quite elementary, therefore we will only mention the main steps. Assumption (52) implies that

$$1-t^2 \leq \frac{2(|\alpha|-1)}{(|\alpha|+1)^3}, \qquad |\alpha+1|^2t^2+|\alpha-1|^2 \geq \frac{2(|\alpha|^3+|\alpha|^2+2)}{|\alpha|+1}.$$

With these estimates we obtain (53). After that, the mean value theorem and (51) provide (54). Inequality (55) follows from (54). □

Recall that $N_4(\alpha)$ and S_α are given by (14) and (15). Define $h_{\alpha,n}$ on S_α as the right-hand side of (12):

$$h_{\alpha,n}(x) := \operatorname{arctanh} \frac{2(|\alpha|^2-1)\tanh\frac{nx}{2}}{|\alpha+1|^2\tanh^2\frac{nx}{2}+|\alpha-1|^2}.$$

Proposition 14 *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and $n > N_4(\alpha)$. Then $h_{\alpha,n}$ is contractive on S_α, and its fixed point is the solution of (46).*

Proof We represent $h_{\alpha,n}$ as the following composition:

$$h_{\alpha,n}(x) = \operatorname{arctanh}\left(\psi_\alpha\left(\tanh\frac{nx}{2}\right)\right).$$

For x in S_α, denote $\tanh\frac{nx}{2}$ by t. Then

$$1-t \le 2e^{-nx} \le 2e^{-N_4(\alpha)\frac{\log|\alpha|}{2}} < \frac{|\alpha|-1}{(|\alpha|+1)^3}.$$

Therefore, by (55) we have $\psi_\alpha(\tanh\frac{nx}{2}) < 1$, and the definition of $h_{\alpha,n}$ makes sense. By (54), $h_{\alpha,n}$ takes values in S_α. Estimate from above the derivative of $h_{\alpha,n}$ using (55), (53), and the elementary inequality $ue^{-u} \le 1/e$:

$$\begin{aligned}
|h'_{\alpha,n}(x)| &\le \frac{|\psi'_\alpha(t)|}{1-\psi_\alpha^2(t)}\cdot\frac{n}{2\cosh^2\frac{nx}{2}} \le (|\alpha|+1)^3 ne^{-nx} \\
&\le (|\alpha|+1)^3 ne^{-\frac{n\log|\alpha|}{2}} = (|\alpha|+1)^3 ne^{-\frac{n\log|\alpha|}{4}}e^{-\frac{n\log|\alpha|}{4}} \\
&\le (|\alpha|+1)^3\cdot\frac{4}{\log|\alpha|}\cdot\frac{\log|\alpha|}{(|\alpha|+1)^5} = \frac{4}{(|\alpha|+1)^2} < 1.
\end{aligned}$$

Obviously, the fixed point of $h_{\alpha,n}$ is the solution of (12) and (46). □

Asymptotic formulas for $\lambda_{\alpha,n,j}$ with $|\alpha| > 1$ and $2 \le j \le n-1$ can be justified in the same manner as for $|\alpha| < 1$, and we are left to prove the exponential convergence (17) and (18).

Proposition 15 *Let $|\alpha| > 1$ and $C_4(\alpha) := |\alpha|^3 e^{\frac{|\alpha|^3}{\log|\alpha|}}$. Then for all $n \ge N_2(\alpha)$*

$$\left|\vartheta_{\alpha,n,1} - \log|\alpha|\right| \le \frac{C_4(\alpha)}{|\alpha|^n}. \tag{56}$$

Proof For brevity, put $x = \vartheta_{\alpha,n,1}$. Apply the mean value theorem to ψ_α, taking into account (53):

$$\left|\tanh(x) - \tanh(\log|\alpha|)\right| = \left|\psi_\alpha\left(\tan\frac{nx}{2}\right) - \psi_\alpha(1)\right| \le 1 - \tan\frac{nx}{2} \le 2e^{-nx}.$$

On the other hand, apply the mean value theorem to tanh:

$$|\tanh(x) - \tanh(\log|\alpha|)| \geq \frac{\big|x - \log|\alpha|\big|}{\cosh^2\frac{3\log|\alpha|}{2}} \geq \frac{2}{|\alpha|^3}\big|x - \log|\alpha|\big|.$$

From this chain of inequalities,

$$|x - \log|\alpha|| \leq |\alpha|^3 e^{-nx}. \tag{57}$$

We already know from Proposition 14 that $x \geq \frac{\log|\alpha|}{2}$. Thus,

$$x \geq \log|\alpha| - |\alpha|^3 e^{-\frac{n\log|\alpha|}{2}}.$$

Using the elementary inequality $ue^{-u} \leq 1/e$ we get

$$nx \geq n\log|\alpha| - |\alpha|^3\, ne^{-\frac{n\log|\alpha|}{2}} \geq n\log|\alpha| - \frac{|\alpha|^3}{\log|\alpha|}. \tag{58}$$

By (57) and (58), inequality (56) holds. □

In a similar manner, $|\vartheta_{\alpha,n,n} - \log|\alpha|| \leq C_4(\alpha)/|\alpha|^n$. Since the derivatives of g_- and g_+ are bounded on $\left[0, \frac{3}{2}\log|\alpha|\right]$, we get limit relations (17) and (18). Thereby we have proven the parts of Theorem 6 related to the extreme eigenvalues $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$.

Proposition 16 (The Eigenvectors for Strong Perturbations) *Let* $\alpha \in \mathbb{C}$, $|\alpha| > 1$, $n \geq N(\alpha)$. *Then the vectors* $v_{\alpha,n,1} := [v_{\alpha,n,1,k}]_{k=1}^n$ *and* $v_{\alpha,n,n} := [v_{\alpha,n,n,k}]_{k=1}^n$ *with components*

$$v_{\alpha,1,n,k} = \sinh(k\vartheta_{\alpha,n,1}) + \overline{\alpha}\sinh((n-k)\vartheta_{\alpha,n,1}), \tag{59}$$

$$v_{\alpha,n,n,k} = (-1)^k\sinh(k\vartheta_{\alpha,n,n})) + (-1)^{k+n}\,\overline{\alpha}\sinh((n-k)\vartheta_{\alpha,n,n}), \tag{60}$$

are the eigenvectors of the matrix $A_{\alpha,n}$ *associated to the eigenvalues* $\lambda_{\alpha,1,n}$ *and* $\lambda_{\alpha,n,n}$, *respectively. For* $2 \leq j \leq n-1$, *the vector* $v_{\alpha,n,j}$ *defined by* (40) *is an eigenvector of* $A_{\alpha,n}$ *associated to the eigenvalue* $\lambda_{\alpha,n,j}$.

Remark 2 It is possible to show that for a fixed α with $|\alpha| > 1$ and large values of n, the norms of the vectors $v_{\alpha,n,1}$ and $v_{\alpha,n,n}$, given by (59) and (60), grow as $|\alpha|^n$. In order to avoid large numbers, we recommend to divide each component of these vectors by $|\alpha|^n$.

6 Numerical Experiments

We use the following notation for different approximations of the eigenvalues.

- $\lambda_{\alpha,n,j}^{\mathrm{gen}}$ are the eigenvalues computed in Sagemath by general algorithms, with double-precision arithmetic.
- $\lambda_{\alpha,n,j}^{\mathrm{fp}}$ are the eigenvalues computed by formulas of Theorems 2 and 5, i.e. solving Eqs. (7), (12), and (13) by the fixed point iteration; these computations are performed in the high-precision arithmetic with 3322 binary digits (≈1000 decimal digits). Using $\lambda_{\alpha,n,j}^{\mathrm{fp}}$ we compute $v_{\alpha,n,j}$ by (40), (59), and (60).
- $\lambda_{\alpha,n,j}^{\mathrm{asympt}}$ are the approximations given by (9) and (19).

In (12) and (13), we compute $\tanh\frac{nx}{2}$ as $1-2e^{-nx}/(1+e^{-nx})$, because $nx/2$ can be large and the standard formula for tanh can produce overflows ("NaN").

We have constructed a large series of examples with random values of α and n. In all these examples, we have obtained

$$\max_{1\le j\le n}|\lambda_{\alpha,n,j}^{\mathrm{gen}}-\lambda_{\alpha,n,j}^{\mathrm{fp}}|<2\cdot 10^{-13},\qquad \frac{\|A_{\alpha,n}v_{\alpha,n,j}-\lambda_{\alpha,n,j}^{\mathrm{fp}}v_{\alpha,n,j}\|}{\|v_{\alpha,n,j}\|}<10^{-996}.$$

This means that the exact formulas from Theorems 2 and 5 are fulfilled up to the rounding errors. Theorems 1 and 4 can be viewed as simple corollaries from Theorems 2 and 5, so they do not need additional tests. For Theorems 3 and 6, we have computed the errors

$$R_{\alpha,n,j}:=\lambda_{\alpha,n,j}^{\mathrm{asympt}}-\lambda_{\alpha,n,j}^{\mathrm{fp}}$$

and their maximums $\|R_{\alpha,n}\|_\infty=\max_{1\le j\le n}|R_{\alpha,n,j}|$. Tables 1 and 2 show that these errors indeed can be bounded by $C_1(\alpha)/n^3$, and $C_1(\alpha)$ has to take bigger values when $|\alpha|$ is close to 1.

Table 1 Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some $|\alpha|<1$

$\alpha=-0.3+0.5i,\quad \lvert\alpha\rvert\approx 0.58$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
64	1.76×10^{-4}	46.05
128	2.49×10^{-5}	52.13
256	3.29×10^{-6}	55.12
512	4.22×10^{-7}	56.58
1024	5.34×10^{-8}	57.31
2048	6.71×10^{-9}	57.67
4096	8.42×10^{-10}	57.84
8192	1.05×10^{-10}	57.93

$\alpha=0.7+0.6i,\quad \lvert\alpha\rvert\approx 0.92$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
64	1.02×10^{-3}	266.71
128	1.59×10^{-4}	333.02
256	2.24×10^{-5}	376.61
512	2.99×10^{-6}	401.28
1024	3.86×10^{-7}	414.29
2048	4.90×10^{-8}	420.94
4096	6.17×10^{-9}	424.30
8192	7.75×10^{-10}	425.99

Table 2 Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some $|\alpha| > 1$

$\alpha = 2+i,\quad |\alpha| \approx 2.23$			$\alpha = 0.8-0.7i,\quad |\alpha| \approx 1.06$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$	n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
64	1.55×10^{-4}	40.59	64	2.19×10^{-4}	57.51
128	2.15×10^{-5}	45.18	128	2.19×10^{-5}	45.90
256	2.82×10^{-6}	47.33	256	1.40×10^{-5}	235.36
512	3.60×10^{-7}	48.36	512	2.99×10^{-6}	401.90
1024	4.55×10^{-8}	48.86	1024	4.55×10^{-7}	488.25
2048	5.72×10^{-9}	49.10	2048	6.16×10^{-8}	528.84
4096	7.16×10^{-10}	49.22	4096	7.98×10^{-9}	548.04
8192	8.97×10^{-11}	49.29	8192	1.01×10^{-9}	557.32

We have also tested (17) and (18). As n grows, $|\alpha|^n|R_{\alpha,n,1}|$ and $|\alpha|^n|R_{\alpha,n,n}|$ approach rapidly the same limit value depending on α. For example,

$$\text{for } \alpha = 2+i, \qquad \lim_{n\to\infty}\left(|\alpha|^n|R_{\alpha,n,1}|\right) \approx 2.86,$$

$$\text{for } \alpha = 0.8-0.7i, \qquad \lim_{n\to\infty}\left(|\alpha|^n|R_{\alpha,n,1}|\right) \approx 1.12\cdot 10^{-2}.$$

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Some Properties of the Kernel of a Transmutation Operator

T. N. Harutyunyan and A. M. Tonoyan

Abstract It is known, that there exist the continuous function $P(x,t)$, which generates the transmutation operator

$$\varphi(x,\lambda) = \frac{\sin \lambda x}{\lambda} + \int_0^x P(x,s)\frac{\sin \lambda t}{\lambda} dt,$$

transforming the solution $\frac{\sin \lambda x}{\lambda}$ of $-y'' = \lambda^2 y$ with initial conditions $y(0) = 0$, $y'(0) = 1$ to the solution $\varphi(x,\lambda)$ of equation $-y'' + q(x)y = \lambda^2 y$ with the same initial values. We have proved that $P(x,t)$ is the solution of integral equation (analogue of Gelfand-Levitan equation).

$$P(x,t) + F(x,t) + \int_0^x P(x,s)F(s,t)ds = 0,$$

where

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} \frac{\sin \lambda_n x \sin \lambda_n t}{\lambda_n^2} - \frac{2}{\pi} \sin\left(n+\frac{1}{2}\right) x \sin\left(n+\frac{1}{2}\right) t \right),$$

where λ_n^2 and a_n are the spectral data of corresponding Sturm-Liouville problem.

Keywords Sturm-Liouville problem · Transmutation operators · Gelfand-Levitan equation

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1 Introduction and Statement of the Main Results

Let us denote by $L(q, \alpha, \beta)$ the following Sturm-Liouville problem

$$ly := -y'' + q(x)y = \mu y, \quad x \in (0, \pi), \quad \mu \in \mathbb{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0, \pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0, \pi), \tag{1.3}$$

where q is a real-valued function, summable on $[0, \pi]$ (we write $q \in L^1_{\mathbb{R}}[0, \pi]$). By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by differential expression $l = -\frac{d^2}{dx^2} + q(x)$ on the domain $D_L = \{y : y' \in AC[0, \pi], ly \in L^2[0, \pi],\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, y(\pi)\cos\beta + y'(\pi)\sin\beta = 0\}$ (for details see [1], §17). It is known, that under these conditions the spectra of the operator $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues (see [1–4]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta) = \lambda_n^2(q, \alpha, \beta), n = 0, 1, 2, \ldots$, emphasizing the dependence of μ_n on q, α and β. We assume that eigenvalues are enumerated in the increasing order, i.e.

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \ldots < \mu_n(q, \alpha, \beta) < \ldots .$$

By $\varphi(x, \mu, \alpha, q)$ we denote the solutions of Eq. (1.1), which satisfy the following initial conditions:

$$\varphi(0, \mu, \alpha, q) = \sin\alpha, \quad \varphi'(0, \mu, \alpha, q) = -\cos\alpha.$$

It is easy to see that eigenvalues μ_n of problem $L(q, \alpha, \beta)$ are the solutions of equation

$$\Phi(\mu) = \Phi(\mu, \alpha, \beta) := \varphi(\pi, \mu, \alpha, q)\cos\beta + \varphi'(\pi, \mu, \alpha, q)\sin\beta = 0,$$

It follows that $\varphi_n(x) := \varphi(x, \mu_n(q, \alpha, \beta), \alpha, q)$ are the eigenfunctions, corresponding to the eigenvalue μ_n.

The square of the L^2 norms of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) := \int_0^\pi |\varphi_n(x)|^2 dx,$$

are called the norming constants. If $\sin\alpha \neq 0$ ($\alpha \neq \pi$), then by $\tilde{\varphi}(x, \mu)$ we denote the solution

$$\tilde{\varphi}(x, \mu, \alpha) = \frac{\varphi(x, \mu, \alpha, q)}{\sin\alpha},$$

which satisfy the initial conditions

$$\tilde{\varphi}(0, \mu, \alpha) = 1, \tilde{\varphi}'(0, \mu, \alpha) = -\cot\alpha = h.$$

The proof of the following assertion can be found in [5] (see also [2, 6, 7]).

Theorem 1.1 *Let $q \in L^1_{\mathbb{R}}[0, \pi]$. There exist a real function $P(x, t)$, which is defined (and continuous) on triangle $0 \leqslant t \leqslant x \leqslant \pi$ and such that the representation*

$$\varphi(x, \lambda^2, \pi, q) = \frac{\sin \lambda x}{\lambda} + \int_0^x P(x, t) \frac{\sin \lambda t}{\lambda} dt \tag{1.4}$$

holds. The function $P(x, t)$ by each of arguments x and t have same smoothness as the function $f(s) = \int_0^s q(t)dt$ (i.e. $P(x, t)$ is absolutely continuous function by each of arguments) and

$$P(x, x) = \frac{1}{2} \int_0^x q(t)dt, \quad P(x, 0) = 0.$$

Also there exist a real function $G(x, t)$ defined and continuous in $0 \leqslant t \leqslant x \leqslant \pi$ and such that the representation

$$\tilde{\varphi}(x, \lambda^2, q) = \cos \lambda x + \int_0^x G(x, t) \cos \lambda t dt$$

holds. The smoothness of $G(x, t)$ same as of $P(x, t)$ and

$$G(x, x) = h + \frac{1}{2} \int_0^x q(t)dt.$$

In the following formulae we denote $\lambda_n = \lambda_n(q, \pi, \beta)$ and $a_n = a_n(q, \pi, \beta)$. Let us define the function

$$F(x, t) = \sum_{n=0}^{\infty} \left[\frac{1}{a_n} \frac{\sin \lambda_n x \sin \lambda_n t}{\lambda_n^2} - \frac{2}{\pi} \sin\left(n + \frac{1}{2}\right) x \sin\left(n + \frac{1}{2}\right) t \right]. \tag{1.5}$$

Theorem 1.2 *For arbitrary fixed $x \in (0, \pi]$, the function $P(x, \cdot)$ satisfies the following integral equation*

$$P(x, t) + F(x, t) + \int_0^x P(x, s) F(s, t) ds = 0, \quad 0 \leqslant t \leqslant x, \tag{1.6}$$

which we will call the Gelfand-Levitan equation (for the problem $L(q, \pi, \beta)$).

We must say that Gelfand and Levitan obtain (see [8]) such integral equation for the kernel $G(x,t)$:

$$G(x,t) + \tilde{F}(x,t) + \int_0^x G(x,s)\tilde{F}(s,t)ds = 0,$$

where the kernel $\tilde{F}(x,t)$ defined by the formula

$$\tilde{F}(x,t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{a_n} - \frac{\cos nx \cos nt}{a_n^0} \right),$$

and where $\lambda_n = \lambda_n(q,\alpha,\beta)$, $a_n = a_n(q,\alpha,\beta)$, $\alpha,\beta \in (0,\pi)$ and

$$a_n^0 = \begin{cases} \frac{\pi}{2}, & n > 1, \\ \pi, & n = 0. \end{cases}$$

We see that in this case Gelfand and Levitan take the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ in the square $(0,\pi] \times [0,\pi)$ as the point of "comparison" (where $\lambda_n\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) = n$, $n = 0, 1, 2, \ldots$).

In our definition of $F(x,t)$ (see (1.5)) we take as the point of "comparison" the point $\left(\pi, \frac{\pi}{2}\right)$, where $\lambda_n\left(0, \pi, \frac{\pi}{2}\right) = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$.

Theorem 1.3 *Let $q \in L^1_{\mathbb{R}}[0,\pi]$. Then the integral equation (1.6) has a unique solution $P(x,\cdot)$, $P(x,\cdot) \in L^2(0,x)$.* □

In what follows, for brevity, by $\varphi(x,\lambda^2)$ we will understand $\varphi(x,\mu,\pi,q)$.

Theorem 1.4 *The kernel $P(x,t)$ admits the following representation*

$$P(x,t) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{\pi} \varphi\left(x, \left(n+\frac{1}{2}\right)^2\right) \sin\left(n+\frac{1}{2}\right)t - \frac{1}{a_n(q,\pi,\beta)} \varphi(x,\lambda_n^2) \frac{\sin \lambda_n t}{\lambda_n} \right) \tag{1.7}$$

where the series converge in the following distributional sense. Let the kernel $P(x,t)$ be extended by zero for $t > x$. Then for any $f \in AC[0,\pi]$, $f(0) = 0$, the following limit exists uniformly with respect to x

$$\lim_{N\to\infty} \int_0^{\pi} f(t) I_N(x,t)dt = -\int_0^{\pi} f(t) \left(F(x,t) + \int_0^x P(x,s)F(s,t)ds \right) dt \tag{1.8}$$

where

$$I_N(x,t) = \sum_{n=0}^{N} \left(\frac{2n+1}{\pi} \varphi\left(x, \left(n+\frac{1}{2}\right)^2\right) \sin\left(n+\frac{1}{2}\right)t \right.$$
$$\left. -\frac{1}{a_n}\varphi(x,\lambda_n^2)\frac{\sin\lambda_n t}{\lambda_n}\right). \tag{1.9}$$

These theorems correspond to the direct Sturm-Liouville problem (when q, α, β are known and we investigate some problems). But the Gelfand-Levitan equations is the principal key for the (constructive) solution of the "inverse problem": we know the spectral data $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ (more precise: we know two sequences, which have the properties, like the spectral data $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ in direct problem), and we want to construct the function $q(x)$ and parameters α and β. This problem in the case $q \in L_{\mathbb{R}}^1[0,\pi]$, $\alpha = \pi$, $\beta \in (0,\pi)$ have been considered in [9], and we will not concern to this aspect here. The authors of books [3] and [10] turned to obtaining an analogue of the Gelfand-Levitan equation for the case $\sin\alpha = 0$, but based on the study of the Goursat problem, i.e. the problem for a second-order partial differential equation, which must be satisfied by the kernel $P(x,t)$ of the transmutation operator (1.4). But in our case ($q \in L_R^1[0,\pi]$), according to Theorem 1.1, $P(x,t)$ may not have second derivative, i.e. their approach is not applicable.

Also we must say that Theorem 1.4 was born under impact of paper [11] and book [12].

Our paper is organized as follows:

In Sect. 2 we present asymptotics of spectral data $\{\lambda_n^2\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ and we use it for study of smoothness of the kernel $F(x,t)$.

In Sect. 3 we prove that $P(x,t)$ satisfies an integral equation, which we call the Gelfand-Levitan equation. After that we prove the uniqueness of the solution of this equation.

In Sect. 4 we give the representation of $P(x,t)$ in term of eigenfunctions.

2 Asymptotics of the Eigenvalues and Norming Constants and Their Application

In paper [4] it was introduced the concept of the function $\delta_n(\alpha,\beta)$, which is defined by the formula

$$\delta_n(\alpha,\beta) := \sqrt{\mu_n(0,\alpha,\beta)} - n = \lambda_n(0,\alpha,\beta) - \lambda_n\left(0,\frac{\pi}{2},\frac{\pi}{2}\right), \quad n \geq 2,$$

and proved that $-1 \leq \delta_n(\alpha,\beta) \leq 1$ and it is a solution of the following (transcendental) equation

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{(n+\delta_n(\alpha,\beta))^2\sin^2\alpha+\cos^2\alpha}}$$
$$-\frac{1}{\pi}\arccos\frac{\cos^2\beta}{\sqrt{(n+\delta_n(\alpha,\beta))^2\sin^2\beta+\cos^2\beta}}$$

In particular $\delta_n(\pi,\beta) = \frac{1}{2} + O\left(\frac{1}{n}\right)$, $\delta_n\left(\pi,\frac{\pi}{2}\right) = \frac{1}{2}$. With the help of $\delta_n(\alpha,\beta)$ for eigenvalues $\mu_n(q,\pi,\beta) = \lambda_n^2(q,\pi,\beta)$ and for norming constants $a_n(q,\alpha,\beta)$ we have obtained (see [4, 13–15], see also [16]) the following asymtptotic formula (when $n \to \infty$):

1.

$$\mu_n(q,\pi.\beta) = [n+\delta_n(\pi,\beta)]^2 + [q] + r_n, \tag{2.1}$$

where $[q] = \frac{1}{\pi}\int_0^\pi q(t)dt$, $r_n = o(1)$, and more precise.

$$\lambda_n(q,\pi,\beta) = n + \delta_n(\pi,\beta) + \frac{c}{2[n+\delta_n(\pi,\beta)]} + l_n \tag{2.2}$$

where $c = [q]$ and $l_n = l_n(q,\pi,\beta) = o(\frac{1}{n})$ are such that the function

$$l(t) := \sum_{n=2}^{\infty} l_n \sin(n+\delta_n(\pi,\beta))t \tag{2.3}$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$ (we will write $l \in AC(0,2\pi)$);

2.

$$a_n(q,\pi,\beta) = \frac{\pi}{2[n+\delta_n(\pi,\beta)]^2}\left(1+\frac{2s_n}{\pi[n+\delta_n(\pi,\beta)]}\right), \tag{2.4}$$

where the remainders $s_n = s_n(q,\pi,\beta) = o(1)$ are such that the function

$$S(t) = \sum_{n=2}^{\infty}\frac{s_n}{n+\delta_n(\pi,\beta)}\cos(n+\delta_n(\pi,\beta))t \tag{2.5}$$

is absolutely continuous on $[0, 2\pi]$, i.e. $S \in AC[0, 2\pi]$. Let us denote by $\lambda_n^0 = \lambda_n(0, \pi, \beta) = n + \delta_n(\pi, \beta)$ and by $a_n^0 = a_n(0, \pi, \beta)$. In particular, if $\beta = \dfrac{\pi}{2}$, then $\lambda_n\left(0, \pi, \dfrac{\pi}{2}\right) = n + \dfrac{1}{2}$ and $a_n\left(0, \pi, \dfrac{\pi}{2}\right) = \dfrac{\pi}{2\left(n + \frac{1}{2}\right)^2}$.

And let us consider the function

$$\begin{aligned}
\hat{F}(x,t) &= \textstyle\sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\sin\lambda_n x}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n} - \frac{1}{a_n^0}\frac{\sin\lambda_n^0 x}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)\\
&= \frac{1}{2}\textstyle\sum_{n=0}^{\infty}\left[\frac{1}{a_n\lambda_n^2}(\cos\lambda_n(x-t) - \cos\lambda_n(x+t))\right.\\
&\left. - \frac{1}{a_n^0(\lambda_n^0)^2}(\cos\lambda_n^0(x-t) - \cos\lambda_n^0(x+t))\right]\\
&= \frac{1}{2}\textstyle\sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\cos\lambda_n(x-t)-1}{\lambda_n^2} - \frac{1}{a_n^0}\frac{\cos\lambda_n^0(x-t)-1}{(\lambda_n^0)^2}\right)\\
&- \frac{1}{2}\textstyle\sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\cos\lambda_n(x+t)-1}{\lambda_n^2} - \frac{1}{a_n^0}\frac{\cos\lambda_n^0(x+t)-1}{(\lambda_n^0)^2}\right)
\end{aligned} \tag{2.6}$$

We must note that adding -1 in the numerators is needed for correctness of the function $f(\xi) = \dfrac{\cos\lambda_n\xi - 1}{\lambda_n^2}$ in the case when $\lambda_n = 0$. It is easy to see that $\hat{F}(x,t) = F(x,t)$ when $\beta = \pi/2$. Now, if we denote

$$H(t) = \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\cos\lambda_n t - 1}{\lambda_n^2} - \frac{1}{a_n^0}\frac{\cos\lambda_n^0 t - 1}{(\lambda_n^0)^2}\right), \tag{2.7}$$

then we see, that

$$\hat{F}(x,t) = \frac{1}{2}\left[H(x-t) - H(x+t)\right]. \tag{2.8}$$

Lemma 2.1 *The function $H(\cdot)$, defined by formula (2.7) is absolutely continuous on arbitrary segment $[a,b] \subset [0, 2\pi)$, ($H(\cdot) \in AC[0, 2\pi)$).*

Proof Denote

$$\begin{aligned}
&k_n := a_n\lambda_n^2,\ k_n^0 := a_n^0(\lambda_n^0)^2,\ \epsilon := \lambda_n(q,\pi,\beta) - \lambda_n(0,\pi,\beta)\\
&= \frac{c}{2(n+\delta_n(\pi,\beta))} + l_n
\end{aligned}$$

Since $\lambda_n^0 := \lambda_n(0, \pi, \beta) = n + \delta_n(\pi, \beta), n = 2, 3, \ldots,$ then, we can write down the general term ($n \geq 2$) of the series (2.6) in the following form:

$$\begin{aligned}
&\frac{1}{a_n}\frac{\cos\lambda_n t - 1}{\lambda_n^2} - \frac{1}{a_n\,(0,\pi,\beta)}\,\frac{\cos\lambda_n\,(0,\pi,\beta)\,t - 1}{\lambda_n^2\,(0,\pi,\beta)} = \frac{\cos\lambda_n t - 1}{k_n} \\
&-\frac{\cos\lambda_n^0 t - 1}{k_n^0} = \frac{1}{k_n}\cos\lambda_n t - \frac{1}{k_n}\cos\lambda_n^0 t + \frac{1}{k_n}\cos\lambda_n^0 t - \frac{1}{k_n^0}\cos\lambda_n^0 t \\
&-\left(\frac{1}{k_n} - \frac{1}{k_n^0}\right) = \frac{1}{k_n}(\cos(\lambda_n^0 + \epsilon_n)t - \cos\lambda_n^0 t) + \left(\frac{1}{k_n} - \frac{1}{k_n^0}\right)\cos\lambda_n^0 t \qquad (2.9)\\
&-\left(\frac{1}{k_n} - \frac{1}{k_n^0}\right) = \frac{1}{k_n}(cos(n + \delta_n(\pi,\beta) + \epsilon_n)t - \cos(n + \delta_n(\pi,\beta))t) \\
&+\left(\frac{1}{k_n} - \frac{1}{k_n^0}\right)\cos(n + \delta_n(\pi,\beta))t) - \left(\frac{1}{k_n} - \frac{1}{k_n^0}\right).
\end{aligned}$$

Using (2.1)–(2.4) and the condition $l_n = o\left(\frac{1}{n}\right)$ we can write $\frac{1}{k_n}$ in the form:

$$\begin{aligned}
&\frac{1}{k_n} = \frac{1}{a_n\lambda_n^2} = \frac{2}{\pi\left(1 + \frac{2s_n}{\pi(n + \delta(\pi,\beta))}\right)\left(1 + \frac{c + r_n}{(n + \delta_n(\pi,\beta))^2}\right)} \qquad (2.10)\\
&= \frac{2}{\pi} - \frac{4}{\pi^2}\frac{s_n}{n + \delta_n(\pi,\beta)} + h_n,
\end{aligned}$$

where

$$r_n = 2(n + \delta_n(\pi,\beta))l_n + \left(\frac{c}{2(n + \delta_n(\pi,\beta))} + l_n\right)^2 = o(1)$$

and

$$\begin{aligned}
h_n = &\frac{-2\frac{c + r_n}{(n + \delta_n(\pi,\beta))^2}}{\pi\left(1 + \frac{2s_n}{\pi(n + \delta(\pi,\beta))}\right)\left(1 + \frac{c + r_n}{(n + \delta_n(\pi,\beta))^2}\right)} \\
&+ \frac{\frac{8s_n^2}{\pi^2(n + \delta_n(\pi,\beta))^2}}{\pi\left(1 + \frac{2s_n}{\pi(n + \delta(\pi,\beta))}\right)} = O\left(\frac{1}{n^2}\right)
\end{aligned}$$

On the other hand,

$$\frac{1}{k_n^0} = \frac{1}{a_n(0,\pi,\beta)(n + \delta_n(\pi,\beta))^2} = \frac{2}{\pi} + g_n,\ g_n = O\left(\frac{1}{n^2}\right)$$

Therefore

$$\frac{1}{k_n} - \frac{1}{k_n^0} = -\frac{4}{\pi^2}\frac{s_n}{n + \delta_n(\pi, \beta)} + h_n - g_n, \tag{2.11}$$

And finally, using elementary trigonometric identities and Maclaurin expansion of cos and sin functions

$$\cos\epsilon_n t = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\epsilon_n^{2k}t^{2k},\ \sin\epsilon_n t = \epsilon_n t + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!}\epsilon_n^{2k+1}t^{2k+1},$$

we get that $\cos\epsilon_n t = 1 + O\left((\epsilon_n t)^2\right) = 1 + O\left(\frac{1}{n^2}\right)$,
$\sin\epsilon_n t = \epsilon_n t + O\left((\epsilon_n t)^3\right) = \epsilon_n t + O\left(\frac{1}{n^3}\right)$ and, consequently,

$$\begin{aligned}&\cos(n + \delta(\pi, \beta) + \epsilon_n)t - \cos(n + \delta_n(\pi, \beta))t = -\epsilon_n t\sin(n + \delta_n(\pi, \beta))t\\&+m_n(t)\cos(n + \delta_n(\pi, \beta))t - z_n(t)\sin(n + \delta_n(\pi, \beta))t,\end{aligned} \tag{2.12}$$

where $m_n(t) = O\left(\frac{1}{n^2}\right)$ and $z_n(t) = O\left(\frac{1}{n^3}\right)$ uniformly with respect to $t \in [0, 2\pi]$.

Now by taking into account (2.9), (2.10), (2.11) and (2.12) we can expand $H(t)$ in the following form:

$$H(t) = \sum_{n=0}^{1}\left(\frac{\cos\lambda_n t - 1}{k_n} - \frac{\cos\lambda_n^0 t - 1}{k_n^0}\right) + \sum_{i=1}^{5}H_i(t) + \sum_{n=2}^{\infty}\left(\frac{1}{k_n} - \frac{1}{k_n^0}\right),$$

where

$$H_1(t) = -\frac{ct}{\pi}\sum_{n=2}^{\infty}\frac{\sin(n + \delta_n(\pi, \beta))t}{n + \delta_n(\pi, \beta)},$$

$$H_2(t) = -\frac{2t}{\pi}\sum_{n=2}^{\infty}l_n\sin(n + \delta_n(\pi, \beta))t = -\frac{2t}{\pi}l(t),$$

$$\begin{aligned}H_3(t) = &-\textstyle\sum_{n=2}^{\infty}\left(-\frac{4}{\pi^2}\frac{s_n}{n + \delta_n(\pi, \beta)} + h_n\right)\epsilon_n t\sin(n + \delta_n(\pi, \beta))t\\&+\textstyle\sum_{n=2}^{\infty}\left(\frac{2}{\pi} - \frac{4}{\pi^2}\frac{s_n}{n + \delta_n(\pi, \beta)} + h_n\right)m_n(t)\cos(n + \delta_n(\pi, \beta))t\\&-\textstyle\sum_{n=2}^{\infty}\left(\frac{2}{\pi} - \frac{4}{\pi^2}\frac{s_n}{n + \delta_n(\pi, \beta)} + h_n\right)z_n(t)\sin(n + \delta_n(\pi, \beta))t,\end{aligned}$$

$$H_4(t) = -\frac{4}{\pi^2}\sum_{n=2}^{\infty}\frac{s_n}{n+\delta_n(\pi,\beta)}\cos(n+\delta_n(\pi,\beta))t = -\frac{4}{\pi^2}s(t),$$

$$H_5(t) = \sum_{n=2}^{\infty}(h_n - g_n)\cos(n+\delta_n(\pi,\beta))t. \tag{2.13}$$

We separate first two term, since we have no $\delta_0(\alpha,\beta)$ and $\delta_1(\alpha,\beta)$ and only after $n \geq 2$ we can use (2.1)-(2.4). Since $T_\beta(x) := \sum_{n=2}^{\infty}\frac{\sin(n+\delta_n(\pi,\beta))x}{n+\delta_n(\pi,\beta)}$ is an absolutely continuous function (see [13] and [16]) on arbitrary segment $[a,b] \subset (0,2\pi)$ and $T_\beta(0) = 0$, then $H_1 \in AC[0,2\pi)$. Similarly, since $l(t)$ is an absolutely continuous function on arbitrary segment $[a,b] \subset (0,2\pi)$ (see (2.3)) and $l(0) = 0$, then $H_2 \in AC[0,2\pi)$. The estimates $s_n = o(1)$, $h_n = O\left(\frac{1}{n^2}\right)$, $\epsilon_n = O\left(\frac{1}{n}\right)$, $m_n(t) = O\left(\frac{1}{n^2}\right)$ and $z_n(t) = O\left(\frac{1}{n^3}\right)$ ensure the absolute and uniform convergence of the series in (2.13) and absolute continuity of the function H_3 on $[0,2\pi]$. The condition (2.5) implies absolute continuity of the function H_4 on $[0,2\pi]$. The estimates $h_n = O\left(\frac{1}{n^2}\right)$ and $g_n = O\left(\frac{1}{n^2}\right)$ ensure the absolute and uniform convergence of the series in (2.13) and absolute continuity of the function H_5 on $[0,2\pi]$. Due to $s(0) = \sum_{n=2}^{\infty}\frac{s_n}{n+\delta_n(\pi,\beta)} < +\infty$ (see (2.5)), we have $\sum_{n=2}^{\infty}\left(\frac{1}{k_n} - \frac{1}{k_n^0}\right) < \infty$ (see (2.11)). Lemma 2.1 is proved. □

Lemma 2.2 *$\hat{F}(x,t)$ is a continuous function in $\{[0,\pi]\times[0,\pi]\}\setminus\{(\pi,\pi)\}$ and $\frac{d}{dx}\hat{F}(x,x) \in L^1_{\mathbb{R}}(0,\pi)$.*

Proof It follows from (2.8) and lemma 2.1. □

3 Derivation of an Analogue of the Gelfand-Levitan Equation

Theorem 3.1 *For each fixed $x \in (0,\pi]$, the kernel $P(x,t)$ of the transmutation operator (see (1.4)) satisfies the following linear integral equation*

$$P(x,t) + \hat{F}(x,t) + \int_0^x P(x,s)\hat{F}(s,t)\,ds = 0,\ \ 0 \leq t < x, \tag{3.1}$$

which is also called Gelfand-Levitan equation.

Proof Since $\mathbb{P}$ defined in (1.4) is a Volterra integral operator with continuous kernel $P(x,t)$, then $\mathbb{I}+\mathbb{P}$ has the inverse operator of the same type (see, for example [2, 5, 17, 18]), which we denote by $(\mathbb{I}+\mathbb{Q})$. Solving Eq. (1.4) with respect to $\dfrac{\sin\lambda x}{\lambda}$ we obtain

$$\frac{\sin\lambda x}{\lambda} = (\mathbb{I}+\mathbb{Q})\varphi(x,\lambda^2) := \varphi(x,\lambda^2) + \int_0^\pi Q(x,t)\varphi(t,\lambda^2)dt, \tag{3.2}$$

where $Q(x,t)$, $0 \le t \le x \le \pi$ is a real continuous function with the same smoothness as $P(x,t)$, and

$$Q(x,x) = -\frac{1}{2}\int_0^x q(t)dt, \quad Q(x,0) = 0.$$

By taking $\lambda = \lambda_n$ in (1.4) and multiplying both sides by $\dfrac{\sin\lambda_n t}{\lambda_n}$, we obtain

$$\begin{aligned}\sum_{n=2}^N \frac{\varphi(x,\lambda_n^2)\dfrac{\sin\lambda_n t}{\lambda_n}}{a_n} &= \sum_{n=2}^N \frac{1}{a_n}\left(\frac{\sin\lambda_n x}{\lambda_n} + \int_0^x P(x,s)\frac{\sin\lambda_n s}{\lambda_n}ds\right)\\ \times\frac{\sin\lambda_n t}{\lambda_n} &= \sum_{n=2}^N \frac{1}{a_n}\left(\frac{\sin\lambda_n x\,\sin\lambda_n t}{\lambda_n^2} + \frac{\sin\lambda_n t}{\lambda_n}\int_0^x P(x,s)\frac{\sin\lambda_n s}{\lambda_n}ds\right).\end{aligned} \tag{3.3}$$

On the other hand, if, on the left side of the (3.3), instead of $\dfrac{\sin\lambda_n t}{\lambda_n}$ substitute the expression (3.2) $(\lambda = \lambda_n)$, we get

$$\begin{aligned}&\sum_{n=2}^N \frac{\varphi(x,\lambda_n^2)\dfrac{\sin\lambda_n t}{\lambda_n}}{a_n} = \sum_{n=2}^N \frac{1}{a_n}\varphi(x,\lambda_n^2)\Bigg(\varphi(t,\lambda_n^2)\\ &+\int_0^t Q(t,s)\varphi(s,\lambda_n^2)ds\Bigg) = \sum_{n=2}^N \frac{1}{a_n}\Bigg(\varphi(x,\lambda_n^2)\varphi(t,\lambda_n^2)\\ &+\varphi(x,\lambda_n^2)\int_0^t Q(t,s)\varphi(s,\lambda_n^2)ds\Bigg).\end{aligned}$$

Let us denote by $\Phi_N(x,t)$ the following expression

$$\Phi_N(x,t) := \sum_{n=0}^N\left(\frac{1}{a_n}\varphi(x,\lambda_n^2)\varphi(t,\lambda_n^2) - \frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 x}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right).$$

It is easy to calculate that the following representation holds (compare with [7])

$$\Phi_N(x,t) = I_{N1}(x,t) + I_{N2}(x,t) + I_{N3}(x,t) + I_{N4}(x,t), \tag{3.4}$$

where

$$I_{N1}(x,t) = \sum_{n=0}^{N} \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{1}{a_n^0} \frac{\sin \lambda_n^0 x}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right),$$

$$I_{N2}(x,t) = \sum_{n=0}^{N} \frac{1}{a_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \int_0^x P(x,s) \frac{\sin \lambda_n^0 s}{\lambda_n^0} ds \Big),$$

$$I_{N3}(x,t) = \textstyle\sum_{n=0}^{N} \int_0^x P(x,s) \left(\frac{1}{a_n} \frac{\sin \lambda_n s}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} \right.$$
$$\left. - \frac{1}{a_n^0} \frac{\sin \lambda_n^0 s}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right),$$

$$I_{N4}(x,t) = -\sum_{n=0}^{N} \frac{\varphi(x,\lambda_n^2)}{a_n} \int_0^t Q(t,s) \varphi(s,\lambda_n^2) ds.$$

□

It is known (see [19]) the following expansion theorem.

Theorem A ([19]) *Let* $q \in L^1_{\mathbb{R}}[0,\pi]$ *and let* f *be an absolutely continuous function on* $[0,\pi]$ *and* $f(0) = 0$. *Then*

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi(x,\lambda_n^2), \quad c_n = \frac{1}{a_n} \int_0^{\pi} f(t) \varphi(t,\lambda_n^2) dt,$$

where the series converge uniformly on $[0,\pi]$, *i.e.*

$$\lim_{N\to\infty} \max_{x\in[0,\pi]} \left| f(x) - \sum_{n=0}^{N} c_n \varphi(x,\lambda_n^2) \right| = 0.$$

If $q(x) \equiv 0$, *then* $\varphi(x, \lambda_n^2) = \varphi(x, \lambda_n^2(0, \pi, \beta), \pi, 0) = \dfrac{\sin \lambda_n(0, \pi, \beta)x}{\lambda_n(0, \pi, \beta)} = \dfrac{\sin \lambda_n^0 x}{\lambda_n^0}$. *In light of Theorem A we receive*

$$\begin{aligned}
&\lim_{N\to\infty} \max_{x\in[0,\pi]} \int_0^\pi f(t)\Phi_N(x,t)dt = \\
&\lim_{N\to\infty} \max_{x\in[0,\pi]} \sum_{n=0}^N \frac{1}{a_n} \int_0^\pi f(t)\phi(t,\lambda_n^2)dt\phi(x,\lambda_n^2) \\
&-\lim_{N\to\infty} \max_{x\in[0,\pi]} \sum_{n=0}^N \frac{1}{a_n^0} \int_0^\pi f(t)\frac{\sin\lambda_n^0 t}{\lambda_n^0}dt\frac{\sin\lambda_n^0 x}{\lambda_n^0} = f(x) - f(x) = 0.
\end{aligned} \tag{3.5}$$

Extending $P(x,t) = Q(x,t) = 0$ for $x < t$ and using that for fixed x $P(x,t)$, $F(x,t)$ and $Q(x,t)$ are absolutely continuous by t and also that $P(x,0) = F(x,0) = Q(x,0) = 0$, we can done limit process under the integral sign and get that uniformly with respect to $x \in [0,\pi]$

$$\begin{aligned}
&\lim_{N\to\infty} \int_0^\pi f(t)I_{N1}(x,t)dt = \lim_{N\to\infty} \int_0^\pi f(t) \sum_{n=0}^N \\
&\left(\frac{1}{a_n}\frac{\sin\lambda_n s}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n} - \frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 s}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)dt \\
&= \int_0^\pi f(t) \lim_{N\to\infty} \sum_{n=0}^N \left(\frac{1}{a_n}\frac{\sin\lambda_n s}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n}\right. \\
&\left.-\frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 s}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)dt = \int_0^\pi f(t)\hat{F}(x,t)dt,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\lim_{N\to\infty} \int_0^\pi f(t)I_{N2}(x,t)dt = \lim_{N\to\infty} \int_0^\pi f(t) \sum_{n=0}^N \frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 t}{\lambda_n^0} \\
&\int_0^x P(x,s)\frac{\sin\lambda_n^0 s}{\lambda_n^0}dsdt = \lim_{N\to\infty} \sum_{n=0}^N \frac{1}{a_n(0,\pi,\beta)} \int_0^\pi \int_0^\pi f(t)\frac{\sin\lambda_n^0 t}{\lambda_n^0} \\
&P(x,s)\frac{\sin\lambda_n^0 s}{\lambda_n^0}dtds = \int_0^\pi \lim_{N\to\infty} \sum_{n=0}^N \frac{1}{a_n(0,\pi,\beta)} \int_0^\pi f(t)\frac{\sin\lambda_n^0 t}{\lambda_n^0}dt \\
&\frac{\sin\lambda_n^0 s}{\lambda_n^0}P(x,s)ds = \int_0^\pi f(s)P(x,s)ds = \int_0^\pi f(t)P(x,t)dt,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\lim_{N\to\infty} \int_0^\pi f(t)I_{N3}(x,t)dt = \lim_{N\to\infty} \int_0^\pi f(t) \sum_{n=0}^N \int_0^x P(x,s) \\
&\left(\frac{1}{a_n}\frac{\sin\lambda_n s}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n} - \frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 s}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)dsdt \\
&= \int_0^\pi f(t) \int_0^x P(x,s) \lim_{N\to\infty} \sum_{n=0}^N \left(\frac{1}{a_n}\frac{\sin\lambda_n s}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n} - \right. \\
&\left.\frac{1}{a_n(0,\pi,\beta)}\frac{\sin\lambda_n^0 s}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)dsdt = \int_0^\pi f(t)\left(\int_0^x P(x,s)\hat{F}(s,t)ds\right)dt,
\end{aligned} \tag{3.8}$$

$$\begin{aligned} &\lim_{N\to\infty} \int_0^\pi f(t) I_{N4}(x,t)dt = -\lim_{N\to\infty} \int_0^\pi f(t) \textstyle\sum_{n=0}^N \dfrac{\varphi(x,\lambda_n^2)}{a_n} \\ &\int_0^t Q(t,s)\varphi(s,\lambda_n^2)dsdt = -\lim_{N\to\infty} \textstyle\sum_{n=0}^N \dfrac{\varphi(x,\lambda_n^2)}{a_n} \int_0^\pi f(t) \\ &\int_0^t Q(t,s)\varphi(s,\lambda_n^2)dsdt = -\lim_{N\to\infty} \textstyle\sum_{n=0}^N \dfrac{\varphi(x,\lambda_n^2)}{a_n} \\ &\int_0^\pi \left(\int_s^\pi Q(t,s) f(t)dt \right)\varphi(s,\lambda_n^2)ds = -\int_0^\pi Q(t,x)f(t)dt, \end{aligned} \tag{3.9}$$

where $Q(t,x) = 0$ for $t < x$. From (3.4) and (3.5)–(3.9), we can conclude that

$$\int_0^\pi f(t)\left(F(x,t) + P(x,t) + \int_0^x P(x,s)F(s,t)ds - Q(t,x)\right)dt = 0.$$

Since the system of eigenfunctions $\{\varphi_m(t)\}_{m=0}^\infty$ of the boundary value problem $L(q,\pi,\beta)$ is complete in $L^2(0,\pi)$ and $\varphi_m(0) = 0,\ m = 0,1,2,\ldots,$ then we can take $f(t) = \varphi_m(t),\ m = 0,1,2,\ldots$ and obtain that for each fixed $x \in (0,\pi]$

$$F(x,t) + P(x,t) + \int_0^x P(x,s)F(s,t)ds - Q(t,x) = 0.$$

For $t < x$, this is equivalent to (3.1). This completes the proof. □

Thus, we prove Theorem 3.1 with kernel $\hat{F}(x,t)$. In particular, when $\beta = \dfrac{\pi}{2}$ it will be proved for $F(x,t)$.

Observe that the solutions $P(x,t)$ of (3.1) has the same smoothness as $\hat{F}(x,t)$.

Theorem 3.2 *For each fixed* $x \in (0,\pi]$, *Eq.* (3.1) *has a unique solution* $P(x,\cdot)$ *in* $L^2[0,x)$.

Proof Since (3.1) is a Fredholm equation, then it is sufficient (see, for example, [18]) to prove that the homogeneous equation

$$p(t) + \int_0^x p(s)F(s,t)ds = 0 \tag{3.10}$$

has only a trivial solution $p(t) \equiv 0$.

Let $p(t) \not\equiv 0$ be a solution of (3.10). Then, multiplying (3.10) by $p(t)$ and integrating from 0 to x, we receive

$$\int_0^x p^2(t)dt + \int_0^x \int_0^x F(s,t)p(s)p(t)dsdt = 0$$

or

$$\int_0^x p^2(t)dt + \sum_{n=0}^{\infty} \frac{1}{a_n} \int_0^x p(s) \frac{\sin \lambda_n s}{\lambda_n} ds \int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt$$
$$- \sum_{n=0}^{\infty} \frac{1}{a_n^0} \int_0^x p(s) \frac{\sin \lambda_n^0 s}{\lambda_n^0} ds \int_0^x p(t) \frac{\sin \lambda_n^0 t}{\lambda_n^0} dt = 0,$$

which is equivalent to

$$\int_0^x p^2(t)dt + \sum_{n=0}^{\infty} \frac{1}{a_n} \left(\int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt \right)^2$$
$$- \sum_{n=0}^{\infty} \frac{1}{a_n^0} \left(\int_0^x p(t) \frac{\sin \lambda_n^0 t}{\lambda_n^0} dt \right)^2 = 0.$$

Since $\left\{ \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right\}_{n=0}^{\infty}$ are the eigenfunctions of the problem $L(0, \pi, \beta)$, then we can extend the function $p(t)$ by zero for $t > x$ and write Parseval's identity in the following form

$$\int_0^{\pi} p^2(t)dt = \int_0^x p^2(t)dt = \sum_{n=0}^{\infty} \frac{1}{a_n^0} \left(\int_0^x p(t) \frac{\sin \lambda_n^0 t}{\lambda_n^0} dt \right)^2 = 0.$$

and therefore,

$$\frac{1}{\lambda_n} \int_0^x p(t) \sin \lambda_n t dt = 0, n \geqslant 0.$$

In paper [20] it was proved that the system of functions $\left\{ \sin \lambda_n t \right\}_{n=0}^{\infty} = \left\{ \sin \lambda_n (q, \pi, \beta) t \right\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$ and accordingly is complete in $L^2[0, \pi]$. Hence, we have $p(t) = 0$. This completes the proof. □

Remark It is networthy that, in contrast to the above-mentioned classical results (see [6, 8, 17, 19], in our case the Levinson's theorem (see [21–23]) cannot be applied for proving completeness of the system of functions $\left\{ \sin \lambda_n t \right\}_{n=0}^{\infty}$, because of asymptotic behavior of λ_n.

4 The Representation of $P(x, t)$ in Terms of Eigenfunctions

If in Eq. (3.1) we substitute the form (2.6) of $\hat{F}(x, t)$, we obtain

$$P(x,t) + \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\sin\lambda_n x}{\lambda_n}\frac{\sin\lambda_n t}{\lambda_n} - \frac{1}{a_n^0}\frac{\sin\lambda_n^0 x}{\lambda_n^0}\frac{\lambda_n^0 t}{\lambda_n^0}\right)$$
$$+ \int_0^x P(x,s)\left(\sum_{n=0}^{\infty}\left(\frac{1}{a_n}\frac{\sin\lambda_n s}{\lambda_n}\frac{\lambda_n t}{\lambda_n} - \frac{1}{a_n^0}\frac{\sin\lambda_n^0 s}{\lambda_n^0}\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right)\right)ds = 0,$$

If we were able to insert the integral under the sign of infinite sum, then we would obtain

$$P(x,t) = -\sum_{n=0}^{\infty}\left\{\frac{1}{a_n}\left(\frac{\sin\lambda_n x}{\lambda_n} + \int_0^x P(x,s)\frac{\sin\lambda_n s}{\lambda_n}ds\right)\frac{\sin\lambda_n t}{\lambda_n}\right.$$
$$\left.-\frac{1}{a_n^0}\left(\frac{\sin\lambda_n^0 x}{\lambda_n^0} + \int_0^x P(x,s)\frac{\sin\lambda_n^0 s}{\lambda_n^0}ds\right)\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right\} =$$
$$-\sum_{n=0}^{\infty}\left\{\frac{1}{a_n}\varphi(x,\lambda_n^2)\frac{\sin\lambda_n t}{\lambda_n} - \frac{1}{a_n^0}\varphi(x,(\lambda_n^0)^2)\frac{\sin\lambda_n^0 t}{\lambda_n^0}\right\}.$$

In particular, if $\beta = \dfrac{\pi}{2}$, then $\lambda_n^0 = n + \dfrac{1}{2}$, $a_n^0 = \dfrac{\pi}{2\left(n+\frac{1}{2}\right)^2}$, and we would obtain

$$P(x,t) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{\pi}\varphi\left(x,\left(n+\frac{1}{2}\right)^2\right)\sin\left(n+\frac{1}{2}\right)t\right.$$
$$\left.-\frac{1}{a_n(q,\pi,\beta)}\varphi(x,\lambda_n)\frac{\sin\lambda_n t}{\lambda_n}\right),$$

i.e. (1.7). But the permutation the integral and infinite sum is possible in sense of (3.5)–(3.9) (was done in Sect. 3). Therefore the representation (1.7) we must understand in sense of (1.8), (1.9).

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On a Riemann Boundary Value Problem with Infinite Index in the Half-plane

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Abstract The paper considers the Riemann boundary value problem in the half-plane in the space $L^1(\rho)$. The weight function $\rho(x)$ has infinite number of zeros on the real axis. The boundary condition is understood in the sense of $L^1(\rho)$. A necessary and sufficient condition is obtained for the normal solvability of the considered problem. The solutions are represented in explicit form.

Keywords Riemann boundary value problem · Weight function · Infinite index · Homogeneous problem

1 Introduction

Let $\Pi^{\pm}$ be the upper and lower half-planes of the complex plane C, and let A be the class of functions Φ analytic in $\Pi^{+} \cup \Pi^{-}$ satisfying the condition

$$|\Phi(z)| \leq C|z|^{n_0}, \quad |Imz| \geq y_0 > 0,$$

where n_0 is a natural number and C is a constant, possibly depending on y_0. If G^{+} is simply connected domain of the complex plane C limited by a curved curve L, and G^{-} is the complement set of $G^{+} \cup L$, then the Riemann boundary value problem investigated by many authors. Here are works [1–7], where you can find detailed

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links: Find an analytic function Φ^+ and Φ^-, $(\Phi^-(\infty) = 0)$ in G^+ and G^- such that

$$\Phi^+(t) - a(t)\Phi^-(t) = f(t), t \in L,$$

where a and f are given functions on L. Investigation of this problem in the classes C^α and L^p, $1 < p < \infty$ is based on the fact that Cauchy type integral is a bounded operator in the spaces C^α and L^p, $1 < p < \infty$.

The Riemann and Dirichlet problem in weighted spaces $L^p(\rho)$, $1 < p < \infty$ was also investigated in papers [8, 9]. Boundary value problems for partial differential equations with weighty boundary conditions were studied in papers [10–13].

For the investigation of the Riemann boundary value problem in the class L^1 another formulation of this problem is proposed, which in the case of the unit circle has the following form [14]: find an analytic function Φ^+ and Φ^- in $\Pi^+ \cup \Pi^-$ such that

$$\lim_{y \to +0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{L^1} = 0.$$

In similar settings the Riemann boundary value problem is investigated in spaces C, L^∞, W (see [15–17]).

In the works [18–20] the Riemann boundary value problem are investigated in the weighted spaces, where weight is concentrated on finite number of singular points. In the work [21] the same problem is investigated in the unit circle when the weight function has a countable number of zeros on the boundary and can be represented as:

$$\rho(t) = \prod_{k=1}^{\infty} |t - t_k|^{\alpha_k},$$

at that $t_k = e^{i\theta_k}$ and $\{\theta_k\}_1^\infty$ is a monotonic sequence tending to zero. In that work it is shown that in particularly, when $a(t) \equiv 1$, if the condition $\sum_{k=1}^{\infty} \alpha_k \ln|1 - t_k| = -\infty$ holds, then the homogeneous problem has an infinite number of linearly independent solutions, which are represented

$$\Phi_0(z) = \frac{A_0}{1-z} + \sum_{k=1}^{\infty} \frac{A_k}{t_k - z},$$

where $\{A_k\} \in l^1$ and A_0 is an arbitrary complex number. Otherwise, when $\sum_{k=1}^{\infty} \alpha_k \ln|1 - t_k| > -\infty$, $\Phi_0(z)$ is represented $\Phi_0(z) = \sum_{k=1}^{\infty} \frac{A_k}{t_k - z}$.

In this work we consider the Riemann boundary value problem in the upper half-plane $\Pi^+ = \{z; Imz > 0\}$ in the following statement:

Problem R Let $f \in L^1(\rho)$. Find an analytic function Φ in $\Pi^+ \cup \Pi^-$ ($\Pi^- = \{z; Imz < 0\}$) such that

$$\lim_{y \to +0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{L^1(\rho)} = 0, \tag{1}$$

where $a(x) \in C^\delta(-\infty; +\infty)$, $a(x) \neq 0$,

$$|a(x) - a(\infty)| < C|x|^{-\delta} \qquad \text{for} \quad |x| \geq C > 0. \tag{2}$$

and

$$\rho(x) = \rho_1(x) \prod_{k=1}^{\infty} \left| \frac{x - x_k}{x + i} \right|^{\alpha_k}, \quad \rho_1(x) = \frac{1}{(1 + |x|)^{\alpha_0}}, \tag{3}$$

at that

$$\sum_{k=1}^{\infty} \alpha_k < \infty,$$

where $0 < \alpha_k < 1$, $k = 1, 2, \ldots$ and $0 < \alpha_0 < 1$.

Here, we will consider the case, when the sequence $\{x_k\}_1^\infty$ has a finite number of condensation points $\{t_k\}_1^N$ which does not belong to it. Take the points t'_k such that $t_k \in (t'_k, t'_{k+1})$, $k = 1, 2, \ldots, N$ and denote

$$T_k = \{x_k\}_1^\infty \cap (t'_k, t'_{k+1}) = \{x_{kj}\}_{j=1}^\infty.$$

Denote

$$X = \cup_{k=1}^N [t'_k, t'_{k+1}) \quad \text{and} \quad X_\infty = (-\infty, t'_1) \cup [t'_{N+1}, \infty). \tag{4}$$

So, we obtain

$$(-\infty, +\infty) = X \cup X_\infty.$$

In that case the sequence $\{x_k\}_1^\infty$ can be represented as $\{x_k\}_1^\infty = \sum_{k=1}^N T_k$.

It should be noted that similar problem in $C(\rho)$ was investigated in the paper [22]. Note that a similar homogeneous problem in $L^p(\rho)$, $p > 1$ has a finite number of linearly independent solutions. It is established that in the case $p = 1$ and $a(x) = 1$ the homogeneous problem R has an infinite number of linearly independent solutions $(x_k - z)^{-1}$, $k = 1, 2, \ldots$.

2 Some Auxiliary Results

1. *First, consider the case* $a(x) \equiv 1$.

Lemma 1 *Let the sequence* $\{A_k\}_0^\infty \in l^1$ *is arbitrary. Then*

$$\Phi(z) = \sum_{k=1}^{\infty} \frac{A_k}{x_k - z} \tag{5}$$

is a solution of the homogeneous problem

$$\lim_{y \to +0} \|\Phi^+(x+iy) - \Phi^-(x-iy)\|_{L^1(\rho)} = 0. \tag{6}$$

Proof Let $\Phi_k(z) = (x_k - z)^{-1}$. Then

$$\Phi_k(x+iy) - \Phi_k(x-iy) = \frac{y}{(x_k - x)^2 + y^2}$$

and

$$\lim_{y \to +0} \int_{-\infty}^{\infty} \frac{|x_k - x|^{\alpha_k} y}{\left((x_k - x)^2 + y^2\right)(1+|x|)^{\alpha_0}|x+i|^{\alpha_k}} dx = 0.$$

Further, we have

$$\Phi(x+iy) - \Phi(x-iy) = \sum_{k=1}^{\infty} \frac{A_k y}{(x_k - x)^2 + y^2}.$$

Since this sum converges uniformly in $\Pi^+ \cup \Pi^-$, then

$$\int_{-\infty}^{\infty} |\Phi(x+iy) - \Phi(x-iy)|\rho(x)dx < C \sum_{k=1}^{\infty} A_k y^{1-\alpha_k}$$

Taking into account, that

$$\lim_{y \to +0} \sum_{k=1}^{\infty} |A_k| y^{1-\alpha_k} = 0$$

we obtain the proof of the lemma. □

Let's present the sequence T_k in the view $T_k = T_k^+ \cup T_k^-$, where T_k^+ the set of those points from T_k are greater than or equal to t_k. Respectively T_k^- the set of those points from T_k which are less than t_k. We say that the limit points $t_k \in T_0$ if

$$|x_j - x_{k_p}| > c|x_j - t_k|, \;\; j \neq k_p, \tag{7}$$

where $c > 0$ is a constant independent of j and k_p.

Lemma 2 *The function $\rho(x)$ is a continuous at the point of condensation t_k if and only if*

$$\prod_{j=1}^{\infty} \left| \frac{x_j - t_k}{x_j + i} \right|^{\alpha_j} = 0. \tag{8}$$

Proof It is enough to establish continuity at the point t_k of the function

$$\rho_k(x) = \prod_{j=1}^{\infty} \left| \frac{x - x_{k_j}}{x + i} \right|^{\alpha_j}$$

where the subsequence x_{k_j} is the corresponding subsequence tending to t_k. Since $\rho(x_k) = 0$, $k = 0, 1, \ldots$, then the sufficiency of the condition (8) is obvious.

Consider the function

$$R(x) = \sum_{k=1}^{\infty} \delta_k \ln \frac{|t_k - x|}{|i + x|}.$$

Let us prove that $\lim_{t \to 0} R(t) = -\infty$. Let $A > 0$ arbitrary number. Choose $N > 0$ such that

$$\sum_{j=1}^{N} \delta_k \ln \frac{|x_j - t_k|}{|x_j + i|} < -A.$$

The number $\delta > 0$ can be chosen so, that for $|t_k - x| < \delta$ holds

$$\sum_{j=1}^{N} \delta_k \ln \frac{|x - t_k|}{|x + i|} < -A.$$

In consequence, we obtain

$$\sum_{j=1}^{\infty} \delta_k \ln \frac{|x - t_k|}{|x + i|} < -\frac{A}{2}.$$

for $|t_k - x| < \delta$. □

Corollary 1 *It needs to note, that if the condition (7) holds for any $k = 1, 2, \ldots, N$, then the function $\rho(x)$ will be continuous at everywhere.*

Corollary 2 *The function $\rho(x)$ will be continuous at points t_k, under the condition*

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|t-t_k|<\Delta t} \rho(t)dt = 0. \tag{9}$$

Lemma 3 *Let the sequence $\{x_j\}_1^\infty$ satisfies the following conditions in a neighborhood of the limit points t_k:*

$$\sum_{j=1}^{\infty} \delta_j \ln \frac{|x_j - t_k|}{|x_j + i|} > -\infty; \quad |x_j - x_m| > c|x_j - t_k|, \quad j \neq m. \tag{10}$$

for some fixed $c > 0$. Then

$$\inf \rho_m = \rho_0 > 0, \quad m = 1, 2, \ldots, \tag{11}$$

where

$$\rho_m = \prod_{j \neq m}^{\infty} \left(\frac{|x_j - x_m|}{|x_j + i|} \right)^{\alpha_j}.$$

Proof It is enough to establish for the subsequence $\{x_{k_j}\}_{j=1}^\infty$.

From the second condition of (10), we have

$$\left| \frac{x_{k_j} - x_{k_m}}{x_{k_j} + i} \right|^{\alpha_j} > c^{\alpha_j} \left| \frac{x_{k_j} - t_k}{x_{k_j} + i} \right|^{\alpha_j}$$

and

$$\prod_{j \neq m}^{\infty} \left(\frac{|x_{k_j} - x_{k_m}|}{|x_{k_j} + i|} \right)^{\alpha_j} > \prod_{j \neq m}^{\infty} c^{\alpha_j} \prod_{j \neq m}^{\infty} \left(\frac{|x_{k_j} - t_k|}{|x_{k_j} + i|} \right)^{\alpha_j}.$$

From the first condition of (10) follows that $\delta > 0$ exists such that $\inf \rho_m = \delta > 0, m = 1, 2, \ldots$. □

Let us denote

$$\delta_{k_n}(x) = \prod_{j \neq n}^{\infty} \left| \frac{x - x_{k_j}}{x + i} \right|^{\alpha_{k_j}}, \quad k = 1, 2, \ldots, N$$

and

$$\delta(x) = \delta_{k_{n+1}}(x) - \delta_{k_n}(x), \qquad x \in [t_k, t_{k+1}).$$

Lemma 4 *There* $x'_{k_n} \in [x_{k_n}, x_{k_{n+1}})$*, where* $k = 1, 2, \ldots, N$ *and* $n = 1, 2, \ldots$ *exist such that* $\delta(x'_{k_n}) = 0$*.*

Proof Consider

$$\delta_{k_{n+1}}(x) - \delta_{k_n}(x) = \prod_{j \neq n+1}^{\infty} \left| \frac{x - x_{k_j}}{x+i} \right|^{\alpha_{k_j}} - \prod_{j \neq n}^{\infty} \left| \frac{x - x_{k_j}}{x+i} \right|^{\alpha_{k_j}} =$$

$$= \prod_{j \neq n, n+1}^{\infty} \left| \frac{x - x_{k_j}}{x+i} \right|^{\alpha_{k_j}} \cdot \left| \left| \frac{x - x_{k_n}}{x+i} \right|^{\alpha_{k_n}} - \left| \frac{x - x_{k_{n+1}}}{x+i} \right|^{\alpha_{k_{n+1}}} \right|.$$

We can choose $c_1, c_2 > 0$ such that

$$\delta(x_{k_{n+1}} - c_1) = |-c_1|^{\alpha_{k_{n+1}}} - |x_{k_{n+1}} - x_{k_n} - c_1|^{\alpha_{k_n}} < 0$$

and

$$\delta(x_{k_n} + c_2) = |x_{k_n} - x_{k_{n+1}} + c_2|^{\alpha_{k_{n+1}}} - |c_2|^{\alpha_{k_n}} > 0.$$

Taking into account that $\delta(x)$ is continuous on $[x_{k_n}, x_{k_{n+1}})$, we see that the equation $\delta(x) = 0$ has a solution. By pointing out those points with x'_{k_n}, we obtain the proof of the lemma. □

Denote $X_{k_n} = [x'_{k_{n-1}}, x'_{k_n}),\ k = 1, 2, \ldots, N,\ n = 1, 2, \ldots$. We will obtain

$$X = \cup_{k=1}^{N} \cup_{n=1}^{\infty} X_{k_n},$$

where X is defined by (4). It is obvious that $X_{k_n} \cap X_{k_{n+1}} = \emptyset$, $n = 1, 2, \ldots$.

Lemma 5 *Let the sequence of points* $\{x_j\}_1^{\infty}$ *satisfies condition* (10)*. Then there exists* $\delta > 0$ *such that for any* $k = 1, 2, \ldots N,\ n = 1, 2, \ldots$:

$$\inf_{x \in X_{k_n}} \delta_{k_n}(x) > \delta > 0.$$

Proof Let $x \in (x'_{k_{n-1}}, x_{k_n})$, then $|x_{k_j} - x| \geq |x_{k_j} - x_{k_n}|$ at $j \geq n+1$. If $j < n$, then we have $|x_{k_j} - x| > |x_{k_j} - x_{k_{n-1}}|$. Using Lemma 3, we get

$$\prod_{j \geq n+1} \left| \frac{x - x_{k_j}}{x+i} \right|^{\alpha_{k_j}} > \prod_{j \geq n+1} \left| \frac{x_{k_j} - x_{k_n}}{x+i} \right|^{\alpha_{k_j}} > \delta > 0$$

and

$$\prod_{j<n}\Big|\frac{x-x_{k_j}}{x+i}\Big|^{\alpha_{k_j}} > \prod_{j<n}\Big|\frac{x_{k_j}-x_{k_{n-1}}}{x+i}\Big|^{\alpha_{k_j}} > \delta.$$

Hence, $\delta_{k_n}(x) \geq \delta^2$, $x \in (x'_{k_{n-1}}, x_{k_n})$.

Let now $x \in (x_{k_n}, x'_{k_n})$, then at $j < n$ we have $|x_{k_j} - x| > |x_{k_j} - x_{k_n}|$ and

$$\prod_{j=1}^{n-1}\Big|\frac{x-x_{k_j}}{x+i}\Big|^{\alpha_{k_j}} \geq \prod_{j=1}^{n-1}\Big|\frac{x_{k_j}-x_{k_n}}{x+i}\Big|^{\alpha_{k_j}} > \delta.$$

At $j \geq n+1$ we have $|x_{k_j} - x| > |x_{k_j} - x_{k_{n+1}}|$. We get

$$\prod_{j>n}\Big|\frac{x-x_{k_j}}{x+i}\Big|^{\alpha_{k_j}} > \prod_{j>n}\Big|\frac{x_{k_j}-x_{k_{n+1}}}{x+i}\Big|^{\alpha_{k_j}} > \delta.$$

Hence, $\delta_k > \delta^2$. □

2. *Now we consider the case, when* $a(x) \in C^{\delta}(\mathbb{R})$, $\delta \in (0, 1]$ *and the condition* (2) *holds for* $a(x)$.

Let $\kappa = inda(t)$, $t \in (-\infty, +\infty)$,

$$S^+(z) = exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^+,$$
$$S^-(z) = \Big(\frac{z+i}{z-i}\Big)^{\kappa} exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z \in \Pi^-, \tag{12}$$

where

$$a_1(t) = \Big(\frac{t+i}{t-i}\Big)^{\kappa} a(t), \qquad inda_1(t) = 0.$$

Let

$$f_{k_j}(x) = \begin{cases} f(x), & x \in X_{k_j} \\ 0, & x \in \mathbb{R}\backslash X_{k_j} \end{cases}.$$

Define the function $\Phi_{k_j}(z)$ as follows

$$\Phi_{k_j}(z) = \frac{S(z)}{2\pi i(x_j - z)}\int_{-\infty}^{+\infty}\frac{f_{k_j}(t)(x_j - t)dt}{S^+(t)(t-z)}, \ j = 1, 2, \ldots \tag{13}$$

Lemma 6 *The estimate*

$$\|\Phi^+_{k_j}(x+iy) - a(x)\Phi^-_{k_j}(x-iy)\|_{L^1(\rho)} \le C\|f_{k_j}\|_{L^1(\rho)},$$

where the constant C *is independent of* y, k *and* j, *is true. The limit relation*

$$\lim_{y\to+0}\|\Phi^+_{k_j}(x+iy) - a(x)\Phi^-_{k_j}(x-iy) - f_{k_j}(x)\|_{L^1(\rho)} = 0$$

also holds.

Proof Note that

$$\Phi^+_{k_j}(x+iy) - a(x)\Phi^-_{k_j}(x-iy) = I_1(f,x,y) + I_2(f,x,y).$$

where

$$I_1(f,x,y) = \frac{y}{\pi}\frac{S(x+iy)}{x_k - x - iy}\int_{X_{k_j}}\frac{f_{k_j}(t)(x_j-t)dt}{S^+(t)((t-x)^2+y^2)},$$

$$I_2(f,x,y) = \frac{yT(x;y)}{2\pi i}\int_{X_{k_j}}\frac{f_{k_j}(t)(x_j-t)dt}{S^+(t)(t-x+iy)},$$

where

$$T(x;y) = \frac{S(x+iy)}{x_j - x - iy} - \frac{a(x)S(x-iy)}{x_j - i + iy}.$$

As

$$\int_{-\infty}^{+\infty}\frac{|x_j-x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}}\frac{y|dx|}{|x_j-x-iy|} \le const,$$

then

$$\|I_1(f,x,y)\|_{L^1(\rho)} =$$

$$= \widetilde{C}_1\int_{-\infty}^{+\infty}\frac{|x_j-x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}}\frac{y|dx|}{(x_j-x-iy)}\int_{X_{k_j}}\frac{|f_{k_j}(t)||x_j-t||dt|}{\big((t-x)^2+y^2\big)}$$

$$\le C_1\|f_{k_j}\|_{L^1(\rho)}\int_{-\infty}^{+\infty}\frac{|x_j-x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}}\frac{y|dx|}{(x_j-x-iy)} \le M_1\|f_{k_j}\|_{L^1(\rho)},$$

where M_1 does not depend on y and k.

As S^+ is bounded, then using the fact (Lemma 3 in [23]) that for sufficiently large R at $|x| > R$ the following estimate we have

$$C_1|S^+(x+iy)|\frac{y}{|x+i|} \leq |S^+(x+iy) - a(x)S^-(x-iy)| \leq C_2|S^+(x+iy)|\frac{y}{|x+i|}, \tag{14}$$

where $C_2 > C_1 > 0$ some constant independent of y, we get

$$|T(x)| \leq \frac{Cy}{(x_j - x)^2 + y^2},$$

where $C > \max_j |x_j|$ is a constant.

Since

$$\int_{-\infty}^{+\infty} \frac{|x_j - x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}} \frac{y|dx|}{\left((x_j - x)^2 + y^2\right)} \leq const,$$

then

$$\|I_2(f, x, y)\|_{L^1(\rho)}$$

$$\leq \widetilde{C}_2 \int_{-\infty}^{+\infty} \frac{|x_j - x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}} \frac{y|dx|}{\left((x_j - x)^2 + y^2\right)} \int_{X_{k_j}} \frac{|f_{k_j}(t)||x_j - t||dt|}{|(t - x + iy)|}$$

$$\leq C_2\|f_{k_j}\|_{L^1(\rho)} \int_{-\infty}^{+\infty} \frac{|x_j - x|^{\alpha_j}}{(1+|x|)^{\alpha_0}|x+i|^{\alpha_j}} \frac{y|dx|}{\left((x_j - x)^2 + y^2\right)} \leq M_2\|f_{k_j}\|_{L^1(\rho)},$$

where M_2 does not depend on y and k.

Therefore

$$\|\Phi^+_{k_j}(x+iy) - a(x)\Phi^-_{k_j}(x-iy)\|_{L^1(\rho)} \leq M\|f_{k_j}\|_{L^1(\rho)}$$

where $M = \max\{M_1, M_2\}$ is a constant independent of y, k, and j. The first estimate of the lemma is proved.

Now let's prove the second statement of the lemma. Let $\varphi_n(x) \in C^\alpha$ be a sequence of finite functions such that

$$\lim_{n\to\infty} \left\|\varphi_n(x) - f_{k_j}(x)\right\|_{L^1(\rho)} = 0. \tag{15}$$

For any n we set

$$\widetilde{\Phi}_n(z) = \frac{S(z)}{2\pi i(x_j - z)} \int_{-\infty}^{+\infty} \frac{\varphi_n(t)(x_j - t)dt}{S^+(t)(t - z)},$$

and from Sokhotski–Plemelj formula we get

$$\lim_{y\to+0} \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - \varphi_n(x) \right\|_{L^1(\rho)} = 0. \tag{16}$$

Using the first estimate of this lemma, we get

$$\lim_{y\to+0} \left\| \Phi_{k_j}^+(x+iy) - a(x)\Phi_{k_j}^-(x-iy) - f_{k_j}(x) \right\|_{L^1(\rho)}$$

$$\leq \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - \varphi_n(x) \right\|_{L^1(\rho)} + \left\| \varphi_n(x) - f_{k_j}(x) \right\|_{L^1(\rho)}$$

$$+ \left\| \left(\widetilde{\Phi}_n^+(x+iy) - \Phi_{k_j}^+(x+iy) \right) - a(x) \left(\widetilde{\Phi}_n^-(x-iy) - \Phi_{k_j}^-(x-iy) \right) \right\|_{L^1(\rho)}$$

$$\leq \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - \varphi_n(x) \right\|_{L^1(\rho)} + 2 \left\| \varphi_n(x) - f_{k_j}(x) \right\|_{L^1(\rho)}.$$

Taking into account (15) and (16), we conclude

$$\lim_{y\to+0} \left\| \Phi_{k_j}^+(x+iy) - a(x)\Phi_{k_j}^-(x-iy) - f_{k_j}(x) \right\|_{L^1(\rho)} = 0.$$

So, we get the proof of the lemma. □

Let $f \in L^1(\rho)$, $x \in X_\infty$, where X_∞ is defined by (4). Define

$$\Phi_\infty(f; z) = \frac{S(z)}{2\pi i} \int_{t\in X_\infty} \frac{f(t)dt}{S^+(t)(t-z)}, \qquad z \in \Pi^+ \cup \Pi^-. \tag{17}$$

Then (see [18])

$$\int_{-\infty}^{\infty} \left| \Phi_\infty^+(f; x+iy) - a(x)\Phi_\infty^-(f; x-iy) \right| \frac{dx}{|x+i|^\alpha} \leq C\|f\|_{L^1(|x+i|^\alpha)},$$

where the constant C is independent of f and y, and besides

$$\lim_{y\to+0} \int_{-\infty}^{\infty} \left| \Phi_\infty^+(f; x+iy) - a(x)\Phi_\infty^-(f; x-iy) \right| \frac{dx}{|x+i|^\alpha} = 0. \tag{18}$$

Indeed, Since

$$\prod_{j=1}^{\infty} \left| \frac{x_k - x}{x+i} \right|^{\alpha_k} > \delta > 0 \quad \text{for} \quad x \in X_\infty,$$

we get (18).

3 The Main Results

3.1 Problem R for $a(x) \equiv 1$

Theorem 1 *Let the sequence* $\{A_k\}_1^\infty$ *such that the series* $\sum_{k=1}^\infty A_k$ *converges (may be not absolute) and* $x_k \to x_0$, $x_k < x_{k+1}$, $k = 1, 2, \ldots$. *Also let*

$$\Phi_0(z) = \sum_{k=1}^{\infty} \frac{A_k}{x_k - z}. \tag{19}$$

Then the following assertions hold.

(a) If $\prod_{k=1}^\infty \left|\frac{x_k - x_0}{x_k + i}\right|^{\alpha_k} = 0$, *then* $\Phi_0(z)$ *converges uniformly in* $\Pi^+ \cup \Pi^-$ *and is a solution of the homogeneous problem R.*

(b) If $\prod_{k=1}^\infty \left|\frac{x_k - x_0}{x_k + i}\right|^{\alpha_k} > 0$ *and* $\sum_{k=1}^\infty A_k = 0$, *then* $\Phi_0(z)$ *is a solution of the homogeneous problem R.*

Proof Let

$$R_N(z) = \sum_{k=1}^{N} \frac{A_k}{x_k - z}.$$

Further

$$\sum_{k=1}^{N} \frac{A_k}{x_k - z} = \sum_{k=1}^{N-1} S_k \frac{x_k - x_{k+1}}{(x_k - z)(x_{k+1} - z)} + \frac{S_N}{x_N - z},$$

where $S_k = \sum_{j=1}^k A_j$. Since $x_k < x_{k+1}, k = 1, 2, \ldots$ and $x_k \to x_0$, taking into account, that the series $\sum_{k=1}^\infty |S_k||x_k - x_{k+1}|$ converges, we conclude that the functional series (19) defines an analytic function out of $\overline{\{x_k\}_1^\infty}$.

Let's prove, that the condition (1) holds. We have

$$\Phi_0(x + iy) - \Phi_0(x - iy) = 2i \sum_{k=1}^{\infty} A_k \frac{y}{(x_k - x)^2 + y^2}.$$

We will set, that this series converges uniformly out of $\overline{\{x_k\}_1^\infty}$. Indeed, denote

$$I_N(x, y) = \sum_{k=1}^{N} A_k \frac{y}{(x_k - x)^2 + y^2}.$$

Similarly, we get

$$I_N(x, y) - I_{N+1}(x, y)$$

$$= \sum_{k=1}^{N-1} S_k \frac{(x_{k+1} - x_k)(x_{k-1} + x_k + 2x)}{\left((x_k - x)^2 + y^2\right)\left((x_{k-1} - x)^2 + y^2\right)} + \frac{S_N y}{(S_N - x)^2 + y^2}.$$

As the series $\sum_{k=1}^{\infty} S_k (x_{k+1} - x_k)$ converges absolutely, then we obtain

$$\Phi_0(x + iy) - \Phi_0(x - iy)$$

$$= \sum_{k=1}^{N-1} S_k y \frac{(x_{k+1} - x_k)(x_{k-1} + x_k + 2x)}{\left((x_k - x)^2 + y^2\right)\left((x_{k-1} - x)^2 + y^2\right)} + \frac{Sy}{(x_0 - x)^2 + y^2},$$

where $x_0 = \lim_{k\to\infty} x_k$, $S = \lim_{k\to\infty} S_k$. Since, the series $\sum_{k=1}^{\infty} (x_{k+1} - x_k)$ converges absolutely and because $S = 0$, then

$$\lim_{y\to+0} \int_{-\infty}^{+\infty} |\Phi_0(x + iy) - \Phi_0(x - iy)|\, \rho(x) dx = 0.$$

Thereby, under the condition $S = 0$, the function $\Phi_0(z)$ is a solution of the homogeneous problem **R**. □

Theorem 2 *Let the sequence* $\{x_k\}_1^{\infty}$ *satisfies condition* (8). *Then the general solution of the homogeneous problem*

$$\lim_{y\to+0} \|\Phi^+(x + iy) - \Phi^-(x - iy)\|_{L^1(\rho)} = 0, \tag{20}$$

may be represented as

$$\Phi_0(z) = \sum_{t_k\in T_0} \frac{B_k}{t_k - z} + \sum_{k=1}^{\infty} \frac{A_k}{x_k - z}, \tag{21}$$

where $\{B_k\}_1^N$ *are any complex numbers and* $\{A_k\}_1^{\infty} \in l^1$.

Proof Since $\rho(x)$ is continuous at $t_k \in T_0$ and $\rho(t_k) = 0$, then

$$\int_{-\infty}^{+\infty} \left|\frac{1}{t_k - x - iy} - \frac{1}{t_k - x + iy}\right| \rho(x) dx =$$

$$= 2 \int_{-\infty}^{+\infty} \frac{y}{(t_k - x)^2 + y^2} \rho(x) dx = \rho(t_k) = 0.$$

Therefore $(t_k - z)^{-1}$ is a solution of the homogeneous problem. So, the function $\sum_{t_k \in T_0} \frac{B_k}{t_k - z}$ is a solution of the problem (20).

Let $\{A_k\}_1^\infty \in l^1$ any sequences and let $\Phi_{0k} = A_k(t_k - z)^{-1}$, $k = 1, 2, \ldots$. As

$$\Phi_{0k}(x+iy) - \Phi_{0k}(x-iy) = 2A_k \frac{iy}{(t_k - x)^2 + y^2},$$

taking into account, that

$$\int_{T_0} \frac{y}{(t_k - x)^2 + y^2} |x_k - x|^{\delta_k} |dx| \to 0$$

we get

$$\lim_{y \to +0} \|\Phi_{0k}^+(x+iy) - a(x)\Phi_{0k}^-(x-iy)\|_{L^1(\rho)} = 0.$$

Further

$$\Phi_0(x+iy) - \Phi_0(x-iy) = 2\sum_{k=1}^\infty \frac{A_k y}{(t_k - x)^2 + y^2}$$

and because this series converges uniformly at $y \to 0$, then

$$\int_{T_0} |\Phi_0(x+iy) - \Phi_0(x-iy)| \rho(x) |dx| \leq$$

$$C \sum_{k=1}^\infty |A_k| \int_{T_0} \frac{y}{(t_k - x)^2 + y^2} |x_k - x|^{\delta_k} |dx| \leq$$

$$\leq C \sum_{k=1}^\infty |A_k| y^{1-\delta_k}.$$

Recalling that $\sum_{k=1}^\infty |A_k| y^{1-\delta_k} \to 0$ at $y \to 0+$, we obtain the proof of the sufficiency of the theorem.

We prove now the necessity of this condition, i.e. that any solution of the homogeneous problem (20) may be represented as in (21). Let $\Phi_0(z)$ be a solution of this problem. From the condition of the theorem the points $\{x_k\}_1^\infty$ are first order poles of the function $\Phi_0(z)$ and if $r_k(z) = A_k(x_k - z)^{-1}$ is the principal part of Laurent series at x_k points, then it is sufficient to establish that

$$\sum_{k=1}^N |A_k| < C, \tag{22}$$

where the number C does not depend on N. Let considered sequence $\{x_{kj}\}_1^\infty$ is taken as follows: $x_{kj} \to t_k$, $j \to \infty$. We put $\zeta_j = 2^{-1}(x_{kj} + x_{kj+1})$, $T'_{kj} = [\zeta_k, \zeta_{k+1})$, $j = 1, 2, \dots$, $T'_k = \sum_{j=1}^{\infty} T'_{kj}$.

Let

$$M = \sup_{y>0} \int_{T_k} |\Phi_0(x+iy) - \Phi_0(x-iy)|\rho(x)|dx|.$$

We will have

$$\int_{T'_k} |R_N(x+iy) - R_N(x-iy)|\rho(x)|dx| \leq 2M,\ 0 < y < y_0, \tag{23}$$

where

$$R_N(z) = \sum_{j=1}^{N} \frac{A_k}{x_{kj} - z}.$$

Since the function $\Phi_0(z) - R_N(z)$ is analytic on the set $\overline{T'}_k$ for any $y < y_0$, we get

$$I_k(y) = \int_{T'_{kj}} |R_N(x+iy) - R_N(x-iy)||x_k - x|^{\delta_k}|dx| >$$

$$> |A_k| \int_{T'_{kj}} \frac{y|x - x_k|^{\delta_k}}{(x_k - x)^2 + y^2}|dx| - \sum_{p \neq j}^{N} |A_p| \int_{T'_{kj}} \frac{y|x - x_k|^{\delta_k}}{(x_k - x)^2 + y^2}|dx|.$$

As $(x_k - x)^2 + y^2 > 2|x_k - x|y$, then

$$\int_{T'_{kj}} \frac{y|x - x_k|^{\delta_k}}{(x_k - x)^2 + y^2}|dx| \geq c'_0|x'_k - x_k|^{\delta_k},$$

where $c'_0 > 0$ does not depend on k, y, N. Taking into account that according to the condition $|x_j - x_{k_p}| > c_1|x_j - t_k|$, where c_1 does not depend on j and p, we get

$$\int_{T'_{kj}} \frac{y|x - x_k|^{\delta_k}}{(x_k - x)^2 + y^2}|dx| \leq \int_{T'_{kj}} \frac{y|x - x_k|^{\delta_k}}{2|x_{kj} - x_{kp}|}|dx| \leq$$

$$\leq \frac{y^2 y^{\delta_k}}{y|x_{kj} - x_{kp}|} = \frac{y^{1+\delta_k}}{y|x_{kj} - x_{kp}|}.$$

Hence from $|x_k' - x_k| > c|x_k|$ we get

$$I_k(y) \geq |A_k| c_0' {x_k}^{\delta_k} - \sum_{p \neq j}^{N} |A_p| \int_{T_{kj}'} \frac{y|x - x_{kp}|^{\delta_k}}{|x_p - x - iy|} |dx|.$$

As

$$\sum_{p \neq j}^{N} |A_j| \frac{y|x - x_{kp}|^{\delta_k}}{|x_p - x - iy|} \to 0$$

converges uniformly at $t \in T_{kj}$, from (11) we get

$$\lim_{y \to 0} I_k(y) \geq c_0''|A_k|$$

where c_0'' is a positive constant independent of y and k. As by (22)

$$\sum_{j=1}^{N} I_{kj}(y) \leq \frac{2M}{c_0''}$$

we get

$$\sum_{j=1}^{N} |A_{kj}| < \frac{2M}{c_0''},$$

that is

$$\sum_{k=1}^{\infty} |A_{kj}| < \frac{2M}{c_0''}.$$

From the last estimate and from the condition of the theorem follows

$$\sum_{k=1}^{\infty} |A_k| < \infty.$$

□

Remark 1 If $T_0 = \emptyset$, then the general solution of the homogeneous problem (20) may be represented as

$$\Phi_0(z) = \sum_{k=1}^{\infty} \frac{A_k}{x_k - z},$$

where $\{A_k\}_1^{\infty} \in l^1$.

Remark 2 It is necessary to note that the Problem (20) in $L^p(\rho)(1 < p < \infty)$ has a finite number of solutions.

3.2 Problem R for $a(x) \in C^\alpha(\mathbb{R})$

Theorem 3 *Let the sequence* $\{t_k\}_1^\infty$ *satisfies the condition* (10).

(a) If $\kappa \geq 0$, *then the general solution of the homogeneous problem* R *may be represented as*

$$\Phi_0(z) = S(z)\left(P_{\kappa-1}(z) + \sum_{t_k \in T_0} \frac{B_k}{t_k - z} + \sum_{k=1}^{\infty} \frac{A_k}{x_k - z}\right),$$

where $P_{\kappa-1}(z)$ *is an arbitrary polynomial of degree* $\kappa - 1$, *and the sequence* $\{A_k\}_{k=1}^\infty \in l^1$.

(b) If $\kappa < 0$, *then the general solution of the homogeneous problem* R *may be represented as*

$$\Phi_0(z) = S(z)\left(\sum_{t_k \in T_0} \frac{B_k}{t_k - z} + \sum_{k=1}^{\infty} \frac{A_k}{x_k - z} + A_0\right),$$

where A_0 *is a constant,* $\{A_k\}_{k=1}^\infty, \{B_k\}_{k=1}^N \in l^1$, $A_{-\kappa+1}, A_{-\kappa+2}, \ldots$ *and* $B_1, B_2, \ldots, B_N$ *are arbitrary, and the numbers* $A_1, A_2, \ldots, A_\kappa$ *are uniquely defined by the system of linear equations*

$$\frac{A_1}{(x_1+i)^j} + \frac{A_2}{(x_1+i)^j} + \ldots + \frac{A_\kappa}{(x_1+i)^j} = -\left(\sum_{k=\kappa+1}^{\infty} \frac{A_k}{(x_1+i)^j} + \sum_{k=1}^{N} \frac{B_k}{(t_k+i)^j}\right), \tag{24}$$

where $j = 1, 2, \ldots, \kappa$.

Proof

(a) Taking into account, that $S^-(z)$ has a zero of order κ at the point $z = i$, then in order the function $S(z)P(z)$ to be a solution of the homogeneous problem R, the order of the polynomial $P(z)$ must be less than or equal to κ. Therefore, $S(z)P_{\kappa-1}(z)$ is a solution of the homogeneous problem R.
So, from Theorem 2, taking into account (14), we get the proof of the point (a) of theorem.

(b) In the case $\kappa < 0$, if $P(z) \equiv C$, then using the inequality (14), we obtain that the function $CS(z)$ is a solution of the homogeneous problem R. If the order of polynomial $P(z)$ is greater than or equal to 1, then taking into account, that

$S^-(z)$ has not zero at any point of $z \in \Pi^-$, we conclude that $S(z)P(z)$ is not a solution of the homogeneous problem R.

Denote

$$\varphi_0(z) = \sum_{k=1}^{\infty} \frac{A_k}{x_k - z} + \sum_{k=1}^{N} \frac{B_k}{t_k - z}.$$

Taking into account, that $S^-(z)$ has a pole of order $-\kappa(\kappa < 0)$ at the point $z = -i$, then in order the function $S(z)\varphi_0(z)$ to be solution of the homogeneous problem R, it must be hold

$$\begin{cases} \varphi_0(-i) = 0 \\ \varphi_0'(-i) = 0 \\ \dots \\ \varphi_0^{\kappa-1}(-i) = 0 \end{cases}$$

Hence, we get the following conditions

$$\frac{A_1}{(x_1+i)^j} + \frac{A_2}{(x_1+i)^j} + \ldots + \frac{A_\kappa}{(x_1+i)^j} = -\left(\sum_{k=\kappa+1}^{\infty} \frac{A_k}{(x_1+i)^j} + \sum_{k=1}^{N} \frac{B_k}{(t_k+i)^j} \right).$$

□

Theorem 4 *Let the sequence* $\{x_k\}_1^\infty$ *satisfies the condition* (10) *and* $\kappa \geq 0$, *then the general solution of the in-homogeneous problem* R *may be represented as* $\Phi(z) = \Phi_0(z) + \Phi_1(z)$, *where* Φ_0 *is the general solution of the homogeneous problem and*

$$\Phi_1(z) = \sum_{k=1}^{N} \Phi_k(z) \tag{25}$$

where

$$\Phi_k(z) = \sum_{j=1}^{\infty} \Phi_{k_j}(z) \tag{26}$$

and

$$\Phi_{k_j}(z) = \frac{S(z)}{2\pi i(x_j - z)} \int_{X_{k_j}} \frac{f_{k_j}(t)(x_j - t)dt}{S^+(t)(t-z)}.$$

Proof Since $\kappa \geq 0$, then from Lemma 6 we have

$$\|\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy)\|_{L^1(\rho)} \leq C\|f\|_{L^1(\rho)},$$

where the constant C is independent of y and k. Therefore

$$\|\Phi_1^+(x+iy) - a(x)\Phi_1^-(x-iy)\|_{L^1(\rho)} \leq NC\|f_k\|_{L^1(\rho)} \leq C\|f\|_{L^1(\rho)}.$$

Similar to the proof of the second part of Lemma 6, we obtain

$$\lim_{y\to+0} \|\Phi_1^+(x+iy) - a(x)\Phi_1^-(x-iy) - f(x)\|_{L^1(\rho)} = 0.$$

Taking into account Theorem 3 we get the proof of the theorem. □

Theorem 5 *Let the sequence $\{x_k\}_1^\infty$ satisfies the condition (10) and $\kappa < 0$, then the general solution of the in-homogeneous problem R may be represented as $\Phi(z) = \Phi_0(z) + \Phi_1(z)$, where $\Phi_1(z)$ is defined by (25) and*

$$\Phi_0(z) = \sum_{t_k \in T_0} \frac{B_k}{t_k - z} + \sum_{k=1}^{\infty} \frac{A_k}{x_k - z},$$

$\{B_k\}_1^N$ and $\{A_k\}_{k=1}^\infty \in l^1$, $A_{-\kappa+1}, A_{-\kappa+2}, \ldots$ are arbitrary complex numbers, and the numbers $A_1, A_2, \ldots, A_{-\kappa}$ are uniquely defined by the system of linear equations

$$\begin{cases} \sum_{k=1}^\infty \frac{A_k}{(x_k+i)} = -\left(\sum_{k=1}^N \frac{B_k}{(t_k+i)} + \widetilde{\Phi}_1(-i)\right) \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^2} = -\left(\sum_{k=1}^N \frac{B_k}{(t_k+i)^2} + \widetilde{\Phi}_1'(-i)\right) \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^3} = -\left(\sum_{k=1}^N \frac{B_k}{(t_k+i)^3} + \widetilde{\Phi}_1''(-i)\right) \\ \ldots \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^{-\kappa}} = -\left(\sum_{k=1}^N \frac{B_k}{(t_k+i)^{-\kappa}} + \widetilde{\Phi}_1^{(-\kappa-1)}(-i)\right) \end{cases}, \quad (27)$$

where $\widetilde{\Phi}_1(z) = \frac{\Phi_1(z)}{S(z)}$.

Proof In the case $\kappa < 0$, $S^-(z)$ has a pole of order $-\kappa(\kappa < 0)$ at the point $z = -i$. Hence in order $\Phi(z)$ to be solution of the in-homogeneous problem R, for $A_1, A_2, \ldots, A_{-\kappa}$ it must be hold (27). Note that the determinant of the linear system (27) is a Vandermonde determinant and is determined by the following formula:

$$det = \prod_{1\leq j<m\leq -\kappa} \left(\frac{1}{x_m + i} - \frac{1}{x_j + i}\right).$$

Since $\frac{1}{x_j+i}$, $j = 1, 2, \ldots, -\kappa$ are distinct, the determinant is non-zero. Hence the numbers $A_1, A_2, \ldots, A_{-\kappa}$ may be uniquely defined by the system of linear equations (27). □

Remark 3 Note that

$$\begin{cases} \Phi_1(-i) = 0 \\ \Phi_1^{'}(-i) = 0 \\ \ldots \\ \Phi_1^{(-\kappa-1)}(-i) = 0 \end{cases}$$

explicitly can be represented as

$$\begin{cases} \sum_{k=1}^{N} \sum_{j=1}^{\infty} I_{1\,1} = 0 \\ \sum_{k=1}^{N} \sum_{j=1}^{\infty} (I_{2\,1} + I_{1\,2}) = 0 \\ \ldots \\ \sum_{k=1}^{N} \sum_{j=1}^{\infty} \sum_{m=1}^{-\kappa} C_m^{-\kappa} I_{m\,-\kappa-m} = 0 \end{cases} .$$

where C_m^n are the binomial coefficients and

$$I_{m\,n} = \frac{1}{2\pi i (x_j + i)^m} \int_{X_{k_j}} \frac{f(t)(x_j - t)}{S^+(t)(t+i)^n} dt, \quad m, n = 1, 2, \ldots, -\kappa\ ,$$

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Repeated Distances and Dot Products in Finite Fields

Alex Iosevich and Charles Wolf

Abstract Let $\mathbb{F}_q^d$, $d \geq 2$, where $\mathbb{F}_q$ is the field with q elements. Let

$$\Delta(E) = \{||x - y|| : x, y \in E\}, \; ||x|| = x_1^2 + \cdots + x_d^2,$$

and

$$\prod(E) = \{x \cdot y : x, y \in E\}, \; x \cdot y = x_1 y_1 + \cdots + x_d y_d.$$

The purpose of this paper is to find the largest possible subset E' of E such that all the distances determined by E' are distinct, and also find a subset E'' of E such that all the dot products determined by E'' are distinct. We provide some number theoretic examples that indicate the degree of sharpness of our results. A general mechanism is outlined that should allow one to study these problems in much greater generality.

Keywords Finite fields · Distance sets · Combinatorics · Operators · Probabilistic method

1 Introduction

Let $\mathbb{F}_q$ be a field of size q. Given $E \subset \mathbb{F}_q^d$, $d \geq 2$, define the analog of the Euclidean distance set

$$\Delta(E) = \{||x - y|| : x, y \in E\},$$

where

$$||x|| = x_1^2 + x_2^2 + \cdots + x_d^2.$$

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In analogy with the Erdős distance problem in Euclidean space, we can ask how many distinct distances are determined by $E \subset \mathbb{F}_q^d$, $d \geq 2$. One can also ask, how large does a subset of $\mathbb{F}_q^d$, $d \geq 2$, needs to be to ensure that $\Delta(E) = \mathbb{F}_q$, or, at least that $|\Delta(E)| > \frac{q}{2}$, where here and throughout, if S is a set, $|S|$ denotes its size. In dimensions three and higher, the best known results are due to Iosevich and Rudnev [9] who proved that if $|E| > 4q^{\frac{d+1}{2}}$, then $\Delta(E) = \mathbb{F}_q$. In odd dimension, the threshold $\frac{d+1}{2}$ is know to be best possible (see e.g. [8]). In two dimensions, the $\frac{d+1}{2} = \frac{3}{2}$ exponent was improved to $\frac{4}{3}$ in [3], and more recently, to $\frac{5}{4}$ when q is prime in [11].

A related question is the number of dot products determined by subset of $\mathbb{F}_q^d$, $d \geq 2$. Given $E \subset \mathbb{F}_q^d$, $d \geq 2$, let $\prod(E) = \{x \cdot y : x, y \in E\}$. Once again, we can ask how large a subset of $\mathbb{F}_q^d$, $d \geq 2$, needs to be to ensure that $\prod(E) = \mathbb{F}_q$, or, at least that $|\prod(E)| > \frac{q}{2}$. The best known results are due to Hart and Iosevich [7], where the threshold $\frac{d+1}{2}$ is established. It was shown to be essentially best possible in [8].

In this paper we investigate a related question. For a set $E \subseteq \mathbb{F}_q^d$, define $\alpha(E)$ to be the largest subset $E' \subseteq E$ such that no distance between points of E' is repeated. In other words, we want the largest subset $E' \subseteq E$ such that for any quadruple of points $x, y, x'y' \in E'$, $||x - y|| = ||x' - y'||$ implies that the pair of points $\{x, y\}$ is the same as the pair $\{x', y'\}$. For an integer n, define $\alpha(n) := \min\{\alpha(E) : |E| = n\}$. Some results in this direction were previously obtained in [12, 14] (see also a detailed discussion below). Let $\beta(n)$ denote the analogous quantity with respect to the dot product problem. We would like to get good bounds on $\alpha(n)$ and $\beta(n)$. In the Euclidean setting, a corresponding result was obtained by Charalambides [4] who proved that every subset of the Euclidean plane of size n contains a subset of size at least $n^{\frac{1}{3}} \log(n)$ such that no distance is repeated.

As the reader shall see, in this paper we obtain very similar results for the distance graph and the dot product graph. While the techniques are, at least superficially, somewhat different due to the lack of translation invariance in the dot product setting, it is reasonable to ask whether a general formalism is possible. We sketch this in the final section of the paper.

1.1 Results

Here are the results for general dimension $d \geq 3$:

Theorem 1

$$\alpha(n) = \begin{cases} \Omega\left(q^{1/3}\right) & \textit{if } n \geq q^{(d+1)/2} \\ \Omega\left(\frac{n^{2/3}}{q^{d/3}}\right) & \textit{if } n \leq q^{(d+1)/2} \end{cases}$$

Note that the lower bounds are equal for $n = q^{(d+1)/2}$. Also, the lower bound of $\frac{n^{2/3}}{q^{d/3}}$ is greater than 1 for $n > q^{d/2}$.

Remark 1 The lower bound of $\Omega(q^{1/3})$ was previously obtained in [14], although their domain is only $n \geq q^{2(q+1)/3}$.

We have stronger bounds for $d = 2$ than for a more general $d \geq 3$.

Theorem 2 *Suppose that the equation $x^2 + 1 = 0$ has no solutions in $\mathbb{F}_q$. Then*

$$\alpha(n) = \begin{cases} \Omega\left(q^{1/3}\right) & \textit{if } n \geq q^{4/3} \\ \Omega\left(\frac{n^{1/2}}{q^{1/3}}\right) & \textit{if } n \leq q^{4/3} \end{cases}$$

Note that the lower bounds are equal for $n = q^{4/3}$. Also, the lower bound of $\frac{n^{1/2}}{q^{1/3}}$ is greater than 1 for $n > q^{2/3}$.

Remark 2 The first bound was previously obtained in [12].

We have improved bounds when $d = 2$ and q is prime.

Theorem 3 *Suppose that the equation $x^2 + 1 = 0$ has no solutions in $\mathbb{F}_q$, and q is prime. Then*

$$\alpha(n) = \begin{cases} \Omega\left(q^{1/2}\right) & \textit{if } n \geq q^{13/8} \\ \Omega\left(\frac{n^{4/9}}{q^{2/9}}\right) & \textit{if } q \leq n \leq q^{13/8} \end{cases}$$

Note that the lower bounds are equal for $n = q^{13/8}$. We have a construction of a set matching the $q^{1/2}$ lower bound when $n \geq q^{13/8}$ in Theorem 3:

Theorem 4 *There exists a set of $\Theta(\sqrt{q})$ points in $\mathbb{F}_q^d$ such that every pair of points has distinct distance.*

Here we present a lower bound on the number of distinct dot products among vectors in $\mathbb{F}_q^d$. Namely, for any 2 points $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ in $\mathbb{F}_q^d$, define the dot product between x and y, denoted $x \cdot y$, as $(x_1 \cdot y_1) + (x_2 \cdot y_2) + \cdots + (x_d \cdot y_d) \mod q$.

For a set $E \subseteq \mathbb{F}_q^d$, define $\beta(E)$ to be the largest subset $E' \subseteq E$ such

$$x \cdot y \neq x' \cdot y'$$

whenever $x, y, x', y' \in E'$ and at least three of the vectors x, y, x', y' are distinct. For an integer n, define $\beta(n) := \min\{\beta(E) : |E| = n\}$. We would like to get bounds on $\beta(n)$.

Theorem 5

$$\beta(n) = \begin{cases} \Omega\left(q^{1/3}\right) & \textit{if } n \geq q^{(d+1)/2} \\ \Omega\left(\frac{n^{2/3}}{q^{d/3}}\right) & \textit{if } n \leq q^{(d+1)/2} \end{cases}$$

Note that the lower bounds are equal for $n = q^{(d+1)/2}$. Also, the lower bound of $\frac{n^{2/3}}{q^{d/3}}$ is greater than 1 for $n > q^{d/2}$.

Remark Requiring at least three of x, y, x', y' to be distinct is reasonable due to the following: If we don't have at least three distinct, the only cases where $x \cdot y = x' \cdot y'$ are (1) $x = x', y = y'$ and (2) $x = y, x' = y'$. Case (1) is the $x \cdot y = x \cdot y$ case, which is not even a distinct pair of vectors. For case (2), $x \cdot x = y \cdot y$, for a set K of k vectors where all dot products are distinct, the total number of dot products is $\binom{k}{2} \approx k^2$. However the set $\{x \cdot x : x \in K\}$ has size k, which is negligible compared to $\binom{k}{2}$.

Similar to the distance case, we present a set with no repeated dot products:

Theorem 6 *There exists a set of* $\Theta(\sqrt{q})$ *points in* $\mathbb{F}_q^d$ *such that every pair of points has distinct dot product.*

2 Repeated Distances in $\mathbb{F}_q^2$

We will use the following theorems from [2] to prove Theorem 2.

First, for a quadruple of distinct points (x, x', y, y'), we have a bound on the number of pairs of points (x, y) and (x', y') with the same distance.

Theorem 7 *Suppose that the equation* $x^2 + 1 = 0$ *has no solutions in* $\mathbb{F}_q$. *Let* $E \subseteq \mathbb{F}_q^2$. *Then*

$$|\{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}| \leq \frac{|E|^4}{q} + q|E|^{5/2}.$$

Second, we consider the triple of points (x, y, y') such that $||x - y|| = ||x - y'||$:

Theorem 8 *Suppose that the equation* $x^2 + 1 = 0$ *has no solution in* $\mathbb{F}_q$. *Let* $E \subseteq \mathbb{F}_q^2$. *Then*

$$|\{(x, y, y') \in E^3 : ||x - y|| = ||x - y'||\}| \leq \frac{|E|^3}{q} + q^{1/2}|E|^2. \tag{1}$$

This follows at once from the main result in [1], which says, in particular, that if $t \neq 0$, then

$$|\{(x^1, \ldots, x^{k+1}) \in E^{k+1} : ||x^{i+1} - x^i|| = t;\ 1 \leq i \leq k\}| = |E|^{k+1} q^{-k} + D_k(E),$$

where

$$|D_k(E)| \leq \frac{2k}{\log(2)} q^{\frac{d+1}{2}} |E|^k q^{-k}.$$

The estimate (1) follows by summing in t. It will be useful for us below to note that the same argument yields a higher dimensional variant of (1), namely

$$|\{(x, y, y') \in E^3 : ||x - y|| = ||x - y'||\}| \leq \frac{|E|^3}{q} + q^{\frac{d-1}{2}} |E|^2. \tag{2}$$

Proof of Theorem 2 For a set E of size n and a probability $0 < p < 1$ to be chosen later, we will choose a subset of $S \subseteq E$ by taking each point of E independently with probability p and putting it into S. The expected size of S, $\mathbb{E}(|S|)$ is pn.

Let $Q = \{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}$, and let $R \subset Q$ be the set of quadruples of Q, where each point of the quadruple is also in S. A quadruple of distinct points (x, x', y, y') appears in S with probability p^4, and a triple of points (x, y, y') appears with probability p^3. By Theorems 7 and 8 the expectation $\mathbb{E}(|R|)$ is

$$\leq O\left(p^4 \frac{n^4}{q} + p^4 q n^{5/2} + p^3 \frac{n^3}{q} + p^3 q^{1/2} n^2\right).$$

Now, for each quadruple of points in R, we create a new set $T \subseteq S$ by deleting one point from S belonging to that quadruple. Therefore, none of the points in T will form a quadruple satisfying $||x - y|| = ||x' - y'||$. By linearity of expectation,

$$\mathbb{E}(|T|) = \mathbb{E}(|S| - |R|) \geq pn - p^4 \frac{n^4}{q} - p^4 q n^{5/2} - p^3 \frac{n^3}{q} - p^3 q^{1/2} n^2. \tag{3}$$

We are ready to choose p: p must be chosen small enough so that removing $|T|$ points from R does not remove all of the points of R. In other words, we must choose p so that np is larger than $\left(p^4 \frac{n^4}{q} + p^4 q n^{5/2} + p^3 \frac{n^3}{q} + p^3 q^{1/2} n^2\right)$ We will split this into cases, depending on which part of the summation $\left(p^4 \frac{n^4}{q} + p^4 q n^{5/2} + p^3 \frac{n^3}{q} + p^3 q^{1/2} n^2\right)$ is largest.

Case 1 For $|E| > q^{3/2}$, $p^4 \frac{|E|^4}{q} > p^4 q |E|^{5/2}$ and $p^3 \frac{n^3}{q} > p^3 q^{1/2} n^2$, so only the $p^4 \frac{|E|^4}{q} + p^3 \frac{n^3}{q}$ term will affect the optimization. For a small constant c, choose $p =$

$c\frac{q^{1/3}}{n}$. Plugging this back into Eq. (3), we get that $\mathbb{E}(|T|) \geq \Omega(q^{1/3})$. Therefore, there exists such a set T of size $\Omega(q^{1/3})$ with no repeated distance.

Case 2 For $q^{3/2} \geq |E| > q^{4/3}$, $p^4\frac{|E|^4}{q} > p^4q|E|^{5/2}$ and $p^3\frac{n^3}{q} < p^3q^{1/2}n^2$, so only the $p^4\frac{|E|^4}{q} + p^3q^{1/2}n^2$ term in Eq. (3) will affect the optimization. For a small constant c, choose $p = c\frac{q^{1/3}}{n}$. Plugging this back into Eq. (3), we get that $\mathbb{E}(|T|) \geq \Omega(q^{1/3})$. Therefore, there exists such a set T of size $\Omega(q^{1/3})$ with no repeated distance.

Case 3 For $|E| < q^{4/3}$, $p^4\frac{|E|^4}{q} < p^4q|E|^{5/2}$ and $p^3\frac{n^3}{q} < p^3q^{1/2}n^2$, so only the $p^4\frac{|E|^4}{q} + p^3q^{1/2}n^2$ term in Eq. (3) will affect the optimization. For a small constant c, choose $p = c\frac{1}{n^{1/2}q^{1/3}}$. Plugging this back into Eq. (3), we get that $\mathbb{E}(|T|) \geq \Omega\left(\frac{n^{1/2}}{q^{1/3}}\right)$. Therefore, there exists such a set T of size $\Omega\left(\frac{n^{1/2}}{q^{1/3}}\right)$ with no repeated distance.

□

We will use the following theorems from [11] to prove Theorem 3.

First, for a quadruple of distinct points (x, x', y, y'), we have a bound on the number of pairs of points (x, y) and (x', y') with the same distance.

Theorem 9 *Suppose that the equation $x^2 + 1 = 0$ has no solutions in $\mathbb{F}_q$, and q is prime. Let $E \subseteq \mathbb{F}_q^2$ and $|E| \geq q$. Then*

$$|\{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}| \leq |E|^{8/3}q^{2/3}.$$

Second, we consider the triple of points (x, y, y') such that $||x - y|| = ||x - y'||$:

Theorem 10 *Suppose that the equation $x^2 + 1 = 0$ has no solution in $\mathbb{F}_q$, and q is prime. Then Let $E \subseteq \mathbb{F}_q^2$, where $|E| \geq q$. Then*

$$|\{(x, y, y') \in E^3 : ||x - y|| = ||x - y'||\}| \leq \frac{|E|^3}{q} + q^{1/2}|E|^{2/3}q^{2/3}. \tag{4}$$

What is actually proved in [11] is Theorem 10. Theorem 9 is deduced from it using Cauchy-Schwarz.

Proof of Theorem 3 For a set E of size n and a probability $0 < p < 1$ to be chosen later, we will choose a subset of $S \subseteq E$ by taking each point of E independently with probability p and putting it into S. The expected size of S, $\mathbb{E}(|S|)$ is pn.

Let $Q = \{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}$, and let $R \subset Q$ be the set of quadruples of Q, where each point of the quadruple is also in S. A quadruple of distinct points (x, x', y, y') appears in S with probability p^4, and a triple of points

(x, y, y') appears with probability p^3. By Theorems 9 and 10 the expectation $\mathbb{E}(|R|)$ is

$$\leq O\left(p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q} + p^3 n^{2/3} q^{2/3}\right).$$

Now, for each quadruple of points in R, we create a new set $T \subseteq S$ by deleting one point from S belonging to that quadruple. Therefore, none of the points in T will form a quadruple satisfying $||x - y|| = ||x' - y'||$. By linearity of expectation,

$$\mathbb{E}(|T|) = \mathbb{E}(|S| - |R|) \geq pn - p^4 n^{8/3} q^{2/3} - p^3 \frac{n^3}{q} - p^3 n^{2/3} q^{2/3}. \tag{5}$$

We are ready to choose p: p must be chosen small enough so that removing $|T|$ points from R does not remove all of the points of R. In other words, we must choose p so that np is larger than $\left(p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q} + p^3 n^{2/3} q^{2/3}\right)$ We will split this into cases, depending on which part of the summation $\left(p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q} + p^3 n^{2/3} q^{2/3}\right)$ is largest.

Case 1 For $n > q^{13/8}$, $\frac{n^3}{q} > n^{2/3} q^{2/3}$, so only the $p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q}$ term will affect the optimization. We choose a p such that np is larger than $p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q}$. In the range $q^{13/8} \geq n$, choose $p = c\frac{q^{1/2}}{n}$ for some constant c. Plugging this back into Eq. (5), we get that $\mathbb{E}(|T|) \geq \Omega(q^{1/2})$. Therefore, there exists such a set T of size $\Omega(q^{1/2})$ with no repeated distance.

Case 2 For $q^{13/8} \geq n > q$, still $\frac{n^3}{q} > n^{2/3} q^{2/3}$, so only the $p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q}$ term will affect the optimization. We choose a p such that np is larger than $p^4 n^{8/3} q^{2/3} + p^3 \frac{n^3}{q}$. In the range $q^{13/8} \leq n$, this is optimized for $p = \frac{c}{n^{5/9} q^{2/9}}$ for some constant c. Plugging this back into Eq. (5), we get that $\mathbb{E}(|T|) \geq \Omega\left(\frac{n^{4/9}}{q^{2/9}}\right)$. Therefore, there exists such a set T of size $\Omega\left(\frac{n^{4/9}}{q^{2/9}}\right)$ with no repeated distance.

□

3 Repeated Distances in $\mathbb{F}_q^d$

We will use the following theorem from [9] to prove Theorem 1.

Theorem 11 *Let $E \subseteq \mathbb{F}_q^d$. Then*

$$|\{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}| \leq \frac{|E|^4}{q} + q^d |E|^2.$$

Proof of Theorem 1 For a set E of size n and a probability $0 < p < 1$ to be chosen later, we will choose a subset of $S \subseteq E$ by taking each point of E independently with probability p and putting it into S. The expected size of S, $\mathbb{E}(|S|)$ is pn.

Let

$$Q = \{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\},$$

and let $R \subset Q$ be the set of quadruples of Q, where each point of the quadruple is also in S. A quadruple of distinct points (x, x', y, y') appears in S with probability p^4, and a triple of points (x, y, y') appears with probability p^3. By Theorem 11 and Eq. (2) the expectation $\mathbb{E}(|R|)$ is

$$O\left(p^4|\{(x, y, x', y') \in E^4 : ||x - y|| = ||x' - y'||\}|\right) \leq O\left(p^4\frac{n^4}{q} + p^4q^dn^2 + p^3\frac{n^3}{q} + p^3q^{\frac{d-1}{2}}n^2\right).$$

Now, for each quadruple of points in R, we create a new set $T \subseteq S$ by deleting one point from S belonging to that quadruple. Therefore, none of the points in T will form a quadruple satisfying $||x - y|| = ||x' - y'||$. By linearity of expectation,

$$\mathbb{E}(|T|) = \mathbb{E}(|S| - |R|) \geq pn - p^4\frac{n^4}{q} - p^4q^dn^2 - p^3\frac{n^3}{q} - p^3q^{\frac{d-1}{2}}n^2. \tag{6}$$

We are ready to choose p: p must be chosen small enough so that removing $|T|$ points from R does not remove all of the points of R. In other words, we must choose p so that np is larger than $p^4\frac{n^4}{q} + p^4q^dn^2 + p^3\frac{n^3}{q} + p^3q^{\frac{d-1}{2}}n^2$. We will split this into 2 cases, depending on which part of the summation $p^4\frac{n^4}{q} + p^4q^dn^2 + p^3\frac{n^3}{q} + p^3q^{\frac{d-1}{2}}n^2$ is largest.

Case 1 For $|E| > q^{(d+1)/2}$, $\frac{|E|^4}{q} > q^d|E|^2$ and $\frac{n^3}{q} > q^{\frac{d-1}{2}}n^2$, so only the $p^4\frac{|E|^4}{q}$ and $p^3\frac{n^3}{q}$ terms in Eq. (6) will affect the optimization. In other words, we can assume the expected remaining number of points is $pn - p^4\frac{n^4}{q} - p^3\frac{n^3}{q}$. For a small constant c, choose $p = c\frac{q^{1/3}}{n}$. Plugging this back into Eq. (6), we get that $\mathbb{E}(|T|) \geq \Omega(q^{1/3})$. Therefore, there exists such a set T of size $\Omega(q^{1/3})$ with no repeated distance.

Case 2 For $|E| < q^{(d+1)/2}$, $\frac{|E|^4}{q} < q^d|E|^2$ and $\frac{n^3}{q} < q^{\frac{d-1}{2}}n^2$, so only the $p^4q^dn^2$ and $p^3q^{\frac{d-1}{2}}n^2$ terms in Eq. (6) will affect the optimization. In other words, we can assume the expected remaining number of points is $pn - p^4q^d|E|^2 - p^3q^{\frac{d-1}{2}}n^2$. For a small constant c, choose $p = c\frac{1}{n^{1/3}q^{d/3}}$. Plugging this back into Eq. (6), we get that $\mathbb{E}(|T|) \geq \Omega\left(\frac{n^{2/3}}{q^{d/3}}\right)$. Therefore, there exists such a set T of size $\Omega\left(\frac{n^{2/3}}{q^{d/3}}\right)$ with no repeated distance.

□

4 Repeated Dot Products in $\mathbb{F}_q^d$

We will use the following theorem from [8] to prove Theorem 5. Let $\ell_k = \{tk : t \in \mathbb{F}_q^*\}$, $E(x) = \mathbb{1}_{\{x \in E\}}$, and

$$\hat{E}(k) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot k) E(x)$$

for some additive character χ.

Theorem 12 *Let $E \subseteq \mathbb{F}_q^d$. Then*

$$|\{(x, y, x', y') \in E^4 : x \cdot y = x' \cdot y'\}| \leq \frac{|E|^4}{q} + |E| q^{2d-1} \sum_{k \neq (0,\ldots,0)} |E \cap \ell_k| |\hat{E}(k)|^2$$

$$+ (q-1) q^{-1} |E|^2 E(0, \ldots, 0).$$

We can estimate the middle term $|E| q^{2d-1} \sum_{k \neq (0,\ldots,0)} |E \cap \ell_k| |\hat{E}(k)|^2$ as follows: $|E \cap \ell_k| \leq q$ for each k, and $\sum_{k \neq (0,\ldots,0)} |\hat{E}(k)|^2 \leq \sum_{x \in \mathbb{F}_q^d} |\hat{E}(k)|^2 = q^{-d} |E|$, so this term is less than or equal to $q^d |E|^2$.

To summarize, we have

$$|\{(x, y, x', y') \in E^4 : x \cdot y = x' \cdot y'\}| \leq \frac{|E|^4}{q} + q^d |E|^2. \tag{7}$$

We include the proof of Theorem 12 from [8] to address triples satisfying

$$x \cdot y = x \cdot y'.$$

Proof of Theorem 12 Let $\nu(t) = |\{(x, y) \in E^2 : x \cdot y = t\}| = \sum_{x \cdot y = t} E(x) E(y)$.

Then $|\{(x, y, x', y') \in E^4 : x \cdot y = x' \cdot y'\}| = \sum_t \nu(t)^2$.

The Cauchy-Schwarz inequality applied to the sum in the variable x yields

$$\sum_t \nu(t)^2 \leq |E| \sum_t \sum_{x \cdot y = t} \sum_{x \cdot y' = t} E(x) E(y) E(y')$$

$$= |E| \sum_{(y - y') \cdot x = 0} E(y') E(y) E(x)$$

$$= |E| q^{-1} \sum_{x, y, y'} \sum_s \chi(s((y' - y) \cdot x)) E(y') E(y) E(x)$$

$$= |E|^4 q^{-1} + |E| q^{2d-1} \sum_{x} \sum_{s \neq 0} E(x) |\hat{E}(sx)|^2$$

$$= |E|^4 q^{-1} + |E| q^{2d-1} \sum_{x} \sum_{s \neq 0} E(sx) |\hat{E}(x)|^2$$

$$= |E|^4 q^{-1} + |E| q^{2d-1} \sum_{s \neq (0,\ldots,0)} |E \cap \ell_x| |\hat{E}(x)|^2 + (q-1) q^{-1} |E|^3 E(0, \ldots 0).$$

In the third line we have used the standard trick that $\sum_{s \in \mathbb{F}_q} \chi(ts)$ equals q for $t = 0$ and zero otherwise. The transition from the fourth to the fifth was after changing variables $sx \to x$ and then $s \to s^{-1}$. □

As in the case for repeated distances when we considered triples points satisfying $||x - y|| = ||x - y'||$, here we consider the number of triples of points satisfying, $x \cdot y = x \cdot y'$ which is counted by the summation $\sum_{(y-y') \cdot x = 0} E(y')E(y)E(x)$. Notice in the second line of the proof of Theorem 12 above is $|E| \sum_{(y-y') \cdot x = 0} E(y')E(y)E(x)$. Following the rest of this proof, we get that

$$|E| \sum_{(y-y') \cdot x = 0} E(y')E(y)E(x) \leq |E|^4 q^{-1} + |E| q^{2d-1} \sum_{s \neq (0,\ldots,0)} |E \cap \ell_x| |\hat{E}(x)|^2$$
$$+ (q-1) q^{-1} |E|^3 E(0, \ldots 0),$$

implying

$$\sum_{(y-y') \cdot x = 0} E(y')E(y)E(x) \leq |E|^3 q^{-1} + q^{2d-1} \sum_{s \neq (0,\ldots,0)} |E \cap \ell_x| |\hat{E}(x)|^2$$
$$+ (q-1) q^{-1} |E|^2 E(0, \ldots 0).$$

As in the argument for Eq. (7), we get that

$$|\{(x, y, x', y') \in E^4 : x \cdot y = x \cdot y'\}| \leq \frac{|E|^3}{q} + q^d |E|. \tag{8}$$

Proof of Theorem 5 For a set E of size n and a probability $0 < p < 1$ to be chosen later, we will choose a subset of $S \subseteq E$ by taking each point of E independently with probability p and putting it into S. The expected size of S, $\mathbb{E}(|S|)$ is pn.

Let $Q = \{(x, y, x', y') \in E^4 : x \cdot y = x' \cdot y'\}$, and let $R \subset Q$ be the set of quadruples of Q, where each point of the quadruple is also in S. A quadruple of distinct points (x, y, x', y') appears in S with probability p^4, and a triple of points (x, y, y') appears with probability p^3. By Theorem 12 the expectation $\mathbb{E}(|R|)$ is

$$O\left(p^4 |\{(x, y, x', y') \in E^4 : x \cdot y = x' \cdot y'\}|\right) \leq O\left(p^4 \frac{n^4}{q} + p^4 q^d n^2 + p^3 \frac{n^3}{q} + p^3 q^d n\right).$$

For each quadruple of points in R, we create a new set $T \subseteq S$ by deleting one point from S belonging to that quadruple. Therefore, none of the points in T will form a quadruple satisfying $x \cdot y = x' \cdot y'$. By linearity of expectation,

$$\mathbb{E}(|T|) = \mathbb{E}(|S| - |R|) \geq pn - p^4 \frac{n^4}{q} - p^4 q^d n^2 - p^3 \frac{n^3}{q} - p^3 q^d n. \tag{9}$$

We are ready to choose p: p must be chosen small enough so that removing $|T|$ points from R does not remove all of the points of R. In other words, we must choose p so that np is larger than $p^4 \frac{n^4}{q} + p^4 q^d n^2 + p^3 \frac{n^3}{q} + p^3 q^d n$. We will split this into 2 cases, depending on which part of the summation $p^4 \frac{n^4}{q} + p^4 q^d n^2 + p^3 \frac{n^3}{q} + p^3 q^d n$ is largest.

Case 1 For $|E| > q^{(d+1)/2}$, $\frac{n^4}{q} > q^d n^2$ and $\frac{n^3}{q} > q^d n$, so only the $p^4 \frac{n^4}{q} + \frac{n^3}{q}$ and term in Eq. (9) will affect the optimization. In other words, we can assume the expected remaining number of points is $pn - p^4 \frac{n^4}{q} - p^3 \frac{n^3}{q}$. For a small constant c, choose $p = c\frac{q^{1/3}}{n}$. Plugging this back into Eq. (6), we get that $\mathbb{E}(|T|) \geq \Omega(q^{1/3})$. Therefore, there exists such a set T of size $\Omega(q^{1/3})$ with no repeated dot product.

Case 2 For $|E| < q^{(d+1)/2}$, $\frac{n^4}{q} < q^d n^2$ and $\frac{n^3}{q} < q^d n$, so only the $p^4 q^d n^2 + q^d n$ term in Eq. (9) will affect the optimization. In other words, we can assume the expected remaining number of points is $pn - p^4 q^d n^2 - q^d n$. For a small constant c, choose $p = c\frac{1}{n^{1/3} q^{d/3}}$. Plugging this back into Eq. (9), we get that $\mathbb{E}(|T|) \geq \Omega\left(\frac{n^{2/3}}{q^{d/3}}\right)$. Therefore, there exists such a set T of size $\Omega\left(\frac{n^{2/3}}{q^{d/3}}\right)$ with no repeated dot product.

□

5 Sets of Size $\Theta(\sqrt{q})$ with no Repetitions

A *Sidon set* S is a set of points in $[n]$ such that every pairwise sum is distinct, i.e. for all $a, b, c, d \in S$, $a + b \neq c + d$. An example of such a set is as follows:

Lemma 1 *For large enough n, we can choose a prime q such that $2 \leq \frac{1}{2}\sqrt{n/2} < q < \sqrt{n/2}$ and define the set of points $a_i = 2qi + (i^2)$, for $i = 0 \ldots q-1$ and (i^2) is the integer modulo q. Then a_i forms a Sidon set of size $\Theta(\sqrt{n})$.*

We include the proof, established by Singer [15].

Proof By Bertrand's Theorem for any integer $n > 1$ there exists a prime q between $\frac{1}{2}\sqrt{n/2}$ and $\sqrt{n/2}$. Also, since $i \leq q - 1$ and $q < \sqrt{n/2}$, then $a_i \in [n]$ for all i.

We will show that if $a_i + a_j = a_k + a_\ell$, then $i = k$ and $j = \ell$. Indeed, $a_i + a_j = a_k + a_\ell$ implies $2qi + (i^2) + 2qj + (j^2) = 2qk + (k^2) + 2q\ell + (\ell^2)$. Since

(i^2), (j^2), (k^2) and (ℓ^2) are all less than q, we have that $2qi + 2qj = 2qk + 2q\ell$, and hence the following 2 equations:

$$i^2 + j^2 \equiv k^2 + \ell^2 \mod q \tag{10}$$

$$i + j = k + \ell \tag{11}$$

These imply $i^2 - \ell^2 \equiv k^2 - j^2 \mod q$ and $i - \ell = k - j$. If $i - \ell = k - j = 0$, then we are done. Otherwise, we get $i + \ell \equiv j + k \mod q$. This together with $i - \ell = k - j$ implies $2i \equiv 2k \mod q$. Since $q > 2$, this implies $i \equiv k \mod q$. Similarly, we also get $j \equiv \ell \mod q$. Since i, j, k, ℓ were chosen between 0 and $q - 1$, we conclude that $i = k$ and $j = \ell$.

Since $q = \Theta(\sqrt{n})$, we have that the number of a_i's is $\Theta(\sqrt{n})$. □

Proof of Theorem 4 For a prime q, we can find a Sidon set $S \subset (\mathbb{F}_{q-1}, +)$ of size $\Theta(\sqrt{q})$ by taking the set constructed in Theorem 1 as a subset of $[q/3]$.

Now consider the set $D = \{(0, \ldots, 0, s) \in \mathbb{F}_q^d, s \in S\}$. Let $x = (0, \ldots, 0, x_d) \in D$. For any points $x, y, x'y' \in D$,

$$||x - y|| = ||x' - y'|| \Leftrightarrow (x_d - y_d)^2 = (x'_d - y'_d)^2.$$

This implies either

$$x_d - y_d = x'_d - y'_d \Leftrightarrow x_d + y'_d = x'_d + y_d,$$

or

$$x_d - y_d = y'_d - x'_d \Leftrightarrow x_d + x'_d = y'_d + y_d.$$

In either case, the set of points $\{x, x', y, y'\}$ cannot be more than 2 distinct points. So this is a set of size $\Theta(\sqrt{q})$ with no repeated distances.

Note that by the pigeonhole principle, every subset of $\mathbb{F}_q^d$ with size $\omega(\sqrt{q})$ will have repeated dot products, so D is an asymptotically tight construction. □

Proof 6 For a prime q, we can find a Sidon set $S \subset (\mathbb{F}_{q-1}, +)$ of size $\Theta(\sqrt{q})$ by taking the set constructed in Theorem 1 as a subset of $[q/3]$. Also, define the isomorphism $\psi : (\mathbb{F}_{q-1}, +) \to (\mathbb{F}_q^*, \cdot)$, $\psi(x) \to 2^x$.

Consider the set $\psi(S)$, which has size $\Theta(q)$. For any distinct $\psi(a), \psi(b), \psi(c), \psi(d) \in \psi(S)$, we'll show that $\psi(a)\psi(b) \neq \psi(c)\psi(d)$:

Suppose not, i.e. there are distinct $a, b, c, d \in S$ such that $\psi(a)\psi(b) = \psi(c)\psi(d)$. Since ψ is an isomorphism, this is the same as

$$\psi(a + b) = \psi(c + d)$$
$$\Leftrightarrow a + b = c + d,$$

which cannot happen since S is a Sidon set.

Now consider the set $D = \{(0, \ldots, 0, s) \in \mathbb{F}_q^d, s \in \psi(S)\}$. For any distinct $x, y, x', y' \in D$, $x \cdot y \neq x' \cdot y'$. So this is a set of size $\Theta(\sqrt{q})$ in $\mathbb{F}_q^d$ with no repeated dot products.

Note that by the pigeonhole principle, every subset of $\mathbb{F}_q^d$ with size $\omega(\sqrt{q})$ will have repeated dot products, so D is an asymptotically tight construction. □

6 General Formulation

The underlying edge operator in the distance graph is

$$A_t f(x) = \sum_{||x-y||=t} f(y),$$

while the edge operator in the dot product case is

$$R_t f(x) = \sum_{x \cdot y = t} f(y) dy.$$

The Euclidean variant of $Af(x)$ is

$$\mathcal{A}_t f(x) = \int f(x - y) d\sigma(y),$$

where σ the surface measure on S^{d-1}, the unit sphere. The Euclidean variant of $Rf(x)$ is the classical Radon transform

$$\mathcal{R}_t f(x) = \int_{x \cdot y = t} \psi(y) f(y) d\sigma_{x,t}(y),$$

where ψ is a smooth cut-off function and $\sigma_{x.t}$ is the surface measure on $\{y \in \mathbb{R}^d : x \cdot y = t\}$. If $t \neq 0$, both $\mathcal{A}_t$ and $\mathcal{R}_t$ map $L^2(\mathbb{R}^d)$ to $H^{\frac{d-1}{2}}(\mathbb{R}^d)$, where $H^s(\mathbb{R}^d)$ is the Sobolev space of $L^2(\mathbb{R}^d)$ function with generalized derivative of order $s > 0$ in $L^2(\mathbb{R}^d)$. See, for example, [16] and the references contained therein.

A significant amount of progress has been made in the Euclidean setting in studying general configuration problems from the point of view of Sobolev estimates. See, for example, [5, 6, 10] for some recent work in this direction. It would be interesting to encode the bounds in the finite field setting using a suitable formalism analogous to their Euclidean counterparts. Both the edge operators A_t and R_t, defined above, satisfy the following bounds that we encode as follows. Let

$$T_t^{\phi} f(x) = \sum_{\phi(x,y)=t} f(y),$$

where $\phi : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$, a function. See [13] for the description of the continuous analog of this family of operators, introduced by Phong and Stein.

Note that if $\phi(x, y) = ||x - y||$, we recover the operator A_t, while if $\phi(\delta, y) = x \cdot y$, we cover R_t. Let

$$T_{t,0}^{\phi} f(x) = T_t^{\phi} f(x) - q^{-d} \sum_{x \in \mathbb{F}_q^d} T_t^{\phi} f(x),$$

which amounts to stripping $Tf(x)$ of its 0'th Fourier coefficient. Both A_t and R_t above satisfy the bound

$$< T_{t,0}^{\phi} f, g > \leq \ Cq^{\frac{d-1}{2}} ||f||_{L^2(\mathbb{F}_q^d)} \cdot ||g||_{L^2(\mathbb{F}_q^d)}, \tag{12}$$

where

$$||f||^2_{L^2(\mathbb{F}_q^d)} = \sum_{x \in \mathbb{F}_q^d} |f(x)|^2,$$

and the inner product on the left hand side above is the $L^2(\mathbb{F}_q^d)$ inner product. The proof that A_t satisfies these bounds is implicit in [9], while the corresponding bound for R_t follows readily from the results in [7] (see also [8]).

The estimate (12) can be viewed as an analog of the $L^2(\mathbb{R}^d) \to H^{\frac{d-1}{2}}(\mathbb{R}^d)$ bound in the Euclidean case since for any ϕ satisfying

$$|\{x \in \mathbb{F}_q^d : \phi(x, y) = t\}| \approx q^{d-1} \text{ for each fixed } y, \tag{13}$$

and

$$|\{y \in \mathbb{F}_q^d : \phi(x, y) = t\}| \approx q^{d-1} \text{ for each fixed } x,$$

the estimate (12) easily holds with $q^{\frac{d-1}{2}}$ replaced by q^{d-1}. It is reasonable to summarize the above using the following notion.

Definition 1 Let $T_t^{\phi}, T_{t,0}^{\phi}$ be as above. Suppose that (13) holds and, in place of (12) we have

$$< T_{t,0}^{\phi} f, g > \leq \ Cq^{d-1} q^{-\alpha} ||f||_{L^2(\mathbb{F}_q^d)} \cdot ||g||_{L^2(\mathbb{F}_q^d)} \tag{14}$$

for some $\alpha > 0$.

Then we say that $T_{t,0}^{\phi}$ is smoothing of order α.

Given a function $\phi : \mathbb{F}_q^d \times \mathbb{F}_q^d$, as above, and $E \subset \mathbb{F}_q^d$, define $\mathcal{G}_t^{\phi}(E)$ to be the graph where the vertices are given by the points of E, and two vertices are connected by an edge of $\phi(x, y) = t$. In the sequel, we shall engage in a systematic study of the properties of this graph under the smoothing assumption (14) and the size assumption (13) above.

An interested reader should have no trouble applying the probabilistic scheme from this paper to a wide variety of setting outlined in this section.

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Banach Spaces of Functions Delta-Subharmonic in the Unit Disc

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Abstract The present work introduces several Banach spaces of delta-subharmonic in the unit disc of the complex plane functions, the M. M. Djrbashian generalized fractional integrals of which possess bounded square integral means. The union of these spaces coincides with the set of all functions delta-subharmonic in the unit disc.

Keywords Banach spaces · Delta-subharmonic functions

1 Preliminaries

We begin by a study of some classes of Green type potentials in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, the M. M. Djrbashian generalized fractional integrals of which possess bounded square integral means. For functions $u(z)$ defined in $\mathbb{D}$, we use the following simplified form of the mentioned fractional integral (see Lemma 1.1 in [6]):

$$L_\omega u(z) = -\int_0^1 u(tz)d\omega(t), \quad z \in \mathbb{D},$$

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and we note that L_ω is defined as the identical operator for $\omega(t) \equiv 1$. Also, we use M. M. Djrbashian's Cauchy type ω-kernel [2, 3]

$$C_\omega(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k}, \quad \Delta_k = k \int_0^1 t^{k-1} \omega(t) dt \quad (k = 0, 1, 2, \dots),$$

which is a holomorphic in $\mathbb{D}$ function under the following choice of the parameter-function $\omega(t)$.

Definition 1 Ω is the class of those functions $\omega(x)$, which are nonincreasing, positive in $[0, 1)$, $\omega \in \mathrm{Lip}_\lambda[0, 1]$ for some $\lambda \in (0, 1]$, and $\omega(0) = 1$, $\omega(1) = 0$ or, alternatively, $\omega(t) \equiv 1$.

Note that for $\omega(t) = \frac{(1-t)^\alpha}{\Gamma(1+\alpha)}$ $(\alpha > 0)$, L_ω becomes the Riemann-Liouville fractional integral of $u(z)$ multiplied by $z^{-\alpha}$, while the kernel $C_\omega(z)$ becomes

$$\frac{\Gamma(1+\alpha)}{(1-z)^{1+\alpha}} := C_\alpha(z), \quad \text{and} \quad L_\omega C_\omega(z) = \frac{1}{1-z} = C_0(z), \quad z \in \mathbb{D}.$$

Further, we deal with the following two Blaschke type factors (see [1–3]) which for $|z| < |\zeta| < 1$ can be written in the forms

$$b_\omega(z, \zeta) = \exp\left\{ - \int_{|\zeta|}^1 \left[C_\omega\left(\frac{z}{\zeta} x\right) + C_\omega\left(\frac{z\overline{\zeta}}{x}\right) - 1 \right] \frac{\omega(x)}{x} dx \right\},$$

$$\widetilde{b}_\omega(z, \zeta) = \exp\left\{ - \int_{|\zeta|^2}^1 C_\omega\left(\frac{z}{\zeta} x\right) \frac{\omega(x)}{x} dx \right\}.$$

When $\omega(t) \equiv 1$, these functions become the ordinary Blaschke factors:

$$b_\omega(z, \zeta)\Big|_{\omega\equiv 1} = \frac{\zeta - z}{1 - \overline{\zeta} z} \frac{|\zeta|}{\zeta} := b_0(z, \zeta), \quad \widetilde{b}_\omega(z, \zeta)\Big|_{\omega\equiv 1} = \frac{\zeta - z}{1 - \overline{\zeta} z} |\zeta| := \widetilde{b}_0(z, \zeta),$$

while for any $\omega(t) \in \Omega$ and any fixed point $0 \neq \zeta \in \mathbb{D}$, the functions $b_\omega(z, \zeta)$ and $\widetilde{b}_\omega(z, \zeta)$ are holomorphic in $\mathbb{D}$, where they have a unique, first order zero at the point $z = \zeta$ (see [1, 3], Lemma 1.5 and formulas (1.9), (1.64)–(1.68) in [2], also see Lemma 4.2 in [7]).

For $\omega(x) \in \Omega$, we consider the Green type potentials

$$P_\omega(z) := \iint_{\mathbb{D}} \log |b_\omega(z, \zeta)| d\nu(\zeta), \quad \widetilde{P}_\omega(z) := \iint_{\mathbb{D}} \log \left|\widetilde{b}_\omega(z, \zeta)\right| d\nu(\zeta)$$

which are convergent and represent some δ-subharmonic functions in $\mathbb{D}$, if the Borel signed measure ν, which is their charge, is such that $\inf\{|\zeta| : \zeta \in \operatorname{supp}\nu\} = d > 0$ and the corresponding Blaschke type conditions is fulfilled:

$$\iint_{\mathbb{D}} \left[\int_{|\zeta|}^{1} \omega(t)dt \right] |d\nu(\zeta)| < +\infty, \quad \iint_{\mathbb{D}} \left[\int_{|\zeta|^2}^{1} \omega(t)dt \right] |d\nu(\zeta)| < +\infty, \tag{1}$$

where $|d\nu(\zeta)|$ means the differential of the complete variation of the measure ν. Besides, it is not difficult to prove that under the above conditions the functions $L_\omega P_\omega(z)$ and $L_\omega \widetilde{P}_\omega(z)$ are continuous, δ-subharmonic in $\mathbb{D}$ and for any $z \in \mathbb{D}$

$$\begin{aligned} L_\omega P_\omega(z) &= \iint_{|\zeta|<1} L_\omega \log |b_\omega(z, \zeta)| d\nu(\zeta), \\ L_\omega \widetilde{P}_\omega(z) &= \iint_{|\zeta|<1} L_\omega \log |\widetilde{b}_\omega(z, \zeta)| d\nu(\zeta), \end{aligned} \tag{2}$$

where the integrals are uniformly convergent inside $\mathbb{D}$. Besides, the following inequalities are true:

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \big[|L_\omega P_\omega(re^{i\vartheta})|d\vartheta < \frac{1}{d} \iint_{\mathbb{D}} \left(\int_{|\zeta|}^{1} \omega(t)dt \right) |d\nu(\zeta)| < +\infty,$$

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|L_\omega \widetilde{P}_\omega(re^{i\theta})\right| d\theta \le \frac{1}{d^2} \iint_{\mathbb{D}} \left(\int_{|\zeta|^2}^{1} \omega(t)\, dt \right) |d\nu(\zeta)| < +\infty.$$

Evidently, it is natural to look for some stronger than (1) conditions which can provide the boundedness of the similar integral means with the squares of the integrands. Such conditions are found in the next section.

2 Banach Spaces of Green Type Charges

In this section, some Banach spaces of Green type potentials formed by the Blaschke type factors $b_\omega(z, \zeta)$ and $\widetilde{b}_\omega(z, \zeta)$ are studied.

2.1 *The Wider Banach Space of Potentials Formed by* b_ω

If $\omega(t) \in \Omega$ and $0 \neq \zeta \in \mathbb{D}$ is a fixed point, then obviously

$$f(z, \zeta) := L_\omega \log b_\omega(z, \zeta) = -\int_{|\zeta|}^{1} \left\{ \frac{1}{1 - \frac{z}{\zeta}x} + \frac{1}{1 - \frac{z\overline{\zeta}}{x}} - 1 \right\} \frac{\omega(x)}{x} dx, \tag{3}$$

for any $z \in \overline{\mathbb{D}}$, except the straight line interval $\{z = \zeta t : 1 \le t \le |\zeta|^{-1}\}$ connecting the points ζ and $\zeta/|\zeta|$, where the integral is to be understood in the sense of its principal value. Using the well-known results on the Cauchy type integrals (see [4], Sec. 5.1), it is not difficult to prove that the function $L_\omega \log|b_\omega(z,\zeta)|$ is harmonic in $\overline{\mathbb{D}}$, except the mentioned interval, is subharmonic in $\mathbb{D}$, nonpositive and continuous in $\overline{\mathbb{D}}$, and $L_\omega \log|b_\omega(e^{i\vartheta},\zeta)| = 0$ $(0 \le \vartheta \le 2\pi)$. Further, note that the following equality is true for any function $\varphi(\zeta)$ and any measure $\mu(\zeta)$, for which the below integrals exist:

$$\left(\mathrm{Re}\int_E \varphi(\zeta)d\mu(\zeta)\right)^2 = \frac{1}{2}\mathrm{Re}\int_E d\mu(\zeta_1)\int_E \left[\varphi(\zeta_1)\varphi(\zeta_2) + \varphi(\zeta_1)\overline{\varphi(\zeta_2)}\right] d\mu(\zeta_2). \tag{4}$$

Thus, if a Borel signed measure ν is such that $\inf\{|\zeta| : \zeta \in \mathrm{supp}\nu\} = d > 0$ and the first of the Blaschke type conditions (1) is fulfilled, then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left[L_\omega P_\omega(re^{i\vartheta})\right]^2 d\vartheta &= \int_0^{2\pi}\left(\iint_{\mathbb{D}} \mathrm{Re} f(re^{i\vartheta},\zeta)\, d\nu(\zeta)\right)^2 d\vartheta \\ &= \iint_{\mathbb{D}} d\nu(\zeta_1)\iint_{\mathbb{D}} J(r,\zeta_1,\zeta_2)\, d\nu(\zeta_2)\end{aligned} \tag{5}$$

with

$$\begin{aligned}J_\omega(r,\zeta_1,\zeta_2) &= \frac{1}{2\pi}\int_0^{2\pi} L_\omega \log|b_\omega(re^{i\vartheta},\zeta_1)| L_\omega \log|b_\omega(re^{i\vartheta},\zeta_2)|\, d\vartheta \\ &= \mathrm{Re}\frac{1}{4\pi}\int_0^{2\pi}\left[f(re^{i\vartheta},\zeta_1)f(re^{i\vartheta},\zeta_2) + f(re^{i\vartheta},\zeta_1)\overline{f(re^{i\vartheta},\zeta_2)}\right]d\vartheta,\end{aligned} \tag{6}$$

provided the above integrals exist and the change of the integration order is true in (5). Formulas (3), (6) make it possible to calculate $J_\omega(r,\zeta_1,\zeta_2)$ by residues. Namely, assuming that $P_\lambda(\psi) = (1-\lambda^2)/|e^{i\psi}-\lambda|^2$ is the Poisson kernel, $\zeta_{1,2} = \rho_{1,2}e^{i\varphi_{1,2}} \in \overline{\mathbb{D}}\backslash\{0\}$, $\varphi_1 - \varphi_2 = \psi$ and $\widetilde{x} = x^{\mathrm{sign}(x-1)}$, we got

$$J_\omega(r,\zeta_1,\zeta_2) = \frac{1}{4}\int_{\rho_1}^{1/\rho_1}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/\rho_2}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2 \tag{7}$$

$$\text{when}\quad 0 \le r \le \rho_{1,2},$$

$$\begin{aligned}J_\omega(r,\zeta_1,\zeta_2) = &-\frac{1}{4}\int_{\rho_1}^{1/r}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{1/r}^{1/\rho_2}\frac{\omega(\rho_2 x_2)}{x_2}P_{x_1/x_2}(\psi)dx_2 \\ &-\frac{1}{4}\int_{1/r}^{1/\rho_1}\frac{\omega(\rho_1 x_1)}{x_1}dx_1\int_{\rho_2}^{1/r}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_2/x_1}(\psi)dx_2\end{aligned}$$

$$+\frac{1}{4}\int_{\rho_1}^{1/r}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/r}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2$$

$$-\frac{1}{4}\int_{1/r}^{1/\rho_1}\frac{\omega(\rho_1x_1)}{x_1}dx_1\int_{1/r}^{1/\rho_2}\frac{\omega(\rho_2x_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2 \quad (8)$$

$$\text{when} \quad d\le\rho_{1,2}\le r,$$

$$J_\omega(r,\zeta_1,\zeta_2)=-\frac{1}{4}\int_{1/r}^{1/\rho_1}\frac{\omega(\rho_1x_1)}{x_1}dx_1\int_{\rho_2}^{1/\rho_2}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_2/x_1}(\psi)dx_2$$

$$+\frac{1}{4}\int_{\rho_1}^{1/r}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/\rho_2}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2 \quad (9)$$

$$\text{when} \quad \rho_1\le r\le\rho_2,$$

$$J_\omega(r,\zeta_1,\zeta_2)=-\frac{1}{4}\int_{\rho_1}^{1/\rho_1}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{1/r}^{1/\rho_2}\frac{\omega(\rho_2x_2)}{x_2}P_{x_1/x_2}(\psi)dx_2$$

$$+\frac{1}{4}\int_{\rho_1}^{1/\rho_1}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/r}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2 \quad (10)$$

$$\text{when} \quad \rho_2\le r\le\rho_1.$$

The calculation leading to the above formulas is of purely technical character, very long and therefore omitted, while these formulas make it possible to prove the following basic lemma.

Lemma 1 *For any $\omega\in\Omega$, the function $J_\omega(r,\zeta_1,\zeta_2)$ is continuous by its variables $\zeta_1,\zeta_2\in\overline{\mathbb{D}}\backslash\{0\}$ and $r\in[0,1]$. Besides, for any $r\in[0,1]$ and $0<d\le|\zeta_{1,2}|\le 1$ the following inequalities are true:*

$$0\le J_\omega(r,\zeta_1,\zeta_2)\le M_d\int_0^{1-|\zeta_1|}\omega(|\zeta_1|(1-x))dx\int_0^{1-|\zeta_2|}\frac{\omega(|\zeta_2|(1-y))}{x+y}dy$$

$$\le\frac{M_d}{2}K_\omega(|\zeta_1|,|\zeta_2|), \quad (11)$$

where $M_d=5/4d^3$ and

$$K_\omega(|\zeta_1|,|\zeta_2|):=\int_0^{1-|\zeta_1|}\omega(|\zeta_1|(1-x))\frac{dx}{\sqrt{x}}\int_0^{1-|\zeta_2|}\omega(|\zeta_2|(1-y))\frac{dy}{\sqrt{y}}.$$

Proof The function $L_\omega|b_\omega(z,\zeta)|$ is nonpositive and continuous in $\overline{\mathbb{D}}$ for any fixed $\zeta\in\mathbb{D}\backslash\{0\}$. Hence, the first inequality in (11) and the continuity of $J_\omega(r,\zeta_1,\zeta_2)$ by $r\in[0,1]$ hold. Further, the continuity of $J_\omega(r,\zeta_1,\zeta_2)$ by $\zeta_1,\zeta_2\in\overline{\mathbb{D}}\backslash\{0\}$ follows from the given above representations (7), (8), (9) and (10). For estimating the function $J_\omega(r,\zeta_1,\zeta_2)$ from above, we denote $\zeta_{1,2}=\rho_{1,2}e^{i\varphi_{1,2}}$, $\psi=\varphi_1-\varphi_2$ and consider several cases.

(i) $0\le r\le \boldsymbol{\rho}_{1,2}$ In (7), we denote

$$J_\omega(r,\zeta_1,\zeta_2)=\frac{1}{4}\Big(\int_{\rho_1}^1\int_{\rho_2}^1+\int_1^{1/\rho_1}\int_1^{1/\rho_2}+\int_{\rho_1}^1\int_1^{1/\rho_2}+\int_1^{1/\rho_1}\int_{\rho_2}^1\Big)$$
$$\times P_{x_1x_2r^2}(\psi)\omega(\rho_1\widetilde{x}_1)\omega(\rho_2\widetilde{x}_2)\frac{dx_1dx_2}{x_1x_2}:=A_1+A_2+A_3+A_4.$$

Making the variable changes $\rho_{1,2}/x_{1,2}=t_{1,2}$, then $t_1=1-x$, $t_2=1-y$ in A_1 and recalling that $\rho_{1,2}\ge d>0$, we get

$$A_1\le\frac{1}{d^2}\int_{\rho_1}^1\int_{\rho_2}^1\frac{\omega(t_1)\omega(t_2)}{1-t_1t_2}dt_1dt_2=\frac{1}{d^2}\int_0^{1-\rho_1}\int_0^{1-\rho_2}\frac{\omega(1-x)\omega(1-y)}{x+y-xy}dt_1dt_2,$$

since $xy\le(1-d)(x+y)$. Then by $\omega(x)\searrow$ we get $A_1\le d^{-3}K_\omega(\rho_1,\rho_2)$.

The change of the variables $\rho_{1,2}x_{1,2}=t_{1,2}$ gives

$$A_2\le\int_1^{1/\rho_1}\int_1^{1/\rho_2}\frac{\omega(\rho_1x_1)\omega(\rho_2x_2)}{1-x_1x_2r^2}\frac{dt_1dt_2}{t_1t_2}<\frac{1}{d^2}\int_{\rho_1}^1\int_{\rho_2}^1\frac{\omega(t_1)\omega(t_2)}{1-\frac{t_1t_2}{\rho_1\rho_2}r^2}dt_1dt_2.$$

Then, using the inequality $\frac{t_1t_2}{\rho_1\rho_2}r^2<t_1t_2$ and changing the variables as $t_1=1-x$, $t_2=1-y$, again we get $A_2\le d^{-3}K_\omega(\rho_1,\rho_2)$.

The change of the variables $\rho_1/x_1=t_1$, $\rho_2x_2=t_2$ gives

$$A_3\le\frac{1}{d^2}\int_{\rho_1}^1\int_{\rho_2}^1\frac{\omega(t_1)\omega(t_2)}{1-\frac{\rho_1t_2}{\rho_2t_1}r^2}dt_1dt_2,\quad\text{where}\quad\frac{\rho_1t_2}{\rho_2t_1}r^2<t_1t_2.$$

Hence, the change of the variables $t_1=1-x$, $t_2=1-y$ gives $A_3\le d^{-3}K_\omega(\rho_1,\rho_2)$, while $A_4\le d^{-3}K_\omega(\rho_1,\rho_2)$, since it is symmetric by $\zeta_{1,2}$. Thus,

$$J_\omega(r,\zeta_1,\zeta_2)\le\frac{1}{d^3}K_\omega(\rho_1,\rho_2),\quad 0\le r\le\rho_{1,2}.$$

(ii) $d\le\boldsymbol{\rho}_{1,2}\le r$ Omitting the first two integrals in the right-hand side of (8), which are nonpositive, we get

$$J_\omega(r,\zeta_1,\zeta_2)\le\frac{1}{4}\int_{\rho_1}^{1/r}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/r}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2$$
$$-\frac{1}{4}\int_{1/r}^{1/\rho_1}\frac{\omega(\rho_1x_1)}{x_1}dx_1\int_{1/r}^{1/\rho_2}\frac{\omega(\rho_2x_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2$$
$$:=B_1+B_2.$$

We separately estimate the integrals

$$B_1 = \frac{1}{4}\left(\int_1^{1/r}\int_1^{1/r} + \int_{\rho_1}^1\int_{\rho_2}^1 + \int_{\rho_1}^1\int_1^{1/r} + \int_1^{1/r}\int_{\rho_2}^1\right)$$
$$\times P_{x_1x_2r^2}(\psi)\omega(\rho_1\widetilde{x}_1)\omega(\rho_2\widetilde{x}_2)\frac{dx_1dx_2}{x_1x_2} := B_1' + B_1'' + B_1''' + B_1^{IV},$$
$$B_2 = -\frac{1}{4}\int_{1/r}^{1/\rho_1}\int_{1/r}^{1/\rho_2} P_{x_1x_2r^2}(\psi)\omega(\rho_1x_1)\omega(\rho_2x_2)\frac{dx_1dx_2}{x_1x_2}. \tag{12}$$

The change of the variables $t_{1,2} = rx_{1,2}$ and the inequalities $t_{1,2}/r > t_{1,2}$ give

$$B_1' \leq \frac{1}{4d^2}\int_r^1\int_r^1 \frac{\omega(\rho_1t_1)\omega(\rho_2t_2)}{1 - t_1t_2}dt_1dt_2,$$

and the change of the variables $t_1 = 1-x$, $t_2 = 1-y$ leads to $B_1' \leq d^{-3}K_\omega(\rho_1, \rho_2)$.

Deleting r^2 in the denominator and using the change of the variables $x_1 = 1/(1-x)$, $x_2 = 1/(1-y)$, we get

$$B_1'' \leq \frac{1}{4d^2}\int_0^{1-\rho_1}\int_0^{1-\rho_2} \frac{\omega\left(\frac{\rho_1}{1-x}\right)\omega\left(\frac{\rho_2}{1-y}\right)}{x + y - xy}dxdy,$$

and the inequalities $1/(1-x) > 1-x$, $1/(1-y) > 1-y$ and $xy \leq (1-d)(x+y)$ lead to $B_1'' \leq d^{-3}K_\omega(\rho_1, \rho_2)$.

The change of the variables $t_1 = 1 - x_1$, $t_2 = rx_2$ gives

$$B_1''' \leq \frac{1}{4d^2}\int_0^{1-\rho_1}\int_r^1 \frac{\omega\left(\frac{\rho_1}{1-t_1}\right)\omega\left(\frac{\rho_2}{r}t_2\right)}{1 - (1-t_1)t_2}dt_1dt_2.$$

Then, the change of the variable $t_1 = x$, $t_2 = 1-y$ and the inequalities $1/(1-x) > 1-x$, $(1-y)/r > 1-y$, $1-\rho_2 > 1-r$ and $x + y - xy > d(x+y)$ give $B_1''' \leq (4d^3)^{-1}K_\omega(\rho_1, \rho_2)$. Also $B_1^{IV} \leq (4d^3)^{-1}K_\omega(\rho_1, \rho_2)$ by symmetry.

As to B_2 of (12), by the variable changes $x_{1,2} = 1/t_{1,2}$ we get

$$B_2 \leq \frac{1}{4d^2}\int_{\rho_1}^r\int_{\rho_2}^r \frac{\omega\left(\frac{\rho_1}{t_1}\right)\omega\left(\frac{\rho_2}{t_2}\right)}{1 - t_1t_2}dt_1dt_2.$$

Then, by the inequalities $\omega(\rho_{1,2}/t_{1,2}) < \omega(\rho_{1,2}t_{1,2})$ and the change of the variables $t_1 = 1-x$, $t_2 = 1-y$ we get $B_2 \leq (4d^3)^{-1}K_\omega(\rho_1, \rho_2)$. Thus,

$$J_\omega(r, \zeta_1, \zeta_2) \leq \frac{5}{4d^3}K_\omega(\rho_1, \rho_2), \quad 0 < d \leq \rho_{1,2} \leq r < 1.$$

(iii) $0 < d \le \boldsymbol{\rho}_1 \le r \le \boldsymbol{\rho}_2$ The first integral in the right-hand side of (9) is nonpositive, and hence

$$\begin{aligned}
J_\omega(r,\zeta_1,\zeta_2) &\le \frac{1}{4}\int_{\rho_1}^{1/r}\frac{\omega(\rho_1\widetilde{x}_1)}{x_1}dx_1\int_{\rho_2}^{1/\rho_2}\frac{\omega(\rho_2\widetilde{x}_2)}{x_2}P_{x_1x_2r^2}(\psi)dx_2\\
&= \frac{1}{4}\left(\int_{\rho_1}^{1}\int_{\rho_2}^{1}+\int_{1}^{1/r}\int_{1}^{1/\rho_2}+\int_{1}^{1/r}\int_{\rho_2}^{1}+\int_{\rho_1}^{1}\int_{1}^{1/\rho_2}\right)\\
&\qquad\times P_{x_1x_2r^2}(\psi)\omega(\rho_1\widetilde{x}_1)\omega(\rho_2\widetilde{x}_2)\frac{dx_1dx_2}{x_1x_2}\\
&:= C'+C''+C'''+C^{IV}.
\end{aligned}$$

By the variable changes $x_1 = 1-x$, $x_2 = 1-y$, we get

$$C_1' \le \frac{1}{4d^2}\int_0^{1-\rho_1}\int_0^{1-\rho_2}\frac{\omega\left(\frac{\rho_1}{1-x}\right)\omega\left(\frac{\rho_2}{1-y}\right)}{x+y-xy}dxdy \le \frac{1}{d^3}K_\omega(\rho_1,\rho_2),$$

since $1/(1-x) > 1-x$ and $1/(1-y) > 1-y$.

Using the variable changes $x_1 = 1-x$, $\rho_2 x_2 = 1-y$ and the inequality $r^2/\rho_2 < 1$, we get

$$C' \le \frac{1}{4d^2}\int_0^{1-\rho_1}\int_0^{1-\rho_2}\frac{\omega\left(\frac{\rho_1}{1-x}\right)\omega(1-y)}{x+y-xy}dxdy \le \frac{1}{4d^3}K_\omega(\rho_1,\rho_2).$$

By the variable changes $x_1 = t_1/r$, $x_2 = t_2/\rho_2$ and the inequalities $r/\rho_2 < 1$, $\rho_1/r < 1$, we get

$$C'' \le \frac{1}{d^2}\int_r^1\omega(t_1)dt_1\int_{\rho_2}^1\frac{\omega(t_2)}{1-t_1t_2}dt_2.$$

Then, by the variable changes $t_1 = 1-x$, $t_2 = 1-y$ and the nonincreasing property of the function $\omega(x)$ we get $C'' \le d^{-3}K_\omega(\rho_1,\rho_2)$.

By the variable changes $x_1 = t_1/r$, $x_2 = t_2$, we get

$$C''' \le \frac{1}{d^2}\int_r^1 dt_1\int_{\rho_2}^1\frac{\omega\left(\frac{\rho_2}{t_2}\right)}{1-t_1t_2}dt_2.$$

Then, by the variable changes $t_1 = 1-x$, $t_2 = 1-y$ we get $C'' \le d^{-3}K_\omega(\rho_1,\rho_2)$. Besides, $C^{IV} \le d^{-3}K_\omega(\rho_1,\rho_2)$ by symmetry. Thus,

$$J_\omega(r,\zeta_1,\zeta_2) \le \frac{1}{d^3}K_\omega(\rho_1,\rho_2), \quad 0 < d \le \rho_1 \le r \le \rho_2 \le 1,$$

and

$$J_\omega(r,\zeta_1,\zeta_2) \le \frac{1}{d^3} K_\omega(\rho_1,\rho_2), \quad 0 < d \le \rho_2 \le r \le \rho_2 \le 1,$$

by symmetry.

The second inequality in (11) holds by the above estimates for $J_\omega(r,\zeta_1,\zeta_2)$, while the last one holds by the inequality $x + y \ge 2\sqrt{xy}$. □

By a Jordan theorem, a finite in any compact from $\mathbb{D}$ Borel signed measure ν is represented in $\mathbb{D}$ as the difference of its positive and negative variations: $\nu = \nu_+ - \nu_-$, where both $\nu_\pm$ are nonnegative Borel measures in $\mathbb{D}$, the supports of which do not intersect. Hence, a condition on both $\nu_\pm$, which provides some properties of the Green type potentials with $\nu_\pm$, provides similar properties of the whole Green type potential with the charge ν.

Definition 2 For any $\omega(t) \in \Omega$ and $0 < d < 1$, $\mathcal{P}_{\omega,d}$ is the class of Green type potentials P_ω in $\mathbb{D}$, such that $\inf\{|\zeta| : \zeta \in \operatorname{supp}\nu\} \ge d > 0$ for the supports of their associated Borel signed measures ν, the first Blaschke type condition in (1) is satisfied and

$$\begin{aligned} \|P_\omega\|_\omega := &\sup_{0<r<1} \int_0^{2\pi} \left[L_\omega P_\omega(re^{i\vartheta})\right]^2 d\vartheta \\ = &\sup_{0<r<1} \left\{ \iint_{\mathbb{D}} |d\nu(\zeta_1)| \iint_{\mathbb{D}} J_\omega(r,\zeta_1,\zeta_2)|d\nu(\zeta_2)| \right\} < +\infty, \end{aligned} \tag{13}$$

where $|d\nu(\zeta)| = d\nu_+(\zeta) + d\nu_-(\zeta)$ is the differential of the complete variation of the measure ν.

Remark 1 The quantity (13) evidently satisfies all the norm axioms including the triangle inequality which follows from that for the complete variations of measures:

$$\big||\nu_1| - |\nu_2|\big| \le |\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|.$$

Note that also the triangle inequalities $(\nu_1+\nu_2)^\pm \le \nu_1^\pm + \nu_2^\pm$ are true for the positive and negative variations $\nu_{1,2}^\pm$ of the measures $\nu_{1,2}$. These lead to the following inequalities:

$$|\nu_1^\pm - \nu_2^\pm| \le |\nu_1 - \nu_2|. \tag{14}$$

Indeed, the inequality $(\nu_2 + \mu)^+ \le \nu_2^+ + \mu^+$ is obvious with any measure μ. Setting $\nu_1 = \nu_2 + \mu$, i.e. $\nu_1 - \nu_2 = \mu$, we get $\nu_1^+ \le (\nu_1 - \nu_2)^+ + \nu_2^+$. Thus $\nu_1^+ - \nu_2^+ \le (\nu_1 - \nu_2)^+ \le |\nu_1 - \nu_2|$ and similarly $\nu_2^+ - \nu_1^+ \le (\nu_2 - \nu_1)^+ \le |\nu_1 - \nu_2|$. Hence $|\nu_1^+ - \nu_2^+| \le |\nu_1 - \nu_2|$, and evidently also $|\nu_1^- - \nu_2^-| \le |\nu_1 - \nu_2|$.

Theorem 1 *For any* $\omega(t) \in \Omega$ *and* $0 < d < 1$, *the set* $\mathcal{P}_{\omega,d}$ *is a Banach space with the norm* (13).

Proof It remains to prove only the completeness of the space $\mathcal{P}_{\omega,d}$ in the norm (13). To this end, suppose $\{P_\omega^{(n)}\}_{n=1}^\infty$ is a sequence of Green type potentials from $\mathcal{P}_{\omega,d}$, which satisfy the Cauchy condition, i.e. for any $\varepsilon > 0$

$$\begin{aligned}&\left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|_\omega \\ &= \sup_{0<r<1} \iint_{\mathbb{D}} |d(\nu_{n+m}-\nu_n)(\zeta_1)| \iint_{\mathbb{D}} J_\omega(r,\zeta_1,\zeta_2)|d(\nu_{n+m}-\nu_n)(\zeta_2)| < \varepsilon,\end{aligned}$$

provided $n \geq N_\varepsilon$ for $N_\varepsilon \geq 1$ large enough and any $m \geq 1$. We shall prove that there is a potential $P_\omega \in \mathcal{P}_{\omega,d}$ such that $\|P_\omega^{(n)} - P_\omega\|_\omega \to 0$ as $n \to \infty$. Note that by the triangle inequalities (14) the same Cauchy condition is true for the positive, negative and complete variations of the measures ν_n, and hence it is sufficient to consider only one of these variations, which we denote again by ν_n for simplicity.

Fixing an arbitrary $\delta \in (d,1)$, denoting $\mathbb{D}_\delta = \{\zeta \in \mathbb{D} : 0 < |\zeta| < \delta\}$ and assuming that $d \leq |\zeta_{1,2}| \leq \delta$, by (3) and (6) we get

$$J_\omega(0,\zeta_1,\zeta_2) > 2\pi\left(\int_\delta^1 \omega(t)dt\right)^2 := K_{\omega,\delta} > 0.$$

Hence, by the continuity of $J_\omega(r,\zeta_1,\zeta_2)$ in $0 \leq r \leq 1$ we obtain

$$\begin{aligned}&\left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|_\omega \\ &\geq \liminf_{\rho\to 0} \sup_{0<r<\rho} \iint_{\mathbb{D}_\delta} |d(\nu_{n+m}-\nu_n)(\zeta_1)| \iint_{\mathbb{D}_\delta} J_\omega(r,\zeta_1,\zeta_2)|d(\nu_{n+m}-\nu_n)(\zeta_2)| \\ &\qquad \geq K_{\omega,\delta}^2 \left(\iint_{\mathbb{D}_\delta} |d(\nu_{n+m}-\nu_n)(\zeta_1)|\right)^2.\end{aligned}$$

Consequently, for any $\varepsilon > 0$ there is some $N_\varepsilon \geq 1$ such that

$$\left(\iint_{\overline{\mathbb{D}}_\delta} \left|d(\nu_{n+m}-\nu_n)(\zeta)\right|\right)^2 \leq (K_{\omega,\delta})^{-2} \left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|_\omega < K_{\omega,\delta}^{-2}\varepsilon$$

for any $n \geq N_\varepsilon$ and $m \geq 1$. Thus, there is some $N'_\varepsilon \geq 1$ such that

$$\left|\nu_{n+m}(\overline{\mathbb{D}}_\delta) - \nu_n(\overline{\mathbb{D}}_\delta)\right| \leq \iint_{\overline{\mathbb{D}}_\delta} |d(\nu_{n+m}-\nu_n)(\zeta)| < \varepsilon$$

for any $n \geq N'_\varepsilon$ and $m \geq 1$. Observe that the Cauchy sequence of numbers $\nu_n(\overline{\mathbb{D}}_\delta)$ is bounded: $0 \leq \nu_n(\mathbb{D}_\delta) \leq A_\delta < \infty$ $(n \geq 1)$. Besides, it is easy to see that the sequence of Borel measures $\{\nu_n\}_{n=1}^\infty$ satisfies the Cauchy condition in the linear manifold Φ_δ of real, finite continuous functions in the closed disc $\overline{\mathbb{D}}_\delta$, which vanish outside $\overline{\mathbb{D}}_\delta$. Hence, by Theorem 0.4′ in [5] there is a Borel measure $\nu^\delta \geq 0$ such that the weak convergence $\nu_n \Rightarrow \nu^\delta$ $(n \to \infty)$ is true in Φ_δ, i.e. for any function $g(\zeta) \in \Phi_\delta$

$$\iint_{\mathbb{D}} g(\zeta) d\nu_n(\zeta) \to \iint_{\mathbb{D}} g(\zeta) d\nu^\delta(\zeta) \quad \text{as} \quad n \to \infty.$$

Since this is true for any $\delta \in (d, 1)$, there is a limit Borel measure ν over the whole $\mathbb{D}$, which coincides with ν^δ in any $\overline{\mathbb{D}}_\delta$. Obviously, for any fixed $\delta \in (d, 1)$ we have $\left|(\nu_n - \nu)(\overline{\mathbb{D}}_\delta)\right| < \varepsilon$, if n is large enough. Hence $\nu_n(\overline{\mathbb{D}}_\delta) \to \nu(\overline{\mathbb{D}}_\delta)$ and $\nu_n \Rightarrow \nu$ as $n \to \infty$ on Φ_δ for any fixed $\delta \in (d, 1)$.

Also the sequence of numbers $\|P_\omega^{(n)}\|$ satisfies the Cauchy condition because of the inequality $\left|\|P_\omega^{(m+n)}\| - \|P_\omega^{(n)}\|\right| \le \|P_\omega^{(m+n)} - P_\omega^{(n)}\|$, and hence

$$\|P_\omega^{(n)}\| \to b \neq +\infty \text{ as } n \to \infty \quad \text{and} \quad \|P_\omega^{(n)}\| \le B < +\infty \text{ for } n \ge 1.$$

Further, assuming that $\{r_k\}_0^\infty \subset [d, 1)$ is a sequence such that $r_0 = d$, $r_k \uparrow 1$, we conclude that

$$\begin{aligned} B &\ge \liminf_{n\to\infty} \sup_{0<r<1} \sum_{k=1}^\infty \sum_{m=1}^\infty \iint_{r_{k-1}<|\zeta|\le r_k} \iint_{r_{m-1}<|\zeta|\le r_m} J_\omega(r, \zeta_1, \zeta_2) d\nu_n(\zeta_1) d\nu_n(\zeta_2) \\ &\ge \sup_{0<r<1} \sum_{k=1}^\infty \sum_{m=1}^\infty \iint_{r_{k-1}<|\zeta|\le r_k} \iint_{r_{m-1}<|\zeta|\le r_m} J_\omega(r, \zeta_1, \zeta_2) d\nu(\zeta_1) d\nu_n(\zeta) = \|P_\omega\|, \end{aligned}$$

where $P_\omega(z)$ is the Green type potential generated by the measure ν, and $\|P_\omega\| < +\infty$. Then, introducing the sequence of nondecreasing in $0 < \rho < 1$ functions

$$\varphi_n(\rho) \equiv \varphi_n(\rho, r) := \iint_{\overline{\mathbb{D}}_\rho \times \overline{\mathbb{D}}_\rho} J_\omega(r, \zeta_1, \zeta_2) |d(\nu - \nu_n)(\zeta_1)||d(\nu - \nu_n)(\zeta_2)|,$$

where $0 < r < 1$ is fixed, we see that

$$\varphi_n(\rho) \to \varphi_n(1) \le \|P_\omega - P_\omega^{(n)}\| \le 2B + 1 \quad \text{as} \quad \rho \to 1 - 0,$$

and evidently $\varphi_n(\rho) \le \|P_\omega - P_\omega^{(n)}\| \le 2B + 1$ for any $\rho \in (0, 1)$, provided n is large enough. Hence, by a Helly theorem there is a subsequence of natural numbers $n_k \uparrow \infty$ such that at all points $0 \le \rho < 1$ there exists a limit function

$$\varphi(\rho) := \lim_{k\to\infty} \varphi_{n_k}(\rho).$$

On the other hand, for any $\rho \in (0, 1)$ the complete variation of the measure $\nu - \nu_n$ in $\overline{\mathbb{D}}_\rho$ weakly tends to zero. Consequently, due to the continuity of $J_\omega(r, \zeta_1, \zeta_2)$ by $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}_\rho$ we conclude that $\varphi_n(\rho) \to 0$ as $n \to \infty$ for any $\rho \in (0, 1)$, and hence

$\varphi(\rho) \equiv 0$ $(0 < \rho < 1)$. Further, $\varphi_{n_k}(\rho)$ is nondecreasing by $0 < \rho < 1$, and hence

$$\lim_{k\to\infty}\Big(\lim_{\rho\to 1-0}\varphi_{n_k}(\rho)\Big) = \lim_{\rho\to 1-0}\lim_{k\to\infty}\varphi_{n_k}(\rho) = 0.$$

Remind that this is true for any $0 \le r < 1$, and hence $\lim_{k\to\infty}\|P_\omega - P_\omega^{(n_k)}\| = 0$, since the function $\sup_{0<r<r'}\big(\lim_{\rho\to 1-0}\varphi_{n_k}(\rho, r)\big)$ is nondecreasing by $0 < r' < 1$. For completing the proof, it remains to see that $\|P_\omega - P_\omega^{(n)}\| \le \|P_\omega - P_\omega^{(n_k)}\| + \|P_\omega^{(n_k)} - P_\omega^{(n)}\|$, where the right-hand side summands are arbitrarily small when n_k and n are sufficiently large. □

2.2 *The Smaller Space of Potentials Formed by b_ω*

In this subsection, we introduce some included in $\mathcal{P}_{\omega,d}$ Banach spaces of Green type potentials, the expression for the norms of which is modular and much more explicit than (13) of $\mathcal{P}_{\omega,d}$, since instead of $J_\omega(r, \zeta_1, \zeta_2)$ of (7), (8), (9), (10), it includes the bigger, modular quantity $K_\omega(|\zeta_1|, |\zeta_2|)$ of (11).

Definition 3 For any $\omega \in \Omega$ and $0 < d < 1$, $\mathcal{P}'_{\omega,d}$ is the set of the Green type potentials P_ω, such that the supports of their Borel associated signed measures ν are located in the ring $d \le |\zeta| < 1$, and

$$\|P_\omega\|' := \iint_{\mathbb{D}}\left(\int_0^{1-|\zeta|}\omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}}\right)|d\nu(\zeta)| < +\infty, \tag{15}$$

where $|d\nu(\zeta)|$ is the differential of the complete variation of the measure ν.

Remark 2 The boundedness of the norm (15) implies the validity of the first Blaschke type condition in (1), which provides the convergence of the Green type potential P_ω. Indeed,

$$\int_0^{1-|\zeta|}\omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}} \ge \int_0^{1-|\zeta|}\omega(1-t)dt = \int_{|\zeta|}^1\omega(t)dt, \quad 0 < |\zeta| < 1,$$

since $\omega(t)$ is a nonnegative, nonincreasing function in $[0, 1]$.

In the case $\omega(t) \equiv 1$ of the ordinary Green potentials P_0, formula (15) takes the form

$$\|P_0\|'_\omega = \iint_{\mathbb{D}}\sqrt{1-|\zeta|}\,|d\nu(\zeta)| < +\infty.$$

Now, we proceed to the main theorem of this subsection.

Theorem 2 *For any* $\omega \in \Omega$ *and* $0 < d < 1$, $\mathcal{P}'_{\omega,d}$ *is a Banach space with the norm* (15).

Proof Only the completeness of the space $\mathcal{P}'_{\omega,d}$ is to be proved, since all other properties of the norm obviously are satisfied. To this end, suppose $\left\{P_\omega^{(n)}\right\}_1^\infty$ is a sequence of Green type potentials from $\mathcal{P}'_{\omega,d}$, which satisfies the Cauchy condition, i.e. for any $\varepsilon > 0$

$$\left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|'_\omega = \iint_{\mathbb{D}} \left(\int_0^{1-|\zeta|} \omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}} \right) \big|d(\nu_{n+m} - \nu_n)(\zeta)\big| < \varepsilon,$$

provided $n \ge N_\varepsilon$ for $N_\varepsilon \ge 1$ large enough and $m \ge 1$. We shall prove that there is a potential $P_\omega \in \mathcal{P}'_{\omega,d}$, such that $\left\| P_\omega^{(n)} - P_\omega \right\|'_\omega \to 0$ as $n \to \infty$. Note that by the triangle inequalities (14) the same Cauchy condition is true for the positive, negative and complete variations of the measures ν_n, and hence it suffices to consider only one of these variations, which we denote again by ν_n (≥ 0) for simplicity.

Fixing an arbitrary $\delta \in (d, 1)$, denoting $\mathbb{D}_\delta = \{\zeta \in \mathbb{D} : 0 < |\zeta| < \delta\}$ and assuming that $d \le |\zeta_{1,2}| \le \delta$, by (15) we obtain that for any $\varepsilon > 0$ there is some $N_\varepsilon \ge 1$ such that for any $n \ge N_\varepsilon$ and $m \ge 1$

$$\iint_{\overline{\mathbb{D}}_\delta} \big|d(\nu_{n+m} - \nu_n)(\zeta)\big| \le (K_{\omega,\delta})^{-2} \left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|'_\omega < K_{\omega,\delta}^{-2}\varepsilon,$$

where $K_{\omega,\delta} = 2\omega(\delta)\sqrt{1-\delta}$. Thus, there is some $N'_\varepsilon \ge 1$ such that

$$\left|\nu_{n+m}(\overline{\mathbb{D}}_\delta) - \nu_n(\overline{\mathbb{D}}_\delta)\right| \le \iint_{\overline{\mathbb{D}}_\delta} |d(\nu_{n+m} - \nu_n)(\zeta)| < \varepsilon$$

for any $n \ge N'_\varepsilon$ and $m \ge 1$. Thus, the sequence of numbers $\left\{\nu_n(\overline{\mathbb{D}}_\delta)\right\}_1^\infty$ satisfies the Cauchy condition, and consequently it is bounded, i.e. $0 \le \nu_n(\overline{\mathbb{D}}_\delta) \le A_\delta < +\infty$ ($n \ge 1$), and hence the sequence $\left\{\nu_n(\overline{\mathbb{D}}_\delta)\right\}_1^\infty$ is bounded and has a finite limit. Besides, the sequence of the corresponding Borel measures $\{\nu_n\}_1^\infty$ satisfies the Cauchy condition in the linear manifold Φ_δ of real, finite continuous functions with compact supports in $\overline{\mathbb{D}}_\delta$. The rest of the proof is the same as that of Theorem 1. □

2.3 Banach Spaces of Green Type Potentials Formed by $\widetilde{b}_\omega(z, \zeta)$

The main results related to the spaces of potentials formed by $\widetilde{b}_\omega(z, \zeta)$ and their proofs are very similar to those on the spaces of potentials formed by $b_\omega(z, \zeta)$, which are given in the previous two sections. Therefore, after some preliminary information we give the main results with proofs, where we shall be focused on the differences from the proofs of the previous two subsections.

If $\omega(t) \in \Omega$ and $0 \neq \zeta \in \mathbb{D}$ is a fixed point, then similar to $L_\omega \log |b_\omega(z, \zeta)|$, the function $L_\omega \log |\widetilde{b}_\omega(z, \zeta)|$ is harmonic in the whole $\mathbb{C}$, except the interval $\{z = t\overline{\zeta} : 1 \leq t \leq 1/|\zeta|\}$. Besides, it is subharmonic in $\mathbb{D}$, nonpositive and continuous in $\overline{\mathbb{D}}$, and

$$\widetilde{f}(z, \zeta) := L_\omega \log \widetilde{b}_\omega(z, \zeta) = -\int_{|\zeta|^2}^1 \frac{1}{1 - \frac{z}{\zeta}t} \frac{\omega(t)}{t} dt, \quad z \in \overline{\mathbb{D}},$$

where the integral is understood in the sense of its principal value on the mentioned interval. Further, note that formula (3) is true for $\widetilde{f}(z, \zeta)$, any function $\varphi(\zeta)$ and any measure $\mu(\zeta)$ for which the integrals exist.

We assume that $\omega(t) \in \Omega$ and a signed Borel measure ν satisfies the second Blaschke type condition in (1). Then, similar to $P_\omega(z)$, the Green type potential $\widetilde{P}_\omega(z)$ with $\log |\widetilde{b}_\omega(z, \zeta)|$ is convergent, and formula (2) is true. Besides,

$$\frac{1}{2\pi} \int_0^{2\pi} \left[L_\omega \widetilde{P}_\omega(re^{i\vartheta})\right]^2 d\vartheta = \iint_{\mathbb{D}} d\nu(\zeta_1) \iint_{\mathbb{D}} \widetilde{J}_\omega(r, \zeta_1, \zeta_2)\, d\nu(\zeta_2) \tag{16}$$

with

$$\begin{aligned}\widetilde{J}_\omega(r, \zeta_1, \zeta_2) &= \frac{1}{2\pi} \int_0^{2\pi} L_\omega \log |\widetilde{b}_\omega(re^{i\vartheta}, \zeta_1)| L_\omega \log |\widetilde{b}_\omega(re^{i\vartheta}, \zeta_2)|\, d\vartheta \\ &= \mathrm{Re} \frac{1}{4\pi} \int_0^{2\pi} \left[\widetilde{f}(re^{i\vartheta}, \zeta_1) \widetilde{f}(re^{i\vartheta}, \zeta_2) + \widetilde{f}(re^{i\vartheta}, \zeta_1) \overline{\widetilde{f}(re^{i\vartheta}, \zeta_2)}\right] d\vartheta,\end{aligned}$$

provided the above integrals exist and the change of the integration order is true in (16). These formulas make it possible to calculate by residues and estimate the integral $\widetilde{J}_\omega(r, \zeta_2, \zeta_2)$. These result on the following lemma which we give without a proof.

Lemma 2 *For any $\omega \in \Omega$, the function $\widetilde{J}_\omega(r, \zeta_1, \zeta_2)$ is continuous by its variables $\zeta_1, \zeta_2 \in \overline{\mathbb{D}} \backslash \{0\}$ and $r \in [0, 1]$. Besides, for any $r \in [0, 1]$ and $0 < d \leq |\zeta_{1,2}| \leq 1$ the following inequalities are true:*

$$\begin{aligned}0 \leq \widetilde{J}_\omega(r, \zeta_1, \zeta_2) &\leq \widetilde{M}_d \int_0^{1-|\zeta_1|^2} \omega(|\zeta_1|(1 - x)) dx \int_0^{1-|\zeta_2|^2} \frac{\omega(|\zeta_2|(1 - y))}{x + y} dy \\ &\leq \frac{\widetilde{M}_d}{2} \widetilde{K}_\omega(|\zeta_1|, |\zeta_2|),\end{aligned} \tag{17}$$

where $\widetilde{M}_d = 5/2d^3$ *and*

$$\widetilde{K}_\omega(|\zeta_1|, |\zeta_2|) := \int_0^{1-|\zeta_1|^2} \omega(|\zeta_1|(1-x))\frac{dx}{\sqrt{x}} \int_0^{1-|\zeta_2|^2} \omega(|\zeta_2|(1-y))\frac{dy}{\sqrt{y}}.$$

The wider spaces of Green type potentials are defined as follows.

Definition 4 For any $\omega(t) \in \Omega$ and $0 < d < 1$, $\widetilde{\mathcal{P}}_{\omega,d}$ is the class of those Green type potentials $\widetilde{P}_\omega$, the Borel signed measures ν of which satisfy the second Blaschke type condition in (1), $\inf\{|\zeta| : \zeta \in \operatorname{supp}\nu\} \geq d$, and

$$\begin{aligned}\|\widetilde{P}_\omega\|_\omega : &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left[L_\omega \widetilde{P}_\omega(re^{i\vartheta})\right]^2 d\vartheta \\ &= \sup_{0<r<1} \left\{ \iint_{\mathbb{D}} \iint_{\mathbb{D}} \widetilde{J}_\omega(r, \zeta_1, \zeta_2)|d\nu(\zeta_1)||d\nu(\zeta_2)| \right\} < +\infty. \qquad (18)\end{aligned}$$

Theorem 3 *For any* $\omega(t) \in \Omega$ *and* $0 < d < 1$*, the set* $\widetilde{\mathcal{P}}_{\omega,d}$ *is a Banach space with the norm* (18)*.*

Proof The proof is the same as that of Theorem 1. The only difference is that $J_\omega(r, \zeta_1, \zeta_2)$ is to be replaced by $\widetilde{J}_\omega(r, \zeta_1, \zeta_2)$. □

2.4 The Smaller Space of Potentials Formed by $\widetilde{b}_\omega$

Here we introduce some contained in $\mathcal{P}_{\omega,d}$ Banach spaces of Green type potentials, the expression for the norms of which is modular and much more explicit than (18) of $\widetilde{\mathcal{P}}_{\omega,d}$, since instead of $\widetilde{J}_\omega(r, \zeta_1, \zeta_2)$ depending on the arguments of $\zeta_{1,2}$, it includes the bigger, modular quantity $\widetilde{K}_\omega(|\zeta_1|, |\zeta_2|)$ of (17).

Definition 5 For any $\omega \in \Omega$ and $0 < d < 1$, $\widetilde{\mathcal{P}}'_{\omega,d}$ is the set of the Green type potentials $\widetilde{P}_\omega$, such that the supports of their Borel associated signed measures are located in the ring $d \leq |\zeta| < 1$, and

$$\|P_\omega\|' := \iint_{\mathbb{D}} \left(\int_0^{1-|\zeta|^2} \omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}} \right)|d\nu(\zeta)| < +\infty, \qquad (19)$$

where $|d\nu(\zeta)|$ is the differential of the complete variation of the measure ν.

Remark 3 The boundedness of the norm (19) implies the validity of the second Blaschke type condition in (1), which provides the convergence of the Green type potential $\widetilde{P}_\omega$. Indeed,

$$\int_0^{1-|\zeta|^2} \omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}} \geq \int_0^{1-|\zeta|^2} \omega(1-t)dt = \int_{|\zeta|^2}^1 \omega(t)dt, \quad 0 < |\zeta| < 1,$$

since $\omega(t)$ is a nonnegative, nonincreasing function in $[0, 1]$.

In the case $\omega(t) \equiv 1$ of the ordinary Green potentials $\widetilde{P}_0$, formula (19) defining the class $\widetilde{\mathcal{P}}'_{\omega,d}$ takes the form

$$\|P_0\|'_\omega = \iint_{\mathbb{D}} \sqrt{1-|\zeta|^2}\,|d\nu(\zeta)| < +\infty.$$

We omit the proof of the next theorem, since it is the same as that of Theorem 2.

Theorem 4 *For any $\omega \in \Omega$ and $0 < d < 1$, $\widetilde{\mathcal{P}}'_{\omega,d}$ is a Banach space with the norm* (19).

3 Banach Spaces of Delta-Subharmonic Functions

A function $U(z)$ is δ-subharmonic in the unit disc $\mathbb{D}$, if it is a difference $U(z) = U_1(z) - U_2(z)$ of two functions subharmonic in $\mathbb{D}$. We assume that the Riesz associated signed measure ν of $U(z)$ is minimally decomposed in the Jordan sense, i.e. $\nu = \nu_+ - \nu_-$, where $\nu_\pm$ are the positive and negative variations of ν. These are some nonnegative Borel measures with non-overlapping supports in $\mathbb{D}$. Two functions $U(z)$ and $V(z) = V_1(z) - V_2(z)$, which are δ-subharmonic in a domain, are said to be equal, i.e. $U(z) = V(z)$, if $U_1(z)+V_2(z) = U_2(z)+V_1(z)$ everywhere in that domain.

Definition 6 Let $\omega(t) \in \Omega$ and $0 < d < 1$. Then $\mathcal{D}_{\omega,d}$ is the set of those δ-subharmonic in $\mathbb{D}$ functions $u(z)$ with associated Borel signed measures ν, the supports of which are located in the ring $\{\zeta : d \leq |\zeta| < 1\}$, and

$$\|u\|_\omega = \sup_{0<r<1} \frac{1}{2\pi}\left\{\int_0^{2\pi} \left[L_\omega u(re^{i\vartheta})\right]^2 d\vartheta\right\}^{1/2} + \iint_{\mathbb{D}}\left(\int_0^{1-|\zeta|} \omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}}\right)|d\nu(\zeta)| < +\infty. \tag{20}$$

Definition 7 Let $\omega(t) \in \Omega$ and $0 < d < 1$. Then $\widetilde{\mathcal{D}}_{\omega,d}$ is the set of those δ-subharmonic in $\mathbb{D}$ functions $u(z)$ with associated Borel signed measures ν, the supports of which are located in the ring $\{\zeta : d \le |\zeta| < 1\}$, and

$$\|u\|_\omega = \sup_{0<r<1} \frac{1}{2\pi}\left\{\int_0^{2\pi} \left[L_\omega u(re^{i\vartheta})\right]^2 d\vartheta\right\}^{1/2} + \iint_{\mathbb{D}} \left(\int_0^{1-|\zeta|^2} \omega\big(|\zeta|(1-t)\big)\frac{dt}{\sqrt{t}}\right)|d\nu(\zeta)| < +\infty. \tag{21}$$

Theorem 5 *For any $\omega \in \Omega$ and $d \in (0,1)$, $D_{\omega,d}$ and $\widetilde{D}_{\omega,d}$ are Banach spaces.*

Proof We give a proof only for the case of $\mathcal{D}_{\omega,d}$, since for $\widetilde{\mathcal{D}}_{\omega,d}$ the proof is almost the same. The quantity (20) evidently satisfies all norm axioms including the triangle inequality. Hence, if $u(z) \in \mathcal{D}_{\omega,d}$, then the harmonic in $\mathbb{D}$ function $u(z) - P_\omega(z) := U(z)$ is such that $L_\omega U(z)$ belongs to the ordinary harmonic Hardy space h^2 in $\mathbb{D}$. Similar to Proposition 2.1 in [8] and Theorem 1.5 in [7], it is not difficult to prove that the inclusion $L_\omega U(z) \in h^2$ means that $U(z)$ is of the Hilbert space $A^2_{\omega_*}$ of harmonic functions in $\mathbb{D}$, which is defined by the condition

$$\|U\|_{A^2_{\omega*}} = \left\{\iint_{\mathbb{D}} \left[U(z)\right]^2 d\mu_{\omega_*}(z)\right\}^{1/2} < +\infty,$$

where $d\mu_{\omega_*}(re^{i\vartheta}) = d\vartheta\, d\omega_*(r^2)$ and $\omega_*(t)$ is the Volterra square of $\omega(t)$, i.e.

$$\omega_*(t) = -\int_t^1 \omega\Big(\frac{t}{\lambda}\Big)d\omega(\lambda), \quad 0 < t < 1.$$

So, $\mathcal{D}_{\omega,d}$ is the direct sum of the Hilbert space $A^2_{\omega_*}$ of harmonic in $\mathbb{D}$ functions and the Banach space of Green type potentials $\mathcal{P}_{\omega,d}$, i.e. any δ-subharmonic function $u(z) \in \mathcal{D}_{\omega,d}$ has a unique representation of the form

$$u(z) = U(z) + P_\omega(z), \quad U(z) \in A^2_{\omega_*}, \ \ P_\omega(z) \in \mathcal{P}_{\omega,d}. \tag{22}$$

Similarly, $\widetilde{D}_{\omega,h}$ is the direct sum of the Hilbert space $h^2_{\omega_*}$ of harmonic in $\mathbb{D}$ functions and the Banach space of Green type potentials $\widetilde{\mathcal{P}}'_{\omega,h}$, i.e. any δ-subharmonic function $u(z) \in \widetilde{D}_{\omega,d}$ has a unique representation of the form

$$u(z) = U(z) + P_\omega(z), \quad U(z) \in h^2_{\omega_*}, \quad P_\omega(z) \in \widetilde{\mathcal{P}}'_{\omega,d}. \tag{23}$$

□

Remark 4 Obviously, similar theorems are true for the Banach spaces of δ-subharmonic functions defined as direct sums of the harmonic Hilbert space A^2_ω

and the large Banach spaces $\mathcal{P}_{\omega,d}$ and $\widetilde{\mathcal{P}}_{\omega,d}$ of Green type potentials (Theorems 1 and 3), where the last summands in the right-hand sides of formulas (20) and (21) for norms are replaced respectively by

$$\|P_\omega\|_\omega = \sup_{0<r<1} \int_0^{2\pi} \left[L_\omega P_\omega(re^{i\vartheta})\right]^2 d\vartheta$$
$$= \sup_{0<r<1} \left\{ \iint_{\mathbb{D}} \iint_{\mathbb{D}} J_\omega(r, \zeta_1, \zeta_2) |d\nu(\zeta_1)||d\nu(\zeta_2)| \right\}$$

and

$$\|\widetilde{P}_\omega\|_\omega = \sup_{0<r<1} \int_0^{2\pi} \left[L_\omega \widetilde{P}_\omega(re^{i\vartheta})\right]^2 d\vartheta$$
$$= \sup_{0<r<1} \left\{ \iint_{\mathbb{D}} \iint_{\mathbb{D}} \widetilde{J}_\omega(r, \zeta_1, \zeta_2) |d\nu(\zeta_1)||d\nu(\zeta_2)| \right\}.$$

Remark 5 Any of the unions

$$\bigcup_{\omega\in\Omega, d>0} \mathcal{D}_{\omega,d}\,, \quad \bigcup_{\omega\in\Omega, d>0} \widetilde{\mathcal{D}}_{\omega,d}\,, \quad \bigcup_{\omega\in\Omega, d>0} \mathcal{D}'_{\omega,d}\,, \quad \bigcup_{\omega\in\Omega, d>0} \widetilde{\mathcal{D}}'_{\omega,d}$$

of the considered Banach spaces coincides with the whole set of functions δ-subharmonic in the unit disc, the supports of the associated signed measures of which are disjoint from the origin. Indeed,

$$\bigcup_{\omega\in\Omega, d>0} \mathcal{D}'_{\omega,d} \subset \bigcup_{\omega\in\Omega, d>0} \mathcal{D}_{\omega,d} \quad \text{and} \quad \bigcup_{\omega\in\Omega, d>0} \widetilde{\mathcal{D}}'_{\omega,d} \subset \bigcup_{\omega\in\Omega, d>0} \widetilde{\mathcal{D}}_{\omega,d},$$

while any function δ-subharmonic in the unit disc, the support of the associated signed measure of which is disjoint from the origin, belongs even to some smaller spaces $\mathcal{D}'_{\omega,d}$ and $\widetilde{\mathcal{D}}'_{\omega,d}$ at some choice of the parameter-function $\omega \in \Omega$ and the number $0 < d < 1$.

Indeed, if $u(z)$ is a delta-subharmonic function in $\mathbb{D}$, the support of which is disjoint from the origin, then for any $r \in (0, 1)$

$$\int_0^{2\pi} \left[L_\omega u(re^{i\vartheta})\right]^2 d\vartheta \le \int_0^1 \left(\int_0^{2\pi} \left[u(r\rho e^{i\vartheta})\right]^2 d\vartheta \right) |d\omega(\rho^2)|$$
$$\le \int_0^1 M(\rho)|d\omega(\rho^2)| < +\infty \tag{24}$$

at some choice of the function-parameter $\omega(x) \in \Omega$, since the function

$$M(\rho) = \max_{0 \le r \le 1} \int_0^{2\pi} \left[u(r\rho e^{i\vartheta})\right]^2 d\vartheta$$

is continuous by $\rho \in [0, 1)$. Further, if ν is the Borel signed measure of the function $u(z)$, then obviously there is a function $\omega(x) \in \Omega$, such that

$$\begin{aligned} \iint_{\mathbb{D}} \left(\int_0^{1-|\zeta|} \omega(|\zeta|(1-t)) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| \\ = \sum_{k=1}^{\infty} \iint_{1-\frac{1}{k} \le |\zeta| < 1 - \frac{1}{k+1}} \left(\int_0^{1-|\zeta|} \omega(|\zeta|(1-t)) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| \\ \le 2 \sum_{k=1}^{\infty} \omega\left(\left(1 - \frac{1}{k}\right)^2 \right) \iint_{1-\frac{1}{k} \le |\zeta| < 1 - \frac{1}{k+1}} |d\nu(\zeta)| < +\infty. \end{aligned} \tag{25}$$

Evidently, the function-parameter $\omega \in \Omega$ can be chosen to provide the validity of both (24) and (25), and hence $u(z) \in \mathcal{D}'_{\omega,d}$ at some choice of $\omega \in \Omega$. The proof for $\widetilde{\mathcal{D}}'_{\omega,d}$ is similar.

Remark 6 Since $\log|f(z)|$ of a meromorphic function $f(z)$ is a particular case of δ-subharmonic function, the considered spaces in particular become Banach spaces of functions meromorphic in the unit disc $\mathbb{D}$. Then, the representation (22), (23) and their two similarities become factorizations in the corresponding four Banach spaces of functions meromorphic in the unit disc, which exhaust the whole set of functions meromorphic in the unit disc.

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Banach Spaces of Functions Delta-Subharmonic in the Half-Plane

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Abstract The present work gives the introduction and some investigation of several Banach spaces of delta-subharmonic in the half-plane of the complex plane functions, the M.M.Djrbashian generalized fractional integrals of which possess bounded square integral means.

Keywords Banach spaces · Delta-subharmonic functions

1 Preliminaries

We investigate some classes of Green type potentials in the upper half-plane $G^+ = \{z : \mathrm{Re} z > 0\}$, the M.M.Djrbashian generalized fractional integrals of which possess bounded square integral means. For functions defined in G^+, the mentioned fractional integral is the operator

$$L_\omega f(z) := \int_0^{+\infty} f(z + i\sigma) d\omega(\sigma), \quad z \in G^+$$

(see [1], Lemma 1.1 in [3], and [6]). Also, we use the Cauchy type ω-kernel for the half-plane

$$C_\omega(z) = \int_0^{+\infty} \frac{e^{itz} dt}{t \int_0^{+\infty} e^{-t\lambda} \omega(\lambda) d\lambda}, \quad z \in G^+,$$

which is a holomorphic function in G^+ under the following choice of the parameter-function $\omega(x)$ (see [4], Subsection 2.1).

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Definition 1 Let Ω_α $(0 < \alpha < +\infty)$ be the set of those functions $\omega(t)$ given in $[0, +\infty)$, which satisfy the following conditions: $\omega(t) > 0$ $(0 < t < +\infty)$, $\omega(t)$ is nondecreasing in $[0, +\infty)$, for some $\lambda \in (0, 1]$, $\omega(t) \in \mathrm{Lip}_\lambda[0, R)$ for any $R > 0$, and $\omega(t) \asymp t^\alpha$ $(t > R)$, $\omega(0) = 0$ ($f(t) \asymp g(t)$ means that $m_1 f(t) \leq g(t) \leq m_2 f(t)$ for some constants $m_{1,2} > 0$).

Note that for $\omega(t) = t^\alpha/\Gamma(1+\alpha)$ $(\alpha > 0)$ the operator L_ω becomes the Liouville fractional integration, and the kernel $C_\omega(z)$ becomes the $1+\alpha$-order of the ordinary Cauchy kernel, i.e. it becomes $\Gamma(1+\alpha)(-iz)^{-(1+\alpha)} := C_\alpha(z)$.

Further, we shall deal with the following two Blaschke type factors which for $\mathrm{Im}z > \mathrm{Im}\zeta > 0$ can be written in the forms

$$b_\omega(z, \zeta) = \exp\left\{ -\int_{-\eta}^{\eta} C_\omega(z - \xi - it)\omega(\eta - |t|)dt \right\},$$

$$\widetilde{b}_\omega(z, \zeta) = \exp\left\{ -\int_{-\eta}^{\eta} C_\omega(z - \xi + it)\omega(\eta + t)dt \right\}.$$

Note that for any $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and any fixed point $\zeta = \xi + i\eta \in G^+$ the functions $b_\omega(z, \zeta)$ and $\widetilde{b}_\omega(z, \zeta)$ are holomorphic in G^+, where they have a unique, first order zero at the point $z = \zeta$ (see formula (2.1) and Lemma 2.1 in [6], also formula (29) and Lemma 3 in [5]). Under the condition $\omega(x) \in \Omega_\alpha$ $(\alpha > 0)$, we consider the Green type potentials

$$P_\omega(z) := \iint_{G^+} \log|b_\omega(z, \zeta)|d\nu(\zeta), \quad \widetilde{P}_\omega(z) := \iint_{G^+} \log\left|\widetilde{b}_\omega(z, \zeta)\right|d\nu(\zeta)$$

which are convergent and represent some δ-subharmonic functions in G^+, if ν is a Borel signed measure satisfying correspondingly to one of the below Blaschke type conditions:

$$\iint_{G^+}\left[\int_0^{\mathrm{Im}\zeta} \omega(t)dt\right]|d\nu(\zeta)| < +\infty, \quad \iint_{G^+}\left[\int_0^{2\mathrm{Im}\zeta} \omega(t)dt\right]|d\nu(\zeta)| < +\infty, \tag{1}$$

where $|d\nu| := d\nu^+ + d\nu^-$, and $\nu^\pm$ are the positive and the negative parts in the Jordan decomposition of ν. Besides, under these conditions we have

$$\sup_{y>0}\int_{-\infty}^{+\infty} \left|L_\omega P_\omega(x+iy)\right|dx < 4\pi \iint_{G^+}\left[\int_0^{\mathrm{Im}\zeta} \omega(t)dt\right]|d\nu(\zeta)| < +\infty,$$

$$\sup_{y>0}\int_{-\infty}^{+\infty} \left|L_\omega \widetilde{P}_\omega(x+iy)\right|dx < \pi \iint_{G^+}\left[\int_0^{2\mathrm{Im}\zeta} \omega(t)dt\right]|d\nu(\zeta)| < +\infty.$$

Note that the above statements on potentials are true due to Theorems 3.1 and 3.3 in [6] and Theorem 2 in [5], except the last estimate which is easy to prove in the same way as the previous one.

2 Banach Spaces of Green Type Potentials

In this section, some Banach spaces of Green type potentials formed by the Blaschke type factors $b_\omega(z,\zeta)$ and $\widetilde{b}_\omega(z,\zeta)$ are introduced, the regularizations of which by means of the operator L_ω possess bounded square integral means.

2.1 The Wide Banach Space of Potentials Formed by b_ω

Note that, if $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$, then by Lemma 2.2 and Theorem 2.2 in [6] the function $L_\omega \log|b_\omega(z,\zeta)|$ is harmonic in $\overline{G^+} = \{z : \mathrm{Im} z \geq 0\}$, except the intercept $\{z = \xi + i\lambda : 0 \leq \lambda \leq \eta\}$, and also it is subharmonic in G^+, nonpositive and continuous in $\overline{G^+}$, and $L_\omega \log|b_\omega(x,\zeta)| = 0$ $(-\infty < x < +\infty)$. Besides, the following representation holds by the results of [6]:

$$f(z,\zeta) := L_\omega \log b_\omega(z,\zeta) = \int_{-\eta}^{\eta} \frac{\omega(\eta - |\tau|)d\tau}{\tau + i(z-\xi)}, \quad z \in G^+, \tag{2}$$

where $\zeta = \xi + i\eta \in G^+$. Besides, by Theorem 3.2 in [6], if $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and the signed Borel measure ν satisfies the first condition in (1), then

$$L_\omega P_\omega(z) = \iint_{G^+} L_\omega \log|b_\omega(z,\zeta)| d\nu(\zeta), \quad z \in G^+.$$

Now, note that the following equality is true for any function $\varphi(\zeta)$ and any measure $\mu(\zeta)$, for which the below integrals exist:

$$\left(\mathrm{Re}\int_E \varphi(\zeta)d\mu(\zeta)\right)^2 = \frac{1}{2}\mathrm{Re}\int_E d\mu(\zeta_1)\int_E \left[\varphi(\zeta_1)\varphi(\zeta_2) + \varphi(\zeta_1)\overline{\varphi(\zeta_2)}\right] d\mu(\zeta_2). \tag{3}$$

Thus, we come to the following representation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[L_\omega P_\omega(x+iy)\right]^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\iint_{G^+} \mathrm{Re} f(x+iy,\zeta)\, d\nu(\zeta)\right]^2 dx$$
$$= \iint_{G^+} d\nu(\zeta_1) \iint_{G^+} J(y,\zeta_1,\zeta_2)\, d\nu(\zeta_2), \tag{4}$$

where

$$J(y,\zeta_1,\zeta_2)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}L_\omega\log|b_\omega(x+iy,\zeta_1)|L_\omega\log|b_\omega(x+iy,\zeta_2)|\,dx$$
$$=\mathrm{Re}\frac{1}{4\pi}\int_{-\infty}^{+\infty}\Big[f(x+iy,\zeta_1)f(x+iy,\zeta_2)+f(x+iy,\zeta_1)\overline{f(x+iy,\zeta_2)}\Big]dx,\tag{5}$$

provided the above integrals exist and the change of the integration order is true in (4).

Lemma 1 *If* $\omega(t)\in\Omega_\alpha$ $(\alpha>0)$*, then the function* $J(y,\zeta_1,\zeta_2)$ *is nonnegative and continuous for any* $0\le y<+\infty$ *and* $\zeta_{1,2}\in\overline{G^+}$.

Proof By the statement (i) of Theorem 2.2 in [6], the function $J(y,\zeta_1,\zeta_2)$ is nonnegative for $0\le y<+\infty$. For proving its continuity, observe that from (2) it follows that there is a constant $K>0$ such that

$$0<F(x,y):=L_\omega\log|b_\omega(x+iy,\zeta_1)|L_\omega\log|b_\omega(x+iy,\zeta_2)|\le\frac{K}{|x+iy|^2}$$

for any $|x+iy|>2\max\{|\xi_1|+\eta_1,|\xi_2|+\eta_2\}$. Hence, for any $\varepsilon>0$

$$\int_{R\le|x|<+\infty}F(x,y)dx<\frac{\varepsilon}{3}$$

when R is large enough. Besides, for any values $y_{1,2}\ge0$

$$\Big|J(y_1,\zeta_1,\zeta_2)-J(y_2,\zeta_1,\zeta_2)\Big|<\frac{2}{3}\varepsilon+\int_{-R\le x\le R}\Big|F(x,y_1)-F(x,y_2)\Big|dx,$$

where the last integral is less than $\varepsilon/3$, provided $|y_1-y_2|$ is small enough, because of the continuity of $L_\omega\log|b_\omega(x+iy,\zeta)|$ holding by the statement (i) of Theorem 2.2 in [6]. The continuity of $J(y,\zeta_1,\zeta_2)$ by $\zeta_{1,2}\in\overline{G^+}$ is proved in the same manner.

□

If a signed Borel measure ν is finite in any compact in G^+, then by the Jordan theorem it can be decomposed in G^+ as the difference of its positive and negative variations: $\nu=\nu_+-\nu_-$, where both $\nu_\pm$ are nonnegative Borel measures in G^+, which supports do not intersect. Therefore, a condition on both $\nu_\pm$, which provides some properties of the Green type potentials by this measures shall provide similar properties of the whole Green type potential.

Definition 2 For any $\omega(t)\in\Omega_\alpha$ $(\alpha>0)$ and $0<h<+\infty$, $\mathcal{P}_{\omega,h}$ is the class of Green type potentials P_ω with associated Borel signed measures ν satisfying the first Blaschke type condition in (1). In addition, the supports of the measures ν are located in the strip $\{|\zeta| : \zeta \in \operatorname{supp}\nu\} < h$, and

$$\begin{aligned}\|P_\omega\|_\omega &:= \sup_{y>0} \int_{-\infty}^{+\infty} |L_\omega P_\omega(x+iy)|^2 dx \\ &= \sup_{y>0} \left\{ \iint_{G^+} \iint_{G^+} J(y,\zeta_1,\zeta_2)|d\nu(\zeta_1)||d\nu(\zeta_2)| \right\} < +\infty, \end{aligned} \tag{6}$$

where $J(y,\zeta_1,\zeta_2)$ is the integral (5) and $|d\nu(\zeta)| = d\nu_+(\zeta) + d\nu_-(\zeta)$ is the differential of the complete variation of the measure ν.

Remark 1 The quantity (6) evidently satisfies all the norm axioms including the triangle inequality which follows from that for the complete variations of measures:

$$\big||\nu_1| - |\nu_2|\big| \le |\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|.$$

Note that also the triangle inequalities $(\nu_1+\nu_2)^\pm \le \nu_1^\pm + \nu_2^\pm$ are true for the positive and negative variations $\nu_{1,2}^\pm$ of the measures $\nu_{1,2}$. These lead to the following inequalities:

$$|\nu_1^\pm - \nu_2^\pm| \le |\nu_1 - \nu_2|. \tag{7}$$

Indeed, the inequality $(\nu_2 + \mu)^+ \le \nu_2^+ + \mu^+$ is obvious with any measure μ. Setting $\nu_1 = \nu_2 + \mu$, i.e. $\nu_1 - \nu_2 = \mu$, we get $\nu_1^+ \le (\nu_1 - \nu_2)^+ + \nu_2^+$. Thus $\nu_1^+ - \nu_2^+ \le (\nu_1 - \nu_2)^+ \le |\nu_1 - \nu_2|$ and similarly $\nu_2^+ - \nu_1^+ \le (\nu_2 - \nu_1)^+ \le |\nu_1 - \nu_2|$. Hence $|\nu_1^+ - \nu_2^+| \le |\nu_1 - \nu_2|$, and evidently also $|\nu_1^- - \nu_2^-| \le |\nu_1 - \nu_2|$.

Theorem 1 *For any $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and $0 < h < +\infty$, the set $\mathcal{P}_{\omega,h}$ is a Banach space with the norm* (6)*.*

Proof It is obvious that only the completeness of the space $\mathcal{P}_{\omega,h}$ is to be proved. To this end, suppose $\left\{P_\omega^{(n)}\right\}_1^\infty$ is a sequence of Green type potentials from $\mathcal{P}_{\omega,h}$, which satisfy the Cauchy condition, i.e. for any $\varepsilon > 0$

$$\begin{aligned}&\left\|P_\omega^{(n+m)} - P_\omega^{(n)}\right\|_\omega \\ &= \sup_{y>0} \iint_{G^+} \iint_{G^+} J(y,\zeta_1,\zeta_2)\big|d(\nu_{n+m}-\nu_n)(\zeta_1)\big|\big|d(\nu_{n+m}-\nu_n)(\zeta_2)\big| < \varepsilon,\end{aligned}$$

provided $n \ge N_\varepsilon$ for some $N_\varepsilon \ge 1$ large enough and any $m \ge 1$. We shall prove that there is a potential $P_\omega \in \mathcal{P}_{\omega,h}$ such that $\left\|P_\omega^{(n)} - P_\omega\right\|_\omega \to 0$ as $n \to \infty$. Note that by the triangle inequalities (7) the same Cauchy condition is true for the positive, negative and complete variations of the measures ν_n, and hence it suffices to consider only one of these variations, which we denote again by ν_n.

By Lemma 1, the function $J(y,\zeta_1,\zeta_2)$ is positive and continuous in $0<y<+\infty$. Further, we fix an arbitrary $\rho\in(0,h]$ and suppose that $|\mathrm{supp}\nu|<h<+\infty$ and $\zeta_{1,2}=\xi_{1,2}+\eta_{1,2}\in\mathrm{supp}\nu$ are such that $\eta_{1,2}\geq\rho>0$. Then, by (2) we get

$$\sup_{y>0}\int_{-\infty}^{+\infty}L_\omega\log|b_\omega(x+iy,\zeta_1)|L_\omega\log|b_\omega(x+iy,\zeta_2)|dx$$
$$>\frac{\omega^2(\rho)h^3}{4}\int_{|x|>2h}\frac{1}{\left[|x|^2+9h^2\right]^4}dx:=K_{\omega,\rho,h}>0.$$

Denoting $G^+_\rho=\{\zeta=\xi+i\eta\in G^+:\rho<\eta<+\infty\}$, observe that for any $\varepsilon>0$ there is some $N_\varepsilon\geq1$ such that

$$\left(\iint_{\overline{G}^+_\rho}\left|d(\nu_{n+m}-\nu_n)(\zeta)\right|\right)^2\leq(K_{\omega,\rho,h})^{-1}\left\|P^{(n+m)}_\omega-P^{(n)}_\omega\right\|_\omega<(K_{\omega,\rho,h})^{-1}\varepsilon$$

for any $n\geq N_\varepsilon$ and $m\geq1$. Thus, there is some $N'_\varepsilon\geq1$ such that

$$\left|\nu_{n+m}(\overline{G}^+_\rho)-\nu_n(\overline{G}^+_\rho)\right|\leq\iint_{\overline{G}^+_\rho}\left|d(\nu_{n+m}-\nu_n)(\zeta)\right|<\varepsilon$$

for any $n\geq N'_\varepsilon$ and $m\geq1$.

Now, suppose $\mathbb{D}_\rho$ is the disc centered at the point $i(\rho+1/\rho)$, with the radius $1/\rho$. Then obviously $\mathbb{D}_\rho\subset G^+_\rho$, and hence also

$$\left|\nu_{n+m}(\overline{\mathbb{D}}_\rho)-\nu_n(\overline{\mathbb{D}}_\rho)\right|<\varepsilon,\quad n\geq N'_\varepsilon,\quad m\geq1.$$

Observe that the Cauchy sequence of numbers $\nu_n(\overline{\mathbb{D}}_\rho)$ is bounded: $0\leq\nu_n(\mathbb{D}_\rho)\leq A_\rho<\infty$ $(n\geq1)$. Besides, it is easy to see that the sequence of Borel measures $\{\nu_n\}_{n=1}^\infty$ satisfies the Cauchy condition in the linear manifold Φ_ρ of real, finite continuous functions in the closed disc $\overline{\mathbb{D}}_\rho$, which vanish outside $\overline{\mathbb{D}}_\rho$. Hence, by Theorem 0.4′ in [2] there is a Borel measure $\nu^\rho\geq0$ such that the weak convergence $\nu_n\Rightarrow\nu^\rho$ $(n\to\infty)$ is true in Φ_ρ, i.e. for any function $g(\zeta)\in\Phi_\rho$

$$\iint_{G^+}g(\zeta)d\nu_n(\zeta)\to\iint_{G^+}g(\zeta)d\nu^\rho(\zeta)\quad\text{as}\quad n\to\infty.$$

Since this is true for any $\rho\in(0,h]$ and the embedded discs $\mathbb{D}_\rho$ exhaust G^+ as $\rho\to0$, there is a limit Borel measure ν over the whole G^+, which coincides with ν^ρ in any $\overline{\mathbb{D}}_\rho$. Obviously, for any fixed $\rho\in(0,h]$ we have $\left|(\nu_n-\nu)(\overline{\mathbb{D}}_\rho)\right|<\varepsilon$, if n is large enough. Hence $\nu_n(\overline{\mathbb{D}}_\rho)\to\nu(\overline{\mathbb{D}}_\rho)$ and $\nu_n\Rightarrow\nu$ as $n\to\infty$ on Φ_ρ for any fixed $\rho\in(0,h]$.

Also the sequence of numbers $\|P_\omega^{(n)}\|$ satisfies the Cauchy condition because of the inequality $\big|\|P_\omega^{(m+n)}\| - \|P_\omega^{(n)}\|\big| \le \|P_\omega^{(m+n)} - P_\omega^{(n)}\|$, and hence

$$\|P_\omega^{(n)}\| \to b \neq +\infty \text{ as } n \to \infty \quad \text{and} \quad \|P_\omega^{(n)}\| \le B < +\infty \text{ for } n \ge 1.$$

Further, assuming that $\{\rho_k\}_0^\infty \subset (0, h]$ is a sequence such that $\rho_0 = h$, $\rho_k \downarrow 0$, we conclude that

$$B \ge \liminf_{n\to\infty} \sup_{y>0} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \iint_{\overline{\mathbb{D}}_{\rho_k}\setminus\mathbb{D}_{\rho_{k-1}}} \iint_{\overline{\mathbb{D}}_{\rho_m}\setminus\mathbb{D}_{\rho_{m-1}}} J_\omega(y,\zeta_1,\zeta_2)|d\nu_n(\zeta_1)||d\nu_n(\zeta_2)|$$

$$\ge \sup_{y>0} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \iint_{\overline{\mathbb{D}}_{\rho_k}\setminus\mathbb{D}_{\rho_{k-1}}} \iint_{\overline{\mathbb{D}}_{\rho_m}\setminus\mathbb{D}_{\rho_{m-1}}} J_\omega(y,\zeta_1,\zeta_2)|d\nu(\zeta_1)||d\nu(\zeta_2)| = \|P_\omega\|,$$

where $P_\omega(z)$ is the Green type potential generated by the measure ν, and $\|P_\omega\| < +\infty$. Then, introducing the sequence of nonincreasing in $0 < \rho \le h$ functions

$$\varphi_n(\rho) \equiv \varphi_n(\rho, y) := \iint_{\overline{\mathbb{D}}_\rho\times\overline{\mathbb{D}}_\rho} J_\omega(y,\zeta_1,\zeta_2)|d(\nu-\nu_n)(\zeta_1)||d(\nu-\nu_n)(\zeta_2)|,$$

where $0 < y < +\infty$ is fixed, we see that

$$\varphi_n(\rho) \to \varphi_n(0) \le \|P_\omega - P_\omega^{(n)}\| \le 2B + 1 \quad \text{as} \quad \rho \to 0,$$

and evidently $\varphi_n(\rho) \le \|P_\omega - P_\omega^{(n)}\| \le 2B + 1$ for any $\rho \in (0, h]$, provided n is large enough. Hence, by a Helly theorem there is a subsequence of natural numbers $n_k \uparrow \infty$ such that at all points $0 < \rho \le h$ there exists a limit function

$$\varphi(\rho) := \lim_{k\to\infty} \varphi_{n_k}(\rho).$$

On the other hand, for any $\rho \in (0, h]$ the complete variation of the measure $\nu - \nu_n$ in $\overline{\mathbb{D}}_\rho$ weakly tends to zero. Consequently, due to the continuity of $J_\omega(y, \zeta_1, \zeta_2)$ by $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}_\rho$ we conclude that $\varphi_n(\rho) \to 0$ as $n \to \infty$ for any $\rho \in (0, 1]$, and hence $\varphi(\rho) \equiv 0$ $(0 < \rho \le h)$. Further, $\varphi_{n_k}(\rho)$ is monotone by $0 < \rho \le h$, and hence

$$\lim_{k\to\infty}\Big(\lim_{\rho\to 0} \varphi_{n_k}(\rho)\Big) = \lim_{\rho\to 0}\lim_{k\to\infty} \varphi_{n_k}(\rho) = 0.$$

Remind that this is true for any $0 < y < +\infty$, and hence $\lim_{k\to\infty} \|P_\omega - P_\omega^{(n_k)}\| = 0$, since the function $\sup_{y'<y<+\infty}\big(\lim_{\rho\to 0}\varphi_{n_k}(\rho, y)\big)$ is nonincreasing by $0 < y' < +\infty$. For completing the proof, it remains to see that $\|P_\omega - P_\omega^{(n)}\| \le \|P_\omega - P_\omega^{(n_k)}\| + \|P_\omega^{(n_k)} - P_\omega^{(n)}\|$, where the right-hand side summands are arbitrarily small when n_k and n are sufficiently large. □

2.2 The Small Space of Potentials Formed by b_ω

In a sense, the condition (6) defining the space $\mathcal{P}_{\omega,h}$ is complicated because the norm is defined by means of a complicated quantity (5). Therefore, below we consider some other, smaller spaces of Green type potentials, possessing more simple forms of norms. We start by calculation and evaluation of the quantity $J(y, \zeta_1, \zeta_2)$.

Lemma 2 *If the integrals are convergent and the change of the integration order is valid in* (4)*, then for any* $0 \le y < +\infty$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}[L_\omega P_\omega(x+iy)]^2\,dx = A(y)+B(y)+C(y)+D(y),$$

where

$$A(y) = \iint_{y\le\eta_1}\iint_{y\le\eta_2} J_A(y,\zeta_1,\zeta_2)\;d\nu(\zeta_2)d\nu(\zeta_1), \tag{8}$$

with

$$\begin{aligned}
J_A(y,\zeta_1,\zeta_2) =& -\int_y^{\eta_1}\omega(\eta_1-|\tau_1|)d\tau_1\int_{-\eta_2}^{y}\frac{(\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(\tau_1-\tau_2)^2}\\
&-\frac{1}{2}\int_y^{\eta_1}\omega(\eta_1-|\tau_1|)d\tau_1\int_y^{\eta_2}\frac{(2y-\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(2y-\tau_1-\tau_2)^2}\\
&+\frac{1}{2}\int_{-\eta_1}^{y}\omega(\eta_1-|\tau_1|)d\tau_1\int_{-\eta_2}^{y}\frac{(2y-\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(2y-\tau_1-\tau_2)^2}\\
:=& J_A'+J_A''+J_A''',
\end{aligned} \tag{9}$$

$$B(y) = \iint_{y\le\eta_1}\iint_{0<\eta_2<y} J_B(y,\zeta_1,\zeta_2)\;d\nu(\zeta_2)d\nu(\zeta_1) \tag{10}$$

with

$$\begin{aligned}
J_B(y,\zeta_1,\zeta_2) =& -\frac{1}{2}\int_y^{\eta_1}\omega(\eta_1-|\tau_1|)d\tau_1\int_{-\eta_2}^{\eta_2}\frac{(\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(\tau_1-\tau_2)^2}\\
&+\frac{1}{2}\int_{-\eta_1}^{y}\omega(\eta_1-|\tau_1|)d\tau_1\int_{-\eta_2}^{\eta_2}\frac{(2y-\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(2y-\tau_1-\tau_2)^2}\\
:=& J_B'+J_B'',
\end{aligned} \tag{11}$$

$$C(y) = \iint_{y\le\eta_1}\iint_{0<\eta_2<y} J_C(y,\zeta_1,\zeta_2)\;d\nu(\zeta_2)d\nu(\zeta_1) \tag{12}$$

with $J_C(y,\zeta_1,\zeta_2) = J_B(y,\zeta_2,\zeta_1)$,

$$D(y) = \iint_{0<\eta_1<y} \iint_{0<\eta_2<y} J_D(y,\zeta_1,\zeta_2)\, d\nu(\zeta_2)d\nu(\zeta_1), \tag{13}$$

with

$$J_D(y,\zeta_1,\zeta_2) = \frac{1}{2}\int_{-\eta_1}^{\eta_1} \omega(\eta_1-|\tau_1|)d\tau_1 \int_{-\eta_2}^{\eta_2} \frac{(2y-\tau_1-\tau_2)\omega(\eta_2-|\tau_2|)d\tau_2}{(\xi_1-\xi_2)^2+(2y-\tau_1-\tau_2)^2}. \tag{14}$$

Proof According to (4) and (5), $J(y,\zeta_1,\zeta_2) = (1/4\pi)\mathrm{Re}\{I_1+I_2\}$ and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} [L_\omega P_\omega(x+iy)]^2\, dx = \frac{1}{4\pi}\mathrm{Re}\iint_{G^+}\iint_{G^+} \{I_1+I_2\} d\nu(\zeta_2)d\nu(\zeta_1),$$

where

$$I_1 := \int_{-\eta_1}^{\eta_1} \omega(\eta_1-|\tau_1|)d\tau_1 \int_{-\eta_2}^{\eta_2} g_1(y,\zeta_1,\zeta_2)\omega(\eta_2-|\tau_2|)d\tau_2,$$

$$I_2 := \int_{-\eta_1}^{\eta_1} \omega(\eta_1-|\tau_1|)d\tau_1 \int_{-\eta_2}^{\eta_2} g_2(y,\zeta_1,\zeta_2)\omega(\eta_2-|\tau_2|)d\tau_2,$$

$$g_1(y,\zeta_1,\zeta_2) := -\int_{-\infty}^{\infty} \frac{dx}{[x-(\xi_1-i(y-\tau_1))]\,[x-(\xi_2-i(y-\tau_2))]},$$

$$g_2(y,\zeta_1,\zeta_2) := \int_{-\infty}^{\infty} \frac{dx}{[x-(\xi_1-i(y-\tau_1))]\,[x-(\xi_2+i(y-\tau_2))]}.$$

Calculating these integrals by residues, we get

$$g_1(y,\zeta_1,\zeta_2) = \begin{cases} 0 \quad , & \text{if} \quad y < \tau_{1,2}, \\ -\frac{2\pi i}{\xi_1-\xi_2+i(\tau_1-\tau_2)}, & \text{if} \quad \tau_2 < y < \tau_1, \\ \frac{2\pi i}{\xi_1-\xi_2+i(\tau_1-\tau_2)}, & \text{if} \quad \tau_1 < y < \tau_2, \\ 0 \quad , & \text{if} \quad y > \tau_{1,2}, \end{cases}$$

$$g_2(y,\zeta_1,\zeta_2) = \begin{cases} \frac{2\pi i}{\xi_1-\xi_2-i(2y-\tau_1-\tau_2)}, & \text{if} \quad y < \tau_{1,2}, \\ 0 \quad , & \text{if} \quad \tau_2 < y < \tau_1, \\ 0 \quad , & \text{if} \quad \tau_1 < y < \tau_2, \\ -\frac{2\pi i}{\xi_1-\xi_2-i(2y-\tau_1-\tau_2)}, & \text{if} \quad y > \tau_{1,2}. \end{cases}$$

Using these equalities, we come to the desired formulas (8)–(14). □

The below lemma gives an estimate form above for the square integral means of the considered Green type potentials.

Lemma 3 *Let* $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ *and let*

$$K_\omega(\eta_1, \eta_2) := \int_0^{\eta_1} \omega(\eta_1 + t_1) dt_1 \int_0^{\eta_2} \frac{\omega(\eta_2 + t_2)}{t_1 + t_2} dt_2,$$

where $\zeta_{1,2} = \xi_{1,2} + i\eta_{1,2} \in G^+$ *are any fixed points. Then*

$$0 < J(y, \zeta_1, \zeta_2) \le \frac{17}{2} K_\omega(\eta_1, \eta_2), \quad 0 < y < +\infty, \tag{15}$$

and hence the requirement

$$\iint_{G^+} \iint_{G^+} K_\omega(\eta_1, \eta_2) |d\nu(\zeta_1)| |d\nu(\zeta_2)| < +\infty$$

posed on a signed Borel measure ν *implies the validity of formulas* (4) *and* (5).

Proof Omitting the first, negative integral J'_A in (9), for the remaining summands we obtain that $J''_A < (1/2) K_\omega(\eta_1, \eta_2)$ for any $0 \le y \le \eta_{1,2}$. Further, one can see that

$$J'''_A \le \frac{1}{2} \int_{-\eta_1}^{y} \omega(\eta_1 - |\tau_1|) d\tau_1 \int_{-\eta_2}^{y} \frac{\omega(\eta_2 - |\tau_2|) d\tau_2}{2y - \tau_1 - \tau_2}.$$

Writing the right-hand side integrals as sums of integrals over $(-\eta_1, 0) \times (-\eta_2, 0)$, $(-\eta_1, 0) \times (0, y)$, $(0, y) \times (-\eta_2, 0)$ and $(0, y) \times (0, y)$, then estimating all these integrals by $K_\omega(\eta_1, \eta_2)$, we come to the inequality $J_A(y, \zeta_1, \zeta_2) < J''_A + J'''_A < (5/2) K_\omega(\eta_1, \eta_2)$ for $0 < y \le \eta_{1,2}$.

Omitting the negative summand J'_B in (11), we see that

$$J''_B \le \frac{1}{2} \int_{-\eta_1}^{y} \omega(\eta_1 + \tau_1) d\tau_1 \int_{-\eta_2}^{\eta_2} \frac{\omega(\eta_2 + \tau_2) d\tau_2}{2y - \tau_1 - \tau_2}.$$

Writing the right-hand side integrals as sums of integrals over $(-\eta_1, 0) \times (-\eta_2, 0)$, $(-\eta_1, 0) \times (0, \eta_2)$, $(0, y) \times (-\eta_2, 0)$ and $(0, y) \times (0, \eta_2)$, then estimating all these integrals by $K_\omega(\eta_1, \eta_2)$, we come to the inequality $J_B(y, \zeta_1, \zeta_2) < J''_B < 2K_\omega(\eta_1, \eta_2)$ for $0 < \eta_2 < y \le \eta_1$. Note that the same inequality is true also for $C(y, \zeta_1, \zeta_2)$, since $J_C(y, \zeta_2, \zeta_1) = J_B(y, \zeta_1, \zeta_2)$.

Further, by (14) for $0 < \eta_{1,2} < y < +\infty$ we get

$$J_D(y, \eta_1, \eta_2) \le \frac{1}{2} \int_{-\eta_1}^{\eta_1} \omega(\eta_1 - |\tau_1|) d\tau_1 \int_{-\eta_2}^{\eta_2} \frac{\omega(\eta_2 - |\tau_2|) d\tau_2}{2y - \tau_1 - \tau_2}.$$

Writing the right-hand side integrals as sums of integrals over $(-\eta_1, 0) \times (-\eta_2, 0)$, $(-\eta_1, 0) \times (0, \eta_2)$, $(0, \eta_1) \times (-\eta_2, 0)$ and $(0, \eta_1) \times (0, \eta_2)$, then estimating all these integrals by $K_\omega(\eta_1, \eta_2)$, we come to the inequality $J_D(y, \zeta_1, \zeta_2) < 2K_\omega(\eta_1, \eta_2)$ for $0 < \eta_{1,2} < y < +\infty$.

Summing the obtained inequalities, we come to the estimate (15). □

Remark 2 If $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and ν is a signed Borel measure in G^+, then by (4) and an application of the inequality $x + y \geq 2\sqrt{xy}$ to (15) we get

$$\sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[L_\omega P_\omega(x+iy)\right]^2 dx \leq \frac{17}{4} \left\{ \iint_{G^+} \left(\int_0^\eta \omega(\eta+t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| \right\}^2. \tag{16}$$

If the support of the measure ν is in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \leq h < +\infty\}$, then the boundedness of the right-hand side integral in (16) implies the validity of the first Blaschke type condition in (1), which provides the convergence of the Green type potential P_ω in G^+. Indeed, since $\omega(t)$ is a positive, nondecreasing function in $(0, +\infty)$,

$$\int_0^\eta \omega(\eta+t) \frac{dt}{\sqrt{t}} \geq \frac{1}{\sqrt{h}} \int_0^\eta \omega(t) dt.$$

The below definition is correct due to Remark 2.

Definition 3 A Green type potential P_ω in G^+ is of the class $\mathcal{P}'_{\omega,h}$, if the support of its associated Borel signed measure ν is located in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \leq h < +\infty\}$ and

$$\|P_\omega\|'_\omega := \iint_{G^+} \left(\int_0^\eta \omega(\eta+t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| < +\infty. \tag{17}$$

Remark 3 In the case $\omega(t) \equiv 1$ of the ordinary Green potentials P_0, formula (17) defining the class $\mathcal{P}'_{\omega,h}$ takes the form

$$\|P_0\|'_\omega = \iint_{G^+} \sqrt{\eta}\, |d\nu(\zeta)| < +\infty.$$

Now, we proceed to the main theorem of this section. Its proof is included for the sake of completeness, though it mainly follows the steps of the proof of Theorem 1.

Theorem 2 *For any $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and any fixed $h \in (0, +\infty)$, the set $\mathcal{P}'_{\omega,h}$ is a Banach space with the norm* (17)*.*

Proof Only the completeness of the space $\mathcal{P}'_{\omega,h}$ is to be proved, since all other properties of the norm obviously are satisfied. To this end, suppose $\{P_\omega^{(n)}\}_1^\infty$ is a sequence of Green type potentials from $\mathcal{P}'_{\omega,h}$, which satisfies the Cauchy condition, i.e. for any $\varepsilon > 0$

$$\left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|'_\omega = \iint_{G^+} \left(\int_0^\eta \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) \left| d(\nu_{n+m} - \nu_n)(\zeta) \right| < \varepsilon,$$

provided $n \geq N_\varepsilon$ for $N_\varepsilon \geq 1$ large enough and any $m \geq 1$. We shall prove that there is a potential $P_\omega \in \mathcal{P}'_{\omega,h}$, such that $\left\| P_\omega^{(n)} - P_\omega \right\|'_\omega \to 0$ as $n \to \infty$. Note that by the triangle inequalities (7) the same Cauchy condition is true for the positive, negative and complete variations of the measures ν_n, and hence it is sufficient to consider only one of these variations, which we denote again by ν_n (≥ 0) for simplicity.

Fixing an arbitrary $\rho \in (0, h]$ and denoting $G_\rho^+ = \{\zeta = \xi + i\eta \in G^+ : \rho < \eta < +\infty\}$, observe that

$$\iint_{\overline{G}_\rho^+} \left| d(\nu_{n+m} - \nu_n)(\zeta) \right| < M_\rho \iint_{\overline{G}_\rho^+} \left(\int_0^\eta \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) \left| d(\nu_{n+m} - \nu_n)(\zeta) \right|$$
$$\leq M_\rho \left\| P_\omega^{(n+m)} - P_\omega^{(n)} \right\|'_\omega < M_\rho \varepsilon,$$

where $M_\rho = [2\sqrt{\rho}\omega(\rho)]^{-1}$. Hence

$$\left| \nu_{n+m}(\overline{G}_\rho^+) - \nu_n(\overline{G}_\rho^+) \right| = \left| \iint_{\overline{G}_\rho^+} d\nu_{n+m}(\zeta) - \iint_{\overline{G}_\rho^+} d\nu_n(\zeta) \right|$$
$$\leq \iint_{\overline{G}_\rho^+} \left| d(\nu_{n+m} - \nu_n)(\zeta) \right| \leq M_\rho \varepsilon.$$

Thus, there is some $N'_\varepsilon \geq 1$ such that

$$\left| \nu_{n+m}(\overline{G}_\rho^+) - \nu_n(\overline{G}_\rho^+) \right| \leq \iint_{\overline{G}_\rho^+} \left| d(\nu_{n+m} - \nu_n)(\zeta) \right| < \varepsilon$$

for any $n \geq N'_\varepsilon$ and $m \geq 1$. Now, suppose $\mathbb{D}_\rho$ is the disc centered at the point $i(\rho + 1/\rho)$, with the radius $1/\rho$. Then obviously $\mathbb{D}_\rho \subset G_\rho^+$, and hence also

$$\left| \nu_{n+m}(\overline{\mathbb{D}}_\rho) - \nu_n(\overline{\mathbb{D}}_\rho) \right| < \varepsilon, \quad n \geq N'_\varepsilon, \ \ m \geq 1.$$

Observe that the Cauchy sequence of numbers $\nu_n(\overline{\mathbb{D}}_\rho)$ is bounded: $0 \leq \nu_n(\mathbb{D}_\rho) \leq A_\rho < \infty$ $(n \geq 1)$. Besides, it is easy to see that the sequence of Borel measures $\{\nu_n\}_{n=1}^\infty$ satisfies the Cauchy condition in the linear manifold Φ_ρ of real, finite continuous functions in the closed disc $\overline{\mathbb{D}}_\rho$, which vanish outside $\overline{\mathbb{D}}_\rho$. Hence, by

Theorem 0.4′ in [2] there is a Borel measure $\nu^\rho \geq 0$ such that the weak convergence $\nu_n \Rightarrow \nu^\rho$ $(n \to \infty)$ is true in Φ_ρ. The rest of the proof is the same as that of Theorem 1. □

2.3 Banach Spaces of Green Type Potentials Formed by $\widetilde{b}_\omega$

The main results related to the spaces of potentials formed by $\widetilde{b}_\omega(z,\zeta)$ and their proof are very similar to those on the spaces of potentials with $b_\omega(z,\zeta)$, which are given in the previous two sections. Therefore, after some preliminary information we give the main results with proofs, where we shall be focused on the differences from the similar proofs given before.

If $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$, then similar to $L_\omega \log|b_\omega(z,\zeta)|$ the function $L_\omega \log|\widetilde{b}_\omega(z,\zeta)|$ is subharmonic in $\overline{G^+} = \{z : \mathrm{Im} z \geq 0\}$, except the intercept $\{z = \xi + ih : 0 \leq h \leq \eta\}$, and also it is subharmonic in G^+, nonpositive and continuous in $\overline{G^+}$. Besides, for any fixed $\zeta = \xi + i\eta \in G^+$

$$\widetilde{f}(z,\zeta) := L_\omega \log \widetilde{b}_\omega(z,\zeta) = -\int_{-\eta}^{\eta} \frac{\omega(\eta+\tau)d\tau}{\tau - i(z-\xi)}, \quad z \in G^+. \tag{18}$$

We assume that $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and the Borel measures $\nu_\pm \geq 0$ satisfy the second Blaschke type condition in (1). Then, similar to $P_\omega(z)$ the Green type potential $\widetilde{P}_\omega(z)$ formed by $\log|\widetilde{b}_\omega(z,\zeta)|$ is convergent, and

$$L_\omega \widetilde{P}_\omega(z) = \iint_{G^+} L_\omega \log|\widetilde{b}_\omega(z,\zeta)| d\nu(\zeta), \quad z \in G^+.$$

Further, note that formula (3) is true for any function $\varphi(\zeta)$ and any measure $\mu(\zeta)$, for which the integrals of (3) exist. Besides, we get the equality

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[L_\omega \widetilde{P}_\omega(x+iy)\right]^2 dx = \iint_{G^+} d\nu(\zeta_1) \iint_{G^+} \widetilde{J}(y,\zeta_1,\zeta_2)\, d\nu(\zeta_2), \tag{19}$$

where

$$\begin{aligned}\widetilde{J}(y,\zeta_1,\zeta_2) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} L_\omega \log|\widetilde{b}_\omega(x+iy,\zeta_1)| L_\omega \log|\widetilde{b}_\omega(x+iy,\zeta_2)|\, dx \\ &= \mathrm{Re}\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left[\widetilde{f}(x+iy,\zeta_1)\widetilde{f}(x+iy,\zeta_2) + \widetilde{f}(x+iy,\zeta_1)\overline{\widetilde{f}(x+iy,\zeta_2)}\right]dx,\end{aligned} \tag{20}$$

provided the above integrals exist and the change of the integration order is true in (19).

The wide spaces of Green type potentials are defined as follows.

Definition 4 Let $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$. Then $\widetilde{\mathcal{P}}_{\omega,h}$ is the class of those Green type potentials $\widetilde{P}_\omega$, the supports of the associated Borel signed measures ν of which are such that $\sup\{|\zeta| : \zeta \in \mathrm{supp}\nu\} < h$, the measures ν are such that their positive and negative variations $\nu_\pm \geq 0$ satisfy the second Blaschke type condition in (1), and

$$\|\widetilde{P}_\omega\|_\omega := \sup_{y>0} \int_{-\infty}^{+\infty} |L_\omega \widetilde{P}_\omega(x+iy)|^2 dx$$
$$= \sup_{y>0} \left\{ \iint_{G^+} \iint_{G^+} \widetilde{J}(y, \zeta_1, \zeta_2) |d\nu(\zeta_1)||d\nu(\zeta_2)| \right\} < +\infty. \tag{21}$$

Remark 4 Similar to Remark 1, it is evident that the quantity (21) satisfies all norm axioms including the triangle inequality.

Theorem 3 *For any $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and $0 < h < +\infty$, the set $\widetilde{\mathcal{P}}_{\omega,h}$ is a Banach space with the norm* (21)*.*

Proof The proof is the same as that of Theorem 1. The only difference is that $J(y, \zeta_1, \zeta_2)$ is to be replaced by $\widetilde{J}(y, \zeta_1, \zeta_2)$. The nonnegativity of $\widetilde{J}(y, \zeta_1, \zeta_2)$ holds by a representation of $L_\omega \log |\widetilde{b}_\omega(z, \zeta)|$ as a sum of $L_\omega \log |b_\omega(z, \zeta)|$ and a harmonic in G^+ function, the nonpositivity of which is easy to prove. To prove the continuity of $\widetilde{J}(y, \zeta_1, \zeta_2)$ by $0 \leq y < +\infty$ and $\zeta_{1,2} \in \overline{G^+}$, first observe that by (18) it follows that there is a constant $C > 0$ such that for any $|x+iy| > 2\max\{|\xi_1|+\eta_1, |\xi_2|+\eta_2\}$

$$0 < \widetilde{F}(x, y) := L_\omega \log |\widetilde{b}_\omega(x+iy, \zeta_1)|\, L_\omega \log |\widetilde{b}_\omega(x+iy, \zeta_2)| \leq \frac{C}{|x+iy|^2}.$$

Hence, for any $\varepsilon > 0$

$$\int_{R \leq |x| < +\infty} \widetilde{F}(x, y) dx < \frac{\varepsilon}{3}$$

when R is large enough. So, for any values $y_{1,2} \geq 0$

$$\left| \widetilde{J}(y_1, \zeta_1, \zeta_2) - \widetilde{J}(y_2, \zeta_1, \zeta_2) \right| < \frac{2}{3}\varepsilon + \int_{-R \leq x \leq R} \left| \widetilde{F}(x, y_1) - \widetilde{F}(x, y_2) \right| dx,$$

where the last integral is less than $\varepsilon/3$ by the continuity of $L_\omega \log |\widetilde{b}_\omega(x+iy, \zeta)|$, if $|y_1 - y_2|$ is small enough. The continuity of $\widetilde{J}(y, \zeta_1, \zeta_2)$ by $\zeta_{1,2} \in \overline{G^+}$ is proved in the same way. □

We omit the calculation and estimation of the quantity $\widetilde{J}(y, \zeta_1, \zeta_2)$, since these are very similar to those of $J(y, \zeta_1, \zeta_2)$, which are given above. So, we start the consideration of the smaller spaces of potentials formed by $\widetilde{b}_\omega$. We begin by the below theorem which we give without its proof.

Theorem 4 *Let* $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ *and let*

$$\widetilde{K}_\omega(\eta_1, \eta_2) = \int_0^{2\eta_1} \omega(\eta_1 + t_1) dt_1 \int_0^{2\eta_2} \frac{\omega(\eta_2 + t_2)}{t_1 + t_2} dt_2, \tag{22}$$

where $\zeta_{1,2} = \xi_{1,2} + i\eta_{1,2} \in G^+$ *are any fixed points. Then*

$$0 < \widetilde{J}(y, \zeta_1, \zeta_2) \le \frac{5}{2} \widetilde{K}_\omega(\eta_1, \eta_2), \quad 0 < y < +\infty, \tag{23}$$

and hence the requirement

$$\iint_{G^+} \iint_{G^+} \widetilde{K}_\omega(\eta_1, \eta_2) |d\nu(\zeta_1)||d\nu(\zeta_2)| < +\infty,$$

on the Borel signed measure ν *implies the validity of formulas* (19) *and* (20).

Remark 5 If $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$, then by (19) and (23) an application of the inequality $x + y \ge 2\sqrt{xy}$ to (22) gives

$$\sup_{y>0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[L_\omega \widetilde{P}_\omega(x + iy) \right]^2 dx \le \frac{5}{4} \left\{ \iint_{G^+} \left(\int_0^{2\eta} \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| \right\}^2.$$

If the support of the measure ν is in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \le h < +\infty\}$, then the boundedness of the right-hand side integral in the above inequality implies the validity of the second Blaschke type condition in (1), which by Theorem 2 of [5] provides the validity of all above statements of this section. Indeed, since $\omega(t)$ is a positive, nondecreasing function in $(0, +\infty)$,

$$\int_0^{2\eta} \omega(\eta + t) \frac{dt}{\sqrt{t}} \ge \frac{1}{\sqrt{2h}} \int_0^{2\eta} \omega(t) dt.$$

The below definition is correct due to Remark 5.

Definition 5 Let $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$. Then a Green type potential $\widetilde{P}_\omega$ in G^+ is of the class $\widetilde{\mathcal{P}}'_{\omega,h}$, if the support of its associated Borel signed measure is located in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \le h < +\infty\}$ and

$$\|\widetilde{P}_\omega\|'_\omega := \iint_{G^+} \left(\int_0^{2\eta} \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| < +\infty. \tag{24}$$

Theorem 5 *For any $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$ and $0 < h < +\infty$, the set $\widetilde{\mathcal{P}}'_{\omega,h}$ is a Banach space with the norm (24).*

Proof We omit the proof, since it is almost the same as that of Theorem 2. □

3 Banach Spaces of Delta-Subharmonic Functions

A function $U(z)$ is δ-subharmonic in the upper half-plane G^+, if it is a difference $U(z) = U_1(z) - U_2(z)$ of two functions subharmonic in G^+. We assume that the Riesz associated signed measure ν of $U(z)$ is minimally decomposed in the Jordan sense, i.e. $\nu = \nu_+ - \nu_-$, where $\nu_\pm$ are the positive and negative variations of the measure ν. These are some nonnegative Borel measures with non-overlapping supports in G^+. Two functions $U(z)$ and $V(z) = V_1(z) - V_2(z)$, which are δ-subharmonic in a domain, are said to be equal, i.e. $U(z) = V(z)$, if $U_1(z) + V_2(z) = U_2(z) + V_1(z)$ everywhere in that domain.

Definition 6 Let $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$. Then $D_{\omega,h}$ is the set of those δ-subharmonic in G^+ functions $u(z)$ with associated Borel signed measures ν, the supports of which are located in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \le h < +\infty\}$, and

$$\|u\|_\omega = \sup_{y>0} \frac{1}{2\pi} \left\{ \int_{-\infty}^{+\infty} |L_\omega u(x+iy)|^2 dx \right\}^{1/2} + \iint_{G^+} \left(\int_0^\eta \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| < +\infty. \tag{25}$$

Definition 7 Let $\omega(t) \in \Omega_\alpha$ $(\alpha > 0)$. Then $\widetilde{D}_{\omega,h}$ is the set of those δ-subharmonic in G^+ functions $u(z)$ with associated Borel signed measures ν, the supports of which are located in a strip $G_h^+ = \{\zeta = \xi + i\eta \in G^+ : 0 < \eta \le h < +\infty\}$, and

$$\|u\|_\omega = \sup_{y>0} \frac{1}{2\pi} \left\{ \int_{-\infty}^{+\infty} |L_\omega u(x+iy)|^2 dx \right\}^{1/2} + \iint_{G^+} \left(\int_0^{2\eta} \omega(\eta + t) \frac{dt}{\sqrt{t}} \right) |d\nu(\zeta)| < +\infty. \tag{26}$$

Theorem 6 *$D_{\omega,h}$ and $\widetilde{D}_{\omega,h}$ are Banach spaces.*

Proof We prove only the case of $D_{\omega,h}$, since for $\widetilde{D}_{\omega,h}$ the proof is almost the same. The quantity (25) evidently satisfies all the norm axioms including the triangle inequality. Hence, if $u(z) \in D_{\omega,h}$, then the harmonic in G^+ function $u(z) - P_\omega(z) := U(z)$ is such that $L_\omega U(z)$ belongs to the ordinary harmonic Hardy space h^2 in G^+. By Proposition 2.1 in [7], this means that $U(z)$ is of the Hilbert space $h^2_{\omega^*}$ of harmonic functions in G^+, which is defined by the condition

$$\|U\|_{h^2_{\omega^*}} = \left\{\iint_{G^+} \left[U(z)\right]^2 d\mu_{\omega^*}(z)\right\}^{1/2} < +\infty,$$

where $d\mu_{\omega^*}(x+iy) = dx d\omega^*(2y)$ and $\omega^*(t)$ is the Volterra square of $\omega(t)$:

$$\omega^*(t) = \int_0^t \omega(t-\lambda)d\omega(\lambda), \quad 0 < t < +\infty, \quad \omega^*(0) = 0.$$

So, $D_{\omega,h}$ is the direct sum of the Hilbert space $h^2_{\omega^*}$ of harmonic in G^+ functions and the Banach space of Green type potentials $\mathcal{P}'_{\omega,h}$, i.e. any δ-subharmonic function $u(z) \in D_{\omega,h}$ has a unique representation of the form

$$u(z) = U(z) + P_\omega(z), \quad U(z) \in h^2_{\omega^*}, \quad P_\omega(z) \in \mathcal{P}'_{\omega,h}. \tag{27}$$

Similarly, $\widetilde{D}_{\omega,h}$ is the direct sum of the Hilbert space $h^2_{\omega^*}$ of harmonic in G^+ functions and the Banach space of Green type potentials $\widetilde{\mathcal{P}}'_{\omega,h}$, i.e. any δ-subharmonic function $u(z) \in \widetilde{D}_{\omega,h}$ has a unique representation of the form

$$u(z) = U(z) + \widetilde{P}_\omega(z), \quad U(z) \in h^2_{\omega^*}, \quad \widetilde{P}_\omega(z) \in \widetilde{\mathcal{P}}'_{\omega,h}. \tag{28}$$

□

Remark 6 Obviously, similar theorems are true for the Banach spaces of δ-subharmonic functions defined as direct sums of the harmonic Hilbert space $h^2_{\omega^*}$ and the large Banach spaces $\mathcal{P}_{\omega,h}$ and $\widetilde{\mathcal{P}}_{\omega,h}$ of Green type potentials (Theorems 1 and 3), where the last summands in the left-hand sides of the definitions of the norms (25) and (26) are replaced respectively by

$$\begin{aligned}\|P_\omega\|_\omega &= \sup_{y>0} \int_{-\infty}^{+\infty} |L_\omega P_\omega(x+iy)|^2 dx \\ &= \sup_{y>0} \left\{\iint_{G^+} \iint_{G^+} J(y,\zeta_1,\zeta_2)|d\nu(\zeta_1)||d\nu(\zeta_2)|\right\},\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{P}_\omega\|_\omega &= \sup_{y>0} \int_{-\infty}^{+\infty} |L_\omega \widetilde{P}_\omega(x+iy)|^2 dx \\ &= \sup_{y>0} \left\{\iint_{G^+} \iint_{G^+} \widetilde{J}(y,\zeta_1,\zeta_2)|d\nu(\zeta_1)||d\nu(\zeta_2)|\right\}.\end{aligned}$$

Remark 7 For any $\alpha > 0$, any of the unions

$$\bigcup_{\omega\in\Omega_\alpha, h>0} \mathcal{D}_{\omega,h}\,, \quad \bigcup_{\omega\in\Omega_\alpha, h>0} \widetilde{\mathcal{D}}_{\omega,h}\,, \quad \bigcup_{\omega\in\Omega_\alpha, h>0} \mathcal{D}'_{\omega,h}\,, \quad \bigcup_{\omega\in\Omega_\alpha, h>0} \widetilde{\mathcal{D}}'_{\omega,h}$$

of the considered Banach spaces includes δ-subharmonic in G^+ functions with any behavior near a finite interval on the real axes, the associated signed measures of which are in a strip $0 < \mathrm{Im}\zeta < h < +\infty$. Indeed,

$$\bigcup_{\omega\in\Omega_\alpha, h>0} \mathcal{D}'_{\omega,h} \subset \bigcup_{\omega\in\Omega_\alpha, h>0} \mathcal{D}_{\omega,h} \quad \text{and} \quad \bigcup_{\omega\in\Omega_\alpha, h>0} \widetilde{\mathcal{D}}'_{\omega,h} \subset \bigcup_{\omega\in\Omega_\alpha, h>0} \widetilde{\mathcal{D}}_{\omega,h},$$

while a function δ-subharmonic in G^+ with any given behavior near a real interval and an associated signed measure in a strip $0 < \mathrm{Im}\zeta < h < +\infty$ belongs to even smaller spaces $\mathcal{D}'_{\omega,d}$ and $\widetilde{\mathcal{D}}'_{\omega,d}$ at some choice of the parameter-function $\omega \in \Omega_\alpha$.

Remark 8 Since $\log|f(z)|$ of a meromorphic function $f(z)$ is a particular case of δ-subharmonic function, all the considered above spaces can be undestood also as Banach spaces of functions meromorphic in the half-plane. Then, the representations (27), (28), along with their two similarities, become factorizations in the corresponding four Banach spaces of functions meromorphic in the half-plane.

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Weighted Estimates Containing Quasilinear Operators

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Abstract Characterizations of weighted estimates for a certain class of quasilinear integral operators with kernels are obtained.

Keywords Integral operator · Weighted inequality · Weight function · Kernel

Mathematics Subject Classification 26D10, 26D15

1 Introduction

Let $I = (0, \infty)$. Let u, v and w be weight functions, i.e., non-negative functions locally summable on I.

For $f \geq 0$, we consider the inequalities

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_t^\infty K(s,t)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}} \tag{1.1}$$

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and

$$\left(\int_0^\infty u(x)\left(\int_x^\infty\left(\int_0^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}} \tag{1.2}$$

with the kernels $K(\cdot,\cdot)\ge 0$ in the classes O_1^- and O_1^+, respectively. Let us present the definitions of these classes.

Definition 1.1 Let $K(\cdot,\cdot)$ be a non-negative function, defined on the set $\{(t,s), 0 < s \le t < \infty\}$, measurable and non-increasing in the second argument. We say that the function $K(t,s)$ belongs to the class O_1^- if there exist non-negative measurable functions $U(\cdot)$ and $Q(\cdot,\cdot)$ and a constant $d \ge 1$ such that

$$d^{-1}(K(t,\tau)+U(t)Q(\tau,s)) \le K(t,s) \le d(K(t,\tau)+U(t)Q(\tau,s)) \tag{1.3}$$

for all $t,\tau,s:\ 0<s\le\tau\le t<\infty$.

Definition 1.2 Let $K(\cdot,\cdot)$ be a non-negative function, defined on the set $\{(t,s), 0 < s \le t < \infty\}$, measurable and non-decreasing in the first argument. We say that the function $K(t,s)$ belongs to the class O_1^+ if there exist non-negative measurable functions $V(\cdot)$ and $R(\cdot,\cdot)$ and a constant $d \ge 1$ such that

$$d^{-1}(R(t,\tau)V(s)+K(\tau,s)) \le K(t,s) \le d(R(t,\tau)V(s)+K(\tau,s)) \tag{1.4}$$

for all $t,\tau,s:\ 0<s\le\tau\le t<\infty$.

In [10–13], criteria for the validity of inequalities (1.1) and (1.2) have been found in the case $0 < r, q < \infty$ and $1 \le p < \infty$, for the kernels $K(\cdot,\cdot)\ge 0$, satisfying the condition (see [2] or [7]): there exists a constant $d > 0$ such that

$$d^{-1}(K(t,\tau)+K(\tau,s)) \le K(t,s) \le d(K(t,\tau)+K(\tau,s)) \tag{1.5}$$

for all $t,\tau,s:\ 0<s\le\tau\le t<\infty$. The method introduced in [10–13] uses an auxiliary function depending on a weight u. In [15], the same problem has been studied by an alternative method, and explicit criteria, without using an auxiliary function, have been obtained for $1 < r < q < \infty$ and $1 < p < \infty$. In [3], another method has been proposed, different from the methods in [10–13] and [15], but for the case $0 < r < \infty$ and $1 < p \le q < \infty$. Here we complete the investigations started in [3]; namely, by the method of [3] we characterize inequalities (1.1) and (1.2) in the case $0 < r < \infty$, $0 < q < p < \infty$ and $p \ge 1$. Moreover, in this paper the kernels involved in inequalities (1.1) and (1.2) satisfy conditions, which are less restrictive than condition (1.5). It is easy to see that the classes O_1^- and O_1^+ are wider than the class of kernels satisfying (1.5), since (1.5) is a partial case of condition (1.3) for $U(t) \equiv 1$, $Q(\tau, s) \equiv K(\tau, s)$ and condition (1.4) for $V(s) \equiv 1$, $R(t, \tau) \equiv K(t, \tau)$, i.e., the kernels satisfying (1.5) belong to the class $O_1^- \cap O_1^+$. As for the general classes O_n^- and O_n^+, $n \geq 0$, they have been introduced in [9] and [8] in connection with establishing conditions for the validity of the following weighted Hardy type inequality with Hardy-Volterra integral operator:

$$\left(\int_0^\infty u(x)\left(\int_x^\infty K(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty v(x)f^p(x)\,dx\right)^{\frac{1}{p}}. \tag{1.6}$$

In [9], characterizations of inequality (1.6) have been obtained for $1 < p \leq q < \infty$. The case $1 < q < p < \infty$ has been investigated in the paper [1] but only for the classes O_1^- and O_1^+. Nevertheless, the characterizations found in [1] are used in the proofs of the main results in the present paper.

Let us note that we have decided not to include motivation for this study, since [10–13, 15] and [3] reveal it completely.

The paper is organized as follows. Section 2 contains all the auxiliary statements and definitions necessary to characterize inequalities (1.1) and (1.2). Section 3 presents the main results. Let us note that the statements of the main results are given for both inequalities (1.1) and (1.2), but the proof is given only for inequality (1.1), since the proof for (1.2) is similar. Moreover, in Sect. 3, as corollaries of the main results, we derive characterizations of (1.1) and (1.2) in the case where involved kernels satisfy condition (1.5). In Sect. 4, we demonstrate an application of the obtained results to bilinear inequalities.

2 Auxiliary Statements and Definitions

Let $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{\mu} = \frac{1}{q} - \frac{1}{p}$. The symbol $A \ll B$ means $A \leq CB$ with some constant C, depending on the parameters p, r and q. Moreover, notation $A \approx B$ means $A \ll B \ll A$, while χ_E stands for the characteristic function of the set E.

Let $L_{p,v}(I)$ be the set of functions f measurable on I, for which the functional

$$\|f\|_{p,v,I} \equiv \|f\|_{p,v} = \begin{cases} \left(\int_I v(t)|f(t)|^p\,dt\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ ess\sup_{t\in I} v(t)|f(t)|, & p = \infty, \end{cases}$$

is finite.

We have to introduce the quantities

$$J^-_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_t^\beta K(s,t) f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}}$$

and

$$J^+_{p,r}(\alpha,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^\beta \left(\int\limits_\alpha^t K(t,s) f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\alpha,\beta)}},$$

where $0 \leq \alpha < \beta \leq \infty$.

In addition to $J^-_{p,r}(\alpha,\beta)$, we also need the quantity

$$H^-_{p,r}(\alpha,\beta_1,\beta) = \sup_{f\geq 0} \frac{\left(\int\limits_\alpha^{\beta_1} \left(\int\limits_{\beta_1}^\beta K(s,t) f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p,v,(\beta_1,\beta)}},$$

where $\alpha < \beta_1 \leq \beta$. Recall that the proof of the main result for inequality (1.2) is similar to the proof of the main result for inequality (1.1), so it is omitted. Since the quantity $H^+_{p,r}(\alpha,\alpha_1,\beta)$ for $\alpha \leq \alpha_1 < \beta$ is used only in the proof, but not in the statement, it is also omitted.

We need the following lemma proved in [14] for linear operators and reproved in [4] for homogeneous operators.

Lemma 2.1 *Let* $D = \bigcup\limits_k D_k$ *and* $V = \bigcup\limits_k V_k$ *be unions of pairwise-disjoint measurable sets. Let* $Tf = \sum\limits_k T_k f_k$, *where* $f = \sum\limits_k f_k$, $f_k = \chi_{V_k} f$, *and* T_k *be a positive homogeneous operator acting from* $L_p(V_k)$ *to the set of measurable functions on* D_k. *Then the inequality*

$$\|Tf\|_{q,D} \leq C\|f\|_{p,V}, \quad f \geq 0, \tag{2.1}$$

holds if and only if the inequality

$$\|T_k f\|_{q,D_k} \leq C_k\|f\|_{p,V_k}, \quad f \geq 0, \tag{2.2}$$

holds for all k and

$$a_{p,q} = \begin{cases} \sup\limits_k C_k < \infty, & 0 < p \le q \le \infty, \\ \left(\sum\limits_k C_k^{\mu}\right)^{\frac{1}{\mu}} < \infty, & 0 < q < p \le \infty. \end{cases}$$

Moreover, $C = a_{p,q}$, *where* C *and* C_k *are the least constants in* (2.1) *and* (2.2), *respectively.*

Now, we give a definition and a lemma of the paper [6].

Definition 2.1 An increasing sequence $\{x_k\} \subset I$ is called a covering sequence of an interval I, if either $\bigcup\limits_k [x_k, x_{k+1}) = I$, $x_k \to 0$ for $k \to -\infty$ and $x_k \to \infty$ for $k \to \infty$, or $\{x_k\} = \{x_k\}_{k=n}^m$ and $x_{n-1} = 0$, $x_{m+1} = \infty$.

Lemma 2.2 *Let* $\{x_k\}$ *and* $\{z_i'\}$ *be two covering sequences. Then there exists a sequence* $\{z_n\} \subset \{z_i'\}$ *such that*

(1) $[z_n, z_{n+1}) \cap \{x_k\}_k \neq \emptyset$ *for all n;*
(2) if $z_i' = z_n \le x_k < z_{n+1}$, *then* $x_k \le z_{i+1}'$.

Moreover, it follows from (1) and (2) that $x_{k+1} \le z_{n+2}$ *and* $z_{n-1} \le z_{i-1}'$.

From the definitions for $J^-_{p,r}(\alpha, \beta)$ and $H^-_{p,r}(\alpha, \beta_1, \beta)$ we have that $J^-_{p,r}(\alpha, \beta) \ge H^-_{p,r}(\alpha, \beta_1, \beta)$. Moreover, we present one more property for these quantities in the following statement.

Lemma 2.3 *Let* $0 \le \alpha < \beta_1 \le \beta \le \infty$. *Then*

$$J^-_{p,r}(\alpha, \beta) \le J^-_{p,r}(\alpha, \beta_1) + J^-_{p,r}(\beta_1, \beta) + H^-_{p,r}(\alpha, \beta_1, \beta), \quad 1 \le r < \infty, \tag{2.3}$$

$$(J^-_{p,r}(\alpha, \beta))^r \le (J^-_{p,r}(\alpha, \beta_1))^r + (J^-_{p,r}(\beta_1, \beta))^r + (H^-_{p,r}(\alpha, \beta_1, \beta))^r, \quad 0 < r < 1. \tag{2.4}$$

Proof Let $1 \le r < \infty$. Then for $f \ge 0$ we have

$$\left(\int_\alpha^\beta \left(\int_t^\beta K(s,t) f(s)\, ds\right)^r w(t)\, dt\right)^{\frac{1}{r}} \le \left(\int_\alpha^{\beta_1} \left(\int_t^{\beta_1} K(s,t) f(s)\, ds\right)^r w(t)\, dt\right)^{\frac{1}{r}}$$
$$+ \left(\int_{\beta_1}^\beta \left(\int_t^\beta K(s,t) f(s)\, ds\right)^r w(t)\, dt\right)^{\frac{1}{r}} + \left(\int_\alpha^{\beta_1} \left(\int_{\beta_1}^\beta K(s,t) f(s)\, ds\right)^r w(t)\, dt\right)^{\frac{1}{r}}. \tag{2.5}$$

The latter and $\|f\|_{p,v,(\alpha,\beta)} \ge \max\{\|f\|_{p,v,(\alpha,\beta_1)}, \|f\|_{p,v,(\beta_1,\beta)}\}$ give (2.3).

In the case $0 < r < 1$ we similarly have

$$\int_\alpha^\beta \left(\int_t^\beta K(s,t)f(s)\,ds\right)^r w(t)\,dt \le \int_\alpha^{\beta_1} \left(\int_t^{\beta_1} K(s,t)f(s)\,ds\right)^r w(t)\,dt$$

$$+\int_{\beta_1}^\beta \left(\int_t^\beta K(s,t)f(s)\,ds\right)^r w(t)\,dt + \int_\alpha^{\beta_1} \left(\int_{\beta_1}^\beta K(s,t)f(s)\,ds\right)^r w(t)\,dt \qquad (2.6)$$

that proves (2.4). The proof of Lemma 2.3 is complete. □

3 Main Results

Let

$$A_1 = \left(\int_0^\infty u(x)\left(\int_x^\infty u(s)ds\right)^{\frac{\mu}{p}} \left(J^-_{p,r}(0,x)\right)^\mu dx\right)^{\frac{1}{\mu}},$$

$$A_2 = \left(\int_0^\infty \left(\int_0^t u(s)\left(\int_0^s w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{p}} \left(\int_t^\infty K^{p'}(s,t)v^{1-p'}(s)\,ds\right)^{\frac{\mu}{p'}} \right.$$

$$\left. \times\, u(t)\left(\int_0^t w(s)ds\right)^{\frac{q}{r}} dt\right)^{\frac{1}{\mu}},$$

$$A_3 = \left(\int_0^\infty \left(\int_0^t u(s)\left(\int_0^s w(z)dz\right)^{\frac{q}{r}} Q^q(t,s)ds\right)^{\frac{\mu}{q}} \right.$$

$$\left. \times \left(\int_t^\infty U^{p'}(s)v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} U^{p'}(t)v^{1-p'}(t)dt\right)^{\frac{1}{\mu}},$$

$$A_4 = \left(\int_0^\infty \left(\int_0^t u(s) \left(\int_0^s Q^r(s,z)w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \right.$$

$$\left. \times \left(\int_t^\infty U^{p'}(s)v^{1-p'}(s)\,ds \right)^{\frac{\mu}{q'}} U^{p'}(t)v^{1-p'}(t)dt \right)^{\frac{1}{\mu}}.$$

Theorem 3.1 *Let* $0 < r < \infty$, $0 < q < p < \infty$ *and* $p \geq 1$. *Let the kernel* $K(\cdot,\cdot)$ *belong to the class* O_1^-. *Then* (1.1) *holds if and only if* $A = \max\{A_1, A_2, A_3, A_4\} < \infty$. *Moreover,* $A \approx C$, *where* C *is the least constant in* (1.1).

Proof If we split the interior integral on the left-hand side of inequality (1.1) into two integrals and take into account (1.3), we obtain

$$\left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^\infty K(s,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$\approx \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_x^\infty K(s,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$\approx \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_t^x K(s,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_x^\infty K(s,x)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^\infty u(x) \left(\int_0^x \left(\int_x^\infty U(s)Q(x,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{q}{r}} dx \right)^{\frac{1}{q}}. \tag{3.1}$$

From (3.1) we derive that the validity of (1.1) is equivalent to the validity of the following three inequalities:

$$\left(\int_0^\infty u(x)\left(\int_0^x\left(\int_t^x K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}} \le C'\,\|f\|_{p,v}, \tag{3.2}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x w(t)\,dt\right)^{\frac{q}{r}}\left(\int_x^\infty K(s,x)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \le C''\,\|f\|_{p,v}, \tag{3.3}$$

$$\left(\int_0^\infty u(x)\left(\int_0^x Q^r(x,t)w(t)\,dt\right)^{\frac{q}{r}}\left(\int_x^\infty U(s)f(s)\,ds\right)^q dx\right)^{\frac{1}{q}} \le C'''\,\|f\|_{p,v}. \tag{3.4}$$

First we prove that inequality (3.2) holds if and only if $A_1 < \infty$. Moreover,

$$C' \approx A_1, \tag{3.5}$$

where C' is the least constant in (3.2).

Necessity. Let (3.2) hold with the least constant $C' > 0$. Let $\{x_k\}_k$ be an arbitrary covering sequence.

Denote by S the left-hand side of inequality (3.2). For $0 \le f \in L_{p,v}$, we have

$$S = \left(\sum_k \int_{x_k}^{x_{k+1}} u(x)\left(\int_0^x\left(\int_t^x K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}}$$

$$\ge \left(\sum_k \int_{x_k}^{x_{k+1}} u(x)dx\left(\int_0^{x_k}\left(\int_t^{x_k} K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}.$$

Combining the latter with each of the estimates

$$\left(\int_0^{x_k}\left(\int_t^{x_k} K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}} \ge \left(\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k} K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}$$

and

$$\left(\int_0^{x_k}\left(\int_t^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}} \geq \left(\int_0^{x_{k-1}}\left(\int_{x_{k-1}}^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}},$$

respectively yields

$$S \geq \left(\sum_k \int_{x_k}^{x_{k+1}} u(x)dx \left(\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \tag{3.6}$$

and

$$S \geq \left(\sum_k \int_{x_k}^{x_{k+1}} u(x)dx \left(\int_0^{x_{k-1}}\left(\int_{x_{k-1}}^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}. \tag{3.7}$$

We assume that

$$u_k(x) = \chi_{(x_k,x_{k+1})}(x)u(x),$$

$$f_k(s) = \chi_{(x_{k-1},x_k)}(s)f(s),$$

$$T_k f_k(x) = u_k^{\frac{1}{q}}(x)\left(\int_{x_{k-1}}^{x_k}\left(\int_t^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}},$$

$$Tf(x) = \sum_k T_k f_k(x),$$

$$\widetilde{T}_k f_k(x) = u_k^{\frac{1}{q}}(x)\left(\int_0^{x_{k-1}}\left(\int_{x_{k-1}}^{x_k}K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}$$

and

$$\widetilde{T}f(x) = \sum_k \widetilde{T}_k f_k(x).$$

Then from (3.2) and (3.6), (3.2) and (3.7) we respectively get

$$\|Tf\|_q \le C'\|f\|_{p,v}, \tag{3.8}$$

$$\|\widetilde{T}f\|_q \le C'\|f\|_{p,v}, \tag{3.9}$$

where C' is the least constant in (3.2). Hence, by Lemma 2.1 for $0 < q < p \le \infty$, $p \ge 1$, we derive from (3.8)

$$\left(\sum_k C_k^\mu\right)^{\frac{1}{\mu}} \le C', \tag{3.10}$$

while from (3.9)

$$\left(\sum_k \widetilde{C}_k^\mu\right)^{\frac{1}{\mu}} \le C', \tag{3.11}$$

where

$$C_k = \sup_{f\ge 0}\frac{\|T_k f_k\|_{q,(x_k,x_{k+1})}}{\|f_k\|_{p,v,(x_{k-1},x_k)}} = \left(\int\limits_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{1}{q}} \sup_{f\ge 0}\frac{\left(\int\limits_{x_{k-1}}^{x_k}\left(\int\limits_t^{x_k} K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f_k\|_{p,v,(x_{k-1},x_k)}}$$

$$= \left(\int\limits_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{1}{q}} J^-_{p,r}(x_{k-1},x_k), \tag{3.12}$$

$$\widetilde{C}_k = \sup_{f\ge 0}\frac{\|\widetilde{T}_k f_k\|_{q,(x_k,x_{k+1})}}{\|f_k\|_{p,v,(x_{k-1},x_k)}} = \left(\int\limits_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{1}{q}} \sup_{f\ge 0}\frac{\left(\int\limits_0^{x_{k-1}}\left(\int\limits_{x_{k-1}}^{x_k} K(s,t)f(s)ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f_k\|_{p,v,(x_{k-1},x_k)}}$$

$$= \left(\int\limits_{x_k}^{x_{k+1}} u(x)dx\right)^{\frac{1}{q}} H^-_{p,r}(0,x_{k-1},x_k). \tag{3.13}$$

Now, replacing (3.12) into (3.10) and (3.13) into (3.11), we obtain

$$\frac{1}{2}(A_{11}+A_{12}) \le C', \tag{3.14}$$

where

$$A_{11} = \sup_{\{x_k\}} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(x_{k-1}, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}} < \infty,$$

$$A_{12} = \sup_{\{x_k\}} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(H^-_{p,r}(0, x_{k-1}, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}} < \infty.$$

Next, let $\int_{x_k}^{\infty} u(x)dx = 2^{-k}$ and $J^-_{p,r}(0, z'_i) = 2^i$. Then by Lemma 2.2 there exists a sequence $\{z_n\} \subset \{z'_i\}$ such that $[z_n, z_{n+1}) \cap \{x_k\}_k \neq \emptyset$ for all n. Let $z'_i = z_n \leq x_k < z_{n+1}$. Hence, $x_k \leq z'_{i+1}$, $x_{k+1} \leq z_{n+2}$ and $z'_{i-1} \geq z_{n-1} > z_{n-2}$.

For $1 \leq r < \infty$, using (2.3), we get

$$J^-_{p,r}(0, z'_{i+1}) = 4(J^-_{p,r}(0, z'_i) - J^-_{p,r}(0, z'_{i-1})) \leq 4(J^-_{p,r}(0, z_n) - J^-_{p,r}(0, z_{n-2}))$$

$$\leq 4(J^-_{p,r}(z_{n-2}, z_n) + H^-_{p,r}(0, z_{n-2}, z_n)),$$

and for $0 < r < 1$, using (2.4), we get

$$J^-_{p,r}(0, z'_{i+1}) \ll J^-_{p,r}(z_{n-2}, z_n) + H^-_{p,r}(0, z_{n-2}, z_n). \tag{3.15}$$

Thus, (3.15) holds for all $0 < r < \infty$.

From (3.15) we obtain

$$A_1 = \left(\sum_k \int_{x_{k-1}}^{x_k} u(x) \left(\int_x^{\infty} u(s)ds \right)^{\frac{\mu}{p}} \left(J^-_{p,r}(0, x) \right)^{\mu} dx \right)^{\frac{1}{\mu}}$$

$$\leq \left(\sum_k \int_{x_{k-1}}^{x_k} u(x) \left(\int_x^{\infty} u(s)ds \right)^{\frac{\mu}{p}} dx \left(J^-_{p,r}(0, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\ll \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(0, z'_{i+1}) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\ll \left(\sum_n \sum_{z_n \le x_k < z_{n+1}} \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{n-2}, z_n) + H^-_{p,r}(0, z_{n-2}, z_n) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\le \left(\sum_n \left(\sum_{z_n \le x_k < z_{n+1}} \int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{n-2}, z_n) + H^-_{p,r}(0, z_{n-2}, z_n) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\le \left(\sum_n \left(\int_{z_n}^{z_{n+2}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{n-2}, z_n) + H^-_{p,r}(0, z_{n-2}, z_n) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\ll \left(\sum_n \left(\int_{z_n}^{z_{n+2}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{n-2}, z_n) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$+ \left(\sum_n \left(\int_{z_n}^{z_{n+2}} u(x)dx \right)^{\frac{\mu}{q}} (H^-_{p,r}(0, z_{n-2}, z_n))^{\mu} \right)^{\frac{1}{\mu}} = A'_{11} + A'_{12}. \tag{3.16}$$

Now, we estimate A'_{11} and A'_{12}. We assume that $z_{2m} = t_m$ and $z_{2m+1} = y_m$. Then

$$A'_{11} \ll \left(\sum_{2m} \left(\int_{z_{2m}}^{z_{2(m+1)}} u(s)ds \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{2(m-1)}, z_{2m}) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$+ \left(\sum_{2m+1} \left(\int_{z_{2m+1}}^{z_{2(m+1)+1}} u(s)ds \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(z_{2m-1}, z_{2m+1}) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$= \left(\sum_m \left(\int_{t_m}^{t_{m+1}} u(s)ds \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(t_{m-1}, t_m) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$+ \left(\sum_m \left(\int_{y_m}^{y_{m+1}} u(s)ds \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(y_{m-1}, y_m) \right)^{\mu} \right)^{\frac{1}{\mu}} \ll A_{11}. \tag{3.17}$$

We similarly get that

$$A'_{12} \ll A_{12}. \tag{3.18}$$

From (3.16), (3.17) and (3.18) we have that $A_1 \ll A_{11} + A_{12}$. Taking into account (3.14), we derive from the last estimate that

$$A_1 \ll C'. \tag{3.19}$$

Sufficiency. Let $A_1 < \infty$. Let the sequence $\{z_k\}$ be such that

$$\left(\int_0^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}} = 2^k \text{ and } \bigcup_k [z_k, z_{k+1}) = I.$$

For $1 \leq r < \infty$, due to (2.5), we have

$$2^{k-1} = \left(\int_0^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}} - \left(\int_0^{z_{k-1}} \left(\int_t^{z_{k-1}} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}}$$

$$\leq \left(\int_{z_{k-1}}^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}} + \left(\int_0^{z_{k-1}} \left(\int_{z_{k-1}}^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}}. \tag{3.20}$$

For $0 < r < 1$ due to (2.6) we have

$$\int_0^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt - \int_0^{z_{k-1}} \left(\int_t^{z_{k-1}} K(s,t) f(s) ds \right)^r w(t)\, dt$$

$$\leq \int_{z_{k-1}}^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt + \int_0^{z_{k-1}} \left(\int_{z_{k-1}}^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt.$$

This also gives

$$2^{k-1} \ll \left(\int_{z_{k-1}}^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}} + \left(\int_0^{z_{k-1}} \left(\int_{z_{k-1}}^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{1}{r}}. \tag{3.21}$$

Using (3.20) and (3.21), we get

$$S \leq \left(\sum_k \int_{z_k}^{z_{k+1}} u(x)dx \left(\int_0^{z_{k+1}} \left(\int_t^{z_{k+1}} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}$$

$$\leq 4 \left(\sum_k 2^{q(k-1)} \int_{z_k}^{z_{k+1}} u(x)dx \right)^{\frac{1}{q}}$$

$$\ll \left(\sum_k \int_{z_k}^{z_{k+1}} u(x)dx \left(\int_{z_{k-1}}^{z_k} \left(\int_t^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}$$

$$+ \left(\sum_k \int_{z_k}^{z_{k+1}} u(x)dx \left(\int_0^{z_{k-1}} \left(\int_{z_{k-1}}^{z_k} K(s,t) f(s) ds \right)^r w(t)\, dt \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} = (\|Tf\|_q + \|\widetilde{T} f\|_q). \tag{3.22}$$

By Lemma 2.1 for $0 < q < p \leq \infty$, $p \geq 1$, we have

$$\|Tf\|_q \leq \left(\sum_k C_k^\mu \right)^{\frac{1}{\mu}} \|f\|_{p,v}, \tag{3.23}$$

$$\|\widetilde{T} f\|_q \leq \left(\sum_k \widetilde{C}_k^\mu \right)^{\frac{1}{\mu}} \|f\|_{p,v}, \tag{3.24}$$

where

$$C_k = \left(\int_{z_k}^{z_{k+1}} u(x)dx \right)^{\frac{1}{q}} J_{p,r}^-(z_{k-1}, z_k),$$

$$\widetilde{C}_k = \left(\int_{z_k}^{z_{k+1}} u(x)dx \right)^{\frac{1}{q}} H_{p,r}^-(0, z_{k-1}, z_k).$$

Thus, from (3.22), (3.23) and (3.24) we obtain

$$S \ll (A_{11} + A_{12})\|f\|_{p,v}. \tag{3.25}$$

Let $\{x_k\}_k$ be an arbitrary covering sequence. Using the estimate

$$J^-_{p,r}(0, x_k) \geq \max\{J^-_{p,r}(x_{k-1}, x_k), H^-_{p,r}(0, x_{k-1}, x_k)\},$$

we have

$$A_1 \geq \sup_{\{x_k\}} \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{\infty} u(s)ds \right)^{\frac{\mu}{p}} dx \left(J^-_{p,r}(0, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\geq \sup_{\{x_k\}} \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{x_{k+1}} u(s)ds \right)^{\frac{\mu}{p}} dx \left(J^-_{p,r}(0, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}}$$

$$\geq \left(\frac{q}{\mu}\right)^{\frac{1}{\mu}} \sup_{\{x_k\}} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(J^-_{p,r}(x_{k-1}, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}} \gg A_{11}$$

and

$$A_1 \gg \left(\frac{q}{\mu}\right)^{\frac{1}{\mu}} \sup_{\{x_k\}} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)dx \right)^{\frac{\mu}{q}} \left(H^-_{p,r}(0, x_{k-1}, x_k) \right)^{\mu} \right)^{\frac{1}{\mu}} \gg A_{12}.$$

The latter yields

$$A_{11} + A_{12} \ll A_1. \tag{3.26}$$

From (3.25) and (3.26) for $f \geq 0$, we have

$$S \ll A_1\|f\|_{p,v},$$

which gives that $C' \ll A_1$. This estimate together with (3.19) proves (3.5). The proof of the statement for inequality (3.2) is complete.

Let us turn to inequalities (3.3) and (3.4). Inequality (3.3) is a weighted Hardy type inequality of form (1.6), whose kernel satisfies condition (1.3). As mentioned in the introduction, it has been investigated in [1]. Thus, by the results of [1] inequality (3.3) holds if and only if $\max\{A_2, A_3\} < \infty$. Moreover,

$$C'' \approx \max\{A_2, A_3\}. \tag{3.27}$$

Inequality (3.4) is the standard Hardy inequality for the function $g(s) \equiv U(s)f(s)$, so that on its right-hand side we have $v(s)f^p(s) = U^{-p}(s)v(s)g^p(s)$. Since $(U^{-p}(s))^{1-p'} = U^{p'}(s)$, inequality (3.4) holds if and only if $A_4 < \infty$ (see, e.g., [5]). Moreover,

$$C''' \approx A_4. \tag{3.28}$$

Thus, inequalities (3.2), (3.3) and (3.4) simultaneously hold if and only if $A = \max\{A_1, A_2, A_3, A_4\} < \infty$. Moreover, from (3.5), (3.27) and (3.28) it follows that

$$A \approx C,$$

where C is the least constant in (1.1). Finally, the simultaneous validity of inequalities (3.2), (3.3) and (3.4) is equivalent to the validity of inequality (1.1). The proof of Theorem 3.1 is complete. □

In the introduction, we have already mentioned that the class of kernels satisfying (1.5) is contained in the class O_1^-, since condition (1.5) is a partial case of condition (1.3). Therefore, if we replace $U(\cdot)$ by 1 and $Q(\cdot,\cdot)$ by $K(\cdot,\cdot)$ in A_3 and A_4 and use condition (1.5) to combine these quantities, we get

$$A_{34} = \left(\int_0^\infty \left(\int_0^t u(s)\left(\int_0^s K^r(t,z)w(z)dz\right)^{\frac{q}{r}} ds\right)^{\frac{\mu}{q}} \left(\int_t^\infty v^{1-p'}(s)\,ds\right)^{\frac{\mu}{q'}} v^{1-p'}(t)dt\right)^{\frac{1}{\mu}}$$

such that $A_{34} \approx A_3 + A_4$. Hence, for kernels satisfying condition (1.5), Theorem 3.1 leads to the following corollary.

Corollary 3.1 *Let* $0 < r < \infty$, $0 < q < p < \infty$ *and* $p \geq 1$. *Let the kernel* $K(\cdot,\cdot)$ *satisfy condition* (1.5). *Then* (1.1) *holds if and only if* $A = \max\{A_1, A_2, A_{34}\} < \infty$. *Moreover,* $A \approx C$, *where* C *is the least constant in* (1.1).

Now, we present the main statement for inequality (1.2). Let

$$B_1 = \left(\int_0^\infty u(x) \left(\int_0^x u(s)ds \right)^{\frac{\mu}{p}} \left(J_{p,r}^+(x,\infty) \right)^\mu dx \right)^{\frac{1}{\mu}},$$

$$B_2 = \left(\int_0^\infty \left(\int_t^\infty u(s) \left(\int_s^\infty w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{p}} \left(\int_0^t K^{p'}(t,s) v^{1-p'}(s)\, ds \right)^{\frac{\mu}{p'}} \right.$$
$$\left. \times\, u(t) \left(\int_t^\infty w(s)ds \right)^{\frac{q}{r}} dt \right)^{\frac{1}{\mu}},$$

$$B_3 = \left(\int_0^\infty \left(\int_t^\infty u(s) \left(\int_s^\infty w(z)dz \right)^{\frac{q}{r}} R^q(s,t)ds \right)^{\frac{\mu}{q}} \right.$$
$$\left. \times \left(\int_0^t V^{p'}(s) v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} V^{p'}(t) v^{1-p'}(t)dt \right)^{\frac{1}{\mu}},$$

$$B_4 = \left(\int_0^\infty \left(\int_t^\infty u(s) \left(\int_s^\infty R^r(z,s) w(z)dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \right.$$
$$\left. \times \left(\int_0^t V^{p'}(s) v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} V^{p'}(t) v^{1-p'}(t)dt \right)^{\frac{1}{\mu}}.$$

Theorem 3.2 *Let* $0 < r < \infty$, $0 < q < p < \infty$ *and* $p \geq 1$. *Let the kernel* $K(\cdot,\cdot)$ *belong to the class* O_1^+. *Then* (1.2) *holds if and only if* $B = \max\{B_1, B_2, B_3, B_4\} < \infty$. *Moreover,* $B \approx C$, *where* C *is the least constant in* (1.2).

Since the kernels satisfying (1.5) belong to the class O_1^+, replacing $V(\cdot)$ by 1 and $R(\cdot,\cdot)$ by $K(\cdot,\cdot)$ in B_3 and B_4 and using condition (1.5) to combine these quantities, we get

$$B_{34} = \left(\int_0^\infty \left(\int_t^\infty u(s) \left(\int_s^\infty K^r(z,t) w(z) dz \right)^{\frac{q}{r}} ds \right)^{\frac{\mu}{q}} \left(\int_0^t v^{1-p'}(s)\, ds \right)^{\frac{\mu}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{\mu}}$$

such that $B_{34} \approx B_3 + B_4$. Therefore, for the kernels satisfying condition (1.5), Theorem 3.2 implies the following corollary.

Corollary 3.2 *Let* $0 < r < \infty$, $0 < q < p < \infty$ *and* $p \geq 1$. *Let the kernel* $K(\cdot,\cdot)$ *satisfy condition* (1.5). *Then* (1.2) *holds if and only if* $B = \max\{B_1, B_2, B_{34}\} < \infty$. *Moreover,* $B \approx C$, *where* C *is the least constant in* (1.2).

4 Application

Let $1 < r, p_1, p_2 < \infty$. Let $\frac{1}{p_1} + \frac{1}{p_1'} = 1$, $\frac{1}{p_2} + \frac{1}{p_2'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. We consider the following bilinear inequality

$$\left(\int_0^\infty w(x) \left(\int_0^x u(t) g(t) dt \right)^r \left(\int_0^x K(x,s) f(s)\, ds \right)^r dx \right)^{\frac{1}{r}} \leq C \left(\int_0^\infty v_1(x) g^{p_1}(x)\, dx \right)^{\frac{1}{p_1}} \left(\int_0^\infty v_2(x) f^{p_2}(x)\, dx \right)^{\frac{1}{p_2}}, \tag{4.1}$$

where u, v_1, v_2 and w are weight functions and the kernel $K(\cdot,\cdot) \geq 0$ is from the class O_1^+. Let $1 < r < p_1$ and $\frac{1}{q_1} = \frac{1}{r} - \frac{1}{p_1}$. For the fixed $f \in L_{p_2,v_2}$, inequality (4.1) is the standard Hardy inequality. Therefore, it is equivalent to the inequality

$$\left(\int_0^\infty \left(\int_z^\infty w(x) \left(\int_0^x K(x,s) f(s)\, ds \right)^r dx \right)^{\frac{q_1}{r}} \times \left(\int_0^z u^{p_1'}(t) v_1^{1-p_1'}(t) dt \right)^{\frac{q_1}{r'}} u^{p_1'}(z) v_1^{1-p_1'}(z) dz \right)^{\frac{1}{q_1}} \leq C_1 \left(\int_0^\infty v_2(x) f^{p_2}(x)\, dx \right)^{\frac{1}{p_2}}, \tag{4.2}$$

with, in addition, $C \approx C_1$, where C and C_1 are the least constants in (4.1) and (4.2), respectively. Inequality (4.2) can be rewritten in the form

$$\left(\int_0^\infty \hat{u}(z)\left(\int_z^\infty\left(\int_0^x K(x,s)f(s)\,ds\right)^r w(x)dx\right)^{\frac{q_1}{r}} dz\right)^{\frac{1}{q_1}} \leq C_1\left(\int_0^\infty v_2(x)f^{p_2}(x)\,dx\right)^{\frac{1}{p_2}}, \tag{4.3}$$

where

$$\hat{u}(z) = \left(\int_0^z u^{p_1'}(t)v_1^{1-p_1'}(t)dt\right)^{\frac{q_1}{r'}} u^{p_1'}(z)v_1^{1-p_1'}(z).$$

In the case $q_1 < p_2$ we can apply Theorem 3.2 to inequality (4.3). By the assumption for $r < p_1$, we have $q_1 = \frac{rp_1}{p_1-r}$. Therefore, for $q_1 < p_2$, we get $\frac{rp_1}{p_1-r} < p_2$ or $r < \frac{p_1p_2}{p_1+p_2}$. Hence, from Theorem 3.2, assuming $\frac{1}{\hat{\mu}} = \frac{1}{q_1} - \frac{1}{p_2}$, $\frac{1}{q_1} + \frac{1}{q_1'} = 1$ and remembering that

$$J^+_{p_2,r}(x,\infty) = \sup_{f\geq 0}\frac{\left(\int_x^\infty\left(\int_x^t K(t,s)f(s)\,ds\right)^r w(t)\,dt\right)^{\frac{1}{r}}}{\|f\|_{p_2,v_2,(x,\infty)}},$$

we get the following theorem.

Theorem 4.1 *Let* $1 < r < \frac{p_1p_2}{p_1+p_2}$ *and* $1 < p_1, p_2 < \infty$. *Let the kernel* $K(\cdot,\cdot)$ *belong to the class* O_1^+. *Then* (4.1) *holds if and only if* $\hat{B} = \max\{\hat{B}_1, \hat{B}_2, \hat{B}_3, \hat{B}_4\} < \infty$. *Moreover,* $\hat{B} \approx C$, *where* C *is the least constant in* (4.1) *and*

$$\hat{B}_1 = \left(\int_0^\infty \hat{u}(x)\left(\int_0^x \hat{u}(s)ds\right)^{\frac{\hat{\mu}}{p_2}}\left(J^+_{p_2,r}(x,\infty)\right)^{\hat{\mu}} dx\right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{B}_2 = \left(\int_0^\infty\left(\int_t^\infty \hat{u}(s)\left(\int_s^\infty w(z)dz\right)^{\frac{q_1}{r}} ds\right)^{\frac{\hat{\mu}}{p_2}}\left(\int_0^t K^{p_2'}(t,s)v_2^{1-p_2'}(s)\,ds\right)^{\frac{\hat{\mu}}{p_2'}} \times \hat{u}(t)\left(\int_t^\infty w(s)ds\right)^{\frac{q_1}{r}} dt\right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{B}_3 = \left(\int_0^\infty \left(\int_t^\infty \hat{u}(s) \left(\int_s^\infty w(z)dz \right)^{\frac{q_1}{r}} R^{q_1}(s,t)ds \right)^{\frac{\hat{\mu}}{q_1}} \right.$$

$$\left. \times \left(\int_0^t V^{p_2'}(s) v_2^{1-p_2'}(s)\, ds \right)^{\frac{\hat{\mu}}{q_1'}} V^{p_2'}(t) v_2^{1-p_2'}(t)dt \right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{B}_4 = \left(\int_0^\infty \left(\int_t^\infty \hat{u}(s) \left(\int_s^\infty R^r(z,s)w(z)dz \right)^{\frac{q_1}{r}} ds \right)^{\frac{\hat{\mu}}{q_1}} \right.$$

$$\left. \times \left(\int_0^t V^{p_2'}(s) v_2^{1-p_2'}(s)\, ds \right)^{\frac{\hat{\mu}}{q_1'}} V^{p_2'}(t) v_2^{1-p_2'}(t)dt \right)^{\frac{1}{\hat{\mu}}}.$$

For the kernel $K(\cdot,\cdot)$ from the class O_1^-, we consider the following bilinear inequality

$$\left(\int_0^\infty w(x) \left(\int_x^\infty u(t)g(t)dt \right)^r \left(\int_x^\infty K(s,x)f(s)\,ds \right)^r dx \right)^{\frac{1}{r}}$$

$$\leq C \left(\int_0^\infty v_1(x) g^{p_1}(x)\,dx \right)^{\frac{1}{p_1}} \left(\int_0^\infty v_2(x) f^{p_2}(x)\,dx \right)^{\frac{1}{p_2}}. \qquad (4.4)$$

In the same way as above, on the basis of Theorem 3.1, remembering that

$$J^-_{p_2,r}(0,x) = \sup_{f \geq 0} \frac{\left(\int_0^x \left(\int_t^x K(s,t)f(s)\,ds \right)^r w(t)\,dt \right)^{\frac{1}{r}}}{\|f\|_{p_2,v_2,(0,x)}}$$

and assuming

$$\hat{A}_1 = \left(\int_0^\infty \hat{u}(x) \left(\int_x^\infty \hat{u}(s)ds \right)^{\frac{\hat{\mu}}{p_2}} \left(J^-_{p_2,r}(0,x) \right)^{\hat{\mu}} dx \right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{A}_2 = \left(\int_0^\infty \left(\int_0^t \hat{u}(s) \left(\int_0^s w(z)dz \right)^{\frac{q_1}{r}} ds \right)^{\frac{\hat{\mu}}{p_2}} \left(\int_t^\infty K^{p_2'}(s,t) v_2^{1-p_2'}(s)\, ds \right)^{\frac{\hat{\mu}}{p_2'}} \right.$$

$$\left. \times\, \hat{u}(t) \left(\int_0^t w(s)ds \right)^{\frac{q_1}{r}} dt \right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{A}_3 = \left(\int_0^\infty \left(\int_0^t \hat{u}(s) \left(\int_0^s w(z)dz \right)^{\frac{q_1}{r}} Q^{q_1}(t,s)ds \right)^{\frac{\hat{\mu}}{q_1}} \right.$$

$$\left. \times \left(\int_t^\infty U^{p_2'}(s) v_2^{1-p_2'}(s)\, ds \right)^{\frac{\hat{\mu}}{q_1'}} U^{p_2'}(t) v_2^{1-p_2'}(t)dt \right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{A}_4 = \left(\int_0^\infty \left(\int_0^t \hat{u}(s) \left(\int_0^s Q^r(s,z) w(z)dz \right)^{\frac{q_1}{r}} ds \right)^{\frac{\hat{\mu}}{q_1}} \right.$$

$$\left. \times \left(\int_t^\infty U^{p_2'}(s) v_2^{1-p_2'}(s)\, ds \right)^{\frac{\hat{\mu}}{q_1'}} U^{p_2'}(t) v_2^{1-p_2'}(t)dt \right)^{\frac{1}{\hat{\mu}}},$$

$$\hat{u}(z) = \left(\int_z^\infty u^{p_1'}(t) v_1^{1-p_1'}(t)dt \right)^{\frac{q_1}{r'}} u^{p_1'}(z) v_1^{1-p_1'}(z),$$

we get an analogue of Theorem 4.1.

Theorem 4.2 *Let* $1 < r < \frac{p_1 p_2}{p_1+p_2}$, $1 < p_1, p_2 < \infty$. *Let the kernel* $K(\cdot,\cdot)$ *belong to the class* O_1^-. *Then* (4.4) *holds if and only if* $\hat{A} = \max\{\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4\} < \infty$. *Moreover,* $\hat{A} \approx C$, *where* C *is the least constant in* (4.4).

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A Neumann Series of Bessel Functions Representation for Solutions of the Radial Dirac System

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Abstract A new representation for a regular solution of the radial Dirac system of a special form is obtained. The solution is represented as a Neumann series of Bessel functions uniformly convergent with respect to the spectral parameter. For the coefficients of the series convenient for numerical computation recurrent integration formulas are given. Numerical examples are presented.

Keywords Radial Dirac system · Transmutation operator · Neumann series of Bessel functions · Dirac oscillator

1 Introduction

We consider the one-dimensional radial Dirac system of the form

$$\begin{pmatrix} \omega_1 & -\dfrac{d}{dr} + \dfrac{\kappa}{r} - p(r) \\ \dfrac{d}{dr} + \dfrac{\kappa}{r} - p(r) & \omega_2 \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0, \tag{1}$$

where p is absolutely continuous complex-valued function on some interval $[0, b]$, $\omega_1, \omega_2 \in \mathbb{C}$, κ is the spin-orbit quantum number, g and f are lower and upper radial wave functions, respectively.

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The system (1) with $\omega_1 = \frac{mc^2+V_s(r)+E-V_v(r)}{\hbar c}$, $\omega_2 = \frac{mc^2+V_s(r)-E+V_v(r)}{\hbar c}$ and $p(r) = \frac{V_{ps}(r)}{\hbar c}$ arises in quantum mechanics when studying the radial Dirac equation. Here V_s is a scalar potential, V_v is the time component of a vector potential and V_{ps} is a pseudoscalar or tensor potential, see, for example, formula 2.1 in [1], formulas (21)–(22) in [6], system (13)–(14) in [20] and (1) in [19]. The system (1) is a special case of the radial Dirac equation in the presence of a tensor or a pseudoscalar potential, and when both scalar and vector potentials are constant. System (1) appears in the recent Jackiw-Pi model of the bilayer graphene [8, 10]. There is a considerable number of publications in which Dirac-type equations (1) are examined, but mostly either exactly solvable potentials are sought (see, e.g., [1, 6, 20]), or an approximate solution is constructed for a concrete potential (see, for example, [7]).

In the present work for an arbitrary potential $p(r)$ we obtain an analytical representation for a regular solution of (1) in the form of a functional series with a simple recurrent integration procedure for calculating its coefficients. The series has the form of a Neumann series of Bessel functions (NSBF) (see, e.g., [2, 23, 24] for more information on NSBF). The following feature of the obtained NSBF representation makes it especially interesting. Its partial sums admit spectral parameter independent error estimates, which guarantee equally accurate approximations of exact solutions both for small and for large values of the spectral parameter. More precisely, when the coupling constants coincide, $\omega_1 = \omega_2$, the estimates are independent of their values, while in the case $\omega_1 \neq \omega_2$ the estimates involve the factor $\left|\sqrt{\omega_2/\omega_1}\right|$, and thus depends on how much the coupling constants differ from each other.

The NSBF representations for solutions of Sturm-Liouville type equations proved to be useful for solving both direct and inverse spectral problems [4, 9, 11–18]. In [15] an NSBF representation was obtained for solutions of the one-dimensional stationary Schrödinger equation. In [13] that result was generalized onto the case of an arbitrary regular Sturm-Liouville equation. Recently in [14] an NSBF representation was obtained for regular solutions of perturbed Bessel equations. In [4, 9, 11, 12, 18] NSBF representations for solutions were used for solving inverse spectral problems.

In the present paper an NSBF representation for regular solutions of (1) is obtained by transforming the system into a couple of perturbed Bessel equations and using results from [14]. We prove the above mentioned error estimates for partial sums of the series representations and discuss the numerical implementation of the NSBF representation. We show that the spectral parameter independent error estimates are evident, indeed, in numerical experiments and show the applicability of the obtained NSBF representation for solving spectral problems for (1).

The paper is organized as follows. In Sect. 2 we obtain the NSBF representation for the regular solution of (1) and prove a convergence result for the approximate solution. In Sect. 3 we summarize the steps required for numerical solution of Eq. (1) and related spectral problems using the proposed representation and show numerical results for the Dirac oscillator.

2 A Representation of the Solution

Consider the following two component radial Dirac system

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + p(r)\right) f = \omega_1 g, \tag{2}$$

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - p(r)\right) g = -\omega_2 f, \tag{3}$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $\kappa \geq \frac{1}{2}$, and $p \in \mathrm{AC}[0, b]$ is in general a complex valued function.

Definition 1 A pair of functions (f_κ, g_κ) is called a **regular solution** of the system (2)–(3) if it satisfies the system as well as the following asymptotic conditions

$$f_\kappa(r) \sim C_f r^\kappa, \qquad g_\kappa(r) \sim C_g r^{\kappa+1}, \quad \text{when } r \to 0$$

where C_f and C_g are some constants.

Together with the potential p the following functions will be considered

$$q_1(r) = p'(r) - \frac{2\kappa}{r} p(r) + p^2(r) \quad \text{and} \quad q_2(r) = -p'(r) - \frac{2\kappa}{r} p(r) + p^2(r). \tag{4}$$

Note that if (f_κ, g_κ) is a regular solution of (2)–(3), the functions f_κ and g_κ are necessarily regular solutions of the equations

$$-f'' + \left(\frac{\kappa(\kappa-1)}{r^2} + q_2(r)\right) f = \omega^2 f, \qquad r \in (0, b] \tag{5}$$

and

$$-g'' + \left(\frac{\kappa(\kappa+1)}{r^2} + q_1(r)\right) g = \omega^2 g, \qquad r \in (0, b], \tag{6}$$

respectively with $\omega^2 = \omega_1\omega_2$.

Note that for $p \in \mathrm{AC}[0, b]$ both potentials q_1 and q_2 are such that $r^\varepsilon q_{1,2}(r) \in L_1(0, b)$ for any small $\varepsilon > 0$, hence the conditions on the potential from [14] are satisfied. In order to apply the results of [14] to Eqs. (5) and (6) we need two solutions of the equations

$$-f_0'' + \left(\frac{\kappa(\kappa-1)}{r^2} + q_2(r)\right) f_0 = 0 \quad \text{and} \quad -g_0'' + \left(\frac{\kappa(\kappa+1)}{r^2} + q_1(r)\right) g_0 = 0, \tag{7}$$

non-vanishing on $(0, b]$ and satisfying the following asymptotics at zero

$$f_0(r) \sim r^{\kappa} \qquad \text{and} \qquad g_0(r) \sim r^{\kappa+1}, \quad \text{when } r \to 0. \tag{8}$$

The solution f_0 can be directly obtained by taking $\omega_1 = 0$ in (2) and is given by

$$f_0(r) = r^{\kappa} \exp\left(-\int_0^r p(s)\,ds\right). \tag{9}$$

To obtain the solution g_0 note that the function $1/f_0$ is a solution of (3) with $\omega_2 = 0$, and hence it is the solution of the second equation in (7) satisfying the asymptotic relation $1/f_0(r) \sim r^{-\kappa}$, $r \to 0$. A second linearly independent solution of (3) with $\omega_2 = 0$ can be chosen in the form $\frac{C}{f_0(r)} \int_0^r f_0^2(s)\,ds$. Chosing $C = 2\kappa + 1$, i.e., taking

$$g_0(r) = (2\kappa+1) r^{-\kappa} \exp\left(\int_0^r p(s)\,ds\right) \int_0^r t^{2\kappa} \exp\left(-2\int_0^t p(s)\,ds\right) dt \tag{10}$$

we obtain the solution of the second equation in (7) satisfying (8).

It can be seen from (9) and (10) that the derivatives of the solutions f_0 and g_0 are given by

$$f_0'(r) = \left(\frac{\kappa}{r} - p(r)\right) f_0(r) \quad \text{and} \quad g_0'(r) = \left(p - \frac{\kappa}{r}\right) g_0(r) + (2\kappa+1) f_0(r). \tag{11}$$

The solution f_0 given by (9) is always non-vanishing on $(0, b]$. The solution g_0 given by (10) is definitely non-vanishing for real valued potentials p but may possess zeros for complex valued functions p. For this reason the following **assumption (A)** concerning the potential p will be made throughout this paper. We assume that the second equation in (7) admits a regular solution g_0 which does not vanish on $(0, b]$. This assumption does not imply any additional restriction on p for the following reason. In [14, Proposition B.1] we show that one can always choose such a constant c that the second equation in (7) with the potential $\widetilde{q}_1(x) := q_1(x) + c$ possesses a non-vanishing solution. Equation (6) can then be written as $-g'' + \left(\frac{\kappa(\kappa+1)}{r^2} + \widetilde{q}_1(r)\right) g = \widetilde{\omega}^2 g$ with $\widetilde{\omega}^2 = \omega^2 + c$ which leads to the same results and conclusions as below. Also, one may construct the regular solution of the system (2)–(3) using only the solution f and its derivative (see Remark 2), however loosing an attractive possibility to verify the accuracy of approximate solutions (see Remark 1 and Sect. 3.1).

Thus, without loss of generality we assume that the regular solution g_0 of the second equation in (7) satisfying (8) does not have zeros in $(0, b]$.

Theorem 1 *Let* $p \in \mathrm{AC}[0, b]$, *and the assumption (A) be fulfilled. Then a regular solution of the system* (2)–(3) *satisfying the asymptotic relations (here* $\omega^2 = \omega_1\omega_2$*)*

$$f_\kappa(r) \sim -\frac{\omega^{\kappa+1}}{\omega_2} d(\kappa - 1) r^\kappa \qquad \text{and} \qquad g_\kappa(r) \sim \omega^{\kappa+1} d(\kappa) r^{\kappa+1}, \quad r \to 0,$$

has the form

$$f_\kappa(r) = -\frac{\omega^2}{\omega_2} r j_{\kappa-1}(\omega r) - \frac{\omega}{\omega_2} \sum_{n=0}^{\infty} \beta_{2,n}(r) j_{\kappa+2n}(\omega r), \tag{12}$$

$$g_\kappa(r) = \omega r j_\kappa(\omega r) + \sum_{n=0}^{\infty} \beta_{1,n}(r) j_{\kappa+2n+1}(\omega r), \tag{13}$$

where $j_\nu(r) = \sqrt{\frac{\pi}{2r}} J_{\nu+\frac{1}{2}}(r)$ *is the spherical Bessel function of the first kind,*

$$d(\kappa) := \frac{\sqrt{\pi}}{2^{\kappa+1}\Gamma(\kappa + 3/2)}.$$

Denote $u_1 := g_0$ *and* $u_2 := f_0$, *where* f_0 *and* g_0 *are solutions of* (7) *satisfying* (8). *Then the functions* $\beta_{j,n}$, $j \in \{1, 2\}$, $n \ge 0$, *can be found from the recurrent formulas*

$$\beta_{j,0}(r) = (2\kappa - 2j + 5)\left(\frac{u_j(r)}{r^{\kappa+2-j}} - 1\right), \qquad j \in \{1, 2\}, \tag{14}$$

$$\beta_{j,n}(r) = -\frac{4n + 2\kappa - 2j + 5}{4n + 2\kappa - 2j + 1}\left[\beta_{j,n-1}(r) + \frac{2(4n + 2\kappa - 2j + 3)u_j(r)\theta_{j,n}(r)}{r^{2n+\kappa-j+2}}\right], \tag{15}$$

$$\theta_{j,n}(r) = \int_0^r \frac{\eta_{j,n}(t) - t^{2n+\kappa-j+1}\beta_{j,n-1}(t)u_j(t)}{u_j^2(t)} dt, \tag{16}$$

$$\eta_{j,n}(r) = \int_0^r \left[t u_j'(t) + (2n + \kappa - j + 1)u_j(t)\right] t^{2n+\kappa-j} \beta_{j,n-1}(t) dt. \tag{17}$$

Proof From (3) we have that

$$f_\kappa = -\frac{1}{\omega_2}\left(g_\kappa' + \kappa g_\kappa/r - p g_\kappa\right).$$

Hence if $g_\kappa(r) \sim d(\kappa)(\omega r)^{\kappa+1}$, $g_\kappa'(r) \sim (\kappa + 1)\omega d(\kappa)(\omega r)^\kappa$ when $r \to 0$, then $f_\kappa(r) \sim -\frac{\omega}{\omega_2}(2\kappa + 1) d(\kappa)(\omega r)^\kappa$. Now we apply Theorem 5.2 from [14] in order to find out that a solution g_κ of (6) satisfying the relation $g_\kappa(r) \sim d(\kappa)(\omega r)^{\kappa+1}$, has the form (13), meanwhile a solution $\widetilde{f}_\kappa$ of (5) satisfying the relation $\widetilde{f}_\kappa(r) \sim$

$d(\kappa-1)(\omega r)^\kappa$, when $r \to 0$, can be written as

$$\widetilde{f}_\kappa(r) = \omega r j_{\kappa-1}(\omega r) + \sum_{n=0}^{\infty} \beta_{2,n}(r) j_{\kappa+2n}(\omega r).$$

And $f_\kappa = -\frac{\omega}{\omega_2}(2\kappa+1)\frac{d(\kappa)}{d(\kappa-1)}\widetilde{f}_\kappa = -\frac{\omega}{\omega_2}\widetilde{f}_\kappa$, which leads to (12). □

For practical use of the representation (12) and (13) the estimates of the difference between the exact solution and its approximation defined as

$$f_{\kappa,N}(r) = --\frac{\omega^2}{\omega_2} r j_{\kappa-1}(\omega r) - \frac{\omega}{\omega_2}\sum_{n=0}^{N} \beta_{2,n}(r) j_{\kappa+2n}(\omega r), \tag{18}$$

$$g_{\kappa,N}(r) = \omega r j_\kappa(\omega r) + \sum_{n=0}^{N} \beta_{1,n}(r) j_{\kappa+2n+1}(\omega r) \tag{19}$$

are needed. From Theorem 1 using [14, Theorem 5.2] the following result follows immediately.

Proposition 1 *Under the conditions of Theorem 1 the following inequalities are valid*

$$|g_\kappa(r) - g_{\kappa,N}(r)| \le \sqrt{r}\varepsilon_N(r) \quad \textit{and} \quad |f_\kappa(r) - f_{\kappa,N}(r)| \le \left|\frac{\omega}{\omega_2}\right| \sqrt{r}\varepsilon_N(r)$$

for all $\omega \in \mathbb{R}$, $\omega_2 \in \mathbb{R}\setminus\{0\}$, *where* ε_N *is a nonnegative function independent on* ω_1 *and* ω_2, *such that* $\max_{r\in[0,b]} \varepsilon_N(r) \to 0$ *as* $N \to \infty$. *Similar result holds for* ω *belonging to a strip* $|\operatorname{Im}\omega| \le C$, *with addition of a multiplicative constant dependent only on the value of* C.

Suppose additionally that $p \in W_1^{2k}[0,b]$, $p(0) = 0$ *and* $\frac{p(r)}{r} \in W_1^{2k-1}[0,b]$ *for some* $k \in \mathbb{N}$. *Here* $W_1^k[0,b]$ *denotes the class of functions having k derivatives, the last one belonging to* $L_1[0,b]$ *space, and* $p(r)/r$ *is assumed to have a finite limit as* $r \to 0$. *Then there exists a constant c, such that*

$$\varepsilon_N(r) \le \frac{c}{N^k}, \qquad 2N > \kappa + k + 1.$$

The independence of ε_N of ω implies that the approximate solution $(f_{\kappa,N}, g_{\kappa,N})$ remains good even for very large values of $|\operatorname{Re}\omega|$.

Remark 1 Even though the derivatives of the regular solutions (f_κ, g_κ) are readily available from (2) and (3), an independent representation (useful, e.g., for verification of accuracy of approximate solutions) for them can be obtained from [14, Theorem 6.3]. Under the conditions and notations of Theorem 1 let $Q_j(r) :=$

$\int_0^r q_j(t)\,dt,\ j \in \{1,2\}$ and let the functions $\gamma_{j,n},\ j \in \{1,2\},\ n \ge 0$ be defined as

$$\gamma_{j,0}(r) = (2\kappa - 2j + 5)\left(\frac{u_j'(r)}{r^{\kappa-j+2}} - \frac{\kappa - j + 2}{r} - \frac{Q_j(r)}{2}\right),$$

$$\gamma_{j,n}(r) = -\frac{4n + 2\kappa - 2j + 5}{4n + 2\kappa - 2j + 1}\Bigg[\gamma_{j,n-1}(r) + (4n + 2\kappa - 2j + 3)\left(\frac{2u_j'(r)\theta_{j,n}(r)}{r^{2n+\kappa-j+2}} + \frac{2\eta_{j,n}(r)}{u_j(r)r^{2n+\kappa-j+2}} - \frac{\beta_{j,n-1}(r)}{r}\right)\Bigg].$$

Then

$$f_\kappa'(r) = -\frac{\omega^3}{\omega_2} r j_{\kappa-2}(\omega r) - \left(\frac{rQ_2(r)}{2} - \kappa + 1\right)\frac{\omega^2}{\omega_2} j_{\kappa-1}(\omega r) - \frac{\omega}{\omega_2}\sum_{n=0}^{\infty}\gamma_{2,n}(r) j_{2n+\kappa}(\omega r), \tag{20}$$

$$g_\kappa'(r) = \omega^2 r j_{\kappa-1}(\omega r) + \left(\frac{rQ_1(r)}{2} - \kappa\right)\omega j_\kappa(\omega r) + \sum_{n=0}^{\infty}\gamma_{1,n}(r) j_{2n+\kappa+1}(\omega r). \tag{21}$$

Remark 2 The regular solution of the system (2)–(3) can be obtained using only the particular solution f_0 and related functions $\beta_{2,n}$ and $\gamma_{2,n},\ n \ge 0$, without the need of the functions g_0, $\beta_{1,n}$ and $\gamma_{1,n}$ at all. Indeed,

$$g_\kappa = \frac{1}{\omega_1}\left(f_\kappa' - \frac{\kappa}{r}f_\kappa + p(r)f_\kappa\right) \qquad \text{and} \qquad g_\kappa' = -\omega_2 f_\kappa - \frac{\kappa}{r}g_\kappa + p(r)g_\kappa,$$

and the representations for f_κ and f_κ' are given by (12) and (20).

3 Numerical Results

3.1 Description of the Algorithm

A numerical method based on the representation (12)–(13) of the regular solution of the system (2)–(3) consists in the following steps.

1. Compute a pair (f_0, g_0) of regular solutions of (7) satisfying (8) using (9) and (10). Compute also their derivatives (f_0', g_0'). In the case that the coefficient p is complex valued, check if the assumption (A) holds, and if not, proceed as

described in Remark 2 or look for a spectral shift (see Appendix B in [14, (8.1)]) such that a pair of solutions (f_0, g_0) becomes non-vanishing.

2. Compute the coefficients $\beta_{j,n}$, $j \in \{1,2\}$, $n \in \{0,1,\dots,N\}$ using the formulas (14)–(17). Note that the coefficients $\beta_{j,n}$ satisfy [14]

$$\sum_{n=0}^{\infty}(-1)^n\beta_{j,n}(r) = \frac{rQ_j(r)}{2}, \qquad r \in [0,b],\ j \in \{1,2\} \tag{22}$$

and decay to zero (however, not necessary monotonously) as $n \to \infty$. The equality (22) can be used to estimate an optimal number of the coefficients N, as a value where the truncated sums cease to decrease when N increases.

3. Compute approximate solutions $f_{\kappa,N}$ and $g_{\kappa,N}$ using (12) and (13).
4. The accuracy of the obtained approximations can be estimated by calculating the discrepancies

$$f'_{\kappa,N} - \frac{\kappa}{r}f_{\kappa,N} + p(r)f_{\kappa,N} - \omega_1 g_{\kappa,N} \quad \text{and} \quad g'_{\kappa,N} + \frac{\kappa}{r}f_{\kappa,N} - p(r)f_{\kappa,N} + \omega_2 g_{\kappa,N}, \tag{23}$$

where $f'_{\kappa,N}$ and $g'_{\kappa,N}$ are computed from the truncated series (20) and (21).

We refer the reader to [15, 17] for implementation details of the proposed algorithm.

3.2 The Dirac Oscillator

As a test example for the proposed algorithm we consider the Dirac oscillator [3, 5, 21].

The large radial component $F(r)$ and the small radial component $G(r)$ of the Dirac wave function are solutions of the following system

$$\left(-\frac{d}{dr} + \left(\frac{\varepsilon(j+1/2)}{r} + m\omega r\right)\right)G(r) = (E-m)F(r), \tag{24}$$

$$\left(\frac{d}{dr} + \left(\frac{\varepsilon(j+1/2)}{r} + m\omega r\right)\right)F(r) = (E+m)G(r), \tag{25}$$

where j is the total angular momentum quantum number, $\varepsilon = \pm 1$, m is the mass of the particle and ω is the frequency. Note that the number

$$l := j + \frac{\varepsilon}{2}$$

is the orbital momentum quantum number and is an integer number, i.e., the fractional part of j is always equal to $1/2$.

The energy spectrum can be obtained [3] from

$$E^2 - m^2 = m\omega\big(2(N+1) + \varepsilon(2j+1)\big)$$

for the positive-energy states, and from

$$E^2 - m^2 = m\omega\big(2(N+2) + \varepsilon(2j+1)\big)$$

for the negative-energy states. Here $N = 2n + l$, $n = 0, 1, 2, \ldots$, is the principal quantum number. The corresponding eigenfunctions are given by

$$F_{n,l}(r) = A\left(r\sqrt{m\omega}\right)^{l+1} \exp(-m\omega r^2/2) L_n^{l+1/2}(m\omega r^2), \tag{26}$$

$$G_{n,l-\varepsilon}(r) = A\left(r\sqrt{m\omega}\right)^{l+1-\varepsilon} \exp(-m\omega r^2/2) L_{n+\varepsilon/2-1/2}^{l-\varepsilon+1/2}(m\omega r^2), \tag{27}$$

where $L_k^s(x)$ is an associated Laguerre polynomial.

The system (24)–(25) is of the type considered in this paper. Since the potential of the problem is increasing, we approximated the semiaxis spectral problem (of finding the values of E for which the regular solution belongs to $L_2(0,\infty)$) by truncating the potential and considering the Dirichlet boundary condition. For any non-trivial solution both f_κ and g_κ can not be equal to zero at one point, so we choose the function f_κ and considered

$$f_\kappa(B) = 0$$

as the boundary condition for the problem truncated onto $[0, B]$ segment. We refer the reader to [22, Section 7.4] for additional details on the convergence of the eigenvalues of truncated problems to the exact ones.

All the computations were performed in machine precision using Matlab 2017. We refer the reader to [17] for the details of the numerical realization. We considered two sets of parameters, having $\varepsilon = \pm 1$ and in both $j = 5/2$ and $m = \omega = 1$. For $\varepsilon = 1$ the corresponding potential is $p = -m\omega r$ in the notations of (2) and (3), and for $\varepsilon = -1$ the corresponding potential is $p = m\omega r$. In the first case the corresponding particular solution f_0 given by (9) is rapidly increasing, for the second case f_0 is rapidly decreasing. We decided to not implement interval subdivision techniques, and utilize the proposed representation directly to illustrate that even straightforward implementation can deliver highly accurate results.

First, we compare approximate solutions with the exact ones for the case $\varepsilon = -1$ for three eigenvalues $E^2 - m^2 \in \{4, 40, 500\}$, corresponding to $n \in \{1, 10, 125\}$. In terms of the system (2) and (3) we have taken $\omega_1 = 2$, $\omega_2 \in \{2, 20, 250\}$.

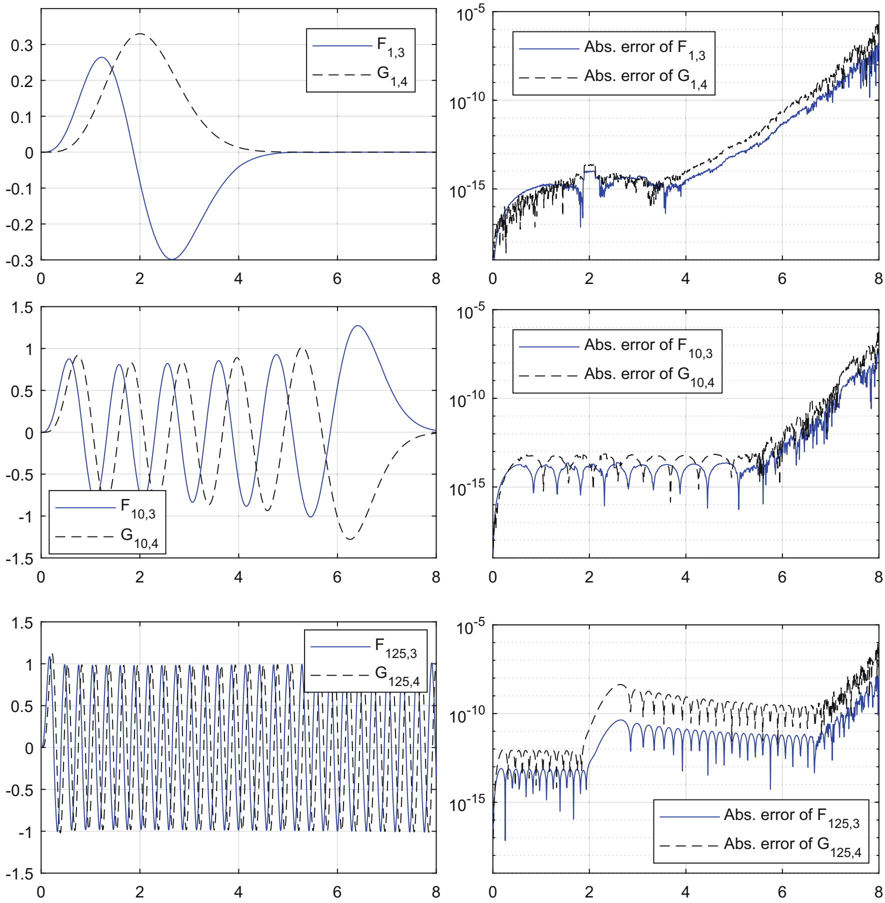

Fig. 1 On the left: components $F_{n,3}$ and $G_{n,4}$ of the eigenfunction of the Dirac oscillator for $n \in \{1, 10, 125\}$ with the parameters $j = 5/2$, $\varepsilon = -1$ and $m = \omega = 1$. On the right: absolute errors of these components

On Fig. 1 we present the solutions and corresponding absolute errors. As one can observe, the error does not increase for large values of ω (corresponding to higher index eigenfunctions) and only increases for large values of r due to machine precision limitations. The approach presented in Remark 2 delivered a more accurate solution component G. This is due to the error near $r = 0$ in the particular solution g_0 computed by (10). For that reason on the plots we present the absolute errors obtained with the aid of the formula from Remark 2.

Approximate solution of the spectral problem requires truncating the interval. A larger interval allows one to compute more eigenvalues and more accurately. However this leads to larger errors in all the coefficients $\beta_{j,n}$ computed, due to machine precision limitations. The equality (22) can be utilized to estimate automatically a truncation parameter B. We took the segment $[0, 20]$, represented all the functions involved by 100001 uniformly spaced on $[0, 20]$ points and computed 100 coefficients $\beta_{2,n}$. After that we checked for each r the convergence of partial sums in (22) to $rQ_2(r)/2$. Due to machine precision limitations, the difference between $\sum_{n=0}^{N}(-1)^n\beta_{2,n}(r)$ and $rQ_2(r)/2$ reaches a plateau at some particular value of $N(r)$, meaning that the difference essentially does not decrease anymore when N increases. Let $e_2(r) := \left|\sum_{n=0}^{N(r)}(-1)^n\beta_{2,n}(r) - rQ_2(r)/2\right|$. We chose as the truncation parameter B the value $0.99 \cdot r_0$, where r_0 is such that for all $r < r_0$ the value $e_2(r)$ is small in comparison with $rQ_2(r)$ (to be more precise, $e_2(r) < |rQ_2(r)|/100$), but for $r > r_0$ the error $e_2(r)$ can be larger than $|rQ_2(r)|/100$. As a result, $B = 7.4786$ was chosen for $\varepsilon = 1$, and $B = 9.0168$ was chosen for $\varepsilon = -1$. See Fig. 2 illustrating this procedure.

In Table 1 we present approximate eigenvalues $E^2 - m^2$ for the parameters $\varepsilon = \pm 1$, $j = 5/2$, $m = \omega = 1$ computed on the truncated intervals $[0, B]$.

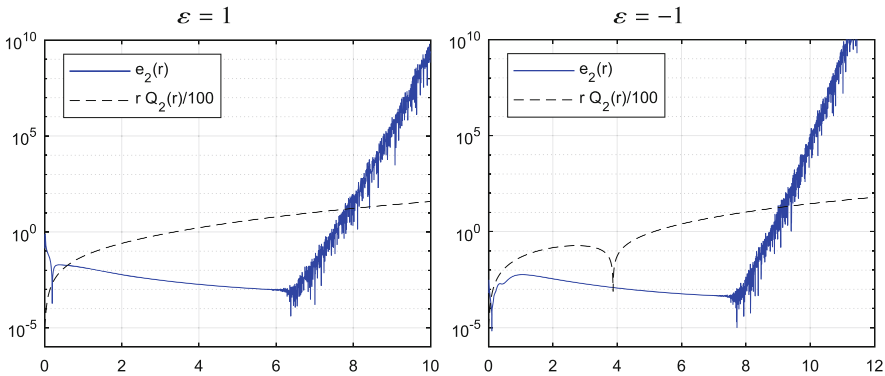

Fig. 2 Application of formula (22) to determine optimal truncation interval for the Dirac oscillator with the parameters $j = 5/2$ and $m = \omega = 1$. On both plots black dashed line shows the value of $|rQ_2(r)/100|$, and the blue solid line shows $e_2(r) := \min_{N(r)\le 100}\left|\sum_{n=0}^{N(r)}(-1)^n\beta_{2,n}(r) - rQ_2(r)/2\right|$ for computed coefficients $\beta_{2,n}$

Table 1 The eigenvalues for the Dirac oscillator problem (24), (25) truncated onto the segment $[0, B]$

$\varepsilon = 1$, on $[0, 7.4786]$

Exact $E^2 - m^2$	Approximate
14	13.999999999987
18	17.9999999998183
22	21.9999999982828
26	25.9999999642871
30	29.9999994276682
34	33.9999942694057
38	37.9999616653564
42	42.0001005681044
46	46.000048366715
50	50.0125330275323
54	54.0378367431326
58	58.2112436119225
Number of $\beta_{2,n}$ used	24

$\varepsilon = -1$, on $[0, 9.0168]$

Exact $E^2 - m^2$	Approximate
0	$7.8 \cdot 10^{-32}$
4	3.99999999999999
8	7.99999999999994
12	12.0000000000002
16	16.0000000000035
20	20.0000000000206
24	24.0000000000766
28	27.9999999997443
32	32.0000000015208
36	35.9999999944383
40	39.9999997918537
44	44.0000015951936
48	48.0000051599966
Number of $\beta_{2,n}$ used	29

Parameters used: $\varepsilon = \pm 1$, $j = 5/2$, $m = \omega = 1$. The last line shows the number of terms used in approximate solution (18)

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Some Algebraic Aspects of Boolean Valued Analysis

Anatoly G. Kusraev

Abstract The article deals with a Boolean valued approach to some algebraic problems arising from functional analysis. The main results are as follows. (1) A universally complete vector lattice without locally one-dimensional bands contains an infinite direct sum of order dense sublattices each of which is a band preserving linear isomorphic (but not lattice isomorphic) copy of the whole lattice. (2) Every separated injective module over a semiprime rationally complete commutative ring admits a direct sum decomposition with homogeneous summands. (3) A semiprime rationally complete commutative ring properly embedded in a ring with projections K is a homogeneity ring of an additive mapping between appropriate K-modules.

Keywords Boolean valued analysis · Universally complete vector lattice · Injective module · Commutative ring · K-module

1 Introduction

The *Boolean valued approach* is a machinery of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras. As these models, one usually takes the classical sets in the shape of the *von Neumann universe* $\mathbb{V}$ and a properly truncated *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ in which the conventional set-theoretic concepts and constructions acquire nonstandard interpretations.

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A general scheme of applying the Boolean valued approach is as follows, see [16, 17]. Assume that $\mathbf{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects, *external* and *internal*, respectively. Suppose we are able to prove the following

Boolean Valued Representation Result: Every external $X \in \mathbf{X}$ embeds into a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ becoming an internal object $\mathscr{X} \in \mathbb{X}$.

Boolean Valued Transfer Principle then tells us that every theorem about $\mathscr{X}$ within Zermelo–Fraenkel set theory with choice ZFC has its counterpart for the original object X interpreted as a Boolean valued object $\mathscr{X}$.

Boolean Valued Machinery enables us to perform some translation of theorems from $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ making use of appropriate general operations and the principles of Boolean valued models.

The paper deals with a Boolean-valued approach to some algebraic problems arising from functional analysis. Section 2 collects some Boolean valued requisites. The main result of Sect. 3 states that a universally complete vector lattice without locally one-dimensional bands contains an infinite direct sum of order dense sublattices each of which is a band preserving linearly isomorphic (but not lattice isomorphic) copy of the whole lattice. This problem is related to band preserving linear operators in vector lattices, see Abramovich and Kitover [2], Kusraev and Kutateladze [17, Chap. 4]. In Sect. 4 it is proved that every separated injective module over a semiprime rationally complete commutative ring admits a direct sum decomposition with homogeneous summands. Problems of this kind arise in the theory of operator algebras, see Chilin and Karimov [5], Ozawa [20]. It is proved in Sect. 5 that a semiprime rationally complete commutative ring properly embedded in a ring with projections K is a homogeneity ring of an additive mapping between appropriate K-modules. This result relates to functional equations, see Wilansky [23].

The reader can find the necessary information on the theory of vector lattices in [1, 3]; Boolean valued analysis, in [4, 15, 16]; rings and modules, in [19]. Troughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. We let := denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

2 Boolean Valued Requisites

Let $\mathbb{B}$ be a complete Boolean algebra. Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ is defined by recursion on α with α running through the class of all ordinals:

$$\mathbb{V}_\alpha^{(\mathbb{B})} = \left\{x : (\exists\,\beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B},\ \mathrm{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})}\right\},$$

$$\mathbb{V}^{(\mathbb{B})} := \bigcup\nolimits_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})} \quad (\mathrm{On}\ \text{is the class of all ordinals}).$$

For making statements about $\mathbb{V}^{(\mathbb{B})}$ take a formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of *Zermelo–Fraenkel set theory* with choice ($\equiv$ ZFC) and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. There is a natural way of assigning to each such statement an element $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$ which acts as the *'Boolean truth-value'* of $\varphi(x_1, \dots, x_n)$ in the universe $\mathbb{V}^{(\mathbb{B})}$. We say that the statement $\varphi(x_1, \dots, x_n)$ *is valid within* $\mathbb{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$.

Theorem 2.1 (Transfer Principle) *All theorems of Zermelo–Fraenkel set theory with choice are true within* $\mathbb{V}^{(\mathbb{B})}$. *More precisely, if* $\varphi(u_1, \dots, u_n)$ *is a theorem of* ZFC *then*

$$(\forall x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})})\ [\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$$

is also a theorem of ZFC.

Given an arbitrary $X \in \mathbb{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ as the set $X{\downarrow} := \{x \in \mathbb{V}^{(\mathbb{B})} : [\![x \in X]\!] = \mathbb{1}\}$. Assume that $X, Y, f, P \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$ and $[\![P \subset X^2]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y and P is a binary relation on X within $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ $(x \in X{\downarrow})$ and $P{\downarrow}$ is a unique binary relation on $X{\downarrow}$ with $(x_1, x_2) \in P{\downarrow} \iff [\![(x_1, x_2) \in P]\!] = \mathbb{1}$. The *ascent* is a transformation acting in the reverse direction. i.e., sending any subset $X \subset \mathbb{V}^{(\mathbb{B})}$ into an element of $\mathbb{V}^{(\mathbb{B})}$. Along with these transformation there is the *canonical embedding* $X \mapsto X^{\wedge}$ of the class of standard sets ($\equiv$ *von Neumann universe*) $\mathbb{V}$ into a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ see [4, 15].

Let $\mathscr{R}$ stands for the field of reals within $\mathbb{V}^{(\mathbb{B})}$ i. e., $\mathscr{R} := (\mathrm{R}, \oplus, \odot, 0, 1, \leqq)$ and $[\![\varphi(\mathscr{R})]\!] = \mathbb{1}$, where $\varphi(\mathscr{R})$ is the conjunction of axioms of the reals. Consider the descent $\mathbf{R} := \mathscr{R}{\downarrow}$ of the algebraic structure $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$. In other words, $\mathbf{R} := (\mathrm{R}{\downarrow}, \oplus{\downarrow}, \odot{\downarrow}, \leqq{\downarrow}, 0, 1)$ is considered as the descent $\mathrm{R}{\downarrow}$ of the underlying set R together with the descended operations $\oplus{\downarrow}$ and $\odot{\downarrow}$ and order relation $\leqq{\downarrow}$ of the structure $\mathscr{R}$. The following fundamental result due to Gordon [7] tells us that the interpretation of reals (complexes) in a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ is a universally complete real (complex) vector lattice with the Boolean algebra of band projections isomorphic to $\mathbb{B}$.

Theorem 2.2 (Gordon) *The algebraic structure* $\mathscr{R}{\downarrow}$ *with the descended operations and order relation is a universally complete real vector lattice and a semiprime* f*-algebra with a ring and order unit* $\mathbb{1} := 1^{\wedge}$. *Moreover,* $\mathbb{R}^{\wedge}$ *is a dense subfield of* $\mathscr{R}$ *within* $\mathbb{V}^{(\mathbb{B})}$.

Proof A detailed proof can be found in [17, Sections 2.2, 2.3]. □

Gordon's theorem 2.2 raises the question of when $\mathbb{R}^{\wedge}$ and $\mathscr{R}$ coincide within $\mathbb{V}^{(\mathbb{B})}$. The answer was obtained by Gutman [10] in terms of the σ-distributivity of the Boolean algebra $\mathbb{B}$. A σ-complete Boolean algebra $\mathbb{B}$ is said to be σ*-distributive* if, for any double sequence $(b^n_m)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$, the equality holds (cf. [22, 19.1]).

$$\bigwedge_{n\in\mathbb{N}}\bigvee_{m\in\mathbb{N}} b^n_m = \bigvee_{m\in\mathbb{N}^\mathbb{N}}\bigwedge_{n\in\mathbb{N}} b^n_{m(n)}$$

Theorem 2.3 (Gutman) *Let $\mathbb{B}$ be a complete Boolean algebra and $\mathscr{R}$ the field of reals within $\mathbb{V}^{(\mathbb{B})}$. The following assertions are equivalent:*

(1) *$\mathbb{B}$ is σ-distributive.*
(2) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^\wedge$.
(3) *$\mathscr{R}\downarrow$ is locally one-dimensional.*

We also need the following result on the structure of cardinals within a Boolean-valued universe. Let $\mathrm{Card}(x)$ symbolize that x is a cardinal.

Theorem 2.4 *Each Boolean valued cardinal is a mixture of some set of relatively standard cardinals. More precisely, Given $x \in \mathbb{V}^{(\mathbb{B})}$, we have $\mathbb{V}^{(\mathbb{B})} \models \mathrm{Card}(x)$ if and only if there are nonempty set of cardinals Γ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma} \subset \mathbb{B}$ such that $x = \mathrm{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ and $\mathbb{V}^{(\mathbb{B}_\gamma)} \models \mathrm{Card}(\gamma^\wedge)$ with $\mathbb{B}_\gamma := [0, b_\gamma]$ for all $\gamma \in \Gamma$.*

Proof See Bell [4, Problem 1.45 and Theorem 1.50], Kusraev and Kutataeladze and [17, Subsections 1.9.7 and 1.9.11]. □

3 Band Preserving Linear Isomorphisms

Abramovich and Kitover in [2, p. 1, Problem B] raised the question as to whether the vector lattices X and Y are lattice isomorphic whenever X and Y are d-isomorphic, that is, there exists a linear disjointness preserving operator $T : X \to Y$ such that T^{-1} is also disjointness preserving? A negative answer was given in the same work, see [2, Theorem 13.4]. The problem has a negative solution even in the class of band preserving operators, see [17, Theorem 4.6.7]. Moreover, if X is a real universally complete vector lattice without locally one-dimensional bands then $X = X_1 \oplus X_2$ for some component-wise closed and laterally complete vector sublattices $X_1 \subset X$ and $X_2 \subset X$ both d-isomorphic to X but neither X_1 nor X_2 is Dedekind complete and hence lattice isomorphic to X, see [18]. The aim of this section is to show that the latter result can be improved to infinite direct sum decomposition.

Definition 3.1 A vector lattice X is said to be *locally one-dimensional* if for every two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2 respectively such that u_1 and u_2 are proportional. Let γ be a cardinal. A vector lattice X is said to be *Hamel γ-homogeneous* whenever there exists a local Hamel basis of cardinality γ in X consisting of strongly distinct weak order units. Two elements $x, y \in X$ are said to be *strongly distinct* if $|x - y|$ is a weak order unit in X.

Lemma 3.2 *Let X be a universally complete vector lattice. There is a band X_0 in X such that $X_0^{\perp}$ is locally one-dimensional and there exists a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(X_0)$ with Γ a set of infinite cardinals such that $\pi_\gamma X_0$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$.*

Proof The proof can be found in [17, Theorem 4.6.9]. □

Lemma 3.3 *The field of reals $\mathbb{R}$ has no proper subfield of which it is a finite extension.*

Proof See, for example, Coppel [6, Lemma 17]. □

Lemma 3.4 *Let $\mathscr{R}$ be the field of reals within $\mathbb{V}^{(\mathbb{B})}$, $X := \mathscr{R}{\downarrow}$, and $b \in \mathbb{B}$. Denote by $\dim(\mathscr{R})$ the internal cardinal with $[\![\dim(\mathscr{R})$ is the algebraic dimension of the vector space $\mathscr{R}$ over $\mathbb{R}^{\wedge}]\!] = \mathbb{1}$. Then bX is strictly Hamel γ-homogeneous if and only if $b \leq [\![\dim(\mathscr{R}) = \gamma^{\wedge}]\!]$.*

Proof This can be proved as in [12, Theorem 8.3.11]. □

Lemma 3.5 *Let $\mathbb{P}$ be a proper subfield of $\mathbb{R}$. There exists an infinite cardinal $\varkappa$ and a family $(\mathscr{X}_\alpha)_{\alpha\leq\varkappa}$ of $\mathbb{P}$-linear subspace in $\mathbb{R}$ such that $\mathbb{R} = \bigoplus_{\alpha\leq\varkappa}\mathscr{X}_\alpha$ and, for every $\alpha \leq \varkappa$, the $\mathbb{P}$-vector spaces $\mathscr{X}_\alpha$ and $\mathbb{R}$ are isomorphic, whilst they are not isomorphic as ordered vector spaces over $\mathbb{P}$.*

Proof It follows from Lemma 3.3 that $\mathbb{R}$ is an infinite dimensional vector space over the field $\mathbb{P}$. Let $\mathscr{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$. Since $\varkappa := |\mathscr{E}|$ is an infinite cardinal, we have the representation $\varkappa = \sum_{\alpha\in\mathrm{A}}\varkappa_\alpha$, where $\varkappa_\alpha = \varkappa$ for all $\alpha \in \mathrm{A}$ and $|\mathrm{A}| \leq \varkappa$. It follows that there is a family of subsets $\mathscr{E}_\alpha \subset \mathscr{E}$ such that $\mathscr{E} = \bigcup_{\alpha\leq\varkappa}\mathscr{E}_\alpha$, $|\mathscr{E}_\alpha| = |\mathscr{E}|$ for all $\alpha \leq \varkappa$, and $\mathscr{E}_\alpha \cap \mathscr{E}_\beta = \varnothing$ whenever $\alpha \neq \beta$. If $\mathscr{X}_\alpha$ denotes the $\mathbb{P}$-subspace of $\mathbb{R}$ generated by $\mathscr{E}_\alpha$, then $\mathscr{X}_\alpha \subsetneq \mathbb{R}$ and $\mathscr{X}_\alpha$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$. If $\mathscr{X}_\alpha$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathscr{X}$ would be order complete and, as a consequence, we would have $\mathscr{X}_\alpha = \mathbb{R}$; a contradiction. □

Definition 3.6 Let X be a vector latice and $u \in X$. An element $v \in X$ is called a *component* of v if $|v| \wedge |u - v| = 0$. The set of all components of u is denoted by $\mathbb{C}(u)$. A subset X_0 is said to be *component-wise closed* if for every $u \in X_0$ the set $\mathbb{C}(u)$ is contained in X_0. A sublattice $X_0 \subset X$ is said to be *laterally complete* if every disjoint family in X_0 has a supremum.

Lemma 3.7 *Let $\mathscr{R}$ — be the field of reals within $\mathbb{V}^{(\mathbb{B})}$ and let us consider a universally complete vector lattice $X := \mathscr{R}{\downarrow}$. For the sublattice $X_0 \subset X$ the following conditions:*

(1) *X_0 is laterally complete, component-wise closed, and $X_0^{\perp\perp} = X$.*
(2) *$X_0 = \mathscr{X}_0{\downarrow}$ for some nonzero vector sublattice $\mathscr{X}_0$ of the field $\mathscr{R}$ considered as vector lattice over the subfield $\mathbb{R}^{\wedge}$.*

Proof It directly follows from [17, Theorem 2.5.1]. □

Denote by $[A]_\sigma$ the set of all elements $x \in X$ representable in the form $x = \sum_{n=1}^{\infty} \pi_k a_k$, where (a_k) is an arbitrary sequence in A and (π_k) is a countable partition of unity in $\mathbb{P}(X)$.

Theorem 3.8 *Assume that a real universally complete vector lattice X is strictly Hamel $\varkappa$-homogeneous for some infinite cardinal $\varkappa$. Then there exists a family $(X_\alpha)_{\alpha\le\varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ satisfying the conditions:*

(1) $X = \left[\bigoplus_{\alpha\le\varkappa} X_\alpha\right]_\sigma$ *and* $X = X_\alpha^{\perp\perp}$ *for all* $\alpha \le \varkappa$.
(2) *The canonical projection* $\pi_\alpha : X \to X_\alpha$ *are all band preserving.*
(3) X_α *is d-isomorphic to* X *for all* $\alpha \le \varkappa$.
(4) X_α *is not Dedekind complete and hence not lattice isomorphic to* X *for all* $\alpha \le \varkappa$.

Proof We can assume without loss of generality that $X = \mathscr{R}{\downarrow}$. Since there is no loally one-dimensional band in X, we have $[\![\mathscr{R} \ne \mathbb{R}^\wedge]\!] = \mathbb{1}$ by Gutman's theorem 2.2. Working within $\mathbb{V}^{(\mathbb{B})}$, we can apply Lemma 3.5 by Transfer Principle and find an infinite cardinal $\varkappa$ and a family $(\mathscr{X}_\alpha)_{\alpha\le\varkappa}$ of $\mathbb{R}^\wedge$-linear subspaces of $\mathscr{R}$ such that $\mathscr{R} = \bigoplus_{\alpha\le\varkappa} \mathscr{X}_\alpha$ and, for every $\alpha \le \varkappa$, there is an $\mathbb{R}^\wedge$-isomorphism τ_α from $\mathscr{X}_\alpha$ onto $\mathscr{R}$, while $\mathscr{X}_\alpha$ and $\mathscr{R}$ are not isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$. Let $p_\alpha : \mathscr{R} \to \mathscr{X}_\alpha$ stand for the projection corresponding to the direct sum decomposition $\mathscr{R} = \bigoplus_{\alpha\le\varkappa} \mathscr{X}_\alpha$. To externalize, put $X_\alpha := \mathscr{X}_\alpha{\downarrow}$, $T_\alpha := \tau_\alpha{\downarrow}$, $S_\alpha := \tau_\alpha^{-1}{\downarrow}$, and $P_\alpha := p_\alpha{\downarrow}$. The maps $S_\alpha : X \to X_\alpha$, $T_\alpha : X_\alpha \to X$, and $P_\alpha : X \to X_\alpha$ are band preserving and $\mathbb{R}$-linear by [17, Theorem 4.3.4]. Moreover, S_α and T_α are injective and $S_\alpha = (\tau_\alpha{\downarrow})^{-1} = T_\alpha^{-1}$, see [17, 1.5.3(2)]. Since $\mathscr{X}_\alpha$ is linearly isomorphic to $\mathscr{R}$, $[\![\mathscr{X}_\alpha \ne \{0\}]\!] = \mathbb{1}$ and hence $X_\alpha^{\perp\perp} = X$. It follows from Lemma 3.6 that X_α is laterally complete and component-wise closed. It remains to observe that a) X_α and X are lattice isomorphic if and only if $\mathscr{X}$ and $\mathscr{R}$ are isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$; b) P_α is order bounded if and only if so is p_α within $\mathbb{V}^{(\mathbb{B})}$. □

Corollary 3.9 *For every universally complete vector lattice X without locally one-dimensional bands there exist an infinite cardinal $\varkappa$ and a family $(X_\alpha)_{\alpha\le\varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ such that X_α is d-isomorphic but not lattice isomorphic to X for all $\alpha \le \varkappa$ and $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \ne \beta$.*

Proof This is immediate from Lemma 3.2 and Theorem 3.8. □

Remark 3.10 As can be seen from the proof of Theorem 3.8, the key role in this section is played by the Boolean valued interpretation of the Hamel basis. In some problems, a similar role belongs to the Boolean valued interpretation of a transcendence basis, see [13].

4 Classification of Injective Modules

Recently, Chilin and Karimov [5] obtained a classification result for laterally complete modules over universally complete f-algebras. They introduced the concept of the *passport* and proved that two such modules are isomorphic if and only if their passport coincide. In this section we will show that these results remain valid for a broader class of separated injective modules over semiprime rationally complete commutative rings.

In what follows, K stands for a commutative semiprime ring with unit and X denotes a unitary K-module. The Boolean valued approach to the above classification problem is based on the following two results (Theorems 4.2 and 4.4) due to Gordon [8, 9].

Definition 4.1 An *annihilator ideal* of K is a subset of the form $S^{\perp} := \{k \in K : (\forall\, s \in S)\ ks = 0\}$ with a nonempty subset $S \subset K$. A subset S of K is called *dense* provided that $S^{\perp} = \{0\}$; i. e., the equality $k \cdot S := \{k \cdot s : s \in S\} = \{0\}$ implies $k = 0$ for all $k \in K$. A ring K is said to be *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K with $h(x) = rx$ for all $x \in J$.

Observe that K is rationally complete if and only if the complete ring of quotients $Q(K)$ is isomorphic to K canonically, see Lambek [19, § 2.3].

Theorem 4.2 *If $\mathscr{K}$ is a field within $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete semiprime ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathbb{A}(\mathscr{K}{\downarrow})$ of the annihilator ideals of $\mathscr{K}{\downarrow}$ such that*

$$b \leq [\![x = 0]\!] \iff x \in \chi(b^*) \quad (x \in K,\ b \in \mathbb{B}).$$

Conversely, if K is a rationally complete semiprime ring and $\mathbb{B}$ stands for the Boolean algebra $\mathbb{A}(K)$ of all annihilator ideals of K, then there is an internal field $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that the ring K is isomorphic to $\mathscr{K}{\downarrow}$.

Proof See [16, Theorem 8.3.1] and [16, Theorem 8.3.2]. □

Definition 4.3 A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $xJ = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0.

The *Baer criterion* says that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$; see Lambek [19]. All modules under consideration are assumed to be *faithful*, that is, $Xk \neq \{0\}$ for any $0 \neq k \in K$, or equivalently, the canonical representation of K by endomorphisms of the additive group X is one-to-one.

Theorem 4.4 *Let $\mathscr{X}$ be a vector space over a field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathbb{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism in Theorem 4.2. Then $\mathscr{X}{\downarrow}$ is a separated unital injective module over $\mathscr{K}{\downarrow}$ such that $b \leq [\![x = 0]\!]$ and $\chi(b)x = 0$ are equivalent for all $x \in \mathscr{X}{\downarrow}$ and $b \in \mathbb{B}$. Conversely, if $K = \mathscr{K}{\downarrow}$ and X is a unital separated injective K-module then there exists an internal vector space $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ over $\mathscr{K}$ such that the K-module X is isomorphic to $\mathscr{X}{\downarrow}$. Moreover if $\jmath : K \to \mathscr{K}{\downarrow}$ is an isomorphism in Theorem 4.2, then one can choose an isomorphism $\imath : X \to \mathscr{X}{\downarrow}$ such that $\imath(ax) = \jmath(a)\imath(x)$ $(a \in K,\ x \in X)$.*

Proof See [16, Theorems 8.3.12] and [16, and 8.3.13]. □

Thus, Theorem 4.4 enables us to apply Boolean valued approach to unital separated injective modules over semiprime rationally complete commutative rings.

Definition 4.5 A family $\mathscr{E}$ in a K-module X is called *K-linearly independent* or symply *linearly independent* whenever, for all $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in K$, and $e_1, \dots, e_n \in \mathscr{E}$, the equality $\sum_{k=1}^{n} \alpha_k e_k = 0$ implies $\alpha_1 = \dots = \alpha_n = 0$. An inclusion maximal K-linearly independent subset of X is called a *Hamel K-basis* for X.

Every unital separated injective K-module X has a Hamel K-basis. A K-linearly independent set $\mathscr{E}$ in X is a Hamel K-basis if and only if for every $x \in X$ there exist a partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(K)$ and a family $(\lambda_{k,e})_{k\in\mathbb{N}, e\in\mathscr{E}}$ in K such that

$$\pi_k x = \sum\nolimits_{e\in\mathscr{E}} \lambda_{k,e} \pi_k e \quad (k \in \mathbb{N})$$

and for every $k \in \mathbb{N}$ the set $\{e \in \mathscr{E} :\ \lambda_{k,e} \neq 0\}$ is finite.

Definition 4.6 Let γ be a cardinal. A K-module X is said to be *Hamel γ-homogeneous* whenever there exists a Hamel K-basis of cardinality γ in X. For $\pi \in \mathbb{P}(X)$ denote by $\varkappa(\pi)$ the least cardinal γ for which πX is Hamel γ-homogeneous. Say that X is *strictly Hamel γ-homogeneous* whenever X is Hamel γ-homogeneous and $\varkappa(\pi) = \gamma$ for all nonzero $\pi \in \mathbb{P}(X)$.

Theorem 4.7 *Let K be a semiprime rationally complete commutative ring and let X be a separated injective module over K. There exists a partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(K)$ with Γ a set of cardinals such that $e_\gamma X$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$. Moreover, X is isomorphic to $\prod_{\gamma\in\Gamma} e_\gamma X$ and the partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ is unique up to permutation.*

Proof According to Theorems 4.2 and 4.4 we may assume that $K = \mathscr{K}{\downarrow}$ and $X = \mathscr{X}{\downarrow}$, where $\mathscr{X}$ is a vector space over the field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\dim(\mathscr{X}) \in \mathbb{V}^{(\mathbb{B})}$, the algebraic dimension of $\mathscr{X}$, is an internal cardinal and, since each Boolean valued cardinal is a mixture of some set of relatively standard cardinals [17, 1.9.11], we have $\dim(\mathscr{X}) = \mathrm{mix}_{\gamma\in\Gamma} b_\gamma \gamma^{\wedge}$ where Γ is a set of cardinals and $(b_\gamma)_{\gamma\in\Gamma}$ is a partition of unity. Thus, for all $\gamma \in \Gamma$ we have $e_\gamma \leq [\![\dim(\mathscr{X}) = \gamma^{\wedge}]\!]$, whence $e_\gamma X$ is strictly Hamel γ-homogeneous. The remaining details are elementary. □

Definition 4.8 The partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(K)$ in Theorem 4.7 is called the passport of K-module X is denoted by $\Gamma(X)$.

Theorem 4.9 *Faithful separated injective K-modules X and Y are isomorphic if and only if* $\Gamma(X) = \Gamma(Y)$.

Remark 4.10

(1) In the particular case that K is a universally complete f-algebra, we get the classification obtained by Chilin and Karimov [5, Theorems 4.2 and 4.3] without using the Boolean valued approach. In this event $\mathscr{X}$ is a vector spaces over the field of reals $\mathscr{K} = \mathscr{R}$ or complexes $\mathscr{K} = \mathscr{C}$ within $\mathbb{V}^{(\mathbb{B})}$. Another particular case of Theorem 4.7 when $\mathscr{X}$ is a vector subspace of $\mathscr{R}$ (considered as a vector space over $\mathbb{R}^\wedge$) was examined by Kusraev and Kutateladze [17, Chap. 4].
(2) The family $(e_\gamma)_{\gamma\in\Gamma}$ in Theorem 16 is called a *decomposition series* if $e_\gamma X$ is (not necessarily strict) Hamel γ-homogeneous for all $\gamma \in \Gamma$. It can be also proved that separated injective modules over $K = L^0(\mathscr{B})$ are isomorphic if and only if their decomposition series are *congruent* in the sense of Ozawa [20].
(3) An *injective Banach lattice* possesses a module structure over some ring of continuous function $C(Q)$ with Q an extremally disconnected Hausdorff topological space [17, Sections 5.10]. This enables one to apply the Boolean valued approach to the classification problem of injective Banach lattices; details can be found in Kusraev [14]. The key role here is played by the concept of the Maharam operator, see [11].

5 Homogeneity Rings of Additive Operators

In this section, a ring K is supposed to have an identity element which is distinct from zero. This implies that K is not the zero ring $\{0\}$. Accordingly, a subring F of K is required to contain the identity element of K. Let X and Y be two unitary K-modules. Then, for any additive mapping $f : X \to Y$, the subset of K defined as

$$H_f := \{k \in K : \ f(kx) = kf(x) \ \text{ for all } \ x \in X\}$$

is a subring of K, the *homogeneity ring* of f, see Rätz [21, Lemma 1]. The problem is to examine what subrings of K have the form H_f for some additive operator f from X to Y? It is proved by Rätz [21, Theorem 3] that, for $X \neq \{0\}$, $Y \neq \{0\}$ and any subring S of K for which X is a *free S-module*, there exists an additive mapping $f : X \to Y$ such that $H_f = S$. The assumption that X is a free S-module seems to be pretty restrictive. However, in a special case of vector spaces over fields this condition is fulfilled so that we have the following.

Lemma 5.1 *Let X and Y be nonzero unitary K-modules and $\{y\}$ is linearly independent for some $y \in Y$. If a subring S of K is a field then there exists an additive mapping $f : X \to Y$ such that $H_f = S$.*

Consider an f-algebra A. Given an additive operator $S : A \to A$, define the homogeneity set $H_S \subset A$ of S as $H_S := \{a \in A : S(ax) = aSx$ forall $x \in A\}$. Then H_S is evidently a subring of A and our problem is to examine what subrings of A have the form H_S for some additive operator S in A?

Definition 5.2 A *projection* of a ring K is an endomorphism π of K with $\pi \circ \pi = \pi$. Say that $\mathscr{B}$ is a *Boolean algebra of projections* in K if $\mathscr{B}$ consists of mutually commuting projections in K under the operations

$$\pi_1 \vee \pi_2 := \pi_1 + \pi_2 - \pi_1 \circ \pi_2, \quad \pi_1 \wedge \pi_2 := \pi_1 \circ \pi_2,$$
$$\pi^* := I_K - \pi \quad (\pi_1, \pi_2, \pi \in \mathscr{B})$$

and in which the zero and the identity operators in K serve as the top and bottom elements of $\mathscr{B}$. Given $x \in K$, the *carrier* of x is defined as the projection $[x] := \bigwedge\{\pi \in \mathscr{B} : \pi x = x\}$.

Definition 5.3 BAP-*ring* is a pair $(K, \mathscr{B})$ where K is a ring with the distinguished complete Boolean algebra of projections $\mathscr{B}$, see [15]. Say that K is a $\mathbb{B}$-complete ring if $\mathbb{B}$ is a complete Boolean algebra isomorphic to $\mathscr{B}$, $(K, \mathscr{B})$ is a BAP-ring, and for every partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathscr{B}$ the following two conditions hold:

(1) If $x \in K$ and $\pi_\xi x = 0$ for all $\xi \in \Xi$ then $x = 0$.
(2) If $(x_\xi)_{\xi \in \Xi}$ is a family in K then there exists $x \in K$ such that $\pi_\xi x = \pi_\xi x_\xi$ for all $\xi \in \Xi$.

Theorem 5.4 *Let $\mathscr{K}$ be a ring within $\mathbb{V}^{(\mathbb{B})}$ and $K := \mathscr{K}{\Downarrow}$. Then K is a $\mathbb{B}$-complete ring and there exists an isomorphism $\jmath$ from $\mathbb{B}$ onto a Boolean algebra of projections $\mathscr{B}$ in K such that*

$$b \le [\![x = 0]\!] \iff \jmath(b)x = 0 \quad (x \in K,\ b \in \mathbb{B}).$$

Conversely, if K is a $\mathbb{B}$-complete ring then there exists $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{K}$ is a ring $]\!] = \mathbb{1}$ and the descent $\mathscr{K}{\downarrow}$ is $\mathbb{B}$-isomorphic to K.

Proof The first part is proved in [15, Theorem 4.2.8], while the second part can be deduced from [15, Theorem 4.3.3]. The reader is referred to [16, Theorems 8.1.4 and 5.1.7] for complete proofs and many pertinent results. □

Lemma 5.5 *Let $\mathscr{K}$ be a ring within $\mathbb{V}^{(\mathbb{B})}$ and $K := \mathscr{K}{\Downarrow}$. For a semiprime commutative subring $F \subset K$ the following are equivalent:*

(1) *F is rationally complete and each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$.*
(2) *F is regular, $\mathbb{B}$-complete, and $xy = 0$ implies $[x] \circ [y] = 0$ for all $x, y \in F$.*
(3) *$F = \mathscr{F}{\downarrow}$ for some field $\mathscr{F}$ which is a subring of $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$.*

Proof Clearly, $[x] \circ [y] = 0$ implies $xy = 0$ for all $x, y \in K$. It follows that $\jmath(b)F$ is an annihilator ideal. Observe that if each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$ or $xy = 0$ implies $[x] \circ [y] = 0$ for all $x, y \in F$, then $\mathbb{B}$ is isomorphic to the Boolean algebra of all annihilator ideals; the isomorphism is given by assigning $b \mapsto \jmath(b)F$ $(b \in \mathbb{B})$. Now, to ensure that the conditions (1) and (2) are equivalent it remains to observe that a semiprime ring is rationally complete if and if it is regular and $\mathbb{B}$-complete with $\mathbb{B}$ the Boolean algebra of annihilator ideals [15, Theorem 4.5.4]. The equivalence of (3) to both (1) and (2) follows from Theorem 4.2. □

Definition 5.6 A separated K-module X is said to be $\mathbb{B}$-complete if K is $\mathbb{B}$-complete and for every partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family$(x_\xi)_{\xi\in\Xi}$ in X there exists $x \in X$ such that $\jmath(b_\xi)x = \jmath(b_\xi)x_\xi$ for all $\xi \in \Xi$.

Theorem 5.7 *Let $\mathscr{X}$ be a modules over a ring $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Then $X := \mathscr{X}{\downarrow}$ is a $\mathbb{B}$-complete module over the $\mathbb{B}$-complete ring $K := \mathscr{K}{\downarrow}$. Conversely, if X is a $\mathbb{B}$-complete K-module then there exists $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ such that $[\![\mathscr{X}$ is a $\mathscr{K}$-module$]\!] = \mathbb{1}$ and there is an isomorphism ι_X from X onto $\mathscr{X}{\downarrow}$ such that $\iota_X(ax) = \iota_K(a)\iota_X(x)$ for all $a \in K$ and $x \in X$, where ι_K is a ring isomorphism from K onto $\mathscr{K}{\downarrow}$ in Theorem 5.4.*

Proof The proof runs along the lines of the proof of Theorems 8.3.1 and 8.3.2 in [16]. □

We are now ready to state and proof the main result of this section.

Theorem 5.8 *Let X and Y be unitary $\mathbb{B}$-complete K-modules. If a subring F of K is rationally complete and each annihilator ideal in F is of the form $\jmath(b)F$ for some $b \in \mathbb{B}$ then there exists an additive mapping $f : X \to Y$ such that $H_f = F$.*

Proof According to Theorem 5.4 we can assume that $K = \mathscr{K}{\downarrow}$, where $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ and $[\![\mathscr{K}$ is a ring$]\!] = \mathbb{1}$. By Lemma 5.5 it follows that $F = \mathscr{F}{\downarrow}$ for some field $\mathscr{F}$ which is a subring of $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Using Theorem 5.7 we can find $\mathscr{K}$-modules $\mathscr{X}$ and $\mathscr{Y}$ within $\mathbb{V}^{(\mathbb{B})}$ such that $X = \mathscr{X}{\downarrow}$ and $Y = \mathscr{Y}{\downarrow}$. The Transfer Principle (Theorem 2.1) guarantees that Lemma 5.1 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exists an additive function $\varphi : \mathscr{X} \to \mathscr{Y}$ with $H_\varphi = \mathscr{F}$. Put $f := \varphi{\downarrow}$ and note that $H_f = H_\varphi{\downarrow}$. It follows that $H_f = F$. □

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Some Questions Related to Extended Eigenvalues and Extended Eigenoperators

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Abstract We survey some results that were originated studying extended eigenvalues of bounded linear operators and their corresponding extended eigenoperators. We cover some aspects of operator theory, such as spectral theory, commutants and bicommutans, orbits of linear operators, etc. We review some recent results and we end up with open questions and future lines of research.

Keywords Extended eigenvalues · Extended eigenoperators · Commutants · Bicommutants · Hypercyclic operators · Cesàro operators · Weighted shifts

1 Introduction

This survey paper is part of the material that we had prepared to present at OTHA, which unfortunately because of COVID 19 lockout, we were unable to present. We would like to thank the organizers of this meeting, and in particular Professor Alexey Karapetyans, for all the effort invested.

In this note, we follow quite closely the recent advances of our research projects. This means that this work is by no means an update or comprehensive report on Extended Spectral Analysis (a very active field of research with many contributors). We focus instead on some new results that have been motivated by the study of extended eigenvalues and their extended eigenoperators [4, 5, 13, 17–20, 22]. For instance, our research of commutants of composition operators constitutes

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a foreplay of subsequent studies on extended eigenoperators for composition operators and this was the origin and the motivation of the research in [19, 20].

What we tried to do is to present some results, to describe some of the tools and to avoid the technicalities each time it was possible. These notes should be seen as an invitation to further reading and a preparation for questions which remain open.

A complex number λ is an *extended eigenvalue* of a bounded linear operator T defined on a Banach space E is there exists a non-zero bounded linear operator X such that $TX = \lambda XT$. The remainder of this paper is structured as follows. In Sect. 2, we will show how to compute extended eigenvalues of some classical operators, such as Cesàro operators or bilateral weighted shift operators (see [17, 18]). We will take a walk through some of the tools that we have used to make these computations, visiting a specific operator, namely, the Cesàro operator $C_1 f(x) = \frac{1}{x}\int_0^x f(s)\,ds$ defined on $L^p[0, 1]$.

In Sect. 3, we will show our results on commutants and bicommutants of composition operators defined on the Hardy space $H^2(\mathbb{D})$ [19, 20]. The study of operators with minimal commutant, or the double bicommutant property were originated in [21] where the authors studied the extended eigenvalues of composition operators defined on the Hardy space $H^2(\mathbb{D})$.

In Sect. 4, we study hypercyclicity of extended eigenoperators associated with an extended eigenvalue. It is well known that the cyclic properties of operators are often transferred to their commutant. However, as we will see, they can not always be carried to extended eigenoperators. Finally, the paper finishes with a brief section devoted to concluding remarks, open questions and related issues.

2 Extended Eigenvalues of Some Classical Operators

As we mentioned, we are interested in the non-trivial solutions of the equation $TX = \lambda XT$. These solutions are called *extended eigenoperators* associated to the extended eigenvalue $\lambda \in \mathbb{C}$. It is also said that T and X λ-commute (this terminology was used by J.B. Conway and G. Prăjitură in [9]), or, equivalently, that the operator X *intertwines* T and λT.

A good source of problems in operator theory is to consider a result that is true for the commutant of an operator T and to try to prove it for the set of extended eigenoperators of T. For instance, Lomonosov [23] proved strikingly that if the commutant of T contains a non-zero compact operator then T has a nontrivial closed hyperinvariant subspace. Some years later, Scott Brown [8] and independently Kim, Moore, and Pearcy [15] proved that if a non-scalar operator T has a non-zero, compact extended eigenoperator then T has a non-trivial, closed hyperinvariant subspace. Ever since, extended eigenoperators have been studied in both, the search of invariant subspaces and the computation of the extended eigenvalues and extended eigenoperators of some specific operators. Moreover, recently, in connection with the first research line, the orbits of extended eigenoperators of a fixed operator T have been studied.

Table 1 Summary: extended eigenvalues of Cesàro operators

Operator	Space	Extended spectrum
C_0	ℓ^2	$[1, \infty)$
C_1	$L^p[0,1] \quad 1 \le p < \infty$	$(0, 1]$
C_∞	$L^p[0,\infty) \quad 1 \le p < \infty$	$\{1\}$

The extended eigenvalues of some classical operators, such as the Volterra operator, any unilateral weighted shift operator, etc. [28, 29], have been computed in the past. In general these sets are neither bounded nor closed. The extended eigenvalues of Cesàro type operators were described in [17]. The discrete Cesàro operator C_0, the finite continuous Cesàro operator C_1 and the infinite continuous Cesàro operator C_∞ defined on the complex Banach spaces ℓ^p, $L^p[0,1]$ and $L^p[0,\infty)$ by the expressions: $(C_0 f)(n) = \frac{1}{n+1}\sum_{k=0}^{n} f(k)$, $(C_1 f)(x) = \frac{1}{x}\int_0^x f(t)\,dt$ and $(C_\infty f)(x) = \frac{1}{x}\int_0^x f(t)\,dt$, respectively. The results are summarized in the Table 1:

Furthermore, in [17] almost all the extended eigenoperators associated with each extended eigenvalue are described. Now, let us explain the tools that we used to get these results. We will focus on a specific example C_1 acting on $L^p[0,1]$. There are many results to discard or to find values in the extended spectrum. For instance, the following result by Biswas and Petrovic [7] can derived from Rosenblum's Theorem.

Theorem 1 (Biswas-Petrovic) *Let us denote by $\sigma(A)$ the spectrum of A, if* $\mathrm{Ext}(A)$ *is the set of extended eigenvalues of A then*

$$\mathrm{Ext}(A) \subseteq \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\}.$$

An example of an application goes as follows. Let us suppose that $A = \alpha I + Q$, where Q is a quasinilpotent operator and $\alpha \neq 0$. In this case, no point $\lambda \neq 1$ can be an extended eigenvalue of A. And since always $1 \in \mathrm{Ext}(A)$, we obtain $\mathrm{Ext}(A) = \{1\}$.

Another very useful result is formulated for operators that have a rich spectral picture. We say that an operator T on a complex Banach space has *rich point spectrum* provided that $\mathrm{int}(\sigma_p(T)) \neq \emptyset$ and for every open disc $D \subseteq \sigma_p(T)$ the family of eigenvectors $\cup_{z\in D}\ker(T - zI)$ is a total set.

Theorem 2 *Let us suppose that an operator T on a complex Banach space has rich point spectrum. If λ is an extended eigenvalue for T then we have* $\lambda \cdot \mathrm{int}(\sigma_p(T)) \subseteq \mathrm{clos}(\sigma_p(T))$.

Proof Let X be an extended eigenoperator of T corresponding to the extended eigenvalue λ, that is, $X \neq 0$ and $TX = \lambda XT$. Let $z \in \mathrm{int}\sigma_p(T)$ and let $n \in \mathbb{N}$ such that $D(z, 1/n) \subset \sigma_p(T)$. Since $X \neq 0$ and T has rich point spectrum, there exist $z_n \in D(z, 1/n)$ and $f_n \in \ker(T - z_n) \setminus \{0\}$ such that $Xf_n \neq 0$. Hence,

$$TXf_n = \lambda XTf_n = \lambda z_n Xf_n,$$

and since $Xf_n \neq 0$, this means that $\lambda z_n \in \sigma_p(T)$. Taking limits as $n \to \infty$ yields $\lambda z \in \operatorname{clos} \sigma_p(T)$, as we wanted. □

For instance, the point spectrum of C_1 on $L^p[0,1]$ is rich and it is the disc $D(q/2, q/2)$ where $1/q = 1 - 1/p$. Thus the only values $\lambda \in \mathbb{C}$ that can be extended eigenvalues are contained in $(0, 1]$.

Here is an example of an application of the above principle. Let us show that $C_1 f(x) = \frac{1}{x} \int_0^x f(s)\, ds$, $f \in L^p[0,1]$ has rich point spectrum. Let $1 < p, q < \infty$ be a pair of conjugate indices, that is, $1/q = 1 - 1/p$.

Leibowitz proved that the point spectrum of the Cesàro operator C_1 on $L^p[0,1]$ is the open disc $D(q/2, q/2)$. Moreover, each $z \in D(q/2, q/2)$ is a simple eigenvalue of C_1 and a corresponding eigenfunction is given by $h_z(x) = x^{(1-z)/z}$.

To prove that C_1 has rich point spectrum, we will use the following version of the Müntz's Theorem, which was conjectured by Borwein and Erdélyi and it was proved by Operstein [26]. We will use also a standard separation argument (Lemma 1).

Theorem 3 (Full Müntz's Theorem on $L^p(0,1)$) *Let $1 < p < \infty$ and let (r_n) be a sequence of distinct real numbers greater than $-1/p$. Then, the linear subspace* $\operatorname{span}\{x^{r_0}, x^{r_1}, \ldots, x^{r_n}, \ldots\}$ *is dense in $L^p[0,1]$ if and only if*

$$\sum_{n=0}^{\infty} \frac{r_n + 1/p}{(r_n + 1/p)^2 + 1} = \infty. \tag{1}$$

Lemma 1 *Let T be a bounded linear operator on a complex Banach space E and let us suppose that there is an analytic mapping $h : \operatorname{in} \sigma_p(T) \to E$ with $h(z) \in \ker(T - zI) \setminus \{0\}$ for all $z \in \operatorname{int} \sigma_p(T)$ and such that $\{h(z) : z \in \operatorname{int} \sigma_p(T)\}$ is a total subset of E. Then T has rich point spectrum.*

Proof Let D be an open disc contained in $\sigma_p(T)$ and let $g^\star \in E^\star$ such that $\langle h(z), g^\star \rangle = 0$ for all $z \in D$. We must show that then $g^\star = 0$. We consider the analytic function $f : \operatorname{int}\sigma_p(T) \to \mathbb{C}$ defined by $f(z) = \langle h(z), g^\star \rangle$. We know by assumption that f vanishes on D. Then, it follows from the principle of analytic continuation that f vanishes on $\operatorname{int}\sigma_p(T)$. Since the family of eigenvectors $\{h(z) : z \in \operatorname{int}\sigma_p(T)\}$ is a total set, it follows that $g^\star = 0$, as we wanted. □

Theorem 4 *The finite continuous Cesàro operator C_1 on $L^p[0,1]$ has rich point spectrum.*

Proof Notice that $\sigma_p(C_1) = D(q/2, q/2)$ is open and connected. Also, the mapping $h : \sigma_p(C_1) \to L^p[0,1]$ defined by $h(z)(x) = x^{(1-z)/z}$ is analytic, and $h(z) \in \ker(C_1 - zI) \setminus \{0\}$. It is a standard consequence of the full Müntz Theorem that the family of eigenfunctions $\{h(z) : z \in D(q/2, q/2)\}$ is total in $L^p[0,1]$. Indeed, it suffices to consider the sequence of distinct real numbers (z_n) defined for every $n \geq 0$ by the expression $z_n = \frac{n+1}{n+2} q$. Since an easy computation shows that the sequence of exponents $r_n = (1 - z_n)/z_n$ satisfies condition (1), the result follows from Lemma 1. □

Now, let us see how the geometric principle of Theorem 2 is applied. We have $\sigma_p(C_1) = D(q/2, q/2)$. If λ is a complex number such that $\lambda D(q/2, q/2) \subset \overline{D(q/2, q/2)}$ then $\lambda \in [0, 1]$. Moreover, 0 cannot be an extended eigenvalue for C_1 because $\ker(C_1) = \{0\}$, hence we conclude that $\mathrm{Ext}(C_1) \subset (0, 1]$. Next we will see that the reverse inclusion also holds.

To prove that a complex number λ is an extended eigenvalue we need to construct an extended eigenoperator. Sometimes we can build an extended eigenoperator directly by deducing it from the equation $TX = \lambda XT$. However, in most cases this is not possible. Sometimes, the fact that an operator is similar to another one which is easy to understand, is used often. In this sense, in the Hilbert space setting we used the classical result by Kriette and Trutt [16], which asserts that there exists a Borel measure μ supported on the closed unit disc, such that $I - C_0$ is unitarily equivalent to multiplication by z on the Hilbert space $H^2(\mu)$, that denotes the closure of complex polynomials on $L^2(\mu)$.

We construct explicitly some particular extended eigenoperators for the operator C_1 acting on the space $L^p[0, 1]$. More precisely, it can be proven that for every $0 < \lambda \leq 1$, the operator X_0 defined by the expression

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda})$$

is an extended eigenoperator for C_1 corresponding to the extended eigenvalue λ. Now we proceed with the heuristic argument to build this operator by necessary conditions. Suppose that

$$C_1 X_0 = \lambda X_0 C_1.$$

Then, for every $n \in \mathbb{N}_0$ we have

$$C_1 X_0 x^n = \lambda X_0 C_1 x^n = \frac{\lambda}{n+1} X_0 x^n,$$

and therefore $X_0 x^n$ is either zero or it is an eigenfunction of C_1, say $X_0 x^n = x^\alpha$. We get

$$\frac{1}{\alpha+1} x^\alpha = C_1 x^\alpha = C_1 X_0 x^n = \frac{\lambda}{n+1} X_0 x^n = \frac{\lambda}{n+1} x^\alpha,$$

and this leads to

$$\frac{1}{\alpha+1} = \frac{\lambda}{n+1}.$$

Now, solving for α in the equation above we obtain $\alpha = (n + 1 - \lambda)/\lambda$, so that

$$X_0 x^n = x^{(n+1-\lambda)/\lambda} = x^{(1-\lambda)/\lambda} (x^{1/\lambda})^n.$$

It follows from linearity that for every polynomial p we have

$$(X_0 p)(x) = x^{(1-\lambda)/\lambda}\, p(x^{1/\lambda}),$$

and since the polynomials are dense in $L^p[0, 1]$, we get

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda}\, f(x^{1/\lambda})$$

for every $f \in L^p[0, 1]$. This shows that the operator X_0 has the desired form. Notice that the argument can be reversed to show that the operator X_0 is indeed an extended eigeoperator.

In the absence of eigenvalues, the problem could be complicated. For instance, C_∞ on $L^p[0, \infty)$ has not eigenvalues, however we were lucky enough. We used a result by Brown, Halmos, and Shields who proved that $I - C_\infty^\star$ is unitarily equivalent to a bilateral shift of multiplicity one. On $L^p[0, \infty)$ we was able to prove the following result which is interesting in its own right.

Theorem 5 *Let us denote by C_∞ the infinite Cesàro operator on $L^p[0, \infty)$. If q is the exponent conjugate of p, the operator $I - \frac{1}{q} C_\infty^\star$ is similar to a bilateral weighted shift.*

The problem of determining the extended eigenvalues of a bilateral weighted shift operator on $\ell_p(\mathbb{Z})$ begun in [17]. Here we were a little excited because this question is related with a fascinating open problem: to know whether a bilateral weighted shift operator has a non-trivial hyperinvariant closed subspace (see the work by S. Atzmon [2] who is an authority on this subject). And the question here is: When a bilateral weighted shift operator has a non trivial compact extended eigenoperator? Because in such a case there exists a non-trivial hyperinvariant closed subspace for the operator. For a bilateral weighted backward shift we obtained in [18] four possible extended spectral pictures in terms of the weight sequences. Namely, first $\overline{\mathbb{D}}$, second $\mathbb{C} \setminus \mathbb{D}$, third $\mathbb{C}$, and fourth $\mathbb{T}$.

We obtained that in the first three cases there exists a non-trivial closed hyperinvariant subspace for the bilateral weighted shift operators, because in such cases there exists a non-trivial compact extended eigenoperator. That is, the hyperinvariant subspace problem for the bilateral weighted shift operators was reduced in terms of its extended spectrum.

3 Extended Eigenoperators, Commutants and Bicommutants

This section is concerned with commutants of composition operators. In general, describing which operators commute with a given operator reveals information about the structure of the operator. So, given an operator T, it is important to know as much as possible about its commutant. In fact, for instance, in order to describe the extended eigenoperators usually we need a description of commutant of the operator. In fact, when we started the paper [21] we took into account that the commutant of a composition operator was not known and we tried to compute it.

Let us denote by φ a holomorphic selfmap of the unit disc. The map φ defines naturally a linear transformation C_φ on the space of holomorphic functions on the unit disc, and it will be also clear that this linear transformation is continuous on the topology of the uniform convergence on compact subsets. What is not so clear is that this linear transformation is bounded on the Hardy space. This follows from the Littlewood Subordination Principle.

The commutant of an operator T is the subalgebra of all bounded linear operators A that commute with T. This algebra contains the algebra generated by T and the identity. It is not difficult to show that the commutant is weakly closed, hence it contains the weak clousure of the algebra generated by T and the identity.

Therefore we will say that an operator T has a *minimal commutant* (MC) if the commutant is equal to the weak clousure of the algebra generated by T and the identity. Also we will say that an operator T has the *minimal commutant property* (MCP) if T has a minimal commutant.

In 1997, in a conference at Laramie, Carl Cowen, and B. MacCluer posed the following question. For which maps φ is the commutant of C_φ minimal? In [19] we answered completely this question when the symbol φ is a linear fractional selfmap of the unit disc. It is well known that in order to obtain general results on composition operators, a good starting point is to try with linear fractional selfmaps, because there is a classical theory developed by Caratheodory, Denjoy, and Wolf, wherein the self maps of the unit disc are shown to be conformally similar to linear fractional selfmaps acting on planar domains. And conformal similarity implies that at the operator level, the composition operator is similar to a composition operator induced by a linear fractional selfmap on the Hardy space of the planar domain. The linear fractional maps of the unit disc are classified in terms of their fixed points.

Recall that a linear fractional map φ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is any map of the form $\varphi(z) = \frac{az+b}{cz+d}$ where the coefficients $a, b, c, d \in \mathbb{C}$ satisfy the condition $ad - bc \neq 0$, to ensure that φ is not constant. We consider those linear fractional maps that take the unit disc into itself. Every such map has either one or two fixed points. Those with one fixed point are called parabolic, and it can be an automorphisms (PA) or not (PNA). In the remaining cases, the linear fractional map φ has two fixed points, one of which must be in the closed unit disc. If it is on $\partial\mathbb{D}$ then φ is said to be hyperbolic (HA and HNA). If it is in the open unit disc, the other one has to lie is in the complement in the Riemann sphere of the closed unit disc. Here we distinguish between two different situations. If φ is an automorphism, then φ is said to be elliptic (E). Otherwise, φ is said to be loxodromic (LOX). The table show a complete answer to the minimal commutant problem for composition operators induced by linear fractional selfmaps of the unit disc, and when they are acting on the Hardy space $H^2(\mathbb{D})$.

Now, let us describe the nature of the problems involved in the Table 2. Let us suppose that T is an operator with a nice point spectrum $\sigma_p(T)$; that is, it contains an open subset U, such that the eigenvalues $\lambda \in U$ are simple, with corresponding eigenvector x_λ. If S is an element in the commutant, so that $TS = ST$, applying

Table 2 Summary of linear fractional composition operators with or without (MC)

Type of φ	Fixed points	Canonical form	MC
EP	$0, \infty$	$\varphi(z) = \omega z$, (ω a root of unity)	no
ENP	$0, \infty$	$\varphi(z) = \omega z$, (ω not a root of unity)	yes
HNA	$1, \infty$	$\varphi(z) = rz + (1 - r)$	no
HA	$1, -1$	$\varphi(z) = \frac{z+r}{1+rz}$	no
PA	$1, 1$	$\varphi(z) = \frac{(2-a)z+a}{-az+2+a}$, $Re(a) = 0$	no
PNA	$1, 1$	$\varphi(z) = \frac{(2-a)z+a}{-az+2+a}$, $Re(a) > 0$	yes
LOX/HNA	$0, c \in \mathbb{C} \setminus \overline{\mathbb{D}}$	$\varphi(z) = \frac{z}{az+b}$, $\lvert b\rvert > 1$, $\lvert a\rvert < 1 - \lvert b\rvert$	yes

on x_λ we obtain $TSx_\lambda = \lambda Sx_\lambda$. Then $Sx_\lambda = 0$ or $Sx_\lambda = f(\lambda)x_\lambda$ for some non-zero complex number $f(\lambda)$. If the map $\lambda \in \sigma_p(T) \to x_\lambda$ is analytic then f is also an analytic function. The question now is: can we define the functional calculus $f(T)$?. In such a case $S = f(T)$. Can we approximate f by polynomials? And these are exactly the problems and the techniques that are involved: approximation and interpolation. In fact, one of the pioneer results on minimal commutants is due to D. Sarason: he proved that the Volterra operator has a minimal commutant, and this is widely known among interpolating theory people.

Next, in our paper [19] we try to extend our work for composition operators induced by univalent general maps that are not linear-fractional. Let φ be a univalent selfmap with an interior fixed point, which is not a rotation. Let $\alpha = \varphi'(0)$. In 1884 G. Königs showed that there exists a function $\sigma : \mathbb{D} \to \mathbb{C}$ such that $C_\varphi \sigma = \alpha\sigma$. In such a case, $G = \sigma(\mathbb{D})$ is called the *Königs domain*. A domain G is said to be *strictly starlike* with respect to the origin if for any $0 \leq t < 1$ we have that $t\overline{G} \subset G$.

Theorem 6 *Let φ be a univalent selfmap of the unit disc, with $\varphi(0) = 0$. If $G = \sigma(\mathbb{D})$ is bounded and strictly starlike with respect to the origin, then C_φ has a minimal commutant.*

In connection with the result above, we answer negatively the following question posed by T. Worner [31]: If φ is univalent and C_φ is compact, does C_φ have a minimal commutant? The counterexample is based on a general criterion for operators on Hilbert spaces to fail the minimal commutant property.

Theorem 7 *Let $A \in \mathcal{B}(H)$. Let us suppose that $A(H) \setminus \overline{A^2(H)}$. Then A^2 has no minimal commutant.*

Moreover, the results above allow us to obtain the following result, which asserts that if we replace the condition "strictly starlike with respect to the origen" by just "starlike with respect to the origen", we cannot obtain that the commutant of the composition operator is minimal.

Theorem 8 *Let φ a univalent selfmap of the unit disc, with $\varphi(0) = 0$, and φ is not a rotation. If $G = \sigma(\mathbb{D})$ is starlike with respect to the origin and C_φ does not have dense range, then C_φ does not have minimal commutant.*

Sketch of the Proof Let $\alpha = \varphi'(0)$ and recall from Königs's theorem that $\varphi(z) = \sigma^{-1}(\alpha\sigma(z))$. Since φ is univalent and it fixes the origin, and in addition, it fails to be a disc automorphism, we have $0 < |\alpha| < 1$.

Since G is starlike with respect to the origin, for every $0 \le r \le 1$, there exists an analytic selfmap φ_r of the unit disc defined by the expression $\varphi_r(z) := \sigma^{-1}(r\sigma(z))$. Notice that $\varphi_r \circ \varphi_s = \varphi_{rs}$, hence $C_{\varphi_s}C_{\varphi_r} = C_{\varphi_{rs}}$ for all $0 \le r, s \le 1$.

Observe that $C_{\varphi_r} \in \{C_\varphi\}'$. Since $\varphi_r(0) = 0$, we have $\|C_{\varphi_r}\| = 1$. It is not hard to show that the map $r \mapsto C_{\varphi_r}$ is continuous from $[0, 1]$ into $\mathcal{B}(H^2(\mathbb{D}))$ endowed with the weak operator topology, and in particular, $C_{\varphi_r} \to I$, as $r \to 1^-$, in the weak operator topology.

Since the range of C_φ fails to be dense, there exists a function $f_0 \in H^2(\mathbb{D})\backslash\{0\}$ such that $C_\varphi^* f_0 = 0$. Let $e_0 \equiv 1$ and notice that $f_0(0) = \langle f_0, e_0\rangle = \langle f_0, C_\varphi e_0\rangle = \langle C_\varphi^* f_0, e_0\rangle = 0$. Then, consider the closed set defined by the expression

$$F := \{r \in [0, 1]\colon C_{\varphi_r}^* f_0 = 0\}.$$

Observe that $0 \in F$ because $C_{\varphi_0}^* f_0 = f_0(0)e_0 = 0$. Also, notice that $1 \notin F$, because $C_{\varphi_1}^* f_0 = f_0 \neq 0$. Moreover, F is an interval. Indeed, if $0 < s < r \in F$ then $C_{\varphi_s}^* f_0 = C_{\varphi_{s/r}}^* C_{\varphi_r}^* f_0 = 0$, hence $s \in F$. Next, let $r_0 = \sup F$. It follows from the continuity of the function $r \mapsto \langle C_{\varphi_r}^* f_0, f_0\rangle$ that there is some $\varepsilon > 0$ such that $\langle C_{\varphi_r}^* f_0, f_0\rangle \neq 0$, and in particular $C_\varphi^* f_0 \neq 0$, whenever $1 - \varepsilon < r < 1$. This shows that $r_0 < 1$.

We claim that $r_0 > 0$. Indeed, since G is open and $0 \in G$, there is some $\delta > 0$ such that $\delta\mathbb{D} \subseteq G$. Further, since G is bounded, there is some $M > 0$ such that $\overline{G} \subseteq M\mathbb{D}$. This allows us to consider, for every $w \in \mathbb{C}$ with $|w| \le \delta/M$, the univalent, analytic selfmap φ_w of the unit disc defined by $\varphi_w(z) := \sigma^{-1}(w\sigma(z))$. Let $r := \delta/M$. It suffices to show that $s := r|\alpha| \in F$. Indeed, let $\theta \in \mathbb{R}$ with $\alpha = |\alpha|e^{i\theta}$, and let $w = re^{-i\theta}$. Observe that $s = \alpha w$, so that $C_{\varphi_s} = C_\varphi C_{\varphi_w}$, so that $C_{\varphi_s}^* f_0 = C_{\varphi_w}^* C_\varphi^* f_0 = 0$, hence, $s \in F$, and this completes the proof of our claim.

Now, the rest of the proof is easy. We know that $F = [0, r_0]$ for some $0 < r_0 < 1$. Then, there is some $0 < r < 1$ such that $r^2 < r_0 < r$, so that $r^2 \in F$, but $r \notin F$.

We finally show that $C_{\varphi_r} \notin \overline{\text{alg}\,(C_\varphi)}^{\sigma}$, and the result follows at once. We proceed by contradiction. Let us suppose for a moment that there exists a net of polynomials $(p_d)_{d\in D}$ such that $p_d(C_\varphi) \to C_{\varphi_r}$, as $d \in D$, in the weak operator topology. Then, for every $f \in H^2(\mathbb{D})$,

$$\langle C_{\varphi_r} f, f_0\rangle = \lim_{d\in D}\langle p_d(C_\varphi) f, f_0\rangle = \lim_{d\in D} p_d(0)\langle f, f_0\rangle. \tag{2}$$

On one hand, if we set $f = C_{\varphi_r}^* f_0$ in Eq. (2), we get

$$\|C_{\varphi_r}^* f_0\|^2 = \lim_{d\in D} p_d(0)\langle C_{\varphi_r}^* f_0, f_0\rangle,$$

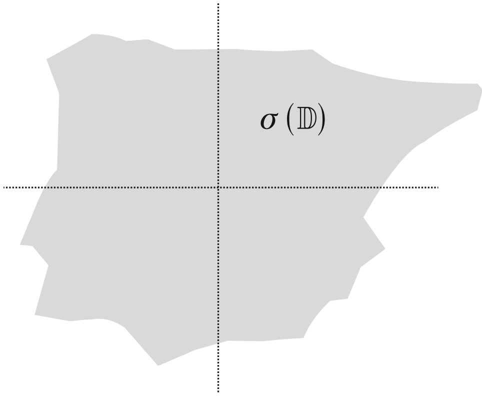

Fig. 1 The Iberian Peninsula strictly starlike with respect to the origin

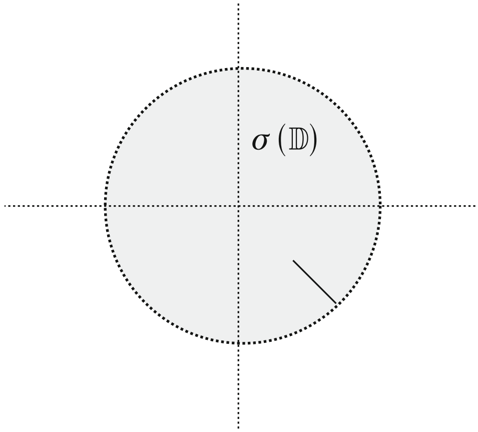

Fig. 2 $\sigma(\mathbb{D})$ is not strictly starlike with respect to the origin

so that $\langle C_{\varphi_r} f_0, f_0\rangle \neq 0$ and $\lim p_d(0) \neq 0$, as $d \in D$, and on the other hand, if we set $f = C_{\varphi_r} f_0$ in Eq. (2), then we get

$$\lim_{d\in D} p_d(0)\langle C_{\varphi_r} f_0, f_0\rangle = 0,$$

and it follows that $\lim p_d(0) = 0$, as $d \in D$. We arrived at a contradiction. □

Let us comment Figs. 1 and 2. Suppose that the map is the one that maps univalently the unit disc into a region similar to the Iberian Peninsula, and that the origin of the coordinates is in Madrid. This region is bounded and strictly starlike with respect to the origin. Then, the composition operator defined by the Königs maps has a minimal commutant. However, in Fig. 2, if σ maps the unit disc onto a slicht disc, then the composition operator defined by the Königs map, does not have minimal commutant.

The double commutant $\{C_\varphi\}''$ of an operator C_φ is defined as the elements T that commute with all operators that commutes with C_φ, in particular the algebra

Table 3 Summary of linear fractional composition operators with or without (MC) or (DCP)

Symbol	Fixed points	Canonical form of the symbol φ	MC	DCP
EP	$0, \infty$	$\varphi(z) = \omega z$, (ω a root of unity)	no	yes
ENP	$0, \infty$	$\varphi(z) = \omega z$, (ω not a root of unity)	yes	yes (trivial)
HNA I	$1, \infty$	$\varphi(z) = rz + 1 - r$, $0 < r < 1$	no	yes
HNA II	$1, 0$	$\varphi(z) = \dfrac{rz}{1-(1-r)z}$, $0 < r < 1$	no	yes
HA	$1, -1$	$\varphi(z) = \dfrac{z+r}{1+rz}$, $0 < r < 1$	no (trivial)	no
PA	$1, 1$	$\varphi(z) = \dfrac{(2-a)z+a}{-az+2+a}$, $\mathrm{Re}\, a = 0$	no (trivial)	no
PNA	$1, 1$	$\varphi(z) = \dfrac{(2-a)z+a}{-az+2+a}$, $\mathrm{Re}\, a > 0$	yes	yes (trivial)
LOX /HNA III	$0, c \in \mathbb{C} \setminus \overline{\mathbb{D}}$	$\varphi(z) = \dfrac{z}{az+b}$, $\lvert b\rvert > 1$, $\lvert a\rvert < 1 - \lvert b\rvert$	yes	yes (trivial)

$\mathrm{alg}(C_\varphi)$ generated by C_φ and the identity is contained in $\{C_\varphi\}''$. Since $\{C_\varphi\}''$ is closed in the weak operator topology, we deduce that $\mathrm{clos}_\sigma(\mathrm{alg})(C_\varphi) \subset \{C_\varphi\}''$.

It was shown in [19] that, if φ is a linear fractional selfmap of the unit disc and C_φ is the induced composition operator on the Hardy space $H^2(\mathbb{D})$, then we characterized when $\mathrm{clos}_\sigma(\mathrm{alg}\,(C_\varphi)) = \{C_\varphi\}'$. Since the double commutant $\{C_\varphi\}''$ lies between these two algebras, it is natural to ask whether it is true that $\mathrm{clos}_\sigma(\mathrm{alg}\,(C_\varphi)) = \{C_\varphi\}''$. Of course, whenever C_φ has a minimal commutant, i.e., $\mathrm{clos}_\sigma(\mathrm{alg}\,(C_\varphi)) = \{C_\varphi\}'$, we know that C_φ enjoys the *double commutant property* (DCP).

A classical result in operator theory, the von Neumann's double commutant theorem, is the assertion that for any self-adjoint, unital subalgebra $\mathcal{A} \subseteq \mathcal{B}(H)$, we have $\mathrm{clos}_\sigma(\mathcal{A}) = \mathcal{A}''$. That is, any unital self-adjoint algebra has the double commutant property. Therefore, it is a very natural question to pose this problem for single generated algebras, in general non self-adjoint (Table 3).

4 Extended Eigenvalues and Cyclic Operators

A bounded linear operator defined on a Banach space E, is said to be *hypercyclic* if there exists $x \in E$ such that the orbit $\{T^n x\}_{n\geq 0}$ is dense in E (thus, separability is required on the space). The study of operator orbits is related to studies of dynamical systems and it is a very active research area (see [10] for more information).

The concept of hypercyclicity can be defined in any topological space. In fact, the notion has its origins in the results of Birkhöff and Maclane, who proved that the translation operator and the differentiation operator are hypercyclic in the space of entire functions endowed with the topology of uniform convergence on compact subsets. Hypercyclic properties are transferred by intertwining relations. In fact, it is usual that hypercyclic operators T on Banach spaces, should have a *hypercyclic commutant* [13], which means that every non-scalar operator that commutes with T, has a multiple that is hypercyclic. Thus, the following general question arises: let T be a hypercyclic operator, is it possible to transfer the hypercyclic properties of T to its extended eigenoperators?

We discovered in [5] that for Banach space operators it is not possible to transfer hypercyclicity to eigenoperators associated with extended eigenvalues of modulus less than 1.

Theorem 9 *Let A and T be two operators on a Banach space E. Assume that T is an extended eigenoperator of A associated to the extended eigenvalues λ. If $|\lambda| < 1$ then T is not hypercyclic.*

Proof Assume by contradiction that T is hypercyclic. Then there exists a nonzero element x of E such that the orbit $\{T^n x\}_{n\geq 1}$ is dense in E. Hence, the orbit $\{A^m T^n x\}_{n\geq 1}$ is also dense in the image $A^m(E)$ for all $m \geq 1$. Since by assumption $|\lambda| < 1$, we can choose $m \geq 1$ such that $|\lambda|^m \|T\| \leq 1$. Observing that $A^m T^n = \lambda^{nm} T^n A^m$, we have:

$$\|A^m T^n x\| = |\lambda|^{nm} \|T^n A^m x\| \leq |\lambda|^{nm} \|T^n\| \|A^m x\| \leq \|A^m x\|.$$

Hence, each element of the orbit $\{A^m T^n x\}_{n\geq 1}$ is bounded whereas this orbit is unbounded because it is dense in the image $A^m(E)$, a contradiction. Thus, T cannot be hypercyclic. □

Moreover, we discovered in [5] an analogous of Theorem 9 for convex cyclic operators. Recall that an operator T is *convex cyclic* if there exists a vector x such that $\{p(T)x \ : \ p$ is a convex polynomial$\}$ is dense (a polynomial p is convex if its coefficients are non-negative and $p(1) = 1$).

In some sense, the Fréchet setting is more interesting for hypercyclicity of extended eigenoperators, because in general the non-transference theorem (Theorem 9) is not true in Fréchet spaces. Indeed, for each $|\omega| < 1$ let us consider the operator $T_{\omega,b} f = f'(\omega^{-1} z + b)$ defined on the space of entire functions $H(\mathbb{C})$. It is known that $T_{\omega,b}$ is hypercyclic (see [22]) and the derivative operator D is an extended eigenoperator of $T_{\omega,b}$ associated with an extended eigenvalue of modulus <1, specifically $T_{\omega,b} D = \frac{1}{\omega} T_{\omega,b} D$. However D continues being hypercyclic.

The hypercyclic behavior of continuous operators in Fréchet spaces is quite different from that of operators in Banach spaces. For example, Godefroy and Shapiro [12], extending the results by Birköff and MacLane, proved that any non-scalar continuous operator on $H(\mathbb{C})$ that commutes with the differentiation operator is hypercyclic. This kind of result is not possible for operators on Banach spaces because contractions are never hypercyclic. In relation to the result of Godefroy and Shapiro, one can ask the following question: Which extended eigenoperators of the differentiation operator are hypercyclic? Let us remark that the set of extended eigenvalues of the differentiation operator D is the whole plane $\mathbb{C}$. Indeed, for each

$\lambda \in \mathbb{C}$, the dilation operator $R_\lambda f(z) = f(\lambda z)$ is an extended eigenoperator of D associated to the extended eigenvalue λ.

We found that hypercyclicity of several operators that are extended eigenoperators of the differentiation operator have been studied in the literature. For instance, it was proved by Bernal and Montes [6] that for any $\lambda, b \in \mathbb{C}$ the operator $C_{\lambda,b} f(z) = f(\lambda z + b)$ is never hypercyclic on the space of entire functions $H(\mathbb{C})$. On the other hand, it was proved in [1, 11, 22] that the operator $T_{\lambda,b} f(z) = f'(\lambda z + b)$ is hypercyclic on $H(\mathbb{C})$ if and only if $|\lambda| \geq 1$. An easy check shows that $C_{\lambda,b}$ and $T_{\lambda,b}$ are extended eigenoperators of the differentiation operator D associated to the extended eigenvalue λ.

At first glance, a more complicated structure appears to emerge with hypercyclic extended eigenoperators of the differentiation operator D. However, the results in [4] basically indicate, that in most cases the only hypercyclic extended eigenoperators of D are those associated with extended eigenvalues of modulus greater than or equal to one, except for multiples of a fixed operator, that in this case are multiples of the operator $C_{\lambda,b}$. In such studies it is very important the following factorization of the extended eigenoperators of D which is reminiscent of the results in [17].

Theorem 10 *If T is an extended eigenoperator of D then there exists an entire function of exponential type $\phi(z)$ such that $T = R_\lambda \phi(D)$ where $R_\lambda f(z) = f(\lambda z)$ is the dilation operator.*

The operator T is said to have a *hypercyclic subspace* if there exists a closed, infinite dimensional subspace whose non-zero vectors are hypercyclic for T. The study of hypercyclic subspaces is a mainstream topic in the theory of hypercyclic operators (see [10, Chapter 10], and [3, Chapter 8]). Similarly, mainly motivated by the invariant subspace problem, it has been studied when an operator T has a *supercyclic subspace*, that is, an infinite dimensional closed subspace whose nonzero vectors are supercyclic for T (see [25]). Petterson, Mennet, and Shkarin (see [24, 27, 30]) proved the following result:

Theorem 11 *Every operator on $H(\mathbb{C})$ commuting with D which is not a multiple of the identity has an hypercyclic subspace.*

Thus, the following question arises: Assume that T is an extended eigenoperator of D. When does T have a hypercyclic subspace? The results in [14] address this direction, obtaining results for the existence of hypercyclic and supercyclic subspaces.

5 Concluding Remarks and Open Questions

In [17] we raised he following problem: which is the extended spectrum of C_0 in ℓ^p. We conjectured that this set is exactly $[1, \infty)$?. However, to obtain something in this direction, we run into the problem of somehow extending Kriette and Trutt result to ℓ^p spaces.

Question 1 Let us consider C_0 on ℓ^p, $1 \leq p < \infty$. Is $\text{Ext}(C_0) = [1, \infty)$?

We have the feeling that the answer is affirmative, because in the continuous case, the extended spectra does not depend on p.

In our work we have also been able to characterize the extended eigenoperators associated with an extended eigenvalue λ. Our contributions show that in most cases the eigenoperators of an operator T can be described by a factorization of the form $X_0 A$ where X_0 is a fixed eigenoperator and A is an element of the commutant of T. For the special case of the Cesàro operator C_1 the following question was posed in [17].

Question 2 If T is an extended eigenoperator of C_1. Is there any factorization of the form $X_0 A$ where X_0 is a fixed extended eigenoperator and A an element of the commutant of C_1?

We know that the question above is affirmative the Hilbert space setting, however we do not the answer in general.

From the results in Sect. 3, we have information about the commutant of a composition operator whose symbol has a bounded Königs domain. However, for unbounded Königs domains, the following question arises:

Question 3 Suppose that the Königs domain of a composition operator is unbounded and strictly starlike with respect to the origin. Does C_φ have a minimal commutant?

We can deduce from Theorems 6 and 8 that there exist compact composition operators with a minimal commutant and other compact composition operators without minimal commutant. It is natural to ask the following question.

Question 4 Which compact composition operators have a minimal commutant?

Let us recall that an operator T is *supercyclic* in $\mathcal{X}$ if there exists $x \in \mathcal{X}$ such that the projective orbit $\{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in $\mathcal{X}$. If an operator T has a multiple which is hypercyclic then the operator is supercyclic. From Theorem 9 it could be deduced that no multiple of T is hypercyclic. At first glance we believed that the above result is true for supercyclic operators. However it was point out in [5] that the same statement is not true for supercyclic operators. However, we believe that under additional conditions, some results can be derive in this direction.

Question 5 Under what conditions can we assert that an extended eigenoperator associated with an extended eigenvalue of modulus less than one is not supercyclic?

In general, in order to understand such structure, it would be interesting to solve these kind of problems.

Question 6 Let us suppose that T is a hypercyclic operator. Which extended eigenoperators of T are hypercyclic?

Finally, we finish this paper with the following general question:

Question 7 Let us suppose that T is a hypercyclic operator. Which extended eigenoperators of T have a hypercyclic subspace?

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Summability to the Hilbert Transform

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Abstract New sufficient conditions for the almost everywhere convergence (at the Lebesgue points) of general summability means of the conjugate integral to the Hilbert transform are obtained in the paper. Consequences and applications are also discussed.

Keywords Fourier integral · Hilbert transform

1991 Mathematics Subject Classification Primary 42A38; Secondary 42A45

1 Introduction

Of course, the two main operators of the one-dimensional harmonic analysis, the Fourier transform and the Hilbert transform, have different "obligations" but certain general features are similar. For example, the idea of summability so natural for the Fourier transform makes sense for the Hilbert transform as well. Recall that the latter is defined in the principal value sense as

$$\begin{aligned}\mathcal{H}g(x) &:= \frac{1}{\pi}\,(\mathrm{P.V.})\int_{\mathbb{R}} g(x-u)\frac{du}{u} = \frac{1}{\pi}\,(\mathrm{P.V.})\int_{\mathbb{R}} \frac{g(u)}{x-u}\,du \\ &= \frac{1}{\pi}\lim_{\delta\downarrow 0}\mathcal{H}_\delta g(x), \qquad x\in\mathbb{R},\end{aligned}$$

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where

$$\mathcal{H}_\delta g(x) := \frac{1}{\pi}\int_\delta^\infty \{g(x-u) - g(x+u)\}\frac{du}{u}$$
$$= \frac{1}{\pi}\int_{|x-t|>\delta} \frac{f(t)}{x-t}\,dt.$$

It is well known that this limit exists for almost every $x \in \mathbb{R}$ if, for example, $g \in L^p(\mathbb{R})$, $1 \le p < \infty$.

The results in this paper are about Cesàro's and more general summability of the conjugate function

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\sin(t-x)u\,dt\,du. \tag{1.1}$$

In his review [3] of Wiener's book "The Fourier Integral and Certain of its Applications" [8], Hille writes: "*An analyst of the older generation would probably understand the term Fourier integral to refer to the formula*" (1.1), "*which is valid for a very restricted class of functions*". This is one of the forms of conjugation; it is used, for example, by Titchmarsh in [6] while showing, in a formal way, how the Hilbert transform appears as a natural conjugation in the non-periodic case.

The following very particular result (see [9, Vol.II, Ch.XVI, Th. 1.22]) demonstrates how Cesàro's $(C,1)$ method works in this setting reducing the conjugation to the Hilbert transform.

Theorem 1.1 *If* $\dfrac{|f(t)|}{1+|t|}$ *is integrable on* $\mathbb{R}$*, then the* $(C,1)$ *means*

$$-\frac{1}{\pi}\int_{-\infty}^\infty f(x+t)\left[\frac{1}{t} - \frac{\sin Nt}{Nt^2}\right]dt$$

converge to the Hilbert transform $\mathcal{H}f(x)$ *almost everywhere as* $N \to \infty$.

It was applied in [5] for establishing re-expansion conditions for the function given by its cosine (sine) Fourier transform in the opposite, sine (cosine) transform. The mentioned conditions were necessary and sufficient and were given in terms of the Hilbert transform of the re-expanded function.

The λ-summability on the half-axis means that we have to deal with (estimate)

$$\lim_{N\to\infty}\int_0^\infty \lambda\Big(\frac{u}{N}\Big)\int_{-\infty}^\infty f(x+t)\sin ut\,dt\,du. \tag{1.2}$$

Indeed, (1.1) resembles the Fourier inverse (for the conjugate function), or, in other words, the inner integral in (1.1) resembles the Fourier transform times modulation. By this, standard summability comes into play by introducing the multiplier $\lambda\big(\frac{u}{N}\big)$, with varying parameter N; the simplest example is $\lambda\left(\frac{u}{N}\right) = \left(1-\frac{u}{N}\right)$ for $0 \le u \le N$, and zero otherwise.

In Proposition 8.2.7 of [1, Section 8.2.3] related results are expressed in terms of the Hilbert transform of λ. We study the convergence of (1.2) in terms of the Fourier transforms of λ and estimates of such Fourier transforms. The conditions we are going to impose on λ slightly have a look of the conditions posed on the functions generating summability methods in [7, Section 5.3], while studying the convergence of the appropriate multidimensional means to the Riesz transforms.

A natural set where the almost everywhere convergence of the means for an integrable function f is usually studied is the set of its Lebesgue points. It will be convenient for us to write the defining relation of a Lebesgue point x of f in the form

$$\frac{1}{\delta}\int_0^{\delta} |f(x+u) - f(x-u)|\,du = o(1)$$

as $\delta \to 0+$.

The structure of the paper is simple. In the following section we formulate and prove the main result and one of the possible corollaries. In the last section, we discuss related settings for the conditions on λ.

We shall use the notation " $\lesssim$ " and " $\gtrsim$ " as abbreviations for " $\le C$ " and " $\ge C$ ", with C being an absolute positive constant, maybe different in different occurrences.

2 Summability to the Hilbert Transform

We will prove a general result, where the summability is defined by (1.2). In the formulation of the theorem as well as in its proof, the cosine and sine Fourier transforms will be involved. For a function $g : [0,\infty) \to \mathbb{C}$, the cosine Fourier transform

$$\widehat{g_c}(x) = \int_0^{\infty} g(t)\cos xt\,dt$$

and the sine Fourier transform

$$\widehat{g_s}(x) = \int_0^{\infty} g(t)\sin xt\,dt,$$

may exist in that or another sense. For example, if g is of bounded variation, both exist in improper sense. As g either λ or λ' will be used. For example, $\widehat{\lambda'_c}$ means the cosine Fourier transform of λ'.

Theorem 2.1 *Let $f \in L^1(\mathbb{R})$. Then (1.1) is λ-summable (1.2) to $-\mathcal{H}f(x)$ at every x, where it exists, and which is a Lebesgue point of f, and so almost everywhere* *provided* λ *is an integrable and locally absolutely continuous function on* $(0,\infty)$*, such that*

$$\lim_{t\to\infty}\lambda(t)=0$$

and

$$\lim_{t\to 0+}\lambda(t)=1,$$

the derivative λ' *is integrable on* $(0,\infty)$ *and*

$$|\widehat{\lambda'_c}\big(t\big)|\le\varphi(t),$$

where φ *is locally absolutely continuous on* $(1,\infty)$*, uniformly bounded on* $[1,\infty)$*, and satisfies*

$$\int_1^\infty\frac{\varphi(t)}{t}\,dt<\infty \tag{2.1}$$

and

$$\int_1^\infty|\varphi'(t)|\,dt<\infty. \tag{2.2}$$

Proof Since both f and λ are Lebesgue integrable, we can change the order of integration in (1.2):

$$\begin{aligned}&\lim_{N\to\infty}\int_0^\infty\lambda\Big(\frac{u}{N}\Big)\int_{-\infty}^\infty f(x+t)\sin ut\,dt\,du\\&=\lim_{N\to\infty}\int_0^\infty[f(x+t)-f(x-t)]\int_0^\infty\lambda\Big(\frac{u}{N}\Big)\sin ut\,du\,dt\\&=\lim_{N\to\infty}\int_0^\infty[f(x+t)-f(x-t)]\,N\widehat{\lambda_s}\big(Nt\big)\,dt.\end{aligned}$$

Integrating by parts, we obtain

$$N\widehat{\lambda_s}\big(Nt\big)=\frac{1}{t}+\frac{1}{t}\widehat{\lambda'_c}\big(Nt\big) \tag{2.3}$$

provided that

(i) λ is locally absolutely continuous on $(0,\infty)$,
(ii) $\lim_{t\to\infty}\lambda(t)=0$,
and

(iii) $\lim_{t\to 0+} \lambda(t) = 1$
hold.

For any $\delta > 0$, we get

$$\lim_{N\to\infty} \int_{\delta}^{\infty} [f(x+t) - f(x-t)]\, N\widehat{\lambda_s}\big(Nt\big)\, dt$$
$$= \int_{\delta}^{\infty} \frac{f(x+t) - f(x-t)}{t}\, dt$$

provided that

(iv) $\lim_{s\to\infty} \widehat{\lambda_c'}(s) = 0$
holds. Furthermore,

$$\int_{\frac{1}{N}}^{\delta} [f(x+t) - f(x-t)]\, N\widehat{\lambda_s}\big(Nt\big)\, dt$$
$$= \int_{\frac{1}{N}}^{\delta} \frac{f(x+t) - f(x-t)}{t}\, dt$$
$$+ \int_{\frac{1}{N}}^{\delta} [f(x+t) - f(x-t)]\frac{1}{t}\widehat{\lambda_c'}\big(Nt\big)\, dt. \tag{2.4}$$

For the last term on the right, the assumption of the theorem implies

$$\int_{\frac{1}{N}}^{\delta} |f(x+t) - f(x-t)|\, \frac{1}{t}|\widehat{\lambda_c'}\big(Nt\big)|\, dt$$
$$\le \int_{\frac{1}{N}}^{\delta} |f(x+t) - f(x-t)|\, \frac{\varphi(Nt)}{t}\, dt. \tag{2.5}$$

Integrating by parts, we obtain

$$\int_{\frac{1}{N}}^{\delta} |f(x+t) - f(x-t)|\, \frac{\varphi(Nt)}{t}\, dt$$
$$= \left[\frac{1}{t}\int_0^t |f(x+u) - f(x-u)|\, du\, \varphi(Nt)\right]_{\frac{1}{N}}^{\delta}$$
$$+ \int_{\frac{1}{N}}^{\delta} \frac{1}{t}\int_0^t |f(x+u) - f(x-u)|\, du\, \frac{\varphi(Nt)}{t}\, dt$$
$$- \int_{\frac{1}{N}}^{\delta} \frac{1}{t}\int_0^t |f(x+u) - f(x-u)|\, du\, N\varphi'(Nt)\, dt. \tag{2.6}$$

The integrated terms are $o(1)$ times $\varphi(N\delta)$ and $\varphi(1)$, respectively. Hence both integrated terms are $o(1)$ provided x is a Lebesgue point, since φ is uniformly bounded on $[1, \infty)$.

If x is a Lebesgue point, then for the last two integrals on the right-hand side of (2.6) to be $o(1)$, it suffices

$$\int_{\frac{1}{N}}^{\delta} \frac{\varphi(Nt)}{t}\, dt = \int_1^{N\delta} \frac{\varphi(t)}{t}\, dt$$

and

$$\int_{\frac{1}{N}}^{\delta} N|\varphi'(Nt)|\, dt = \int_1^{N\delta} |\varphi'(t)|\, dt$$

to be uniformly bounded. This is the case if

(v) $\int_1^\infty \frac{\varphi(t)}{t}\, dt < \infty$
and
(vi) $\int_1^\infty |\varphi'(t)|\, dt < \infty$,
respectively, hold true.

Finally,

$$\left|\int_0^{\frac{1}{N}} [f(x+t) - f(x-t)]\, N\widehat{\lambda_s}\big(Nt\big)\, dt\right|$$

$$\lesssim N \int_0^{\frac{1}{N}} |f(x+t) - f(x-t)|\, dt = o(1),$$

provided that

(vii) $|\widehat{\lambda_s}|$ is bounded,
since x is a Lebesgue point.

We now have to check that (i)–(vii) are valid under assumptions of the theorem. Indeed, (i)–(iii) are assumed. Since the Fourier transform of an integrable function vanishes at infinity, by the Riemann-Lebesgue lemma, (iv) holds. Similarly, the Fourier transform of an integrable function is bounded, which implies (vii). Finally, (v) and (vi) are just (2.1) and (2.2), respectively, which completes the proof. □

Back to Titchmatsh's book, the result of [6, Th.107] is given for any (C, α) method. In turn, Theorem 1.1 follows from it by taking $\alpha = 1$ and integrating then by parts in the integral representation of $\widehat{\lambda_s}\big(Nt\big)$. In fact, [6, Th.107] is a corollary of the latter theorem.

Corollary 2.2 *Let $f \in L^1(\mathbb{R})$. Then, for any positive α, (1.1) is summable (C, α) to $\mathcal{H}f(x)$ at every x, where it exists, and which is a Lebesgue point of f, and so almost everywhere.*

Proof First, it suffices to suppose that $0 < \alpha < 1$. The (C, α) summability means that we estimate

$$\lim_{N\to\infty} \int_0^N \big(1 - \frac{u}{N}\big)^\alpha \int_{-\infty}^\infty f(x+t) \sin ut \, dt \, du$$
$$= \lim_{N\to\infty} \int_0^\infty [f(x+t) - f(x-t)] \int_0^N \big(1 - \frac{u}{N}\big)^\alpha \sin ut \, du \, dt.$$

We have to check the assumptions of Theorem 2.1 for $\lambda(t) = (1-t)_+^\alpha$, where the standard subindex $+$ means that λ takes the indicated value only when non-negative, and vanishes if negative. All the conditions on λ are checked readily. Since the Fourier transform of its derivative is $O(t^{-\alpha})$, we can take $\varphi(t) = t^{-\alpha}$. It remains to observe that (2.1) and (2.2) reduce to the integrability of $t^{-1-\alpha}$ over $(1, \infty)$. □

3 Remarks on the Assumptions

The proof of Theorem 2.1 shows that the assumptions of the theorem are not very restrictive and seem quite natural. However, it is worth discussing certain options for these assumptions and reasonable substitutions.

By the Riemann-Lebesgue lemma, (iv) holds, for example, if λ' is integrable. This is the case if, say, λ is of bounded variation. One more option for this is assuming λ to be general monotone. Additionally assuming then

$$\int_0^\infty \frac{|\lambda(t)|}{t} \, dt < \infty$$

leads to the boundedness of total variation; for all these notions and facts, see, e.g., [4]. However, since we assume λ to be integrable, a natural way to have λ' integrable is to assume λ to belong to the Sobolev space $W^{1,1}$.

It is interesting that not proceeding from (2.4) to (2.5) and then to (2.6) but integrating by parts directly in (2.4), which seems to be more precise, leads to worse results and conditions. For example, Corollary 2.2 is not verifiable then.

There is an interesting way to guarantee (2.1), or more precisely, the same integral with $|\widehat{\lambda_c'}(t)|$ in place of $\varphi(t)$. In this case, the finiteness of the integral can be ensured by belonging of λ' to the local Hardy space. For many details about this space, see a recent paper [2].

In conclusion, there are many interesting combinations to substitute for the assumptions of Theorem 2.1.

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Applying Wavelet Transforms as a Solution for Convolution Type Equations

Vladimir A. Lukianenko

Abstract By analogy with the method of solving a convolution-type equation regarding Fourier transforms, the work presents similar properties for the continuous wavelet transform. Such as formulas for the transform of Fourier convolution with the kernel in form of parent wavelet, shift in the original, scaling, representation of the generalized delta functions via parent wavelets, etc. The expression of continuous wavelet transform through convolution regarding Fourier transform, that is used to prove Parseval's equality. Further results include: A convolution operator regarding continuous wavelet transform. A showcase, based on the example of a convolution-type equation with two kernels, of how with use of wavelet transform the equation comes down to a solution for the Riemann boundary value problem in the theory of analytic functions. A demonstration of the fact that Noether's theorems are analogous to theorems on the solvability of convolution type equations of Fourier transform. Showcase of that the advantage of using the wavelet transform increases with the availability of approximate data and for improper tasked problems with operators of convolution type.

Keywords Wavelet Transform · Convolution Tape Equations · Regularizing algorithms

1 Introduction

Convolution type equations have different representations depending on the type of transform that turns the convolution of two functions into the product of transforms that are included in the convolution of the function [1]. The convolutions are most commonly used regarding transforms of Fourier, Laplace, Mellin, etc. In applications, convolution is the integral operator with a kernel that depends on

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the difference (or sum, of relationship between the arguments). Operators, invariant relative to the shift, are one of such. In case of boundary value problems that come down to convolution type equations, aka integral equations of convolution type (with two kernels, paired, of smooth transition, of Wiener-Hopf, etc.), their discrete analogs are presented in the monographs of Cherskiy Yu. I., Gakhov F. D. [1, 2], and many other original works [3–5]. However, in scientific literature there are practically no works, especially in Russian, in which continuous the wavelet transform is used to solve a convolution-type equation regarding Fourier and wavelet transforms. The continuous wavelet transform is calculated by convolution of the signal and a wavelet function. Wavelet function is a small oscillatory wave which contains both the analysis and the window function. Wavelet analysis has a wide range of applications [6–15]. By analogy with the properties of Fourier transform, we'll demonstrate the corresponding properties of the wavelet transform that are required for the solution of convolution type equation. We will confine ourselves mainly to spaces $L_p(\mathbb{R})$ for p=1,2, although the results remain valid for $1 \leqslant p < \infty$ (with obvious revisions). The obtained properties are used to study convolution-type equations.

2 Continuous Wavelet Transform and Its Properties

2.1 Background Information and Results

We devote this section to define some known topics about Fourier transform (FT) and continuous wavelet transform (CWT) and introduce the notation we will use in next section. Most results in this section come from [15–28].

With the further purpose of applying continuous wavelet transform to solve the Equation of convolution type (ECT), we give analogs of the statements for the ECT theory [1].

Definition 1 The continuous wavelet transform (CWT) of the function $f(t) \in L_2(\mathbb{R})$ with the wavelet function $\varphi(t) \in L_1(\mathbb{R}) \bigcap L_2(\mathbb{R})$ is defined by the following formula

$$(W_\varphi f)(a,b) = F_\varphi(a,b) = \int_{\mathbb{R}} f(t)\overline{\varphi_{a,b}}(t)dt = (f, \varphi_{ab})_{L_2}, \tag{1}$$

provided that the integral exists, where

$$\varphi_{a,b}(t) = |a|^{-\frac{1}{2}}\, \varphi\left(\frac{t-b}{a}\right),\ a \in \mathbb{R}_+,\ a \neq 0,\ b \in \mathbb{R}, \tag{2}$$

and $\varphi(t)$ satisfies the following condition:

$$C_\varphi = \int_{\mathbb{R}} \frac{|\Phi(\xi)|^2}{|\xi|} d\xi < \infty, \tag{3}$$

$$|(W\varphi f)(a,b)| \leqslant ||f||_{L_2}\,||\varphi||_{L_2},$$

where $\Phi(\xi)$ there is a Fourier transform (FT) of the function $\varphi(t)$:

$$\Phi(\xi) = (\mathcal{F}\varphi)\,(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t) e^{it\xi} dt \tag{4}$$

The wavelet function $\varphi(t)$ it is called a basis (parent) wavelet.

The inverse FT is defined by the formula

$$\varphi(t) = (\mathcal{F}^{-1}\,\Phi\,)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(\xi) e^{-it\xi} d\xi \tag{5}$$

From (3) follows that $\Phi(\xi) = (\mathcal{F}\varphi)(\xi)$ is continuous in vicinity of the point $\xi = 0$ and $\Phi(0) = 0$ (wavelets in which this condition is violated are also considered):

$$\Phi(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t) dt = 0.$$

Let's consider the properties of basis wavelet functions. We shall insert the convolution of functions $k(t) \in L_1(\mathbb{R})$ and $f(t) \in L_2(\mathbb{R})$ regarding Fourier transform:

$$h(t) = (k * f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t-s) f(s) ds. \tag{6}$$

Lemma 1 *If the function $\varphi(t)$ is the parent wavelet and $\psi(t) \in L_1(\mathbb{R})$, then the convolution of functions $(\varphi * \psi)(t) = h(t)$ is the parent wavelet.*

Proof Write down

$$\|H(\xi)\|^2_{L_2} = \int_{\mathbb{R}} |h(t)|^2 dt = \int_{\mathbb{R}} |(\varphi * \psi)(t)|^2 dt = \int_{\mathbb{R}} |\Phi(\xi) \cdot \Psi(\xi)|^2 d\xi =$$

$$= \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t-s)\psi(s) ds \right|^2 dt \leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t-s)||\psi(s)|^{\frac{1}{2}} |\psi(s)|^{\frac{1}{2}} ds \right)^2 dt \leqslant$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\psi(s)| ds \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t-s)|^2 |\psi(s)| ds \right) dt =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\psi(s)| ds \right)^2 \int_{\mathbb{R}} |\varphi(t)|^2 dt = \frac{1}{\sqrt{2\pi}} \|\psi\|^2_{L_1} \cdot \|\varphi\|^2_{L_2}.$$

Thus, convolution $(\varphi * \psi)(t) = h(t) \in L_2(\mathbb{R})$.

Further, the inequality is true:

$$C_h = C_{\varphi*\psi} = \int_{\mathbb{R}} \frac{|\mathcal{F}(\varphi*\psi)(\xi)|^2}{|\xi|} d\xi = \int_{\mathbb{R}} \frac{|\Phi(\xi)\Psi(\xi)|^2}{|\xi|} d\xi \leqslant$$

$$\leqslant (max|\Psi(\xi)|)^2 \int_{\mathbb{R}} \frac{|\Phi(\xi)|^2}{|\xi|} d\xi = C_\varphi \|\Psi(\xi)\|^2_{L_\infty(\mathbb{R})} < \infty,$$

which proves the Lemma. □

Note that if it is necessary to associate the parent wavelet $\varphi(t)$ with another parent wavelet $h(t)$ using the convolution operator $\varphi * \psi = h$, then it is necessary to indicate the function ψ, which is the kernel of the integral convolution operator, i.e. to solve the integral equation of convolution type of the first kind. In the general case, such a solution can be obtained only approximately, for which it is necessary to use to solve equations (in Fourier cases) $H(\xi) = \Psi(\xi)\Phi(\xi)$ and $|\Phi(\xi)| \to 0,\ |\xi| \to \infty$, for example, with first order regularizer

$$\Psi_\alpha(\xi) = R_\alpha(\xi) H(\xi),\ R_\alpha(\xi) = \frac{\overline{\Phi}(\xi)}{\alpha\left(1+\xi^2\right) + |\Phi(\xi)|^2}.$$

The regularization parameter $\alpha > 0$ depends on the infelicity level of the convolution operator and the right-hand side of $h(t)$. Choosing different functions $\psi(t)$ one can get the parent wavelets $h(t)$ with the necessary set of properties, i.e., $h(t)$ can be found as a result of the solution a direct task.

The Fourier transform of the function $\varphi_{a,b}(t)$:

$$\mathcal{F}\{\varphi_{a,b}(t)\}(\xi) = \mathcal{F}\left\{\frac{1}{\sqrt{a}}\varphi\left(\frac{t-b}{a}\right)\right\} = |a|^{\frac{1}{2}} e^{ib\xi}\Phi(a\xi). \tag{7}$$

Indeed,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{a}}\, \varphi\left(\frac{t-b}{a}\right) e^{it\xi} dt = \left\{\frac{t-b}{a} = \tau,\ t = b + a\tau,\ dt = a d\tau\right\} =$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{|a|}{\sqrt{a}} \int_{\mathbb{R}} \varphi(\tau) e^{ib\xi} e^{ia\xi\tau} d\tau = e^{ib\xi} \frac{|a|^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\tau) e^{ia\xi\tau} d\tau.$$

Reverse FT:

$$\mathcal{F}^{-1}\{|a|^{\frac{1}{2}} e^{ib\xi}\Phi(a\xi)\} = \frac{1}{\sqrt{a}}\, \varphi\left(\frac{t-b}{a}\right) = \varphi_{a,b}(t).$$

Let's consider the main properties of CWT.

1. Linearity:

$$W_\varphi(\alpha_1 f_1 + \alpha_2 f_2)(a,b) =$$

$$= \alpha_1 (W_\varphi f_1)(a,b) + \alpha_2 (W_\varphi f_2)(a,b) = \alpha_1 F_{1\varphi}(a,b) + \alpha_2 F_{2\varphi}(a,b),$$

where α_1, α_2 are conctants.

2. Shift in the original:

$$W_\varphi\{f(t-\tau)\}(a,b) = W_\varphi(f)(a,b-\tau) = F_\varphi(a,b-\tau), \tag{8}$$

$$W_\varphi^{-1}\{F_\varphi(a,b-\tau)\}(t) = f(t-\tau)$$

$$W_\varphi\{f(t-\tau)\}(a,b) = \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(t-\tau)\,\overline{\varphi\left(\frac{t-b}{a}\right)}dt = \begin{Bmatrix} t-\tau = s,\ dt = ds \\ t = s+\tau \end{Bmatrix} =$$

$$= \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(s)\,\overline{\varphi\left(\frac{s-(b-\tau)}{a}\right)}ds.$$

3. Scaling, $\sigma \neq 0$:

$$W_\varphi\left\{f\left(\frac{t}{\sigma}\right)\right\}(a,b) = |\sigma| W_\varphi(f)\left(\frac{a}{\sigma},\frac{b}{\sigma}\right) = |\sigma| F_\varphi\left(\frac{a}{\sigma},\frac{b}{\sigma}\right) \tag{9}$$

$$W_\varphi\left\{f\left(\frac{t}{\sigma}\right)\right\}(a,b) = \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f\left(\frac{t}{\sigma}\right)\overline{\varphi\left(\frac{t-b}{a}\right)}dt = \begin{Bmatrix} \frac{t}{\sigma} = \tau,\ dt = \sigma d\tau \\ t = \tau\sigma \quad \sigma > 0 \end{Bmatrix} =$$

$$= \frac{\sigma}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(\tau)\,\overline{\varphi\left(\frac{\sigma\tau-b}{a}\right)}d\tau = \frac{\sigma}{|\sigma|^{\frac{1}{2}}|\frac{a}{\sigma}|^{\frac{1}{2}}}\int_{\mathbb{R}} f(\tau)\,\overline{\varphi\left(\frac{\tau-\frac{b}{\sigma}}{\frac{a}{\sigma}}\right)}d\tau = |\sigma| F_\varphi\left(\frac{a}{\sigma},\frac{b}{\sigma}\right).$$

$$W_\varphi(f(-t))(a,b) = W_\varphi(f)(-a,-b) \tag{10}$$

For $Q\varphi = \varphi(-t)$ we have:

$$W_{Q\varphi}(Qf)(a,b) = W_\varphi(f)(a,-b).$$

$$W_{Q\varphi}(Qf)(a,b) = \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(-t)\,\overline{\varphi\left(\frac{-t+b}{a}\right)}dt =$$

$$= \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(\tau)\,\overline{\varphi\left(\frac{\tau+b}{a}\right)}d\tau = W_\varphi(f)(a,-b).$$

4. Symmetry:

$$(W_\varphi f)(a,b) = (W_f\varphi)\left(a, \frac{-b}{a}\right) \tag{11}$$

Truly, $(W_\varphi f)(a,b) = \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(t)\,\overline{\varphi\left(\frac{t-b}{a}\right)}dt = \left\{\begin{matrix} \frac{t-b}{a} = s, & dt = ads \\ t = b + as \end{matrix}\right\} =$

$$= \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(b+as)\,\overline{\varphi(s)}\,a\,ds = \frac{1}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} \overline{\varphi(s)} f\left(\frac{s+\frac{b}{a}}{a}\right)ds = \overline{(W_f\varphi)}\left(a, \frac{-b}{a}\right).$$

Lemma 2 *It is fair to represent CWT as a convolution regarding the Fourier transform*

$$(W_\varphi f)(a,b) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\sqrt{2\pi}}{|a|^{\frac{1}{2}}}\,\overline{\varphi\left(\frac{t-b}{a}\right)} f(t)dt = \left(\frac{\sqrt{2\pi}}{|a|^{\frac{1}{2}}}\,\overline{\varphi\left(\frac{t}{a}\right)} * f(t)\right)(b) =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} k(b-t)f(t)dt = (k*f)(b), \quad k(t) = \frac{\sqrt{2\pi}}{|a|^{\frac{1}{2}}}\,\overline{\varphi\left(-\frac{t}{a}\right)}. \tag{12}$$

Lemma 3 *Suppose the function $f(t)$ is a homogeneous function of degree m, that is $f(\lambda t) = \lambda^m f(t)$, then*

$$(W_\varphi f)(\lambda a, \lambda b) = |\lambda|^{\frac{1}{2}}\lambda^m (W_\varphi f)(a,b) = |\lambda|^{\frac{1}{2}}\lambda^m F_\varphi(a,b). \tag{13}$$

Proof $(W_\varphi f)(\lambda a, \lambda b) = \frac{1}{|\lambda a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(t)\,\overline{\varphi\left(\frac{t-\lambda b}{\lambda a}\right)}dt = \{t = \lambda s, dt = \lambda ds\} =$

$$= \frac{|\lambda|}{|\lambda|^{\frac{1}{2}}|a|^{\frac{1}{2}}}\int_{\mathbb{R}} f(\lambda s)\,\overline{\varphi\left(\frac{\lambda s - \lambda b}{\lambda a}\right)}ds = \frac{|\lambda|^{\frac{1}{2}}}{|a|^{\frac{1}{2}}}\int_{\mathbb{R}} \lambda^m f(s)\,\overline{\varphi\left(\frac{s-b}{a}\right)}ds =$$

$= |\lambda|^{\frac{1}{2}}\lambda^m (W_\varphi f)(a,b)$, that's what I needed to prove. □

The Parseval (Plancherel) equality in the continuous wavelet transform is basic, as in the theory of the Fourier transform.

Theorem 1 *Suppose the functions $f, g \in L_2(\mathbb{R})$. Then we have*

$$\big((W_\varphi f)(a,b), (W_\varphi g)(a,b)\big)_{L_2(\mathbb{R}\times\mathbb{R})} = 2\pi C_\varphi (f,g)_{L_2(\mathbb{R})}. \tag{14}$$

where the constant C_φ is defined by the formula (3).

Proof The proof follows from the representation of the Fourier wavelet transform and Parseval's theorem on the Fourier transform.

$$(W_\varphi f)(a,b) = \int_{\mathbb{R}} f(t)\ \frac{1}{|a|^{\frac{1}{2}}}\ \overline{\varphi\left(\frac{t-b}{a}\right)} dt = (f, \varphi_{a,b}) = (F, \Phi_{ab}) =$$

$$= \int_{\mathbb{R}} F(\xi)\overline{|a|^{\frac{1}{2}}\ e^{ib\xi}\ \Phi(a\xi)} d\xi = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\xi)\overline{\Phi(a\xi)}\ e^{-ib\xi} d\xi.$$

Similarly

$$(W_\varphi g)(a,b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} G(\xi)\overline{\Phi(a\xi)}\ e^{-ib\xi} d\xi.$$

$$\big((W_\varphi f)(a,b), (W_\varphi g)(a,b)\big) = (F_\varphi(a,b), G_\varphi(a,b)) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (W_\varphi f)(a,b)\overline{(W_\varphi g)(a,b)}\ \frac{da\ db}{|a|^2} =$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(|a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\xi)\overline{\Phi(a\xi)}\ e^{-ib\xi}\ d\xi\ \cdot\ |a|^{\frac{1}{2}} \overline{\int_{\mathbb{R}} G(\eta)\overline{\Phi(a\eta)}\ e^{-ib\eta}\ d\eta}\right)\ \frac{da\ db}{|a|^2} =$$

$$= 2\pi \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{F}^{-1}\left\{F(\xi)\overline{\Phi(a\xi)}\right\}(b)\ \overline{\mathcal{F}^{-1}\left\{G(\eta)\overline{\Phi(a\eta)}\right\}(b)}\ db\right) \frac{da}{|a|} =$$

$$= 2\pi \int_{\mathbb{R}} \left(F(\xi)\overline{\Phi(a\xi)}, G(\xi)\overline{\Phi(a\xi)}\right)_{L_2} \frac{da}{|a|} =$$

$$= 2\pi \int_{\mathbb{R}} F(\xi)\overline{G(\xi)} \left(\int_{\mathbb{R}} \overline{\Phi(a\xi)}\Phi(a\xi)\frac{da}{|a|}\right) d\xi =$$

$$= 2\pi \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\Phi(a\xi)|^2 da}{|a|}\right) F(\xi)\overline{G(\xi)} d\xi = 2\pi \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\Phi(\omega)|^2 d\omega}{|\omega|}\right) F(\xi)\overline{G(\xi)} d\xi =$$

$$= 2\pi C_\varphi (F, G) = 2\pi C_\varphi (f, g)_{L_2(\mathbb{R})},$$

Which proves the theorem. □

Using Parseval's equality, the inversion formula is found—the inverse wavelet transform (Inversion Formula).

Theorem 2 *Let $f(t) \in L_2(\mathbb{R})$, then*

$$f(t) = ((W_\varphi^{-1} F_\varphi(a,b))(t) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}}\int_{\mathbb{R}} (W_\varphi f)(a,b)\ \varphi_{ab}(t)\frac{da\ db}{|a|^2} \tag{15}$$

Proof Let $f(t) \in L_2(\mathbb{R})$, then we have

$$2\pi C_\varphi (f, g)_{L_2} = \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a, b)\, \overline{(W_\varphi g)(a, b)}\, \frac{da\, db}{|a|^2} =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a, b)\, \varphi_{a,b}(t)\, \overline{g(t)}\, \frac{da\, db}{|a|^2}\, dt =$$

$$= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a, b)\, \varphi_{a,b}(t)\, \frac{da\, db}{|a|^2},\ g(t) \right)_{L_2},$$

whence the required result follows. □

Theorem 3 *The wavelet transform provides isomorphism of space* $L_2(\mathbb{R})$ *at* $L_2(\mathbb{R}^2)$.

Proof The proof follows from Theorem 2 and the formula

$$\|f\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} |(W_\varphi f)(a, b)|^2 \frac{da\, db}{|a|^2}, \tag{16}$$

which is obtained from the Parseval equality (14) if $f(t) = g(t)$, end proof. □

Wavelet transform of one-way functions. Consider the one-way functions

$$f_\pm(t) = \frac{1}{2}(\pm f(t) + sign\ f(t))$$

or

$$f_+(t) = \begin{cases} f(t), & t > 0, \\ 0, & t < 0, \end{cases} \qquad f_-(t) = \begin{cases} 0, & t > 0, \\ -f(t), & t < 0. \end{cases} \tag{17}$$

Let's find the wavelet transform

$$(W_\varphi f_\pm)(a, b) = \int_{\mathbb{R}} f_+(t)\, \overline{\varphi_{a,b}(t)}\, dt = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F^\pm(\xi)\, \overline{\Phi(a\xi)}\, e^{-ib\xi}\, d\xi. \tag{18}$$

Here the function $F^+(z)$ is analytic in the upper half-plane, and $F^-(z)$ is analytic in the lower half-plane.

Representation of generalized Delta functions. The integral representation of the Delta function is used to solve the convolution equation

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{ib(\xi - \eta)} db = \delta(\xi - \eta).$$

A similar representation is true for CWT [4], its identity operator is represented as an inverse CWT applied to the CWT. Indeed, if we consider the formula in more detail for treatment of CWT (15):

$$f(t) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\varphi f)(a,b)\, \varphi_{ab}(t)\, \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(s)\, \overline{\varphi_{ab}(s)}\, ds \right) \varphi_{ab}(t)\, \frac{da\, db}{|a|^2} =$$

$$= \int_{\mathbb{R}} f(s) \left(\frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{ab}(s)}\, \varphi_{ab}(t) \frac{da\, db}{|a|^2} \right) ds = \int_{\mathbb{R}} \delta(t-s) f(s) ds = f(t).$$

Thus

$$\delta(t-s) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_{ab}(t)\, \overline{\varphi_{ab}(s)}\, \frac{da\, db}{|a|^2}. \tag{19}$$

Truly, we transform the integral (19) and use the classical representation of generalized function through the Fourier transform:

$$I = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|a|^{\frac{1}{2}}}\, \varphi\left(\frac{(t-b)}{a} \right) \frac{1}{|a|^{\frac{1}{2}}}\, \overline{\varphi\left(\frac{(s-b)}{a} \right)}\, \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}^{-1} \left\{ |a|^{\frac{1}{2}} e^{ib\xi} \Phi(a\xi) \right\} (t)\, \overline{\mathcal{F}^{-1} \left\{ |a|^{\frac{1}{2}} e^{ib\eta} \Phi(a\eta) \right\} (s)}\, \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |a|^{\frac{1}{2}} \Phi(a\xi) e^{ib\xi} e^{-i\xi t} d\xi \right) \times$$

$$\times \overline{\left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |a|^{\frac{1}{2}} \Phi(a\eta) e^{ib\eta} e^{-i\eta s} d\eta \right)}\, \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(a\xi) e^{-i\xi t}\, \overline{\Phi(a\eta)} e^{i\eta s} \left(\frac{1}{2\pi} e^{ib\xi} e^{-ib\eta} db \right) d\xi\, d\eta\, \frac{da}{|a|} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(a\xi)\, \overline{\Phi(a\eta)}\, e^{-i\xi t}\, e^{i\eta s}\, \delta(\xi - \eta)\, d\xi\, d\eta\, \frac{da}{|a|} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} |\Phi(a\xi)|^2\, e^{-i\xi(t-s)}\, d\xi\, \frac{da}{|a|} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\Phi(\omega)|^2}{|\omega|} \, d\omega \right) e^{-i\xi(t-s)} \, d\xi =$$

$$= \frac{C_\varphi}{2\pi C_\varphi} \int_{\mathbb{R}} e^{-i\xi(t-s)} \, d\xi = \delta(t-s).$$

To justify the decision of ECT in the class of generalized functions, it is convenient to use the approach of Yu. I. Chersky for the definition of generalized functions [1, 2, 6].

2.2 *Convolution of Functions Relative to CWT*

Let's consider the product of wavelet transform of the functions $k(t)$ and $f(t)$ and find the inverse wavelet transform, as a result we obtain the convolution of the functions $k(t)$ and $f(t)$. Operation of convolution regarding CWT will be denoted by "$\#''$", in contrast to the "$*''$" for the Fourier transform, if both convolutions will be performed in the calculations. Let's review several different representations for convolution regarding CWT

$$(k\#f)(t) = W_\varphi^{-1}\{\left(W_\varphi k\right)(a,b) \cdot \left(W_\varphi\right)(a,b)\} \equiv W_\varphi^{-1}\left(K_\varphi(a,b)\, F_\varphi(a,b)\right) =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} K_\varphi(a,b)\, F_\varphi(a,b)\, \varphi_{a,b}(t) \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(\tau)\, \overline{\varphi_{a,b}(\tau)} d\tau \right) \left(\int_{\mathbb{R}} f(s)\, \overline{\varphi_{a,b}(s)}\, ds \right) \varphi_{a,b}(t) \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|a|^{\frac{1}{2}} \int_{\mathbb{R}} K(\xi)\, \overline{\Phi(a\xi)} e^{-ib\xi} d\xi \right) \times$$

$$\times \left(|a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\eta)\, \overline{\Phi(a\eta)} e^{-ib\eta} d\eta \right) \varphi_{a,b}(t) \frac{da\, db}{|a|^2} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(\xi) \overline{\Phi(a\xi)} e^{-ib\xi} d\xi \right) \left(\int_{\mathbb{R}} F(\eta) \overline{\Phi(a\eta)} e^{-ib\eta} d\eta \right) \times$$

$$\times \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |a|^{\frac{1}{2}} \Phi(a\zeta) e^{ib\zeta} e^{-i\zeta t} d\zeta \right) \frac{da\, db}{|a|} =$$

$$= \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi)\, \overline{\Phi(a\xi)} F(\eta)\, \overline{\Phi(a\eta)} \Phi(a\zeta) \times$$

$$\times \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} e^{-ib(\xi+\eta-\zeta)}\, db \right) d\xi\, d\eta e^{-i\zeta t}\, d\zeta \frac{da}{|a|^{\frac{1}{2}}} =$$

$$= \frac{1}{2\pi C_{\varphi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K\,(\xi)\, \overline{\Phi\,(a\xi)} F(\zeta-\xi) \overline{\Phi(a(\zeta-\xi))}\, \Phi(a\zeta)\, e^{-i\zeta t}\, d\xi\, d\zeta\, \frac{\sqrt{2\pi}\, da}{|a|^{\frac{1}{2}}} =$$

$$= \frac{\sqrt{2\pi}}{2\pi C_{\varphi}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\zeta-\xi) \int_{\mathbb{R}} \overline{\Phi(a\xi)}\, \overline{\Phi(a(\zeta-\xi))}\, \Phi(a\zeta) \frac{da}{|a|^{\frac{1}{2}}}\, e^{-i\zeta t}\, d\xi\, d\zeta =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\zeta-\xi) Q(\xi,\zeta) e^{-i\zeta t} d\xi d\zeta = (k\#f)(t),$$

where

$$Q(\xi,\zeta) = \frac{1}{\sqrt{2\pi} C_{\varphi}} \int_{\mathbb{R}} \overline{\Phi(a\xi)}\, \overline{\Phi(a(\zeta-\xi))} \Phi(a\zeta) \frac{da}{|a|^{\frac{1}{2}}}.$$

Thus, we have obtained an expression for the convolution in terms of the Fourier transform $K(\xi)$, $F(\xi)$ functions $k(t)$, $f(t)$. Further,

$$(k\#f)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} k(\tau) f(s) \left(\frac{1}{2\pi C_{\varphi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\, \overline{\varphi_{a,b}(s)}\, \varphi_{a,b}(t) \frac{da\, db}{|a|^2} \right) d\tau\, ds =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} q(t,\tau,s) k(\tau) f(s) d\tau\, ds = \int_{\mathbb{R}} n(t,s)\, f(s)\, ds, \tag{20}$$

where

$$n(t,s) = \int_{\mathbb{R}} q(t,\tau,s)\, k(\tau)\, d\tau,$$

$$q(t,\tau,s) = \frac{1}{2\pi C_{\varphi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi_{a,b}(\tau)}\, \overline{\varphi_{a,b}(s)}\, \varphi_{a,b}(t) \frac{da\, db}{|a|^2}.$$

If instead of one parent function $\varphi_{a,b}\,(t)$ we take different $\varphi_{a,b},\ \psi_{a,b},\ \theta_{a,b}$,

$$F_{\psi}(a,b) = (W_{\psi} f)(a,b), \quad K_{\theta}(a,b) = (W_{\theta} k)(a,b)$$

$$H_{\varphi}(a,b) = (W_{\varphi} h)(a,b) = W_{\varphi}(k\#f))(a,b),$$

then, similarly to formula (20), we obtain.

$$(k\#f)(t) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\theta k)\,(a,b) \left(W_\psi f\right)(a,b)\, \varphi_{a,b}\,(t)\, |a|^{-2}\, da\, db =$$

$$= W_\varphi^{-1}\{K_\theta(a,b) F_\psi(a,b)\}(t) = W_\varphi^{-1}\{(W_\theta k)(a,b)(W_\psi f)(a,b)\}(t) =$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} q(t,\tau,s) k(\tau) d\tau \right) f(s) d\tau ds \tag{21}$$

$$q(t,\tau,s) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\theta_{a,b}(\tau)}\; \overline{\psi_{a,b}(s)}\; \varphi_{a,b}(t)\; |a|^{-2}\, da\, db$$

Integral operator $(Qk)(t,s) = \int_{\mathbb{R}} q(t,\tau,s)\, k(\tau)\, d\tau = n(t,s) =$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\theta_{a,b}(\tau)}\; \overline{\psi_{a,b}(s)}\; \varphi_{a,b}(t)\; |a|^{-2}\, da\, db\; k(\tau)\, d\tau$$

The wavelet transform of the function $q(t,\tau,s)$ is found by the formula:

$$(W_\varphi q(t,\tau,s))(a,b) = \int_{\mathbb{R}} q(t,\tau,s)\; \overline{\varphi_{a,b}(t)}\; dt =$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\theta_{a,b}(\tau)}\; \overline{\psi_{a,b}(s)}\; \varphi_{a,b}(t)\; \overline{\varphi_{a,b}(t)}\; dt\; |a|^{-2}\, da\, db = \overline{\theta_{a,b}(\tau)} \cdot \overline{\psi_{a,b}(s)}.$$

Comment Estimates of CWT norms depend on the choice of functions. Let

$$\theta \in L_1(\mathbb{R}),\;\; \psi \in L_2(\mathbb{R}),\;\; \varphi \in L_1(\mathbb{R}),\;\; k \in L_1(\mathbb{R}),\;\; f \in L_2(\mathbb{R}),$$

then

$$||k\#f||_{L_1} = \int_{\mathbb{R}} |(k\#f)(t)| dt \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} q(t,\tau,s) f(s) ds \right| |k(\tau)| d\tau dt \leqslant$$

$$\leqslant \int_{\mathbb{R}} |k(\tau) d\tau| \int_{\mathbb{R}} \int_{\mathbb{R}} |q(t,\tau,s)|\, |f(s)| ds dt \leqslant$$

$$\leqslant C ||\psi||_{L_2}\, ||\theta||_{L_1}\, ||\varphi||_{L_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|k(\tau)|\, |f(s)|\, d\tau ds}{|\tau - s|^{\frac{1}{2}}} \leqslant$$

$$\leqslant C ||\psi||_{L_2}\, ||\theta||_{L_1}\, ||\varphi||_{L_1}\, ||k||_{L_1}\, ||f||_{L_2}.$$

In the assessment $\int_{\mathbb{R}} |q(t,\tau,s)|\,dt \leqslant C|\tau - s|^{-\frac{1}{2}} ||\psi||_{L_2}\, ||\theta||_{L_1}\, ||\varphi||_{L_1}$ it turns out as Hardy-Littlewood-Sobolev inequality.

The choice of spaces depends on the class of problems to be solved. For convolution type equations, for example, the function class f is convenient:

$$(\mathcal{F}\varphi) = F \in L_2(\mathbb{R}) \cap H_\lambda(\mathbb{R}),\ 0 < \lambda \leqslant 1,$$

where $H_\lambda(\mathbb{R})$—the class of Hölder-continuous functions [1].

2.3 Convolution of Wavelet Transforms

Assuming

$$F_\varphi(a,b) = (W_\varphi f)(a,b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(t) \overline{\varphi\left(\frac{t-b}{a}\right)} dt$$

the wavelet transform of the function $f(t)$ using the wavelet $\varphi(t)$. If the notation F_φ is irrelevant, we will drop index φ in the function $F_\varphi(a,b)$ (that is, simply $F(a,b)$). Consider the convolution of the functions $k(t)$ and $f(t)$ regarding Fourier transform (6)

$$h(t) = (k * f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t-s) f(s) ds. \tag{22}$$

Similarly we can consider the convolution of the functions $K(a,b)$ and $F(a,b)$. If we suggest that

$$(Wf)(a,b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F(\xi) \overline{\Phi(a\xi)} e^{-ib\xi} d\xi = F(a,b),$$

$$(Wk)(a,b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} K(\xi) \overline{\Phi(a\xi)} e^{-ib\xi} d\xi = K(a,b),$$

then formulas for convolution of wavelet transform can be obtained in the following form

$$H(a,b) = (K * F)(a,b) = C \int_{\mathbb{R}} K(a, b-u) F(a,u) du =$$

$$= \frac{C}{|a|} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(\xi) \overline{\Phi(a\xi)} e^{-i(b-u)\xi} d\xi \right) \left(\int_{\mathbb{R}} F(\eta) \overline{\Phi(a\eta)} e^{-iu\eta} d\eta \right) du =$$

$$= \frac{C}{|a|} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\eta) \overline{\Phi(a\xi)}\ \overline{\Phi(a\eta)} \left(\int_{\mathbb{R}} e^{-i(b-u)\xi} e^{-iu\eta} du \right) d\xi d\eta.$$

Using

$$\int_{\mathbb{R}} e^{-i(b-u)\xi} e^{-iu\eta} du = \int_{\mathbb{R}} e^{-ib\xi} e^{iu(\xi-\eta)} du = e^{-ib\xi} \int_{\mathbb{R}} e^{iu(\xi-\eta)} du = 2\pi e^{-ib\xi} \delta(\xi - \eta),$$

get

$$H(a,b) = \frac{2\pi C}{|a|} \int_{\mathbb{R}} \int_{\mathbb{R}} K(\xi) F(\eta) \overline{\Phi(a\xi)}\, \overline{\Phi(a\eta)}\, e^{-ib\xi} \delta(\xi - \eta) d\eta d\xi =$$

$$= \frac{2\pi C}{|a|} \int_{\mathbb{R}} K(\xi) F(\xi) \overline{\Phi(a\xi)}\, \overline{\Phi(a\xi)}\, e^{-ib\xi} d\xi =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k_{\varphi}(b-t) f_{\varphi}(t) dt = k_{\varphi} * f_{\varphi},$$

where

$$k_{\varphi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K(\xi)\, \overline{\Phi(a\xi)}\, e^{-it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t-s) \frac{1}{|a|}\, \overline{\varphi} \left(\frac{s}{a}\right) ds.$$

We have found a function $g(t)$ as the convolution of another function $f(t)$ and a filter (kernel) $k(t)$. Let $H(a,b)$ and $F(a,b)$ denote the CWT with same wavelet of $h(t)$ and $f(t)$, respectively. Then it is shown that

$$H(a,b) = \int_{\mathbb{R}} k(b-t) F(a,t) dt.$$

This means that the CWT of the convolution between some function and a filter is the convolution in the time of space variable, at every fixed scale, of the CWT of the function and the very filter.

In [29] given the convolution $h(t)$ as examined above, let $F(a,b)$ denote the CWT of the function $f(t)$ with wavelet $\varphi(t)$ and $K(a,b)$ denote the CWT of the filter $k(t)$ with wavelet $\varphi(-t)$. Then, it is shown that

$$h(t) = \frac{1}{C_{\varphi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(a,\xi) K(a, t-\xi) d\xi \right) \frac{da}{a^2},$$

where C_{φ} is the normalization constant.

Therefore, the convolution between some function and a filter is the sum over all scales of the convolution in the time or space variable, at every fixed scale, of their respective CWT, but with different wavelets.

3 Equations of Convolution Type

3.1 Equations of Convolution Type with Two Kernels

When using the Fourier Transforms, convolution equations (Wiener-Hopf, twokernel, paired, smooth transition, and others) are reduced to solving boundary value problems in the theory of analytic functions (Riemann and Carleman problems). This allows us to formulate solvability theorems (Noether's theorems). It will be shown later that similar results can be obtained using CWT. In this case, the solvability (Noether) theorems follow directly from the well-known Noether theorems for the corresponding boundary value problems in the theory of analytic functions. Using CWT allows us to select the desired characteristics of the desired solutions (filtering, image recovery based on indirect measurement data).

In [30] paper, wavelet transform and distributional wavelet transform are employed in evaluating solution of certain integral equations [21, 31] (Volterra integral equation, Abel integral equation in terms of an operator defined in that space of distributions).

In [14] two algorithms are presented, the blind iterative Wiener deconvolution used as a pint spread function estimator and the blind wavelet-regularized deconvolution algorithm. Method is based on regularized Wiener filter and redundant discrete wavelet transform.

Let's consider the integral equation [1]:

$$f(t) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(t-s) f(s) ds + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 k_2(t-s) f(s) ds - g(t) = 0, \quad t \in \mathbb{R}, \tag{23}$$

which, by using one-sided functions $f_\pm(t)$ comes down to the equation

$$f_+(t) - f_-(t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_1(t-s) f_+(s) ds - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_2(t-s) f_-(s) ds - g(t) = 0.$$

Applying the wavelet transform, we arrive at the equation

$$|a|^{1/2} \int_{\mathbb{R}} \left[F^+(\xi) - F^-(\xi) + K_1(\xi) F^+(\xi) - K_2(\xi) F^-(\xi) - G(\xi)\right] \overline{\Phi}(a\xi) e^{-ib\xi} d\xi = 0$$

using the Fourier transform in the variable b, we obtain

$$\left([1 + K_1(\xi)] F^+(\xi) - [1 + K_2(\xi)] F^-(\xi) - G(\xi)\right) |a|^{\frac{1}{2}} \overline{\Phi}(a\xi) = 0$$

or we end up with the Riemann boundary value problem [1, 2]

$$[1 + K_1(\xi)]F^+(\xi) - [1 + K_2(\xi)]F^-(\xi) = G(\xi), \quad \xi \in \mathbb{R}. \tag{24}$$

Under the solution of necessary and sufficient conditions of solvability

$$1 + K_j(\xi) \neq 0, \quad \xi \in \mathbb{R}, \quad j = 1, 2$$

the solution of the Riemann problem (24) depends on the index $\chi = ind A(\xi)$ of the coefficient

$$A(\xi) = [1 + K_1(\xi)][1 + K_2(\xi)]^{-1}$$

Theorem 4 (Noether) *Let $\chi = ind A(x)$. Then for $\chi > 0$ the homogeneous equation (23) has exactly χ linearly independent solutions, and the inhomogeneous equation is unconditionally resolvable and its solution contains χ arbitrary constants. In case of $\chi \leq 0$, the homogeneous equation has only a trivial solution. But inhomogeneous equation in case of $\chi = 0$ is unconditionally solvable and has only one solution. The necessary and sufficient conditions of the inhomogeneous equation for $\chi < 0$*

$$\int_{\mathbb{R}} \frac{F(\xi)d\xi}{X^+(\xi)\,[1 + K_1(\xi)]\,(\xi + i)^k} = 0, \; k = 1, 2, \ldots, |\chi|,$$

where factorization of the coefficient $A(\xi)$ is presented in the form $A(\xi) = \frac{X^+(\xi)}{X^-(\xi)}$.In all cases, when the Riemann problem (24) is solvable, the solution is given in quadratures.

The proof of this theorem and the corresponding formulas can be found in [1, pp. 35, 50, 51].

In the case of solvability, the solution to integral equation (23) in Fourier cases are found by the formula

$$F(\xi) = F^+(\xi) - F^-(\xi) \tag{25}$$

$$f(t) = f_+(t) - f_-(t) \tag{26}$$

Here $F^{\pm}(\xi)$ is the solution to the Riemann problem [1, 2]. In wavelet transform cases

$$F_\varphi(a, b) = F_\varphi^+(a, b) - F_\varphi^-(a, b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} [F^+(\xi) - F^-(\xi)]\overline{\Phi}(a\xi)e^{-ib\xi}d\xi \tag{27}$$

and the solution to the integral equation of convolution type with two kernels (23) is found using inverse wavelet transform

$$f(t) = (W_\varphi^{-1} F_\varphi(a,b))(t). \tag{28}$$

3.2 Convolution Equations Regarding the Wavelet Transform

Let's consider a convolution-type equation

$$Af \equiv f(t) + (k\#f)(t) = g(t),\ t \in \mathbb{R}, \tag{29}$$

where $(k\#f)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} q(t,\tau,s)k(\tau)f(s)\, d\tau\, ds$,

$$q(t,\tau,s) = \frac{1}{2\pi C_\varphi} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_{a,b}(t)\overline{\varphi_{a,b}}(\tau)\overline{\varphi_{a,b}}(s)|a|^{-2}\, da\, db.$$

We apply the CWT W_φ to Eq. (29), we obtain the algebraic equation

$$\left[1 + K_\varphi(a,b)\right] F_\varphi(a,b) = G_\varphi(a,b). \tag{30}$$

Provided that the conditions of solvability are satisfied

$$1 + K_\varphi(a,b) \neq 0,\ a,b \in \mathbb{R},\ a \neq 0 \tag{31}$$

we find the solution of Eq. (29) through wavelet transforms in the form

$$F_\varphi(a,b) =$$

$$= \left[1 + K_\varphi(a,b)\right]^{-1} G_\varphi(a,b) = G_\varphi(a,b) - K_\varphi(a,b)\left[1 + K_\varphi(a,b)\right]^{-1} G_\varphi(a,b) =$$

$$= G_\varphi(a,b) + R_\varphi(a,b)G_\varphi(a,b)$$

or

$$K^{-1}g \equiv f(t) = g(t) + (r\#g)(t),\ t \in \mathbb{R} \tag{32}$$

where

$$r(t) = W_\varphi^{-1}\left[R_\varphi(a,b)\right] = -W_\varphi^{-1}\left\{K_\varphi(a,b)\left[1 + K_\varphi(a,b)\right]^{-1}\right\},$$

i.e., the solution is represented through the resolvent $r(t)$ in the same form as the original equation.

Comment The function properties

$$R_{\varphi}(a,b) = K_{\varphi}(a,b)[1+K_{\varphi}(a,b)]^{-1},\; K_{\varphi}(a,b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} K(\xi)\overline{\Phi(a\xi)}e^{-ib\xi}d\xi$$

are determined by the kernel selection $k(t)$. For example, $k \in L_1 \cap L_2$ or

$$K(\xi) \in \{\{0\}\} = L_2(\mathbb{R}) \cap H_{\lambda}(\mathbb{R}),$$

$0 < \lambda \leqslant 1$ ($K(\xi) = (\mathcal{F}k)(\xi)$—continuous bounded function $K(\xi) \to 0$, $|\xi| \to \infty$), then the inverse wavelet transform is applied to the function $K_{\varphi}(a,b)$: $(CWT^{-1}R_{\varphi}G_{\varphi})(t) \in L_2(\mathbb{R})$ for $g \in L_2(\mathbb{R})$. In more general cases, the theory of generalized functions is used [1, pp. 254-283].

Consider Eq. (29) on the semi axis, i.e., an analogue of the Wiener-Hopf-type equation $Af = g$, $t > 0$. Representing the function $f(t)$ in terms of the one-sided functions (17) and (18), we arrive to the equation

$$f_{+} + k\# f_{+} = f_{-} + g_{+},\; t \in \mathbb{R},$$

or in form of CWT $\left[1+K_{\varphi}(a,b)\right]F_{\varphi}^{+}(a,b) = F_{\varphi}^{-}(a,b) + G_{\varphi}^{+}(a,b)$, where

$$F_{\varphi}^{\pm}(a,b) = |a|^{\frac{1}{2}} \int_{\mathbb{R}} F^{\pm}(\xi)\overline{\Phi(a\xi)}e^{-ib\xi}\, d\xi.$$

$$|a|^{\frac{1}{2}} \int_{\mathbb{R}} \left(\left[1+K_{\varphi}(a,b)\right]F^{+}(\xi) - F^{-}(\xi) - G^{+}(\xi)\right)\overline{\Phi(a\xi)}e^{-ib\xi}\, d\xi = 0.$$

The resulting problem is no longer the classical Riemann boundary value problem of the theory of analytic functions. Integral operators

$$\int_{\mathbb{R}} F^{\pm}(\xi)\overline{\Phi(a\xi)}\, e^{-ib\xi}\, d\xi =$$

$$= \sqrt{2\pi}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F^{\pm}(\xi)\, \overline{\Phi(a\xi)}\, e^{-ib\xi}d\xi = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{|a|}\, \overline{\varphi\left(\frac{b-s}{a}\right)} f_{\pm}(s)\, ds =$$

$$= |a|^{-1} \int_{\mathbb{R}} f_{\pm}(s)\, \overline{\varphi\left(\frac{b-s}{a}\right)}\, ds.$$

It is taken into account here that

$$F^{-1}\left\{\overline{\Phi(a\xi)}\right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{\Phi(a\xi)}\, e^{-ib\xi}d\xi = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{\Phi(s)}e^{-i\frac{b}{a}s}\, ds = \frac{1}{|a|}\overline{\varphi\left(\frac{b}{a}\right)}.$$

We obtain an equation representing the ECT with two kernels (23)

$$|a|^{-\frac{1}{2}}\int_{\mathbb{R}}\left(\left[1+K_{\varphi}\left(a,b\right)\right]f_{+}\left(s\right)-f_{-}\left(s\right)-g_{+}\left(s\right)\right)\overline{\varphi\left(\frac{b-s}{a}\right)}ds=0$$

or

$$[1+K_{\varphi}(a,b)]|a|^{\frac{1}{2}}\int\limits_{0}^{\infty}f(s)\overline{\varphi\left(\frac{b-s}{a}\right)}ds+|a|^{\frac{1}{2}}\int\limits_{-\infty}^{0}f(s)\,\overline{\varphi\left(\frac{b-s}{a}\right)}ds=$$

$$=|a|^{\frac{1}{2}}\int\limits_{0}^{\infty}g\left(s\right)\overline{\varphi\left(\frac{b-s}{a}\right)}ds. \tag{33}$$

The ECT theory [1, 2] with essential generalizations is applicable to such equations.

3.3 Equations of Convolution Type of the First Kind

Reviewing the problems of solving the ECT of the first kind regarding the Fourier transform

$$Af=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}k(t-s)f(s)\,ds=u(t),\ t\in\mathbb{R}, \tag{34}$$

$$k(t)\in L_1(\mathbb{R}),\ f(t)\in L_2(\mathbb{R}),\ u(t)\in L_2(\mathbb{R}).$$

In applied problems, the exact value of the operator A and the right-hand side u are unknown.

Given that: (A_h, u_δ, η), where A_h is an approximately specified operator A, u_δ is an approximately specified right u, $\eta=(h,\delta)$ is an infelicity extent

$$\|A_h-A\|\le h,\quad \|u_\delta-u\|\le\delta$$

$$A_hf=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}k_h(t-s)f(s)ds=u_\delta(t),\ t\in\mathbb{R}. \tag{35}$$

Regularizing algorithms for solving such equations using transform of Fourier's theory is widely represented in many works [5]. Let's use the wavelet transform for solving ECT of the first kind.

The wavelet transform of the functions $k_h\,(t)$ and $u_\delta\,(t)$ denoted by

$$K_h(a,b) = W_\varphi\left\{k_h(t)\right\}(a,b),\ \ U_\delta(a,b) = W_\varphi\left\{u_\delta(t)\right\}(a,b).$$

It is required to find the wavelet transform of the function $f : F(a,b) = W\{f\}$ (a,b), which is solution of Eq. (35).

Let's consider a formal scheme for solving (35) without regularization. We apply a wavelet transform to Eq. (35), and obtain

$$W\left\{A_h f - u_\delta\right\}(a,b) = |a|^{\frac{1}{2}}\int_{\mathbb{R}}\left[K_h(\xi)F(\xi) - U_\delta(\xi)\right]\overline{\Phi(a\xi)}\, e^{-ib\xi}\,d\xi = 0.$$

This condition will be satisfied if $[K_h(\xi)F(\xi) - U_\delta(\xi)]\,\overline{\Phi(a\xi)}|a|^{\frac{1}{2}} = 0$ or

$$|a|^{\frac{1}{2}}\int_{\mathbb{R}} K_h(\xi)F(\xi)\overline{\Phi}(a\xi)e^{-ib\xi}\,d\xi = |a|^{\frac{1}{2}}\int_{\mathbb{R}} U_\delta(\xi)\overline{\Phi}(a\xi)e^{-ib\xi}\,d\xi$$

We will assume that the original function $F\,(a,b)$ can be represented in the form

$$F(a,b) = R(a,b)U_\delta(a,b),$$

then the problem is reduced to finding the function $R\,(a,b)$. Formally

$$F\,(\xi) = \frac{U_\delta(\xi)}{K_h(\xi)},\ \ K_h(\xi) \neq 0,\ \ \xi \in \mathbb{R}$$

taking into account regularization of the first order

$$F(\xi) \approx F_\alpha(\xi) = \frac{\overline{K_h}(\xi)}{\alpha(1+\xi^2) + |K_h(\xi)|^2}\,U_\delta(\xi) = R_\alpha(\xi)\cdot U_\delta(\xi)$$

$$F_\alpha(\xi)|a|^{\frac{1}{2}}\overline{\Phi(a\xi)}e^{-ib\xi} = R_\alpha(\xi)U_\delta(\xi)|a|^{\frac{1}{2}}\overline{\Phi(a\xi)}e^{-ib\xi}$$

$$F_\alpha(a,b) = |a|^{\frac{1}{2}}\int_{\mathbb{R}} R_\alpha\,(\xi)\cdot U_\alpha\,(\xi)\,\overline{\Phi\,(a\xi)}\,e^{-ib\xi}d\xi, \tag{36}$$

where

$$R_\alpha(\xi) = \frac{\overline{K_h(\xi)}}{\alpha\left(1+\xi^2\right) + |K_h\,(\xi)|^2}.$$

Equations of the first kind regarding the wavelet transform. The problem of solving the equation

$$Af \equiv k\#f = g \tag{37}$$

is set incorrectly. The regularizing algorithm corresponds to the solution of the extreme problem $J(f(\cdot)) = \alpha \|f\|_{L_2}^2 + \|k\#f - g\|_{L_2}^2 \to \inf$, where α is the regularization parameter and is determined through the inaccuracy level of the kernel and the right part $\eta = (h, \delta)$.

$$(k\#f)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} q(t, \tau, s)k(\tau)\, d\tau \;\, f(s)\, ds = \int_{\mathbb{R}} n(t, s) f(s)\, ds$$

Let's find the Frechet derivative of the functional

$$J(f) = \|k\#f - g\|_{L_2}^2 = (k\#f - g,\ k\#f - g)\,,$$

$$\left(J' f\right) h = 0 \;\Leftrightarrow\; \frac{C_\varphi^{-1}}{2\pi}\left(K_\varphi(a,b)F_\varphi(a,b) - G_\varphi(a,b),\, K_\varphi(a,b)H_\varphi(a,b)\right)_{L_2(\mathbb{R}^2)} = 0,$$

$$\left[\alpha + \left|K_\varphi(a,b)\right|^2\right] F_\varphi(a,b) = \overline{K_\varphi}(a,b)\, G_\varphi(a,b)\,.$$

Hence it follows that

$$F_\varphi(a,b) = \overline{K_\varphi}(a,b)\left[\alpha + \left|K_\varphi(a,b)\right|^2\right]^{-1} G_\varphi(a,b) = R_\alpha(a,b)\, G_\varphi(a,b)\,,$$

or

$$f_\alpha(t) = W_\varphi^{-1}\left\{R_\alpha(a,b)\, G_\varphi(a,b)\right\}(t) = (r_\alpha\#g)(t)\,. \tag{38}$$

In contrast to the solution of the convolution-type equation (35), the solution of the ECT (37) is expressed through the wavelet transform of the kernel and the right side. From Parseval's equality (14) to the theory regularization methods [5].

Theorem 5 *Let D be a closed, convex set of a priori constraints of the problem (for example, $D = W_2^1 \subset L_2$), A is a one-to-one operator, $\overline{g} = A\overline{f}$, $\overline{f}$ is an exact solution (37), $\overline{f} \in D$. Approximate solution $f_\eta^{\alpha(\eta)}$ belongs to the set D for a given set (A_h, g_δ, η), $\eta = (\delta, h)$, where δ is the infelicity of the right-hand side of Eq. (37), $\|A - A_h\| \leqslant h$, $h \geqslant 0$, $\|g_\delta - \overline{g}\| \leqslant \delta$, then $f_\eta^{\alpha(\eta)} \to \overline{z}$ as $\eta \to 0$ in such a way that $(\eta + \delta)^2/\alpha(\eta) \to 0$. The solution can be chosen according to the generalized residual principle (for example, according to formula (38) to $f \in L_2$, $\|f\|_{L_2}^2 \to inf$).*

4 Conclusion

It is shown in the work that the systematic transfer of the theory of equations of the convolution type regarding the Fourier transform to equations of convolution type regarding wavelet conversion is possible, but not always advisable. It's a perspective to use discrete wavelet transforms. There is an even greater variety of tasks, methods and solution algorithms.

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Boundary Behaviour of Partial Derivatives for Solutions to Certain Laplacian-Gradient Inequalities and Spatial QC Maps

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Abstract Mostly we study mappings between smooth domains which satisfy some inequalities related to Laplacian and gradient. As an application, if in addition considered mappings are quasiconformal we get some results on the boundary behaviour of their partial derivatives.

Keywords PDE of the second order · Laplacian-gradient inequalities · Quasiconformal harmonic mappings · Boundary behaviour of partial derivatives

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1 Introduction

As we indicated in the abstract, the subject of our study is quasiconformal mappings in the plane and space between smooth domains which satisfy an inequality which we call the Laplacian-gradient inequality (see Definition 1(4) below; in terminology of [10], it was also called Poisson differential inequality). As an application, we get some results which we can consider as spatial versions of Kellogg's theorem. We also outline some results of this type for harmonic maps and maps which satisfy PDE of second order. As far as we know the ideas developed in this manuscript have origins in [10] and author communication at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10–19, 2009 [16]. In particular, we further develop and prove some result announced and outlined at this communication.

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In order to discuss the subject we first need some definitions.

Definition 1

(1) By e we denote the Euclidean distance and for $x \in \mathbb{R}^n$ denote by $|x|$ (or $|x|_e$ if there is possibility of confusion) the Euclidean norm of x.

For $R > 0$, by $B(a, R)$ and $S(a, R)$ we denote ball in $\mathbb{R}^n$ with center at a of radius R and its boundary respectively.

We simply write $\mathbb{B}_R$ and $\mathbb{S}_R$ for ball in $\mathbb{R}^n$ with center at 0 of radius R and its boundary respectively. If $R = 1$, then we omit R from the notations but also write $\mathbb{B}_n$ and $\mathbb{S}_{n-1}$ for the unit ball and the unit sphere.

For a set M in $\mathbb{R}^n$ by bM we denote the boundary of M.

Throughout this paper by D, D', D_1, D_2, G and Ω we denote domains in $\mathbb{R}^n$ and use notation $x = (x_1, x_2, \dots, x_n)$ for $x \in \mathbb{R}^n$. If X and Y are subsets of $\mathbb{R}^n$ and $f : X \to Y$, we denote by f_i the coordinate function and write $f = (f_1, f_2, \dots, f_n)$. By $D_i f(x)$ we denote partial derivative with respect to x_i at x. Thus, if $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are fixed and g is a function of the variable x_i, $g(x_i) = f(x)$, then $D_i f(x) = g'(x_i)$.

(2) Let $\Omega \subset \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$ and $f, g : \Omega \to \mathbb{R}^+$. If there is a positive constant c such that $f(x) \le c\, g(x)\,,\ x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that

$$\frac{1}{c}\, g(x) \le f(x) \le c\, g(x)\,, \quad x \in \Omega\,,$$

we write $f \approx g$ (or $f \asymp g$) on Ω.

(3) Let D, D_1 be domains in $\mathbb{R}^n$ and $w = (w_1, \dots, w_n) : D \to D_1$. We can identify ∇w with the matrix $[D_j w_i]$ and define $|\nabla w| := (\sum_{1 \le i, j \le n} |D_j w_i|^2)^{1/2}$.

(4) Let D be a domain in $\mathbb{R}^n$ and $s : D \to \mathbb{R}$. If

$$|\Delta s| \le a|\nabla s|^2 + b$$

on D, then we we say that s satisfies a, b-Laplacian-gradient (in terminology of [10] Poisson differential inequality) inequality on D. If $w : D \to \mathbb{R}^n$ satisfies the above inequality with w instead of s, then we say that w satisfies a, b-Laplacian-gradient inequality on D.

(5) We say that a bounded domain Ω in $\mathbb{R}^n$ and its boundary belong to the class $C^{k,\alpha}$, $0 \le \alpha \le 1$, if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and we have mapping ψ from B onto D such that (cf. [6, page 95])

(a) $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$
(b) $\psi(B \cap \partial\Omega \subset \partial\mathbb{R}^n_+)$
(c) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

We refer to ψ in the above definition as a local coordinate diffeomorphism flattering the boundary in a neighboorhod of x_0.

(6) Suppose that $f : D \to D'$ is differentiable at a point $x \in D$. By $f'(x)$ (or $(df)_x$) we denote the linear operator which can be identified with the matrix $[D_j f_i(x)]$ and maps the tangent space at x into the tangent space at $f(x)$.

(7) If $f : G \to \mathbb{R}^n$, where $G \subset \mathbb{R}^n$, is a domain, we define

$$|f'(x)| = \max_{|h|=1} |f'(x)h| \quad \text{and} \quad l(f'(x)) = \min_{|h|=1} |f'(x)h| \ ,$$

when f is differentiable at $x \in G$. We adopt the standard terminology and notation for K-quasiconformal (K-qc) mappings [20]. If in addition f is a C^1 homemorphism on G and there is a constant $K \in [1, \infty)$ such that $|f'(x)|^n \le K J(x, f)$, $x \in G$, where $J(x, f)$ denotes the Jacobian of f, then we say that it is a K-quasiconformal (shortly K-qc) mapping. A map is called quasiconformal (shortly qc) if it is K-quasiconformal with some K.

For harmonic quasiconformal mappings we use short notation HQC mappings.

We give first some comments concerning the above definitions. We use mainly Arabic numerals to list all the conditions like (1), (2), etc., and for some specific hypothesis notation like (h1), (h-1), h_1, i_1, etc.

For bi-Lipshitz mappings we use a short name bi-Lip.

Note that in the setting (5) of Definition 1, we can choose the ball B such that

Claim 1 ψ is bi-Lip on B if $k \ge 1$ and that $|D^2_{ij}\psi|$, $1 \le i, j \le n$ are bounded on B if $k \ge 2$.

The analytic definition for K-qcty in [20] is a more general than our definition and it is reduced to our definition if f is C^1. Next, in the setting (6) of Definition 1 it is known that f is qc if and only if there is a constant $K_1 \in [1, \infty)$ such that $|f'(x)| \le K_1 l(f'(x))$, $x \in G$. Hence we first conclude that for every $1 \le i \le n$, $l(f'(x) \le |\nabla f_i(x)| \le |f'(x)|$ and therefore

Claim 2 $|f'(x)| \approx l(f'(x)) \approx |\nabla f_i(x)|$, $x \in G$.

For the convenience of the readers we include a proof of Claim 2 in Sect. 3.1. Namely we prove that Claim 2 holds a.e. on G for qc mappings (in general sense without hypothesis that f is C^1 on G) and as a corollary we get Claim 2.

1.1 Local C^2-coordinate Method Flattering the Boundary

In this subsection we will try brifely (without technical details) to outline an approach on which we refer as local C^2 -coordinate method flattering the boundary (related to functions which satisfy the Laplacian-gradient inequality).

To continue, let D, D_1 and D_2 be domains in $\mathbb{R}^n$ and

(1) $w = (w_1, \dots, w_n) : D \to D_1$, $H : D_1 \to \mathbb{R}$ be C^2 functions and set $\hat{w} = H \circ w$. Using the chain rule formula to compute the derivative of a composite function we first have

(2) $D_k \hat{w} = \sum_{i=1}^{n} D_i H D_k w_i$,

and hence

$$\Delta(H \circ w) = \Sigma_{i=1}^n D^2_{ii} H |\nabla w_i|^2 + 2\Sigma_{i<j}^n D^2_{ij} H \, \langle \nabla w_i, \nabla w_j \rangle + \Sigma_{i=1}^n D_i H \Delta w_i. \tag{1.1}$$

Using the change of variables formula (1.1) for Laplacian we can prove

Claim 3 if $w = (w_1, \dots, w_n) : D \to D_1$ is a C^2 mapping, Δw and $|\nabla w|$ are bounded on D and the partial derivatives of H of the second order $D^2_{ij} H$, where $1 \le i, j \le n$, are bounded, then $\Delta(H \circ w)$ is bounded on D.

In order to outline our approach concerning spatial versions of Kellogg's theorem suppose in addition that

(3) the considered mapping w in (1) is harmonic (more generally Δw is bounded on D or w satisfies Laplacian-gradient inequality on D),
(4) the codomain D_1 is a C^2 domain,
(5) w is proper and it has continuous extension on $\overline{D}$.
For simplicity we suppose in addition a stronger hypothesis than (5), that
(6) w is homeomorphisms of $\overline{D}$ onto $\overline{D}_1$.

Here in general even with the continuity hypothesis (6) we can not conclude a priori that the boundary functions w^* has some kind of smoothness. To get locally a smooth boundary function we use the hypothesis (4) which provides local coordinates.

Namely, let $x_0 \in bD$ and $y_0 = w(x_0)$ and let ψ be the local coordinate around y_0 from Definition 1 (5) defined on a ball B and $\tilde{w} = \psi \circ w$. If $W = f^{-1}(B \cap D_1)$, then by Claim 1 there is a ball $W_1 = B(x_0, r_1)$ with center x_0 such that $\overline{W_1} \subset W$ and $|\nabla \tilde{w}| \approx |\nabla w|$ on $V = W_1 \cap \mathbb{B}_n$. Hence we conclude:

Claim 4 If the considered mapping w in (1) satisfies the Laplacian-gradient inequality on D and partial derivatives of H of the second order $D^2_{ij} H$, where $1 \le i, j \le n$, are bounded, then $\tilde{w}$ satisfies the Laplacian-gradient inequality on V.

Now we consider $\tilde{w}_n = H \circ w$, where $H = \psi_n$. Note first that n-th coordinate ψ_n is 0 on some part T_1 of the neighboorhud of y_0 with respect to bD_1. Hence we conclude that

(1'): $\tilde{w}_n$ is 0 on some part T of the neighboorhud of x_0 with respect to bD, and that by Claim 4, we have
(2'): $\tilde{w}_n$ satisfies the Laplacian-gradient inequality on $V_1 = B_2 \cap \mathbb{B}_n$, where $B_2 = B(x_0, r_2)$ for every $r_2 \in (0, r_1)$.

This approach leads us to study the boundary behavior of gradient of real valued functions which satisfy the Laplacian-gradient inequality with smooth boundary condition. In this setting we can apply Claim 5 below which states that $|\nabla \tilde{w}_n|$ is bounded on V.

We refer to this approach as local special C^2 -coordinate method (for functions which satisfy the Laplacian-gradient inequality). It seems useful to get a local version of the above approach. Namely, we can consider hypothesis:

(7) D is a domain in $\mathbb{R}^n$, $B = B(x_0, r_0)$, $V = B \cap D$, $T = B \cap bD$, w is C^2 and it satisfies the Laplacian-gradient inequality on V, w is homeomorphisms on $V \cup T$ and $f(T)$ is a C^2 hyper-surface.

We leave the interested readers to state and derive the corresponding versions of the statements (1') and (2') above if the hypothesis (7) holds.

1.2 Main Results

Now we present a short content of the manuscript. Recall that we study mappings in plane and space which satisfy the Laplacian-gradient inequality. The proof of Lemma 9 and $9'$ of Heinz's paper [7] clearly applies to $n \geq 2$. We can use Heinz's approach (cf. also Kalaj paper [9]) to prove Lemmma 2.2 stated here as

Lemma A (Local Gradient Lemma Version 1) *Consider the hypothesis:*

(h_1) *For a given $x_0 \in \mathbb{S}_{n-1}$ the real-valued function u is defined and continuous on $B(x_0, r_0) \cap \overline{\mathbb{B}}_n$, and C^2 on $V_0 = V_0(r_0) := B(x_0, r_0) \cap \mathbb{B}_n$.*

(h_2) *Δu is bounded on V_0.*

(h_3) *u is $C^{1,\alpha}$ on $B(x_0, r_0) \cap \mathbb{S}_{n-1}$.*

Conclusion (I): Then (h_1), (h_2) and (h_3) imply that for every $r < r_0$ partial derivatives of u are bounded on on $B_r \cap \mathbb{B}_n$, where $B_r = B(x_0, r)$.

Further in order to make an auxiliary statement that is interesting in itself let us consider the hypothesis (h_4): u satisfies a, b-Laplacian-gradient inequality on V_0.

Claim 5 The hypothesis (h_1), (h_3) and (h_4) imply that Δu is bounded on $V_0(r)$ for every $r < r_0$.

Thus under hypothesis (h_1) and (h_3) we have (h_2) is equivalent with (h_4). It is interesting that in this setting (h_4) is only a priori more general than (h_2). We can refine Heinz's [7] approach to prove Claim 5, and also note that in all results in this paper the hypothesis that a function has bounded Laplacian can be replaced by the hypothesis that the function satisfies the Laplacian-gradient inequality which is only a priori more general. Note also here that if ∇u is bounded and u satisfies the Laplacian-gradient inequality on a domain D, then Δu is bounded on D.

In this paper we show that Lemma A together with the local special C^2 -coordinate method as an application give a short proof of Theorem B below.

We can also improve the conclusion of Lemma A that partial derivatives of u are continuous on $B_1 \cap \overline{\mathbb{B}}_n$. Namely, in Sect. 2 we prove Theorem 2.3 stated here as:

Theorem A *Suppose that*

(1) *u is real valued function continuous on $\overline{\mathbb{B}}_n$*
(2) *the restriction u_b of u on $\mathbb{S}_{n-1}$ is $C^{1,\alpha}$*
(3) *Δu is bounded*

Conclusion (II)*: Then the partial derivatives of u have continuous extension on $\overline{\mathbb{B}}_n$ (more generally on $\mathbb{R}^n$).*

The proof of Theorem B is based on Lemma 2.3 below which states that partial derivatives of the Green potential are continuous on the closed unit ball and the representation formula related to the Green function on the ball which is proved in Sect. 4.

Recall that we need further technical consideration to prove the Claim 5 above and we outline a proof in Sect. 5 and plan to consider the subject related to Claim 5 in a forthcoming paper. Since our primary purpose here is to explain our idea in a simple situation and in particular to give a simple proof of Theorem 2.1 stated here as:

Theorem B *Let D be a C^2 domain in $\mathbb{R}^n$ and let $f : \mathbb{B}_n \xrightarrow{onto} D$ be a C^2 K-qc mapping. If Δf is bounded (more generally f satisfies the Laplacian-gradient inequality) on $\mathbb{B}_n$, then f is Lip on $\mathbb{B}_n$.*

We discussed this result at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10–19, 2009, and in [14], where a proof of more general results related to PDE and inner estimates, was outlined; cf. [16] and Sect. 3 for more details.

The idea is as follows. For every point $y_0 \in \partial D$ there exists a ball $B = B(y_0)$ and C^2 mapping ψ from B onto V such that (cf. [6], page 95) $\psi(B \cap D) \subset \mathbb{R}^n_+$ and $\psi(B \cap bD) \subset b\mathbb{R}^n_+$. If $\tilde{f} = \psi \circ f$ and $W = f^{-1}(B)$, then $\tilde{w}_n = 0$ on $W \cap \mathbb{S}$, and $\tilde{f}$ satisfies Laplacian-gradient inequality on $W \cap \mathbb{B}_n$. Recall we refer to this approach as local flattering the boundary C^2-coordinate method. This together with Claim 5 yields the proof.[1]

In Kalaj's paper [9] (see also [8] arXiv:0712.3580v1 (2007)[2] and arXiv:0712.3580v3 (2009)) and later in Astala-Manojlović paper [1], Theorem B is proved. More precisely, in [9] (2013) it is proved that a quasiconformal mapping of the unit ball onto a domain with C^2 smooth boundary, satisfying the Laplacian-gradient inequality (the Poisson differential inequality), is Lipschitz continuous.

In [1](2015) among the other things it is proved: A harmonic K-quasiconformal mapping from $\mathbb{B}$ to $\mathbb{B}$ is Lipschitz with the Lipschitz constant depending on the

[1] Although a considerable number of works have appeared since then, it seems that this approach is not noted in the literature.

[2] As far as we know this is the first paper tightly related to Theorem B.

value of K, dimension of n and $dist(f(0), \mathbb{S})$. Their approach which yields a simple proof of this result can be related to a special form of the iterating procedure based on Sobolev embedding and Lap-grad estimates[3] in HQC theory on the unit ball. Similar arguments are also appeared in [12] and [11], and in a more general form in the author communication [18, 19].

Note here that it seems that O. Martio [13] was the first one who considered HQC mapping of the unit disk related to the context of this paper, and the subject related to HQC mappings is now an active area of research; in particular it has been intensively studied with Belgrade Analysis group,[4] see for example the literature cited in this paper and [3, 14, 15] and the literature cited there.

In Sect. 3 we consider further results concerning the growth of gradient of mappings which satisfy certain PDE equations (or inequalities) A self-contained elementary exposition of the representation formula for the Green function on the ball is given in Sect. 4. Finally in Sect. 5 we consider the local versions of the bootstrap argument and outline a proof of Claim 5.

2 Proofs of Main Results

Recall we can refine Heinz's approach (cf. also [9]) to prove:

Lemma 2.1 *Consider the hypothesis:*

(h1) *a real-valued function u is defined and continuous on $\overline{\mathbb{B}}_n$, and C^2 on $\mathbb{B}_n$.*
(h2) *Δu is bounded on $\mathbb{B}_n$.*
(h3) *u is $C^{1,\alpha}$ on $\mathbb{S}_{n-1}$.*

Conclusion (III)*: Then ∇u is bounded on $\mathbb{B}_n$.*

Note here that Theorem 2.3 gives a stronger conclusion:

Claim 6 Under the above hypothesis (h1)-(h3), ∇u has continuous extension on $\overline{\mathbb{B}}$.

Next we can prove local version of Lemma 2.1.

Lemma 2.2 (Local Gradient Lemma Version 1) *Consider the hypothesis:*

(h_1) *For a given $x_0 \in \mathbb{S}_{n-1}$ a real-valued function u is defined and continuous on $B(x_0, r_0) \cap \overline{\mathbb{B}}_n$, and C^2 on $V_0(r_0) := B(x_0, r_0) \cap \mathbb{B}_n$.*
(h_2) *Δu is bounded on V_0.*
(h_3) *u is $C^{1,\alpha}$ on $B(x_0, r_0) \cap \mathbb{S}_{n-1}$.*

Conclusion (IV)*: Then (h_1), (h_2) and (h_3) imply that for every $r < r_0$, the partial derivatives of u are bounded on $V_0(r) := B_r \cap \mathbb{B}_n$, where $B_r = B(x_0, r)$.*

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[3]In the literature also known as "the bootstrap argument".

[4]By M. Pavlović, V. Marković, D. Kalaj, V. Božin, M. Arsenović, M. Marković, N. Lakić, D. Šarić, M. Knežević, V. Todorčević, M. Laudanović, M. Svetlik, I. Anić, etc,. . . .

Recall that Conclusion (IV) also holds if we replace the hypothesis (h_2) with a priori more general hypothesis (h_4): u satisfies a, b-Laplacian-gradient inequality on V_0. Namely, if we suppose (h_1), (h_3) and (h_4) we have by Claim 5 and Lemma 2.2 above, that there is a constant $M > 0$ such that $|\nabla u| \le M$ on $V_0(r)$, $r < r_0$. Thus we have:

Claim 7 (Local Gradient Lemma Version 2)

- Under hypothesis (h_1) and (h_3) the hypothesis (h_2) and (h_4) are equivalent.
- In particular, the hypothesis (h_1), (h_3) and (h_4) imply that $|\nabla u|$ is bounded on $V_0(r)$, $r < r_0$.

Now we return to the proof of Lemma 2.2.

Proof Let $0 < r < r_0$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ with $supp\, \eta \subset B(x_0, r_0)$ and $\eta = 1$ on $B(x_0, r_0)$. Since $D_k(\eta u) = u D_k \eta + \eta D_k u$ and $\Delta(\eta u) = u\Delta\eta + 2\langle D\eta, Du\rangle + \eta\Delta u$, we find that $\Delta(\eta u)$ is bounded on $\mathbb{B}_n$ and that ηu is $C^{1,\alpha}$ on $\mathbb{S}_{n-1}$. By Lemma 2.1, $\nabla(\eta u)$ is bounded on $\mathbb{B}_n$. Hence since $\eta u = u$ on $V_0(r)$, we conclude that ∇u is bounded on $V_0(r)$ for every $r < r_0$. □

Remark 1 Note here it seems that very simple approach in the proof of Lemma 2.2 does not apply to a proof of Claim 5. Concerning the use $\eta \in C_0^\infty(\mathbb{R}^n)$ in the proof of Lemma 2.2 (we refer to this method as reducing the local claim to a global one using a test function) note here that in general (h_4) does not imply that ηu satisfies the Laplacian-gradient inequality. However (h_2) implies that $\Delta(\eta u)$ is bounded.

We leave to the readers to prove the local version of Theorem 2.3:

Claim 8 Under the above hypothesis (h_1), (h_2) and (h_3) there is $r_1 < r_0$ such that partial derivatives of u have continuous extension on $B_1 \cap \overline{\mathbb{B}}_n$, where $B_1 = B(x_0, r_1)$.

Recall from the introduction:

Let D, D_1 and D_2 be domains in $\mathbb{R}^n$ and $w = (w_1, \dots, w_n) : D \to D_1$, $H : D_1 \to \mathbb{R}$ be C^2 functions and set $\hat{w} = H \circ w$.

Using the chain rule formula to compute the derivative of a composite function $\hat{w} = H \circ w$ we first have

(i) $D_k \hat{w} = \sum_{i=1}^{n} D_i H D_k w_i$,

and hence

$$\Delta(H \circ w) = \Sigma_{i=1}^n D_{ii}^2 H |\nabla w_i|^2 + 2\Sigma_{i<j}^n D_{ij}^2 H \,\langle \nabla w_i, w_j\rangle + \Sigma_{i=1}^n D_i H \Delta w_i. \tag{2.1}$$

Using change of variables formula (2.1) for Laplacian we can prove

Claim 9 if $w = (w_1, \dots, w_n) : D \to D_1$ is a C^2 function, Δw and $|\nabla w|$ are bounded on D and the partial derivatives of H of the second order, i.e., $D_{ij}^2 H$ are bounded, then $\Delta(H \circ w)$ is bounded on D.

Theorem 2.1 *Consider the hypothesis:*

(h_5) *Let D be a C^2 domain in $\mathbb{R}^n$ and $f : \mathbb{B}_n \xrightarrow{onto} D$ a C^2 K-qc mapping.*
(h_6) *Δf is bounded (more generally f satisfies the Laplacian-gradient inequality) on $\mathbb{B}_n$.*

Conclusion (V): Then f is Lip on $\mathbb{B}_n$.

Proof Suppose first that Δf is bounded. Let $x_0 \in \mathbb{S}$ and $y_0 = f(x_0)$. For every point $y_0 \in \partial D$ there exists a ball $B = B(y_0, r_0)$ and a mapping ψ from B onto B^* such that [6, p. 95] $\psi(B \cap D) \subset \mathbb{R}^n_+$ and $\psi(B \cap bD) \subset b\mathbb{R}^n_+$. By Claim 1 (see the introduction) we can choose B such ψ is Bi-Lip on B and that $D^2_{ij}\psi$ are bounded on B. If $\tilde{f} = \psi \circ f$ and $W = f^{-1}(B)$, then $\tilde{w}_n = 0$ on $W \cap \mathbb{S}$ and by Claim 9 $\Delta \tilde{w}_n$ is bounded on $W \cap \mathbb{B}_n$. Hence an application of Lemma 2.2, Local gradient lemma Version 1, shows that there is a ball W_1 with center x_0 such that $\overline{W_1} \subset W$ and that $\tilde{w}_n$ is Lip on $V = W_1 \cap \mathbb{B}_n$. Hence since ψ is Bi-Lip on B and f is K-qc, $\tilde{f}$ is K_1-qc on V. Next using that $\tilde{f}_n$ is Lip on V, and $\tilde{f}$ is K_1-qc on V and Claim 2 property of a qc mapping from the introduction, we conclude that $\tilde{f}$ is Lip on V and therefore since $f = \psi^{-1} \circ \tilde{f}$ it is Lip on V. Since x_0 is an arbitrary point we conclude f is Lip on $\mathbb{B}_n$. If f satisfies the Laplacian-gradient inequality on $\mathbb{B}_n$ the proof can be based on Claim 7, Local gradient lemma Version 2. □

The Newtonian potential $u = N[f]$ of a compactly supported integrable function f is defined as the convolution

$$u(x) = \Gamma * f(x) = \int_{\mathbb{R}^n} \Gamma(x - y) f(y)\, dy$$

where the Newtonian kernel Γ in dimension n is defined by

$$\Gamma(x) = \Gamma_n(x) = \Gamma(|x|) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x|^{2-n} & \text{if } n > 2 \\ \frac{1}{2\pi} \ln |x| & \text{if } n = 2, \end{cases}$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$. Note here that if σ_n is the volume of unit sphere, then $\sigma_n = n\omega_n$.

It is convenient to introduce the inversion with respect to the sphere S_R :

$$y^* = \frac{R^2}{|y|^2} y,\ y \neq 0.$$

Set $c_n = \frac{1}{n(2-n)\omega_n}$, $G_1(x, \xi) = c_n|x - \xi|^{2-n}$ and $G_2(x, \xi) = -\Gamma(|\xi||x - \xi^*|/R)$.

The Poisson kernel P_R and the Green function $\overline{g}_R$ for ball B_R are given respectively by

$$P_R(x, \eta) = \frac{R^2 - |x|^2}{n\omega_n R|x - \eta|^n}\,,$$

and $\overline{g}_R(x, \xi) := G_1(x, \xi) + G_2(x, \xi)$.

We also use a short notation $P_R[\varphi](x)$ for $\int\limits_{\mathbb{S}_R} P_R(x,y)\varphi(y)ds_y$. Concerning the notation P_R, $\overline{g}_R$ and $P_R[\varphi]$ if $R=1$ we omit R from the notation.

For the proof of Theorem 2.3 below we need Lemma 2.3 below and:

Theorem 2.2 *If $u : \overline{\mathbb{B}_R} \to \mathbb{R}$ is continuous, C^2 on $\mathbb{B}_R$, and $u = \varphi$ on $\mathbb{S}_R$ and $f = \Delta u$ bounded and locally Holder continuous on $\mathbb{B}_R$, then*

$$u(x) = \int\limits_{\mathbb{S}_R} P_R(x,y)\varphi(y)ds_y + \int\limits_{\mathbb{B}_R} \overline{g}_R(x,y)f(y)dy. \tag{2.2}$$

Here ds is (n-1)-dimensional surface measure on $\mathbb{S}_R$ inherited from $\mathbb{R}^n$, P_R is the Poisson kernel and $\overline{g}_R$ is the Green function for ball B_R.

We define Green potential with $G[f](x) := \int\limits_{\mathbb{B}_n} \overline{g}(x,y)f(y)dy$.

If we introduce $N^*[f] = \int\limits_{\mathbb{B}_n} \Gamma(|\xi||x-\xi^*|)f(\xi)dV(\xi)$, then we have $G[f] = N[f] - N^*[f]$. Note if $x \in \mathbb{S}_R$, since $\overline{g}_R$ is the Green function for ball B_R, then $\overline{g}(x,y) = 0$ for every $y \in \mathbb{B}_R$ and therefore $G[f](x) = 0$. Thus the Green potential is 0 on $\mathbb{S}_R$.

Lemma 2.3 *If f is a bounded function on $\mathbb{B}_n$. Then the partial derivatives of $G[f]$ are continuous on $\overline{\mathbb{B}}_n$.*

Proof Recall we set $G[f](x) := \int\limits_{\mathbb{B}} \overline{g}(x,y)f(y)dy$, and then we have $G[f] = N[f] - N^*[f]$.

In the proof we also need to use the inversion with respect to the unit sphere $\mathbb{S}$ which we denote by $R^*(y) = \frac{y}{|y|^2}$ and with abuse of notation we write shortly y^* instead of $R^*(y)$ in formulas below.

It is convenient to introduce $K_1(x,y) = \nabla_x G_1(x,y)$, $K_2(x,y) = \nabla_x G_2(x,y)$, $K_1[f](x) = \int\limits_{\mathbb{B}} K_1(x,y)f(y)dy$ and $K_2[f](x) = \int\limits_{\mathbb{B}} K_2(x,y)f(y)dy$.

Then we have $\nabla_x G[f](x) = K_1[f](x) + K_2[f](x)$ for $x \in \mathbb{B}$.

Let us prove that $K_2[f]$ is continuous on $\overline{\mathbb{B}}_n$. In a similar way we can prove that $K_1[f]$ is continuous on $\overline{\mathbb{B}}_n$. We will use that

$$K_2(x,y) = \nabla_x G_2(x,y) = c_n |y^*|^{n-2} \frac{x-y^*}{|x-y^*|^n}.$$

Next, we introduce

$$K_3[f](x) = \int\limits_{B(0,1/2)} K_2(x,y)f(y)dy \quad \text{and} \quad K_4[f](x) = \int\limits_{A(1/2,1)} K_2(x,y)f(y)dy.$$

and note that

$$K_2[f] = K_3[f] + K_4[f]. \tag{2.3}$$

Case 1 Continuity of $K_3[f]$.

Since f is bounded function on $\mathbb{B}_n$ there is $m_0 > 0$ such that $|f| \le m_0$ a.e. on $\mathbb{B}_n$.

Note that

$$M(x, y) := |K_2(x, y) f(y)| \le m_0 c_n |y^*|^{n-2} \frac{1}{|x - y^*|^{n-1}}$$

and that R^* maps $B(0, 1/2)$ onto $B(0, 2)^c$. Hence for $|x| \le 1$ and $|y| \le 1/2$, $|x - y^*| \ge 1$ and therefore $M(x, y) \le m_0 c_n |y^*|^{n-2}$. Since $|y^*|^{n-2}$ is integrable on $B(0, 1/2)$, by Lebesgue dominated convergence theorem we conclude that $K_3[f]$ is continuous on the closed unit ball $\overline{\mathbb{B}}_n$.

Case 2 Continuity of $K_4[f]$. By change of variables $y = R^*(z) = z^*$ (i.e. $z = R^*(y)$), we have $K_2(x, y) f(y) dy = B(x, z) dz$, where

$$B(x, z) := c_n |z|^{n-2} f(z^*) \frac{1}{|x - z|^{n-1}} J_{R^*}(z).$$

It is convenient to write

$$B(x, z) := c_n C(x, z) \frac{1}{|x - z|^{n-1}},$$

where

$$C(x, z) := |z|^{n-2} f(z^*) J_{R^*}(z)$$

is bounded on $A(1, 2)$ by M_0. Hence

$$K_4[f](x) = \int_{A(1,2)} B(x, z) dz = c_n \int_{A(1,2)} C(x, z) \frac{1}{|x - z|^{n-1}} dz.$$

Next for $z \in A(1, 2)$, $|B(x, z)| \le c_n M_0 \frac{1}{|x-z|^{n-1}}$.

Using change of variables $x - z = u$, we find

$$K_4[f](x) = \int_{A(x;1,2)} \frac{1}{|u|^{n-1}} C(x, x - u) du,$$

where $A(x;1,2)=\{u:1<|u-x|<2\}$. Next if we set

$$D(x,u):=\frac{1}{|u|^{n-1}}C(x,x-u),$$
$$|D(x,u)|\le c_n M_0\frac{1}{|u|^{n-1}}$$

and since $|u|^{1-n}$ is integrable on $B(0,3)$, by Lebesgue dominated convergence theorem we conclude that $K_4[f]$ is continuous on the closed unit ball $\overline{\mathbb{B}}_n$. Finally by (2.3) $K_2[f]$ is continuous on the closed unit ball $\overline{\mathbb{B}}_n$. □

Theorem 2.3 *Under the hypothesis of Lemma 2.1, partial derivatives of u have continuous extension on $\mathbb{R}^n$.*

Proof Set $f=\Delta u$ and $f_0=u_b$. Then $u=G[f]+P[u_b]$. Use that $G[f]=N[f]-N^*[f]$.

By the hypothesis (h3) in Lemma 2.1, $P[u_b]$ is $C^{1,\alpha}$ on $\mathbb{B}_n$. □

We leave the reader to prove vector-valued version of the previous theorem:

Theorem 2.4 *Suppose that*

(h_7) *$u:\mathbb{B}_n\to\mathbb{R}^m$ is a vector valued function which is continuous on $\overline{\mathbb{B}}_n$*
(h_8) *the restriction u_b of u on $\mathbb{S}_{n-1}$ is $C^{1,\alpha}$*
(h_9) *Δu is bounded*

Conclusion (VI): Then the partial derivatives of u have continuous extension on $\mathbb{R}^n$.

3 Further Results

In order to discuss subject related to qr mapping we need a few definitions.

3.1 qc and qr

Here we suppose that D and D' are domains in $\mathbb{R}^n$.

Definition 2

(1) A homeomorphism $f:D\to D'$ satisfies the condition (N) if $m(A)=0$ implies $m(fA)=0$.
(2) The analytic definition for K-q.c. in [20]: A homeomorphism $f:D\to D'$ is a K-qc (in the analytic sense) if f is ACL, f is differentiable a.e. in D, and $|f'(x)|^n\le K|J(x,f)|$ a.e. on D.

(3) Let Ω be a domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be continuous. We say that f is quasiregular (shortly qr) if

   (a) f belongs to Sobolev space $W^n_{1,loc}(\Omega)$,
   (b) there exists K, $1 \le K < \infty$, such that

$$|f'(x)|^n \le K\, J_f(x) \;\; a.e. \tag{3.1}$$

The smallest K in (3.1) is called the outer dilatation $K_O(f)$.
If f is qr, also

$$J_f(x) \le K'\, l(f'(x))^n \quad a.e. \tag{3.2}$$

for some K', $1 \le K' < \infty$. The smallest K' in (3.2) is called the inner dilatation $K_I(f)$ and $K(f) = max(K_O(f), K_I(f))$ is called the maximal dilatation of f. If $K(f) \le K$, then f is called K-quasiregular.

Here, we state only a few basic results.

(i_1) If $f : D \xrightarrow{onto} D'$ is qc, then $f^{-1} : D' \to D$ is qc and both satisfies the condition (N).

(i_2) Change of variables formula: If $f : D \xrightarrow{onto} D'$ is qc and $A \subset D$ is measurable, then fA is measurable, and

$$m(fA) = \int_A |J_f(x)| dm(x).$$

Furthermore, $J_f(x) \ne 0$ a.e.

(i_3) Reshetnyak's main theorem: every nonconstant qr map is discrete and open.

We leave to the readers to state and prove a version of Theorem B for qr mappings.
Let us prove:

Claim 10 If $f : D \xrightarrow{onto} D'$ is qc, then $l(f'(x)) \le |\nabla f_i(x)| \le |f'(x)|$ a.e. on D.

Proof Let $x \in D$ and $|\nabla f_i(x)| \ne 0$. Then the vector $h_0 = \nabla f_i(x)/|\nabla f_i(x)|$ is unit and $(df_i)_x(h_0) = |\nabla f_i(x)|$. By definition of qc mapping $l(f'(x)) \le |\nabla f_i(x)| \le |f'(x)|$. If $E = \{x \in G : \nabla f_i(x) = 0\}$, then the Jacobian J_f equals 0 on E. Next if $E' = f(E)$ we conclude by (i_2) that $m(E') = 0$ and therefore since f^{-1} is qc satisfies the condition (N), $m(E) = 0$. Hence $l(f'(x)) \le |\nabla f_i(x)| \le |f'(x)|$ a.e. on D. □

If in addition f is C^1, then the mappings $x \mapsto l(f'(x))$, $x \mapsto |\nabla f_i(x)|$ and $x \mapsto |f'(x)|$, where $x \in D$, are continuous and therefore Claim 2 holds.

In [11] it was proved that every K-quasiconformal mapping w of the unit ball B in $\mathbb{R}^n$, $n \ge 2$ onto a C^2-Jordan domain G is Hölder continuous with constant $a =$

$2 - n/p$, provided its weak Laplacian Δw is in $L^p(B^n)$ for some $n/2 < p < n$. In particular it is Hölder continuous for every $0 < a < 1$ provided that $\Delta w \in L^n(\mathbb{B}^n)$. Finally for $p > n$, they prove that w is Lipschitz continuous, a result, whose proof has been already sketched in [12].

Although author's approach outlined in communication at [16] is not noticed in the literature, papers appear that consider the generalizations of harmonic functions in this subject. For example, the Lipschitz continuity for solutions of the $\overline{\alpha}$-Poisson equation is considered in [5].

3.2 Hyperbolic Poisson Equation

In [4], the authors investigate solutions of the hyperbolic Poisson equation $\Delta_h u(x) = \psi(x)$, where $\psi \in L^\infty(\mathbb{B}^n, \mathbb{R}^n)$ and

$$\Delta_h u(x) = (1 - |x|^2)^2 \Delta u(x) + 2(n-2)(1 - |x|^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x)$$

is the hyperbolic Laplace operator in the n-dimensional space $\mathbb{R}^n$ for $n \geq 2$. We show that if $n \geq 3$ and $u \in C^2(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}^n}, \mathbb{R}^n)$ is a solution to the hyperbolic Poisson equation, then it has the representation (1) $u = P_h[\phi] - G_h[\psi]$ provided that $u \mid_{\mathbb{S}^{n-1}} = \phi$ and

$$\int_{\mathbb{B}^n} (1 - |x|^2)^{n-1} |\psi(x)| \, d\tau(x) < \infty.$$

Here P_h and G_h denote Poisson and Green integrals with respect to Δ_h, respectively. Furthermore, they prove that functions of the form $u = P_h[\phi] - G_h[\psi]$ are Lipschitz continuous.

Further, in this section we outline how to put their considerations in more general concept, cf. [16] and [14].

3.3 Riemannian Manifold

In this subsection the Einstein notation or Einstein summation convention is used on some places.

Let $N = N^m$ be m-dimensional manifold, $M = M^n$ an n-dimensional Riemannian manifold and $\varphi : N^m \to M^n$ a smooth immersion.

Let us recall some known facts:

If y^α and x^i are local coordinates on N and M respectively then the pull back of a scalar product g on M is given by

$$g^*_{\alpha\beta} := (\varphi^* g)_{\alpha\beta} := g_{ij} D_\alpha \varphi^i D_\beta \varphi^j,$$

where $\varphi^i = x^i \circ \varphi$.

If φ is C^2, then g^* is C^1 and so the Christoffel symbols Γ^k_{ij} are continuous.

If the element of length is given as $ds = (g_{ij} dx^i dx^j)^{1/2}$, we have

$$\langle \xi, \eta \rangle = g_{ij} \xi^i \eta^j \text{ and } \langle \xi, \xi \rangle = g_{ij} \xi^i \xi^j.$$

Let g^{ij} be the inverse matrix of the matrix g_{ij} and $g = \det[g_{ij}]$.

If f is a scalar function on M, then $\nabla_i f = D_i f$ is a covariant vector and $\nabla^i f = g^{ij} D_j f$ is a contra variant vector, and

$$|\nabla f|^2 = g^{ij} D_i f D_j f. \tag{3.3}$$

Next $\text{div} v = \frac{1}{\sqrt{g}} D_i(\sqrt{g} v^i)$. Let $v = \nabla f$. Then $v^i = g^{ij} D_j f$ and combining the definitions of the gradient and the divergence, the formula for the Laplace-Beltrami operator $\triangle_g f$ applied to a scalar function f, in local coordinates, is

$$\triangle_g f = \text{div}\,\text{grad}\, f = \frac{1}{\sqrt{g}} D_i(\sqrt{g} v^i) = \frac{1}{\sqrt{g}} D_i(\sqrt{g} g^{ij} D_j f). \tag{3.4}$$

We say that D is a C^2 domain in general sense if for every $x_0 \in bD$ there are a ball $B = B(x_0, r_0)$, $y_0 \in \mathbb{S}$ and a neighborhood $V \ni y_0$ and C^2 function $\phi : V \cap \mathbb{B}$ into $B \cap D$ with $\phi(y_0) = x_0$ such that the following properties hold:

(1) ϕ is continuous on $V \cap \overline{\mathbb{B}}$
(2) ϕ is $C^{1,\alpha}$, $0 < \alpha < 1$, on $V \cap \mathbb{S}$
(3) the second derivative of ϕ are bounded on $\phi : V \cap \mathbb{B}$
(4) ϕ maps $V \cap \mathbb{S}$ into $B \cap bD$

In the next statement we use the hypothesis:

(H_1) Let $\phi : \overline{\mathbb{B}}_n \to \overline{D}$ be continuous, $\phi(\mathbb{S}) \subset bD$, $\phi : \mathbb{B}_n \xrightarrow{onto} D$ be a C^2 injective mapping and define Rimaniann metric $g = G_\phi$ on D such that ϕ is an isometry between $(\mathbb{B}_n, e)$ and (D, g).

In differential geometry, a smooth map f from one Riemannian manifold (M, g) to another Riemannian manifold (N, h) is called harmonic if its coordinate "representations" satisfy a certain nonlinear partial differential equation whose coefficients are related to the Christoffel symbols $\Gamma(g)^k_{ij}$ on M and the composition of the Christoffel symbol $\Gamma(h)^\gamma_{\alpha\beta}$ on N with f, $\Gamma(h)^\gamma_{\alpha\beta} \circ f$, and $[g^{ij}]$, where the element of length on M is given by $ds = (g_{ij} dx^i dx^j)^{1/2}$ and the matrix $[g^{ij}]$

is the inverse matrix of the matrix $[g_{ij}]$. This partial differential equation for a mapping also arises as the Euler-Lagrange equation of a functional generalizing the Dirichlet energy (which is often itself called "Dirichlet energy"). As such, the theory of harmonic maps encompasses both the theory of unit-speed geodesics in Riemannian geometry, and the theory of harmonic functions on open subsets of Euclidean space and on Riemannian manifolds; for details see [21].

Suppose that a domain D satisfies the hypothesis (H_1) and let $u : D \to \mathbb{R}$ be a C^2 function. We say that u satisfies a, b-Laplacian-gradient inequality with respect to the metric g on D (described in the hypothesis (H_1)) if

(H_2) $|\triangle_g u| \leq a|\nabla_g u|^2 + b$ on D.

Now we can give a more general form of Theorem 2.3:

Proposition 3.1 *Let D be a domain in $\mathbb{R}^n$ which satisfies hypothesis* (H_1), and $s : D \to \mathbb{R}$. Suppose that s satisfies a, b-Laplacian-gradient inequality with respect to g and s has continuous extension on $\overline{D}$ which is $C^{1,\alpha}$, $0 < \alpha < 1$, on bD. Then ∇s has continuous extension on $\overline{D}$, and in particular ∇s is bounded on D.

Whether this Proposition remains true if we replaced (H_1) with the hypothesis that D is C^2 domain in general sense (or in particular C^2) ?

Proof A proof can be based on Lemma 2.2 using local coordinates. If $\tilde{s} = s \circ \phi$, then $\Delta_e \tilde{s} = \Delta_g s$ and therefore $\tilde{s}$ satisfies Laplacian-gradient inequality on $\mathbb{B}_n$. Hence by Theorem 2.3, $\nabla \tilde{s}$ has continuous extension on $\overline{\mathbb{B}}_n$ and therefore s has continuous extension on $\overline{D}$. □

We say that $D \subset \mathbb{R}^n$ has a $C^{1,\alpha}$ boundary in strong sense if it is the image of the unit ball $B \subset \mathbb{R}^n$ under a $C^{1,\alpha}$ diffeomorphism up to the boundary.

3.4 The Growth of Gradient of Mappings Which Satisfy Certain PDE Equations (or Inequalities)

Consider the following setting: Let D and G be domains in $\mathbb{R}^n$, and suppose that

(h-0) $f : D \xrightarrow{onto} G$ is a harmonic injective map with respect to Euclidean scalar product on both domains or more generally

(h-1) $Lf = 0$, where L is a strongly elliptic operator of the second order.

Here we roughly outline how to apply Theorem 6.18 [6] and the method of local coordinates in this setting. We try to use an approach as in the proof of Theorem 2.1.

Let us to consider the following setting. Next suppose that $f : D \xrightarrow{onto} G$ and let G be C^2 and $y_0 \in bG$. Then there is a ball $B = B(y_0, r_0)$, $r_0 > 0$, and a local coordinate ψ on B. Set $\tilde{f} = \psi \circ f$, $y = f(x)$, $z = \psi(y)$, $B^* = \psi(B)$ and $V^* = \psi(B \cap G)$. Recall that by e we denote euclidean scalar product and let g be a scalar product on B^* such that ψ is an isometry of (B, e) onto (B^*, g). If $W = f^{-1}(B \cap G)$, then we conclude:

Claim 11 The hypothesis (h-0) implies that $\tilde{f}$ is a harmonic map from (W, e) into (V^*, g).

It seems that using the method of local coordinates one can get further results. In particular we announce the following result:

Claim 12 The hypothesis (h-1) implies that there is a strongly elliptic operator L' of the second order such that $L'\tilde{f} = 0$.

Here, in addition, we only outline a few results:

Theorem 3.1 *Consider the following hypothesis*

(1) *Let D be $C^{1,\alpha}$ domain and D' be $C^{2,\alpha}$, $0 < \alpha < 1$, (more generally C^2) domain in $\mathbb{R}^n$ and*
(2) *$f : D \xrightarrow{onto} D'$ is a harmonic K-qc map.*

Conclusion (VII): Then f is Lip on D.

Proof If f is harmonic, then we apply [6, Theorem 6.18] and local coordinate approach. For every point $y_0 \in \partial D'$ there exists a ball $B = B(y_0)$ and a mapping ψ from B onto V such that $\psi(B \cap D') \subset \mathbb{R}^n_+$ and $\psi(B \cap bD') \subset b\mathbb{R}^n_+$ (cf. [6, p. 95]). If $\tilde{f} = \psi \circ f$ and $W = f^{-1}(B \cap D)$, then $\tilde{w}_n = 0$ on $W \cap bD$. Therefore by Claim 11 and Theorem 6.18 [6], $\tilde{w}_n$ is Lip on $W_1 \cap \overline{D}$, where $\overline{W_1} \subset W$. Hence since f is K-qc, $\tilde{f}$ is K_1-qc and $\tilde{f}_j$ are Lip and therefore $\tilde{f}$ is Lip and $f = \psi^{-1} \circ \tilde{f}$ is Lip on $W_1 \cap \overline{D}$. Since x_0 is an arbitrary point f is Lip on D. □

In addition, as we mention above instead of the hypothesis that f is harmonic we can consider more general hypothesis (h-1) stated here as:

(H_3) $Lf = 0$, where L is strongly elliptic operator of the second order on D.

Note that in this setting we can also prove the corresponding version of the above result. Namely, we can apply [6, Theorem 6.18] and the method of local coordinates as we described the above; see [14] and [16] where a proof has been outlined. Further suppose that domains D and Ω are bounded domains in $\mathbb{R}^n$ and its boundaries belong to class $C^{k,\alpha}$, $0 \le \alpha \le 1$, $k \ge 2$ (more generally C^2). Suppose further that g and g' are C^1 metric on $\overline{D}$ and $\overline{\Omega}$ respectively. Using inner estimate (cf. [6, Theorem 6.14]) we can prove

Theorem C (Theorem 6.9 [14]) *If $u : D \to \Omega$ is a (g, g')-harmonic qc map, then u is Lipschitz on D.*

Let C^k $(\overline{\Omega})$ denote family of functions (mappings) which belong to $C^k(\Omega)$ and all derivatives of order $\le k$ have continuous extension to $\overline{\Omega}$.

We study the growth of gradient of mappings which satisfy certain PDE equations (or inequalities) using the Green-Laplacian formula for functions and their derivatives. If in addition the considered mappings are quasiconformal (qc) between C^2 domains, we show that they are Lipschitz. Some of the obtained results can be considered as versions of Kellogg-Warshawski type theorem for qc-mappings.

More precisely, developing further methods from Heinz paper [7], we prove[5]

Theorem D ([17]) *Hypothesis*

(1) *Let* Ω *be a Jordan domain in* $\mathbb{R}^n$ *with* C^2 *boundary and* $f : \mathbb{B}_n \xrightarrow{onto} \Omega$ *be* C^2*, which has continuous extension on* $\overline{\mathbb{B}}$.
(2) *Suppose that* f *satisfy the Laplacian-gradient inequality on* $B_0 = B(z_0, r_0) \cap \mathbb{B}$*, where* $x_0 \in \mathbb{S}$ *and* $r_0 > 0$ *and* f *maps* $B_0 \cap \mathbb{S}$ *into* $b\Omega$.

Conclusion (VIII)*: There is* $0 < r_1 < r_0$*,* $c > 0$ *and a unit vector fields* X *on* $B_1 = B(x_0, r_1) \cap \mathbb{B}$ (*i.e. to each* x *we associate a unit vector* $h = h(x)$ *with initial point at* x) *such that* $|df_x(h(x))| \leq c$ *for every* $x \in B_1$.

If in addition f is qc in B_0, then f is Lipschitz continuous on B_1.

Every Lyapunov domain in plane is exhausted by a monotonous sequence of C^∞-domains which are Lyapunov-uniformly bounded. Hence we can prove that qc harmonic mappings between Lyapunov-domains (in particular C^2-domains) are Lipschitz.

At Belgrade Analysis seminar we also consider

Conjecture A Let D and D' be $C^{1,\alpha}$, $0 < \alpha < 1$, domains (bounded) in $\mathbb{R}^n$ and $f : D \xrightarrow{onto} D'$ a C^2 K-qc. If f satisfies the Laplacian-gradient inequality on D (in particular Δf is bounded), then f is Lip on D.

It seems that local special C^2-coordinate method which works for C^2 domains needs to be modified. Namely if we work with $C^{1,\alpha}$, $0 < \alpha < 1$, domains the local coordinate ψ is $C^{1,\alpha}$, where $0 < \alpha < 1$, and therefore in general $\tilde{f}$ does not satisfy Laplacian-gradient inequality. It seems that we can use fractional Laplacian in this setting. See [2] for a fractional Laplace equation and a self-contained elementary exposition of the representation formula for the Green function on the ball. In a forthcoming paper we plan to consider

(1) a version of Theorem 3.1 if codomain is C^2 or more generally Lyapunov domain, and
(2) to give more details to justify application of [6, Theorem 6.18] in the above context, and
(3) to prove Theorem C and D.

4 The Representation Formula for the Green Function on the Ball

In this section we give a self-contained elementary exposition of the representation formula for the Green function on the ball. This formula is well known, however we believe that the exposition is partially original and easy to follow, hence we hope

[5]See also Cmft2017, Jule 10–15, Lublin, Poland (see http://cmft2017.umcs.lublin.pl), plenary speakers.

that this approach will be accessible to a wide audience of young researchers and graduate students that want to approach the subject.

Definition 3 Suppose that v is smooth at $B'(a,r) := B(a,r) \setminus \{a\}$, $n = n_y = y - a/|y-a|$ (n is outward normal with respect to $B(a,s)$ for $y \in S(a,s)$, $s > 0$) and

(1) $D_n v(a + \varepsilon y)\varepsilon^{n-1}$ tends to A uniformly with respect to $y \in \mathbb{S}$.
(2) $M_v(\varepsilon) = \max_{y \in S(a,\varepsilon)} |v(y)| = o(\varepsilon^{1-n})$

If v satisfies (1) and (2) we say that v has admissible singularity at a with limit A.

Recall $E_n(x) = |x|^{2-n}$. In particular, in the proof of The Mean -Value Property we have verified that $v(y) = E_n(x-y) = |y-x|^{2-n}$ satisfies the above condition (1) at x with limit $A = 2 - n$.

4.1 Green Identity Related Admissible Singularity

In this subsection by $d\sigma$ we denote (n-1)-dimensional surface measure on the corresponding hyper-surface inherited with euclidean scalar product from $\mathbb{R}^n$.

We use the following lemma to prove a more general version of Green identity.

Lemma B

(1) *Suppose that (i1): u is continuous at a, v is smooth at $B'(a,r)$ and $D_n v(a + \varepsilon y)\varepsilon^{n-1}$ tends to $A = L(a) = L_v(a)$ uniformly wrt $y \in \mathbb{S}$.*
Then

$$\int_{S(a,\varepsilon)} u D_n v d\sigma$$

tends $Au(a)\sigma_n$, where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.

(2) *Suppose that (i2): u is C^1 on $B(a,r)$ and $M_v(\varepsilon) = M_v(a,\varepsilon) = \max_{y \in S(a,\varepsilon)} |v(y)| = o(\varepsilon^{1-n})$. Then $\int_{S(a,\varepsilon)} v D_n u d\sigma$ tends to* 0.

A domain on which Stokes formula holds we call an admissible domain.

By $S(a,\varepsilon)^+$ we denote the sphere $S(a,\varepsilon)$ oriented by the outward unit normal $\mathbf{n}$ (defined with respect to $B(a,\varepsilon)$). Let D be a admissible domain. By S^+ we denote the boundary of D oriented with the outward unit normal. It means that $\frac{\partial}{\partial n}$ (the notation $D_{\mathbf{n}}$ is also frequently used) denotes the derivative with respect to the outward unit normal vector. Set

$$I_S := \int_{S^+} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma \quad \text{and} \quad I(a,\varepsilon) := \int_{S(a,\varepsilon)^+} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma.$$

Proposition 4.1 *Let D be an admissible domain and $a \in D$. If $u \in C^2$ in $\overline{D}$, $v \in C^2(\overline{D} \setminus \{a\})$ and v has admissible singularity at $a \in D$ with limit A. Then*

$$Au(a)\sigma_n = I_S - v.p. \int_D (u\Delta v - v\Delta u)\, dx. \tag{4.1}$$

If Δv is Lebesgue integrable on D, then we can omit v.p. in the previous formula.

Proof An application Green's identity with $A_\varepsilon = D \setminus B(a, \varepsilon]$ yields

$$I_S - I(a, \varepsilon) = \int_{A_\varepsilon} (u\Delta v - v\Delta u)\, dx. \tag{4.2}$$

Since by Lemma A, $I(a, \varepsilon) \to bu(a)\sigma_n$ as $\varepsilon \to 0$, we have

$$I_S - bu(a)\sigma_n = v.p. \int_D (u\Delta v - v\Delta u)\, dx.$$

□

Proposition 4.2 *Let D be an admissible domain and $a, b \in D$. Suppose that $u \in C^2(\overline{D} \setminus \{b\})$ and u has admissible singularity at $b \in D$ with limit B, and $v \in C^2(\overline{D} \setminus \{a\})$ and v has admissible singularity at $a \in D$ with limit A. Then*

$$Au(a)\sigma_n - Bv(b)\sigma_n = I_S - v.p. \int_D (u\Delta v - v\Delta u)\, dx. \tag{4.3}$$

If Δu and Δv are Lebesgue integrable on D, then we can omit v.p. in the previous formula.

We first as an application of this proposition derive a well known result. In particular, if $\varphi \in C_0^2(\mathbb{R}^n)$, then there is $R > 0$ such that $supp\varphi \subset B(0, R)$ and by an application of Proposition 4.2,

$L_E(0)\varphi(0)\sigma_n = I_{S(0,R)} - v.p. \int_{B(0,R)} E\Delta\varphi.$

Since $I_{S(0,R)} = 0$ and $L_E(0) = 2 - n$,

$\Delta E = (2 - n)\sigma_n\delta_0.$

Proof An application of Green's identity with $A_\varepsilon = D \setminus (B(a, \varepsilon] \cup B(b, \varepsilon])$ yields

$$I_S - I(a, \varepsilon) + I(b, \varepsilon) = \int_{A_\varepsilon} (u\Delta v - v\Delta u)\, dx. \tag{4.4}$$

Since by Lemma A, $I(a, \varepsilon)$ tends $Au(a)\sigma_n$ and $I(b, \varepsilon)$ tends $Bv(b)\sigma_n$ if $\varepsilon \to 0$, we have

$I_S - Au(a)\sigma_n + Bv(b)\sigma_n = v.p. \int_D (u\Delta v - v\Delta u)\, dx.$ □

Now we outline a proof of the formula which is known as the representation of harmonic functions by Green's function. Suppose that $u, h \in C^1(\overline{D}) \cap C^2(D)$ and

$\Delta h = 0$ in D. Then by the second Green identity, since D is an admissible domain, if S is the oriented boundary of D, we have

$$\int_S \left(u\frac{\partial h}{\partial n} - h\frac{\partial u}{\partial n} \right) d\sigma = -\int_D h\Delta u.$$

The notation $\Gamma(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$, where $n > 2$, is also used in the literature. Set $\overline{g} = \Gamma + h$. By Proposition 4.1

$$u(x) = \int_S \left(u\frac{\partial \overline{g}}{\partial n} - \overline{g}\frac{\partial u}{\partial n} \right) d\sigma + \int_D \overline{g}\Delta u.$$

If $\overline{g} = 0$ on S, then

$$u(x) = \int_S u\frac{\partial \overline{g}}{\partial n}\, d\sigma + \int_D \overline{g}\Delta u. \tag{4.5}$$

In addition if u is harmonic in D, we get the representation of harmonic function by boundary values:

$$u(x) = P[\overline{g}](x) = \int_S u\frac{\partial \overline{g}}{\partial n}\, d\sigma. \tag{4.6}$$

For $\eta \in S$, $\frac{\partial g}{\partial n}(y,\eta)$ is harmonic in y as a limit of harmonic functions. Hence, if $g \in L_1(\mathbb{S})$, $P[g]$ is harmonic in $\mathbb{B}$.

A function $\overline{g}(x,\xi)$ defined on $\overline{D} \times D$ with the following properties:

(1) $\overline{g}$ is harmonic in x in D except for $x = \xi$
(2) $\overline{g}$ is continuous in $\overline{D}$ except for $x = \xi$ and $\overline{g} = 0$ on bD.
(3) $\overline{g} - |x-\xi|^{m-2}$ harmonic for $x = \xi$

is called Green's function for D. Recall the definition of Green's function for ball B_R and the inversion with respect to the sphere S_R(which are given in Sect. 2). Let $B_R = B_R(0)$ and for $x \in B_R$, $x \neq 0$, define

$$x^* = I_R x = \frac{R^2}{|x|^2}x, \tag{4.7}$$

and for $x = 0$ set $I_R 0 = \infty$. I_R is reflection through the sphere S_R. By computation,

1. $A^\xi(x) := |x-\xi|^2 = |x|^2 + |\xi|^2 - 2x\cdot\xi$ and
2. $B^\xi(x) := (|\xi||x-\xi^*|/R)^2 = \frac{|x||\xi|}{R} + R^2 - 2x\cdot\xi$.

Hence for $|x| = R$, $A^\xi = B^\xi$ and therefore

$|x| = R$, then $|\xi||x-\xi^*| = R|x-\xi|$. Motivated by this equality, we define

$$\overline{g}(x,\xi) = \Gamma(x-\xi) - \Gamma(|\xi||x-\xi^*|/R),\ \xi \neq 0; \tag{4.8}$$

$$\overline{g}(x,\xi) = \Gamma(x) - \Gamma(R),\ \xi = 0. \tag{4.9}$$

It is not difficult to check that $\overline{g}$ is the Green function for B_R.

If $u : \overline{\mathbb{B}}_n \to R$ is continuous, C^2 on $\mathbb{B}_R$, and $u = \varphi$ on $\mathbb{S}_R$ and $\Delta u = f$ bounded and locally Hölder continuous on $\mathbb{B}_R$, then

$$u(x) = \int_{\mathbb{S}_R} P_R(x, y)\varphi(y)ds_y + \int_{\mathbb{B}_R} \overline{g}(x, y) f(y)dy. \tag{4.10}$$

We conclude with the following comments:The Newtonian potential w of a given function f on a domain D is a solution of the Poisson equation (i) $\Delta w = f$. The solution is not unique, since addition of any harmonic function to w will not affect the equation and it is also a solution. Because of this roughly speaking we consider the operation of taking the Newtonian potential of a function as a partial inverse to the Laplace operator. This fact can be used to prove existence and uniqueness of solutions to the Dirichlet problem for the Poisson equation in suitably regular domains, and for suitably well-behaved functions f and the boundary data φ. The strategy is that we first apply a Newtonian potential to obtain a solution of (i) on the domain D, and then adjusts by adding a harmonic function to get the correct boundary data, which is continuous on $\overline{D}$, harmonic on D and equals φ on bD.

5 Local Versions Based on Sobolev Embedding and the Iterating Procedure

In this section we consider the local versions of some results obtained in [1, 9]. In particular we outline two different methods related to proofs and application of

(1) The local version of "the bootstrap argument", and
(2) Claim 5 .

Both methods give short proofs of some results which generalize Theorem B from the introduction.

In order to discuss the subject we first state some results and definitions.

Theorem E ([9]) *Suppose that*

(i) *u is a real valued function continuous on $\overline{\mathbb{B}}_n$*
(ii) *the restriction u_b of u on $\mathbb{S}_{n-1}$ is $C^{1,\alpha}$*
(iii) *u satisfies Laplacian-gradient inequality on $\mathbb{B}_n$*

Conclusion (II)*: Then partial derivatives of u are bounded on $\mathbb{B}_n$*

Definition 4 Let D be a domain in space and χ be a real valued function $\chi : D \to \mathbb{R}$ of class C^2 on D. If there exists a finite positive number c such that

$$|\nabla\chi(z_0)| \le c(1 + \frac{\max\{|\chi(z) - \chi(z_0)| : z \in \overline{B(z_0, R_0)}\}}{R_0}) \tag{5.1}$$

whenever $\overline{B(z_0, R_0)} \subset D$, we say that χ satisfies the mean Gradient-Oscillation estimate on D.

For example, the next result shows that harmonic functions (and more generally functions for which the Laplacian-Gradient inequality hold) satisfy the mean Gradient-Oscillation estimate:

Theorem F (Gradient-Oscillation Estimate, [9]) *Let D be a bounded domain and χ be a real function $\chi : D \to [-1, 1]$ of class C^2 on D.*

Suppose that χ satisfy the Laplacian-Gradient inequality with parameters a, b on D. Then χ satisfies the mean Gradient-Oscillation estimate on D with $c = c(a, b, n, \chi)$.

More precisely Theorem 3.5 in [9] states that $c = c(a, b, n, \chi)$ and if $a \leq C_n$ then $c = c_5(a, b, n, \chi)$ can be chosen independently of χ. In the planar case c is independent of χ, cf. [7, Theorem 2']. In order to give a relatively simple proof of Theorem E, which is not based on Heinz's approach [7] we add hypothesis (iv) in the next theorem.

Theorem 5.1 *Suppose that*

(i) *$u : \mathbb{B}_n \to \mathbb{R}^m$ is a vector valued function which is continuous on $\overline{\mathbb{B}}_n$*
(ii) *the restriction u_b of u on $\mathbb{S}_{n-1}$ is $C^{1,\alpha}$*
(iii) *u is C^2 function which satisfies Laplacian-gradient inequality on $\mathbb{B}_n$*
(iv) *∇u is in L^{q_0},*

Conclusion (VI)*: (a) If $0 < q_0 \leq n$, then ∇u is in L^{q_1}, where $q_1 = \frac{nq_0}{2n-q_0}$.*

(b) If $q_0 > n$, then partial derivatives of u have continuous extension on $\overline{\mathbb{B}}_n$ (and therefore on $\mathbb{R}^n$). Note here if $u : D \to \mathbb{R}^n$ is qc and G compact subset of D then Gehring showed that ∇u is in $L^{q_0}(G)$ for some $q_0 > n$, and for example if $f : \mathbb{B}_n \xrightarrow{onto} \mathbb{B}_n$ is a qc then ∇u is in $L^{q_0}(\mathbb{B}_n)$ for some $q_0 > n$, so in the context related to qc mappings the hypothesis (iv) is natural. It turns out that the following lemma has found applications in the context relate to HQC mappings:

Lemma C *Suppose that h belongs $L^p(\mathbb{B}_n)$, $1 \leq p < \infty$.*

(1) *If $p < n$, $\nabla G[h]$ belongs L^q on $\mathbb{B}_n$, where $q = \frac{np}{n-p}$.*
(2) *If $p = n$, $\nabla G[h]$ belongs L^q on $\mathbb{B}_n$, for every $q < \infty$.*
(3) *If $p > n$, $\nabla G[h]$ is bounded on $\mathbb{B}_n$*

This result can be considered as a simple version of the Sobolev embedding theorem. The proof of this lemma can be based on the Calderon-Zygmund inequality (see for example [6, Theorem 9.9]) and Morrey theorem; or on the Riesz potential Lemma 7.12 [6] and the Green formula. Recall that the results which can be related to Lemma C are used in [1, 11, 12] to prove the Lipschitz continuity of the HQC mappings in the corresponding context. We refine their approaches to prove Theorem 5.1 and Theorem 5.2.

Proof of Theorem 5.1.

Under the hypothesis of Theorem 5.1 we have Green's formula

$$u = P[u] + v, \text{ where } \quad v = G[\Delta u],$$

and

Claim 13 $\nabla P[u]$ is bounded on $\mathbb{B}_n$.

By Claim 13 and Lemma C(1), ∇u is in L^{q_1} and the part (a) follows.

The part b) follows from the bootstrap argument.

We will use the following notation. For a $x_0 \in \mathbb{S}_{n-1}$, set $V(r, x_0) = B_r \cap \mathbb{B}_n$ and $T(r, x_0) = B_r \cap \mathbb{S}_{n-1}$, where $B_r = B(x_0, r)$, $r > 0$. If from the context it is clear on which point x_0 we refer we can omit x_0 from the notation and we write simply $V(r)$ and $T(r)$.

Now we consider local gradient a priori estimates based on an approuch which we call the local bootstrap argument. To get a local version of Theorem 5.1 for $u : V(r_0) \to \mathbb{R}$ we construct a decreasing sequence r_k, $k \geq 0$, which converges to $s_0 > 0$ and sequences of functions $u_k = \eta_k u$, on which we apply a version of the bootstrap argument, where $B_k = B(x_0, r_k)$, $\eta_k \in C_0^2(B_k)$, $\eta_k = 1$ on B_{k+1}. This approach leads to the proof of the following result(the proof with all the details will appear in a forthcoming paper):

Theorem 5.2 (Local Gradient a Priori Estimate) *For a $x_0 \in \mathbb{S}_{n-1}$, set $V = B_0 \cap \mathbb{B}_n$, where $B_0 = B(x_0, r_0)$, $r_0 > 0$, and $T = B_0 \cap \mathbb{S}_{n-1}$. Suppose that*

(1) *$u : \mathbb{B}_n \to \mathbb{R}$ is a real valued function continuous on $V \cup T$*
(2) *u is C^2 function which satisfies the Laplacian-gradient inequality on $V = B_0 \cap \mathbb{B}_n$*
(3) *the restriction u_b of u is $C^{1,\alpha}$ on T*
(4) *∇u is in $L^{q_0}(V)$, $q_0 > n$.*

Conclusion (VI): Then for every $s_0 \in (0, r_0)$ the partial derivatives of u have continuous extension on $\overline{W_0}$, where $W_0 = B_{s_0} \cap \mathbb{B}_n$, $B_{s_0} = B(x_0, s_0)$.

Remark 2 We refer to Theorem 5.2 as the local gradient estimate Version 1. This result has some interesting applications. For example this result together with the local C^2 -coordinate method flattering the boundary give short proofs of some results which generalize Theorem B from the introduction.

Although the work related to this section is in progress we can outline some results. Note that it is interesting to check whether the condition (4) in Theorem 5.2 is superfluous. Here we state and outline a proof of a version of Claim 5 stated here as

Claim 14 The hypothesis (1),(2) and (3) in Theorem 5.2, imply that there is a $s_0 \in (0, r_0)$ such that partial derivatives of u are bounded on $\overline{W_0}$, where $W_0 = B_{s_0} \cap \mathbb{B}_n$, $B_{s_0} = B(x_0, s_0)$.

Outline of proof. Let $0 < r_1 < r_0$, $B_0 = B_{r_0}$, $B_1 = B_{r_1}$, $\eta \in C_0^2(B_0)$, $\eta = 1$ on B_1, and $v = \eta u$. Next we improve approach in [9] (see Lemma 2.7 there) to show that there is $0 < r_2 < r_1$ such that $|v(x) - v(\eta)| \leq L|x - \eta|$, $(x, \eta) \in V(r_2) \times T(r_2)$ and finally we apply Theorem F(the Gradient -Oscillation estimate).

Finally, note that the author communicated some generalizations of Theorem 5.2 that apply to functions satisfying weaker inequalities than the Laplacian-gradient inequality at "Congressio-Mathematica" 2020, cf. [19]. Hence for example we can derive

Theorem 5.3 *Suppose*

(h1) *G Jordan planar domain having C^1 boundary*
(h2) *f be a harmonic quasi-conformal mapping between the unit disk $\mathbb{U}$ and G Conclusion:*
 (1) *∇f belongs $L^s(\mathbb{U})$ for all $s > 0$.*
 (2) *f is Hölder for all $\alpha \in (0, 1)$.*

Instead of (h1) and (h2) respectively we also consider more general hypothesis

(h3) f is quasi-conformal mapping between the unit disk $\mathbb{U}$ and G which satisfies the Laplacian-gradient inequality, and
(h4) G is quasi-disk which is C^1 at a boundary point

and generalize Theorem 5.3.

In a privative communication D. Kalaj informed the author that he also proved Theorem 5.3 recently.

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Is Every Nontrivial Hausdorff Operator on Lebesgue Space a Non-Riesz Operator?

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Abstract We consider a generalization of Hausdorff operators on Lebesgue spaces and under natural conditions prove that such an operator is not a Riesz operator provided it is non-zero. In particular, it cannot be represented as a sum of a quasinilpotent and compact operators.

Keywords Hausdorff operator · Riesz operator · Quasinilpotent operator · Compact operator · Lebesgue space

1 Introduction and Preliminaries

The one-dimensional Hausdorff transformation

$$(\mathcal{H}_1 f)(x) = \int_{\mathbb{R}} f(xt)d\chi(t), \tag{1}$$

where χ is a measure on $\mathbb{R}$ with support $[0, 1]$, was introduced by Hardy [1, Section 11.18] as a continuous variable analog of regular Hausdorff transformations (or Hausdorff means) for series. Its modern n-dimensional generalization looks as follows:

$$(\mathcal{H} f)(x) = \int_{\mathbb{R}^m} \Phi(u) f(A(u)x)du, \tag{2}$$

where $\Phi : \mathbb{R}^m \to \mathbb{C}$ is a locally integrable function, $f : \mathbb{R}^n \to \mathbb{C}$, $A(u)$ stands for a family of non-singular $n \times n$-matrices with entries defined on $\mathbb{R}^m$ with values in $\mathbb{R}$, $x \in \mathbb{R}^n$ is a column vector. The modern theory of Hausdorff operators begins

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with the paper by Liflyand and Moricz [2]. See survey articles [3, 4] for historical remarks and the state of the art up to 2014.

To justify the operator to be defined by (2), the following approach may be suggested. Hardy [1, Theorem 217] proved that (if χ is a probability measure) transformation (1) gives rise to a regular generalized limit at infinity of the function f in a sense that if f is continuous on $\mathbb{R}$ and $f(x) \to l$, then $\mathcal{H}_1 f(x) \to l$ as $x \to \infty$. Note that the map $x \mapsto xt$ $(t \neq 0)$ is the general form of automorphisms of the additive group $\mathbb{R}$. This observation leads to the definition of a (generalized) Hausdorff operator on a general group G via the automorphisms of G introduced and studied by the author in [5], and [6]. For the additive group $\mathbb{R}^n$, this definition looks as follows.

Definition 1 Let (Ω, μ) be a σ-compact topological space endowed with a positive regular Borel measure μ, let Φ be a locally integrable function on Ω, and let $(A(u))_{u\in\Omega}$ be a μ-measurable family of $n \times n$-matrices, non-singular for μ-almost every u, with $\Phi(u) \neq 0$. We define the *Hausdorff operator* with the kernel Φ by (recall that $x \in \mathbb{R}^n$ is a column vector)

$$(\mathcal{H}_{\Phi,A} f)(x) = \int_\Omega \Phi(u) f(A(u)x) d\mu(u).$$

The general form of a Hausdorff operator given by Definition 1 (with an arbitrary measure space (Ω, μ) instead of $\mathbb{R}^m$) gives us, for example, the opportunity to consider (in the case $\Omega = \mathbb{Z}^m$) discrete Hausdorff operators (see [7, 8], and Example 3 below).

As mentioned above, Hardy proved that the Hausdorff operator (1) possesses some regularity property. For the operator given by Definition 1, the multidimensional version of his result is also true as the next proposition shows.

Proposition 1 ([8]) *Let all the conditions of Definition 1 be fulfilled. In order for the transformation $\mathcal{H}_{\Phi,A}$ be regular, i.e., f measurable and locally bounded on $\mathbb{R}^n$, with $f(x) \to l$ as $x \to \infty$ should imply $\mathcal{H}_{\Phi,A} f(x) \to l$, it is necessary and sufficient that $\int_\Omega \Phi(u) d\mu(u) = 1$.*

So, as for the classical transformation considered by Hardy the Hausdorff transformation in the sense of Definition 1 gives rise to a family (for various (Ω, μ), Φ, and $A(u)$) of regular generalized limits at infinity for functions of n variables.

(For a different approach to justify definition (2), see [9].)

The problem of compactness of Hausdorff operators was posed by Liflyand [10] (see also [3]). There is a conjecture that nontrivial Hausdorff operator on $L^p(\mathbb{R}^n)$ is non-compact. For the case $p = 2$ and for commuting $A(u)$ this hypothesis was confirmed in [7] (and for the diagonal $A(u)$—in [5]). Moreover, we conjecture that (at least for commuting $A(u)$) a nontrivial Hausdorff operator on $L^p(\mathbb{R}^n)$ is non-Riesz.

Recall that a *Riesz operator* T is a bounded operator on some Banach space with spectral properties like those of a compact operator; i. e., T is a non-invertible operator whose nonzero spectrum consists of eigenvalues of finite multiplicity with no limit points other then 0. This is equivalent to the fact that $T - \lambda$ is Fredholm for all scalars $\lambda \neq 0$ [11]. For example, a sum of a quasinilpotent and compact operator is Riesz [12, Theorem 3.29]. In [12], other interesting characterizations of Riesz operators can be found as well.

In this note we prove the above mentioned conjecture in the case where the family $A(u)$ consists of permutable and positive (negative) definite matrices. The result has been announced in [13].

2 The Main Result

We shall employ three lemmas to prove our main result.

Lemma 1 ([5] (cf. [1, (11.18.4)], [14])) *Let* $|\det A(u)|^{-1/p}\Phi(u) \in L^1(\Omega)$. *Then the operator* $\mathcal{H}_{\Phi,A}$ *is bounded in* $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ *and*

$$\|\mathcal{H}_{\Phi,A}\| \leq \int_\Omega |\Phi(u)||\det A(u)|^{-1/p} d\mu(u).$$

This estimate is sharp (see Theorem 1 in [8]).

Lemma 2 ([8] (cf. [14]).) *Under the assumptions of Lemma 1 the adjoint for the Hausdorff operator on* $L^p(\mathbb{R}^n)$ *is of the form*

$$(\mathcal{H}^*_{\Phi,A} f)(x) = \int_\Omega \overline{\Phi(v)}|\det A(v)|^{-1} f(A(v)^{-1}x) d\mu(v).$$

Thus, the adjoint for a Hausdorff operator is also Hausdorff.

Lemma 3 *Let* S *be a ball in* $\mathbb{R}^n$, $q \in [1, \infty)$, *and let* $R_{q,S}$ *denote the restriction operator* $L^q(\mathbb{R}^n) \to L^q(S)$, $f \mapsto f|S$. *If we identify the dual* $(L^q)^*$ *of* L^q *with* L^p $(1/p + 1/q = 1)$, *then the adjoint* $R^*_{q,S}$ *is the operator of natural embedding* $L^p(S) \hookrightarrow L^p(\mathbb{R}^n)$.

Proof For $g \in L^p(S)$, let

$$g^*(x) = \begin{cases} g(x), & \text{for } x \in S, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus S. \end{cases}$$

Then the map $g \mapsto g^*$ is the natural embedding $L^p(S) \hookrightarrow L^p(\mathbb{R}^n)$.

By definition, the adjoint $R^*_{q,S} : L^q(S)^* \to L^q(\mathbb{R}^n)^*$ acts according to the rule

$$(R^*_{q,S}\Lambda)(f) = \Lambda(R_{q,S} f) \quad (\Lambda \in L^q(S)^*, f \in L^q(\mathbb{R}^n)).$$

If we (by the Riesz theorem) identify the dual of $L^q(S)$ with $L^p(S)$ via the formula $\Lambda \leftrightarrow g$, where

$$\Lambda(h) = \int_S g(x)h(x)dx \quad (g \in L^p(S), h \in L^q(S)),$$

and analogously for the dual of $L^q(\mathbb{R}^n)$, then the definition of $R^*_{q,S}$ is of the form

$$\int_{\mathbb{R}^n} (R^*_{q,S}g)(x)f(x)dx = \int_S g(x)(f|S)(x)dx.$$

But

$$\int_S g(x)(f|S)(x)dx = \int_{\mathbb{R}^n} g^*(x)f(x)dx \quad (f \in L^q(\mathbb{R}^n)).$$

The right-hand side of the last formula is the linear functional from $L^q(\mathbb{R}^n)^*$. If we (again by the Riesz theorem) identify this functional with the function g^*, the result follows.

Theorem 1 *Let $A(v)$ be a commuting family of real positive definite $n \times n$-matrices (v runs over the support of Φ), and $(\det A(v))^{-1/p}\Phi(v) \in L^1(\Omega)$. Then every nontrivial Hausdorff operator $\mathcal{H}_{\Phi,A}$ in $L^p(\mathbb{R}^n)$ ($1 \le p \le \infty$) is a non-Riesz operator (and in particular it is non-compact).*

Proof Assume the contrary. Since $A(u)$ form a commuting family, there are an orthogonal $n \times n$-matrix C and a family of diagonal non-singular real matrices $A'(u) = \operatorname{diag}(a_1(u), \dots, a_n(u))$ such that $A'(u) = C^{-1}A(u)C$ for $u \in \Omega$. Consider the bounded and invertible operator $\widehat{C}f(x) := f(Cx)$ on $L^p(\mathbb{R}^n)$. Because of the equality $\widehat{C}\mathcal{H}_{\Phi,A}\widehat{C}^{-1} = \mathcal{H}_{\Phi,A'}$, operator $\mathcal{H} := \mathcal{H}_{\Phi,A'}$ is Riesz and nontrivial, too.

Noting that each open hyperoctant in $\mathbb{R}^n$ is $A'(u)$-invariant, we choose such an open n-hyperoctant U that $\mathcal{K} := \mathcal{H}|L^p(U) \ne 0$. Then $L^p(U)$ is a closed $\mathcal{K}$-invariant subspace of $L^p(\mathbb{R}^n)$, and $\mathcal{K}$ is a nontrivial Riesz operator in $L^p(U)$ by [12, p. 80, Theorem 3.21].

Let $1 \le p < \infty$. To get a contradiction, we shall use the modified n-dimensional Mellin transform for the n-hyperoctant U in the form

$$(\mathcal{M}f)(s) := \frac{1}{(2\pi)^{n/2}} \int_U |x|^{-\frac{1}{q}+is} f(x)dx, \quad s \in \mathbb{R}^n.$$

Here and below we assume that

$$|x|^{-\frac{1}{q}+is} := \prod_{j=1}^{n} |x_j|^{-\frac{1}{q}+is_j},$$

where

$$|x_j|^{-\frac{1}{q}+is_j} := \exp\left(\left(-\frac{1}{q}+is_j\right)\log|x_j|\right).$$

The map $\mathcal{M}$ is a bounded operator between $L^p(U)$ and $L^q(\mathbb{R}^n)$ for $1 \le p \le 2$ $(1/p+1/q=1)$. It can easily be obtained from the Hausdorff–Young inequality for the n-dimensional Fourier transform by using the exponential change of variables (see [15]). Let $f \in L^p(U)$. First assume that $|y|^{-1/q}f(y) \in L^1(U)$. Then, as in the proof of Theorem 1 from [8] (or [7]), using Fubini–Tonelli's theorem and integrating by substitution $x = A(u)'^{-1}y$ yield

$$(\mathcal{M}\mathcal{K}f)(s) = \varphi(s)(\mathcal{M}f)(s)\ (s \in \mathbb{R}^n),$$

where the function

$$\varphi(s) := \int_\Omega \Phi(u)|a(u)|^{-1/p-is}d\mu(u)$$

("the symbol of a Hausdorff operator" [8, Definition 2], [7]) is bounded and continuous on $\mathbb{R}^n$.

Thus,

$$\mathcal{M}\mathcal{K}f = \varphi\mathcal{M}f. \tag{3}$$

By continuity, the last equality is valid for all $f \in L^p(U)$.

Let $1 \le p \le 2$. There exists a constant $c > 0$ such that the set $\{s \in \mathbb{R}^n : |\varphi(s)| > c\}$ contains an open ball S. Formula (3) implies that

$$M_\psi R_{q,S}\mathcal{M}\mathcal{K} = R_{q,S}\mathcal{M},$$

where $\psi = (1/\varphi)|S$, M_ψ denotes the operator of multiplication by ψ, and $R_{q,S}: L^q(\mathbb{R}^n) \to L^q(S)$, $f \mapsto f|S$ is the restriction operator. Let $T = R_{q,S}\mathcal{M}$. Passing to the conjugates gives

$$\mathcal{K}^*T^*M_\psi^* = T^*.$$

By [16, Theorem 1] this implies that the operator $T^* = \mathcal{M}^*R_{q,S}^*$ has finite rank. By Lemma 3, $R_{q,S}^*$ is the operator of natural embedding $L^p(S) \hookrightarrow L^p(\mathbb{R}^n)$.

To compute $\mathcal{M}^*$, for $g \in L^p(\mathbb{R}^n)$ consider the operator

$$(\mathcal{M}'g)(x) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}|x|^{-\frac{1}{q}+is}g(s)ds,\ x \in U.$$

This is a bounded operator taking $L^p(\mathbb{R}^n)$ into $L^q(U)$. Indeed, since

$$|x|^{-\frac{1}{q}+is} = \prod_{j=1}^{n} |x_j|^{-\frac{1}{q}} \exp(is_j \log |x_j|),$$

we have

$$(\mathcal{M}'g)(x) = |x|^{-\frac{1}{q}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(is \cdot (\log |x_j|)) g(s) ds, \ x \in U,$$

where $|x| := |x_1| \dots |x_n|$, $(\log |x_j|) := (\log |x_1|, \dots, \log |x_n|)$, and the dot denotes the inner product on $\mathbb{R}^n$. Thus, we can express the function $\mathcal{M}'g$ via the Fourier transform $\widehat{g}$ of g as

$$(\mathcal{M}'g)(x) = |x|^{-1/q} \widehat{g}(-(\log |x_j|)), \quad x \in U,$$

and therefore

$$\|\mathcal{M}'g\|_{L^q(U)} = \left(\int_U |x|^{-1} |\widehat{g}(-(\log |x_j|))|^q dx \right)^{1/q}.$$

Putting in the last integral $y_j := -\log |x_j|$ $(j = 1, \dots, n)$ and taking into account that the Jacobian module of this transformation is

$$\left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right| = \det \operatorname{diag}(e^{-y_1}, \dots, e^{-y_n}) = \exp\left(-\sum_{j=1}^{n} y_j \right),$$

we get by the Hausdorff–Young inequality that

$$\|\mathcal{M}'g\|_{L^q(U)} = \|\widehat{g}\|_{L^q(\mathbb{R}^n)} \le \|g\|_{L^p(\mathbb{R}^n)}.$$

If $f \in L^p(U)$, and $f(x)|x|^{-1/q} \in L^1(U)$, $g \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the Fubini–Tonelli's theorem implies

$$\int_{\mathbb{R}^n} (\mathcal{M}f)(s) g(s) ds = \int_U f(x) (\mathcal{M}'g)(x) dx.$$

Since the bilinear dual pairing $(\varphi, \psi) \mapsto \int \varphi\psi d\nu$ is continuous on $L^p(\nu) \times L^q(\nu)$, the last equality is valid for all $f \in L^p(U)$, $g \in L^p(\mathbb{R}^n)$ by continuity. So, $\mathcal{M}' = \mathcal{M}^*$.

It has been shown above that the restriction of the operator $\mathcal{M}^*$ to $L^p(S)$ has finite rank. Since $\mathcal{M}^*$ can easily be reduced to the Fourier transform, this contradicts the Paley–Wiener theorem on the Fourier image of the space $L^2(S)$, see, e. g., [17, Theorem III.4.9] (in our case $L^2(S) \subseteq L^p(S)$).

Finally, if $2 < p \le \infty$, one can use duality arguments. Indeed, by Lemma 2 the adjoint operator $\mathcal{H}^*_{\Phi,A'}$ (as an operator on $L^q(\mathbb{R}^n)$) is also of Hausdorff type. More precisely, it equals $\mathcal{H}_{\Psi,B}$, where

$$B(u) = A(u)'^{-1} = \mathrm{diag}(1/a_1(u), \ldots, 1/a_n(u))$$

and

$$\Psi(u) = \Phi(u)|\det A(u)'^{-1}| = \Phi(u)/\prod_j a_j(u).$$

It is easy to verify that $\mathcal{H}_{\Psi,B}$ satisfies all the conditions of Theorem 1 (with q, Ψ and B in place of p, Φ and A, respectively). Since $1 \le q < 2$, the operator $\mathcal{H}_{\Psi,B}$ is not a Riesz operator on $L^q(\mathbb{R}^n)$. The same is for $\mathcal{H}_{\Phi,A}$, because T is a Riesz operator if only if its conjugate T^* is a Riesz operator [12, p. 81, Theorem 3.22].

3 Corollaries and Examples

For the next corollary, we need the following

Lemma 4 *Let J be a bounded and invertible operator on a Banach space X. A bounded operator T on X which commutes with J is a Riesz operator if and only if JT is a similar operator.*

Proof We use the fact that an operator T is a Riesz operator if and only if it is asymptotically quasi-compact [11] (see also [12, Theorem 3.12]). This means that

$$\lim_{n\to\infty}\left(\inf_{C\in\mathcal{K}(X)}\|T^n - C\|^{1/n}\right) = 0,$$

where $\mathcal{K}(X)$ denotes the ideal of compact operators on X (Ruston condition). Since $(JT)^n = J^nT^n$ and

$$\inf_{C\in\mathcal{K}(X)}\|(JT)^n - C\|^{1/n} \le \|J\|\inf_{C\in\mathcal{K}(X)}\|T^n - J^{-n}C\|^{1/n}$$
$$= \|J\|\inf_{C'\in\mathcal{K}(X)}\|T^n - C'\|^{1/n},$$

the result follows.

Corollary 1 *Let $A(v)$ be a commuting family of real negative definite $n \times n$-matrices (v runs over the support of Φ), and* $(\det A(v))^{-1/p}\Phi(v) \in L^1(\Omega)$. *Then every nontrivial Hausdorff operator $\mathcal{H}_{\Phi,A}$ in $L^p(\mathbb{R}^n)$ ($1 \le p \le \infty$) is non-Riesz (and, in particular, non-compact).*

Proof Let $Jf(x) := f(-x)$. Since $-A(v)$ form a commuting family of real positive definite $n \times n$-matrices, and $\mathcal{H}_{\Phi,A} = J\mathcal{H}_{\Phi,(-A)}$, this corollary follows from Lemma 4 and Theorem 1.

Corollary 2 *Under the conditions of Theorem 1 or Corollary 1, the Hausdorff operator $H_{\Phi,A}$ is not the sum of a quasinilpotent and compact operators.*

Indeed, as is mentioned in the introduction, the sum of a quasinilpotent and compact operators is a Riesz operator.

Corollary 3 *Let $n = 1$, $\phi : \Omega \to \mathbb{C}$, and let $a(u)$ be a real and positive (negative) function on Ω (u runs over the support of ϕ), and $|a(u)|^{-1/p}\phi(u) \in L^1(\Omega)$. Then every nontrivial Hausdorff operator*

$$(\mathcal{H}_{\phi,a} f)(x) = \int_{\Omega} \phi(u) f(a(u)x) d\mu(u) \ (x \in \mathbb{R})$$

in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$ is a non-Riesz operator (and, in particular, is not the sum of a quasinilpotent and compact operators).

Example 1 Let $t^{-1/q}\psi(t) \in L^1(0, \infty)$. Then, by Corollary 3, the operator

$$(\mathcal{H}_{\psi} f)(x) = \int_0^{\infty} \frac{\psi(t)}{t} f\left(\frac{x}{t}\right) dt$$

is a non-Riesz operator in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$ provided it is non-zero.

Example 2 Let $(t_1 t_2)^{-1/p}\psi_2(t_1, t_2) \in L^1(\mathbb{R}^2_+)$. Then, by Theorem 1, the operator

$$(\mathcal{H}_{\psi_2} f)(x_1, x_2) = \frac{1}{x_1 x_2} \int_0^{\infty} \int_0^{\infty} \psi_2\left(\frac{t_1}{x_1}, \frac{t_2}{x_2}\right) f(t_1, t_2) dt_1 dt_2$$

is a non-Riesz operator in $L^p(\mathbb{R}^2_+)$ $(1 \le p \le \infty)$ provided it is non-zero.

Example 3 (Discrete Hausdorff Operators, cf. [7, Example 3]) Let $\Omega = \mathbb{Z}^m_+$, and μ be a counting measure. Then Definition 1 turns into ($f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$)

$$(\mathcal{H}_{\Phi,A} f)(x) = \sum_{k \in \mathbb{Z}^m_+} \Phi(k) f(A(k)x),$$

provided that the series on the right-hand side converges absolutely ($A(k)$ form a commuting family of real non-singular $n \times n$ matrices). Assume that $\sum_{k \in \mathbb{Z}^m_+} |\Phi(k)| |\det A(k)|^{-1/p} < \infty$. Then the operator $\mathcal{H}_{\Phi,A}$ is bounded on $L^p(\mathbb{R}^n)$ by Lemma 1, and is a non-Riesz operator by Theorem 1 and Corollary 1 if it is non-zero and matrices $A(k)$ are positive (negative) definite.

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Homogenization, Drift, Stabilization and Destabilization for the Patlak-Keller-Segel Systems Driven by the Indirect Taxis and the Short-Wavelength External Signals

Andrey Morgulis and Konstantin Ilin

Abstract We study a model of the active continuous media driven by the interspecific taxis and also by an external signal. We employ the Patlak-Keller-Segel law for modelling the tactical motions. We address the short-wavelength signals with the use the homogenization. We show that such a signal exerts the effect on the large-scale dynamics by the drift that arises from the homogenization. We analyse in detail the signals which have the form of travelling waves. We find out that they are capable of producing the quasi-equilibria—that is, the short-wavelength patterns which stay in equilibrium on average. We examine the stability of the quasi-equilibria and compare the results to the case when the signal is off. The effect of the signal turns out to be not single-valued but depending on the speed at which the signal-producing wave propagates. Namely, there is an independent threshold value such that increasing the amplitude of the wave destabilizes the quasi-equilibria provided that the wave speed is above this value. Otherwise, the same action exerts the opposite effect. It is worth to note that the effect is exponential in the amplitude of the signal in both cases.

Keywords Patlak-Keller-Segel systems · Prey-taxis · Indirect taxis · External signal production · Stability · Instability · Poincare-Andronov-Hopf bifurcation · Averaging · Homogenization

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1 Introduction

Mathematical modelling the communities of living species, cells, biological tissues and similar objects often treats them as some active continuous media which are capable of responding to the changes in the states of themselves or the external environment by directed macroscopic movement. Such an ability is known as taxis. Modelling the taxis often relies on seeing it as a kind of cross-diffusion the very popular example of which is the Patlak-Keller-Segel (PKS) law. The latter states that the tactical flux of medium in response to the changes of some scalar field called stimulus or signal is parallel to the gradient of this scalar. Interest in PKS models is due to the role of taxis in the processes of self-organization of active media into non-trivial space-time structures [1].

A homogeneous or, equivalently, a translation-invariant PKS system can stay in an equilibrium which is also homogeneous in the sense that the density of each species is constant. One of the possible reasons for the system to leave the equilibrium is the instability of the latter to the small perturbations. It is known that such instabilities, accompanied by local bifurcations, are often the first links in chains of dynamic transitions leading to complex space-time structures [2–6].

In this communication, we discuss the communities endowed with both an interspecific taxis and the taxis driven by an external signal, the part of that can be played by an environmental characteristic such as temperature, salinity or the terrain relief. Modelling such communities leads to inhomogeneous systems, and then the quasi-equilibria—that is, the inhomogeneous patterns staying 'in equilibrium on average', substitute the homogeneous equilibria. With the help of homogenization [7, 8], we will study the effect of a short-wave signal on the stability of the quasi-equilibria following the line of the seminal Kapitza's theory of the upside-down pendulum.

Although considering PKS systems on the inhomogeneous environments seems to be quite natural, mentioning them in the literature is much less often than mentioning the homogeneous ones.There are several articles, e.g., [9] or [10] aimed at the topics like the global boundedness, extinction or coexistence but not at the issues we raise here. Also, we would like to mention article [11] focused on the effect of the terrain relief on the excitation of waves in a spatially distributed living community. Although this article does not consider taxis, it employs homogenization, which we use too.

2 The Governing Equations

We'll be considering a system of PKS family that is as simple as possible but still capable of forming the non-trivial spatio-temporal patterns due to the instabilities and local bifurcations of the homogeneous equilibria. Dimensionless form of this system reads as

$$u_t = (\kappa q + f + \delta_1 u_x)_x - \nu u; \;\; p_t = (\delta_2 p_x - pu)_x; \;\; q_t = q(1 - q - p) + \delta q_{xx} \tag{1}$$

Here the notation of x, t stand for a spatial and temporal coordinates; while being used as subscripts, they denote partial differentiation; the notation of δ_2, δ_1, δ, κ, ν stand for positive parameters. The second and third equations in the line (1) (from left to right) describe the balances of densities of two interacting species for which we use the notations of p and q. At that, only one of these equations includes the advective term, $(pu)_x$, in which the notation of u stands for the advecting velocity. For the sake of definiteness, we'll be calling the predator (prey) the species that is capable (not capable) of advective moving. Setting the balance equations presumes that the reproduction and losses of the prey due to predation obey the logistic and Lotka-Volterra laws correspondingly and that the contribution from the reproduction and mortality of the predators is negligible. The last assumption makes sense if reproducing-dying the predators goes on much slower than the other processes considered. The first (from left) equation in the line (1) governs the advective velocity of the predator species in response to the prey density and the external signal. Also, it takes into account the diffusion of velocity and the resistance to predators' motion due to the environment. The coefficients of the diffusion and resistance are δ_1 and ν correspondingly. The coefficient denoted as κ measures intensity of the prey-taxis.

In the articles [2, 3], Govorukhin, Arditi et al. introduced homogeneous version of the system (1) (in which $f = 0$) for taking into account the inertia of species' transport while studying the spatio-temporal pattern formation in the living communities. In particular, they reported the transitions to the complex wave motions due to the destabilization of the homogeneous equilibria. It's worth to note that the inertial system (1) is allied in some respects to the Cattaneo model of chemosensitive motions and hyperbolic limits for kinetic equations discussed in the articles [12, 13].

Although the inertial system (1) does not employ the PKS law explicitly, it is equivalent to a PKS system. This equivalence follows from a simple ansatz $u = \kappa \partial_x \phi$ found by Tyutyunov et al. [5]. At that, ϕ plays the part of the stimulus that drives the taxis. The equation governing it arises from integrating the equation for the advective velocity after putting $u = \kappa \partial_x \phi$. The prey' density, in turn, enhances the production of the stimulus, to that the external signal contributes too. Such kind of taxis is indirect in the sense that the stimulus is not the prey density but the intensity of a field emitted by prey. Tello and Wrozhek and also Li and Tao have addressed such kind of taxis in their recent articles [14, 15] making focus upon the existence of the non-trivial steady states and the global boundedness of solutions. We'll be considering other issues for which the inertial form of the system is more convenient, and henceforth we'll be using only it.

3 Summary of the Results

In this section, we outline the results of our study. In the forthcoming sections, we briefly discuss justifying them. A reader can find the details in the preprint [16].

In what follows, let's consider the fast variables (ξ, τ) as the angular coordinates on 2-torus $\mathbb{T}^2$. Let

$$\langle g \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} g(x, t, \xi, \tau)\, d\xi d\tau.$$

Theorem 1 *Assume that the external signal takes the following form* $f = f(x, t, \xi, \tau)$*, where* $\xi = \omega x$*,* $\tau = \omega t$*, and let* $\delta_i = \nu_i \omega^{-1}$*,* $\nu_i = \text{const} > 0$*,* $i = 1, 2$*. Then the asymptotic of system* (1) *for* $\omega \to \infty$ *reads as*

$$(p, q, u)(x, t) = (\bar{p}(x, t) P(x, t, \tau, \xi), \bar{q}(x, t), \bar{u}(x, t) + \tilde{u}(x, t, \tau, \xi)) + O(\omega^{-1}), \tag{2}$$

$$\tilde{u}_{0\tau} = (f + \nu_1 \tilde{u}_{0\xi})_\xi,\ \langle \tilde{u} \rangle = 0;\ P_\tau = (\nu_2 P_\xi - P(\bar{u} + \tilde{u}))_\xi,\ \langle P \rangle = 1; \tag{3}$$

$$\bar{u}_t = (\kappa \bar{q} + \bar{f})_x - \nu \bar{u},\ \partial \bar{p}_t + (\bar{p}(\bar{u} + \langle \tilde{u} P \rangle))_x = 0, \bar{q}_t = \bar{q}(1 - \bar{p} - \bar{q}) + \delta \bar{q}_{xx} \tag{4}$$

where $\bar{f} = \langle f \rangle$*, and Eqs. (3) have to be solved on* $\mathbb{T}^2$*.*

In what follows, the system (4) is called *homogenized*. It's worth to stress that the averaged velocity denoted as $\bar{u}$ is not the actual velocity of the averaged transport of the predators—that is, there is a drift the velocity of which is equal to $\langle \tilde{u} P \rangle$, and only it collects all the remembrances about the external signal and conducts them to the homogenized system.

The homogenized system is closed relative to the unknowns $\bar{p}, \bar{q}$, and $\bar{u}$ for every given external signal. Indeed, once we have specified the external signal, f, resolving the problems (3) determines *the drift mapping* $\mathcal{V}(f) : \bar{u} \mapsto \langle \tilde{u} P \rangle$, the structure of which is as follows. Solving the equations for $\tilde{u}$ determines a linear mapping $\mathcal{G} : f \mapsto \tilde{u}$. Let now a real constant σ substitute $\bar{u}$ in the equations for the unknown P. Then solving them for a given $\tilde{u}$ induces a non-linear mapping $\mathcal{P}(\tilde{u}) : \sigma \mapsto \langle \tilde{u} P \rangle$. Finally,

$$(\mathcal{V}(f)\bar{u})\,(x, t) = \mathcal{P}(\mathcal{G} f)\bar{u}(x, t).$$

In what follows, we'll be calling *quasi-equilibrium (equilibrium)* a stationary solution to the homogenized system (4) (to exact system (1)) such that $\bar{u} = 0$ ($u = 0$). We'll be calling *homogeneous* the quasi-equilibria (equilibria) in which the distribution of each species is uniform.

The homogeneous version of system (1) (in which $f = 0$) allows a family of homogeneous equilibria that reads as

$$p \equiv p_e, \quad q \equiv q_e, \; u = 0, \quad p_e = \text{const} > 0, \quad q_e = \text{const} > 0, \quad p_e + q_e = 1. \tag{5}$$

The first inequality is a natural restriction necessary for getting the positiveness of the equilibrium densities of both species.

Let a homogenized system allow a quasi-equilibrium and the notation of p_e and q_e stand for the corresponding densities of species. Set $\bar{p} = p_e$ and $\bar{q} = q_e$ and let $\tilde{u}, \tilde{P}$ are the solutions to problems (3) where $\bar{u} = 0$. Then the formula (2) gives the leading approximation for the quasi-equilibrium pattern.

Let the external signal has the form of a travelling wave—that is,

$$f = A\tilde{f}(\eta), \; \eta = \xi - c\tau, \; A = \text{const} \geq 0, \; c = \text{const} \geq 0, \; \langle \tilde{f} \rangle = 0. \tag{6}$$

Since the signals of this class are invariant to translations in variables x, t, the corresponding homogenized systems are invariant too. Given with such an invariance, we again arrive at the family (5), the members of that, this time, stand as the homogeneous quasi-equilibria—that is, $\bar{p} = p_e, \bar{q} = q_e$ and $\bar{u} = 0$.

We'll be using the following notation. Let ∂_η^{-1} stand for the right inverse to differentiation ∂_η—that is,[1]

$$\partial_\eta \partial_\eta^{-1} w = w, \; \int_0^{2\pi} \partial_\eta^{-1} w \, d\eta = 0 \quad \forall\, w : \; w(\eta) = w(\eta + 2\pi), \; \int_0^{2\pi} w \, d\eta = 0. \tag{7}$$

For every signal specified by the expressions (6), we have got the explicit expression for the quasi-equilibrium pattern. The latter is the travelling wave propagating at the same speed as the forcing wave. The stationary signals—that is, those determined by the formulae (6) in which $c = 0$, $\eta = \xi$ and $\tilde{f} = \tilde{f}(\xi)$, gives rise to the simplest patterns. Namely,

$$\tilde{u} = \tilde{u}_e = -A(\nu_1 \partial_\xi)^{-1} \tilde{f}, \; P = P_e = \langle \mathrm{e}^{as} \rangle^{-1} \mathrm{e}^{as}, \; s = -\partial_\xi^{-2} \tilde{f}, \; a = (\nu_1 \nu_2)^{-1} A, \tag{8}$$

Such a pattern does not produce the drifting since $\langle \tilde{u}_e P_e \rangle = \text{const}\langle (P_e)_\xi \rangle = 0$. However, this is not the case for $c \neq 0$. Then there is a *residual drifting* at a constant velocity $v_e \overset{\text{def}}{=} \mathcal{V}(f)0 = \text{const} \neq 0$. The residual drifting greatly matters despite the uniformity of it. We'll be seeing that below. The matter of fact is that we are no

[1] While considering the signals (6), we require $2\pi/c$-periodicity in τ instead of $2\pi-$periodicity. In accordance with this, we re-define the averaging, $\langle \cdot \rangle$.

more free to choose the reference frame since we already have distinguished that one relative to which the signal-producing wave propagates at the specified speed.

Now we are about presenting the outcomes of the linear stability analysis for the equilibria and quasi-equilibria. At this point, several terminological remarks and comments seem to be useful. Basically, our understanding of the notion of stability comes from Liapunov's theory. In particular, the linear stability analysis for an equilibrium of real-valued system $\dot{y} = F(y)$ means studying the behaviour of the small perturbations having the form $\exp(\lambda t)y$, or, for more generality, the semi-group $\exp(tA)$ generated by the differential A of the mapping F evaluated at this equilibrium. In particular, the solutions of the spectral problem $Ay = \lambda y$ are in one-to one correspondence to the eigenmodes of the small perturbations having the above form. We call such a spectral problem the *stability problem*. We say that an eigenmode is *stable (unstable, neutral)* if the real part of the corresponding eigenvalue, λ, is negative (positive, equal to zero).

Let now the system, together with the equilibrium of it, smoothly depend on several parameters. The space of the parameters naturally separates itself into *the areas of instability or stability.* When the point of parameters' space belongs to the former (latter) area, there exist (no) unstable eigenmodes. Every smooth path embedded in the space of the parameters parametrizes the family of equilibria, that, in turn, generates the family of the stability problems. We call a submanifold in the space of the parameters as *the neutral submanifold* if the transversal intersection of it by a smooth path generically entails the transversal intersection of the imaginary axis by the branch of the eigenvalues of the corresponding stability problems. In general, the neutral submanifold of a specific equilibrium is multi-foliated or stratified. Disappearing or arising the unstable eigenmodes upon crossing the neutral submanifold is what we call *the occurrence of instability.* We say that the instability is *oscillatory (monotonic)* if the corresponding branch of the eigenvalues is not real (is real). We call *critical* the subset of neutral submanifold that separates the areas of stability and instability one from another one.

It is well-known that an occurrence of instability in the family of equilibria of a smooth family of vector fields indicates the local bifurcations. If there are no additional degenerations, branching the equilibria family accompanies the monotone instability, and branching the limit cycle off the family accompanies the oscillatory instability. This is the so-called Poincare-Andronov-Hopf bifurcation. More complex bifurcations happen in the case of additional degeneracy, e.g. when the neutral spectrum is multiple. For more information on this subject, a reader could refer to monographs [17–19].

From the results reported in articles [2, 3], it follows that the occurrence of the oscillatory instability of the homogeneous equilibria of the homogeneous version of the system (1) entails exciting the time-periodical waves, which branch off towards the area of instability. They keep the periodicity as long as the parameters vary themselves near the neutral submanifold but on the unstable side of it, and they reveal more and more complex dynamics, which, in the end, becomes chaotic, upon extending deeper inside the domain of instability.

In what follows, we'll be considering only the signals having the form of the travelling waves (6). Then the expressions (5) give the homogeneous quasi-equilibria. Let's choose out of the family (5) a quasi-equilibrium with some specific densities $p_e > 0$ and $q_e = 1 - p_e > 0$. Given the invariance of homogenized system (4) and homogeneous version of the exact system to the spatio-temporal translations, we'll be considering only the eigenmodes having the following form

$$(\hat{u}, \hat{p}, \hat{q}) \exp(i\alpha x + \lambda t), \ \lambda \in \mathbb{C}, \ \alpha \in \mathbb{R}. \tag{9}$$

Thus, the homogenized system (4) explicitly depends on the positive parameters κ, ν, δ, and two more independent parameters arise upon setting the stability problem for the homogeneous quasi-equilibria. These are the wavenumber of the perturbation, α, and the equilibrium density of the predators, p_e. Also, the homogenized system depends on the signal, f. The following Lemma 1 explains how this dependence exerts the effect on the stability problem. The formulation of the lemma involves the quantities denoted c and a. Informally, they stand as the speed of the signal-producing wave and the effective amplitude of it, correspondingly. Formally, we have been defining them in the lines (6) and (8).

Lemma 1 *The stability problem for the eigenmodes* (9) *linked to the homogeneous quasi-equilibria* (5) *depends on the action of two functionals. These are the velocity of residual drift,* $v_e = v_e(f)$*, and one more functional denoted as* $\mu(f)$*. The values of* $v_e = v_e(f)$ *and* $\mu(f)$ *enter the stability problem as parameters. At that,* $\mu(f) > 0$ *for every* f*, and neither parameter* κ *nor* $\mu(f)$ *appear separately but only as the aggregate* $\kappa\mu(f) > 0$*. All other parameters of the stability problem are independent of* f*. Further,* $\mu(0) = 1$*,* $v_e(0) = 0$*, and* $\mu \to +0$ *and* $v_e - c \to 0$ *exponentially when* $a \to \infty$ *provided that the signal is non-degenerate in the sense explained in Sect. 5.*

We'll be calling the quantity $\bar{\kappa} \stackrel{def}{=} \mu(f)\kappa$ *the effective intensity of taxis*. Henceforth we assume that the parameters $(p_e, \alpha, \nu, \delta)$ obey the following restrictions $0 < p_e < 1, \alpha^2 > 0, \nu > 0, \delta > 0$. The first inequality is a natural restriction necessary for getting the positiveness of the equilibrium densities of both species. The other inequalities are for getting rid of the excessively degenerated cases.

Theorem 2 *Consider a specific homogeneous quasi-equilibria of the family* (5) *and the eigenmodes* (9)*. Then the corresponding neutral manifold is bifoliated. The leaves of it are the graphs of the algebraic functions* $\bar{\kappa} = \kappa_c^{\pm}(v_e, p_e, \beta, \nu, \delta)$ *where* $\beta = \alpha^2$*. At that,* $\kappa_c^- < \kappa_c^+$ *provided that* $v_e = v_e(f) \neq 0$*, and* $\kappa_c^- = \kappa_c^+$ *otherwise.On the neutral manifold, the instability is always oscillatory. Every eigenmode having the wavenumbers* $\alpha = \pm\sqrt{\beta}$ *and linked to quasi-equilibrium with density* p_e *is stable provided that*

$$\bar{\kappa} = \mu(f)\kappa < \kappa_c^-(v_e, p_e, \beta, \nu, \delta) \tag{10}$$

There is an unstable eigenmode if $\bar{\kappa} > \kappa_c^-(v_e, p_e, \beta, \nu, \delta)$. *Let* $c > c_*$ *where* $c_* = \sqrt{\nu(q_e + \delta\beta)\beta^{-1}}$. *Then it turns out that* $v_e > \min_\beta c_* = \sqrt{\nu\delta}$ *and* $\kappa_c^- < 0$ *for every sufficiently large value of a, and the inequality (10) necessarily fails to hold, and therefore, the considered eigenmode is unstable for every admissible set of the problem parameters. Moreover, every eigenmode is unstable provided that the squared wavenumber of it,* $\beta = \alpha^2$, *satisfies the inequality* $\beta > \beta_* = q_e\nu(v_e^2 - \nu\delta)^{-1} > 0$. *Let now* $c < c_*$. *Then the inequality (10) holds for every sufficiently high values of a. Moreover,* $\kappa_*^-(a, c, \nu, \delta) = \inf\{\kappa_c^-(v_e, p_e, \beta, \nu, \delta), 0 < p_e < 1, \beta > 0\} > 0$, *and the inequality (10) holds for every specific equilibrium density and every specific wave number of eigenmode provided that the value of a is sufficiently large and* $c < \min_\beta c_* = \sqrt{\nu\delta}$. *The threshold value of a depends only on* ν, δ, c *and on the signal profile,* $\tilde{f}$.

4 Discussion and Comments

In our considerations, the case of homogeneity (in which $f = 0$) appears in the limit $A \to +0$ (or $a \to +0$). There exist the homogeneous equilibria which form the same family as the quasi-equilibria, namely, that we have been determining in the line (5). This circumstance allows us to compare the stability or instability of the equilibria to the quasi-equilibria in the most direct manner.

In the case of homogeneity, $v_e = v_e(0) = 0$ and $\mu = \mu(0) = 1$. Hence the stability condition (10) takes the form

$$\kappa < \kappa_c(p_e, \beta, \nu, \delta) \stackrel{def}{=} \kappa_c^-(0, p_e, \beta, \nu, \delta) = \kappa_c^+(0, p_e, \beta, \delta), \tag{11}$$

We stress that $\kappa_*(\nu, \delta) = \inf\{\kappa_c(p_e, \beta, \nu, \delta), 0 < p_e < 1, \beta > 0\} > 0$. It is important as neither an instability of any homogeneous equilibria nor the accompanying bifurcations are possible provided that $\kappa < \kappa_*(\nu, \delta)$, no matter which values of predators' density and the disturbances wavelengths are specified. In this sense, the value of $\kappa_*(\nu, \delta)$ is the threshold of the collective stability of the whole family of the homogeneous equilibria.

The residual drift also vanishes for the stationary signals defined by (6) with $c = 0, \eta = \xi$ and $\tilde{f} = \tilde{f}(\xi)$. Therefore we set $v_e = 0$ in the stability condition (10). Then the criteria for the stability of a specific homogeneous quasi-equilibrium and for the collective stability of such quasi-equilibria read as

$$\kappa < \langle e^{-as}\rangle\langle e^{as}\rangle\kappa_c(p_e, \beta, \nu, \delta),\ \kappa < \langle e^{-as}\rangle\langle e^{as}\rangle\kappa_*(\nu, \delta) \tag{12}$$

correspondingly where function $s = s(\xi)$ have been defined in the line (8). The use of the expressions listed there makes deriving the criteria (12) straightforward. By Lemma 1, the multiplier $\langle e^{-as}\rangle\langle e^{as}\rangle = \mu^{-1}(f)$ grows exponentially for $a \to \infty$. Hence, we are capable of satisfying both inequalities from the line (12) by increasing the amplitude of the signal, A. Comparing these observations to those concerned with the case of homogeneity, we conclude that a suitable increase in the amplitude of the signal causes stabilizing or even the collective stability of the homogeneous quasi-equilibria. At that, the threshold for the collective stability compared to the case of homogeneity grows exponentially, and in this sense, the stabilizing effect is exponentially sharp. For example, let $f = A \sin \xi$. Then $s = \sin \xi$, and $\langle e^{-as} \rangle \langle e^{as} \rangle = I_0^2(a)$ where I_0 is the modified Bessel function of the first kind that grows exponentially. The criterion for the collective stability reads as $\kappa < I_0^2(a)\kappa_*(\nu, \delta)$. This is the simplest realization of the scenario of stabilization outlined in Theorem 2.

The simple example of stabilization that we just have been discussing demonstrates losing the taxis ability due to amplifying the external signal. It is quite natural to suppose perceiving the intensive external signals to be capable of distracting the predators from pursuing the prey. Nevertheless, the exponential acuity of the effect is surprising. The case becomes less straightforward if the signal-producing wave propagates at a non-zero speed. Then, from Theorem 2, it follows that amplifying the signal exerts opposite effects depending on the relation of the wave speed, c, to an independent threshold denoted as c_*. Namely, for $c > c_*$, amplifying the signal leads to the total destabilization of the short-wavelength modes. Compared to the case of homogeneity, this is an utterly destabilizing effect. These observations sharply contrast to those reported regarding the case of the stationary signals. For $c < c_*$, amplifying the signal stabilizes the quasi-equilibria in the same way as in the case of the stationary signal. The left and right frames in Fig. 1 illustrate the mentioned scenarios. The case of homogeneity corresponds to $a = 0$.

The short-wavelength stabilization or destabilization of the quasi-equilibria resemble the effects of high-frequency vibration widely known in classical mechanics or the continuous media mechanics. The celebrated examples are the upside-down pendulum or the counterparts of it emergent from the dynamics of the stratified fluid. The vibrations exert the effect on such kind of systems via the so-called effective potential energy that arises upon averaging, see, e.g., articles [20, 21]. Our examples show that the short-wavelength fluctuations of the environment exert their effects on the PKS systems quite differently, namely, by giving rise to the drift. Studying the drift arising upon averaging the transport of some density by an oscillating medium goes back to Stokes. Now this area is the subject of continuing researches. A reader can find more details and references, e.g., in the article [22]. Usually, Stokes' drift turns out to be relatively weak, and the effect of it reveals itself only regarding the long-term transport or mixing. In the PKS system, we have been observing stabilizing or destabilizing the quasi-equilibria only due to Stokes's drift right in the leading approximation.

In our considerations, the unstable modes occur upon crossing the graphs of two critical values of the tactical intensity denoted as $\kappa_c^{\pm}$. Right on each graph, there exists a neutral wave. The neutral wave occurring at the lower critical value that is equal to κ_c^- propagates upstream, i.e. faster than the residual drift, and the other one propagates downstream, i.e. slower than the residual drift. Since the value of κ_c^- is the stability threshold, the wave propagating upstream destabilizes the

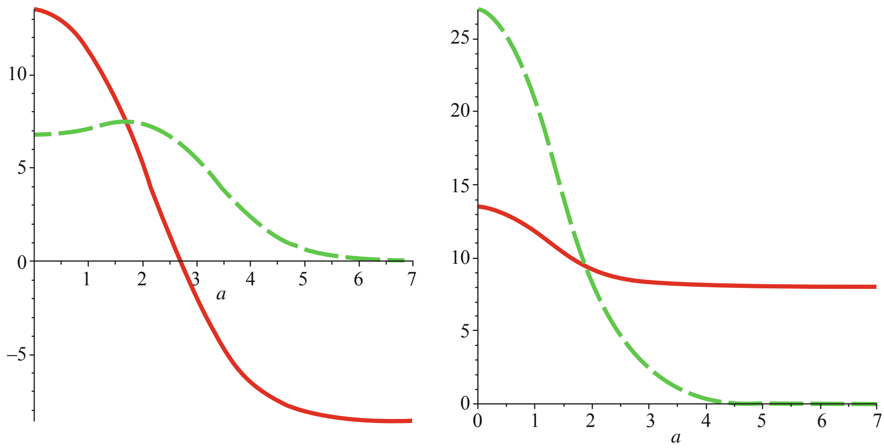

Fig. 1 The figure displays the plots κ_c^- vs a and $\bar{\kappa}$ vs a built with the use of the explicit expressions, which we derive in Sect. 5. The signal profile is $\sin(\eta)$. Solid and long-dashed lines indicates the plots κ_c^- vs a and $\bar{\kappa}$ vs a correspondingly. The right (left) frame displays the plots drawn for $c = 1/2$ ($c = 2$). The case of homogeneous system corresponds to $a = 0$. The values of the other parameters are as follows:$\nu_1 = \nu_2 = \delta = \nu = 1$, $p_e = \beta = 2/3$. For such parameters, $c_* = \sqrt{3/2}$ and $\kappa_c = \kappa_* = 27/2$. Calculating these values repeats the course of proving Theorem 2. Thus, $c > c_*$ ($c < c_*$) for the plots shown in the left (right) frame. For $a \to +0$, the limit value of $\bar{\kappa}$ is equal to $\kappa = 2\kappa_* = 27$ ($\kappa = \kappa_*/2 = 27/4$) for the right frame (left) frame. By Theorem 2, the eigenmode with squared wavenumber $\beta = 2/3$ associated to the quasi-equilibrium with the value of predator's density equal to $p_e = 2/3$ is unstable (stable) for every value of a for which long-dashed line is above (below) the solid one. Hence we see in the left frame that the growth of a destabilizes the eigenmode, which is stable for the values of a close to zero. Further increase in a entails the occurrence of the negative values of κ_c^- (that causes the total destabilizing). In the right frame, we see that the growth of a stabilizes the eigenmode, which is unstable for the values of a close to zero

quasi-equilibria. These features are due to breaking the mirror symmetry by the drift. When the drift vanishes, the upper and lower critical intensities merge. The corresponding neutral modes become conjugated by the mirror symmetry and span two-dimensional spectral subspace. When the drift arises, the double neutral mode that exists due to the symmetry splits itself into two simple ones. Thus we arrive at a two-parametric phenomenon that deserves detailed nonlinear analysis.

The neutral spectrum that we face upon considering the instability of the equlibria or quasi-equlibria is always multiple. Namely, there is pure imaginary pair of eigenvalues and null eigenvalue. The imaginary pair can be even double due to the mirror symmetry if the external signal does not produce the drift. The null eigenvalue is due to the conservation law for the predators' density, because of which the homogeneous equilibria are not isolated but form the continuous 1-parametric family. Let the imaginary pair be simple, say, due to the drift, or because of setting the boundary conditions. For example, Govorukhin et al. and Arditi et al. considered the initial-boundary value problem for the homogeneous system (1) in a bounded spatial domain with the use of Neumann's boundary conditions (also known as the no-flux boundary conditions). However, the mentioned conservation law persists itself despite the drift or Neumann's boundary conditions. The equilibrium family and the null eigenvalue persist too. One can get rid of this residual degeneration by restricting the system on a level set of the conserved quantity. As a result, the common theory of the Poincare-Andronov-Hopf bifurcation becomes applicable to the restricted system. Alternatively, one can apply the general results on the bifurcation accompanying the oscillatory instability for the vector fields, which possess the so-called cosymmetry [23].[2]

Several authors [4–6] explored the oscillatory instability and the spatiotemporal patterns it creates in more general but still homogeneous PKS systems. They considered the finite domains and employed Neumann's boundary conditions. Given such boundary conditions and the kinetic of species, the mentioned degenerations disappear, and the oscillatory instability follows the generic Poincare-Andronov-Hopf scenario.

Considering the PKS systems in the bounded spatial domains allows us to avoid some difficulties emergent from the continuous spectra on the stage of the nonlinear analysis, but brings us at the problem of choosing the boundary conditions. For instance, imposing Neumann's conditions as done in the cited articles leads to some technical issues upon homogenizing the problem and even upon doing the linear analysis provided that there is the drift. Addressing these technicalities does not seem to be essential since it is not clear whether the Neumann conditions are less artificial than the others. Besides, reducing the symmetry by the boundary conditions can cut out some solutions. In such circumstances, the softer conditions of spatial periodicity seem to be the best. Under such conditions, the homogeneous system (1) and more general PKS systems possess the translational-reflectional symmetry. Although it doubles the neutral modes, the corresponding bifurcations can be addressed using the general theory given in article [24].

Two more apparent topics for continuing the research are the short standing waves and the slowly modulated travelling waves. For instance, expressions $f = A\sin\tau\sin\xi$ and $f = A\sin t\sin(\xi - c\tau)$ specify representatives of the former and the latter kinds correspondingly. Both classes of signals produce the homogeneous quasi-equilibria but the linearization near them lead to the systems with variable coefficients. In the case of time-periodic slow modulation, one can address the corresponding spectral problem using Floquet's theory.

Also, it is of interest to what extent the shape of the signal influences its effect in analogy with the issues addressed in [25]. In particular, there arises an optimization problem for the functional $\mu(f)$ subject to restriction $\langle f^2\rangle = 1$.

[2]Given the cosymmetry, the limit cycle generically does not branch off from the critical equilibrium except for integrable cosymmetry that is equivalent to a conservation law. In case of such an exception, the cycle branches off 'as usual' provided that there is no additional degeneration.

5 Proofs

In this section we sketch the proofs of the assertions claimed above.

Proof of Theorem 1 We write the exact system (1) in the fast variables ξ, τ, and then seek the power asymptotic expansion in the negative integer powers of $\omega \to \infty$. We require the coefficients of the asymptotic series to be $2\pi-$periodic in ξ and τ. Replacing the unknowns by the power series in the exact system yields a chain of equations, and we solve them one by one. It is a quite routine procedure modulo the following lemma. □

Lemma 2 *On* $\mathbb{T}^2$, *consider equation* $Q_\tau + (wQ - Q_\xi)_\xi = 0$ *where* $w = w(\xi, \tau)$ *is a smooth function, and* $\epsilon = \text{const} > 0$. *There exists a unique solution to this equation satisfying the additional condition* $\langle Q \rangle = 1$.

Proof of Lemma 2 Let H be the space of the Fourier series in ξ, τ with square-summable coefficients and let $\mathcal{L} : \mathrm{H} \to \mathrm{H}$, be operator linked to the left-hand side of equation $Q_\tau + (wQ - Q_\xi)_\xi = 0$. Let us prove that $\dim \operatorname{Ker} \mathcal{L} = 1$ and $\langle \chi \rangle \neq 0 \,\forall\, \chi \in \operatorname{Ker} \mathcal{L} \setminus \{0\}$. Let $\mathcal{L}^*$ denote the operator adjoint to $\mathcal{L}$ and let $\mathcal{J} : \mathrm{H} \to \mathrm{H}$ be the action of inversion $(\xi, \tau) \mapsto (-\xi, -\tau)$. Define $\breve{\mathcal{L}}^* = \mathcal{J}\mathcal{L}^*\mathcal{J}$. Then $\breve{\mathcal{L}}^* : \varphi \mapsto \varphi_\tau - \epsilon\varphi_{\xi\xi} + w\varphi_\xi$. Notice that PDE $\breve{\mathcal{L}}^*\varphi = 0$ obeys the strong maximum and minimum principles (see, e.g. [26] or [27]). Hence $\operatorname{Ker} \breve{\mathcal{L}}^* = \{\varphi \equiv \text{const}\} = \operatorname{Ker} \mathcal{L}^*$. Applying the unilateral strong maximum/minimum principles to PDE $\breve{\mathcal{L}}^*\breve{\psi} = 1$ shows that neither equation $\breve{\mathcal{L}}^*\breve{\psi} = 1$ nor equation $\mathcal{L}^*\psi = 1$ has a solution belonging to H. Consequently, the resolvent $(\mathcal{L}^* - \lambda\mathcal{I})^{-1}$, $\lambda \in \mathbb{C}$, has a simple pole at the origin. Since this resolvent is compact, the pair of operators $\mathcal{L}^*$ and $\mathcal{L}$ obeys the Fredholm theorems. Hence $\dim \operatorname{Ker} \mathcal{L} = 1$. Furthermore, conjecturing that $\langle \chi \rangle = 0$ for some $\chi \in \operatorname{Ker} \mathcal{L} \setminus \{0\}$ would imply the existence of solution to equation $\mathcal{L}^*\psi = \text{const} \neq 0$ but this contradicts to what we have proved above. □

Proof of Lemma 1 The system governing the small perturbations of a quasi-equilibrium with densities p_e, q_e reads as

$$\bar{u}_t + \nu\bar{u} - \kappa\bar{q}_x = 0;\; \bar{p}_t + (v_e\bar{p} + p_e(\bar{u} + \mathcal{V}'(\tilde{f})\bar{u}))_x = 0;\; \bar{q}_t + q_e(\bar{p} + \bar{q}) - \delta\bar{q}_{xx} = 0,$$

where $\mathcal{V}(f)$ denotes the drift operator, $v_e = \mathcal{V}(f)0$, and $\mathcal{V}'(f)$ stands for the differential of the drift operator evaluated at $\bar{u} = 0$. Let's show that $\bar{u} + \mathcal{V}'(\tilde{f})\bar{u} = \mu(f)\bar{u}$ for some functional μ. It will allow us to substitute κ by $\mu(f)\bar{\kappa}$ after a suitable renormalization. As we have been seeing in Sect. 2, $\mathcal{V}(f) : \bar{u} \mapsto \langle \tilde{u}P \rangle$, where the notations of $\tilde{u}$ and P stand for the solutions to the problems (3). Let $\mathcal{L} : \mathrm{H} \to \mathrm{H}$ be the operator that we have been defining in the course of proving Lemma 2. Let w be the coefficient of PDE to which we have been linking this operator. The kernel of $\mathcal{L}^*$ consists of the identically constant functions for every w. We put $w = \sigma + \tilde{w}$, where $\sigma = \text{const}$, and $\langle \tilde{w} \rangle = 0$. Let $\Pi^*(\sigma)$ be the spectral projector onto $\ker\mathcal{L}^*$. Then $\Pi^*(\sigma) : g \mapsto \langle gQ \rangle$ where Q is the solution to equation $\mathcal{L}Q = 0$ satisfying the condition $\langle Q \rangle = 1$. Hence, the well-known result in the perturbation theory for the linear operators implies that the operator-valued function $\Pi^*(\sigma)$ is analytic in some neighborhood of the origin. Hence, the mapping $\sigma \mapsto \Pi^*(\sigma)\tilde{w}$ is analytic too. Finally, we arrive at the analyticity of the drift mapping since $\langle \tilde{u} P\rangle = \Pi^*(\bar{u})\tilde{u}$. By this conclusion, we calculate $\mathcal{V}'(\tilde{f})$ by the perturbation method and identify the action of it with multiplying by a scalar. Namely, $\mathcal{V}'(\tilde{f}) : g \mapsto \langle \tilde{u} P_1\rangle g$ where P_1 is the solution to the following problem

$$P_{1\tau} = (\nu_2 P_{1\xi} - P - \tilde{u}_0 P_1)_\xi,\ \langle P_1\rangle = 0, \quad P_\tau = (\nu_2 P_\xi - \tilde{u}_0 P)_\xi,\ \langle P\rangle = 1$$

Note that function P determines the quasi-equilibrium pattern in accordance with the formula (2). Since we consider the signals depending only on the fast variables ξ, τ, the multiplier $\langle \tilde{u} P_1\rangle$ is just a constant. Thus, $\mu(f) = 1 + \langle \tilde{u} P_1\rangle$.

By assumption, $f = A\tilde{f}(\eta)$, where $\eta = \xi - c\tau$, $A = \text{const} \geq 0$, $c = \text{const} \geq 0$, and function $\tilde{f}$ is 2π-periodic and equal to zero on average. Let $g * h$ denote the convolution of functions g and h on $\mathbb{R}$. Define

$$\exp_\pm(\sigma) = \begin{cases} e^\sigma, & \pm\sigma > 0, \\ 0, & \mp\sigma < 0, \end{cases} \quad s = -\partial_\eta^{-1} \exp_+^{-\frac{c}{\nu_1}} * \tilde{f}; \tag{13}$$

where the superscript indicates raising to a power, and the convolution acts in variable η. Then $\tilde{u}_0 = A\mathcal{G}\tilde{f} = \nu_1^{-1} A\partial_\eta s$. Further, we put

$$E = e^{as},\ a = (\nu_1\nu_2)^{-1}A,\ R(\sigma) = \langle E E^{-1}(\cdot - \sigma)\rangle,\ z = (c - \bar{u})\nu_2{}^{-1}.$$

Let $z \neq 0$ and P be the periodic solution to the Eq. (3). Then

$$P(z,\cdot) = \Gamma_\pm^{-1}(z)E(\cdot)\left(\exp_\pm^{-z} * E^{-1}\right)(\cdot),\ \Gamma_\pm(z) = \int_{\mathbb{R}} \exp_\pm^{-z}(\sigma)R(\sigma)\,d\sigma,\ \pm z > 0.$$

Using the Fourier series matches two expressions for P displayed here one to another one. Indeed, let $\hat{E}_k$ and $\check{E}_k$ be the Fourier coefficients of functions E and E^{-1} correspondingly. Then

$$P(z,\eta) = \Gamma^{-1}(z)E(\eta)\left(\check{E}_0 + z\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\check{E}_k e^{ik\eta}}{z+ik}\right), \tag{14}$$

$$\Gamma(z) = (\check{E}_0\hat{E}_0) + z\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(\hat{E}_k^*\check{E}_k)}{z+ik} = \pm z\Gamma_\pm(z),\ \pm z > 0. \tag{15}$$

Given the formula (15), we express the drift velocity as follows

$$\mathcal{V}(A\tilde{f})\bar{u} = \langle \tilde{u}_0 P\rangle = -\nu_2 z\Gamma^{-1}(z)\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{ik(\hat{E}_k^*\check{E}_k)}{z+ik} = \nu_2 z(1 - \Gamma^{-1}(z)). \tag{16}$$

(Deriving the second equality in this chain uses identity $\sum_{k\in\mathbb{Z}} \hat{E}_k^* \check{E}_k = \langle EE^{-1}\rangle = 1$.)
Then the total advective velocity that enters the homozenized system (4) reads as

$$v \stackrel{\text{def}}{=} \bar{u} + \mathcal{V}(A\tilde{f})\bar{u} = c - \frac{z\nu_2}{\Gamma(z)}. \tag{17}$$

Besides, there is one more representation, namely

$$v(z) = c \mp \frac{\nu_2}{\Gamma_\pm(z)}, \ \pm z > 0. \tag{18}$$

The listed expressions give the analytic continuation to the advective velocity to some strip parallel to the real axis in the complex plane of variable z. This conclusion agrees with the analyticity of the drift operator, that we have been discussing earlier. Formula (18) (where one has to put $z = c/\nu_2 > 0$) shows that the residual drifting does exist. Indeed, let $\bar{u} = 0$ Then $z = c/\nu_2$, and

$$v = \mathcal{V}(A\tilde{f})0 \stackrel{def}{=} v_e = c - \frac{\nu_2}{\Gamma_+(c/\nu_2)}, \ c > 0. \tag{19}$$

Hence, the velocity of the residual drift, v_e, generally takes a nonzero value. Inspecting the case of $c = 0$ with the use of the above formula (17) shows that $v_e = 0$—that is, no residual drift occurs if the signal-producing wave is stationary.

Straightforward evaluating the expression (19) for the values of c tending to zero or infinity and for the fixed values of the other variables shows that

$$v_e \to 0, \ c \to +0, \quad v_e \to 0, \ c \to \infty. \tag{20}$$

Consider now expression (19) for the value of a tending to zero or to infinity and for the fixed values of all other quantities. Then

$$v_e \to 0, \ a \to +0, \quad v_e \to c, \ a \to \infty, \tag{21}$$

in the generic case of the signal-producing wave. The first limit is obvious while the second requires some estimates by the Laplace method.

Let's estimate the integrals denoted as $\Gamma_\pm$ when the signal amplitude, a, tends to infinity. For definiteness, let's consider $\Gamma_+ = \Gamma_+(z)$, $z > 0$. We put this integral into a slightly different form, namely,

$$\frac{1}{2\pi} \int\limits_{(0,2\pi)\times(0,\infty)} \mathrm{e}^{-z\sigma}\mathrm{e}^{aS(\eta,\sigma)} d\eta d\sigma, \quad S(\eta,\sigma) = s(\eta) - s(\eta-\sigma), \ z > 0 \tag{22}$$

Assume that every critical point of function s is non-degenerate. The periodicity allows us to reduce the domain of integration of integral (22) to the cylinder $C =$

$\{(\eta, \sigma) \in \mathbb{S} \times (0, 2\pi)\}$, and we arrive at estimating the following integral

$$\int_C e^{-z\sigma} e^{aS(\eta,\sigma)} d\eta d\sigma. \tag{23}$$

The local maximizers of S on the whole $(\eta, \sigma)-$plain are

$$(\eta, \sigma) \in \mathbb{R}^2 : \ s'(\eta) = s'(\eta - \sigma) = 0, \quad s''(\eta)s''(\eta - \sigma) < 0.$$

The last inequality means that no maximizers belong to ∂C. Then the leading term in Laplace's asymptotics of integral (23) for $a \to \infty$ reads as

$$\frac{2\pi e^{a\operatorname{osc}(s)}}{a} \sum_{(x,y)\in M} \frac{e^{-z(x-y)}}{\sqrt{-s''(x)s''(y)}}$$

where $M = \{(x, y) : \ 0 < x \le 2\pi, 0 < x - y < 2\pi, \operatorname{osc}(s) = s(x) - s(y)\}$, $\operatorname{osc}(s) = \sup_{\mathbb{R}} s - \inf_{\mathbb{R}} s$. Revisiting the integral (22) gives the following estimate

$$\Gamma_+(z) \sim \frac{e^{a\operatorname{osc}(s)}}{a(1 - e^{-2\pi z})} \sum_{(x,y)\in M} \frac{e^{-z(x-y)}}{\sqrt{-s''(x)s''(y)}}, \quad a \to \infty. \tag{24}$$

This estimate proofs the second limit equality in (21), and shows that approaching the limit value is exponential therein. However, we also need to prove the exponential decay for the functional μ when $a \to \infty$. This assertion follows from the equality

$$\mu(f) = \frac{1}{\Gamma_+^2(\frac{c}{\nu_2})} \int_0^\infty R(\sigma) e^{-\frac{c\sigma}{\nu_2}} \sigma \, d\sigma = \left. \frac{d\Gamma_+^{-1}(z)}{dz} \right|_{z=\frac{c}{\nu_2}}. \tag{25}$$

This completes the proof. □

Remark 1 Let $s(\eta) = \sin \eta$. Then $\operatorname{osc}(s) = 2$, $M = \{(\pi/2, -\pi/2)\}$, and estimate (24) reads as

$$\Gamma_+(z) \sim \frac{e^{2a} e^{-\pi z}}{a(1 - e^{-2\pi z})}, \quad a \to \infty.$$

The asymptotic of the value of μ entailed by the last estimate for $z = c/\nu_2 \to +0$ matches the asymptotic of the function $I_0^{-2}(a)$, $a \to \infty$, in consistence with the example of stabilization discussed in Sect. 4.

Proof of Theorem 2 Let a homogeneous quasi-equilibrium be specified by density p_e. Link to it an eigenmode that has the form specified by the expression (9).

The characteristic equation for the corresponding stability problem is polynomial. Namely, $\lambda^3 + (iU + d_1)\lambda^2 + (d_2 + iUd_1)\lambda + iUd_2 + r = 0$, where $U = \alpha v_e$, $d_1 = \nu + q_e + \delta\beta$, $d_2 = \nu(q_e + \delta\beta)$, $r = \beta p_e q_e \bar{\kappa}$, $q_e = 1 - p_e$. We employ the complex-valued version of Hurwitz's theorem to count the roots belonging to the right complex semi-plane. It brings us at the chain of Hankel's matrix minors, which allows only four generic distributions of signs of the minors, namely $++++$, $+++-$, $++--$, $++-+$. Direct inspection shows that there exists three algebraic functions $\kappa_c^{\pm}(v_e, p_e, \beta, \nu, \delta)$, and $\kappa_c(p_e, \beta, \nu, \delta)$ between which the inequalities claimed by Theorem 2 hold. When the parameter $\bar{\kappa}$ passes through the values of κ_c^-, κ_c, and κ_c^+ in the ascending order, switching between the above diagrams happens from left to right provided that the values of all other parameters stay unaltered. Hence the number of roots in the right complex semi-plane takes the values of 0, 1, and 2 provided that the value of $\bar{\kappa}$ belongs to intervals $(0, \kappa_c^-)$, (κ_c^-, κ_c^+) and (κ_c^+, ∞) correspondingly. Changing this number is due to changing the sign of the senior minor of Hankel's matrix. Therefore, passing the parameter $\bar{\kappa}$ through the values of $\kappa_c^{\pm}$ entails crossing the imaginary axis by the root of the characteristic polynomial at some non-zero point. Hence the oscillatory instability occurs. Note that passing the parameter $\bar{\kappa}$ through the value of κ_c does not change the number of unstable roots except for the case $v_e = 0$, in which $\kappa_c^- = \kappa_c = \kappa_c^+$. Appearing the negative values of κ_c^- for a specific wavelength excludes the diagram $+ + ++$. Hence the eigenmodes with such a wavelength are always unstable. Checking of the other assertions of Theorem 2 is fully straightforward. □

Remark 2 For $\kappa = \kappa_c^{\pm}$ and $v_e > 0$, the neutral eigenmode is proportional to $\mathrm{e}^{i\sqrt{\beta}(x \pm c_* t)}$. For $v_e < 0$, it suffices to substitute $x \pm c_* t$ by $x \mp c_* t$.

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Nontangential Summability of Fourier Series

A. D. Nakhman and B. P. Osilenker

Abstract Some problems of nontangential summability of trigonometric Fourier series by semicontinuous methods are considered. Upper bounds for the family of means of the series in terms of the Hardy-Littlewood maximal function are obtained. Under a Sz.-Nagy-type condition, a weak-type estimate for the corresponding maximal operator and a statement about nontangential summability almost everywhere are proved. In particular, the exponential means of Fourier series, including the generalized Poisson-Abel summation method, are considered. Applications to the theory of strongly continuous semigroups of operators and to the solution of the generalized Dirichlet problem in a half-plane are given.

Keywords Fourier series · Maximal estimates · Nontangential trajectories · Summability almost everywhere

1 Introduction

Let

$$s[f] = \sum_{k=-\infty}^{\infty} c_k(f) \exp(ikx)$$

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be the Fourier series of an arbitrary 2π-periodic function *f*, summable on $Q = (-\pi, \pi]$ (in notation: $f \in L(Q)$), where

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt) \, dt, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.1}$$

We study behavior of the family of means

$$U_h(f) = U(f, x; \Lambda, h) = \sum_{k=-\infty}^{\infty} \lambda_{|k|}(h) c_k(f) \exp(ikx), \tag{1.2}$$

constructed by using of an infinite, generally speaking, sequence

$$\Lambda = \{\lambda_k(h), h > 0, k = 0, 1, \ldots; \ \lambda_0(h) = 1\}. \tag{1.3}$$

Let $\| f \|_L = \int_{-\pi}^{\pi} | f(t) | \, dt$ be a norm in $L = L(Q)$. For any $f \in L(Q)$ and almost all *x*, the maximal Hardy-Littlewood function is defined as

$$f^* = f^*(x) = \sup_{\eta > 0} \frac{1}{2\eta} \int_{x-\eta}^{x+\eta} | f(t) | \, dt,$$

and the following "weak type" estimate holds ([10], v. 1, p. 60):

$$\mu \left\{ x \in Q | \ f^*(x) > \varsigma > 0 \right\} \leq C \frac{\| f \|_L}{\varsigma}. \tag{1.4}$$

Throughout this paper, the letter *C* will be used for positive constants, not necessarily the same at each occurrence. In (1.4), *C* does not depend on *f* and ς; by μ we denote the Lebesgue measure of the corresponding set. In [6], the following result was obtained.

Let *m* be an arbitrary non-negative integer (*m* may depend on *h*) and

$$\sum\nolimits^{\#} (h; \Lambda, m) = \sup_{k=0,1,\ldots} | \lambda_k(h) | + \sum_{k=0}^{\infty} \frac{(k+1)(|m-k|+1)}{m+k+1} \ln \frac{2(m+k+1)}{|m-k|+1} \, | \lambda_k(h) - 2\lambda_{k+1}(h) + \lambda_{k+2}(h) | \, .$$

Let, further,

$$U_{\#}^* (f, x; \Lambda) = \sup_{h,m} \frac{| U(f, x; \Lambda, h) |}{\sum^{\#} (h; \Lambda, m)}.$$

Theorem 1.1 If the terms of sequence (1.3) satisfy the conditions

$$\lambda_N(h) = \mathrm{O}_h\left(\frac{1}{\ln\ N}\right), N \to \infty$$

and

$$\sum\nolimits^{\#}(h;\ \Lambda, m) < C, \tag{1.5}$$

for each $h > 0$, then for almost all x, the series in (1.2) converges and the estimate

$$U^*_{\#}(f, x;\ \Lambda) \le C\ f^*(x) \tag{1.6}$$

holds.

The constants C in (1.5) and (1.6) are different and depend only on Λ.

A consequence of (1.6) is a weak-type estimate for $U^*_{\#}(f, x;\ \Lambda)$ and convergence $U(f, x; \Lambda, h) \to f(x)$ almost everywhere under the additional condition $\lambda_k(h) \to 1$ $(h \to +0, k = 0, 1, \ldots)$. Relation (1.6) is an extension to the semicontinuous case of known results on the summability of Fourier series almost everywhere (see, for example, [4]).

Our goal is to obtain analogs of the cited results for the case of "nontangential" summability of Fourier series. In Sects. 2–4, we obtain the "nontangential" estimates of the means in terms of the maximal function $f^*(x)$. In Sect. 5, we present weak-type estimates for maximal operators associated with means (1.2). In Sect. 6, we consider some exponential methods for summing the Fourier series.

2 Maximal Estimates

Put $\Delta\lambda_k(h) = \lambda_k(h) - \lambda_{k+1}(h)$, and $\Delta^2\lambda_k(h) = \Delta(\Delta\lambda_k(h))$, $k = 0, 1, \ldots$ For each x, we consider the domain of "nontangential trajectories" tending (y, h) to $(x, 0)$

$$\Gamma_d(x) = \{(y, h) \mid\ y \in (-\pi, \pi], h > 0,\ \frac{|\ y - x\ |}{h} \le d\}, d = const, d > 0.$$

Let $m = \left[\frac{1}{2dh}\right]$; it could be assumed that $m \ge 1$. Denote

$$\sum(h;\ \Lambda, d) = \sup_{k=0,1,\ldots}\ |\ \lambda_k(h)\ | + \sum_{k=0}^{\infty} \tfrac{(k+1)(|m-k|+1)}{m} \ln \tfrac{2(m+k+1)}{|m-k|+1}\ |\ \Delta^2\lambda_k(h)\ |,$$

$$U^*(f, x; \Lambda, d) = \sup_{(y,h)\in\Gamma_d(x)} \frac{| U(f, y; \Lambda, h) |}{\sum(h; \Lambda, d)}.$$

Theorem 2.1 Assume that the terms of sequence (1.3) for each $h > 0$, satisfy the conditions

$$N \mid \lambda_N(h) \mid + N^2 \mid \Delta\lambda_N(h) \mid = o_h(1), \;\; N \to +\infty \tag{2.1}$$

and

$$\sum(h; \Lambda, d) < C. \tag{2.2}$$

Then for all $(y, h) \in \Gamma_d(x)$, the series $U(f, y; \Lambda, h)$ converges and the estimate

$$U^*(f, x; \Lambda, d) \leq C \; f^*(x) \tag{2.3}$$

holds at every point, where $f^*(x)$ exists.

Constant C in (2.2) does not depend on h; C in (2.3) does not depend on f and x.

Remarks

1. If parameter h takes the discrete values $h_m = \frac{1}{m+1}, m = 0, 1, \ldots,$ so that $\lambda_k(h) = \lambda_k^m$ in (1.3) and $\lambda_k^m = 0$ for $k > m$, then the sum in condition (2.2) turns into the sum in Sz.-Nagy [4]

$$\sup_{k=0,1,\ldots,m} |\lambda_k^m| + \sum_{k=0}^{m} \frac{(k+1)(m-k+1)}{m} \ln\frac{2(m+k+1)}{m-k+1} |\Delta^2\lambda_k^m|.$$

2. Condition (2.2), being applied in the case of nontangential summability, is more restrictive than condition (1.5) in the «radial» case.

3 Estimates for Integrals Containing de la Vallée-Poussin Kernels

We begin the proof of Theorem 2.1 with estimates for integrals containing de la Vallée-Poussin kernels

$$V_{m,k}(t) = \frac{\sin\frac{m-k+1}{2}t\cdot\sin\frac{m+k+1}{2}t}{2(m-k+1)\sin^2\frac{t}{2}}, \qquad V_{m,m+1}(t) = \frac{t\sin(m+1)t}{4\sin^2\frac{t}{2}}; \; k = 0, 1, \ldots; \quad m = 1, 2, \ldots$$

and, in particular, the Dirichlet and Fejér kernels $D_k(t) = V_{k,k}(t)$, $F_k(t) = V_{k,0}(t)$ respectively.

It is obvious that for the kernel $V_{m,k}(t)$, the inequalities

$$|V_{m,k}(t) \leq \begin{cases} C\,(m+k+1)\,, 0 \leq \mid t \mid \leq \frac{3}{2}\pi; \\ \frac{C}{|t|}, 0 < \mid t \mid \leq \frac{3}{2}\pi; \\ \frac{C}{(|m-k|+1)t^2}, 0 < \mid t \mid \leq \frac{3}{2}\pi \end{cases} \tag{3.1}$$

hold for all $k = 0, 1, \ldots, m = 1, 2, \ldots$.

Lemma 3.1 Put $m = \left[\frac{1}{2dh}\right]$. If $1 \leq k \leq 2m - 1$, then, for all $(y, h) \in \Gamma_d(x)$, the following estimates hold:

$$\int_{-\pi}^{\pi} \mid f(t) \mid \cdot \mid V_{m,k}\,(y-t) \mid dt \leq C \ln \frac{2\,(m+1)}{\mid m-k \mid +1} f^*(x), 1 \leq k \leq 2m-1; \tag{3.2}$$

$$\int_{-\pi}^{\pi} \mid f(t) \mid \cdot \mid V_{m,k}\,(y-t) \mid dt \leq C \frac{k+1}{m} f^*(x), k \geq 2m. \tag{3.3}$$

Remark Combining the inequalities (3.2) and (3.3), we obtain

$$\int_{-\pi}^{\pi} | f(t) | \cdot | V_{m,k}(y-t) | dt \leq C \frac{k+m}{m} \ln \frac{2(m+k+1)}{| m-k | +1} f^*(x). \tag{3.4}$$

for all $k = 1, 2, \ldots$. In particular, if $\rho = \left[\frac{m}{2}\right]$ and $k = 0, 1, \ldots, \rho + 1$, then

$$\int_{-\pi}^{\pi} \mid f(t) \mid \cdot F_k\,(y-t)\;\; dt \leq C\;\; f^*(x). \tag{3.5}$$

The constants C in relations (3.2)–(3.5) do not depend on the function f, variable x and parameters m and k.

Proof of Lemma 3.1 If $|x - t| \geq \frac{1}{m} \geq 2dh$, then the inequality $|y - t| \geq |x - t| - |y - x|$ implies the relation $|y - t| \geq |x - t| - dh \geq \frac{1}{2}\,|x - t|$ for all $(y, h) \in \Gamma_d(x)$. Therefore, the estimate

$$\mid y - t \mid \geq \frac{1}{2} \mid x - t \mid$$

turns out to be valid for all x and t, satisfying the condition $\mid x - t \mid \geq \frac{1}{m}$.

While evaluating the integral on the left-hand side of (3.4), due to the periodicity of the integrand, we can assume $|y-t| \leq \frac{3\pi}{2}$; then $|x-t| \leq 3\pi$.

1. If $1 \leq k \leq 2m-1$, then

$$\frac{1}{|m-k|+1} \geq \frac{1}{m}.$$

Choose natural numbers $S_1 = S_1(k,m)$, $S_2 = S(k,m)$, so that

$$\frac{2^{S_1-1}}{m} \leq \frac{1}{|m-k|+1} < \frac{2^{S_1}}{m} \text{ and } \frac{2^{S_2-1}}{|m-k|+1} \leq 3\pi < \frac{2^{S_2}}{|m-k|+1}.$$

By virtue of (3.1), we have

$$\int_{-\pi}^{\pi} |f(t)| \cdot |V_{m,k}(y-t)|\, dt \leq C\Big(m \int_{|x-t|\leq \frac{1}{m}} |f(t)|\, dt + \int_{\frac{1}{m}\leq |x-t| \leq \frac{1}{|m-k|+1}} |f(t)| \frac{1}{|y-t|} dt +$$
$$+ \int_{\frac{1}{|m-k|+1}\leq |x-t| \leq 3\pi} |f(t)| \frac{1}{(|m-k|+1)(y-t)^2} dt).$$

Now, we get

$$\int_{-\pi}^{\pi} |f(t)| \cdot |V_{m,k}(y-t)|\, dt \leq C\, m \int_{|x-t|\leq \frac{1}{m}} |f(t)|\, dt +$$
$$+ C \sum_{j=1}^{S_1} \frac{m}{2^{j-1}} \int_{\frac{2^{j-1}}{m} \leq |x-t| \leq \frac{2^j}{m}} |f(t)|\, dt + C \sum_{j=1}^{S_2} \frac{1}{2^{j-1}} \left(\frac{|m-k|+1}{2^{j-1}} \int_{\frac{2^{j-1}}{|m-k|+1} \leq |x-t| \leq \frac{2^j}{|m-k|+1}} |f(t)| dt \right) \leq$$
$$\leq C\,(1+S_1)\, f^*(x) \leq C \ln \frac{2(m+1)}{|m-k|+1} f^*(x).$$

2. If $k \geq 2m$, then $k-m \geq m$. Choosing a natural number $S_3 = S_3(k,m)$, so that

$$\frac{2^{S_3-1}}{m} \leq 3\pi < \frac{2^{S_3}}{m},$$

we obtain

$$\int_{-\pi}^{\pi} |f(t)| \cdot |V_{m,k}(y-t)|\, dt \leq$$
$$\leq C \left((m+k) \int_{|x-t|\leq \frac{1}{m}} |f(t)| dt + \int_{\frac{1}{m} \leq |x-t| \leq \pi} |f(t)| \frac{1}{(|m-k|+1)(y-t)^2} dt \right) \leq$$
$$\leq C \left(\frac{2k}{m}\, m \int_{|x-t|\leq \frac{1}{m}} |f(t)| dt + \sum_{j=1}^{S_3} \frac{m}{(k-m+1)2^j} \cdot \frac{m}{2^j} \cdot \int_{\frac{2^{j-1}}{m} \leq |x-t| \leq \frac{2^j}{m}} |f(t)| dt \right) \leq C \frac{k}{m} f^*(x).$$

Estimates (3.2) and (3.3) are established. Estimate (3.5) follows from (3.4) for $k = 1, \ldots, \rho + 1$, and (3.5) is obvious for $k = 0$. Lemma 3.1 is proved.

Remark In particular case $d = 0$, the following result is valid [6]: for all $m = 0, 1, \ldots, k = 0, 1, \ldots,$ the estimate

$$| \int_{-\pi}^{\pi} f(t)\ V_{m,k}\,(\mathrm{x} - t)\ dt\ | \leq C\ \ln \frac{2\,(m+k+1)}{|\,m-k\,|+1} f^*(x)$$

holds at every point, where $f^*(x)$ exists.

4 Proof of Theorem 2.1

For all $k = 0, 1, \ldots, \quad m = 0, 1, \ldots,$ we obtain the obvious relations

$$D_k(t) = (k+1)\ F_k(t) - kF_{k-1}(t), \qquad D_k(t) = (m-k+1)\,V_{m,k}(t) - (m-k)\,V_{m,k+1}(t), \tag{4.1}$$

Further, we turn to the integral form (1.2), using representation (1.1) of the Fourier coefficients and the Abel transformation ([10], vol. 1, p. 15). We get

$$\begin{aligned}
U\,(f, \mathrm{y}; \Lambda, h) &= \lim_{N\to+\infty} \sum_{k=-N}^{N} \lambda_{|k|}(h) \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp\left(ik\,(x-t)\right) dt = \\
&= \lim_{N\to+\infty} \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \tfrac{1}{2} + \sum_{k=1}^{N} \lambda_k(h) \cos\ k\,(\mathrm{y}-t) \right\} dt = \\
&= \lim_{N\to+\infty} \tfrac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \tfrac{1}{2} + \sum_{k=1}^{N} \lambda_k(h) \left(D_k\,(\mathrm{y}-t) - D_{k-1}\,(\mathrm{y}-t) \right) \right\} dt = \\
&= \lim_{N\to+\infty} \tfrac{1}{\pi} \lambda_N(h) \int_{-\pi}^{\pi} f(t) D_N\,(\mathrm{y}-t)\ dt + \tfrac{1}{\pi} \lim_{N\to+\infty} \int_{-\pi}^{\pi} f(t) \sum_{k=0}^{N-1} \Delta\lambda_k(h) D_k\,(\mathrm{y}-t)\, dt.
\end{aligned} \tag{4.2}$$

The first term under the limit sign on the right-hand side of (4.2) tends to zero for all (y, h) due to condition (2.1) and the estimate $|D_N(y-t)| \leq CN$; this estimate follows from (3.1). Hence,

$$U\,(f, \mathrm{y}; \Lambda, h) = \frac{1}{\pi} \lim_{N\to+\infty} U_N\,(f, \mathrm{y}; \Lambda, h) \tag{4.3}$$

where

$$U_N\,(f, \mathrm{y};\, \Lambda, h) = \int\limits_{-\pi}^{\pi} f(t) \sum_{k=0}^{N-1} \Delta\lambda_k(h) D_k\,(\mathrm{y}-t)\,dt. \tag{4.4}$$

Lemma 4.1 Let the terms of the sequence in (1.3) satisfy conditions (2.1) and (2.2) for each $h > 0$. Then for all $(y, h) \in \Gamma_d(x)$, the estimate

$$\sup_{N|N\geq 2m} |U_N(f, y; \Lambda, h)| \leq C f^*(x) \sum(h, \Lambda, d). \tag{4.5}$$

is valid; C in (4.5) does not depend on f, x and parameter m.

Proof Let, as in Sect. 3, $\rho = \left[\frac{m}{2}\right]$. Using the Abel transformation, we derive from equalities (4.1) and (4.4) that

$$\begin{aligned}
U_N\,(f, \mathrm{y};\, \Lambda, h) &= \int\limits_{-\pi}^{\pi} f(t) \left\{ \sum_{k=0}^{\rho} \Delta\lambda_k(h)\,((k+1)\,F_k\,(\mathrm{y}-t) - kF_{k-1}\,(\mathrm{y}-t)) \right\} dt + \\
&+ \int\limits_{-\pi}^{\pi} f(t) \left\{ \sum_{k=\rho+1}^{N} \Delta\lambda_k(h)\,\big((m-k+1)\,V_{m,k}\,(\mathrm{y}-t) - (m-k)\,V_{m,k+1}\,(\mathrm{y}-t)\big) \right\} dt = \\
&= \int\limits_{-\pi}^{\pi} f(t) \left\{ \Delta\lambda_\rho(h)\,(\rho+1)\,F_\rho\,(\mathrm{y}-t) + \sum_{k=0}^{\rho-1} \Delta^2\lambda_k(h)\,(k+1)\,F_k\,(\mathrm{y}-t) \right\} dt + \\
&+ \int\limits_{-\pi}^{\pi} f(t) \Big\{ \Delta\lambda_{\rho+1}(h)\cdot(m-\rho)\,V_{m,\rho+1}\,(\mathrm{y}-t) \Big\} dt + \\
&+ \int\limits_{-\pi}^{\pi} f(t) \left\{ -\Delta\lambda_N(h)\,(m-N)\,V_{m,N+1}\,(\mathrm{y}-t) - \sum_{k=\rho+1}^{N-1} \Delta^2\lambda_k(h)\,(m-k)\,V_{m,k+1}\,(\mathrm{y}-t) \right\} dt.
\end{aligned} \tag{4.6}$$

Applying estimate (3.5) for the case where the integrand is the Fejér kernel and estimate (3.4) for the case of de la Vallée-Poussin kernel, we get from (4.6), for all $(y, h) \in \Gamma_d(x)$,

$$\begin{aligned}
&|\,U_N\,(f, \mathrm{y};\, \Lambda, h)\,| \leq \\
&\leq \mathrm{C} f^*(x) \Big\{ |\Delta\lambda_\rho(h)|\,(\rho+1) + \sum_{k=0}^{\rho-1} |\,\Delta^2\lambda_k(h)\,|\,(k+1) + |\,\Delta\lambda_{\rho+1}(h)\,|\cdot(m-\rho) + \\
&+ |\,\Delta\lambda_N(h)\,|\,(|m-N|+1)\,\ln\tfrac{2(m+N+1)}{|m-N|+1} + \sum_{k=\rho+1}^{N-1} |\,\Delta^2\lambda_k(h)\,|\,(|m-k|+1)\ln\tfrac{2(m+k+1)}{|m-k|+1} \Big\}
\end{aligned} \tag{4.7}$$

The terms $|\Delta\lambda_{\rho}(h)|(\rho+1)$ and $|\Delta\lambda_{\rho+1}(h)|\cdot(m-\rho)$ in (4.7) will be estimated from above by $\sum(h;\Lambda,d)$, due to the relation

$$(\rho+1)\ \Delta\lambda_{\rho+1}(h)=\lambda_0(h)-\lambda_{\rho+1}(h)+\sum_{k=0}^{\rho}(k+1)\,\Delta^2\lambda_k(h).$$

Further, it is obvious that

$$\sum_{k=0}^{\rho-1}|\,\Delta^2\lambda_k(h)\,|\,(k+1)+\sum_{k=\rho+1}^{N-1}|\,\Delta^2\lambda_k(h)\,|\,(|m-k|+1)\ln\frac{2(m+k+1)}{|m-k|+1}\le \mathrm{C}\sum(h;\,\Lambda,d)\,.$$

Finally, proceed to the estimate

$$|\,\Delta\lambda_N(h)\,|\,(|m-N|+1)\ \ln\frac{2\,(m+N+1)}{|\,m-N\,|+1}.$$

We have

$$|\,\Delta\lambda_N(h)\,|\,(|m-N|+1)\ln\frac{2(m+N+1)}{|m-N|+1}\le \mathrm{C}\ \ (N+1)\,|\,\Delta\lambda_N(h)\,|\le \mathrm{C}\sum_{k=N}^{\infty}(k+1)\,|\,\Delta^2\lambda_k(h)\,|\,.$$

The last sum does not exceed

$$\sum_{k=N}^{\infty}\frac{(k+1)\,(|m-k|+1)}{m+1}\ln\frac{2\,(m+k+1)}{|\,m-k\,|+1}\quad|\,\Delta^2\lambda_k(h)\,| \tag{4.8}$$

due to the obvious, for $k\ge 2m$, estimate

$$k+1\le C\ \frac{(k+1)\,(|m-k|+1)}{m+1}\ \ln\ \frac{2\,(m+k+1)}{|\,m-k\,|+1}.$$

Hence,

$$|\,\Delta\lambda_N(h)\,|\,(|m-N|+1)\ln\frac{2\,(m+N+1)}{|\,m-N\,|+1}\le \mathrm{C}\sum(h;\,\Lambda,d)\,.$$

Moreover, (4.8) implies the equality

$$\lim_{N\to\infty}|\,\Delta\lambda_N(h)\,|\,(|m-N|+1)\ \ln\frac{2\,(m+N+1)}{|\,m-N\,|+1}=0, \tag{4.9}$$

which we will need later.

Returning to (4.7), we obtain assertion (4.5), and Lemma 4.1 is proved.

Further, according to (4.4), the means $U_N(f, y; \Lambda, h)$ can be represented as the sum of a block of terms independent of N, infinitesimal (4.9) and the partial sum of the dominating (see (3.4)) series

$$\sum_{k=\rho+1}^{\infty} \mid \Delta^2 \lambda_k(h) \mid (|m-k|+1) \int_{-\pi}^{\pi} \mid f(t) \mid \cdot \mid V_{m,k+1}(y-t) \mid dt.$$

Thus, the sequence $U_N(f, y; \Lambda, h)$, for chosen x and all $(y, h) \in \Gamma_d(x)$, converges. Moreover, it follows by (4.3) and (4.5) that

$$\pi \mid U(f, y; \Lambda, h) \mid = \lim_{N \to +\infty} \mid U_N(f, y; \Lambda, h) \mid \le \sup_{N | N \ge 2m} \mid U_N(f, y; \Lambda, h) \mid \le C\ f^*(x) \sum (h; \Lambda, d) \tag{4.10}$$

Estimate (4.10) obviously implies (2.3). This completes the proof of Theorem 2.1.

5 Weak Type Estimate: Nontangential Summability

Theorem 5.1 Under conditions of (2.1) and (2.2) we have that the relation

$$\mu \left\{ x \in Q |\ U^*(f, x; \Lambda, d) > \varsigma > 0 \right\} \le C \frac{\| f \|_L}{\varsigma} \tag{5.1}$$

holds; C in (5.1) depends only on Λ and d.

If the condition

$$\lim_{h \to 0} \lambda_k(h) = 1; \quad k = 0, 1, \ldots, \tag{5.2}$$

is carried out too, then

$$\lim_{\substack{(y,h) \to (x,0) \\ (y,h) \in \Gamma_d(x)}} U(f, y; \Lambda, h) = f(x) \tag{5.3}$$

for almost every x.

Proof The weak-type estimate in (5.1) is a direct consequence of inequality (2.3) if we take into account (1.4). The proof of convergence almost everywhere (that is, relation (5.3)) follows from the corresponding estimates of weak type and condition (5.2) in a standard way (see [10], vol. 2, pp. 464, 465).

6 Exponential Means

Denote

$$\sum\nolimits^{\#\#}(h;\,\Lambda,d)=\sup_{k=0,1,\ldots}\mid\lambda_k(h)\mid+\sum_{k=0}^{\infty}\frac{(k+1)\,(k+m)}{m}\mid\Delta^2\lambda_k(h)\mid. \tag{6.1}$$

It is obvious that

$$\sum(h;\,\Lambda,d)\le\sup_{k=0,1,\ldots}\mid\lambda_k(h)\mid+\sum_{k=0}^{\infty}\tfrac{(k+1)(|m-k|+1)}{m}\cdot\tfrac{2(m+k+1)}{|m-k|+1}\mid\Delta^2\lambda_k(h)\mid\le C\sum\nolimits^{\#\#}(h;\,\Lambda,d). \tag{6.2}$$

Let now

$$\lambda_0(h)=1,\,\lambda_k(h)=\lambda\Big(x,h\Big)\Big|_{x=k},\quad k=1,2,\ldots,$$

where $\lambda(x,h)=\exp(-h\phi(x))$. Suppose that

(A) $\phi\in C^2(0,+\infty)$; $\phi(x)\ge 0$, $\phi'(x)\ge 0$, $\phi''(x)\ge 0$, $x\in(0,+\infty)$;
(B) $x^2(\phi'(x))^2\exp(-h\phi(x))$ and $x^2\mid\phi''(x)\mid\exp(-h\phi(x))$ (or some of their majorants) decrease to zero as x increases.

We note that

$$\lambda'_{\mathrm{x}}(x,h)=-h\phi'(x)\exp(-h\phi(x)),$$

$$\lambda''_{xx}(x,h)=h\exp(-h\phi(x))\left\{h\big(\phi'(x)\big)^2-\phi''(x)\right\},$$

and apply twice the Lagrange theorem to the second finite differences in (6.1). According to conditions (B), the right-hand side of (6.1) is dominated by the sum of the corresponding improper integrals

$$\begin{aligned}I_\phi(h)&=h\int_0^\infty\left\{h\big(\phi'(x)\big)^2+\phi''(x)\right\}\,x\,(1+hx)\exp(-h\phi(x))\,dx\\&=h^2\int_0^\infty x\big(\phi'(x)\big)^2\exp(-h\phi(x))\,dx+h^3\int_0^\infty x^2\big(\phi'(x)\big)^2\exp(-h\phi(x))\,dx+\\&+h\int_0^\infty x\phi''(x)\exp(-h\phi(x))\,dx+h^2\int_0^\infty x^2\phi''(x)\exp(-h\phi(x))\,dx\end{aligned} \tag{6.3}$$

For condition (2.2) to be satisfied (see (6.2)), it suffices to require that

$$I_\phi(h) \leq C. \tag{6.4}$$

Condition (2.1) will be satisfied if

$$\left(x + hx^2\phi'(x)\right)\exp\left(-h\phi(x)\right) \to 0 \quad \text{as} \quad x \to +\infty. \tag{6.5}$$

We now consider the case where $\phi(x) = x^{\alpha}$, $\alpha \geq 1$; then

$$\lambda_k(h) = \exp\left(-hk^{\alpha}\right), \quad k = 0, 1, \ldots; \; \alpha \geq 1\,. \tag{6.6}$$

Corollary 6.1 The assertions of Theorems 2.1 and 5.1 are valid for generalized Poisson-Abel means

$$\sigma\,(f, \mathrm{y}; \alpha, h) = \sum_{k=-\infty}^{\infty} \exp\left(-h|k|^{\alpha}\right) c_k(f) \exp\left(iky\right) \tag{6.7}$$

for all $\alpha \geq 1$. In particular, the relation

$$\lim_{\substack{(y,h) \to (x,0) \\ (y,h) \in \Gamma_d(x)}} \sum_{k=-\infty}^{\infty} \exp\left(-h|k|\right)\; c_k(f)\; \exp\left(iky\right) = f(x),\; f \in L(Q)$$

(non-tangential convergence of the Poisson-Abel means) holds for almost all x.

We note that the maximal functions associated with nontangential directions are of considerable interest in various issues of harmonic analysis: ([2], p. 37; [8], Chapter XIV; [7], pp. 75, 76]). Further, in the "radial" case ($d = 0$) a generalization of the Poisson-Abel summation method in the other direction and by other methods was obtained by R.M. Trigub [9].

Proof Condition (6.5), for (6.6), is obviously satisfied by virtue of L'Hôpital's rule. Conditions (A) and (B) in this case can be easily verified. To verify condition (6.4), we transform the integral

$$J = h^s \int_0^{\infty} x^{\beta} \exp\left(-hx^{\alpha}\right) dx$$

(the values of the parameters s and β will be specified later) by changing variables $t = hx^{\alpha}$ to be of the form

$$J = \frac{1}{\alpha}\, h^{s-\frac{1+\beta}{\alpha}} \int_0^{\infty} t^{\frac{1+\beta}{\alpha}-1}\, \exp(-t)\, dt = \frac{1}{\alpha} h^{s-\frac{1+\beta}{\alpha}}\, \Gamma\left(\frac{1+\beta}{\alpha}\right),$$

where Γ is Euler's gamma-function. In the following four cases (respectively, the four integrals in (6.3)) we will have:

1. $s = 2, \quad \beta = 2\alpha - 1$; hence $J = \frac{1}{\alpha}\, \Gamma(2)$;
2. $s = 3, \quad \beta = 2\alpha$; then $J = \frac{1}{\alpha}\, h^{1-\frac{1}{\alpha}}\, \Gamma\left(2 + \frac{1}{\alpha}\right)$;
3. $s = 1, \quad \beta = \alpha - 1$; hence $J = \frac{1}{\alpha}\, \Gamma(1)$;
4. $s = 2, \quad \beta = \alpha$; hence $J = \frac{1}{\alpha}\, h^{1-\frac{1}{\alpha}}\, \Gamma\left(1 + \frac{1}{\alpha}\right)$.

Now, the sum in (6.3) does not exceed

$$C\left(1 + h^{1-\frac{1}{\alpha}}\right),$$

so that the sequence (6.6) satisfies condition (6.4), whence the assertion of Corollary 6.1 follows.

Consider now the exponential-polynomial summation methods, that is, the case where $\phi(x)$ is a polynomial of degree *n:*

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \quad a = a_n > 0.$$

One may assume $n = 2, 3, \ldots$, since the case $n = 1$ corresponds to the already considered Poisson-Abel means.

Corollary 6.2 The assertions of Theorems 2.1 and 5.1 are valid for exponential-polynomial means

$$\sigma(f, y; n, h) = \sum_{k=-\infty}^{\infty} \exp(-hP_n(|k|)) c_k(f) \exp(iky) \quad \text{for all } n = 1, 2, \ldots .$$

Proof It is enough to verify conditions (A) and (B) and (6.4); in this case, the summation in (6.1) can be considered with respect to $k \in [\nu, +\infty)$, where ν is a fixed natural number, which we will choose later. Relation (6.5) obviously holds due to L'Hôpital's rule.

Since $a = a_n > 0$ and

$$P_n(x) = x^n \left(a + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n}\right),$$

We conclude that for sufficiently large values of x the inequality

$$P_n(x) > \frac{a}{2} x^n. \tag{6.8}$$

is valid. In the same way, for sufficiently large x, we have

$$P_n'(x) > 0 \quad \text{and} \quad P''_n(x) > 0. \tag{6.9}$$

Let now a natural ν be such that relations (6.8) and (6.9) hold simultaneously at $x \geq \nu$; we can assume that $\nu < m$. Under these conditions, (A) will be satisfied.

Let us pass to conditions (B). For $\phi(x) = P_n(x)$, the decreasing of majorant functions easily follows from the obvious inequalities

$$(P_n'(x))^2 \leq C_n x^{2n-2}, \ |P_n''(x)| \leq C_n x^{n-2} \quad (x \geq 1) \tag{6.10}$$

and (see (6.8))

$$\exp(-hP_n(x)) < \exp(-h\frac{a}{2} x^n). \tag{6.11}$$

The proof of estimate (6.4) now follows from estimates (6.10) and (6.11), for which one should repeat the same argument as that above for the generalized Poisson means.

Remark If A_p-condition of Muckenhoupt [3] is satisfied, then the maximal estimate (2.3) allows one to obtain for $U^*(f, x; \Lambda, d)$ the weighted L^p-inequalities of weak ($p \geq 1$) and strong ($p > 1$) types [6]. Conditions (2.1), (2.2) and (5.2) ensure the convergence of (1.2) in metric of $L^p (p \geq 1)$ weighted space.

7 Applications

1. The family of operators ([1], p. 698)

$$T(h)(f, x) = \sum_{k=-\infty}^{\infty} c_k(f) \exp\left(-hk^2 + ikx\right)$$

is a strongly continuous semigroup on $[0, +\infty)$. The result of Corollary 6.1, in addition to the well-known properties of this semigroup, allows us to assert the convergence $T(h)(f, y)$ to the identity operator along nontangential trajectories:

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} T(h)(f,y) = f(x),\ f\in L(Q); \tag{7.1}$$

here (7.1) is valid for almost all x

2. In [5], we have studied the generalized Dirichlet problem in the half-plane

$$i^{2-2\alpha}\frac{\partial^{2\alpha}U}{\partial^{2\alpha}x}+\frac{\partial^2 U}{\partial^2 h}=0, \tag{7.2}$$

$$U(f,x;\alpha;0)=f(x), \tag{7.3}$$

where differentiation with respect to x is the corresponding fractional differentiation, and (7.3) is understood as the limit relation

$$\lim_{h\to+0} U(f,x;\alpha;h)=f(x). \tag{7.4}$$

In a more general formulation of the problem, $U(f, x; \alpha; 0)$ can be considered as the limit along nontangential trajectories, i.e. (7.4) is replaced by the more general requirement

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} U(f,y;\alpha;h)=f(x). \tag{7.5}$$

According to Corollary 6.1 the solution of the problem (7.2), (7.5) is the family of means (6.7)

$$U(f,\mathrm{y};\alpha,h)=\sigma(f,\mathrm{y};\alpha,h)$$

for all $\alpha \geq 1$; here (7.5) holds for almost each x.

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On Some Fast Implementations of Fourier Interpolation

Anry Nersessian

Abstract This article presents new results related to the author's recent research on accelerated use of classical Fourier techniques. In the case of implementation of the Fourier interpolation of a smooth function on a uniform grid of a finite segment, some algorithms are proposed similar to the "over-convergence" algorithms for fast summation of truncated Fourier series.

Numerical results confirm the effectiveness of the applied approach.

Keywords Fourier series and interpolations · Acceleration of convergence · Approximation · Over-ñonvergence phenomenon · Adaptive algorithms · Signal processing · Detection of periodicity · Quadrature formulas

1 Introduction

One of the classical tools of mathematics is the apparatus of Fourier series based on the orthogonal system $\{e^{i\pi kx}\}$, $k = 0, \pm 1, \pm 2, \dots$, complete in $L_2[-1, 1]$. However, in practice it is extremely limited because of the poor approximation of piecewise smooth functions. Thus, in the case where a function has discontinuity points (taking also into account discontinuities at the ends of the interval $[-1, 1]$), an intense oscillation arises in their neighborhood, and the uniform convergence is absent (the Gibbs phenomenon). This leads to a slow L_2-convergence on the entire segment $[-1, 1]$. The corresponding phenomenon is observed in the case of Fourier interpolation. Classical methods of summation of truncated Fourier series (see, for example, [1, Chapter 3]) do not actually change the situation.

In the last century, a number of studies appeared devoted to the methods of effective summation of the Fourier series. The pioneer of "overcoming the Gibbs

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phenomenon" is A.N. Krylov, who, at the dawn of the twentieth century (see [2]), proposed the methods that he later developed in the monograph [3]. In particular, he proposed the following approach.

Let a piecewise smooth function f be given on the segment $[-1, 1]$ with the Fourier coefficients $\{f_s\}$, $s = 0, \pm 1, \ldots, \pm n,\ n \geq 1$, and with the following jumps of the function f or its derivatives up to the order $q \geq 1$ at the points $\{a_k\}$, $-1 < a_1 < \cdots < a_m = 1,\ 1 \leq m < \infty$:

$$A_{p,k}(f) = f^{(k)}(a_p - 0) - f^{(k)}(a_p + 0),$$
$$k = 0, 1, \cdots, q \geq 0,\ p = 1, \cdots, m. \tag{1}$$

In the neighborhoods of other points we assume that $f \in C^{q+1}$. Let us construct a function $g = g(x), x \in [-1, 1]$ with Fourier coefficients $\{g_s\}, 0 \leq |s| \leq n$, which has the same jumps at the same points, and $g \in C^{q+1}$ at the neighborhoods of other points. With jumps (1) in hand, one can construct, e.g., piecewise-polynomial g. As a result, 2-periodic extension of the function $F = (f - g)$ is q times continuously differentiable on the whole axis, and therefore

$$f(x) - g(x) = \sum_{s=-n}^{n} (f_s - g_s)e^{i\pi sx} + r_n(x),$$

where $r_n(x) = o(n^{-q})$, as $n \to \infty$, for $x \in [-1, 1]$.

Thus, taking into account only first $(2n + 1)$ Fourier coefficients and truncating the remainder term r_s, it is possible to approximate f in the form

$$f(x) \simeq f_n(x) \stackrel{def}{=} g(x) + \sum_{s=-n}^{n} (f_s - g_s)e^{i\pi sx}. \tag{2}$$

This method is widely known in Russian literature (see, for example, [4]). However, computing the jumps $\{A_{sk}(f)\}$ directly by the function f seriously limits the scope of practical application of A. N. Krylov's method.

The "spectral" method proposed by Knut Eckhoff in [5] (1993) turned out to be more practical, since it is based only on the use of the coefficients $\{f_s\}$. By integration by parts, it is easy to obtain the following asymptotic representation of the Fourier coefficients:

$$f_s = -\frac{1}{2}\sum_{p=1}^{m} e^{-i\pi s\, a_p} \sum_{k=0}^{q-1} \frac{A_{p,k}}{(i\pi s)^{k+1}} + r_s,\ r_n = o(s^{-q}),\ s \to \infty. \tag{3}$$

As a function g from (2), K. Eckhoff used those Bernoulli polynomials $\{B_k(x)\},\ x \in [-1, 1],\ k \geq 0$, which Fourier coefficients $\{b_{k,s}\}$ are of the following simple form:

$$b_{k,s} = \begin{cases} 0, & s = 0, \\ \dfrac{(-1)^{s+1}}{2\,(i\pi s)^{k+1}}, & s \neq 0,\ k = 1, 2, \ldots. \end{cases} \tag{4}$$

Denoting $B_0(x) = 1$, we conclude that the polynomials $\{B_k(x)\},\ k = 0, 1, \ldots, n,\ x \in [-1, 1]$ compose a basis on the space of polynomials of degree n. Bernoulli polynomials extend to the real axis with period 2 as piecewise-smooth functions.

According to Krylov's scheme, the sequence

$$F_n(x) \stackrel{def}{=} \sum_{p=1}^{m} \sum_{k=0}^{q} A_{p,k} B_k(x - a_p - 1) + \sum_{s=-n}^{n} \omega_s\, e^{i\pi s x}, \tag{5}$$

where the quantities $\{\omega_s\}$ are given explicitly, converges to f with the rate $o(n^{-q})$, as $n \to \infty$.

K. Eckhoff suggests to find approximate values of the jumps $\{\tilde{A}_{p,k} \approx A_{p,k}\}$ by solving the following system of linear equations with the Vandermonde matrix, derived from the principal part (1) by choosing the indexes $s = s_k,\ k = 1, 2, ..., m(q+1),\ \theta n \leq |s_k| \leq n,\ 0 < \theta = const < 1$:

$$f_s = -\frac{1}{2} \sum_{p=1}^{m} e^{-i\pi s a_p} \sum_{k=0}^{q} \frac{\tilde{A}_{p,k}}{(i\pi s)^{k+1}},\ s = s_1, s_2, ..., s_{m(q+1)} \tag{6}$$

In fact, K. Eckhoff implemented the "acceleration" for Fourier interpolation in a similar way (see [6]). It is natural to call this acceleration scheme the Krylov-Eckhoff method **(KE-method)**. Over the past two to three decades, in numerous works dealing with the KE-method scheme, not only polynomials have been used as functions of g (see, for example, [6–9] and references therein).

In [10], for a smooth function f, a combination of the KE-method and the *Padé* approximation was realized. As a result of a number of experiments, it turned out that, as a rule, such an algorithm is "almost exact" on certain infinite-dimensional spaces, although only a finite number of Fourier coefficients is used. In [11], a similar phenomenon was discovered for Bessel series.

However, the results of numerical experiments can only serve as a basis for a hypothesis about the accuracy of the summation of truncated Fourier series in some infinite-dimensional space. Theoretical substantiation of this hypothesis was obtained in [12–14].

To show that the same ideas can be used in the case of Fourier interpolation, we begin with a brief overview of the results of the recent works.

2 Preliminary

2.1 Basic Definitions

The classical definition of the partial sums of a Fourier series is focused on the traditional forms of summation of series. Here we use more general notation in [14].

Definition 1 We call any sum of the form

$$S_n(x) \stackrel{def}{=} \sum_{k \in D_n} f_k \exp(i\,\pi\,k\,x),\ \ x \in [-1, 1], \tag{7}$$

the truncated Fourier series, where

$$f_s = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-i\,\pi\,s\,x}\,dx, \quad s = 0, \pm 1, \pm 2, \dots \tag{8}$$

are Fourier coefficients of f, and $D_n = \{d_k\}, k = 1, \dots, n$, is a set of n different integers ($n \geq 1$). We will assume that $D_0 = \emptyset$.

Definition 2 Let $n \geq 1$ be a fixed integer. Consider a system of functions
$U_n = \{\exp(i\,\pi\,\lambda_k\,x)\},\ \ \lambda_k \in \mathbb{C},\ x \in [-1, 1],\ k = 1, 2, \dots, n,$
where $\{\lambda_k\}$ are arbitrary parameters. Consider the linear span $Q_n = span\{U_n\}$. We call a function $q \in Q_n$ a quasi-polynomial of degree at most n.

It is easy to see that $q \in Q_n$ if and only if either $q(x) \equiv 0$ or $q(x) = \sum_k P_{\beta_k}(x)\ \exp(i\,\pi\,\lambda_k\,x)$, where the polynomials $P_{\beta_k}(x) \not\equiv 0$ are of degree exactly β_k, and $m = \sum_k (1 + \beta_k) \leq n$. The number m will be understood below as the degree of the quasi-polynomial q.

Definition 3 We call the system $\{\lambda_k\} \subset \mathbb{C}$ parameters of the quasi-polynomial $q(x) = \sum_k P_{\beta_k}(x) \exp(i\,\pi\,\lambda_k\,x) \in Q_n$ and the number $1 + \beta_k$ the multiplicity of the parameter λ_k.

Definition 4 Let the parameters $\{\lambda_k\}, k = 1, \dots, n$ be fixed. We denote by $Q_n(\{\lambda_k\})$ the set of all corresponding $q(x) \in Q_n$.

2.2 A Parametric Biorthogonalization

For fixed integers $m \geq 1$ and $r \in D_m$, consider a set of m non-integer parameters $\{\lambda_k\} \subset \mathbb{C},\ k \in D_m$, and the infinite sequence (for details, see [14])

$$t_{r,s} \stackrel{def}{=} (-1)^{s-r}\,\Big(\prod_{\substack{p \in D_m \\ p \neq r}} \frac{s-p}{r-p}\Big) \prod_{k \in D_m} \frac{r-\lambda_k}{s-\lambda_k},\ s = 0, \pm 1, \dots \tag{9}$$

Further we denote (x∈[-1,1])

$$T_r(x) \stackrel{def}{=} \exp(i\,\pi\,r\,x) + \sum_{s\notin D_m} t_{r,s} \exp(i\,\pi\,s\,x),\ \ r \in D_m,$$

$$f(x) \simeq F_m(x) \stackrel{def}{=} \sum_{r\in D_m} f_r T_r(x),\ \ R_m(x) \stackrel{def}{=} f(x) - F_m(x). \tag{10}$$

It follows from (9) that for $s \to \infty,\ |t_{r,s}| = O(|s|^{-1})$.

It is obvious that system $\{T_r(x), 1/2\exp(i\,\pi\,r\,x)\},\ r \in D_m$, is biorthogonal on the segment $x \in [-1, 1]$ and L_2-error of approximation $f(x) \simeq F_m(x)$ can be derived from the formula

$$||R_m||^2 = \sum_{s\notin D_m} \|f_s - \sum_{r\in D_m} f_r t_{r,s}\|^2. \tag{11}$$

The following obvious formula ($\lambda \in \mathbb{C},\ x \in [-1, 1]$)

$$\exp(i\,\pi\,\lambda\,x) = \sum_{s=-\infty}^{\infty} \operatorname{sinc}(\pi\,(s-\lambda)) \exp(i\,\pi\,s\,x), \tag{12}$$

where $\operatorname{sinc}(z) = \sin(z)/z,\ \operatorname{sinc}(0) = 1,\ z \in \mathbb{C}$, will play a key role in the future. Comparing (12) with (9), we arrive at the conclusion that the system $\{T_r\}$ consists of a linear combination of functions $\{\exp(i\,\pi\,\lambda_k\,x)\},\ k \in D_m$.

2.3 The Traditional Algorithm

Formulas (10) contain a description of the traditional algorithm corresponding to parameters $\{\lambda_k\}$. Note that for $\lambda_k = k$ this is the truncated Fourier series (7) itself. But this is the worst choice (the Gibbs phenomenon will appear, see **Introduction**). In the case $\lambda_k = 0, \forall k$, the Bernoulli polynomials are used, which leads to an effective summation of the Fourier series (see formulas (2)–(5)).

Consider the usual case $D_m = \{j\},\ j = -m, -m+1, \ldots, m$. Let $\{\lambda_k\}$ be real and depend on both m and k. In our numerical experiments, we used the choice according to the formula

$$\lambda_k = \frac{2\,\arctan\left(\frac{m}{9} + \frac{1}{5}\right)}{\pi}\,k,\quad k = 0, \pm 1, \ldots, \pm m.$$

Note that here $\lambda_k/k < 1$ and for $m \to \infty,\ \lambda_k/k \to 1$. This summation method gives good results (see [12]), but there is no reason to believe that such a choice is optimal.

2.4 The Adaptive Algorithm

In the main theorem of the paper [14], the "phenomenon of over-convergence" was theoretically substantiated. This result is implemented by the following **Algorithm** $\mathfrak{A}$.

For a fixed integer $n > 0$, let $S_n(x)$ be a truncated Fourier series (see (1)). Consider formulas (2)–(4) with symbolic (not numerical) parameters $\{\lambda_q\}$, $q \in D_n$. Let us now choose a new set of integer numbers $\tilde{D}_n = \{\tilde{d}_k\}, k = 1, \ldots, n$, $\tilde{D}_n \cap D_n = \emptyset$. To determine the parameters $\{\lambda_q\}$, $q \in D_n$, we additionally use Fourier coefficients $\{f_s\}$, $s \in \tilde{D}_n$, and solve the system of equations

$$f_s - \sum_{r \in D_n} f_r t_{r,s} = 0, \ s \in \tilde{D}_n. \tag{13}$$

The solution to this essentially nonlinear equation is quite natural, since it reduces the error (11). This is the essence of the adaptation of parameters $\{\lambda_q\}$ to the given function f. If the solution exists, then the system $\{T_r\}$ in (4) is used to approximate f by F_n.

We now show how Algorithm $\mathfrak{A}$ can step-by-step be realized by given $2n$ Fourier coefficients $\{f_{d_k}\} \bigcup \{\{f_{\tilde{d}_k}\}\}$ (see[14]).

Step 1 Using representation (9), we formally bring the left-hand side of (13) to a common denominator, and fix the conditions $\lambda_q \neq \tilde{d}_k, k = 1, 2, \ldots, n$. Then Eq. (13) will take the form

$$f_s \prod_{q \in D_n} (s - \lambda_q) = \sum_{r \in D_n} (-1)^{s-d_r} f_r \ (\prod_{\substack{p \in D_n \\ p \neq d_r}} \frac{s - d_p}{d_r - d_p}) \prod_{q \in D_n} (d_r - \lambda_q), \quad s \in \tilde{D}_n. \tag{14}$$

Step 2 Using Vieta's formula, we decompose the products in (8) containing the parameters $\{\lambda_k\}$ and denote

$$e_k = (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2}, \ldots, \lambda_{i_k}, \ k = 1, 2, \ldots, n.$$

The reader can make sure that, as a result, Eq. (14) will take the form of a linear system of equations with respect to the variables $\{e_k\}$.

Step 3 Solve resulting equation **by the least square method**.

Step 4 Again, according to Vieta's formula, find all roots $\{z_k\}$, $k = 1, 2, \ldots, n$ of the corresponding polynomial $P(z) = \sum_{p=0}^{n} e_p \, z^{n-p}$, $e_0 = 1$, and put $\{\lambda_k\} = \{z_k\}$.

Step 5 If for each integer value of λ_k we have $d_k \in D_n \cup \tilde{D}_n$, then realize the approximation $f \simeq F_n$ (see conditions (**ii**) and (**iii**) of *Lemma 2* of [14] and Theorem 1 below).

Remark 1 The least square method provides a unique solution in **Step 3**. Even if on the right of Eq. (5) there is a nonzero discrepancy (L_2 minimization of the left-hand side), error (11) cannot increase.

Remark 2 If condition in **Step 5** is not met, then this is manifested by stopping the algorithm due to dividing a nonzero value by zero. One of the ways out of the situation is to add two new Fourier coefficients and restart the algorithm.

The following theorem is the main result of the paper [14].

Theorem 1 (The Over-Convergence Phenomenon) *Let $f \in Q_n$ (see Definition 2) and the sets D_n, $\tilde{D}_n$ and the Fourier coefficients $\{f_s\}$, $s \in D_n \cup \tilde{D}_n$, of the f be given. Denote by Λ the set of integer parameters in the representation $f = \sum_k P_{m_k}(x) \exp(i\,\pi\,\mu_k\,x)$.*

In order for the approximation by Algorithm $\mathfrak{A}$ to be exact (that is, $f(x) \equiv F_n(x),\ x \in [-1, 1]$), it is necessary and sufficient that $\Lambda \subseteq D_n \cup \tilde{D}_n$.

2.5 Summary of the Method

Let us indicate the main features of the above approach.

For a truncated Fourier system $\{e^{i\pi kx}\}, k \in D_m$, a biorthogonal family $\{T_k(x)\}, k \in D_m$, is explicitly constructed depending on m parameters $\{\lambda_k\}, k \in D_m$. In particular, the system $\{T_k\}$ can coincide both with $\{e^{i\pi kx}\}$ (for integer parameters) and with a system of polynomials (for all zero parameters). For fast summàtion of the corresponding truncated Fourier series $\sum_{r \in D_m} f_r\, e^{i\pi rx}$, two algorithms are proposed.

The traditional algorithm is based on the biorthogonal approximation (see (10))

$$f(x) \simeq \sum_{r \in D_m} f_r\, T_r(x)$$

It uses prefetching of $\{\lambda_k\}$ (usually depending on m) parameters and is, in fact, *an explicit implementation* of the **KE method** using functions from Q_m.

This algorithm is usually quite efficient (see Sects. 2.3 and 5.3). In addition, it is very simple and does not require solving any equations. Moreover, it is effective for multidimensional Fourier series.

It is exact for the corresponding n-dimensional set of functions $g \in Q_n(\{\lambda_k\})$ *with preset parameters* $\{\lambda_k\}$ (see Definition 4).

The adaptive algorithm is based on a special search for parameters $\{\lambda_k\}$ by solving the nonlinear system of Eqs. (14), which contains the given Fourier coefficients.

In addition, you need to find all roots of the corresponding polynomial (see step 4 in Sect. 2.4). As a result, an optimal choice of parameters is achieved.

This algorithm is much more complex but much more efficient. *It is exact for the infinite-dimensional set* Q_n (see Theorem 1). This phenomenon is discovered for the first time numerically in [10] and theoretically confirmed in [12]. To present it briefly, let for even $m = 2n$ the Fourier coefficients $\{f_q\},\ q \in D_m$ be somehow split into two equal parts $D_m = D_n \cup \tilde{D}_n$. Then the corresponding adaptive algorithm produces an approximation $f(x) \simeq \sum_{r\in D_n} f_r\, T_r(x)$, in which there are no Fourier coefficients $\{f_r\}$, $r \in \tilde{D}_n$. These coefficients are used in Algorithm $\mathfrak{A}$ for an optimal choice of parameters $\{\lambda_k\}$, $k \in D_n$.

3 Fast Interpolation Methods for Finite Fourier Series

A periodic trigonometric interpolation on a uniform finite grid is commonly called Fourier interpolation (see, for example, [15]). As in the case of the Fourier series, it is practically ineffective due to the presence of the Gibbs phenomenon. Overcoming this phenomenon ("acceleration of convergence") is associated with the use of non-periodic functions. In the literature, relatively few works are devoted to this topic (see, for example, [6, 7, 9, 15] and references therein). It will be shown below that the methods and results of Sect. 2 lead to the case of "overcoming the Gibbs phenomenon" for Fourier interpolation.

3.1 The Finite Fourier Series

A discrete analogue of the Fourier series is (see for example [16]) the expansion in eigenfunctions of the boundary value problem for the finite difference equation

$$y_{n+1} = \lambda\, y_n,\ n = 0, \dots, m-1,\ y_0 = y_m,\ m \ge 2. \tag{15}$$

The eigenvalues of this problem are the roots of the equation $\lambda^m = 1$ and the eigenvectors $\{\exp(2i\,\pi\,k\,(m-1)/m)\}, r = 0, \dots, m-1$ satisfy the following discrete orthogonality relation

$$\frac{1}{m}\sum_{k=0}^{m-1} \exp(2i\,\pi\,k\,r/m)\exp(-2i\,\pi\,k\,s/m) = \delta_{rs},\ r, s = 0, \dots, m-1.$$

The eigenvectors expansion (finite Fourier series for the vector $\{f_r\}$) is of the form

$$f_r = \sum_{k=0}^{m-1} \hat{f}_k \exp(2i\,\pi\,k\,r/m),\ r = 0, \dots, m-1, \tag{16}$$

where discrete Fourier coefficients $\{\hat{f}_k\}$ are of the form

$$\hat{f}_k = \frac{1}{m} \sum_{s=0}^{m-1} f_s \exp(-2i\,\pi\, s\, k/m),\ k = 0, \ldots, m-1. \tag{17}$$

3.2 Fourier Interpolation Improvements

Now, let a continuous function $f(x)$ be given on the segment $x \in [-1, 1]$. If we denote $f_r = f(\frac{r}{m}),\ r = 0, \ldots, m-1$, then the classical Fourier interpolation of the function f is of the form

$$I_m(f) \stackrel{def}{=} \sum_{k=0}^{m-1} \hat{f}_k\, e^{2i\pi kx},\ \ x \in [0, 1]. \tag{18}$$

On the other hand, it is not difficult to verify that

Lemma 1 *For any* $\lambda \in \mathbb{C}$

$$I_m(e^{2i\pi\lambda x}) = \frac{\left(1 - e^{2i\pi\lambda}\right)}{m} \sum_{s=0}^{m-1} \frac{e^{\frac{2i\pi s}{m}}\, e^{2i\pi s x}}{\left(e^{\frac{2i\pi s}{m}} - e^{\frac{2i\pi\lambda}{m}}\right)},\ x \in [0, 1]. \tag{19}$$

We note that for an integer $\lambda = \lambda_0$, one can go to the limit $\lambda \to \lambda_0$.

Remark 3 This formula, similar to relation (12) for Fourier series, is crucial for what follows. By differentiating (19) with respect to λ several times we get on the left an interpolation formula for a function of the form $P(x)\, e^{2i\pi\lambda x}$, where $P(x)$ is the corresponding polynomial and the sum on the right will already contain the powers of the denominators.

It may seem that here the set of functions Q_m on [0,1] (see Definition 2) is now naturally applicable, but it is necessary to take into account that $I_m(\exp(i\,\pi\,\lambda\, x)) = I_m(\exp(i\,\pi\,\mu\, x))$ if $\Re(\lambda - \mu)/m)$ is integer. Therefore, it is clear that here we are dealing with a factor set Q_m over this identification.

Definition 5 $\hat{Q}_m \stackrel{def}{=} Q_m / \sim$.

In particular, interpolation (18) is only one of the possible (see below, other classical interpolation (26)).

The above analogy with the Fourier series case allows us to construct the following scheme. Consider a set of parameters $\{\lambda_p\}$, $p = 0. \ldots, m-1$. Similarly to (9), we denote (r, s=0,. . . ,m-1)

$$\hat{t}_{r,s} \stackrel{def}{=} e^{\frac{2i\pi(r-s)}{m}} \Big(\prod_{\substack{p=0\\p\neq r}}^{m-1} \frac{e^{\frac{2i\pi s}{m}} - e^{\frac{2i\pi p}{m}}}{e^{\frac{2i\pi r}{m}} - e^{\frac{2i\pi p}{m}}}\Big) \prod_{q=0}^{m-1} \frac{e^{\frac{2i\pi r}{m}} - e^{\frac{2i\pi\lambda_q}{m}}}{e^{\frac{2i\pi s}{m}} - e^{\frac{2i\pi\lambda_q}{m}}}. \tag{20}$$

If the denominators on the right are not zeros, we have $\hat{t}_{r,r} = 1$ and $\hat{t}_{r,s} = 0$ for $r \neq s$. The expression on the right is a rational function with respect to $w = e^{\frac{2i\pi s}{m}}$. Using the method of residues, we can expand $\hat{t}_{r,s}$ into simple fractions with respect to w.

Let us first consider the case of non-integer parameters $\{\lambda_k\},\ \lambda_k \neq \lambda_q,\ \forall(k,q)$ in more detail. In this case, these fractions will have simple poles. Taking into account the formula (19) of Lemma 1, we see that for each k the coefficients $\hat{f}_s$ of the interpolation $I_m(g_k(x))$ of the functions

$$g_k(x) = \frac{m\exp(2i\pi\lambda_k x)}{(1-\exp(2i\pi\lambda_k))},\ \ k = 0,\dots,m-1,$$

have the following form of simple fractions:

$$\hat{g}_{k,s} = \frac{e^{\frac{2i\pi s}{m}}}{\left(e^{\frac{2i\pi s}{m}} - e^{\frac{2i\pi\lambda_k}{m}}\right)},\ \ k,s = 0,\dots,m-1. \tag{21}$$

Hence it follows that for a fixed r, decomposition of $\hat{t}_{r,s}$ into simple fractions is a linear combination of the fractions $\hat{g}_{k,s}$.

Although the interpolated function f is not uniquely determined by its coefficients $\{\hat{f}_s\}$, we will **assign** to each system $\{\hat{g}_{k,s}\}$, with fixed k, one function $g_k(x)$. This allows us to insert exponential functions in $\hat{Q}_m$ into the interpolation process at the same nodes $x_k = \frac{k}{m}$. The implementation of the operations described above, based on the decomposition into simple fractions of the right-hand side of the formula (20), leads to the following interpolation scheme. For different parameters $\{\lambda_p\}$, $p = 0.\dots,m-1$ consider the functions

$$T_r(x) \stackrel{def}{=} \sum_{k=0}^{m-1} \hat{c}_{r,k}\, e^{2i\pi\lambda_k x}, r = 0,\dots,m-1, x \in [0,1], \tag{22}$$

where

$$\hat{c}_{r,k} = \frac{m\, e^{-\frac{2i\pi r}{m}}\left(e^{\frac{2i\pi r}{m}} - e^{\frac{2i\pi\lambda_k}{m}}\right)}{1-e^{2i\pi\lambda_k}}$$

$$\Big(\prod_{\substack{p=0\\p\neq k}}^{m-1} \frac{e^{\frac{2i\pi r}{m}} - e^{\frac{2i\pi\lambda_p}{m}}}{e^{\frac{2i\pi\lambda_k}{m}} - e^{\frac{2i\pi\lambda_p}{m}}}\Big) \prod_{\substack{q=0\\q\neq r}}^{m-1} \frac{e^{\frac{2i\pi\lambda_k}{m}} - e^{\frac{2i\pi q}{m}}}{e^{\frac{2i\pi r}{m}} - e^{\frac{2i\pi q}{m}}}.$$

Definition 6 Let the parameters $\Lambda = \{\lambda_k\}$ and coefficients $\{\hat{f}_k\}$ of a continuous function $f(x)$ (see (18)) be given. We define the interpolation

$$f(x) \simeq F_m(\Lambda, x) \stackrel{def}{=} \sum_{r=0}^{m-1} \hat{f}_r \, T_r(x) = \sum_{k=0}^{m-1} d_k e^{2i\pi\lambda_k x}, \tag{23}$$

where $d_k = \sum_{r=0}^{m-1} \hat{f}_r \hat{c}_{r,k}$.

Remark 4 The explanation after formula (19) and Remark 3 allow us to define interpolation $F_m(\Lambda, x)$ in a similar way for the general case, where the parameters are either integer or coincident.

4 Two Interpolation Algorithms

4.1 The Traditional Algorithm

As in the case of Fourier series (see above), formulas (20)–(23) define the implementation of the traditional algorithm. This traditional algorithm is exact *on an m-dimensional subset of functions from* $\hat{Q}_n$.

We can recommend an algorithm similar to that described above for the Fourier series (see Sect. 2.3). A slightly different choice is used below in a numerical experiment (see Sect. 5).

4.2 The Adaptive Algorithm

This algorithm is also similar to the adaptive algorithm for the Fourier series (see Sect. 2.4). It is factually based on the use of the same **Algorithm** $\mathfrak{A}$. The situation here is even simpler, since interpolation is performed in an m-dimensional space.

Let m be even (m=2 n). To clarify the details, we denote $D_n = \{0, 1, \dots, n-1\}$, $\tilde{D}_n = \{n, n+1, \dots, 2n-1\}$. Let the coefficients $\{\hat{f}_k\}, k = 0, \dots, m-1$, of f be given. First, we construct approximation $f(x) \simeq F_n(\Lambda, x)$ only by coefficients $\{\hat{f}_k\}, k \in D_n$, where n parameters $\Lambda = \{\lambda_p\}$ are given in symbolic form (i.e., in formulas (20)–(23), we replace m with n). We find these undefined parameters by solving the system of equations for $\{z_q\}$ (compare with formula (9) and Eqs. (13))

$$\hat{f}_s - \sum_{r=0}^{n-1} \hat{f}_r \, \frac{a(s)}{a(r)} \left(\prod_{\substack{p=0 \\ p \neq r}}^{n-1} \frac{a(s) - a(p)}{a(r) - a(p)} \right) \prod_{q=0}^{n-1} \frac{a(r) - z_q}{a(s) - z_q} = 0, \tag{24}$$

where (see (20)) $a(s) = e^{\frac{2i\pi s}{m}}$, $z_q = e^{\frac{2i\pi\lambda_q}{m}}$ and $s \in \tilde{D}_n$. Noticing that Eq. (13) can also be formally rewritten as (24), it is easy to see that the scheme of Algorithm $\mathfrak{A}$ is fully implemented here (see Sect. 2.4).

Here we point out one important issue. The discrepancy in at least one of the equations of systems (13) and (24) can be nonzero due to the use of the least square method in Algorithm $\mathfrak{A}$ (see Remark 1). In this regard, when implementing adaptive algorithms, it is necessary to make appropriate corrections in the form of residual decompositions (they will usually be lacunary). Otherwise, the precision at the interpolation nodes may be impaired. In the case of the Fourier series, this is practically not that important.

The main theoretical result of this work is

Theorem 2 (The Phenomenon of the Over-Interpolation) *Let $m = 2n$ is even (m), $f \in \hat{Q}_n$ (see Definition 5) and the sets D_n, $\tilde{D}_n$ and the Fourier interpolation coefficients $\{\hat{f}_s\}$, $s = 0, \ldots, m-1$, of the f be given. Then **Algorithm** $\mathfrak{A}$ is sharp, that is, $f(x) \equiv F_n(\Lambda, x)$, $x \in [0, 1]$.*

The proof of this theorem is quite similar to the proof of Theorem 1 (see [14]).

As you can see, interpolation using Algorithm $\mathfrak{A}$ is exact *on the infinite-dimensional set of functions emph* $\hat{Q}_n$.

Remark 5 Compared with Theorem 1, the absence of a condition on $D_n \cup \tilde{D}_n$ in Theorem 2 is due to the fact that all possible coefficients $\{\hat{f}_s\}$ have already been used here.

5 Numerical Results

The corresponding algorithms are implemented here in the Wolfram Mathematica code with extended working precision (see [17]).

Below, the Fourier interpolation produced by the classical algorithm is compared with interpolation produced by the traditional and adaptive algorithms (see Sects. 4.1 and 4.2). Therefore, an even $m = 2\,n$ is chosen.

5.1 *Implementation Notes*

As compared with the case of the Fourier series, here for the traditional algorithm (see Sect. 2.3), the following different choice of parameters, for fixed m, is used:

$$\lambda_k = \frac{2\,(-1)^k}{\pi} \arctan\left(\frac{1}{15} + \frac{m}{15\log(m+5)}\right) k, \quad k = 0, \ldots, m-1. \tag{25}$$

Algorithm codes have been implemented provided that parameters $\Lambda = \{\lambda_k\}$ of $g \in \{\hat{Q}_m\}$ (see Definition 5) were installed as $\{-n \leq \Re(\lambda_k) \leq n\}$. This actually minimizes the frequencies of all parameters Λ. One can select a different frequency range. In particular, interpolation (18) here is of the form

$$I_m(f) \stackrel{def}{=} \sum_{k=0}^{n-1} \hat{f}_k e^{2i\pi kx} + \sum_{k=n}^{2n-1} \hat{f}_k e^{2i\pi x(k-2n)}. \tag{26}$$

Remark 6 The interpolation nodes are located at points $\{\frac{k}{m}\}, k = 0, \dots, m-1$. However, approximation $f(x)$ on interval $(\frac{m-1}{m}, 1]$ is rather an extrapolation. Real interpolation takes place on segment $[0, \frac{m-1}{m}]$ there must be a point (i.e. this must be erased). Therefore, it is initially convenient to transfer the values of the function f, given on $x \in [0, 1]$, to segment $[0, \frac{m-1}{m}]$ by linear change of the variable and actually use interpolation involving edge points $x = 0, 1$.

5.2 *Test Functions*

Numerical experiments were carried out with the following four complex-valued smooth functions.

$$f_1 = 3\left|\sin\left(\frac{2}{3}\pi\left(x - \frac{1}{3}\right)\right)\right|^5 + 2\,i\left|\sin\left(\frac{2}{3}\pi\left(x - \frac{3}{4}\right)\right)\right|^5;$$
$$f_2 = J_{\frac{7}{5}}\left(7x^2 + (5 - i)\right); \quad f_3 = \mathrm{erf}\left(e^{3\,i\,x}\right);$$
$$f_4 = \mathrm{sinc}((4\pi + 2i)x),\ x \in [0, 1]. \tag{27}$$

Here $f_1 \in C^5$ and $f_1^{(6)}$ is a piecewise smooth function. The other three functions are analytic in a neighborhood of segment [0,1]. The function $f_2(z)$ has two branch points $z = \pm\sqrt{-\frac{5}{7} + \frac{i}{7}}$, while f_3 and f_4 are entire functions. The function f_4 is of exponential type.

5.3 *Numerical Results*

In the Tables 1, 2, 3, and 4 the L_2-errors on the segment $[0, 1]$ are given on the left, and on the segment $[0, \frac{m-1}{m}]$ on the right (see Remark 6).

Table 1 L_2- errors for function $f_1(x)$

$Alg \downarrow$	m=10	m=22	m=34
$Class.$	4.6e−1; 1.2e−1	3.2e−2; 9e−2	2.6e−1; 7.5e−2
$Trad.$	1.5e−1; 6.8e−3	2e−2; 4.2e−4	8.2e−3; 1.8e−4
$Adapt.$	4.4e−2; 2.1e−3	1e−2; 3.1e−4	1.4e−4; 4.9e−6

Table 2 L_2- errors for function $f_2(x)$

$Alg \downarrow$	m=8	m=16	m=24
$Class.$	1.2e−1; 9e−2	1.2e−1; 1.3e−1	1.3e−1; 1.e−1
$Trad.$	1.2e−1; 4.7e−3	1.8e−4; 3.9e−6	7.6e−6; 1.4e−7
$Adapt.$	1.3e−2; 7.8e−4	2.9e−5; 3.2e−7	8.9e−10; 6.3e−12

Table 3 L_2- errors for function $f_3(x)$

$Alg \downarrow$	m=8	m=16	m=24
$Class.$	5.1e−1; 2.6e−1	4.7e−1; 2.9e−1	4.5e−1; 3.1e−1
$Trad.$	1.4e−0; 7.6e−2	1.4e−1; 3.8e−3	6.6e−4; 1.5e−5
$Adapt.$	3e−2; 1.7e−2	3.9e−6; 1.2e−6	1.6e−10; 1.1e−11

Table 4 L_2- errors for function $f_4(x)$

$Alg \downarrow$	m=4	m=12	m=20
$Class.$	4e−1; 1.8e−1	1.9e−1; 6.2e−2	1.4e−1; 4.6e−2
$Trad.$	7.4e−2; 1.7e−1	2.5e−4; 5.1e−6	5.7e−6; 8.7e−8
$Adapt.$	1.8e−1; 1.8e−1	1.1e−5; 3.4e−6	4e−12; 2.2e−13

6 Conclusion

The presented numerical results confirm the effectiveness of the proposed interpolation algorithms. This is especially true for the adaptive algorithm, which demonstrates clear superiority. The results confirm that removing the extrapolation interval $(\frac{m-1}{m}, 1)$ gives more objective data (see Remark 6).

The above method for constructing fast algorithms was proposed quite recently. Therefore, many theoretical and applied aspects of similar algorithms have not yet been studied. For example, formula (25) is selected by using insufficient number of experiments. It is not clear what ways are beneficial for choosing sets D_n and $\tilde{D}_n$ in the adaptive algorithm (see Sect. 2.5 and Remark 4).

This method can easily be extended to such Fourier tools as sine (cosine) series and interpolations. This can be done both as a consequence of the above, and directly, by developing similar algorithms of the traditional and adaptive types.

From a practical point of view, the results of item **4** on Fourier interpolation are more diversified than on Fourier series (Sect. 2). Let us point out, for example, such areas as signal and image processing, quadrature formulas, detection of hidden periodicity, and approximate methods for ODE.

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Boundary Value Problems for 3D-Dirac Operators and MIT Bag Model

Vladimir Rabinovich

Abstract We consider the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ of boundary value problem

$$\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}\,\boldsymbol{u} = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{u} \text{ on } \Omega \\ \mathfrak{B}\boldsymbol{u} = \boldsymbol{u}_{\partial\Omega}^{(2)} - \mathfrak{b}\boldsymbol{u}_{\partial\Omega}^{(1)} \text{ on } \partial\Omega \end{cases} \tag{1}$$

for $3D-$Dirac operators $\mathfrak{D}_{\boldsymbol{A},\Phi}$ with the magnetic potential $\boldsymbol{A} \in L^{\infty}(\Omega, \mathbb{R}^3)$, and electric potential $\Phi \in L^{\infty}(\Omega)$ in domain $\Omega \subset \mathbb{R}^3$ with C^2-uniformly regular boundary, where $\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \end{pmatrix} \in \mathbb{C}^4$ be a vector with $\boldsymbol{u}^{(j)} \in \mathbb{C}^2$, $j = 1, 2$, $\boldsymbol{u}_{\partial\Omega}^{(j)}$ is the boundary value of $\boldsymbol{u}^{(j)}$ on $\partial\Omega$, where $\mathfrak{b}$ is a 2×2 matrix with entries $\mathfrak{b}_{ij}$ belonging to the space of bounded continuous functions on $\partial\Omega$. We associate with this boundary value problem an unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ in $L^2(\Omega, \mathbb{C}^4)$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi}$ on Ω with domain

$$dom\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} = \left\{\boldsymbol{u} \in H^1(\Omega, \mathbb{C}^4) : \boldsymbol{u}_{\partial\Omega}^{(2)} - \mathfrak{b}\boldsymbol{u}_{\partial\Omega}^{(1)} = \boldsymbol{0} \text{ on } \partial\Omega\right\}.$$

We obtain conditions of the self-adjointness of $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ and the discreteness of its spectrum. We give applications of this results to the operator of MIT bag model of relativistic quantum mechanics.

Keywords Dirac operators · Boundary value problems · Self-adjointness · Fredholmness

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1 Introduction

The study of massive relativistic particles of spin-$\frac{1}{2}$ such as electrons or quarks involves the Dirac operators which are of great importance in elementary particle physics (see for instance [19, 26]). There is also extensive mathematical literature devoted to the boundary value problems for the Dirac type operators (see for instance [10–13, 16, 20, 24] and references cited there).

We consider in the paper the boundary value problem

$$\begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{u}(x) = \boldsymbol{f}(x), x \in \Omega \\ \mathfrak{B}\boldsymbol{u}(x') = \boldsymbol{\varphi}(x'), x' \in \partial\Omega \end{cases} \tag{2}$$

in a domain $\Omega \subset \mathbb{R}^3$ with C^2-unformly regular boundary $\partial\Omega$ (for the definition of uniformly regular manifolds see for instance [3, 18]), where $\mathfrak{D}_{\boldsymbol{A},\Phi}$ is the Dirac operator

$$\begin{aligned} \mathfrak{D}_{\boldsymbol{A},\Phi} &= \alpha \cdot (i\nabla + \boldsymbol{A}) + \alpha_0 m + \Phi I_4 \\ &= \sum_{j=1}^{3} \alpha_j \left(i\partial_{x_j} + A_j\right) + \alpha_0 m + \Phi I_4, \end{aligned} \tag{3}$$

$\boldsymbol{u}$ is a vector-function on Ω with values in $\mathbb{C}^4$, α_j, $j = 0, 1, 2, 3$ are the 4×4 Dirac matrices satisfying the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4 \ (j, k = 0, 1, 2, 3),$$

I_n is the $n \times n$ unit matrix, $\boldsymbol{A} = (A_1, A_2, A_3)$ is the vector-valued potential of the magnetic field $\boldsymbol{H}$, Φ is the potential of the electrostatic field $\boldsymbol{E}$, m is the mass of particle. We use the system of coordinates for which the Planck constant $h = 1$, the light speed $c = 1$, and the charge of electron $e = 1$. We assume that the functions A_j, $j = 1, 2, 3$, and Φ belong to the space $L^\infty(\Omega)$. The boundary operator $\mathfrak{B}$ is defined as follows. Let $\boldsymbol{u} = \begin{pmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \end{pmatrix} \in \mathbb{C}^4$ where $\boldsymbol{u}^{(j)} \in \mathbb{C}^2$, $j = 1, 2$.

Then

$$\mathfrak{B}\boldsymbol{u} = \boldsymbol{u}^{(2)}_{\partial\Omega} - \mathfrak{b}\boldsymbol{u}^{(1)}_{\partial\Omega} \text{ on } \partial\Omega \tag{4}$$

where $\mathfrak{b} = \left(b^{ij}_k\right)^2_{i,j=1}$ is a $2 \times 2-$matrix with elements $b_{ij} \in L^\infty(\partial\Omega)$, $\boldsymbol{u}^{(j)}_{\partial\Omega}$, $j = 1, 2$ are the traces of $\boldsymbol{u}^{(j)}$ on the boundary $\partial\Omega$. The boundary condition (4) allows one to obtain Lopatinsky's condition in an explicit and easily verifiable form. Moreover, the class of boundary value problems (2) and (4) contains the so-called

MIT bag model described by the boundary condition

$$\boldsymbol{u}^{(2)}_{\partial\Omega} - i\,(\sigma\cdot\boldsymbol{\nu})\,\boldsymbol{u}^{(1)}_{\partial\Omega} = \mathbf{0} \text{ on } \partial\Omega \tag{5}$$

where $\sigma\cdot\boldsymbol{\nu} = \sigma_1\nu_1+\sigma_2\nu_2+\sigma_3\nu_3$ and σ_j are the Pauli matrices, and $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal vector to $\partial\Omega$. Note that the MIT bag model is used to the study of the confinement of particles of spin-$\frac{1}{2}$ into domains of $\mathbb{R}^3$ (see for instance [14, 15, 19, 21], Chap. 77). Recently, the interest in the MIT bag model has arisen in the connection with problems of interactions of relativistic particles with thin charged shells (see [4–6, 25], and reference cited there). In the $2D$-dimension the Dirac operators with special boundary conditions similar to the MIT bag model used in the description of graphene. Some spectral properties such operators have been considered in [7, 8] (see also references cited in these papers). It should be noted that the *MIT bag model* has been investigated also in the book [22] by means of the quaternion analysis methods.

Let $H^1(\Omega, \mathbb{C}^4)$ be the Sobolev space of $4-$dimensional vector-valued functions $\boldsymbol{u}$ on Ω. We associate with the boundary value problem (2) the operator

$$\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4) \oplus H^{1/2}(\partial\Omega, \mathbb{C}^2) = \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4).$$

We consider also the realization of boundary value problem (2) as an unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ in $L^2(\Omega, \mathbb{C}^4)$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi}$ with domain

$$H^1_{\mathfrak{B}}(\Omega, \mathbb{C}^4) = \left\{u \in H^1(\Omega, \mathbb{C}^4) : \mathfrak{B}\boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega\right\}$$

and study the self-adjointness of $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ for domains with C^2-uniformly regular boundary $\partial\Omega$.

Our investigation of self-adjointness of $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is based on the study of the parameter-dependent problem

$$\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}(\mu)\boldsymbol{u}(x) = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi}(\mu)u(x) = \left(\mathfrak{D}_{\boldsymbol{A},\Phi} - i\mu I_4\right)\boldsymbol{u}(x) = \boldsymbol{f}(x), x \in \Omega \\ \mathfrak{B}\boldsymbol{u}(x') = \boldsymbol{\varphi}(x'), x' \in \partial\Omega \end{cases}, \mu \in \mathbb{R} \tag{6}$$

in domains with C^2-uniformly regular boundary. We prove that if the parameter-dependent Lopatinsky condition are satisfied for $\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}(\mu) : H^1(\Omega, \mathbb{C}^4) \to \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ uniformly on the boundary $\partial\Omega$, then the operator $\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}(\mu)$ is invertible for large enough values of $|\mu|$. Using this result we prove that if $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is a symmetric operator and the Lopatinsky condition holds uniformly on $\partial\Omega$, then $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is a self-adjoint operator. Moreover, if the domain Ω is bounded then $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} : H^1(\Omega, \mathbb{C}^4) \to \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ is a Fredholm operator, and unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ has the discrete spectrum, only.

Applying these results we obtain the self-adjointness of the operator of MIT bag problem in domains with C^2-uniformly regular boundary, and the discreteness of the spectrum of this operator if Ω is a bounded domain.

Note that the operators of the MIT bag problems for bounded domains have been considered in [4, 5, 25], where the authors proved, in particular, its self-adjointness. We note also the very recent paper [6] devoted to the spectral properties of the MIT bag problems and some of its generalizations for domains with bounded C^2-boundary. The approach of mentioned papers is based on the abstract boundary triple techniques from the extension theory of symmetric operators and the study of certain classes of (boundary) integral operators, that appear in a Krein type resolvent formula.

It should be noted that our approach allows to study self-adjointness of operators of boundary problems for 3D-Dirac operators in domains with bounded and unbounded boundaries. The class of boundary conditions, proposed in our paper, contains the boundary conditions of the MIT bag model and its generalizations considered in the paper [6].

1.1 Notations

- If X, Y are Banach spaces then we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators acting from X into Y with the uniform operator topology. In the case $X = Y$ we write shortly $\mathcal{B}(X)$.
- An operator $A \in \mathcal{B}(X, Y)$ is called a Fredholm operator if $ker A = \{x \in X : Ax = 0\}$, and $coker A = Y/ \operatorname{Im} A$ are finite dimensional spaces. Let $\mathcal{A}$ be a closed unbounded operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain $dom\mathcal{A}$. Then $\mathcal{A}$ is called a Fredholm operator if $ker\mathcal{A} = \{x \in dom\mathcal{A} : \mathcal{A}x = 0\}$ and $coker A = \mathcal{H}/ \operatorname{Im} \mathcal{A}$ where $\operatorname{Im} \mathcal{A} = \{y \in \mathcal{H} : y = \mathcal{A}x, x \in dom\mathcal{A}\}$ are finite-dimensional spaces. Note that $\mathcal{A}$ is a Fredholm operator as unbounded operator in $\mathcal{H}$ if and only if $\mathcal{A} : dom\mathcal{A}. \to \mathcal{H}$ is a Fredholm operator as a bounded operator where $dom\mathcal{A}$ is equipped by the graph norm

$$\|u\|_{dom\mathcal{A}} = \left(\|u\|_{\mathcal{H}}^2 + \|\mathcal{A}u\|_{\mathcal{H}}^2\right)^{1/2}, u \in dom\mathcal{A}$$

  (see for instance [9]).
- The essential spectrum $sp_{ess}\mathcal{A}$ of an unbounded operator $\mathcal{A}$ is a set of $\lambda \in \mathbb{C}$ such that $\mathcal{A} - \lambda I$ is not Fredholm operator as the unbounded operator, and the discrete spectrum $sp_{dis}\mathcal{A}$ of $\mathcal{A}$ is a set of isolated eigenvalues of finite multiplicity. It is well known that if $\mathcal{A}$ is a self-adjoint operator then $sp_{dis}\mathcal{A} = sp\mathcal{A} \setminus sp_{ess}\mathcal{A}$.

- We denote by $L^2(\Omega, \mathbb{C}^4)$, $L^2(\partial\Omega, \mathbb{C}^4)$ the Hilbert spaces of 4−dimensional vector-functions $\boldsymbol{u}(x) = (u^1(x), u^2(x), u^3(x), u^4(x))$, $x \in \Omega$, $x \in \partial\Omega$ with the scalar products

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Omega,\mathbb{C}^4)} = \int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x) dx,$$

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{L^2(\partial\Omega,\mathbb{C}^4)} = \int_{\partial\Omega} \boldsymbol{\varphi}(x') \cdot \boldsymbol{\psi}(x') dx'$$

  where $\boldsymbol{h} \cdot \boldsymbol{g} = \sum_{j=1}^4 h_j \bar{g}_j$.
- We denote by $H^s(\mathbb{R}^3, \mathbb{C}^4)$ the Sobolev space on $\mathbb{R}^3$ of distributions $\boldsymbol{u} \in \mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4)$ with the norm

$$\|\boldsymbol{u}\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^s \left\| \hat{\boldsymbol{u}}(\xi) \right\|_{\mathbb{C}^4}^2 d\xi \right)^{1/2} < \infty, s \in \mathbb{R}$$

  where $\hat{\boldsymbol{u}}$ is the Fourier transform of $\boldsymbol{u}$ in the sense of distributions. If Ω is a domain in $\mathbb{R}^3$ then $H^s(\Omega, \mathbb{C}^4)$ is the space of restrictions of $\boldsymbol{u} \in H^s(\mathbb{R}^3, \mathbb{C}^4)$ on Ω with the norm

$$\|\boldsymbol{u}\|_{H^s(\Omega,\mathbb{C}^4)} = \inf_{l\boldsymbol{u} \in H^s(\mathbb{R}^3,\mathbb{C}^4)} \|l\boldsymbol{u}\|_{H^s(\mathbb{R}^3,\mathbb{C}^4)}$$

  where $l\boldsymbol{u}$ is an extension of $\boldsymbol{u}$ on $\mathbb{R}^3$. If $s > 1/2$ then the distributions in $H^s(\Omega, \mathbb{C}^4)$ have the traces on $\partial\Omega$, and we denote by $H^{s-1/2}(\partial\Omega)$ the Sobolev space on $\partial\Omega$ consisting of these traces.
- We denote by $C_b(\partial\Omega)$ the class of bounded continuous functions on the surface $\partial\Omega$.
- We say that the domain $\Omega \subset \mathbb{R}^3$ has a C^2−uniformly regular boundary $\partial\Omega$ if $\partial\Omega$ is a C^2− surface, and the following conditions hold: (i) for fixed $r > 0$ and for every point $x_0 \in \partial\Omega$ there exists a ball $B_r(x_0) = \left\{ x \in \mathbb{R}^3 : |x - x_0| < r \right\}$ and the homeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ such that

$$\varphi_{x_0}(B_r(x_0) \cap \Omega) = B_1(0) \cap \mathbb{R}^3_+, \mathbb{R}^3_+ = \left\{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 > 0 \right\},$$

$$\varphi_{x_0}(B_r(x_0) \cap \partial\Omega) = B_1(0) \cap \mathbb{R}^2_{y'}, \mathbb{R}^2_{y'} = \left\{ y = (y', y_3) \in \mathbb{R}^3 : y_3 = 0 \right\};$$

  (ii) if $\varphi^i_{x_0}$, $i = 1, 2, 3$ are the coordinate functions of the mappings φ_{x_0} then

$$\sup_{x_0 \in \partial\Omega} \sup_{|\alpha| \le 2, x \in B_r(x_0)} \left| \partial^\alpha \varphi^i_{x_0}(x) \right| < \infty, i = 1, 2, 3.$$

  Note if Ω is a domain with C^2-compact boundary, then $\partial\Omega$ is uniformly regular.

1.2 Free Dirac Operators

Let

$$\mathfrak{D}_0 = i\boldsymbol{\alpha} \cdot \nabla + \alpha_0 m = \sum_{j=1}^{3} i\alpha_j \partial_{x_j} + \alpha_0 m$$

be the free Dirac operator (see for instance [26]) where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, α_j, $j = 0, 1, 2, 3$ are the 4×4 Dirac matrices

$$\alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, j = 1, 2, 3, \tag{7}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

are the 2×2 Pauli matrices satisfying the relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I_2, \; j, k = 1, 2, 3. \tag{9}$$

Relations (9) yield that

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \; j, k = 0, 1, 2, 3, \tag{10}$$

where I_n is the $n \times n$ unit matrix. Equality (10) implies that

$$(i\boldsymbol{\alpha} \cdot \nabla)^2 = -\Delta I_4$$

where Δ is $3D-$Laplacian. Moreover

$$\mathfrak{D}_0^2 = \left(-\Delta + m^2\right) I_4.$$

It is well-known that the unbounded operator $\mathfrak{D}_0$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$ is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ (see for instance [26]), and

$$sp\mathfrak{D}_0 = sp_{ess}\mathfrak{D}_0 = (-\infty, -|m|] \bigcup [|m|, +\infty).$$

2 Parameter-Dependent Boundary Problems for $3 - D$ Dirac Operators in Domains with C^2-regular Boundary

We consider the parameter-dependent boundary value problem

$$\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}\ \boldsymbol{u} = \begin{cases} (\mathfrak{D}_{\boldsymbol{A},\Phi} - i\mu I_4)\boldsymbol{u} = \mathfrak{D}_{\boldsymbol{A},\Phi}(\mu)\boldsymbol{u} = \boldsymbol{f} \ \text{ on } \Omega, \mu \in \mathbb{R}, \\ \mathfrak{B}\boldsymbol{u} = u^{(2)}_{\partial\Omega} - \mathfrak{b}u^{(1)}_{\partial\Omega} = \boldsymbol{g} \text{ on } \partial\Omega \end{cases}, \tag{11}$$

where $\Omega \subset \mathbb{R}^3$ is a domain with C^2-uniformly regular boundary. In what follows we assume that

$$\boldsymbol{A} \in L^\infty(\Omega, \mathbb{C}^4), \Phi \in L^\infty(\Omega), \mathfrak{b} = \left(b^{ij}\right)^2_{i,j=1}, b^{ij} \in C_b(\partial\Omega). \tag{12}$$

We introduce the Sobolev spaces $H^s_\mu(\mathbb{R}^3, \mathbb{C}^4)$ of distributions $\boldsymbol{u} \in H^s(\mathbb{R}^3, \mathbb{C}^4)$ on $\mathbb{R}^3$ with values in $\mathbb{C}^4$ and with norm depending on the parameter $\mu \in \mathbb{R}$

$$\|u\|_{H^s_\mu(\mathbb{R}^n,\mathbb{C}^4)} = \left(\int_{\mathbb{R}^3} (1 + |\xi|^2 + \mu^2)^s \left|\hat{u}(\xi)\right|^2 d\xi\right)^{1/2} < \infty, s \in \mathbb{R}, \mu \in \mathbb{R}$$

where $\hat{\boldsymbol{u}}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\boldsymbol{u}(x)dx$ is the Fourier transform of $\boldsymbol{u}$ in the distributions sense. We denote by $H^s_\mu(\Omega, \mathbb{C}^4)$ the space of the restrictions on the domain Ω of distributions in $H^s_\mu(\mathbb{R}^3, \mathbb{C}^4)$ with the norm

$$\|\boldsymbol{u}\|_{H^s_\mu(\Omega,\mathbb{C}^4)} = \inf_{l\boldsymbol{u}\in H^s_\mu(\mathbb{R}^3,\mathbb{C}^4)} \|l\boldsymbol{u}\|_{H^s_\mu(\mathbb{R}^3,\mathbb{C}^4)}$$

where $l\boldsymbol{u}$ is the extension of $\boldsymbol{u} \in H^s_\mu(\Omega, \mathbb{C}^4)$ on $\mathbb{R}^3$. Note that

$$\|\boldsymbol{u}\|_{H^s_\mu(\Omega,\mathbb{C}^4)} \le (1 + \mu^2)^{-\frac{(r-s)}{2}} \|\boldsymbol{u}\|_{H^r_\mu(\Omega,\mathbb{C}^4)}, r \ge s, \tag{13}$$

and the trace operator $\gamma_{\partial\Omega}$ is bounded from $H^1_\mu(\Omega, \mathbb{C}^4)$ into $H^{1/2}_\mu(\partial\Omega, \mathbb{C}^4)$ if $\partial\Omega$ is C^2-surface. We consider the operator $\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}$ as acting from $H^1_\mu(\Omega, \mathbb{C}^4)$ into $L^2(\Omega, \mathbb{C}^4) \oplus H^{1/2}_\mu(\partial\Omega, \mathbb{C}^4) = \mathcal{H}_\mu(\Omega, \partial\Omega, \mathbb{C}^4)$.

Let $\mathfrak{D}^0(\mu) = i\boldsymbol{\alpha} \cdot \nabla - i\mu I_4$ be the main part of the parameter-dependent operator $\mathfrak{D}_{\boldsymbol{A},\Phi}(\mu) = \mathfrak{D}_{\boldsymbol{A},\Phi} - i\mu I_4$. Since

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\xi} + i\mu I_4)(\boldsymbol{\alpha} \cdot \boldsymbol{\xi} - i\mu I_4) = \left(|\boldsymbol{\xi}|^2 + \mu^2\right) I_4$$

the operator $\mathfrak{D}_{\boldsymbol{A},\Phi} - i\mu I_4$ is the uniformly elliptic operator with parameter $\mu \in \mathbb{R}$ (see for instance [1, 2]).

For a fix point $x_0 \in \partial\Omega$ we use the local system of orthogonal coordinates $y = (y_1, y_2, y_3)$ where the axis y_1, y_2 belong to the tangent plane to $\partial\Omega$ at the point x_0 and the axis $y_3 = z$ is directed along the inward normal vector $\boldsymbol{\nu}_{x_0}$ to $\partial\Omega$ at the point x_0. Going to the local coordinates and taking the main part of the operator $\mathfrak{D}_{\boldsymbol{A},\Phi} - i\mu I_4$ we obtain the operator

$$\mathbb{D}^0_{\mathfrak{B}_{x_0}}(\mu)\boldsymbol{\psi}(y) = \left(\mathbb{D}^0_{\mathfrak{B}_{x_0}} - i\mu\right)\boldsymbol{\psi}(y) \tag{14}$$

$$= \begin{cases} \left(i\alpha_1\partial_{y_1} + i\alpha_2\partial_{y_2} + i\alpha_3\partial_{y_3} - i\mu I_4\right)\boldsymbol{\psi}(y) = \boldsymbol{f}(y), \, y \in \mathbb{R}^3_+ \\ \mathfrak{B}_{x_0}\boldsymbol{\psi}(y') = \boldsymbol{\psi}^{(2)}(y', 0) - \mathfrak{b}(x_0)\boldsymbol{\psi}^{(1)}(y', 0) = \boldsymbol{\varphi}(y'), \, y' \in \mathbb{R}^2 \end{cases}.$$

We consider the operator $\mathbb{D}^0_{\mathfrak{B}_{x_0}}(\mu)$ as acting from $H^1_\mu(\mathbb{R}^3_+, \mathbb{C}^4)$ into $\mathcal{H}_\mu(\mathbb{R}^3_+, \mathbb{R}^2, \mathbb{C}^4) = L^2(\mathbb{R}^3_+, \mathbb{C}^4) \oplus H^{1/2}_\mu(\mathbb{R}^2, \mathbb{C}^2)$. Applying the Fourier transform with respect to $y' = (y_1, y_2) \in \mathbb{R}^2$ we obtain the family of $1 - D$ parameter-dependent operators on $\mathbb{R}_+$

$$\hat{\mathbb{D}}^0_{\mathfrak{B}_{x_0}}(\boldsymbol{\xi}', \mu)\boldsymbol{\psi}(\boldsymbol{\xi}', \mu, z) \tag{15}$$

$$= \begin{cases} \left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + i\alpha_3\frac{d}{dz} - i\mu I_4\right)\boldsymbol{\psi}(\boldsymbol{\xi}', \mu, z), \, z \in \mathbb{R}_+, \\ \boldsymbol{\psi}^{(2)}(\boldsymbol{\xi}', \mu, 0) - \mathfrak{b}(x_0)\boldsymbol{\psi}^{(1)}(\boldsymbol{\xi}', \mu, 0), \end{cases}$$

where $\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' = \alpha_1\xi_1 + \alpha_2\xi_2$.

We are looking for the exponentially decreasing solutions of the equation

$$\hat{\mathbb{D}}^0_{\mathfrak{B}_{x_0}}(\boldsymbol{\xi}', \mu)\boldsymbol{\psi} = \boldsymbol{0} \text{ on } \mathbb{R}_+. \tag{16}$$

Since

$$\left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + i\alpha_3\frac{d}{dz} + i\mu I_4\right)\left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + i\alpha_3\frac{d}{dz} - i\mu I_4\right) = \left(\left|\boldsymbol{\xi}'\right|^2 + \mu^2 - \frac{d^2}{dz^2}\right) I_4 \tag{17}$$

the equation

$$\left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + i\alpha_3\frac{d}{dz} - i\mu I_4\right)\boldsymbol{\psi}(\boldsymbol{\xi}', \mu, z) = 0 \tag{18}$$

has the following exponentially decreasing solutions on $\mathbb{R}_+$

$$\boldsymbol{\psi}(\boldsymbol{\xi}', \mu, z) = \boldsymbol{h}\left(\boldsymbol{\xi}', \mu\right)e^{-\rho z}, \, \rho = \sqrt{\left|\boldsymbol{\xi}'\right|^2 + \mu^2}, \, z > 0 \tag{19}$$

where the vector $\boldsymbol{h}\left(\boldsymbol{\xi}', \mu\right)$ satisfies the equation

$$\left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' - i\rho\alpha_3 + i\mu I_4\right) \boldsymbol{h}\left(\boldsymbol{\xi}', \mu\right) = 0. \tag{20}$$

Taking into account (17) we obtain that the vectors $\boldsymbol{h}\left(\boldsymbol{\xi}', \mu\right) \in \mathbb{C}^4$ have the form

$$\boldsymbol{h}\left(\boldsymbol{\xi}', \mu\right) = \Theta(\boldsymbol{\xi}', \mu)\boldsymbol{f} = \left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' - i\rho\alpha_3 + i\mu I_4\right) \boldsymbol{f} \tag{21}$$

where $\boldsymbol{f}$ is a vector in $\mathbb{C}^4$. Let

$$\Lambda(\boldsymbol{\xi}', \mu) = \sigma' \cdot \boldsymbol{\xi}' - i\rho\sigma_3 = \begin{pmatrix} -i\rho & \bar{\varsigma} \\ \varsigma & i\rho \end{pmatrix}, \varsigma = \xi_1 + i\xi_2, \tag{22}$$

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We set

$$\boldsymbol{h}_1(\boldsymbol{\xi}', \mu) = \Theta(\boldsymbol{\xi}', \mu) \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} i\mu\boldsymbol{e}_1 \\ \Lambda(\boldsymbol{\xi}', \mu)\boldsymbol{e}_1 \end{pmatrix}, \text{ where}$$

$$\Lambda(\boldsymbol{\xi}', \mu)\boldsymbol{e}_1 = \begin{pmatrix} -i\rho & \bar{\varsigma} \\ \varsigma & i\rho \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -i\rho \\ \varsigma \end{pmatrix}$$

$$\boldsymbol{h}_2(\boldsymbol{\xi}', \mu) = \Theta(\boldsymbol{\xi}', \mu) \begin{pmatrix} 0 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{pmatrix} \Lambda(\boldsymbol{\xi}', \mu)\boldsymbol{e}_2 \\ i\mu\boldsymbol{e}_2 \end{pmatrix}, \text{ where} \tag{23}$$

$$\Lambda(\boldsymbol{\xi}', \mu)\boldsymbol{e}_2 = \begin{pmatrix} -i\rho & \bar{\varsigma} \\ \varsigma & i\rho \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{\varsigma} \\ i\rho \end{pmatrix}.$$

The vectors $\boldsymbol{h}_1(\xi', \mu), \boldsymbol{h}_2(\xi', \mu)$ are orthogonal and satisfy Eq. (20). Hence

$$\left\{\boldsymbol{h}_1(\xi', \mu)e^{-\rho z}, \boldsymbol{h}_2(\xi', \mu)e^{-\rho z}\right\}$$

is the fundamental system of solutions of Eq. (18) in $L^2(\mathbb{R}_+, \mathbb{C}^4)$.

Hence every exponentially decreasing solutions of Eq. (18) on $\mathbb{R}_+$ are of the form

$$\boldsymbol{\psi}(\boldsymbol{\xi}', \mu, z) = C_1\boldsymbol{h}_1(\boldsymbol{\xi}', \mu)e^{-\rho z} + C_2\boldsymbol{h}_2(\boldsymbol{\xi}', \mu)e^{-\rho z} \tag{24}$$

where C_1, C_2 are arbitrary constants. Substituting $\boldsymbol{\psi}$ in the boundary condition

$$\boldsymbol{\psi}^{(2)}(\boldsymbol{\xi}', \mu, 0) - \mathfrak{b}(x_0)\boldsymbol{\psi}^{(1)}(\boldsymbol{\xi}', \mu, 0) = \boldsymbol{0}$$

we obtain the system of linear equations

$$C_1\left(\boldsymbol{h}_1^{(2)}-\mathfrak{b}(x_0)\boldsymbol{h}_1^{(1)}\right)+C_2(\boldsymbol{h}_2^{(2)}-\mathfrak{b}(x_0)\boldsymbol{h}_2^{(1)})=\mathbf{0} \tag{25}$$

with respect to C_1, C_2. Let $\mathcal{L}(x_0,\boldsymbol{\xi}',\mu)$ be the matrix with columns $\{b_1(x_0,\boldsymbol{\xi}',\mu), b_2(x_0,\boldsymbol{\xi}',\mu)\}$ where

$$\begin{aligned} b_1(x_0,\boldsymbol{\xi}',\mu)&=\boldsymbol{h}_1^{(2)}\left(\boldsymbol{\xi}',\mu\right)-\mathfrak{b}(x_0)\boldsymbol{h}_1^{(1)}\left(\boldsymbol{\xi}',\mu\right), \\ b_2(x_0,\boldsymbol{\xi}',\mu)&=\boldsymbol{h}_2^{(2)}\left(\boldsymbol{\xi}',\mu\right)-\mathfrak{b}(x_0)\boldsymbol{h}_2^{(1)}\left(\boldsymbol{\xi}',\mu\right). \end{aligned} \tag{26}$$

System (25) has the trivial solution if and only if

$$\det\mathcal{L}(x_0,\boldsymbol{\xi}',\mu)\neq 0$$

- Condition

$$\det\mathcal{L}(x_0,\boldsymbol{\xi}',\mu)\neq 0 \text{ for every } \left(\boldsymbol{\xi}',\mu\right):\left|\boldsymbol{\xi}'\right|^2+\mu^2=1 \tag{27}$$

  is called the local parameter-dependent Lopatinsky condition at the point $x_0\in\partial\Omega$ for the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)$.
- We say that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)$ satisfies the uniform parameter-dependent Lopatinsky condition if

$$\inf_{x\in\partial\Omega,\mu^2+\left|\boldsymbol{\xi}'\right|^2=1}\left|\det\mathcal{L}(x,\boldsymbol{\xi}',\mu)\right|>0. \tag{28}$$

- It should be noted that if condition (27) is satisfied then the operator

$$\mathbb{D}^0_{\mathfrak{B}_{x_0}}(\mu):H^1_\mu(\mathbb{R}^3_+,\mathbb{C}^4)\to\mathcal{H}_\mu(\mathbb{R}^3_+,\mathbb{R}^2,\mathbb{C}^4),\, x_0\in\partial\Omega$$

  is invertible for every $\mu\in\mathbb{R}\diagdown[-\varepsilon,\varepsilon]\,,\varepsilon>0$ and

$$\sup_{\mu\in\mathbb{R}\diagdown[-\varepsilon,\varepsilon]}\left\|\left(\mathbb{D}^0_{\mathfrak{B}_{x_0}}(\mu)\right)^{-1}\right\|_{\mathcal{B}(\mathcal{H}_\mu(\mathbb{R}^3_+,\mathbb{R}^2,\mathbb{C}^4),H^1(\mathbb{R}^3_+,\mathbb{C}^4))}\leq C(x_0) \tag{29}$$

  with a constant $C(x_0)>0$ independent of $\mu\in\mathbb{R}\diagdown[-\varepsilon,\varepsilon]$, and if condition (28) holds, then

$$\sup_{\mu\in\mathbb{R}\diagdown[-\varepsilon,\varepsilon],x\in\partial\Omega}\left\|\left(\mathbb{D}^0_{\mathfrak{B}_{x}}(\mu)\right)^{-1}\right\|_{\mathcal{B}(\mathcal{H}_\mu(\mathbb{R}^3_+,\mathbb{R}^2,\mathbb{C}^4),H^1_\mu(\mathbb{R}^3_+,\mathbb{C}^4))}<\infty. \tag{30}$$

Theorem 1 *Let $\partial\Omega \subset \mathbb{R}^3$ be a C^2−uniformly regular surface, the magnetic potential $\boldsymbol{A} = (A_1, A_2, A_3) \in L^\infty(\Omega, \mathbb{C}^3)$, the electrostatic potential $\Phi \in L^\infty(\Omega)$, $\mathfrak{b} \in C_b(\partial\Omega)\otimes\mathcal{B}(\mathbb{C}^2)$, and the uniformly parameter-dependent Lopatinsky condition (30) for $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)$, $\mu \in \mathbb{R}$ be satisfied. Then there exists $\mu_0 > 0$ such that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) : H^1(\Omega, \mathbb{C}^4) \to \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ is invertible for every $\mu \in \mathbb{R} : |\mu| > \mu_0$.*

Proof The proof uses the ideas of the paper [2] (see also [1, Sec.3]). However, we consider the parameter-dependent boundary problems for unbounded domains, and therefore we need an infinite partition of unity and estimates associated with them. Since the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi}(\mu)$ is a uniformly elliptic parameter-depending operator on $\bar{\Omega}$, and the Lopatinsky condition (30) are satisfied uniformly on $\partial\Omega$, and $\partial\Omega$ is the uniformly regular surface, then there exists $r > 0$ and $\mu_0 > 0$ such that there exists a countable covering $\cup_{j\in\mathbb{N}} B_r(x_j) \supset \bar{\Omega}$ of the finite multiplicity $N \geq 1$ such that for every x_j, $j \in \mathbb{N}$ there exist locally inverse operators

$$L_{x_j}(\mu), R_{x_j}(\mu) \in \mathcal{B}(\mathcal{H}_\mu(\Omega, \partial\Omega, \mathbb{C}^4), H^1_\mu(\Omega, \mathbb{C}^4))$$

such that

$$\sup_{j\in\mathbb{N},|\mu|\geq\mu_0} \left\|L_{x_j}(\mu)\right\| = d_L < \infty, \quad \sup_{j\in\mathbb{N},|\mu|\geq\mu_0} \left\|R_{x_j}(\mu)\right\| = d_R < \infty, \tag{31}$$

and

$$\begin{aligned} L_{x_j}(\mu)\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)\eta_j I &= \eta_j I, \\ \eta_j\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)R_{x_j}(\mu) &= \eta_j I \end{aligned} \tag{32}$$

for every $\eta_j \in C_0^\infty(B_r(x_j))$, $j \in \mathbb{N}$. Let the system $\left\{\theta_j\right\}_{j\in\mathbb{N}}$ be a partition of unity subordinated to the covering $\cup_{j\in\mathbb{N}} B_r(x_j)$, that is $\theta_j \in C_0^\infty(B_r(x_j))$, $0 \leq \theta_j(x) \leq 1$, and

$$\sum_{j\in\mathbb{N}} \theta_j(x) = 1, x \in \bar{\Omega}. \tag{33}$$

Moreover, the sum $\sum_{j\in\mathbb{N}} \theta_j(x)$ contains for every $x \in \bar{\Omega}$ not more than N non zero terms. Let $\varphi_j \in C_0^\infty(B_r(x_j)$, $0 \leq \varphi_j(x) \leq 1$, and $\theta_j\varphi_j = \theta_j$. We set

$$L(\mu)\boldsymbol{v} = \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\varphi_j \boldsymbol{v}, \ \boldsymbol{v} \in C_0^\infty(\Omega, \mathbb{C}^4) \oplus C_0^\infty(\partial\Omega, \mathbb{C}^2), \tag{34}$$

$$R(\mu)\boldsymbol{v} = \sum_{j\in\mathbb{N}} \varphi_j R_{x_j}(\mu)\theta_j \boldsymbol{v}, \ v \in C_0^\infty(\Omega, \mathbb{C}^4) \oplus C_0^\infty(\partial\Omega, \mathbb{C}^2). \tag{35}$$

Taking into account that the covering $\left\{B_r(x_j)\right\}_{j\in\mathbb{N}}$ has the finite multiplicity N we obtain for every $\mu \geq \mu_0$ the estimates

$$\|L(\mu)\boldsymbol{v}\|_{H^1_\mu(\Omega,\mathbb{C}^4)} \leq C \sup_{j\in\mathbb{N}} \left\|L_j(\mu)\right\| \|v\|_{\mathcal{H}_\mu(\Omega,\partial\Omega,\mathbb{C}^4)} \leq Cd_L \|v\|_{\mathcal{H}_\mu(\Omega,\partial\Omega,\mathbb{C}^4)}, \tag{36}$$

$$\|R(\mu)\boldsymbol{v}\|_{H^1_\mu(\Omega,\mathbb{C}^4)} \leq C \sup_{j\in\mathbb{N}} \left\|R_j(\mu)\right\| \|v\|_{\mathcal{H}_\mu(\Omega,\partial\Omega,\mathbb{C}^4)} \leq Cd_R \|v\|_{\mathcal{H}_\mu(\Omega,\partial\Omega,\mathbb{C}^4)} \tag{37}$$

for every $\boldsymbol{v} \in C_0^\infty(\Omega,\mathbb{C}^4) \oplus C_0^\infty(\partial\Omega,\mathbb{C}^2)$ with the constant $C > 0$ independent of $\boldsymbol{v}$**.** Estimates (36) and (37) yield that the operators $L(\mu)$, $R(\mu)$ are continued to bounded operators acting from $L^2(\Omega,\mathbb{C}^4) \oplus H_\mu^{1/2}(\partial\Omega,\mathbb{C}^2)$ into $H^1_\mu(\Omega,\mathbb{C}^4)$). Let $\varphi_j\psi_j = \varphi_j$, $\psi_j \in C_0^\infty(B_r(x_j))$, $0 \leq \psi_j(x) \leq 1$. Then

$$L(\mu)\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) = \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\varphi_j\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)\psi_j I = I + T_1(\mu), \tag{38}$$

where

$$T_1(\mu) = \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\left[\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu), \varphi_j I\right]\psi_j,$$

and

$$\left[\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu), \varphi_j I\right] = \mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)\varphi_j I - \varphi_j\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu).$$

Applying estimate (13) we obtain that

$$\left\|\left[\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu), \varphi_j I\right]\right\|_{\mathcal{B}(H^1_\mu(\Omega,\mathbb{C}^4),\mathcal{H}_\mu(\Omega,\partial\Omega,\mathbb{C}^4))} \leq \frac{C}{|\mu|}, |\mu| \geq \mu_0, \tag{39}$$

with the constant $C > 0$ independent of $j \in \mathbb{N}$. Again applying the finiteness of the covering $\left\{B_r(x_j)\right\}_{j\in\mathbb{N}}$ we obtain from (36) and (39) the estimate

$$\|T_1(\mu)\| \leq \frac{C}{|\mu|} \sup_{j\in\mathbb{N}} \left\|L_{x_j}(\mu)\right\|, |\mu| \geq \mu_0. \tag{40}$$

Hence there exists $\mu_1 \geq \mu_0$ such that

$$\sup_{|\mu| \geq \mu_1} \|T_1(\mu)\| < 1. \tag{41}$$

Thus the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)$ has the left inverse operator $\mathbb{L}(\mu) = (I + T_1(\mu))^{-1} L(\mu)$ for every $\mu \in \mathbb{R} : \mu \geq \mu_1$. In the same way we prove that there exists a right inverse operator $\mathbb{R}(\mu)$ of $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu)$ for $|\mu| \geq \mu_2 > \mu_1$. Hence the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) : H^1_\mu(\Omega, \mathbb{C}^4) \to \mathcal{H}_\mu(\Omega, \partial\Omega, \mathbb{C}^4)$ is invertible for every $|\mu| \geq \mu_2$. □

Corollary 2 *Let conditions of Theorem 1 be satisfied. Then there exists* $\tilde{\mu} > 0$ *such that the operator* $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) : H^1(\Omega, \mathbb{C}^4) \to \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ *is invertible for every* $\mu \in \mathbb{R} : |\mu| \geq \tilde{\mu}$.

Proof For every fix $\mu \in \mathbb{R}$ the norms in the spaces $H^1_\mu(\Omega, \mathbb{C}^4)$, $\mathcal{H}_\mu(\Omega, \partial\Omega, \mathbb{C}^4)$ are equivalent to the norms in the spaces $H^1(\Omega, \mathbb{C}^4)$, $\mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ without parameter μ. It implies the invertibility $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) : H^1(\Omega, \mathbb{C}^4) \to \mathcal{H}(\Omega, \partial\Omega, \mathbb{C}^4)$ for every $\mu : |\mu| \geq \tilde{\mu}$. □

3 Fredholmness and the a Priori Estimate

We say that $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ satisfies the local Lopatinsky condition at the point $x_0 \in \partial\Omega$ if

$$\det \mathcal{L}(x_0, \xi', 0) \neq 0 \text{ if } \left|\xi'\right| = 1,$$

and $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ satisfies the uniform Lopatinsky condition if

$$\inf_{x \in \partial\Omega, |\xi'|=1} \left|\det \mathcal{L}(x, \xi', 0)\right| > 0. \tag{42}$$

Proposition 3 *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with* C^2-*uniformly regular boundary, the magnetic potential* $\boldsymbol{A} = (A_1, A_2, A_3) \in L^\infty(\Omega, \mathbb{C}^3)$, *the electrostatic potential* $\Phi \in L^\infty(\Omega)$, $\mathfrak{b} \in C_b(\partial\Omega) \otimes \mathcal{B}(\mathbb{C}^2)$, *and the uniform Lopatinsky condition (42) be satisfied. Then there exists* $C > 0$ *such that for every* $\boldsymbol{u} \in H^1(\Omega, \mathbb{C}^4)$ *the a priori estimate*

$$\|u\|_{H^1(\Omega,\mathbb{C}^4)} \leq C \left(\left\| \mathfrak{D}_{\boldsymbol{A},\Phi} \boldsymbol{u} \right\|_{L^2(\Omega,\mathbb{C}^4)} + \|\mathfrak{B}\boldsymbol{u}_{\partial\Omega}\|_{H^{1/2}(\partial\Omega,\mathbb{C}^2)} + \|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{C}^4)} \right) \tag{43}$$

holds.

Proof In the proof of a priori estimate we follow the standard elliptic theory: the ellipticity of the Dirac operator in Ω implies the local a priori estimate at every point $x \in \Omega$. The ellipticity of the Dirac operator and Lopatinsky condition at the points $x \in \partial\Omega$ imply the local a priori estimates at the points of the boundary. Since the operator $\mathfrak{D}_{\boldsymbol{A},\Phi}$ is uniformly elliptic and the Lopatisky conditions are fulfilled uniformly on $\partial\Omega$ the constants in the local a priori estimates can be chosen independent of the point $x \in \bar{\Omega}$. Since $\partial\Omega$ is a C^2-uniformly regular surface there exists a countable covering $\cup_{j=1}^{\infty} B(x_j, \varepsilon)$ of $\bar{\Omega}$ of finite multiplicity and a partition of unity $\sum_{j\in\mathbb{N}} \varphi_j(x) = 1, x \in \bar{\Omega}$ subordinated to this covering which allows one to glue the local estimates and to obtain a priori estimate (43). □

Theorem 4 *Let Ω be a bounded domain with C^2-boundary, and other conditions of Theorem 1 be fulfilled. Then the operator*

$$\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4) \oplus H^{1/2}(\partial\Omega, \mathbb{C}^2)$$

is Fredholm, and the spectrum of the unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is discrete.

Proof As it follows from the standard elliptic theory (see for instance [1, 23]) $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} - \lambda I, \lambda \in \mathbb{C}$ is the Fredholm family analytically depending on the parameter $\lambda \in \mathbb{C}$. Moreover, by Theorem 1 the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} - i\mu I$ is invertible for $\mu \in \mathbb{R}$ with $|\mu|$ is large enough. Then the Theorem on the analytical family of Fredholm operators (see for instance [17]) yields the discreteness of the spectrum of $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$. □

4 Self-adjointness of the Unbounded Operator $\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\mathfrak{B}}$

Now we consider the self-adjointness in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ of the unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ defined by the Dirac operator

$$\mathfrak{D}_{\boldsymbol{A},\Phi} = \boldsymbol{\alpha} \cdot (i\nabla + \boldsymbol{A}) + \alpha_0 m + \Phi I_4$$

with $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{C}^4)$, $\Phi \in L^\infty(\Omega)$ and with the domain

$$H^1_{\mathfrak{B}}(\Omega, \mathbb{C}^4) = \left\{ \begin{array}{c} \boldsymbol{u} \in H^1(\Omega, \mathbb{C}^4) : \\ \mathfrak{B}\boldsymbol{u}(x') = \boldsymbol{u}^{(2)}_{\partial\Omega}(x') - \mathfrak{b}(x')\boldsymbol{u}^{(1)}_{\partial\Omega}(x') = 0, x' \in \partial\Omega. \end{array} \right\} \tag{44}$$

where $\mathfrak{b} \in C_b(\partial\Omega, \mathcal{B}(\mathbb{R}^2))$.

Theorem 5 *Let: (i) $\Omega \subset \mathbb{R}^3$ be a domain with the C^2-uniformly regular boundary, (ii) the vector potential $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^3)$ and the electrostatic potential $\Phi \in L^\infty(\Omega)$ be real-valued, (iii) the uniformly parameter-dependent Lopatinsky condition on $\partial\Omega$ hold, (iv) $\mathfrak{b} \in C_b(\partial\Omega) \otimes \mathcal{B}(\mathbb{C}^2)$, and*

$$\mathfrak{b}^* (\sigma \cdot \boldsymbol{\nu}) + (\sigma \cdot \boldsymbol{\nu}) \, \mathfrak{b} = 0 \text{ on } \partial\Omega \tag{45}$$

where $\boldsymbol{\nu}$ is the outward unit normal vector to $\partial\Omega$. Then the operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is self-adjoint in $L^2(\Omega,\mathbb{C}^4)$.

Proof At first, we prove that the operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is symmetric. Integrating by parts we obtain

$$\left\langle \mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{u},\boldsymbol{v}\right\rangle_{L^2(\Omega,\mathbb{C}^4)} - \left\langle \boldsymbol{u},\mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{v}\right\rangle_{L^2(\Omega,\mathbb{C}^4)} = \langle(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{u}_{\partial\Omega},\boldsymbol{v}_{\partial\Omega}\rangle_{L^2(\partial\Omega,\mathbb{C}^4)}\,,$$
$$\boldsymbol{u},\boldsymbol{v}\in H^1_{\mathfrak{B}}(\Omega,\mathbb{C}^4).$$

Taking into account (44) we obtain that

$$\begin{aligned}(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{u}_{\partial\Omega}\cdot\boldsymbol{v}_{\partial\Omega} &= (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\,\boldsymbol{u}^{(2)}_{\partial\Omega}\cdot\boldsymbol{v}^{(1)}_{\partial\Omega} + (\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\,\boldsymbol{u}^{(1)}_{\partial\Omega}\cdot\boldsymbol{v}^{(2)}_{\partial\Omega}\\ &= \big((\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\,\mathfrak{b} + \mathfrak{b}^*\,(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\big)\,\boldsymbol{u}^{(1)}_{\partial\Omega}\cdot\boldsymbol{v}^{(1)}_{\partial\Omega}.\end{aligned} \tag{46}$$

Equalities (45) and (46) yield that $(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{u}_{\partial\Omega}\cdot\boldsymbol{v}_{\partial\Omega} = 0$ for every $\boldsymbol{u},\boldsymbol{v}\in H^1_{\mathfrak{B}}(\Omega,\mathbb{C}^4)$. Thus the operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is symmetric. It follows from a priori estimate (43) that the operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is closed. Moreover, Corollary 2 yields that the deficiency indices of $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ are equal 0. Hence (see for instance [9], page 100) the operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is self-adjoint. □

5 MIT Bag Model

We consider the operator of *MIT bag model* (see [4–6, 19], Chap.77, [14, 15, 21])

$$\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}\boldsymbol{u}(x) = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{u}(x),\ x\in\Omega\\ \mathfrak{B}\boldsymbol{u} = (I_4 + i\alpha_0(\alpha\cdot\boldsymbol{\nu}))\boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega\end{cases} \tag{47}$$

where $\boldsymbol{\nu}$ is the unit outward normal vector to $\partial\Omega$. The boundary condition in (47) can be written as

$$\begin{cases}\boldsymbol{u}^{(1)} + i\,(\sigma\cdot\boldsymbol{\nu})\,\boldsymbol{u}^{(2)} = \mathbf{0}\\ \boldsymbol{u}^{(2)} - i\,(\sigma\cdot\boldsymbol{\nu})\,\boldsymbol{u}^{(1)} = \mathbf{0}\end{cases}. \tag{48}$$

Taking into account that $(\sigma\cdot\boldsymbol{\nu})^2 = I_2$ we obtain that the second equation in (48) is equivalent to the first equation in (48). Hence the operator of MIT bag model can be written as

$$\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{b}}\boldsymbol{u}(x) = \begin{cases}\mathfrak{D}_{\boldsymbol{A},\Phi}\boldsymbol{u}(x),\ x\in\Omega\\ \boldsymbol{u}^{(2)} - \mathfrak{b}\boldsymbol{u}^{(1)} = \mathbf{0}\ \text{ on } \partial\Omega,\end{cases} \tag{49}$$

where $\mathfrak{b} = i\,(\sigma\cdot\boldsymbol{\nu})$ satisfies condition (45).

We denote by $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ the unbounded operator defined by $\mathfrak{D}_{\boldsymbol{A},\Phi}$ with domain

$$dom\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} = \left\{ \boldsymbol{u} \in H^1(\Omega, \mathbb{C}^4 : \boldsymbol{u}^{(2)} - \mathfrak{b}\boldsymbol{u}^{(1)} = \boldsymbol{0} \text{ on } \partial\Omega \right\}.$$

We check the parameter-dependent Lopatinsky conditions for the operator $\mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}(\mu) = \mathbb{D}_{\boldsymbol{A},\Phi,\mathfrak{B}} - i\mu I_4$. According formulas (26) the parameter-dependent Lopatinsky matrix is

$$\begin{aligned}\mathcal{L}(\xi', \mu) &= (\boldsymbol{h}_1^{(1)} + i\sigma_3\boldsymbol{h}_1^{(2)}, \boldsymbol{h}_2^{(1)} + i\sigma_3\boldsymbol{h}_2^{(2)}) \\ &= (i\mu\mathbf{e}_1 + i\sigma_3\Lambda\mathbf{e}_1, \Lambda\mathbf{e}_2 - \mu\sigma_3\boldsymbol{e}_2).\end{aligned}$$

Taking into account formulas (22) and (23) we obtain that

$$\begin{aligned}\det \mathcal{L}(\xi', \mu) &= \det \begin{pmatrix} i(\mu - i\rho) & \bar{\varsigma} \\ -i\varsigma & \mu + i\rho \end{pmatrix} \\ &= i\left(\mu^2 + \rho^2 + |\varsigma|^2\right) = 2i\rho^2, \varsigma = \xi_1 + i\xi_2 \in \mathbb{C}.\end{aligned} \tag{50}$$

Thus the parameter-dependent Lopatinsky condition is satisfied on $\partial\Omega$ uniformly. Moreover, formula (50) for $\mu = 0$ yields the uniformly Lopatinsky condition on $\partial\Omega$ and a priori estimate (43).

Thus Theorems 5 and 4 yield the following result.

Theorem 6 *Let $\Omega \subset \mathbb{R}^3$ be a domain with C^2-uniformly regular boundary, the potentials $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{C}^3)$, and $\Phi \in L^\infty(\Omega)$ are real-valued, and $\mathfrak{b} = i(\sigma \cdot \boldsymbol{\nu})$. Then: (i) the unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ of the MIT bag model is self-adjoint in $L^2(\Omega, \mathbb{C}^4)$; (ii) If Ω is a bounded domain with C^2-boundary, then $\mathcal{D}_{\boldsymbol{A},\Phi,\mathfrak{B}}$ is the Fredholm operator with the discrete spectrum.*

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A Refinement of the Adams Theorem on the Riesz Potential

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Abstract The goal of this note is to consider the application of the complex interpolation space of Morrey spaces. Actually, the boundedness of Riesz potentials acting on Morrey spaces, which is obtained by Adams, is refined by means of the complex interpolation.

Keywords Morrey space · Riesz potential · Complex interpolation · Calderón–Lozanovsky product

2010 Classification 26A33, 42B35

1 Introduction

In this paper, we investigate the boundedness of the Riesz potential I_α (the fractional integral operator) of order $0 < \alpha < n$ which acts on Morrey spaces. Our result will be an endpoint case of the Adams theorem and refines some existing results.

We write $Q(x,r) \equiv \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{j=1,2,\ldots,n} |x_j - y_j| \le r \right\}$ when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. Denote by $\mathcal{Q}$ the set of all cubes of the form $Q(x,r)$ for some $x \in \mathbb{R}^n$ and $r > 0$. Let $1 \le q \le p < \infty$. Then the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} |Q(x,r)|^{\frac{1}{p} - \frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}}$$

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is finite. We investigate the boundedness property of the Riesz potential I_α, which is given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^n).$$

The celebrated Adams theorem [1] asserts that I_α maps $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ boundedly as long as $1 < q \le p < \infty$ and $1 < t \le s < \infty$ satisfies $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}$ and $\frac{q}{p} = \frac{t}{s}$. It is known that the boundedness fails in the endpoint case where $q = 1$. In this note, we recall a counterexample disproving the boundedness in the endpoint case above because this counterexample can be used to prove the sharpness of our results.

There are several ways to compensate for this failure. One of the ways is to use the weak Morrey space $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$. Recall that the weak Morrey space $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathrm{W}\mathcal{M}^s_t} \equiv \sup\limits_{\lambda>0} \lambda \|\chi_{[\lambda,\infty]}(|f|)\|_{\mathcal{M}^s_t}$ is finite. It is known also as the Adams theorem that $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathrm{W}\mathcal{M}^s_t(\mathbb{R}^n)$ boundedly as long as $1 \le q \le p < \infty$ and $1 < t \le s < \infty$ satisfies $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}$ and $\frac{q}{p} = \frac{t}{s}$. Another way is to use a function space strictly embedded into $\mathcal{M}^p_1(\mathbb{R}^n)$. We write

$$\|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)} = \inf\left\{\lambda > 0 : \frac{1}{|Q(x,r)|}\int_{Q(x,r)} \frac{|f(y)|}{\lambda} \log\left(3 + \frac{|f(y)|}{\lambda}\right) \mathrm{d}y \le 1\right\}$$

for a measurable function f. The quantity $\|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)}$ is called the Orlicz average of f. The Orlicz–Morrey space $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$, $p > 1$, is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} \equiv \sup\limits_{x\in\mathbb{R}^n, r>0} |Q(x,r)|^{\frac{1}{p}} \|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)}$ is finite. It is known that $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ boundedly as long as $1 = q < p < \infty$ and $1 < t \le s < \infty$ satisfies $\frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}$ and $\frac{q}{p} = \frac{t}{s}$. See [11, 12] for more.

The goal of this note is to investigate how strongly we need the space $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ by the use of the Calderón-Lozanovsky product. We improve this boundedness result via the Calderón-Lozanovsky product. Let $0 < \theta < 1$ and $1 < p < \infty$. Then the Calderón-Lozanovsky product $(\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ is the set of all measurable functions f for which $|f| \le |f_0|^{1-\theta}|f_1|^\theta$ for some $f_0 \in \mathcal{M}^p_1(\mathbb{R}^n)$ and $f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$. The norm of $f \in (\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ is given by

$$\|f\|_{(\mathcal{M}^p_1)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta} = \inf\{(\|f_0\|_{\mathcal{M}^p_1})^{1-\theta}(\|f_1\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}})^\theta\},$$

where f_0 and f_1 move over all functions in $\mathcal{M}^p_1(\mathbb{R}^n)$ and $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ satisfying $|f| \le |f_0|^{1-\theta}|f_1|^\theta$. See [12] for more about the Calderón–Zygmund product.

We will prove the following theorem:

Theorem 1 *Let* $0 < \alpha < n$ *and* $1 < p < \frac{n}{\alpha}$. *Write*

$$s = \frac{np}{n - p\alpha}, \quad t = \frac{n}{n - p\alpha}, \quad \theta_0 = 1 - \frac{p\alpha}{n} = \frac{p}{s} = \frac{1}{t}.$$

(i) *The fractional integral operator* I_α *maps* $(\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta_0}(\mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))^{\theta_0}$ *into* $\mathcal{M}^s_t(\mathbb{R}^n)$.

(ii) *If the fractional integral operator* I_α *maps* $(\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))^{\theta}$ *into* $\mathcal{M}^s_t(\mathbb{R}^n)$*, then* $\theta \geq \theta_0$.

Recall that the fractional maximal operator M_α of order $\alpha \in [0, n)$ is defined by

$$M_\alpha f(x) \equiv \sup_{Q \in \mathcal{Q}} \chi_Q(x)\ell(Q)^{\alpha - n} \int_Q |f(y)|\mathrm{d}y$$

for $f \in L^0(\mathbb{R}^n)$, so that $M = M_0$ is the Hardy–Littlewood maximal operator.

Theorem 2 *The conclusion of Theorem 1 remains valid if we replace* I_α *by* M_α.

We use the space $\mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)$ because we have

$$\|Mf\|_{\mathcal{M}^p_1} \sim \|f\|_{\mathcal{M}^p_{\mathrm{L\,log\,L}}} \tag{1}$$

for all $f \in \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)$ [15]. Thus, (1) covers Theorem 2 in the case where $\alpha = 0$.

We will explain why Theorems 1 and 2 are related to the complex interpolation.

Following [2], we write $S \equiv \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and let $\bar{S}$ be its closure. For $j = 0, 1$, set $j + i\mathbb{R} \equiv \{z \in \mathbb{C} : \mathrm{Re}(z) = j\}$.

Definition 1 Let $1 < p < \infty$.

1. The space $\mathcal{F}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ is defined as the set of all functions $F : \bar{S} \to \mathcal{M}^p_1(\mathbb{R}^n)$ such that
   (a) F is continuous on $\bar{S}$ and $\sup_{z \in \bar{S}} \|F(z)\|_{\mathcal{M}^p_1} < \infty$,
   (b) F is holomorphic on S,
   (c) the function $t \in \mathbb{R} \mapsto F(1 + it) \in \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)$ is bounded and continuous.

   The space $\mathcal{F}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ is equipped with the norm

$$\|F\|_{\mathcal{F}(\mathcal{M}^p_1, \mathcal{M}^p_{\mathrm{L\,log\,L}})} \equiv \left(\sup_{t \in \mathbb{R}} \|F(it)\|_{\mathcal{M}^p_1}, \sup_{t \in \mathbb{R}} \|F(1 + it)\|_{\mathcal{M}^p_{\mathrm{L\,log\,L}}} \right).$$

2. Let $\theta \in (0,1)$. The *first complex interpolation space* $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]_\theta$ with respect to $(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ is defined as the set of all functions $f \in \mathcal{M}^p_1(\mathbb{R}^n)$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$. The norm on $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]_\theta$ is defined by the trace norm. The space $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]_\theta$ is also called the *Calderón's first complex interpolation space*, or the *lower complex interpolation space* of $(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$.

Definition 2 Let $1 < p < \infty$. Let $\theta \in (0,1)$.

1. Define $\mathcal{G}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ as the set of all functions $G : \bar{S} \to \mathcal{M}^p_1(\mathbb{R}^n)$ such that
   (a) G is continuous on $\bar{S}$ and $\sup_{z \in \bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{M}^p_1} < \infty$,
   (b) G is holomorphic on S,
   (c) the function $t \in \mathbb{R} \mapsto G(it) \in \mathcal{M}^p_1(\mathbb{R}^n)$ is Lipschitz continuous on $\mathbb{R}$.
   (d) the function $t \in \mathbb{R} \mapsto G(1+it) - G(1) \in \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)$ is Lipschitz continuous on $\mathbb{R}$ for each $j = 0, 1$.

   The space $\mathcal{G}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ is equipped with the norm

$$\|G\|_{\mathcal{G}(\mathcal{M}^p_1, \mathcal{M}^p_{\mathrm{L\,log\,L}})} \equiv \max \left\{ \|G(i\,\cdot)\|_{\mathrm{Lip}(\mathbb{R}, \mathcal{M}^p_1)},\ \|G(1+i\,\cdot)\|_{\mathrm{Lip}(\mathbb{R}, \mathcal{M}^p_{\mathrm{L\,log\,L}})} \right\}. \tag{2}$$

2. The *second complex interpolation space* $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta$ with respect to $(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$ is defined as the set of all functions $f \in \mathcal{M}^p_1(\mathbb{R}^n)$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$. The norm on $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta$ is defined by

$$\begin{aligned} &\|f\|_{[\mathcal{M}^p_1, \mathcal{M}^p_{\mathrm{L\,log\,L}}]^\theta} \\ &\equiv \inf\{\|G\|_{\mathcal{G}(\mathcal{M}^p_1, \mathcal{M}^p_{\mathrm{L\,log\,L}})} : f = G'(\theta) \text{ with } G \in \mathcal{G}(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))\}. \end{aligned}$$

   The space $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta$ is also called the *Calderón's second complex interpolation space*, or the *upper complex interpolation space* of $(\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n))$.

   See [2] for these definitions. According to the general theory in [2],

$$[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta \supset [\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]_\theta.$$

More precisely, $\mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)$ is dense in $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]_\theta$ under the norm of $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta$. The spaces $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L\,log\,L}}(\mathbb{R}^n)]^\theta$ and

$[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ have the same norm once we restrict the norm of $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ to $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ thanks to [3]. Furthermore, according to [10, 16], $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ and $(\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ are the same function spaces.

We will rephrase and slightly refine Theorems 1 and 2 with this in mind.

Theorem 3 *Let $1 < p < \frac{n}{\alpha}$. Define s, t and θ_0 as in Theorem 1.*

- (i) *The fractional integral operator I_α maps $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^{\theta_0}$ into $\mathcal{M}^s_t(\mathbb{R}^n)$.*
- (ii) *If the fractional integral operator I_α maps $[\mathcal{M}^p_1(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ into $\mathcal{M}^s_t(\mathbb{R}^n)$, then $\theta \geq \theta_0$.*

The same conclusion remains valid if we replace I_α by M_α.

Since our counterexamples used in (ii) in our thoerems are compactly supported, the proof of Theorem 3 is essentially included in those of Theorems 1 and 2. So we do not prove Theorem 3.

Before we conclude this section, we record a historical remark on the recent development of the complex interpolation of Morrey spaces. In 2013, Lemarié-Rieusset established that the first and the second complex interpolation spaces are different in [7, 8]. Based on his work, Lu, Yang and Yuan developed the complex interpolation of smoothness Morrey spaces [9]. A passage to the generalized Morrey spaces together with various closed subspaces is done by Hakim and the present author [4–6].

The rest of this paper is devoted to the proof of Theorems 1 and 2.

2 Proofs

The proofs of Theorems 1 and 2 parallel. We prove Theorem 1(i) and Theorem 2(i) and then prove Theorem 1(ii) and Theorem 2(ii).

2.1 Proof of (i)

We concentrate on M_α. In fact, by a similar way using the sparse family, we can handle I_α as we did in [13]. Let $f \in (\mathcal{M}^p_1(\mathbb{R}^n))^{1-\theta_0}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^{\theta_0}$, so that there exist $f_0 \in \mathcal{M}^p_1(\mathbb{R}^n)$ and $f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ satisfying $|f| \leq |f_0|^{1-\theta_0}|f_1|^{\theta_0}$. Choose such f_0 and f_1 arbitrarily. By Hölder's inequality, we have $M_\alpha f(x) \leq$

$(\|f_0\|_{\mathcal{M}^p_1})^{1-\theta_0} M f_1(x)^{\theta_0}$ for all $x \in \mathbb{R}^n$. By the use of (1), we conclude

$$\begin{aligned}\|M_\alpha f\|_{\mathcal{M}^s_t} &\le (\|f_0\|_{\mathcal{M}^p_1})^{1-\theta_0} \|(Mf_1)^{\theta_0}\|_{\mathcal{M}^s_t} \\ &\le (\|f_0\|_{\mathcal{M}^p_1})^{1-\theta_0} (\|Mf_1\|_{\mathcal{M}^p_1})^{\theta_0} \\ &\lesssim (\|f_0\|_{\mathcal{M}^p_1})^{1-\theta_0} (\|f_1\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}})^{\theta_0}\end{aligned}$$

Since f_0 and f_1 are arbitrary, we obtain

$$\|M_\alpha f\|_{\mathcal{M}^s_t} \lesssim \|f\|_{(\mathcal{M}^p_1)^{1-\theta_0}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^{\theta_0}},$$

as required.

2.2 *Proof of* (*ii*)

We have only to concentrate on the fractional maximal operator M_α in view of the pointwise estimate $M_\alpha f(x) \lesssim I_\alpha[|f|](x)$.

Let $R > 1$ satisfy $2(R+1)^{\frac{1}{p}-1} = 1$. For $e \in \{0,1\}^n$, write

$$f_e(x) = \frac{1}{R+1}x + \frac{R}{R+1}e \quad (x \in \mathbb{R}^n).$$

Let

$$E_0 \equiv [0,1]^n, \quad E_{j+1} \equiv \bigcup_{e\in\{0,1\}^n} f_e(E_j) \quad (j = 0, 1, \ldots).$$

Note that $\{E_j\}_{j=0}^\infty$ is a decreasing sequence of subsets in $[0,1]^n$. As we showed in [14, Proposition 4.1], $\|\chi_{E_j}\|_{\mathcal{M}^p_1} \sim \|\chi_{[0,(1+R)^{-j}]^n}\|_{L^p}$. We start with the following estimate:

Lemma 1 *For all $x \in \mathbb{R}^n$, we have*

$$M_\alpha \chi_{E_j}(x) \gtrsim \sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha} \left(\frac{2}{R+1}\right)^{kn} \chi_{E_{j-k}}(x). \tag{3}$$

Proof of of Lemma 1 We may assume that $x \in E_0$: otherwise the right-hand side is 0.

Denote by j_0 the largest number $j \in \{0, 1, \dots, j\}$ such that $x \in E_j$. Suppose that $j_0 < j$ for the time being. Since $x \in E_{j_0} \setminus E_{j_0+1}$, we obtain

$$\sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha} \left(\frac{2}{R+1}\right)^{kn} \chi_{E_{j-k}}(x) = \sum_{k=j-j_0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha} \left(\frac{2}{R+1}\right)^{kn}.$$

Since

$$2(R+1)^{\alpha-1} = (R+1)^{\alpha-\frac{1}{p}} < 1,$$

we have

$$\begin{aligned}\sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha} \left(\frac{2}{R+1}\right)^{kn} \chi_{E_{j-k}}(x) &= \sum_{k=j-j_0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha} \left(\frac{2}{R+1}\right)^{kn} \\ &\sim \left(\frac{1}{R+1}\right)^{j_0\alpha} \left(\frac{2}{R+1}\right)^{(j-j_0)n}.\end{aligned}$$

Meanwhile, if we denote by Z the connected component of E_{j_0} to which x belongs, then Z is a cube of side-length $(1+R)^{-j_0}$ and hence

$$M_\alpha \chi_{E_j}(x) \geq (1+R)^{-j_0\alpha} \frac{|E_j \cap Z|}{|Z|} = 2^{(j-j_0)n}(1+R)^{-jn}(1+R)^{j_0(n-\alpha)}.$$

Thus, (3) is proved for the case where $j_0 < j$.

If $j_0 = j$, we can go through the similar argument and we omit the details. □

We refer back to the proof of (ii). Our proof consists of two inequalities:

$$\frac{\|M_\alpha \chi_{E_j}\|_{\mathcal{M}_t^s}}{\|\chi_{E_j}\|_{\mathcal{M}_1^p}} \gtrsim j^{\frac{1}{t}} = j^{\frac{n-p\alpha}{n}} = j^{\theta_0}. \tag{4}$$

$$\|\chi_{E_j}\|_{\mathcal{M}^p_{\mathrm{L\log L}}} \sim j\|\chi_{E_j}\|_{\mathcal{M}_1^p}. \tag{5}$$

Once (4) and (5) are proven, then we will have

$$\begin{aligned} j^{\theta_0}\|\chi_{E_j}\|_{\mathcal{M}_1^p} &\lesssim \|M_\alpha \chi_{E_j}\|_{\mathcal{M}_t^s} \\ &\lesssim \|\chi_{E_j}\|_{(\mathcal{M}_1^p)^{1-\theta}(\mathcal{M}^p_{\mathrm{L\log L}})^\theta}\end{aligned}$$

$$\leq (\|\chi_{E_j}\|_{\mathcal{M}^p_1})^{1-\theta}(\|\chi_{E_j}\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}})^{\theta}$$
$$\sim j^{\theta}\|\chi_{E_j}\|_{\mathcal{M}^p_1},$$

which forces $j^{\theta_0} \lesssim j^{\theta}$ for all $j \in \mathbb{N}$, or equivalently $\theta_0 \leq \theta$.

Let us prove (4).

Once we fix $x \in \mathbb{R}^n$, the right-hand side is a geometric series. Thus,

$$M_\alpha \chi_{E_j}(x)^t \gtrsim \sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha t} \left(\frac{2}{R+1}\right)^{knt} \chi_{E_{j-k}}(x).$$

We integrate this inequality over $[0, 1]^n$ and use the definition of $R > 1$ to have

$$\begin{aligned}\int_{[0,1]^n} M_\alpha \chi_{E_j}(x)^t dx &\gtrsim \sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha t} \left(\frac{2}{R+1}\right)^{knt} \left(\frac{2}{R+1}\right)^{(j-k)n} \\ &= \sum_{k=0}^{j} \left(\frac{1}{R+1}\right)^{(j-k)\alpha t} (R+1)^{-\frac{knt}{p}} (R+1)^{-\frac{(j-k)n}{p}}.\end{aligned}$$

Arithmetic shows

$$\left(\frac{1}{R+1}\right)^{-k\alpha t} (R+1)^{-\frac{knt}{p}} (R+1)^{\frac{kn}{p}} = (R+1)^{\frac{kn}{p}-\frac{knt}{p}+k\alpha t} = 1.$$

Thus,

$$\left(\int_{[0,1]^n} M_\alpha \chi_{E_j}(x)^t dx\right)^{\frac{1}{t}} \gtrsim j^{\frac{1}{t}} \left(\frac{1}{R+1}\right)^{j\alpha} (R+1)^{-\frac{jn}{pt}} = j^{\frac{1}{t}} (R+1)^{-\frac{jn}{p}}.$$

Thus, (4) follows.

It remains to prove (5). Since the invese φ^{-1} of a convex function $\varphi(r) = r\log(3+r)$, $r > 0$, satisfies

$$\varphi^{-1}(r) \sim \frac{r}{\log(3+r)} \quad (r > 0),$$

we calculate

$$\begin{aligned}&(1+R)^{-\frac{kn}{p}} \|\chi_{E_j}\|_{\mathrm{L}\log\mathrm{L};[0,(1+R)^{-k}]^n} \\ &= (1+R)^{-\frac{kn}{p}} \inf\left\{\lambda > 0 : \frac{2^{(j-k)n}|[0,(1+R)^{-j}]^n|}{\lambda|[0,(1+R)^{-k}]^n|} \log\left(3+\frac{1}{\lambda}\right) \leq 1\right\}\end{aligned}$$

$$= (1+R)^{-\frac{kn}{p}} \inf\left\{\lambda > 0 \,:\, \varphi\left(\frac{1}{\lambda}\right) \le \frac{(1+R)^{(j-k)n}}{2^{(j-k)n}}\right\}$$

$$\sim (1+R)^{-\frac{kn}{p}} \frac{2^{(j-k)n}}{(1+R)^{(j-k)n}} \log\left(3 + \frac{(1+R)^{(j-k)n}}{2^{(j-k)n}}\right)$$

$$= \frac{2^{jn}}{(1+R)^{jn}} \log\left(3 + \frac{(1+R)^{(j-k)n}}{2^{(j-k)n}}\right).$$

Thus, since $R > 1$,

$$\sup_{k=0,1,2,\ldots,j} (1+R)^{-\frac{kn}{p}} \|\chi_{E_j}\|_{\mathrm{L}\log\mathrm{L};[0,(1+R)^{-k}]^n}$$

$$= \|\chi_{E_j}\|_{\mathrm{L}\log\mathrm{L};[0,1]^n}$$

$$\sim \frac{2^{jn}}{(1+R)^{jn}} \log\left(3 + (2^{-1} + 2^{-1}R)^{jn}\right),$$

which implies

$$\|\chi_{E_j}\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} \sim j \frac{2^{jn}}{(1+R)^{jn}}.$$

Finally, recall that

$$\|\chi_{E_j}\|_{\mathcal{M}^p_1} \sim \frac{2^{jn}}{(1+R)^{jn}}.$$

Thus, we obtain (5).

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Inversion of the Weighted Spherical Mean

Elina Shishkina

Abstract The paper contains the inversion formula for the weighted spherical mean. The interest to reconstruction a function by its integral by sphere grews tremendously in the last six decades, stimulated by the spectrum of new problems and methods of image reconstruction. We consider a generalization of the classical spherical mean and its inverse in the case when generalized translation acts to function instead of regular. As a particular case this problem includes action of spherical means on radially symmetric functions.

Keywords Inverse problem · Weighted spherical mean · Riesz B-potential · Bessel operator

1 Introduction

Reconstruction of a function from a known subset of its spherical means is widely developed in pure and applied mathematics. Its connection with photoacoustic images is as follows. Let the speed of sound propagation in the medium be a constant value. Then the pressure at a certain point in time is expressed in terms of the spherical mean pressure and its time derivative at some previous point in time [1]. Therefore, this imaging technique requires the inversion of spherical means.

The problem of reconstruction a function f supported in a ball $B \in \mathbb{R}^n$, if the spherical means of f are known over all geodesic spheres centered on the boundary ∂B was solved using different approach in [1–8]. It is remarkable that reconstruction formulas in [1–8] are different for even and odd dimension of Euclidean space n.

Classical spherical mean has the form

$$M(x,r,u)=\frac{1}{|S_n(1)|}\int\limits_{S_n(1)} u(x+\beta r)dS, \qquad x=(x_1,\dots,x_n), \tag{1}$$

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where $S_n(1)$ is unit sphere centered at the origin, β is a coordinate of the sphere $S_n(1)$.

Operator (1) intertwines Laplace operator and one-dimensional Bessel operator with index $n-1$. Great interest among various researchers is a generalization of the spherical mean (1). So, in paper [9] was considered a spherical mean in space with negative curvature, in [10] and [11] was studied a generalization of the spherical mean generated by the Dunkl transmutation operator. In this paper we consider the spherical weighted mean (see (7)), which is the transmutation operator intertwining the multidimensional operator

$$(\Delta_\gamma)_x=\sum_{i=1}^{n}(B_{\gamma_i})_{x_i}, \qquad (B_{\gamma_i})_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad \gamma_i\geq 0, \qquad i=1,\ldots,n \tag{2}$$

and one-dimensional Bessel operator with index $n+|\gamma|-1$ of the form

$$(B_{n+|\gamma|-1})_t=\frac{d^2}{dt^2}+\frac{n+|\gamma|-1}{t}\frac{d}{dt}, \qquad t>0, \qquad |\gamma|=\gamma_1+\ldots+\gamma_n. \tag{3}$$

Such spherical mean is closely related to B-ultra-hyperbolic equation of the form (see [12, 13])

$$\sum_{j=1}^{n}(B_{\gamma_i})_{x_j}u=\sum_{j=1}^{n}(B_{\gamma_i})_{y_j}u, \qquad u=u(x_1,\ldots,x_n,y_1,\ldots,y_n). \tag{4}$$

Not so long ago, various new methods for solving inverse problems with the Bessel operator appeared. The inverse problem involving recovery of initial temperature from the information of final temperature profile in the case of heat equation with Bessel operator was studied in [14]. The solution of the inverse spectral problem for Bessel-type differential operators on noncompact star-type graphs under generalized Neumann-type boundary conditions was considered in [15]. V. V. Kravchenko with coauthors introduced a new method to solution of the inverse Sturm-Liouville problem (see [16–19]) which can be adapted to inverse problem with Bessel operator. Their idea is based on the observation that the potential can be recovered from the very first coefficient of the Fourier–Legendre series, and to find this coefficient a system of linear algebraic equations can be obtained directly from the Gel'fand-Levitan equation.

The paper contains an inversion formula for the weighed spherical mean (7) using the properties of the mixed Riesz hyperbolic B-potential (13).

2 Basic Definitions

In this section we give a summary of the basic notations, terminology and results which will be used in this article.

Suppose that $\mathbb{R}^{n+1}$ is the $n+1$-dimensional Euclidean space,

$$\mathbb{R}^{n+1}_{+}=\{(t,x)=(t,x_1,\dots,x_n)\in\mathbb{R}^{n+1},\quad x_1>0,\dots,x_n>0\},$$

$\gamma=(\gamma_1,\dots,\gamma_n)$ is a multiindex consisting of fixed real numbers $\gamma_i\geq 0$, $i=1,\dots,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$. Let Ω be a finite or infinite open set in $\mathbb{R}^{n+1}$ symmetric with respect to each hyperplane $x_i=0$, $i=1,\dots,n$, $\Omega_+=\Omega\cap\mathbb{R}^{n+1}_{+}$ and $\widetilde{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^{n+1}_{+}$ where

$$\overline{\mathbb{R}}^{n+1}_{+}=\{(t,x)=(t,x_1,\dots,x_n)\in\mathbb{R}^{n+1},x_1\geq 0,\dots,x_n\geq 0\}.$$

We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\widetilde{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all existing derivatives of these functions with respect to x_i for any $i=1,\dots,n$ are continuous up to $x_i=0$ and all existing derivative with respect to t are continuous for $t\in\mathbb{R}$. Class $C^m_{ev}(\widetilde{\Omega}_+)$ consists of all functions from $C^m(\widetilde{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all nonnegative integer $k\leq\frac{m-1}{2}$ and for $i=1,\dots,n$ (see [20] and [21, p. 21]). In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^{n+1}_{+})$ by C^m_{ev}. We set

$$C^\infty_{ev}(\widetilde{\Omega}_+)=\bigcap C^m_{ev}(\widetilde{\Omega}_+)$$

with intersection taken for all finite m. Let $C^\infty_{ev}(\overline{\mathbb{R}}^{n+1}_{+})=C^\infty_{ev}$. Assuming that $\overset{\circ}{C}{}^\infty_{ev}(\widetilde{\Omega}_+)$ is the space of all functions $f\in C^\infty_{ev}(\widetilde{\Omega}_+)$ with a compact support. We will use the notation $\overset{\circ}{C}{}^\infty_{ev}(\widetilde{\Omega}_+)=\mathcal{D}_+(\widetilde{\Omega}_+)$.

Let $\mathcal{L}^\gamma_p(\Omega_+)$, $1\leq p<\infty$ be the space of all measurable in Ω_+ functions such that

$$\int\limits_{\Omega_+}|f(t,x)|^p x^\gamma\,dt\,dx<\infty,$$

where and further

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p \geq 1$, the $\mathcal{L}_p^\gamma(\Omega_+)$–norm of f is defined by

$$||f||_{\mathcal{L}_p^\gamma(\Omega_+)} = \left(\int\limits_{\Omega_+} |f(t,x)|^p x^\gamma dt dx \right)^{1/p}.$$

Let $\mathcal{L}_p^\gamma = \mathcal{L}_p^\gamma(\mathbb{R}_{n+1}^+)$.

We will use the generalized convolution product defined by the formula

$$(f * g)_\gamma(x,t) = \int\limits_{\mathbb{R}_+^{n+1}} f(\tau, y)({}^\gamma\mathbb{T}_x^y g)(t-\tau, x) y^\gamma d\tau dy, \tag{5}$$

where ${}^\gamma\mathbb{T}_x^y$ is multidimensional generalized translation

$$({}^\gamma\mathbb{T}_x^y f)(t,x) = ({}^{\gamma_1}T_{x_1}^{y_1} \ldots {}^{\gamma_n}T_{x_n}^{y_n} f)(t,x). \tag{6}$$

Each of one-dimensional generalized translations ${}^{\gamma_i}T_{x_i}^{y_i}$ is defined for $i=1,\ldots,n$ by the next formula (see [22, p. 122, formula (5.19)])

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(t,x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int\limits_0^\pi \sin^{\gamma_i-1}\varphi_i \times$$

$$\times f(t, x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \ldots, x_n)\, d\varphi_i,$$

$\gamma_i > 0$, $i = 1, \ldots, n$ and for $\gamma_i = 0$ generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ is

$${}^0T_{x_i}^{y_i} = \frac{f(x+y) - f(x-y)}{2}.$$

As the space of basic functions we will use the subspace of rapidly decreasing functions:

$$S_{ev}(\mathbb{R}_+^{n+1}) = \left\{ f \in C_{ev}^\infty : \sup_{(t,x)\in R_+^{n+1}} \left| t^{\alpha_0} x^\alpha D^\beta f(t,x) \right| < \infty \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$ are arbitrary integer nonnegative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $D^\beta = D_t^{\beta_0} D_{x_1}^{\beta_1} \ldots D_{x_n}^{\beta_n}$, $D_t = \frac{\partial}{\partial t}$, $D_{x_j} = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$.

The multidimensional Fourier–Bessel transform of a function $f\in\mathcal{L}_1^\gamma(\mathbb{R}_+^{n+1})$ is

$$\mathcal{F}_\gamma[f](\tau,\xi)=\widehat{f}(\tau,\xi)=\int\limits_{\mathbb{R}_+^{n+1}} f(t,x)\,e^{-it\tau}\,\mathbf{j}_\gamma(x;\xi)x^\gamma dtdx,$$

where

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i),\qquad \gamma_1\geq 0,\ldots,\gamma_n\geq 0.$$

Now we introduce a weighted spherical mean. When constructing a weighted spherical mean, instead of the usual shift, a multidimensional generalized translation (6) is used.

Weighted spherical mean (see [12, 13, 23]) of function $f(x)$, $x\in\overline{\mathbb{R}}_+^n$ for $n\geq 2$ is

$$(M_t^\gamma f)(x)=(M_t^\gamma)_x[f(x)]=\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)} {}^\gamma\mathbb{T}_x^{t\theta}f(x)\theta^\gamma dS, \tag{7}$$

where $\theta^\gamma=\prod\limits_{i=1}^n\theta_i^{\gamma_i}$, $S_1^+(n)=\{\theta:|\theta|=1,\theta\in\mathbb{R}_+^n\}$ is a part of a sphere in $\mathbb{R}_+^n$, and $|S_1^+(n)|_\gamma$ is given by

$$|S_1^+(n)|_\gamma=\int\limits_{S_1^+(n)}x^\gamma dS=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{8}$$

For $n=1$ let $M_t^\gamma[f(x)]={}^\gamma T_x^t f(x)$.

Let $\nu>0$. One-dimensional Poisson operator is defined for integrable function f by the equality

$$\mathcal{P}_x^\nu f(x)=\frac{2C(\nu)}{x^{\nu-1}}\int\limits_0^x\left(x^2-t^2\right)^{\frac{\nu}{2}-1}f(t)\,dt,\qquad C(\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}, \tag{9}$$

or

$$\mathcal{P}_x^\nu f(x)=C(\nu)\int\limits_0^\pi f(x\cos\varphi)\sin^{\nu-1}\varphi\,d\varphi,. \tag{10}$$

The constant $C(\nu)$ is chosen so that $\mathcal{P}_x^\nu[1]=1$. For $\nu=0$ one-dimensional Poisson operator turns into an identical operator: $\mathcal{P}_x^0=I$.

3 Mixed Hyperbolic Riesz B-potential and Its Inversion

The given supplementary information in this section will be used in the 4th section where the main results are presented.

Mixed hyperbolic Riesz B–potential was studied in [23–25]. Here we provide a definition of the mixed hyperbolic Riesz B–potential and give some separation results that we will use in the next section.

Let $|x|=\sqrt{x_1^2+\ldots+x_n^2}$. First for $(t,x)\in\mathbb{R}_+^{n+1}$, $\lambda\in C$ we define function s^λ by the formula

$$s^\lambda(t,x)=\begin{cases}\frac{(t^2-|x|^2)^\lambda}{N(\alpha,\gamma,n)}, & \text{when } t^2\geq|x|^2 \text{ and } t\geq 0;\\ 0, & \text{when } t^2<|x|^2 \text{ or } t<0,\end{cases}\tag{11}$$

where

$$N(\alpha,\gamma,n)=\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right).\tag{12}$$

Regular weighted distribution corresponding to (11) we will denote by s_+^λ.

We introduce the *mixed hyperbolic Riesz B–potential* $I_{s,\gamma}^\alpha$ *of order* α as a generalized convolution product (5) with a weighted distribution $s_+^{\frac{\alpha-n-|\gamma|-1}{2}}$ and $f\in S_{ev}$ (see [25]):

$$(I_{s,\gamma}^\alpha f)(t,x)=\left(s_+^{\frac{\alpha-n-|\gamma|-1}{2}}*f\right)_\gamma(t,x).\tag{13}$$

The precise definition of the constant $N(\alpha,\gamma,n)$ allows to obtain the semigroup property or index low of the potential (13).

We can rewrite formula (13) as

$$(I_{s,\gamma}^\alpha f)(t,x)=\int\limits_{\mathbb{R}_+^{n+1}} s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma\mathbb{T}_x^y)f(t-\tau,x)y^\gamma d\tau dy.\tag{14}$$

Integral (14) converges absolutely for $n+|\gamma|-1<\alpha$ for integrable with weight y^γ on the part of the cone $\{|y|<\tau\}^+=\{y\in\mathbb{R}_+^n:|y|<\tau\}$, $0<\tau<t$ function $f(\tau,y)$.

For $0 \leq \alpha \leq n+|\gamma|-1$

$$(I^\alpha_{s,\gamma} f)(t,x) = \left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^q (I^{\alpha+2q}_{s,\gamma} f)(t,x)$$

where $q = \left[\frac{n+|\gamma|-\alpha+1}{2}\right]$.

In the case when $f(t,x) = h(t)F(x)$ we get

$$(I^\alpha_{s,\gamma} hF)(t,x) =$$

$$= \frac{1}{N(\alpha,\gamma,n)} \int\limits_0^\infty h(t-\tau)d\tau \int\limits_{|y|<\tau} (\tau^2 - |y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}} ({}^\gamma \mathbb{T}^y_x) F(x) y^\gamma dy. \qquad (15)$$

Theorem 1 ([24]) *Let* $n+|\gamma|-1<\alpha<n+|\gamma|+1$, $1 \leq p<\frac{n+|\gamma|+1}{\alpha}$. *For the next estimate*

$$||I^\alpha_{s,\gamma} f||_{q,\gamma} \leq M||f||_{p,\gamma}, \qquad f \in S_{ev} \qquad (16)$$

to be valid it is necessary and sufficient that $q = \frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$. *Constant* M *does not depend on* f.

Remark By virtue of (16) there is unique extension of $I^\alpha_{s,\gamma}$ to all $\mathcal{L}^\gamma_p$, $1 < p < \frac{n+|\gamma|+1}{\alpha}$ preserving boundedness when $n+|\gamma|-1 < \alpha < n+|\gamma|$. It follows that this extension is introduced by the integral (14) from its absolute convergence.

Theorem 2 ([24]) *For* $f \in S_{ev}$ *the Fourier–Hankel transform of mixed hyperbolic Riesz potential* $I^\alpha_{s,\gamma} f$ *is*

$$\mathcal{F}_\gamma[I^\alpha_{s,\gamma} f](\tau,\xi) = \mathcal{Q}\left|\tau^2 - |\xi|^2\right|^{-\frac{\alpha}{2}} \cdot \mathcal{F}_\gamma[f(t,x)](\tau,\xi), \qquad (17)$$

where

$$\mathcal{Q} = \begin{cases} 1, & |\xi|^2 \geq \tau^2; \\ e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2 < \tau^2, \tau \geq 0; \\ e^{\frac{\alpha\pi}{2}i}, & |\xi|^2 < \tau^2, \tau < 0. \end{cases}$$

For the inversion of the potential (13) approach based on the idea of approximative inverse operators (see [26]) was used. This method gives an inverse operator as a limit of regularized operators. Namely, taking into account the formula (17) we will construct inverse operator for the potential (13) in the form

$$(I^\alpha_{s,\gamma})^{-1} f = \lim_{\varepsilon\to 0} \left(\mathcal{F}^{-1}_\gamma(\mathcal{Q}|\tau^2 - |\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau|-\varepsilon|\xi|}) * f\right)_\gamma,$$

where the limit is understood in the norm $\mathcal{L}_p^\gamma$ or almost everywhere.

Let

$$g_{\alpha,\gamma,\varepsilon}(t,x) = \mathcal{F}_\gamma^{-1}(\mathcal{Q}|\tau^2 - |\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau|-\varepsilon|\xi|})(t,x),$$

then

$$(I_{s,\gamma}^\alpha)^{-1} f = \lim_{\varepsilon\to 0} (g_{\alpha,\gamma,\varepsilon} * f)_\gamma. \tag{18}$$

Theorem 3 ([24]) *The function $g_{\alpha,\gamma,\varepsilon}(t,x)$ belongs to the space $\mathcal{L}_p^\gamma$, $1 < p < \infty$ with additional restriction $\frac{2(n+|\gamma|)-1}{2(n+|\gamma|)-2} < p$ for $n+|\gamma|-1 < \alpha < n+|\gamma|$ when $n+|\gamma|+1$ is odd.*

Theorem 4 ([24]) *Let $n+|\gamma|-1 < \alpha < n+1+|\gamma|$, $1 < p < \frac{n+1+|\gamma|}{\alpha}$ with the additional restriction $p < \frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+1+|\gamma|+2\alpha(n+|\gamma|)}$ when $n+|\gamma|-1 < \alpha < n+|\gamma|$ and n is odd. Then*

$$((I_{s,\gamma}^\alpha)^{-1} I_{s,\gamma}^\alpha f)(t,x) = f(t,x), \qquad f(t,x) \in \mathcal{L}_p^\gamma,$$

where $(I_{s,\gamma}^\alpha)^{-1} f = \lim\limits_{\varepsilon\to 0} (I_{s,\gamma}^\alpha)_\varepsilon^{-1} f$.

It is easy to see that if $\alpha = 2m$, $m \in \mathbb{N}$ the inverse to $I_{s,\gamma}^{2m}$ is $\left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m$:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m (I_{s,\gamma}^{2m} f)(t,x) = f(t,x), \qquad f(t,x) \in \mathcal{L}_p^\gamma.$$

The inverse Fourier–Hankel transform of $\mathcal{Q}|\tau^2 - |\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau|-\varepsilon|\xi|}$ can be represented in the form

$$g_{\alpha,\gamma,\varepsilon}(t,x) = \mathcal{C}(n,\gamma,\alpha) \int_0^\infty |1-r^2|^{\frac{\alpha}{2}} r^{n+|\gamma|-1} \times$$

$$\times \left[e^{-\frac{\alpha\pi i}{2}\theta(1-r)} \mathcal{F}_{n,\gamma,\alpha,\varepsilon}^{+}(r,x,t) + e^{\frac{\alpha\pi i}{2}\theta(1-r)} \mathcal{F}_{n,\gamma,\alpha,\varepsilon}^{-}(r,x,t) \right] dr, \tag{19}$$

where

$$\mathcal{C}(n,\gamma,\alpha) = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}} \frac{\Gamma(n+|\gamma|+1+\alpha)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}$$

$$\mathscr{F}^{+}_{n,\gamma,\alpha,\varepsilon}(r,x,t)=\frac{{}_2F_1\left(\frac{n+|\gamma|+1+\alpha}{2},\frac{n+|\gamma|+2+\alpha}{2};\frac{n+|\gamma|}{2};-\frac{|x|^2r^2}{(\varepsilon+\varepsilon r+it)^2}\right)}{(\varepsilon+\varepsilon r+it)^{n+|\gamma|+1+\alpha}},$$

$$\mathscr{F}^{-}_{n,\gamma,\alpha,\varepsilon}(r,x,t)=\frac{{}_2F_1\left(\frac{n+|\gamma|+1+\alpha}{2},\frac{n+|\gamma|+2+\alpha}{2};\frac{n+|\gamma|}{2};-\frac{|x|^2r^2}{(\varepsilon+\varepsilon r-it)^2}\right)}{(\varepsilon+\varepsilon r-it)^{n+|\gamma|+1+\alpha}}.$$

4 Inverse Problem

In this section we consider the recovery of a function f from a knowledge of its weighted spherical mean $M^{\gamma}_{\rho} f$.

Let $f = f(x) \in C^2(\mathbb{R}^n_+)$, such that $\left.\frac{\partial f}{\partial x_i}\right|_{x_i=0} = 0$, $i = 1,\dots,n$. The weighted spherical mean $M^{\gamma}_t f$ is the transmutation operator intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_t$ (see [27]):

$$(B_{n+|\gamma|-1})_t(M^{\gamma}_t f)(x)=(M^{\gamma}_t(\Delta_\gamma)_x f)(x). \tag{20}$$

Let consider the integral operator

$$(\mathscr{M}^{\gamma,k}_t f)(x)=\frac{1}{|S^+_1(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}({}^\gamma\mathbf{T}^y_x f(x))(t^2-|y|^2)_+^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy. \tag{21}$$

Operator $t^{1-k}\mathscr{M}^{\gamma,k}_t$ intertwines $(\Delta_\gamma)_x$ and $(B_k)_t$ when $k>n+|\gamma|-1$:

$$(B_k)_t(t^{1-k}\mathscr{M}^{\gamma,k}_t f)(x)=(t^{1-k}\mathscr{M}^{\gamma,k}_t(\Delta_\gamma)_x f)(x). \tag{22}$$

We'll tend to use spherical coordinates in (21) when $k>n+|\gamma|-1$, then, using (9) we can write

$$(\mathscr{M}^{\gamma,k}_t f)(x)=$$

$$=\frac{1}{|S^+_1(n)|_\gamma}\int\limits_{\{|y|<t\}^+}({}^\gamma\mathbf{T}^y_x f(x))(t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy=\{y=\rho\theta\}=$$

$$=\frac{1}{|S^+_1(n)|_\gamma}\int\limits_0^t(t^2-\rho^2)^{\frac{k-n-|\gamma|-1}{2}}\rho^{n+|\gamma|-1}d\rho\int\limits_{S^+_1(n)}({}^\gamma\mathbf{T}^{\rho\theta}_x f(x))\theta^\gamma dS=$$

$$= \int_0^t (t^2-\rho^2)^{\frac{k-n-|\gamma|-1}{2}} \rho^{n+|\gamma|-1} (M_\rho^\gamma f)(x) d\rho =$$

$$= \frac{t^{k-n-|\gamma|}}{2C(k-n-|\gamma|+1)} \left(\mathcal{P}_t^{k-n-|\gamma|+1} t^{n+|\gamma|-1} (M_\rho^\gamma f)(x)\right)(t).$$

Now let find the inverse operator for $\mathscr{M}^{\gamma,k}$. Let multiply (21) by $h(t-\tau)$ and integrate by τ from 0 to ∞. The function $h(t)$ should be chosen such that the function $h(t-\tau)(\mathscr{M}_\tau^{\gamma,k} f)(x)$ is an integrable by τ by the interval from 0 to ∞.

We obtain

$$\int_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,k} f)(x) d\tau =$$

$$= \frac{1}{|S_1^+(n)|_\gamma} \int_0^\infty h(t-\tau) d\tau \int_{\{|y|<\tau\}^+} (\tau^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} ({}^\gamma \mathbf{T}_x^y f)(x) y^\gamma dy.$$

Taking into account (15) we get

$$\frac{|S_1^+(n)|_\gamma}{N(k,\gamma,n)} \int_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,k} f)(x) d\tau = (I_{s,\gamma}^k hf)(t,x),$$

where $N(k,\gamma,n)$ defined by (12) and $I_{s,\gamma}^k$ is the mixed hyperbolic Riesz B–potential (14) of order $k>0$ acting to function $h(t)f(x)$. Therefore, using theorem 4, we obtain

$$h(t)f(x) = \frac{|S_1^+(n)|_\gamma}{N(k,\gamma,n)} \left((I_{s,\gamma}^k)_{\phi,y}^{-1} \int_0^\infty h(\phi-\tau)(\mathscr{M}_\tau^{\gamma,k} f)(y) d\tau \right)(t,x), \tag{23}$$

where $n+|\gamma|-1 < k < n+1+|\gamma|$ and $(I_{s,\gamma}^k)^{-1}$ is given by (18). So, in the inverse formula (23) we have an arbitrary parameter $k \in (n+|\gamma|-1, n+1+|\gamma|)$ and an arbitrary non-zero function h (such that the function $h(t-\tau)(\mathscr{M}_\tau^{\gamma,k} f)(x)$ is an integrable by τ) depending on one variable.

In order to find the inverse to weighed spherical mean the formula (23) can be simplified. We can take $k=2m>n+|\gamma|-1$, $m\in\mathbb{N}$. In this case

$$(I_{s,\gamma}^{2m})^{-1} = \left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m$$

and

$$h(t)f(x)=\frac{|S_1^+(n)|_\gamma}{N(2m,\gamma,n)}\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)^m\int\limits_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,2m}f)(x)d\tau. \tag{24}$$

So we obtain the main statement.

Theorem 5 *Let* $f=f(x)\in C^2(\mathbb{R}_+^n)$*, such that* $\left.\frac{\partial f}{\partial x_i}\right|_{x_i=0}=0$*,* $i=1,\ldots,n$ *and*

$$(\mathscr{M}_t^{\gamma,k}f)(x)=\frac{t^{k-n-|\gamma|}}{2C(k-n-|\gamma|+1)}\left(\mathcal{P}_t^{k-n-|\gamma|+1}t^{n+|\gamma|-1}(M_\rho^\gamma f)(x)\right)(t),$$

where $M_\rho^\gamma f$ *is the weighted spherical mean (7) of the function* f*, is* $\mathcal{P}_t^\nu$ *the one-dimensional Poisson operator (9),* $C(\nu)$ *is the constant defined in (9). Then the function* f *can be reconstructed by its weighted spherical mean by the formula*

$$h(t)f(x)=\frac{|S_1^+(n)|_\gamma}{N(2m,\gamma,n)}\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)^m\int\limits_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,2m}f)(x)d\tau,$$

where function $h(t)$ *is arbitrary such that the function* $h(t-\tau)(\mathscr{M}_\tau^{\gamma,k}f)(x)$ *is an integrable by* τ *by the interval from* 0 *to* ∞*,* $|S_1^+(n)|_\gamma$ *is given by (8),* $N(2m,\gamma,n)$ *is given by (12).*

Example Let $h(t)=e^t$, $(M_\rho^\gamma f)(x)=\mathbf{j}_\gamma(x,\xi)\ j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)$, where $\xi=(\xi_1,\ldots,\xi_n)$ some vector.

$$(\mathscr{M}_\tau^{\gamma,2m}f)(x)=\int\limits_0^\tau(\tau^2-\rho^2)^{\frac{2m-n-|\gamma|-1}{2}}\rho^{n+|\gamma|-1}(M_\rho^\gamma f)(x)d\rho=$$

$$=\mathbf{j}_\gamma(x,\xi)\int\limits_0^\tau(\tau^2-\rho^2)^{\frac{2m-n-|\gamma|-1}{2}}\rho^{n+|\gamma|-1}\ j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)d\rho=$$

$$=2^{\frac{n+|\gamma|}{2}-1}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)|\xi|^{1-\frac{n+|\gamma|}{2}}\mathbf{j}_\gamma(x,\xi)\times$$

$$\times\int\limits_0^\tau\left(\tau^2-\rho^2\right)^{\frac{2m-n-|\gamma|-1}{2}}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\,d\rho.$$

Using formula 2.12.4.6 from [28] we obtain

$$(\mathscr{M}_\tau^{\gamma,2m} f)(x) =$$

$$= 2^{\frac{n+|\gamma|}{2}-1}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)|\xi|^{1-\frac{n+|\gamma|}{2}}\mathbf{j}_\gamma(x,\xi)\frac{2^{\frac{2m-n-|\gamma|-1}{2}}\tau^{m-\frac{1}{2}}}{|\xi|^{\frac{2m-n-|\gamma|+1}{2}}}\times$$

$$\times\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)J_{m-\frac{1}{2}}(|\xi|\tau) =$$

$$= \frac{2^{m-\frac{3}{2}}\tau^{m-\frac{1}{2}}}{|\xi|^{m-\frac{1}{2}}}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)\mathbf{j}_\gamma(x,\xi)J_{m-\frac{1}{2}}(|\xi|\tau).$$

Taking into account that $h(t)=e^t$ we obtain

$$\int\limits_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,2m} f)(x)d\tau =$$

$$= e^t\,\frac{2^{m-\frac{3}{2}}}{|\xi|^{m-\frac{1}{2}}}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)\times$$

$$\times\mathbf{j}_\gamma(x,\xi)\int\limits_0^\infty e^{-\tau}\tau^{m-\frac{1}{2}}J_{m-\frac{1}{2}}(|\xi|\tau)d\tau$$

$$= e^t\,\frac{2^{m-\frac{3}{2}}}{|\xi|^{m-\frac{1}{2}}}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)\times$$

$$\times\mathbf{j}_\gamma(x,\xi)\frac{2^{m-\frac{1}{2}}|\xi|^{m-\frac{1}{2}}\left(|\xi|^2+1\right)^{-m}\Gamma(m)}{\sqrt{\pi}} =$$

$$= \frac{\Gamma(m)\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)}{2^{2-2m}\sqrt{\pi}\left(1+|\xi|^2\right)^m}\,e^t\mathbf{j}_\gamma(x,\xi).$$

Let calculate the constant

$$\frac{|S_1^+(n)|_\gamma}{N(2m,\gamma,n)}=$$

$$=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\frac{\sqrt{\pi}}{2^{2m-n-1}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)\Gamma(m)}=$$

$$=\frac{2^{2-2m}\sqrt{\pi}}{\Gamma(m)\,\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2m-n-|\gamma|+1}{2}\right)}.$$

Therefore

$$\frac{|S_1^+(n)|_\gamma}{N(2m,\gamma,n)}\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)^m\int\limits_0^\infty h(t-\tau)(\mathscr{M}_\tau^{\gamma,2m}f)(x)d\tau=$$

$$=\frac{1}{\left(1+|\xi|^2\right)^m}\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)^m e^t\mathbf{j}_\gamma(x,\xi)=e^t\mathbf{j}_\gamma(x,\xi),\qquad(25)$$

that gives $f(x)=\mathbf{j}_\gamma(x,\xi)$. In the formula (25) we used the fact that $\Delta_\gamma\mathbf{j}_\gamma(x;\xi)=-|\xi|\mathbf{j}_\gamma(x;\xi)$ [12].

This result confirmed by the formula (see [23])

$$(M_\rho^\gamma)_x\mathbf{j}_\gamma(x,\xi)=\mathbf{j}_\gamma(x,\xi)\;j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|).$$

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On Subadditivity for Subspace-Valued Maps

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Abstract Let X be a linear space. A function F assigning to each element of a group G a subspace $F(g)$ of X is called subadditive if $F(gh) \subset F(g) + F(h)$ for any $g, h \in G$. Assuming that the dimension of each image $F(g)$ does not exceed a common constant we are interested to show that all $F(g)$ are contained in a finite-dimensional subspace X_0 of X. It was proved in Shulman (J Funct Anal 263(5):1468–1484, 2012) that this is possible if X is a Banach Γ-space, for some group Γ, all subspace $F(g)$ are Γ-invariant and the action of Γ satisfies some additional conditions (uniform boundedness and finite multiplicity). Here we considerably improve estimations of $\dim X_0$ presented in the paper mentioned above, and show that the results extend to the case when F is "almost" subadditive or all $F(g)$ are "almost" invariant.

Keywords Set-valued functions on groups · Subadditivity · Stability problems · Representations of topological groups

1 Introduction and Preliminary Results

It is known [1, 2] that matrix elements of finite-dimensional representations of groups can be characterized as functions admitting an addition theorem of the form

$$f(gh) = \sum_{i=1}^{n} u_i(g)v_i(h) \tag{1.1}$$

This equality can be regarded as a functional equation with $2n + 1$ unknown functions $f, u_1, \ldots, u_n, v_1, \ldots, v_n$. Such equations (the Levi-Civita functional

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equations) was studied in a number of works starting with the classical papers of Stéphanos [3], Levi-Cività [4], and Stäkel [5].

In a series of works [6–10] there were studied various related functional equations in several variables, for example

$$f(x+y+z) = 3f(x) + 3f(y) + 3f(z) - f(x-y) - f(y-z) - f(z-x),$$

$$f(x_1+x_2+x_3+x_4) = \sum_{i\neq j} f(x_i+x_j) - 2\sum_{i=1}^{4} f(x_i),$$

$$nC_{n-2}^{k-2} f(\frac{x_1+x_2+\cdots+x_n}{n}) = k \sum_{1\le i_1<\ldots<i_k\le n} f(\frac{x_{i_1}+\cdots+x_{i_k}}{k}) - C_{n-2}^{k-1}\sum_{i=1}^{n} f(x_i).$$

In a general form all of these equations can be written as follows:

$$f(g_1g_2\cdots g_n) = \sum_E \sum_{j=1}^{N_E} u_j^E v_j^E \tag{1.2}$$

(see [11–14]). Here E runs through all proper non-empty subsets of $\{1, 2, \ldots, n\}$, $N_E \in \mathbb{N}$ and for each E, the functions u_j^E only depend on variables g_i with $i \in E$, while the v_j^E only depend on g_i with $i \notin E$.

The work [14] gives complete solution of the Eq. (1.2) for commutative unital semigroups. The case of approximately finitely generated semigroups was considered in [12]. It was shown in [13] that bounded solutions of (1.2) on arbitrary groups are, as for (1.1), the matrix functions, while in general (even for commutative G) this is not the case. So the group-theoretic nature of arbitrary solutions of (1.2) is still not completely understood.

It was shown in [12, 13] that solutions of (1.2) are related to locally invariant finite-dimensional subspaces in G-spaces, that is such subspaces L that gL is close to L, for each $g \in G$ (precise definitions will be given in Sect. 5). Furthermore it was proved in [13] that the study of locally invariant subspaces, at least for uniformly bounded representations, can be based on the theory of subspace-valued subadditive maps on G. In this section we present a short introduction to this theory and list some known results. Then in Sect. 2 we will establish a stronger version of the result on subadditive subspace-valued functions proved in [13, Theorem 4.2]. In the next two sections we study stability of this result: in Sect. 3 the subadditivity condition is perturbed while in Sect. 4 the restrictions on image subspaces are weakened. The last Sect. 5 is devoted to the study of locally invariant subspaces; it will be shown in Theorem 5.1 that under some assumptions on the representation, every such a subspace is contained in a finite-dimensional invariant subspace. We conclude the work by a corollary of Theorem 5.1 that has direct applications to the functional equation (1.2); we will describe them in subsequent publications.

1.1 Set-valued Subadditive Functions

Given a semigroup G and an arbitrary set Ω we call a set-valued map $F : G \to 2^{\Omega}$ *subadditive* if

$$F(gh) \subset F(g) \cup F(h) \qquad \forall g, h \in G. \tag{1.3}$$

As an example let $G = \mathbb{N}$, the multiplicative semigroup of natural numbers, and Ω the set of all prime numbers. It is easy to see that the map F that corresponds to each $n \in \mathbb{N}$ the set of all primes dividing n, is subadditive.

In what follows, for any set M we denote by $|M|$ its "cardinality" in the primitive sense ($|M| = \infty$, if M is infinite).

In [13] the following problem was raised: *let*

$$|F(g)| \leq n \qquad \text{for each } g \in G. \tag{1.4}$$

Is it true that

$$N := |\cup_{g\in G} F(g)| < \infty \ ?$$

The affirmative answer was obtained for groups:

Theorem 1.1 ([13, Theorem 2.2]) *Let F be a subadditive function on a group G and let $\sup_{g\in G} |F(g)| = n$. Then*

$$|\cup_{g\in G} F(g)| \leq 4n - 1.$$

If, in addition, $F(g^{-1}) = F(g)$ then

$$|\cup_{g\in G} F(g)| \leq 2n - 1$$

and the estimate is sharp.

This result was extended to subadditive maps from a group G to the σ-algebra $\mathcal{M}(\Omega, \mu)$ of measurable subsets of a measure space (Ω, μ). For $A, B \in \mathcal{M}(\Omega, \mu)$, we write $A \subset_0^{\mu} B$ if $\mu(A \setminus B) = 0$. In this sense we understand the subadditivity condition (1.3) for a map $F : G \to \mathcal{M}(\Omega, \mu)$.

Theorem 1.2 ([13, Theorem 3.1]) *Let G be a group, $F : G \to \mathcal{M}(\Omega, \mu)$ a subadditive map:*

$$F(gh) \subset_0^{\mu} F(g) \cup F(h), \ \textit{for all } g, h \in G.$$

If $\mu(F(g)) \leq a$, for some $a > 0$ and all $g \in G$, then there is $A \in \mathcal{M}(\Omega, \mu)$ such that $\mu(A) \leq 4a$ and $F(g) \subset_0^{\mu} A$ for all $g \in G$.

1.2 Almost Subadditive Set-valued Maps

Now let us consider the situation when subadditivity holds only "approximately".

Let $\delta > 0$; for $A, B \in \mathcal{M}(\Omega, \mu)$, we write $A \subset_{\delta}^{\mu} B$ if $\mu(A \setminus B) < \delta$.

Let us call a map $F : G \to \mathcal{M}(\Omega, \mu)$ δ-*subadditive* if

$$F(gh) \subset_{\delta}^{\mu} F(g) \cup F(h) \qquad \text{for all } g, h \in G. \tag{1.5}$$

The question is whether for a δ-subadditive map a uniform boundedness of all $\mu(F(g))$ still implies finiteness of the measure of the full image $F(G)$. The answer will be positive if the "image" will be also approximate.

Theorem 1.3 ([13, Theorem 3.2]) *Let $F : G \to \mathcal{M}(\Omega, \mu)$ be a δ-subadditive function on a group G and $\mu(F(g)) < a$ for all $g \in G$. Assume that $\delta < a/3$. Then there is a set $K \subset \Omega$ such that $\mu(K) \leq 6a$ and $F(g) \subset_{8\delta}^{\mu} K$.*

Applying this to the "counting measure" μ_c on Ω ($\mu_c(M) = |M|$, for all $M \subset \Omega$), we obtain an approximate version of Theorem 1.1:

Corollary 1.4 *Let $k, n \in \mathbb{N}$ with $n \geq 3k$. Suppose that a map $F : G \to 2^{\Omega}$ satisfy the conditions*

$$|F(gh) \setminus (F(g) \cup F(h))| \leq k \text{ for all } g, h \in G.$$

and

$$\sup_{g \in G} |F(g)| = n$$

Then there exists a subset M of Ω with $|M| \leq 6n$ such that

$$|F(g) \setminus M| \leq 8k \quad \forall g \in G.$$

1.3 Subadditive Subspace-Valued Maps

Let us consider a version of subadditivity for maps to the lattice $\mathcal{S}(X)$ of subspaces in a linear space X (an analogue of 2^{Ω} in non-commutative probability theory and non-commutative geometry).

We call a map $F : G \to \mathcal{S}(X)$ subadditive if

$$F(gh) \subset F(g) + F(h) \tag{1.6}$$

where by the sum of a (finite or infinite) family of subspaces we call the linear span of their union.

Assume that all subspaces $F(g)$, for $g \in G$, are finite-dimensional, and set

$$n(F) = \sup_{g \in G} \dim F(g), \quad N(F) = \dim \sum_{g \in G} F(g) \tag{1.7}$$

(a priori $N(F)$ can be equal ∞). It is easy to see that $N(F) \leq Cn(F)$ if G is finitely generated. But, in distinction from the commutative case, this does not extend to all groups. To see this let us consider the following example.

Example 1.5 Let $G = X$, an infinite-dimensional linear space considered as a group with respect to the addition. For each $0 \neq x \in G$, let $F(x) = \mathbb{C}x$, the one-dimensional subspace of X containing x. Let also $F(0) = \{0\}$. Then clearly F is a subadditive map of G to $\mathcal{S}(X)$, $n(F) = 1$, $N(F) = \infty$. ■

To obtain a working version we have to restrict the range of a map. We will do this, considering the maps to the lattice of invariant subspaces of some group representations.

Let T be a representation of a group Γ on a linear space X and let π be a finite-dimensional irreducible representation of Γ. Let us say that π *occurs* in T if there is an invariant subspace $Y \subset X$ such that the restriction $T|_Y$ of T to Y is equivalent to π. In this case we call Y *a π-subspace*. Furthermore π *occurs with finite multiplicity in* T if the linear span Y_π of all π-subspaces in X is finite-dimensional; equivalently, the number of elements in any family of linearly independent π-subspaces does not exceed some number $m = m(\pi, T)$.

Let us say that T is *fm-representation* if each irreducible representation of Γ which occurs in T occurs with finite multiplicity.

Moreover if the multiplicity $m(\pi, T)$ of each occurring in T irreducible representation π can be evaluated via the dimension of this representation:

$$m(\pi, T) \leq \phi(\dim \pi),$$

where $\phi : \mathbb{N} \to \mathbb{R}$ is an increasing function, then we say that T has uniform multiplicity ϕ.

An important example is the regular representation $\gamma \mapsto R_\gamma$ of Γ on the space $X = \mathbb{C}^\Gamma$ of all complex-valued functions on Γ:

$$R_\gamma f(\tau) = f(\tau\gamma).$$

Indeed let π be an irreducible finite-dimensional representation of Γ, and $(a_{ij}(\gamma))_{i,j=1}^n$ be the matrix of $\pi(\gamma)$ in some basis. If Y is π-subspace of X, then there is a basis $e_1, \ldots, e_n$ of Y with $R_\gamma e_i = \sum_{j=1}^n a_{ij}(\gamma) e_j$. In other words

$$e_i(\tau\gamma) = \sum_{j=1}^n a_{ij}(\gamma) e_j(\tau),$$

for all $\gamma, \tau \in \Gamma$. Setting $\tau = 1$, the unit of Γ, we get

$$e_i(\gamma) = \sum_{j=1}^{n} a_{ij}(\gamma) e_j(1),$$

whence $Y \subset L_\pi$, the linear span of the matrix functions $a_{ij}(\gamma)$. Since $\dim L_\pi = (\dim \pi)^2$, we see that $m(\pi, R) = \dim \pi$, so the representation R has uniform multiplicity id, where $id(n) = n$, for all $n \in \mathbb{N}$.

We will consider representations by bounded operators on Banach spaces; a representation T is *uniformly bounded* if $\sup_{\gamma\in\Gamma} \|T_\gamma\| < \infty$. For example the restriction of the regular representation to the space of all bounded functions on Γ gives an example of a uniformly bounded representation with uniform multiplicity id.

Theorem 1.6 ([13, Theorem 4.2]) *Let T be a uniformly bounded fm-representation of a group Γ and $\mathcal{E}$ be the lattice of all finite-dimensional T-invariant subspaces. Let G be a group and $F : G \to \mathcal{E}$ be a map such that $F(gh) \subset F(g) + F(h)$ and $\dim F(g) \le n$ for each $g, h \in G$. Then $\sum_{g\in G} F(g)$ is finite-dimensional. Moreover if T has uniform multiplicity ϕ then*

$$\dim(\sum_{g\in G} F(g)) \le (4n - 1)n\phi(n). \tag{1.8}$$

Remark 1.7 The only property of uniformly bounded representations which was used in the proof of this theorem in [13] is complete reducibility: every finite-dimensional invariant subspace decomposes in direct sum of minimal invariant subspaces. So it would be sufficient to assume that each finite-dimensional subrepresentation of T is uniformly bounded. Moreover if the group Γ is reductive then the restriction of uniform boundedness can be removed.

2 Strengthening of Theorem 1.6

As above we deal with a Banach space X where some uniformly bounded representation T of a group Γ acts. Here and in what follows we denote by Ω the set of all equivalence classes of irreducible finite-dimensional representations of Γ. Let us define an atomic measure μ on Ω by setting $\mu(\{\pi\}) = \dim \pi$, for each $\pi \in \Omega$. We will prove a stronger version of Theorem 1.6 applying Theorem 1.2 with this choice of the measure space (Ω, μ).

Theorem 2.1 *In the conclusion of Theorem 1.6 the estimate (1.8) can be replaced by*

$$\dim(\sum_{g\in G} F(g)) \le (4n - 1)\phi(n).$$

Proof For each $g \in G$, let $M(g)$ be the set of those $\pi \in \Omega$ which occur in $T|_{F(g)}$. The assumption $\dim F(g) \leq n$ implies that $\mu(M(g)) \leq n$, for each $g \in G$.

Since each finite-dimensional bounded representation can be uniquely decomposed in a direct sum of irreducible ones, we have

$$M(gh) \subset M(g) \cup M(h).$$

Applying Theorem 1.2 to the set-valued function $g \mapsto M(g)$ we conclude that there is a subset $N \subset \Omega$ such that $\mu(N) \leq 4n - 1$ and $M(g) \subset N$, for each $g \in G$ (in our case $\subset_0^\mu = \subset$).

We denote by $\pi_1, \ldots, \pi_k$ the elements of N and let Y_{π_i} be the linear span of all π_i-subspaces in X ($i = 1, \ldots, k$). Then

$$\sum_{g \in G} F(g) \subset Y_{\pi_1} + \cdots + Y_{\pi_k},$$

which is a finite-dimensional subspace due to the finite multiplicity of the representation T. Furthermore, if T has the uniform multiplicity ϕ, then $\dim Y_{\pi_i} \leq \dim \pi_i \cdot \phi(\dim \pi_i) \leq \dim \pi_i \cdot \phi(n)$, whence

$$\dim \sum_{g \in G} F(g) \leq \phi(n) \sum \dim \pi_i = \phi(n)\mu(N) \leq (4n-1)\phi(n).$$

■

3 Almost Subadditive Subspace-Valued Maps

For subspaces Y, Z of a linear space and a positive integer k, we write

$$Y \subset_k Z,$$

if $Y \subset Z + V$ for some k-dimensional subspace V.

Theorem 3.1 *Let T be a uniformly bounded fm-representation with uniform multiplicity ϕ of a group Γ on a Banach space X, and $\mathcal{E}$ be the lattice of all finite-dimensional T-invariant subspaces.*

Let G be a group and $F : G \to \mathcal{E}$ be a map such that

$$\dim F(g) \leq n, \text{ for each } g \in G,$$

and

$$F(gh) \subset_k F(g) + F(h), \text{ for any } g, h \in G,$$

for some $k, n \in \mathbb{N}$ with $n \geq 3k$.

Then there is a subspace $E \in \mathcal{E}$ such that

$$\dim E \leq 6n\phi(n)$$

and

$$F(g) \subset_{8k\phi(8k)} E, \text{ for each } g \in G.$$

Proof Let the measure space (Ω, μ) be as above. For $L \in \mathcal{E}$, let $M(L)$ denote the set of those $\pi \in \Omega$ which occur in $T|_L$. Then

$$M(L_1 + L_2) = M(L_1) \cup M(L_2), \text{ for all } L_1, L_2 \in \mathcal{E}.$$

Note also that $\mu(M(L)) \leq \dim L$, for each $L \in \mathcal{E}$.

Suppose now that $L_1, L_2 \in \mathcal{E}$ and let $K \subset X$ be a finite-dimensional subspace such that $L_1 \subset L_2 + K$. We claim that there is a subspace $R \in \mathcal{E}$ such that

$$L_1 \subset L_2 + R \text{ and } \dim R \leq \dim K.$$

Indeed, for each $\gamma \in \Gamma$, one has

$$L_1 = T_\gamma L_1 \subset T_\gamma L_2 + T_\gamma K = L_2 + T_\gamma K.$$

It follows that

$$L_1 \subset L_3 \text{ where } L_3 = \cap_{\gamma \in \Gamma}(L_2 + T_\gamma K).$$

Clearly $L_3 \in \mathcal{E}$ and $L_2 \subset L_3 \subset L_2 + K$, whence $L_3 = L_2 + R$ (we again use complete reducibility of T), where $R \in \mathcal{E}$ and $\dim R \leq \dim K$. Thus $L_1 \subset L_3 = L_2 + R$. The claim is proved.

Applying this to the subspaces $L_1 = F(gh)$ and $L_2 = F(g) + F(h)$ we may write the condition of almost subadditivity in the form

$$F(g_1g_2) \subset F(g_1) + F(g_2) + R, \text{ where } R = R(g_1, g_2) \in \mathcal{E} \text{ and } \dim R \leq k. \tag{3.1}$$

For $g \in G$, set $\Phi(g) = M(F(g))$. It follows from (3.1) that

$$\Phi(g_1g_2) \subset M(F(g_1) + F(g_2) + R) = \Phi(g_1) \cup \Phi(g_2) \cup M(R), \text{ for all } g_1, g_2 \in G. \tag{3.2}$$

Since $\mu(M(R(g_1, g_2))) \leq \dim R(g_1, g_2) \leq k$, it follows from (3.2) that

$$\Phi(g_1g_2) \subset_k^\mu \Phi(g_1) \cup \Phi(g_2).$$

By our assumption, $\mu(\Phi(g)) \leq \dim F(g) \leq n$, for all $g \in G$. Applying Theorem 1.3 to the set-valued function $\Phi(g)$ we get that there is a subset $N \subset \Omega$ such that $\mu(N) \leq 6n$ and

$$\Phi(g) \subset^{\mu}_{8k} N, \qquad \text{with} \quad \mu(N) \leq 6n,$$

for each $g \in G$.

Let E be the sum of all π-subspaces with $\pi \in N$. Clearly each $\pi \in M(E) = N$ has the multiplicity $\leq \phi(\dim \pi)$, and since $\dim \pi \leq \mu(N) \leq 6n$, the multiplicity of all representations in $M(E)$ does not exceed $\phi(6n)$. Therefore $\dim E \leq \phi(6n)\mu(N) \leq 6n\phi(6n)$.

By complete reducibility, $F(g) = E \cap F(g) + D$, for some $D = D(g) \in \mathcal{E}$. If π is an irreducible representation acting on a subspace of D, then the corresponding class belongs to $\Lambda = \Phi(g) \setminus N$. Since $\mu(\Lambda) \leq 8k$, $\dim \pi \leq 8k$ whence the multiplicity of π does not exceed $\phi(8k)$. Therefore

$$\dim D \leq \phi(8k)\mu(\Lambda) \leq 8k\phi(8k)$$

and the statement follows from the inclusion $F(g) \subset E + D$. ■

4 Subadditive Subspace-Valued Maps with Non-Invariant Values

Our next aim is to extend Theorem 1.6 to subspace-valued maps with non-necessarily invariant values. We preserve notations introduced in the previous chapters.

Let L be a finite-dimensional subspace of X. A subspace E of X is called *invariantly L-majorized* if

$$E \subset L + W, \text{ where } W \in \mathcal{E}. \tag{4.1}$$

If $\dim W \leq n$ then we say that E is *(L, n)-majorized.*

Theorem 4.1 *Let T be a uniformly bounded fm-representation of a group Γ on a Banach space X. We fix a finite-dimensional subspace L of X, and a number $n \in \mathbb{N}$. Let F be a subadditive map of a group G to the lattice of subspaces of X:*

$$F(g_1 g_2) \subset F(g_1) + F(g_2),$$

such that all subspaces $F(g)$ are (L, n)-majorized. Then $\sum_{g \in G} F(g)$ is invariantly L-majorized, and therefore finite-dimensional. Moreover, if T has uniform multi-

plicity ϕ then

$$\dim\left(\sum_{g\in G} F(g)\right) \le 3n \dim L + (4n-1)\phi(n).$$

Proof We will use induction on $\dim L$. If $\dim L = 0$, then there is nothing to prove. Let us suppose that the statement holds if $\dim L < k$, and consider the case that $\dim L = k$.

Case 1 First assume that L has only trivial intersection with any invariant subspace of dimension $\le 3n$.

For each L-invariant subspace E, let us denote by $U(E)$ the set of all finite-dimensional invariant subspaces W satisfying (4.1).

We claim that for each W_1, W_2 in $U(E)$ such that $\dim W_1 + \dim W_2 \le 3n$, their intersection also belongs to $U(E)$. Indeed, if $W_1, W_2 \in U(E)$, then each element $\xi \in E$ can be written as $\xi = l_1 + w_1$ and $\xi = l_2 + w_2$, where $l_i \in L$, $w_i \in W_i$ $(i = 1, 2)$. Hence $w_1 - w_2 = l_2 - l_1 \in L$. The vector $w_1 - w_2$ belongs to the invariant subspace $W_1 + W_2$, which has dimension $\le 3n$ and therefore, by our assumption, has trivial intersection with L. We obtain that $w_1 - w_2 = 0$, whence $w_1 = w_2 \in W_1 \cap W_2$. Since $\xi \in T_g L$ is arbitrary we conclude $T_g L \subset L + W_1 \cap W_2$, which means $W_1 \cap W_2 \in U(E)$. Thus $U(E)$ is closed under finite intersections; since $U(E)$ consists of finite-dimensional subspaces, it is closed under arbitrary intersections.

Assuming that E is (L, n)-invariant, so $U(E)$ contains subspaces of dimension $\le n$, set

$$Z(E) = \cap\{K \in U(E) : \dim K \le n\}.$$

By the above, $Z(E) \in U(E)$. Moreover, $Z(E)$ is contained in any subspace $V \in U(E)$ with $\dim V \le 2n$. Indeed, since $\dim Z(E) + \dim V \le 3n$, we conclude that $V \cap Z(E) \in U(E)$ and $\dim(V \cap Z(E)) \le n$. Therefore $V \cap Z(E)$ contains $Z(E)$, so $Z(E) \subset V$.

Now return to the map $g \mapsto F(g)$. For $g \in G$, set $Q(g) = Z(F(g))$. Since $F(gh) \subset F(g) + F(h) \subset L + Q(g) + L + Q(h) = L + Q(g) + Q(h)$, we get that $Q(g) + Q(h) \in U(F(gh))$. Since $\dim(Q(g) + Q(h)) \le 2n$, this subspace contains $Q(gh)$. Thus the map $g \mapsto Q(g)$ is subadditive, and its values are invariant subspaces of dimension $\le n$. By Theorem 1.6, there is a finite-dimensional invariant subspace R, containing all $Q(g)$. It follows that $L + R$ contains all $F(g)$. If T has uniform multiplicity ϕ then $\dim R \le (4n-1)\phi(n)$ and we are done since $\dim(L + R) \le \dim L + (4n-1)\phi(n)$ which is even better than what we need.

Case 2 Suppose now that L has non-zero intersection with an invariant subspace S such that $\dim S \le 3n$. Changing X by $\overline{X} = X/S$ and dealing with the quotient representation $\overline{T}$ of G on $\overline{X}$, denote by $\overline{L}$ and $\overline{F(g)}$ the images of the subspaces L

and $F(g)$ under the natural epimorphism $p : X \to \overline{X}$. Then all subspaces $\overline{F(g)}$ are $(\overline{L}, n)$-invariant and the map $g \to \overline{F(g)}$ is subadditive. Furthermore, $\overline{T}$ is uniformly bounded fm-representation, and if T has uniform multiplicity ϕ then the uniform multiplicity of the quotient representation $\overline{T}$ does not exceed $\phi(n)$. Indeed, suppose that there exist more than $\phi(n)$ linearly independent π-subspaces in $\overline{X}$. Let H be their linear span, and E be a p-preimage of H in X. Since the representation T is bounded the finite-dimensional invariant subspace S has an invariant complement S' in E and the action of G on S' is equivalent to the representation $\overline{T}$. So, S' contains more than $\phi(n)$ linearly independent π-subspaces which is impossible.

Since $\dim \overline{L} < k$, we obtain, by the induction assumption, that all $\overline{F}(g)$ are contained in a finite-dimensional invariant subspace $W \subset \overline{X}$, and if T has uniform multiplicity ϕ then

$$\dim W \le 3n \dim \overline{L} + (4n-1)\phi(n).$$

Set $\widetilde{W} = p^{-1}(W)$, then $\widetilde{W}$ is T-invariant, $F(g) \subset \widetilde{W}$, for each $g \in G$, and

$$\dim \widetilde{W} \le 3n + \dim W \le 3n + 3n \dim \overline{L} + (4n-1)\phi(n) \le 3n \dim L + (4n-1)\phi(n). \tag{4.2}$$

■

5 Locally Invariant Subspaces

Let a representation T of a group G on a linear space X be given and suppose that we have to prove that some finite-dimensional subspace L of X is contained in a finite-dimensional invariant subspace $\widetilde{L}$. Sometimes for this it suffices to know that the image of L under each operator T_g is close to L in some sense. Using Theorem 1.6 we will establish a result of this kind. This statement was proved in [13, Theorem 5.1]; here we will correct a small gap in that proof and obtain a much better estimation of $\dim \widetilde{L}$.

Theorem 5.1 *Let T be a uniformly bounded fm-representation of a group G on a Banach space X, and let $n \in \mathbb{N}$. Assume that a finite-dimensional subspace $L \subset X$ has the property that for any $g \in G$, there is an invariant subspace K such that $T_g L \subset L + K$ and $\dim K \le n$. Then L is contained in a finite-dimensional invariant subspace $\widetilde{L}$ of X. Moreover, if T has uniform multiplicity ϕ then*

$$\dim \widetilde{L} \le 3n \dim L + (4n-1)\phi(n). \tag{5.1}$$

Proof Using induction on $\dim L$ we may assume that for subspaces of smaller dimensions the result is proved. Denote by $\mathcal{E}$ the set of all finite-dimensional T-invariant subspaces in X.

(1) First, let us prove that L is contained in a finite-dimensional invariant subspace.

If L has a non-trivial intersection with a subspace $W \in \mathcal{E}$ then, as in the proof of Theorem 4.1, one can reduce the dimension of L by coming to the quotient representation on X/W and then use induction on $\dim L$.

So we may assume that L has zero intersection with any T-invariant subspace. Denote

$$U(g) = \{K \in \mathcal{E} \mid T_g L \subset L + K\},$$

$$V(g) = \{K \in U(g) \mid \dim K \le n\}.$$

Claim If L does not intersect finite-dimensional invariant subspaces of dimension $\le 3n$ then $V(g)$ has a smallest element which is contained in any $K \in U(g)$ with $\dim K \le 2n$.

Let K_1, K_2 in $U(g)$ be such that $\dim K_1 + \dim K_2 \le 3n$, then their intersection also belongs to $U(g)$. Indeed, each element $\xi \in T_g L$ can be written as $\xi = l_1 + k_1$ and $\xi = l_2 + k_2$ where $l_i \in L$, $k_i \in K_i$. Hence $k_1 - k_2 = l_2 - l_1 \in L$. Since $k_1 - k_2$ belongs to the invariant subspace $K_1 + K_2$, which has dimension $\le 3n$ and therefore, by our assumption, has trivial intersection with L, we conclude that $k_1 - k_2 = 0$, whence $k_1 = k_2 \in K_1 \cap K_2$. Since $\xi \in T_g L$ is arbitrary, $T_g L \subset L + K_1 \cap K_2$, that is $K_1 \cap K_2 \in U(g)$.

It follows that the set $V(g)$ is closed under intersections, so the space

$$L(g) = \cap\{K \in V(g)\}$$

is the smallest element of $V(g)$. Furthermore, $L(g)$ is contained in any subspace $K \in U(g)$ with $\dim K \le 2n$. Indeed, since $\dim L(g) + \dim K \le 3n$, we get that $K \cap L(g) \in U(g)$ and $\dim(K \cap L(g)) \le n$. Hence $K \cap L(g) \in V(g)$ and therefore contains $K \cap L(g) \supseteqq L(g)$. Thus $L(g) \subset K$. The claim is proved.

Now for every $g, h \in G$, we have

$$T_{gh} L = T_g(T_h(L)) \subset T_g(L + L(h)) \subset T_g L + L(h) \subset L + L(g) + L(h).$$

In other words $L(g) + L(h) \in U(gh)$. Since $\dim(L(g) + L(h)) \le 2n$ we conclude that $L(gh) \subset L(g) + L(h)$. In other words the map $g \mapsto L(g)$ is subadditive. Applying Theorem 2.1, we get that the invariant subspace $M = \sum_{g \in G} L(g)$ is finite dimensional, and $\dim M \le (4n-1)\phi(n)$ if T has uniform multiplicity ϕ. We get

$$T_g(L + M) \subset T_g L + M \subset L + L(g) + M \subset L + M.$$

Thus $L + M$ is a finite-dimensional invariant subspace containing L.

(2) We established that L is contained in a finite-dimensional invariant subspace Y. Assuming that T has uniform multiplicity ϕ we have to prove the inequality (5.1).

The above arguments show that if L has trivial intersections with invariant subspaces of dimension $\leq 3n$ then L is contained in an invariant subspace $L + M$ of dimension

$$\dim(L + M) \leq \dim L + \dim M \leq \dim L + (4n - 1)\phi(n),$$

which is even better than we need.

Suppose now that there is an invariant subspace W with $\dim W \leq 3n$ and $L_1 := L \cap W \neq 0$. Let Z be an invariant complement of W in X, and let P be a projection on Z with $\ker P = W$. Then the subspace $L_2 = PL$ satisfies the same conditions as L: $T_g L_2 = T_g PL = PT_g L \subset L_2 + PK$, for each $K \in V(g)$. By the induction assumption there is an invariant subspace Y_2 of Z, containing L_2 with $\dim Y_2 \leq 3n \dim(L_2) + (4n - 1)\phi(n)$. Therefore $L \subset R := W + L_2$ and

$$\dim R \leq 3n + 3n \dim(L_2) + (4n - 1)\phi(n) =$$
$$3n(\dim L_2 + 1) + (4n - 1)\phi(n) \leq 3n \dim L + (4n - 1)\phi(n).$$

■

The following result, proved with another estimation in [13], had direct applications in theory of functional equations. We include it for correcting the constant.

Corollary 5.2 *Let T be a uniformly bounded fm-representation of a group G on a Banach space X, and let $n \in \mathbb{N}$. Suppose that a vector $\xi \in X$ and a finite-dimensional subspace $L \subset X$ have the property that for each $g \in G$,*

$$T_g \xi \in L + R(g), \tag{5.2}$$

where $R(g)$ is an invariant subspace of X with $\dim R(g) \leq n$.

Then ξ belongs to a finite-dimensional invariant subspace E of X. If T has uniform multiplicity ϕ then

$$\dim E \leq 4m^2 n + (8mn - 1)\phi(2mn), \tag{5.3}$$

where $m =$ $\dim L$.

Proof We may assume that L is a minimal subspace that satisfies our assumptions.

For each $g \in G$, choose $u(g) \in L$ and $r(g) \in R(g)$ such that $T_g \xi = u(g) + r(g)$. It follows that

$$T_h u(g) = T_{hg}\xi - T_h r(g) \in L + R(hg) + R(g) \subset L + M_{h,g},$$

where $M_{h,g}$ is an invariant subspace of X with $\dim M_{h,g} \leq 2n$.

The linear span of all vectors $u(g)$ generates L because of the minimality assumption. Choosing a basis $u(g_1), \ldots, u(g_m)$ in L we obtain that

$$T_h L \subset L + M(h, g_1) + M(h, g_2) + \cdots + M(h, g_m).$$

Denoting $M(h, g_1) + M(h, g_2) + \cdots + M(h, g_m)$ by L_h we get that, for each h, L_h is an invariant subspace of X with $\dim L_h \le 2nm$ and $T_h L \subset L + L_h$. Applying Theorem 5.1 we conclude that L and all $R(g)$ are contained in a finite-dimensional invariant subspace E, and $\xi = T_e \xi \in L + R(e) \subset E$.

If T has uniform multiplicity ϕ then (5.3) follows from (5.1) (with replacement of n by $2nm$). ■

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Generalized Cauchy–Riemann Equations with Power-law Singularities in Coefficients of Lower Order

A. P. Soldatov and A. B. Rasulov

Abstract The generalized Cauchy-Riemann equation $u_{\bar{z}} + au + b_{\bar{u}} = f$ is considered with coefficients having power singularities at the origin. More exactly it is assumed that $a(z) = O(|z|^{-\alpha-1})$, $\alpha \in \mathbb{R}$, and $b(z) = O(|z|^{-1})$ as $z \to 0$. With the help of so called modified Pompeiu integral a general solution of this equation is described.

Keywords Generalized Cauchy-Riemann equation · Modified Pompeiu integral · Power singularities · General solution · Integral representation

1 Introduction

In a finite domain $D \subseteq \mathbb{C} \setminus \{0\}$, containing a punctured circle centered at $z = 0$, we consider the elliptic equation

$$\frac{\partial u}{\partial \bar{z}} + au + b\bar{u} = f, \tag{1.1}$$

with complex-valued functions $a = a(z), b = b(z)$ and $f = f(z)$ belonging to $C(\overline{D} \setminus \{0\})$. Its coefficients a, b admit power singularities at the point $z = 0$. According to the theory of generalized analytic functions [1] in every closed disc $B \subseteq D$, the solution of this equation belongs to the Sobolev class $W^{1,p}(B)$ with any $p > 2$. It is well known that this class is embedded in the Hölder space $C^{\mu}(B)$ with exponent $\mu = 1 - 2/p$. Therefore, μ can be chosen arbitrarily close to 1. Functions

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φ of this type, i.e. belonging to $W^{1,p}(B)$ in every closed disc $B \subseteq D$ with any $p > 1$ are called regular in the domain D. Obviously, together with such a function, the function e^{φ} will be regular, too.

We denote by $C_\lambda(\overline{D}, 0)$, $\lambda \in \mathbb{R}$, the space of all continuous on $\overline{D} \setminus \{0\}$ functions $\varphi = \varphi(z)$ with behavior $O(|z|^\lambda)$ at $z \to 0$. It is supplied with the norm

$$|\varphi| = \sup_{z \in D} |z|^{-\lambda} |\varphi(z)|, \tag{1.2}$$

it becomes, obviously, a Banach space. The notation $\varphi \in C_{\lambda+0}(\overline{D}, 0)$ means that $\varphi \in C_{\lambda+\varepsilon}(\overline{D}, 0)$ with some $\varepsilon > 0$. Similarly, $\varphi \in C_{\lambda-0}(\overline{D}, 0)$, if φ belongs to $C_{\lambda-\varepsilon}(\overline{D}, 0)$ with any $\varepsilon > 0$. In particular, functions $\varphi \in C_{-0} = C_{0-0}(\overline{D}, 0)$ admit singularities of logarithmic type at the point $z = 0$.

In this notation, with respect to the coefficients of Eq. (1.1), we assume that

$$a \in C_{-\alpha-1}(\overline{D}, 0), \ \alpha \in \mathbb{R}; \quad b \in C_{-1}(\overline{D}, 0). \tag{1.3}$$

The main purpose of the paper is to describe all solutions of this equation which are regular in the domain D under these assumptions. The coefficient a plays an essential role for the behavior of these solutions. Therefore, an additional requirement is imposed on it that, up to a term from the class $C_{-\alpha-1+0}(\overline{D}, 0)$, the function a is homogeneous of degree $-\alpha - 1$. More precisely, let it be representable in the form

$$a(z) = |z|^{-\alpha-1} \sum\nolimits_{k \in \mathbb{Z}} a_k e^{ik\theta} + a^0(z), \quad a^0 \in C_{-\alpha-1+0}(\overline{D}, 0), \tag{1.4}$$

where $\theta = \arg z$ and the series is absolutely convergent.

The classical theory of I.N. Vekua[1] of generalized analytic functions covers the case when the coefficients a, b belong to a space $L^p(D)$ with exponent $p > 2$. The coefficients of such systems can admit weak singularities, limited by the requirement of p-integrability. In particular, equations with coefficients $a \in C_{-\alpha-1}(\overline{D}, 0)$, $\alpha \geq 0$, and $b \in C_{-1}(\overline{D}, 0)$ do not satisfy this condition. In this regard, I.N. Vekua [1] introduced the class of quasi-summable coefficients $a = a(z)$ and $b = b(z)$ (that is summable with a weight that is a meromorphic function), for which he proposed a formula for presenting the solution in the form of an integral representation of the second kind. However, this formula does not cover coefficients with common power-law singularities. The need to study generalized Cauchy–Riemann systems with coefficients admitting singularities of at least first order was first pointed out by I.N. Vekua [1] and A.V. Bitsadze [2].

In the monograph by L.G. Mikhailov [3] (see also [4]), solutions of Eq. (1.1) with coefficients a, $b \in C_{-1}(\overline{D}, 0)$ are sought in the class $C_\beta(\overline{D}, 0)$, $0 < \beta < 1$. The solvability of the integral equation, to which Eq. (1.1) is reduced, is proved under certain conditions for the smallness of these coefficients.

Z.D. Usmanov [5] introduced a solvability theory for Eq. (1.1) for $a = 0$, $b(z) = \bar{z}^{-1}\beta e^{ik\varphi}$, $k \in Z$. However, the case when $b(z) = \bar{z}^{-1}(\beta_1 e^{ik\varphi} + \beta_2 e^{im\varphi})$, where

$\beta_1 \neq \beta_2$, leads to infinite systems of ordinary differential equations that seem to be difficult to solve. Therefore, in the future, [6] the main goal of Z.D. Usmanov was to find a connection between solutions of Eqs. (1.1) and the coefficients $a(z) = 0$, $b(z) \in C_{-1}(\overline{D}, 0)$ by means of a linear integral equation with a completely continuous operator. This connection allows us to reduce the general case to equations of a particular form with a constant coefficient. It is shown [6] that for the case $a = 0$, $b = \lambda|z|^{-\alpha}$, $\alpha > 0$, there exist solutions of the equation in the form of Fourier series, whose coefficients are determined through the Bessel and MacDonald functions. In [7], analytical solutions of some elliptic equations with singular coefficients are found.

In recent years, numerous studies have been devoted to the construction of a general solution to Eq. (1.1), as well as to more general equations and systems with singular coefficients (see, for example [8–20]). In the paper [8] H. Begehr and Dao-Ding studied Eq. (1.1) with $a \in C_{-1}(\overline{D}, 0)$, $b \in L^p(D)$, $p > 2$. M. Reissig and A. Timofeev [12] considered the case when the coefficient b belongs to a special weighted space S_p. The class of continuous solutions of Eq. (1.1), for $a, b \in C_{-1}(\overline{D}, 0)$ was studied in the works of A.B. Tungatarov [13].

Solutions of Eq. (1.1) with coefficients $a, b \in C_{-1}(\mathbb{C}, 0)$ were constructed in the works of A. Meziani [16]. In [10, 11] it is shown that for the model Eq. (1.1), for $a(z) = \lambda\bar{z}^{-1}$, $b \in L^p(D)$, $p > 2$, the number of continuous solutions depends on the size and sign of the constant λ. This observation is implicitly confirmed by results of the paper [15], where Eq. (1.1) was studied for $a(z) = \lambda z^{-1}$, $b(z) = \mu\bar{z}^{-1}$, where λ and μ are complex constants.

In the paper [21], when the coefficients $a, b \in C_{-1}(D, l)$, an explicit solution of the problem is obtained, where singular segment l is a carrier of conditions for a linear conjugation problem, and ∂D is a carrier of conditions for a Hilbert problem. In [22–24] it is considered the case, where the coefficient $b \in C_\beta(\overline{D}, 0)$, $\beta > -1$.

Equation (1.1) arises in the theory of infinitesimal bendings of a surface of positive curvature with a general structure at a point of flattening [25–28]. Equations with singular coefficients are also found in other applications. For example, in the monograph by A.V. Bitsadze [2] it is shown that the Maxwell-Einstein equations in Ernst's version are reduced to a solution of Eq. (1.1) with singular line.

2 Modified Pompeiu Operator

In the theory of generalized analytic functions of I.N. Vekua [1] the Pompeiu integral plays an important role

$$(Tg)(z) = \frac{1}{\pi}\int_D \frac{g(t)}{t - z} d_2 t, \quad z \in D. \tag{2.1}$$

As it is known, if the density g is summable in the domain D and in every closed disc $B \subseteq D$ it belongs to $L^p(B)$ with any $p > 2$, then the function $u = Tg$ belongs to $W^{1,p}(B)$ and satisfies the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial u}{\partial \bar{z}} = g. \tag{2.2}$$

In particular, all regular solutions to this equation have the form $u = Tg + \phi$, where the function ϕ is analytic in the domain D.

In the sequel, it is more convenient to deal with a modified Pompeiu integral

$$(T_0\varphi)(z) = \frac{1}{\pi}\int_D \frac{\varphi(t)}{t(t-z)} d_2t, \qquad z \in D, \tag{2.3}$$

which obviously makes sense for functions $\varphi \in C_\lambda(\overline{D}, 0)$ with $\lambda > -1$.

Lemma 2.1

(a) *The operator T_0 is bounded in the space $C_\lambda(\overline{D}, 0)$, $-1 < \lambda < 0$, and its norm $|T_0|_{C_\lambda}$ coincides with the constant*

$$\sigma(\lambda) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{|t|^\lambda d_2t}{|t||t-1|}. \tag{2.4}$$

(b) *The function $\sigma = \sigma(\lambda)$ has the symmetry property $\sigma(\lambda) = \sigma(-\lambda-1)$ on the interval $(-1, 0)$. It increases from $\sigma(-1/2)$ to $+\infty$ on the interval $(-1/2, 0)$.*

Proof

(a) Let $\varphi \in C_\lambda(\overline{D}, 0)$, $-1 < \lambda < 0$. Then

$$|z|^{-\lambda}|(T_0\varphi)(z)| \leq |\varphi|_{C_\lambda}\int_D \frac{|t|^\lambda}{|z|^\lambda}\frac{d_2t}{|t(t-z)|}.$$

It is easy to see that here the integral does not exceed the constant (2.4). Consequently, the function $T_0\varphi \in C_\lambda(\overline{D}, 0)$ and, by definition (1.2), it follows

$$|T_0\varphi|_{C_\lambda} \leq \sigma(\lambda)|\varphi|_{C_\lambda}. \tag{2.5}$$

Let $0 < r < 1$ be so small that the punctured disc $0 < |z| \leq \sqrt{r}$ is contained in D. Consider the unit norm function $\varphi = \varphi(z) \in C_\lambda(\overline{D}, 0)$ which in the circle $|z| \leq r$ coincides with

$$\varphi(z) = |z|^\lambda \frac{z(z-r)}{|z(z-r)|}.$$

Then

$$(T_0\varphi)(r) \geq \frac{1}{\pi}\left(\int_{|t|\leq\sqrt{r}} - \int_{|t|\geq\sqrt{r}}\right)\frac{|t|^{\lambda}d_2t}{|t||t-r|} = \frac{r^{\lambda}}{\pi}\left(\int_{|t|\leq 1/\sqrt{r}} - \int_{|t|\geq 1/\sqrt{r}}\right)\frac{|t|^{\lambda}d_2t}{|t||t-1|}.$$

Hence,

$$r^{-\lambda}(T_0\varphi)(r) \geq \left(\sigma(\lambda) - \frac{2}{\pi}\int_{|t|\geq 1/\sqrt{r}}\frac{|t|^{\lambda}d_2t}{|t||t-1|}\right),$$

what in correspondence with (2.5) means that the norm of the operator T_0 in the space $C_\lambda(\overline{D}, 0)$ exactly coincides with $\sigma(\lambda)$.

(b) The equality $\sigma(\lambda) = \sigma(-\lambda-1)$ is obtained by replacing $t = 1/t'$ in the integral (2.4). Similarly, we make sure that the derivative

$$\sigma'(\lambda) = \int_{\mathbb{C}}\frac{(\ln|t|)|t|^{\lambda}d_2t}{|t||t-1|} = \int_{|t|<1}\frac{(\ln|t|)(|t|^{\lambda} - |t|^{-\lambda-1})d_2t}{|t||t-1|},$$

which proves the second statement in (b).

Note that the positive constant $\sigma(-1/2)$ is expressed in a rather complicated way in terms of elliptic integrals of the first and third kind.

Consider the integral (2.3) for functions $\varphi \in C_\lambda(\overline{D}, 0)$ with $0 < \lambda < 1$. In this case, it also makes sense for $z = 0$. Since $\widetilde{\varphi}(z) = z^{-1}\varphi(z) \in C_{\lambda-1}$, by Lemma 2.1 the function $T_0\widetilde{\varphi}$ belongs to $C_{\lambda-1}(\overline{D}, 0)$. Given the obvious identity

$$(T_0\varphi)(z) - (T_0\varphi)(0) = z(T_0\widetilde{\varphi})(z)$$

we conclude that operator $(T_0\varphi)(z) - (T_0\varphi)(0)$ is bounded in the space $C_\lambda(\overline{D}, 0)$ and its norm coincides with $\sigma(\lambda - 1)$.

Obviously, the operator of multiplication by a function

$$\rho_k(z) = z^k, \quad k = 0, \pm 1, \ldots,$$

realizes an isometric isomorphism $C_\lambda(\overline{D}, 0) \to C_{k+\lambda}(\overline{D}, 0)$. This fact allows us to apply it to the description of general solutions of Eq. (2.2) in a family of weighted spaces.

Lemma 2.2 *The general solution of Eq.* (2.2) *with right-hand side* $g \in C_{-\lambda-1}(\overline{D}, 0)$, $\lambda \in \mathbb{R}$, *is given by the representation*

$$u = -(\rho_{-[\lambda]}T_0\rho_{[\lambda]+1})g + \phi, \tag{2.6}$$

where $[\lambda]$ *is the integer part of* λ *and the function* ϕ *is analytic in the domain* D.

If the number λ is non-integer (integer), then this solution belongs to the class $C_{-\lambda}(\overline{D},0)$ ($C_{-\lambda-0}$) if and only if the function $\phi_0 = \rho_{[\lambda]}\phi$ is analytic in $D\cup\{0\}$ and continuous on $\overline{D}$.

Proof The function ρ_k is analytic in the domain D, therefore, multiplying equation (2.2) by it the factor $\rho_k = \rho_k(z)$ can be shifted under the differentiation sign. Due to the inequality $-2 < [\lambda] - \lambda - 1 \le -1$, the function $\rho_{[\lambda]}g$ is summable in the domain D, so that the general solution of Eq. (2.2) is given by the formula

$$u = (\rho_{-[\lambda]}T\rho_{[\lambda]})g + \rho_{-[\lambda]}\phi,$$

where ϕ is analytic in the domain D. According to (2.1) and (2.3) operators T and T_0 are connected by the relation $T = -T_0\rho_1$, so this formula turns into (2.6).

The last statement follows from Lemma 2.1 and the fact that a function ϕ which is analytic in D with behavior $O(|z|^{1-\lambda})$ (or with behavior $O(|z|^{-\lambda-\varepsilon})$ for any $\varepsilon > 0$) at the point $z = 0$ actually behaves like $O(|z|^{-[\lambda]})$.

Lemma 2.1 can be somehow supplemented. For this purpose, we introduce the space $C^\mu_0(\overline{D},0)$, $0<\mu<1$, of all bounded functions φ with finite norm

$$|\varphi| = |\varphi|_0 + \{\varphi\}_\mu, \quad \{\varphi\}_\mu = \sup_{z_1,z_2\in D} \frac{||z_1|^\mu\varphi(z_1) - |z_2|^\mu\varphi(z_2)|}{|z_1 - z_2|^\mu}, \tag{2.7}$$

where here and below $|\varphi|_0$ means the sup $-$norm. Note that this norm is equivalent to the norm

$$|\varphi| = |\varphi|_0 + \{\varphi\}'_\mu, \tag{2.8}$$

where $\{\varphi\}'_\mu$ is determined by the upper bound by the points $z_1, z_2 \in D$, satisfying the condition $1/2 \le |z_1|/|z_2| \le 2$. In other words, it consists of all bounded functions φ, where the function $\psi = \psi(z) = |z|^\mu\varphi(z)$ satisfies a Hölder condition with exponent μ.

We define the weighted space C^μ_λ as image of C^μ_0 under the weighted transformation $\varphi_0(z) \to |z|^\lambda\varphi_0(z)$. It is equipped with the transferred norm $|\varphi| = |\varphi_0|_{C^\mu_0}$.

Theorem 2.1 *For* $-1<\lambda<0$, *and* $0<\mu<1$ *the operator* $T_0 : C_\lambda(\overline{D},0) \to C^\mu_\lambda(\overline{D},0)$ *is bounded.*

Proof It is enough to make sure that the operator

$$(\widetilde{T}_0\varphi)(z) = \frac{1}{\pi}\int_D \frac{|t|^\lambda\varphi(t)d_2t}{|z|^\lambda t(t-z)}, \quad z \in D,$$

is bounded as a mapping $C_0(\overline{D},0) \to C^\mu_0(\overline{D},0)$. Let us extend the function φ by zero to the whole plane (preserving the notation), then the function $\psi = \widetilde{T}_0\varphi$ can also be considered in the whole plane. Following to the proof of Lemma 2.1 we receive the estimate

$$|\psi|_0 \le c|\varphi|_0 \tag{2.9}$$

for sup −norm of this function.

For an integer k we consider the function $\psi_k(z) = \psi(2^k z)$ in the ring $K = \{1/2 \le |z| \le 2\}$. It is obvious that

$$\psi_k(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{|t|^\lambda \varphi(2^k t) d_2 t}{|z|^\lambda t(t-z)}, \quad z \in K. \tag{2.10}$$

We state that the function $\psi_k(z),\ z \in K$, belongs to the Hölder space $C^\mu(K)$ and satisfies the estimate

$$|\psi_k|_{C^\mu(K)} \le c|\varphi|_0, \tag{2.11}$$

where the constant $c > 0$ is independent of φ and k.

Indeed, we write $\psi_k = \psi_k^0 + \psi_k^1$ where ψ_k^0 is determined by integration over the ring $\widetilde{K} = \{1/4 \le |t| \le 4\}$. In this ring, the function $\varphi(2^k t)$ belongs to the space L^p with any $p > 2$. Its L^p- norm is estimated in terms of the sup − norm $|\varphi|_0$ of the function φ. Choosing p by the condition $1 - 2/p = \mu$ and using the corresponding result for the Pompeiu integral, we prove the validity of estimate (2.11) for the function ψ_k^0.

As for the function ψ_k^1, it is bounded together with its gradient on the ring K and estimated in terms of the sup − norm $|\varphi|_0$ of function φ. Therefore, estimate (2.11) is also valid for ψ_k^1. Taking into account (2.10), it also holds for the function ψ_k.

Now let $z_1, z_2 \in D$ and $1/2 \le |z_1|/|z_2| \le 2$. Then there is an integer k such that both points $z'_j = 2^k z_j$ belong to K. Therefore

$$\frac{||z_1|^\mu \psi(z_1) - |z_2|^\mu \psi(z_2)|}{|z_1 - z_2|^\mu} = \frac{||z'_1|^\mu \psi_k(z'_1) - |z'_2|^\mu \psi_k(z'_2)|}{|z'_1 - z'_2|^\mu}$$

and based on inequality (2.11) we obtain the estimate

$$\{\psi\}'_\mu \le 2c|\varphi|_0$$

for the semi-norm appearing in (2.8). Together with (2.9), it also leads to an estimate for the norm (2.8) equivalent to (2.7), which completes the proof of the theorem.

Note that the paper [29] contains necessary estimates for generalized operators of Cauchy and Pompeiu types in various weighted spaces that can be used in the theory of elliptic equations and systems on the plane.

In accordance with Lemma 2.1, we consider the equation $\varphi + Q\varphi = f$ in the class $C_\lambda(\overline{D}, 0)$, $-1 < \lambda < 0$, so that the right-hand side $f \in C_\lambda(\overline{D}, 0)$ and the solution u should be determined in the same class.

Based on the bounded function $q = q(t) \in C_0(\overline{D}, 0)$, we consider the $\mathbb{R}-$ linear integral operator

$$Q\varphi = T_0(q\overline{\varphi}), \tag{2.12}$$

and the corresponding integral equation $\varphi + Q\varphi = f$. The conditions for the solvability of the last equation can be conveniently written by using the bilinear form

$$(\varphi, \psi) = \mathrm{Re} \int_D \varphi(t)\psi(t)d_2t, \tag{2.13}$$

which makes sense for any complex-valued functions $\varphi \in C_\lambda(\overline{D}, 0)$, $\psi \in C_{\nu-1}(\overline{D}, 0)$ for $\lambda + \nu > -1$. We recall that the class $C_{\delta-0}(\overline{D}, 0)$ consists of functions belonging to $C_{\delta-\varepsilon}$ for any $\varepsilon > 0$. For $\delta = 0$ this class is denoted by C_{-0}. In particular, the expression (2.13) also makes sense for any $\varphi \in C_{\lambda-0}(\overline{D}, 0)$, $\psi \in C_{\nu-1+0}(\overline{D}, 0)$ for $\lambda + \nu > -1$.

It is easy to see that the operator Q^* acting by the formula

$$\overline{Q^*\psi} = -q(\rho_{-1}T_0\rho_1)\psi, \tag{2.14}$$

based on Lemma 2.1 is bounded in the space $C_{\nu-1}(\overline{D}, 0)$, $-1 < \nu < 0$ and is conjugate to Q in the sense of the identity

$$(Q\varphi, \psi) = (\varphi, Q^*\psi), \tag{2.15}$$

valid for any $\varphi \in C_\lambda(\overline{D}, 0)$, $\psi \in C_{\nu-1}(\overline{D}, 0)$ for $-1 < \lambda, \nu < 0$, $\lambda + \nu > -1$.

Suppose the upper limit relation

$$q_0 = \overline{\lim_{t\to 0}}|q(t)| < \frac{1}{\sigma(-1/2)}. \tag{2.16}$$

So on the basis of Lemma 2.1(b) we can define in the interval $(-1/2, 0]$ the point δ from the equation

$$q_0\sigma(\delta) = 1, \ -1/2 < \delta \le 0. \tag{2.17}$$

The case $\delta = 0$ arises here for $q_0 = 0$ or, which is equivalent, when $q(z) \to 0$ for $z \to 0$. Lemma 2.1(b) also implies that $q_0\sigma(\lambda) < 1$ for $-\delta - 1 < \lambda < \delta$ and the interval $(-\delta - 1, \delta) \subseteq (-1, 0)$ is symmetric with respect to transformation $\lambda \to -\lambda - 1$.

Theorem 2.2

(a) The homogeneous equation $\varphi + Q\varphi = 0$ has in the class $C_\lambda(\overline{D}, 0)$, $-\delta - 1 < \lambda < \delta$, a finite number of linear independent solutions $\varphi_1, \dots, \varphi_n$ that belong to the function space $C_{\delta-0}(\overline{D}, 0)$.

(b) The homogeneous equation $\psi + Q^*\psi = 0$ has in the class $C_{\lambda-1}(\overline{D}, 0)$ the same number of linear independent solutions $\psi_1, \dots, \psi_n$ that belong to the function space $C_{\delta-1-0}(\overline{D}, 0)$.

(c) The nonhomogeneous equation $\varphi + Q\varphi = f$ is solvable in the class $C_\lambda(\overline{D}, 0)$, $-\delta - 1 < \lambda < \delta$ if and only if

$$(f, \psi_j) = 0, \quad 1 \le j \le n. \tag{2.18}$$

(d) Let the functions $\tilde{\varphi}_1, \dots, \tilde{\varphi}_n \in C_\delta(\overline{D}, 0)$ be chosen by the condition

$$(\tilde{\varphi}_i, \psi_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad 1 \le i, j \le n. \tag{2.19}$$

Then the operator K with

$$K\varphi = \varphi + Q\varphi + \sum\nolimits_{i=1}^{n} (\varphi, \varphi_i)\tilde{\varphi}_i, \quad \varphi \in C_\lambda(\overline{D}, 0), \tag{2.20}$$

is invertible in the space $C_\lambda(\overline{D}, 0)$, $-\delta - 1 < \lambda < \delta$, and under assumption (2.18) the function $\varphi = K^{-1} f$ is one of the solutions of equation $\varphi + Q\varphi = f$.

Proof We fix λ in the interval $(-\delta - 1, \delta)$ and write the function $q \in C_0(\overline{D}, 0)$ as a sum of two functions $q^0, q^1 \in C_0(\overline{D}, 0)$ so that for some $r > 0$ the next conditions are fulfilled:

$$q^0(z) = 0, \ |z| \le r; \quad |q^1|_0 \sigma(\lambda) < 1, \tag{2.21}$$

where $|\varphi|_0$ means the sup – norm of the function φ. Since

$$q_0 \sigma(\lambda) < 1, \ -\delta - 1 < \lambda < \delta, \tag{2.22}$$

this choice of λ is possible. According to this choice, we write $Q = Q_0 + Q_1$, where Q_j is defined similarly to (2.12) by the aid of q^j. Then in view of Theorem 2.1 the operator Q_0 is compact in the space $C_\lambda(\overline{D}, 0)$, and by Lemma 2.1(a) the norm of the operator Q_1 in this space is less than 1. Consequently, the operator $1 + Q$ is Fredholm and its index is zero.

Thus, its kernel $\ker(1 + Q)$ is finite-dimensional and has some dimension n. The image $\operatorname{im}(1 + Q) = (1 + Q)(C_\lambda)$ is closed in the Banach space $C_\lambda(\overline{D}, 0)$, the quotient space $C_\lambda / \operatorname{im}(1 + Q)$ is finite-dimensional and its dimension is also n.

Due to the symmetry of the function σ in inequality (2.22), we can replace λ by $-\lambda - 1$. Therefore, for a sufficiently small $\varepsilon > 0$, it also holds for $\nu = -\lambda - 1 + \varepsilon$.

Consider the adjoint operator $Q^* = Q_0^* + Q_1^*$ in the space $C_{\nu-1}(\overline{D}, 0)$. As above, the operator Q_0^* is compact, as for Q_1^*, from its expression similar to (2.14) and the choice of ν it is clear that its norm in the space $C_{\nu-1}(\overline{D}, 0)$ is also strictly less than 1. Thus, the operator $1 + Q^*$ is also Fredholm in the space $C_{\nu-1}(\overline{D}, 0)$ and its index is zero. Let n^* be the dimension of the space $\ker(1 + Q^*)$.

By the choice of ν the bilinear form (2.13) is bounded on the product space $C_\lambda \times C_{\nu-1}(\overline{D}, 0)$. Let the closed space $[\ker(1+Q^*)]^\perp \subseteq C_\lambda$ consist of functions $\varphi \in C_\lambda(\overline{D}, 0)$ which are orthogonal to the kernel $\ker(1 + Q^*)$ with respect to this form and $[\ker(1 + Q)]^\perp \subseteq C_{\nu-1}(\overline{D}, 0)$ has a similar meaning with respect to $\ker(1 + Q)$. Then, by virtue of (2.15), we have the relations

$$\operatorname{im}(1 + Q) \subseteq [\ker(1 + Q^*)]^\perp, \quad \operatorname{im}(1 + Q^*) \subseteq \ker(1 + Q)]^\perp. \tag{2.23}$$

In terms of dimensions, these relations mean that

$$n = \dim[C_\lambda/\operatorname{im}(1 + Q)] \geq n^*, \quad n^* = \dim[C_{\nu-1}/\operatorname{im}(1 + Q^*)] \geq n.$$

Therefore, $n = n^*$ and the relations (2.23) are in fact exact equalities.

Changing now λ and $\nu = -\lambda - 1 + \varepsilon$ in the interval $(-\delta - 1, \delta)$, we may conclude that the kernel $\ker(1 + Q)$ in the space $C_\lambda(\overline{D}, 0)$ does not depend on λ and the kernel $\ker(1 + Q^*)$ in the space $C_{-\lambda-1}(\overline{D}, 0)$ has a similar property. As a result, we prove the statements (a)–(c) of the theorem.

Consider the operator K from (2.20) and the homogeneous equation $K\varphi = 0$ in the class $\varphi \in C_\lambda(\overline{D}, 0)$. It can be rewritten as $\varphi + Q\varphi = f$ with the right-hand side

$$f = -\sum\nolimits_{i=1}^{n} (\varphi, \varphi_i)\tilde{\varphi}_i,$$

which has to satisfy the solvability condition (2.18). But due to the choice (2.19) this is possible only when

$$(\varphi, \varphi_i) = 0, \ 1 \leq i \leq n, \tag{2.24}$$

i.e. when $f = 0$. But then φ is a linear combination of functions φ_j. Substituting this linear combination into (2.24) and taking into account that the Gram matrix with elements (φ_i, φ_j) is invertible, we obtain $\varphi = 0$.

Since the operator K differs from $1 + Q$ by a finite-dimensional term, it is also a Fredholm operator of index zero. Therefore, this operator is invertible.

Furthermore, let $\varphi = K^{-1} f$, where the function f satisfies the conditions (2.18). Since the function $\varphi + Q\varphi$ also satisfies these conditions, they are also satisfied with respect to the function

$$\sum\nolimits_{i=1}^{n} (\varphi, \varphi_i)\tilde{\varphi}_i = f - (\varphi + Q\varphi).$$

which, in view of (2.19), implies $(\varphi, \varphi_i) = 0,\ 1 \le i \le n$. therefore, $\varphi + Q\varphi = f$, which completes the proof of (d) and that of the theorem.

Note that Theorem 2.1 allows us to extend Theorem 2.2 to the spaces $C^\mu_\lambda(\overline{D}, 0)$ with $0 < \mu < 1$. Indeed, based on this theorem, any solution $\varphi \in C_\lambda(\overline{D}, 0),\ -\delta - 1 < \lambda < \delta$ of equation $\varphi + Q\varphi = f$ with the right-hand side $f \in C^\mu_\lambda(\overline{D}, 0)$ belongs to the same class. In particular, the functions φ_j belong to $C^\mu_{\delta-0}(\overline{D}, 0)$. The functions $\tilde{\varphi}_j$ in condition (2.18) can obviously be chosen from the class C^μ_δ. Therefore, the operator K is bounded in the space C^μ_λ and the above reasoning shows that it is invertible in these spaces, too.

3 Original Equation

We construct explicitly one regular solution of Eq. (2.2) with the right-hand side $\varphi = -a$. For this purpose, we denote by $a^1 = a - a^0$ the absolutely convergent series in (1.4). If the number α is not an integer, then by the obvious relations

$$2\frac{\partial}{\partial \bar{z}}(|z|) = \frac{z}{|z|}, \quad 2\frac{\partial}{\partial \bar{z}}(z^k|z|^{-\alpha-k}) = -(\alpha + k)z^{k+1}|z|^{-\alpha-k-2}$$

the function

$$\omega^1 = \omega^1(z) = |z|^{-\alpha} \sum\nolimits_{k=-\infty}^{\infty} \frac{2a_{k+1}}{\alpha + k} e^{ik\theta} \tag{3.1a}$$

is a solution to Eq. (2.2) with the right-hand side $-a^1$.

If α is an integer, then by the relation

$$2\frac{\partial}{\partial \bar{z}}(\ln |z|) = \frac{\partial}{\partial \bar{z}}(\ln \bar{z}) = \frac{z}{|z|^2}$$

a solution to this equation is the function

$$\omega^1(z) = |z|^{-\alpha} \sum\nolimits_{k \ne -\alpha} \frac{2a_{k+1}}{\alpha + k} e^{ik\theta} - 2a_{1-\alpha} z^{-\alpha} \ln |z|. \tag{3.1b}$$

Therefore, by Lemma 2.2, the function

$$\omega = \omega^1 + \omega^0, \quad \omega^0 = (\rho_{-[\alpha]} T_0 \rho_{[\alpha]+1}) a^0 \in C_{1-\alpha+0}(\overline{D}, 0), \tag{3.2}$$

is a regular solution to the equation

$$\frac{\partial \omega}{\partial \bar{z}} = -a. \tag{3.3}$$

Let us turn to the original equation (1.1).

Theorem 3.1 *Let the coefficients of* (1.1) *satisfy the conditions* (1.3)*,* (1.4) *and* $\delta \in (-1/2, 0)$ *be the root of the equation*

$$\sigma(\delta)\ \left[\overline{\lim}_{z\to 0}|zb(z)|\right] = 1$$

(putting $\delta = 0$ *in the case* $zb(z) \to 0$ *as* $z \to 0$*). Let* ω *is defined by* (3.1) *and* (3.2)*. Furthermore,* f *belongs to the class of functions for which* $e^{\omega} f \in C_{-\lambda-1}(\overline{D}, 0)$ *with non-integer weight order* λ*.*

Then, by using the notations of Theorem 2.2 *with respect to*

$$q = \rho_1 b e^{-2i\,\mathrm{Im}\,\omega}, \quad g_0 = T_0[\rho_{[\lambda]+1}(e^{\omega} f)], \tag{3.4}$$

a general regular solution of Eq. (1.1) *in the class of functions for which* $e^{\omega}u \in C_{-\lambda}(\overline{D}, 0)$ *and* $-\delta - 1 < [\lambda] - \lambda < \delta$*, is described by the formula*

$$u = \rho_{[\lambda]}\left[K(\phi_0 - g_0) + \sum\nolimits_1^n \xi_j \varphi_j\right] \tag{3.5}$$

with arbitrary $\xi_1, \ldots, \xi_n \in \mathbb{R}$*, where the function* $\phi_0 \in C(\overline{D})$ *is analytic in domain* $D \cup \{0\}$ *and satisfies the conditions*

$$(\phi_0, \psi_j^*) = (g_0, \psi_j^*), \quad 1 \le j \le n. \tag{3.6}$$

Proof As we note in the beginning of the paper the function e^{ω} is regular together with ω and, by virtue of (3.3), its derivative

$$\frac{\partial}{\partial \bar{z}}(e^{\omega}) = -ae^{\omega}.$$

Therefore, taking account of the notations (3.4), equality (1.1) can be rewritten in the form

$$\frac{\partial}{\partial \bar{z}}(e^{\omega}u) = e^{\omega}(f - b\bar{u}) = (e^{\omega} f) - \rho_{-1} q(e^{\bar{\omega}}\bar{u}).$$

Based on Lemma 2.2, we conclude that with respect to $v = e^{\omega}u \in C_{-\lambda}(\overline{D}, 0)$ this equality is equivalent to the equation

$$v + (\rho_{-[\lambda]} T_0 \rho_{[\lambda]})v = \rho_{-[\lambda]}(-g + \phi_0),$$

where $\phi_0 = \rho_{[\lambda]}\phi$ is analytic in $D \cup \{0\}$ and g_0 appears in (3.4). Therefore, after substituting $\varphi = \rho_{[\lambda]}v \in C_{[\lambda]-\lambda}(\overline{D}, 0)$ we conclude the assumptions of Theorem 2.2, from which the relations (3.5) and (3.6) are obtained directly.

Note that the remark to Theorem 2.2 extends to Theorem 3.1. In other words, if $e^{\omega} f \in C^{\mu}_{-\lambda}(\overline{D}, 0)$, then formula (3.5) with functions $\phi_0 \in C^{\mu}(\overline{D})$ describes the general solution of Eq. (1.1) in the class of functions for which $e^{\omega} u \in C^{\mu}_{-\lambda}(\overline{D}, 0)$. It is only necessary to take into account that, by Theorem 2.1, the function g_0 in (3.4) belongs to the class $C^{\mu}_{[\lambda]-\lambda}(\overline{D}, 0)$.

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Norms of Linear Functionals, Summability of Trigonometric Fourier Series and Wiener Algebras

R. M. Trigub

Abstract Let $L_{0,1}(\mathbb{R}_+)$ be the Banach space endowed with the norm

$$\|f\|_{0,1} = \int_0^\infty f^*, \qquad f^*(x) = \operatorname*{ess\,sup}_{t \ge x} |f(t)|,$$

while $L_{0,\infty}(\mathbb{R}_+)$ with the norm $\sup\limits_{x>0} \dfrac{1}{x} \displaystyle\int_0^x |f|$.

First, the norms of continuous linear functionals are calculated in these spaces (Theorems 1 and 2). Secondly, these theorems are applied to the problems of summability of trigonometric Fourier series, as $\varepsilon \to 0$, by the linear means of type

$$\sum_{k \in \mathbb{Z}} \varphi(k\varepsilon) \widehat{f_k} e^{ikx},$$

where $\widehat{f_k}$ are the Fourier coefficients of f, $k \in \mathbb{Z}$, according to the function $\varphi : \mathbb{R} \to \mathbb{C}$.

Theorem 6 is a criterion, that is, a necessary and sufficient condition for the summability at every Lebesgue point (almost everywhere).

In Theorem 3, a general sufficient condition for the boundedness in ε on a set wider than that of Lebesgue points of the norms of these means as functionals is obtained. As a consequence, a general sufficient condition for the convergence of the mentioned means as $\varepsilon \to 0$ at all the points of differentiability of indefinite integral of f (d-points) is proved (Theorem 4), while for compactly supported φ a necessary condition is given.

The results are accompanied by examples.

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1 Introduction

Let f and $g : \mathbb{R} \to \mathbb{C}$ be locally integrable (written $L_{1,\mathrm{loc}}(\mathbb{R})$). We introduce two Banach spaces

$$L_{0,\infty} = L_{0,\infty}(\mathbb{R}) = \left\{ f : \|f\|_{0,\infty} = \sup_{x>0} \frac{1}{2x} \int_{-x}^{x} |f| \right\},$$

$$L_{0,1} = L_{0,1}(\mathbb{R}) = \left\{ g : \|g\|_{0,1} = 2 \int_{0}^{\infty} g^*,\ g^*(x) = \operatorname*{ess\,sup}_{|t|\geq x>0} \left|g(t)\right| \right\}.$$

Obviously, $\|f\|_{0,\infty} \leq \|f\|_{L_\infty}$, while $\|g\|_{0,1} \geq \|g\|_{L_1}$.

The following two theorems proved in this paper are the main results.

Theorem 1 *In the adopted terms, we have*

$$\textbf{I.} \quad \sup_{f:\|f\|_{0,\infty}\leq 1} \left| \int_{\mathbb{R}} fg \right| = \|g\|_{0,1}, \qquad \text{and} \qquad \textbf{II.} \quad \sup_{g:\|g\|_{0,1}\leq 1} \left| \int_{\mathbb{R}} fg \right| = \|f\|_{0,\infty}.$$

Theorem 2 *If $f \in L_1(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, +\infty)$, and g is uniformly continuous on $\mathbb{R}_+$, then*

$$\sup_{f:\left|\int_0^x f\right|\leq x} \left| \int_{\mathbb{R}_+} fg \right| = \lim_{\delta\to+0} \delta \sum_{k\in\mathbb{Z}_+} (k+1)\left|g(k\delta) - g((k+1)\delta)\right|.$$

We shall write the Fourier series of a 2π-periodic function $f \in L_1(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$, in the form

$$f \sim \sum_{k\in\mathbb{Z}} \widehat{f}_k e_k, \qquad e_k = e^{ikx}, \qquad \widehat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt} dt.$$

As the celebrated Kolmogorov's example shows, a Fourier series may diverge everywhere. This is the reason why for a long time the convergence to f, as $\varepsilon \searrow 0$, of the linear means of the form $\Phi_\varepsilon \sim \sum_k \varphi(k\varepsilon) e_k$ has been studied:

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(x-t)\Phi_\varepsilon(t)dt \sim \sum_{k\in\mathbb{Z}} \varphi(k\varepsilon) \widehat{f}_k e^{ikx} \tag{1}$$

in accordance with the function $\varphi : \mathbb{R} \to \mathbb{C}$. Usually, $\frac{1}{\varepsilon} = n \in \mathbb{N}$. For the convergence of the constant functions, $\varphi(0) = 1$ should hold. If φ is bounded and continuous almost everywhere, then for the convergence of the corresponding means to f in $L_1(\mathbb{T})$ norm (or $C(\mathbb{T})$ norm for continuous periodic functions) the following criterion holds true: φ, *after possible correction by continuity belongs to the Wiener algebra*

$$W(\mathbb{R}) = \left\{ f : f(x) = \int_{-\infty}^{\infty} e^{-itx} d\mu(t),\ \|f\|_W = |\mu|(\mathbb{R}) \right\},$$

with $\varphi(0) = 1$. See [11, **8.1.2**]; for basics on the finite Borel measures μ and positive measures $|\mu|$, where $|\mu| = \operatorname{var} \mu$, see, e.g., [8, Ch. XI]. In the case where $d\mu(t) = g(t)dt$, $g \in L_1(\mathbb{R})$, with $f = \widehat{g}$, we have an ideal $W_0(\mathbb{R})$. The properties of these algebras are collected in the survey [7] (see also A–E in Sect. 4).

Lebesgue introduced the l-points, that is, the points x for which $l_f(x)$ exists such that

$$\lim_{|h|\to 0} \frac{1}{h} \int_0^h \left| f(x+t) - f(x) \right| dt = 0.$$

He proved that for each $f \in L_1(\mathbb{T})$ almost all the points are its Lebesgue points. Besides this, he proved that the arithmetic means of the partial Fourier sums ($(C, 1)$-means), corresponding to $\varphi(x) = \big(1 - |x|\big)_+$, converge at all the l-points. There is a criterion for the summability at Lebesgue points (see Theorem 6 below). For the proof, see [2, Th.5].

There is a wider set of the points of differentiability of integral $F(x) = \int_0^x f$, that is, the points for which the limit

$$\lim_{|h|\to 0} \frac{1}{h} \int_0^h f(x+t)dt = \lim_{|h|\to 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$

exists. These are called the d-points. Convergence at all the d-points implies the convergence at all the l-points, while convergence at all the l-points yields the convergence in norm on the whole space $C(\mathbb{T})$ ($L_1(\mathbb{T})$) as well as on $L_p(\mathbb{T})$ for $p > 1$. Hahn in [4] detected that $(C, 1)$-means sum a Fourier series not at every d-point, while Hardy proved in [5, Th.253] that a (C, α)-method, with $\alpha > 1$ sums at every d-point. To ensure the convergence of the $(C, 1)$-means at all the d-points, the following criterion is given in [10, Th.1]:

In order for the $(C, 1)$*-means to converge at a d-point* x, *it is necessary and sufficient for the Fourier series of the continuous function*

$$F_x(t) = \frac{1}{t} \int_x^{x+t} f(u)\, du$$

to be convergent at zero.

In general, convergence of a family of linear operators on a Banach space, say as $\varepsilon \searrow 0$, is proved by means of the Banach-Steinhaus criterion. It contains two conditions: boundedness of the norms of these operators in ε and convergence on a dense set. Back to the condition $\varphi(0) = 1$, we mention that it means the convergence of the mentioned means (1) on a dense set of functions.

The following theorem is our main new application and is connected with the problem of summability of Fourier series of any function $f \in L_1(\mathbb{T})$ at all the points of differentiability of the integral $F(x) = \int_0^x f$, that is, d-points. We mention that they satisfy the condition

$$\sup_{0<|h|\le\pi} \left| \frac{1}{h} \int_x^{x+h} f(t)\,dt \right| = \sup_{0<|h|\le\pi} \left| \frac{F(x+h) - F(x)}{h} \right| < \infty. \tag{2}$$

There are points that satisfy this condition other than d-points, for instance, those with finite one-sided derivatives of F.

Theorem 3 *Let $\varphi \in W_0(\mathbb{R})$ (which is also necessary).*

I. *If φ is compactly supported, $\Phi_\varepsilon \sim \sum_k \varphi(k\varepsilon)e_k$, and*

$$\sup_{\varepsilon>0} \sup_{\substack{f:\left|\frac{1}{t}\int_0^t f\right|\le 1,\\ |t|\le\pi}} \left| \int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)\,dt \right| < \infty,$$

then $\varphi_1' \in W_0(\mathbb{R})$, where $\varphi_1(x) = x\varphi(x)$.

II. *If $\varphi_1' \in W_0(\mathbb{R})$, then*

$$\sup_{\varepsilon>0} \sup_{\substack{f:\left|\frac{1}{t}\int_x^{x+t} f\right|\le 1,\\ |t|\le\pi}} \left| \int_{\mathbb{T}} f(x-t)\Phi_\varepsilon(t)\,dt \right| < \infty.$$

Applying Theorem 3 to summability at d-points, we derive the following

Theorem 4

I. *If $\varphi_1' \in W_0(\mathbb{R})$ and $\varphi(0) = 1$, then at every d-point of any $f \in L_1(\mathbb{T})$ there holds*

$$\lim_{\varepsilon\to 0} \sum_k \varphi(k\varepsilon)\widehat{f}_k e^{ikx} = F'(x) \qquad \left(F(x) = \int_0^x f\right).$$

II. *Condition $\varphi_1' \in W(\mathbb{R})$ is not sufficient.*

III. *Condition $\varphi_1' \in W_0(\mathbb{R})$ cannot be necessary in general.*

The structure of the paper is as follows. Further in Sect. 2, Theorem 1 is proven; in Sect. 3, the proof of Theorem 2 is given; in Sect. 4 these theorems are applied to summability of the Fourier series (for a general sufficient condition for summability at d-points, see Theorem 4). Finally, in Sect. 5 some properties of the introduced spaces are discussed.

2 Proof of Theorem 1

Integrating by parts, we get

$$\Big|\int_{-\infty}^{\infty} fg\Big| \le \int_{-\infty}^{\infty} |f|g^* = \Big[\int_0^x fg^*(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\int_0^x |f|dg^*(x).$$

If

$$\|g\|_{0,1} = 2\int_0^{\infty} g^* < \infty,$$

then $\lim xg^*(x) = 0$ as $x \to 0$ and $x \to \infty$. Therefore, if $\|f\|_{0,\infty} < \infty$, then $[\ldots]_{-\infty}^{\infty} = 0$ and

$$\Big|\int_{\mathbb{R}} fg\Big| \le -\int_{\mathbb{R}} |x|dg^*(x)\cdot\|f\|_{0,\infty} = \|f\|_{0,\infty}\int_{\mathbb{R}} g^* = \|f\|_{0,\infty}\|g\|_{0,1}.$$

To prove (**II**) in the theorem, we choose

$$g = g_N, \qquad g_N(x) = \frac{\operatorname{sign} f(x)}{2N} \ (|x| \le N), \qquad g_N(x) = 0 \ (|x| > N),$$

where $\operatorname{sign} z = \frac{|z|}{z}$ $(z \in \mathbb{C}\setminus\{0\})$ and $\operatorname{sign} 0 = 0$. This gives

$$\|g_N\|_{0,1} = 1, \qquad \Big|\int_{-\infty}^{\infty} fg_N\Big| = \frac{1}{2N}\int_{-N}^{N}|f|, \qquad \sup_{N>0}\Big|\int_{-\infty}^{\infty} fg_N\Big| = \|f\|_{0,\infty}.$$

More complicated is the proof of (**I**) in this theorem. If $g(x) = g^*(x)$ (here and in what follows almost everywhere), then we set $f = \operatorname{sign} g$. Let

$$M_g = \sup_{\|f\|_{0,\infty}\le 1}\Big|\int_{\mathbb{R}} fg\Big|. \tag{3}$$

We first prove that $M_g < \infty \Rightarrow xg(x) \in L_\infty(\mathbb{R})$. For x and $h > 0$, we set $f_h(t) = \frac{x+h}{h}$ if $\big(t \in [x, x+h]\big)$, $f_h(t) = 0$ if $t > 0$ and $t \notin [x, x+h]$, and

$f_h(-t) = f_h(t)$. Then for $y \in [x, x+h]$, and correspondingly for any $y > 0$,

$$\int_{-y}^{y} f_h = 2\int_0^y f_h = 2\int_x^y f_h = 2\frac{x+h}{h}(y-x) \le 2y.$$

Here we check the linear equality with respect to y at $y = x$ and $y = x+h$. By this,

$$\int_{\mathbb{R}} \left|f_h g\right| = \frac{x+h}{h}\int_{x\le|t|\le x+h} \left|g(t)\right| dt \le Mg.$$

Passing to the limit as $h \to +0$, we obtain $x\left|g(x)\right| \le Mg$ $(x > 0)$. This implies $g^*(x) < \infty$ for $x \ne 0$. Setting $f = \operatorname{sign} g$, one can assume f and g to be positive.

If $g(x) = g^*(x)$, we assume $f \equiv 1$. Let us consider the case $g(x) \ne g^*(x)$ at a set of positive measure. Then, an interval $[x_1, x_2]$ exists, for definiteness, $[x_1, x_2] \subset \mathbb{R}_+$, on which

$$g^*(x_2) \le g^*(x_1), \qquad g^*(x) < g^*(x_1) \qquad \big(x \in (x_1, x_2)\big).$$

Let us derive from this that

$$g(x) \le g^*(x_2) \qquad \big(x \in (x_1, x_2)\big). \tag{4}$$

If $g^*(x_2) = g^*(x_1)$, this is obvious. If $g^*(x_2) < g^*(x_1)$, we assume the contrary, that is, that a set $E \subset (x_1, x_2)$ of positive measure exists on which $g(x) > g^*(x_2)$ (and also $g(x) \le g^*(x_1)$). But in this case, for any $\delta > 0$ small enough, we have on $E_\delta = E \cap [x_1 + \delta, x_2 - \delta]$ that $g(x) > g^*(x_2)$. If the measure of E_δ were zero for small $\delta > 0$, the measure E would also be zero. Therefore, there is $x_3 \in [x_1 + \delta, x_2 - \delta]$ such that

$$g^*(x_1) > g^*(x_3) = \operatorname*{ess\,sup}_{|x|\in E_\delta} g(x) > g^*(x_2).$$

This contradiction proves (4).

We now take f to be smaller in the integral $\displaystyle\int_{x_1}^{x_2} fg$, whereafter the norm does not increase, and enlarge g to g^* so that the product fg leaves unchanged. If $g \ne 0$, we replace f by $f_1 = f \cdot \frac{g}{g^*}$ $(fg = f_1 g^*)$. And if $g = 0$, we put $f_1 = f = 0$ and $g = g^*$. Substituting f_1 for f in this way only on (x_1, x_2), we obtain

$$\int_{\mathbb{R}} fg = \int_{\mathbb{R}} f_1 g_1, \qquad g_1 = g^* \,\big(x \in [x_1, x_2]\big), \qquad g_1 = g \,\big(x \in \mathbb{R} \setminus [x_1, x_2]\big).$$

To complete the proof of Theorem 1, we replace g by g^* without changing the integral on each of the at most countable number of intervals and set $f \equiv 1$.

3 Proof of Theorem 2

We first prove an auxiliary lemma.

Lemma 1 (Discrete Version) *Let* $M_\beta = \sup\limits_{\left|\sum_{k=0}^{n} \alpha_k\right| \le n+1} \left|\sum_{k\in\mathbb{Z}_+} \alpha_k \beta_k\right|.$

If $M_\beta < \infty$, *then* $\beta_n = O\left(\frac{1}{n}\right)$, *while for* $\beta_n = O\left(\frac{1}{n}\right)$, *we have*

$$M_\beta = \sum_{k\in\mathbb{Z}_+} (k+1)|\Delta\beta_k| \qquad (\Delta\beta_k = \beta_k - \beta_{k+1}).$$

Proof If $M_\beta < \infty$, then for $\alpha_n = n+1$ and $\alpha_k = 0$ if $k \ne n$, we have

$$(n+1)|\beta_n| \le M_\beta.$$

If $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} \alpha_k$, $n \in \mathbb{Z}_+$, we obtain, for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \alpha_k\beta_k = \sum_{k=0}^{n} (A_k - A_{k-1})\beta_k = \sum_{k=0}^{n} A_k\beta_k - \sum_{k=0}^{n-1} A_k\beta_{k+1} = \sum_{k=0}^{n-1} A_k\Delta\beta_k + A_n\beta_n. \tag{5}$$

Assuming further that $\beta_n = O\left(\frac{1}{n}\right)$, we derive from $\sum_{k=0}^{\infty}(k+1)|\Delta\beta_k| < \infty$ that

$$\beta_n = \sum_{k=n}^{\infty} \Delta\beta_k, \qquad |\beta_n| \le \frac{1}{n+1}\sum_{k=n}^{\infty}(k+1)|\Delta\beta_k| = o\left(\frac{1}{n}\right).$$

Let us now check that if also $\beta_{k+1} \ne \beta_k$ for $k \ge 0$, then

$$M_\beta = \sum_{k\in\mathbb{Z}_+} (k+1)|\Delta\beta_k|.$$

Let, for definiteness, $\Delta\beta_0 = \beta_0 - \beta_1 \ge 0$. If $\Delta\beta_0 > 0$, we choose $A_0 = \alpha_0 = \text{sign}\Delta\beta_0 = 1$ and $\alpha_1 = 2\text{sign}\Delta\beta_1 - \alpha_0$. Thus

$$A_1 = \alpha_0 + \alpha_1 = 2\text{sign}\Delta\beta_1, \qquad |A_1| = 2, \qquad \alpha_1 = 1.$$

Choosing further

$$\alpha_k = (k+1)\text{sign}\Delta\beta_k - A_{k-1} \quad (k \ge 2),$$

we conclude, by induction, that

$$A_k = (k+1)\operatorname{sign}\Delta\beta_k, \quad |A_k| = k+1, \quad \alpha_k = 1, \quad A_k\Delta\beta_k = (k+1)|\Delta\beta_k|. \tag{6}$$

It follows from (5) that $M_\beta \le \sum_{k=0}^{\infty} |A_k\Delta\beta_k|$, while for the chosen sequence $\{\alpha_k\}_0^\infty$, we get the equality.

Let now $\beta_{k+1} = \beta_k$ for some k. Let us fix $\varepsilon > 0$. If $\beta_1 = \beta_0 = \tilde{\beta}_0$, we then replace β_1 by $\tilde{\beta}_1$, which satisfies the condition

$$0 < \left|\tilde{\beta}_1 - \beta_1\right| < \varepsilon.$$

If $\beta_{k+1} = \tilde{\beta}_k$, we choose $\tilde{\beta}_{k+1}$ $(k \ge 1)$, which satisfies the condition

$$0 < \left|\tilde{\beta}_{k+1} - \beta_{k+1}\right| < \frac{\varepsilon}{k^3}.$$

Hence

$$\Big|\sum_{k\in\mathbb{Z}_+} \alpha_k\beta_k - \sum_{k\in\mathbb{Z}_+} \alpha_k\tilde{\beta}_k\Big| \le \varepsilon \sum_{k=1}^{\infty} \frac{|\alpha_k|}{k^3} = \varepsilon \sum_{k=1}^{\infty} \frac{1}{k^3}|A_k - A_{k-1}| \le \varepsilon \sum_{k=1}^{\infty} \frac{2k+1}{k^3} = c\varepsilon,$$

and $|M_\beta - M_{\tilde{\beta}}| \le c\varepsilon$. But it has already been proved that $M_{\tilde{\beta}} = \sum_{k\in\mathbb{Z}_+} (k+1)|\Delta\tilde{\beta}_k|$ in case of convergent series. Therefore, $M_\beta = \sum_{k\in\mathbb{Z}_+} (k+1)|\Delta\beta_k|$ $(\varepsilon \to 0)$.

Let now

$$\sum_{k\in\mathbb{Z}_+} (k+1)|\Delta\beta_k| = \infty.$$

As follows from (5),

$$\Big|\sum_{k\in\mathbb{Z}_+} A_k\Delta\beta_k\Big| = \infty \qquad \Leftrightarrow \qquad \Big|\sum_{k\in\mathbb{Z}_+} \alpha_k\beta_k\Big| = \infty.$$

For $\Delta\beta_0 \ge 0$ and $\Delta\beta_k \ne 0$ $(k \ge 0)$, the sequence $\{\alpha_k\}_0^\infty$ has been constructed above such that $A_k\Delta\beta_k = (k+1)|\Delta\beta_k|$. Therefore $\sum_k (k+1)|\Delta\beta_k| = \infty$ yields $M_\beta = \infty$. If $\beta_{k+1} = \beta_k$ for some k, then we replace β_k by $\tilde{\beta}_k$ whenever $\varepsilon = 1$ (see above). The proof of the lemma is complete.

We are now in a position to proceed to the proof of Theorem 2.

Proof As in the proof of Theorem 1, we have

$$M_g = \sup_{f:\left|\int_0^x f\right|\le x} \left|\int_0^\infty fg\right| < \infty \qquad \Rightarrow \qquad xg(x) \in L_\infty(\mathbb{R}_+).$$

Fixing $\delta > 0$, we replace g by the step-wise function (k is integer)

$$g_\delta(t) = g(k\delta) \qquad \big(t \in [k\delta, (k+1)\delta)\big).$$

Then

$$\lim_{\delta\to 0} \max_t \left|g(t) - g_\delta(t)\right| = 0 \qquad \Rightarrow \qquad \lim_{\delta\to 0} \int_0^\infty fg_\delta = \int_0^\infty fg,$$

since g is uniformly continuous and $f \in L_1(\mathbb{R}_+)$. Therefore, $\lim_{\delta\to 0} M_{g_\delta} = M_g$ and

$$M_{g_\delta} = \sup_{\left|\int_0^x f\right|\le x} \left|\int_0^\infty fg_\delta\right| = \sup\left|\sum_{k=0}^\infty \int_{k\delta}^{(k+1)\delta} fg_\delta\right| = \sup\left|\sum_{k=0}^\infty g(k\delta)\frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} f\right|\delta.$$

Applying the lemma for $\alpha_k = \frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} f$ and $\beta_k = g(k\delta)$, we get $\left|\sum_{k=0}^n \alpha_k\right| = \left|\frac{1}{\delta}\int_0^{(n+1)\delta} f\right| \le n+1$.

It remains to check that for any sequence $\{\alpha_k\}_0^\infty$ satisfying $\left|\sum_{k=0}^n \alpha_k\right| \le n+1$ ($n \in \mathbb{Z}_+$), there is a function satisfying the conditions

$$\frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} f = \alpha_k \quad (k \in \mathbb{Z}_+), \qquad \frac{1}{x}\left|\int_0^x f\right| \le 1 \quad (x > 0).$$

Setting $f(t) = \alpha_0$ on $[0, \delta)$ taking into account that $|\alpha_0| \le 1$, we have

$$\frac{1}{\delta}\int_0^\delta f = \alpha_0, \qquad \left|\int_0^x f\right| = |\alpha_0|x \le x \quad \big(x \in [0, \delta)\big).$$

Assuming that the function is already constructed on $[0, n\delta)$ for $n \in \mathbb{N}$, which satisfies the conditions

$$\frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} f = \alpha_k \quad (0 \le k \le n-1), \qquad \left|\int_0^x f\right| \le x \quad \big(x \in [0, n\delta)\big),$$

we set $f(t) = \alpha_n$ on $[n\delta, (n+1)\delta)$. Then $\frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} f = \alpha_n$, and if $x = \delta(n+\varepsilon)$, where $\varepsilon \in [0, 1)$, we have

$$\int_0^x f = \int_0^{n\delta} f + \int_{n\delta}^{(n+\varepsilon)\delta} f = \delta\Big(\sum_{k=0}^{n-1} \alpha_k + \varepsilon\alpha_n\Big).$$

It has to be proved that for $\varepsilon \in [0, 1)$,

$$\Big|\sum_{k=0}^{n-1} \alpha_k + \varepsilon\alpha_n\Big| \le n + \varepsilon.$$

This equality is valid for $\varepsilon = 0$ $\Big(\Big|\sum_{k=0}^{n-1} \alpha_k\Big| \le n\Big)$ and $\varepsilon = 1$ $\Big(\Big|\sum_{k=0}^{n} \alpha_k\Big| \le n+1\Big)$. It suffices to check the inequality of the form $|z_1 + \varepsilon z_2| \le a + \varepsilon|z_2|$ only for $\varepsilon = 0$ and $\varepsilon = 1$, since after squaring, a linear inequality with respect to ε, with real coefficients, holds, which completes the proof of Theorem 2.

We can now derive a consequence for functions on the whole axis.

Let $f \in L_1(\mathbb{R})$, let g be uniformly continuous on $\mathbb{R}$, and M_g be defined in (3). By f_c and f_s we denote the even and odd parts of f, respectively. Correspondingly, $(f = f_c + f_s)$

$$M_g = \sup\Big|\int_{\mathbb{R}} fg\Big| \le \sup\Big|\int_{\mathbb{R}} f_c g\Big| + \sup\Big|\int_{\mathbb{R}} f_s g\Big|.$$

Replacing in the first integral g by g_c and in the second one g by g_s, we obtain

$$M_g \le 2\sup\Big|\int_0^\infty f_c g_c\Big| + 2\sup\Big|\int_0^\infty f_s g_s\Big|.$$

Applying Theorem 2, with $\Big|\int_0^x f_c\Big| \le x$ and $\Big|\int_0^x f_s\Big| \le x$, we get

$$M_g \le 2\Big(\lim_{\delta\to 0} \delta \sum_{k=0}^\infty \big|g_c(k\delta) - g_c((k+1)\delta)\big|(k+1) + \lim_{\delta\to 0} \delta \sum_{k=0}^\infty \big|g_s(k\delta) - g_s((k+1)\delta)\big|(k+1)\Big).$$

Proceeding to the lower estimate of M_g, we obtain

$$\sup\Big|\int_{\mathbb{R}} f_c g\Big| \le \frac{1}{2}\sup\Big|\int_{\mathbb{R}} f(t)g(t)dt\Big| + \frac{1}{2}\sup\Big|\int_{\mathbb{R}} f(-t)g(t)dt\Big| = M_g,$$

along with the same inequality for f_s. Thus,

$$M_g \geq \frac{1}{2}\Big(\sup\Big|\int f_c g\Big| + \sup\Big|\int f_s g\Big|\Big) = \sup\Big|\int_0^\infty f_c g_c\Big| + \sup\Big|\int_0^\infty f_s g_s\Big| =$$

$$= \lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}(k+1)\big|g_c(k\delta) - g_c((k+1)\delta)\big| + \lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}(k+1)\big|g_s(k\delta) - g_s((k+1)\delta)\big|.$$

By this, we have proved the following

Corollary 5 *For* $M_g = \sup_{f:\left|\int_0^x f\right|\leq|x|}\Big|\int_{\mathbb{R}} fg\Big| < \infty$*, it is necessary and sufficient that*

$$\lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}(k+1)\big|g_c(k\delta)-g_c((k+1)\delta)\big| + \lim_{\delta\to 0}\delta\sum_{k=0}^{\infty}(k+1)\big|g_s(k\delta)-g_s((k+1)\delta)\big| < \infty,$$

or there are the constants $c_1 > 0$ *and* $c_2 > 0$ *such that for all uniformly continuous* g*, there holds*

$$c_1 M_g \leq \overline{\lim_{\delta\to+0}}\,\delta\sum_{k\in\mathbb{Z}}\big(|k|+1\big)\big|g(k\delta) - g((k+1)\delta)\big| \leq c_2 M_g.$$

We observe that for passing to the upper limit, we use $g = g_c + g_s$, while for $g = g_c$ or g_s, we have

$$\sum_{k\in\mathbb{Z}}\big(|k|+1\big)\big|g(k\delta)-g((k+1)\delta)\big| = \sum_{k=0}^{\infty}(k+1)\big|g(k\delta)-g((k+1)\delta)\big| + \sum_{k=1}^{\infty}(k+2)\big|g(k\delta)-g((k+1)\delta)\big|.$$

It remains to take into account that the upper limit of a sum does not exceed the sum of the upper limits.

For the lower estimate of M_g, we mention that

$$\sum_{k=0}^{\infty}(k+1)\big|g_c(k\delta) - g_c((k+1)\delta)\big| \leq \frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Big(\big|g(k\delta) - g((k+1)\delta)\big|$$

$$+\big|g(-k\delta) - g(-(k+1)\delta)\big|\Big) \leq \sum_{k\in\mathbb{Z}}\big(|k|+2\big)\big|g(k\delta) - g((k+1)\delta)\big|.$$

4 Applications

As mentioned in Sect. 1, assertion I of Theorem 1 was used in [11, p.353] for proving the following criterion of summability of Fourier series at all the Lebesgue points:

Theorem 6 *If* $\varphi(0) = 1$ *and* $\varphi(x) = \int_{-\infty}^{\infty} e^{ixt} d\mu(t)$*, i.e.,* $\varphi \in W(\mathbb{R})$ *(this is also necessary if* $\varphi \in C(\mathbb{R})$*), then for*

$$\lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \varphi(k\varepsilon) \widehat{f}_k e^{ikx} = f(x)$$

at every Lebesgue point of any function $f \in L_1(\mathbb{T})$*, it is necessary and sufficient that for the measure* μ *to be absolutely continuous with respect to the Lebesgue measure, that is,* $d\mu = g dx$ *(*$\varphi \in W_0(\mathbb{R})$*), and*

$$\int_0^{\infty} g^*(x)dx < \infty, \qquad g^*(x) = \operatorname*{ess\,sup}_{|t| \geq x} \big| g(t) \big|.$$

We are now in a position to prove Theorem 3.

Proof The following properties of the algebra $W(\mathbb{R})$ will be needed; some of them are obtained quite recently.

A. *For any* $h \in \mathbb{R}$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$*, we have* $\|f(\lambda \cdot +h)\|_W = \|f(\cdot)\|_W$*. If* $\lim\limits_{n\to\infty} f_n(x) = f(x)$*,* $\|f_n\|_W \leq 1$*, and* $f \in C(\mathbb{R})$*, then* $\|f\|_W \leq 1$ (see, e.g., [11, 6.1.4 b]).

B. *If* $f \in W(\mathbb{R})$*, then* f *is uniformly continuous on* $\mathbb{R}$ *and at* $x \in \mathbb{R}$ *the integral*

$$\lim_{\substack{A \to \infty \\ \varepsilon \to +0}} \int_{\varepsilon}^{A} \frac{f(x+t) - f(x-t)}{t} dt$$

converges, not necessarily absolutely. If $f \in W_0(\mathbb{R})$*, then* $f(\infty) = \lim\limits_{|x|\to\infty} f(x) = 0$.

C. *The functions in* $W(\mathbb{R})$ *possess the local property and may differ from functions in* $W_0(\mathbb{R})$ *only near infinity.*

If $f \in W(\mathbb{R})$*,* $f(\infty) = 0$ *and for some* N *and* $|x| \geq N$*, this function is of bounded variation, then* $f \in W_0(\mathbb{R})$ (see [11, **6.1.3c**]).

D. *If* $f \in W_0 \cap L_1(\mathbb{R})$*, then* $\widehat{f} \in L_1(\mathbb{R})$ and $\|f\|_{W_0} = \frac{1}{2\pi} \|\widehat{f}\|_{L_1(\mathbb{R})}$ [10].

E. *If* $f \in AC_{\text{loc}}(\mathbb{R})$*,* f *and* $f' \in L_2(\mathbb{R})$*, then* $\|f\|_{W_0} \leq c\big(\|f\|_{L_2} + \|f'\|_{L_2}\big)$ (due to Titchmarsh and Beurling, see, e.g., [7]).

If $f(x) = O\Big(\frac{1}{|x|^{\alpha}}\Big)$*,* $\alpha > 0$*,* $f'(x) = O\Big(\frac{1}{|x|^{\beta}}\Big)$ ($\beta \in \mathbb{R}$)*, and* $\alpha + \beta > 1$*, then* $f \in W_0(\mathbb{R})$ ([6, Corollary]; see, e.g., also Lemma 2).

The necessity of the condition $\varphi \in W_0(\mathbb{R})$ in Theorem 3 readily follows from Theorem 6.

Lemma 2 *In order for the series* $\sum_k c_k e_k$ *to be a Fourier-Stieltjes series (Fourier series), it is necessary and sufficient a function* $\varphi \in W(\mathbb{R})$ $\big(\varphi \in W_0(\mathbb{R})\big)$ *to exist such that* $\varphi(k) = c_k$, $k \in \mathbb{Z}$. *Moreover, if* $d\mu \sim \sum_k c_k e_k$, *then the total variation of the measure on* $\mathbb{T}$ *equals*

$$|\mu|(\mathbb{T}) = \int_{\mathbb{T}} d|\mu| = \min_{\varphi_c} \|\varphi_c\|_W, \qquad \varphi_c(k) = c_k \quad (k \in \mathbb{Z}),$$

and the minimum is attained for $\varphi_{c_0}(x) = \int_{\mathbb{T}} e^{-itx} d\mu(t)$.

For the Fourier series, where μ is absolutely continuous with respect to the Lebesgue measure, this assertion is proved in [9, Th.8]. In the general case, the proof is the same.

We observe that a point x satisfies condition (2) for a function f if and only if the point 0 satisfies the same condition for $f(x-t)$ as a function of t.

Lemma 3 *There holds*

$$\sup_{\varepsilon} \int_{\mathbb{T}} \Big| \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big) e^{ikt} \Big| dt < \infty \qquad \Rightarrow$$

$$\sup_{\varepsilon} \sup_{f : \left|\frac{1}{t}\int_0^t f\right| \le 1} \Big| \int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt \Big| < \infty.$$

Proof We introduce the function $F_0(t) = \dfrac{i}{1-e^{-it}} \int_0^t \big(f(-u) - \widehat{f_0}\big)du$, with $F_0(\pi) = F_0(-\pi)$, and the integral

$$\int_{\mathbb{T}} F_0(t) \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big) e^{ikt} dt = i \int_{\mathbb{T}} F_0(t) \big[(1-e^{-it}) \sum_k ik\varphi(k) e^{ikt}\big] dt$$

$$= i \int_{\mathbb{T}} F_0(t)(1-e^{-it})\Phi_\varepsilon'(t)dt = -\int_{\mathbb{T}} \int_0^t \big(f(-u) - \widehat{f_0}\big)du \cdot \Phi_\varepsilon'(t)\, dt.$$

Integrating by parts and taking into account that the integrated terms vanish by periodicity, we reduce the latter to

$$\int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt + \widehat{f_0} \cdot 2\pi\varphi(0).$$

We observe that f is determined uniquely by F_0, and the least upper bound of the absolute value of the introduced integral over all the functions F_0, for which

$\|F_0\|_{L_\infty} \le M$, is equal to

$$M \int_{\mathbb{T}} \Big| \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big)e^{ikt} \Big| \, dt.$$

Setting $\|f\|_0 = \sup_{|t|\le\pi} \Big|\frac{1}{t}\int_0^t f\Big|$, we get

$$\Big|\frac{1}{t}\int_0^t \big(f(-u) - \widehat{f_0}\big)du\Big| \le \|f\|_0 + |\widehat{f_0}| \le 2\|f\|_0$$

and

$$\sup_{\|f\|_0\le 1} |F_0(t)| = \sup\Big|\frac{1}{2\sin\frac{t}{2}}\int_0^t \big(f(-u)-\widehat{f_0}\big)du\Big| \le \frac{\pi}{2}\sup\Big|\frac{1}{t}\int_0^t \big(f(-u)-\widehat{f_0}\big)du\Big| \le \pi.$$

Hence,

$$\sup_{\|f\|_0\le 1} \Big| \int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt \Big| \le 2\pi|\varphi(0)|$$

$$+ \sup_{\|f\|_0\le 1} \Big| \int_{\mathbb{T}} F_0(t) \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big)e^{ikt} \Big| \, dt$$

$$\le \big(2\pi|\varphi(0)| + \pi\big) \int_{\mathbb{T}} \Big| \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big)e^{ikt} \Big| \, dt,$$

which completes the proof of the lemma.

Lemma 4 *If* $\varphi_1' \in W_0(\mathbb{R})$, *then*

$$\int_{\mathbb{T}} \Big| \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big)e^{ikt} dt \Big| \le \|\varphi_1'\|_{W_0}.$$

Proof If $f(x) = \int_{-\infty}^{\infty} g(y)e^{-ixy}dy$ and $g \in L_1(\mathbb{R})$ then for the function

$$f_h(x) = \frac{1}{h}\int_x^{x+h} f(t)dt = \frac{1}{h}\int_0^h f(x+t)dt = \int_{-\infty}^{\infty} e^{-ixy}\frac{1-e^{-ihy}}{ihy}g(y)dy,$$

we have

$$\|f_h\|_{W_0} \le \int_{-\infty}^{\infty} |g(y)|dy = \|f\|_{W_0}.$$

Taking into account that $k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon) = -\frac{1}{\varepsilon}\int_{\varepsilon x}^{\varepsilon(x+1)} \varphi_1'(t)dt$ and applying Lemma 2 and **A** yield

$$\int_{\mathbb{T}} \Big| \sum_k \big(k\varphi(k\varepsilon) - (k+1)\varphi((k+1)\varepsilon)\big)e^{ikt}\Big| dt \le \Big\| \frac{1}{\varepsilon}\int_0^{\varepsilon} \varphi_1'\big(\varepsilon(\cdot)+t\big)\, dt \Big\|_{W_0} \le \|\varphi_1'\|_{W_0},$$

which completes the proof.

The sufficiency of conditions (**II**) in Theorem 3 follows from Lemmas 4 and 3. Let us now proceed to the proof of the necessity of condition $\varphi_1' \in W_0$ for compactly supported functions.

Let

$$\sup_{\|f\|_{0,\infty}\le 1} \Big| \int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt \Big| \le M_0 \quad \big(\varepsilon \in (0,1]\big).$$

We set $\varphi(0) = 0$. Then the integral is independent of $\widehat{f_0}$, moreover, one can take $\widehat{f_0} = 0$ if desired. If

$$g_\varepsilon(t) = \sum_{k\ne 0} \varphi(k\varepsilon)e^{ikt} - \sum_{k\ne 0}(-1)^k\varphi(k\varepsilon) \quad \big(g_\varepsilon(\pi) = g_\varepsilon(-\pi) = 0\big),$$

then

$$\sup_{\|f\|_{0,\infty}\le 1} \Big| \int_{\mathbb{T}} f(-t)g_\varepsilon(t)dt \Big| \le M_0 \qquad \Rightarrow \qquad \int_{\mathbb{T}} |g_\varepsilon(t)|dt \le M_0. \tag{7}$$

Considering $f(t) = g_\varepsilon(t) = 0$ for $|t| \ge \pi$, and applying the Corollary 5 of Theorem 2, we obtain

$$\overline{\lim_{\delta\to 0}}\, \delta \sum_{k\in\mathbb{Z}} \big(|k|+1\big)\big|g_\varepsilon(k\delta) - g_\varepsilon((k+1)\delta)\big| \le M_1.$$

Therefore, both M_1 and M_0 are independent of ε.

Further, we take into account that if $g_{1,\varepsilon}(t) = tg_\varepsilon(t)$ and $k \ge 0$, then

$$\delta(k+1)\big|g_\varepsilon(k\delta) - g_\varepsilon((k+1)\delta)\big| = \big|g_{1,\varepsilon}(k\delta) - g_{1,\varepsilon}((k+1)\delta) + \delta g_\varepsilon(k\delta)\big|,$$

and

$$\delta(k+1)\big|g_\varepsilon(-k\delta) - g_\varepsilon((1-k)\delta)\big| = \big|g_{1,\varepsilon}(-k\delta) - g_{1,\varepsilon}((1-k)\delta) + 2\delta g_\varepsilon((1-k)\delta)\big|.$$

The limits of the Riemannian integral sums can be calculated and estimated. For example, by (7),

$$\lim_{\delta\to 0}\sum_{k\geq 1}\delta\left|g_\varepsilon((1-k)\delta)\right| = \int_{-\pi}^{0}|g_\varepsilon(t)|dt \leq M_0.$$

Thus,

$$\overline{\lim_{\delta\to 0}}\sum_{k\in\mathbb{Z}}\left|g_{1,\varepsilon}(k\delta) - g_{1,\varepsilon}((k+1)\delta)\right| \leq M_1 + 4M_0 = M_2.$$

By the continuity of $g_{1,\varepsilon}$, this upper limit equals to the total variation of $g_{1,\varepsilon}$ on $\mathbb{T}$, and since also $g_{1,\varepsilon} \in C^1$ on $\mathbb{T}$, we have

$$\int_{\mathbb{T}}\left|g'_{1,\varepsilon}\right| \leq M_2 \quad \big(\varepsilon\in(0,1]\big).$$

Taking into account $g'_{1,\varepsilon}(t) = tg'_\varepsilon(t) + g_\varepsilon(t)$ and (7), we obtain

$$\int_{\mathbb{T}}|t|\left|\sum_k k\varphi(k\varepsilon)e^{ikt}\right|dt = \int_{\mathbb{T}}\left|tg'_\varepsilon(t)\right|dt \leq 2\pi M_2 + M_0 = M_3.$$

Substituting $t = \varepsilon x$, we get

$$\int_{|x|\leq\frac{\pi}{\varepsilon}}|x|\left|\varepsilon\sum_k\varphi_1(k\varepsilon)e^{ik\varepsilon x}\right|dx \leq M_3.$$

Fixing $\varepsilon_1 > 0$ within the limits of integration $\left(|x| \leq \dfrac{\pi}{\varepsilon_1}\right)$ and passing to the limit as $\varepsilon\to 0$ and further as $\varepsilon_1\to 0$, we obtain

$$\int_{-\infty}^{\infty}\left|x\widehat{\varphi}_1(x)\right|dx \leq M_3. \tag{8}$$

The continuity of $\widehat{\varphi}_1$ allows us to derive from this that $\widehat{\varphi}_1 \in L_1(\mathbb{R})$.

By the Fourier inversion formula,

$$\varphi_1(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_1(x)e^{ixy}dx.$$

Now (8) implies $\varphi'_1 \in W_0(\mathbb{R})$, which completes the proof of Theorem 3.

We now prove Theorem 4.

Proof

I. By Theorem 3, the norms of the functionals are bounded in ε:

$$\sup_{\|f\|_{0,\infty}\le 1}\left|\int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt\right| \le M.$$

In virtue of the condition $\varphi(0) = 1$, it suffices to indicate a set of functions dense in the space for which

$$\lim_{x\to 0}\left|\frac{1}{x}\int_0^x f - F'(0)\right| = 0.$$

For $\varepsilon > 0$, there is $\delta > 0$ such that for $|x| < \delta$

$$\left|\frac{1}{x}\int_0^x f - F'(0)\right| < \varepsilon.$$

We introduce a function $f_\delta \in C(\mathbb{T})$ such that it is equal to $F'(0)$ if $|x| < \delta$, while for $\delta < |x| \le \pi$,

$$\left|\frac{1}{x}\int_{\delta\le|x|\le\pi}(f - f_\delta)\right| \le \frac{1}{\delta}\int_{\delta\le|x|\le\pi}|f - f_\delta| < \varepsilon.$$

Then the distance between f and f_δ is estimated by

$$\sup_x\left|\frac{1}{x}\int_0^x(f-f_\delta)\right| \le \sup_{|x|\le\delta}\left|\frac{1}{x}\int_0^x f-F'(0)\right|+\sup_{\delta\le|x|\le\pi}\left|\frac{1}{x}\int_0^x(f-f_\delta)\right| < 2\varepsilon.$$

Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{T}} f(-t)\Phi_\varepsilon(t)dt = F'(0).$$

Hence, we have convergence at the origin for all the functions $f \in L_1(\mathbb{T})$, and, correspondingly, at all the d-points, as required.

II. One can see that the condition $\varphi_1' \in W(\mathbb{R})$ is not sufficient from the case $\varphi(x) = \varphi_1'(x) \equiv 1$.

III. To prove this, let us consider the Lebesgue summability method $\left(\varphi(t) = \dfrac{\sin t}{t}\right)$ (see, e.g., [1, Ch.VII, §5]). In this case,

$$\sum_k \frac{\sin k\varepsilon}{k\varepsilon}\widehat{f}_k e^{ikx} = \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} \underset{\varepsilon\to 0}{\to} F'(x),$$

and $\varphi_1' \in W(\mathbb{R})$ but not in $W_0(\mathbb{R})$.

Sufficient conditions for summability at d-points obtained earlier in [10] are essentially weaker, since the constraints φ_1 and $\varphi_1' \in L_1(\mathbb{R})$ and $\varphi \in V \cap L_1(\mathbb{R})$ are removed. The summability at d-points of basic classical methods is checked there as well. The only non-considered method is that of Picard.

Example 1 Picard's method, with $\varphi(t) = \left(\frac{1}{1+|t|^\alpha}\right)^\beta$, satisfies the corollary provided α and $\beta > 0$, and only in this case.

See **D** above.

Example 2 For Riesz's method, with $\varphi(t) = (1 - |t|^\alpha)_+^\beta$, where $\alpha > 0$, $\beta > 0$, summability at l-points holds always, while at d-points only for $\alpha > 0$, $\beta > 1$ (see the necessity in Theorem 3).

Remark 7 We have $\|\varphi\|_W = \varphi(0)$ if φ is positive definite, and only in this case. Correspondingly, the kernel of the integral convolution operator $\Phi_\varepsilon \geq 0$; see Lemma 2. Functions $\varphi \in W_0(\mathbb{R})$ are positive definite in Picard's method ($\alpha \in (0, 2]$, $\beta > 0$, and only in this case), as well as in the Lebesgue method, while in Riesz's methods provided at least that $\alpha \in (0, 1]$ and $\beta \geq 1$.

It is also worth noting that Lebesgue studied the convergence of general singular integrals (see also [3]).

5 Appendices

I. We begin with the space $L_{0,1}(\mathbb{R}_+)$ endowed with the norm

$$\|f\|_{0,1} = \int_0^\infty f^*, \qquad f^*(x) = \operatorname*{ess\,sup}_{t \geq x} |f(t)|.$$

Three properties of norm are obvious. We will prove the completeness of $L_{0,1}$.

Proof It is evident that for $x > 0$,

$$x \cdot \operatorname*{ess\,sup}_{t \geq x} |f(t)| \leq \int_0^x f^*(t)dt.$$

Therefore, for any $\delta > 0$,

$$\|f\|_{L_\infty[\delta,+\infty)} = \operatorname*{ess\,sup}_{t \geq \delta} |f(t)| \leq \frac{1}{\delta}\|f\|_{0,1}.$$

If for $m \geq n$ and $\varepsilon_n \to 0$, we have $\|f_m - f_n\|_{0,1} \leq \varepsilon_n$, then

$$\|f_m - f_n\|_{L_\infty[\delta,+\infty)} \leq \frac{1}{\delta}\varepsilon_n.$$

By the completeness of L_∞, there is $f \in L_\infty[\delta, +\infty)$, for which

$$\|f - f_n\|_{L_\infty[\delta,+\infty)} \le \frac{1}{\delta}\varepsilon_n.$$

By the Fatou lemma, we pass to the limit as $m \to \infty$ in the inequality

$$\left\| f_m(\cdot + \delta) - f_n(\cdot + \delta)\right\|_{0,1} \le \varepsilon_n,$$

and then to the limit as $\delta \to +0$. Completeness is proved.

($\boldsymbol{\alpha}$) In the space $L_{0,1}$, the functions vanishing in a neighborhood of the origin and taking a finite number of values form a dense set.

($\boldsymbol{\beta}$) It is impossible to approximate the function f_1, which is 1 on $[0, 1]$ and zero for $x > 1$, by continuous functions, while it is impossible to approximate the function f_2, equal $\sin\dfrac{1}{x}$ on $(1, 2)$ and zero otherwise, by step-wise functions.

($\boldsymbol{\gamma}$) The Banach space $L_{0,1}$ is not separable, since the distance between two functions f_y, where $f_y(x) = 1$ if $x < 1 + y$ and $f_y(x) = 0$ if $x > 1 + y$, is not less than 1 provided $y > 0$.

II. In connection with the summability criteria at Lebesgue points a space has appeared endowed with the norm $\sup\limits_{0<|x|\le\pi}\left|\dfrac{1}{x}\displaystyle\int_0^x |f|\right|$, while in connection with the summability criteria at d-points a space endowed with the norm $\sup\limits_{0<|x|\le\pi}\left|\dfrac{1}{x}\displaystyle\int_0^x f\right|$ has appeared. The former is complete. It turns out that the latter is not. Indeed, otherwise comparison of these norms and the Banach theorem on invertible operator would imply the equivalence of these norms. But this is not the case, for example, for $f(t) = \frac{1}{t}\sin\frac{1}{t}$ on $(0, 1]$. More precisely, the sequence of functions $f_n(t) = 0$ if $0 < t \le \frac{1}{n}$ and $f_n(t) = \frac{1}{t}\sin\frac{1}{t}$ if $\frac{1}{n} < t \le 1$ satisfies the Cauchy condition but does not converge in this space. This argument is suggested by Vit. Volchkov.

What space is complete is the one with the norm

$$\|f\|_{L_1(\mathbb{T})} + \sup_{0<|x|\le\pi}\left|\frac{1}{x}\int_0^x f(t)\,dt\right|;$$

however, if $\lim\limits_{x\to 0}\frac{1}{x}\int_0^x f(t)dt$ does not exist, then, for $\varepsilon > 0$ small enough, no function g continuous at zero exists for which $\sup\limits_{0<x\le 1}\frac{1}{x}\left|\int(f(t) - g(t)\,dt\right| < \varepsilon$. The same assertion is true if f is replaced by $|f|$ and g by $|g|$.

We mention that both spaces are reflexive: $L^*_{0,1} = L_{0,\infty}$ and $L^*_{0,\infty} = L_{0,1}$ (see Theorem 1). We also mention that for $\lambda > 0$,

$$\left\| f(\lambda\cdot)\right\|_{0,\infty} = \|f(\cdot)\|_{0,\infty}, \qquad \left\| f(\lambda\cdot)\right\|_{0,1} = \lambda\|f(\cdot)\|_{0,1}.$$

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Unilateral Ball Potentials in Grand Lebesgue Spaces

S. M. Umarkhadzhiev and M. U. Yakhshiboev

Abstract Necessary and sufficient conditions for the boundedness of unilateral ball potentials (ball fractional integrals) in grand Lebesgue spaces on $\mathbb{R}^n$ were obtained for various classes of grandizers. Lower bounds are proved for the norm of these operators in grand spaces with integrable grandizers and upper bounds in spaces with radial grandizers; the norms of the operators in question are found in the case of a power grandizer: $a(x) = \frac{1}{|x|^n}$.

Keywords Unilateral ball potential (ball fractional integrals) · Grand Lebesgue space · Grandizer · Norm bound · Operator boundedness

1 Introduction

Unilaterals ball potentials (ball fractional integrals) of order α by $\mathbb{R}^n$ are determined by the equalities

$$\left(B_{+}^{\alpha}\varphi\right)(x) = \gamma_{n,\alpha} \int\limits_{y<|x|} \frac{\left(|x|^2 - |y|^2\right)^{\alpha}}{|x-y|^n} \varphi(y)\, dy,$$

$$\left(B_{-}^{\alpha}\varphi\right)(x) = \gamma_{n,\alpha} \int_{y>|x|} \frac{\left(|y|^2 - |x|^2\right)^{\alpha}}{|x-y|^n} \varphi(y)\, dy,$$

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where $\gamma_{n,\alpha} = \frac{2}{\omega_{n-1}\Gamma(\alpha)} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi^n}\,\Gamma(\alpha)}$. These operators were studied in [1–3]. A number of properties of unilateral ball potentials could be found in [4, 5], [6, § 29].

A theorem on the boundedness of operators B_+^α and B_-^α in weighted Lebesgue spaces is proved in [1]:

$$L^p(\Omega, w) := \left\{ f : \|f\|_{L^p(\Omega,w)} := \left(\int\limits_\Omega |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

The following theorem is true.

Theorem 1.1 *Let* $1 < p < \infty$*,* $r = |x|$*. Operators* B_+^α *and* B_-^α *are bounded in the following spaces:*

(1) $B_+^\alpha : L_p\left(\mathbb{R}^n, r^\lambda\right) \to L_p\left(\mathbb{R}^n, r^{\lambda-2\alpha}\right)$*, where* $\lambda < n(p-1)$*;*
(2) $B_-^\alpha : L_p\left(\mathbb{R}^n; r^\lambda\right) \to L_p\left(\mathbb{R}^n; r^{\lambda-2\alpha}\right)$*, where* $\lambda > 2\alpha p - n$*.*

In this paper, unilaterals ball potentials of the following form are considered:

$$\left(\mathbb{B}_+^\alpha \varphi\right)(x) = |x|^{-2\alpha} \int\limits_{|y|<|x|} \frac{\left(|x|^2 - |y|^2\right)^\alpha}{|x-y|^n} \varphi(y)\, dy,$$

$$\left(\mathbb{B}_-^\alpha \varphi\right)(x) = \int\limits_{|y|>|x|} |y|^{-2\alpha} \frac{\left(|y|^2 - |x|^2\right)^\alpha}{|x-y|^n} \varphi(y)\, dy.$$

Theorem 1.1 directly implies that the operators $\mathbb{B}_+^\alpha$ and $\mathbb{B}_-^\alpha$ are bounded in space $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The action of operators $\mathbb{B}_+^\alpha$ and $\mathbb{B}_-^\alpha$ in the so-called grand Lebesgue spaces $L_a^{p)}(\mathbb{R}^n)$ was studied. Our goal was to obtain necessary and sufficient conditions on the grandizer that ensure the boundedness of the operators in question in grand spaces.

Basic statements for operator $\mathbb{B}_+^\alpha$ are given in Theorems 3.1, 3.2 and 3.3, and the results for operator $\mathbb{B}_-^\alpha$ are formulated in Theorem 3.4.

2 Preliminaries

The grand Lebesgue spaces $L^{p)}(\Omega)$ on bounded sets $\Omega \subset \mathbb{R}^n$ were introduced in [7] when solving the problem of integrability of the Jacobian of mapping. On sets of arbitrary (not necessarily finite) measure, these spaces are extended in [8] by introducing a functional parameter of small power:

$$L_a^{p)}(\Omega) := \left\{ f : \|f\|_{L_a^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \varepsilon \|f\|_{L^{p-\varepsilon}(\Omega, a^{\frac{\varepsilon}{p}})} < \infty \right\}, \tag{2.1}$$

where $1 < p < \infty$ and $a(x)$—are the arbitrary measurable nonnegative functions on Ω, called a grandizer. The choice of a grandizer to define a grand space may be dictated by the tasks for exploring such spaces. In [8–10] it was assumed that $a \in L^1(\Omega)$, which guarantees the embedding of $L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega)$. The grand space thus defined depends, generally speaking, on the choice of grandizer, although a different choice of grandizers can lead to the same grand space (see [11]).

For $a \in L^1(\Omega)$, there is a sequence of embeddings

$$L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega) \hookrightarrow L^{p-\varepsilon_1}(\Omega, a^{\frac{\varepsilon_1}{p}}) \hookrightarrow L^{p-\varepsilon_2}(\Omega, a^{\frac{\varepsilon_2}{p}}),\ 0 < \varepsilon_1 < \varepsilon_2 < p-1. \tag{2.2}$$

In the commonly used definition of the grand Lebesgue space on bounded sets with $a(x) \equiv 1$, the embedding $L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega)$ is always true, i.e. in this sense, the grand space is an extension of the classical Lebesgue space. According to (2.2), a similar embedding on sets of infinite measure is guaranteed by condition $a \in L^1(\Omega)$. This condition is often assumed when considering grand spaces on unbounded sets, see, for example, [8, 12], although the grand spaces on such sets can be considered without this condition (see, for example, Theorem 3.4).

3 The Main Statement

We introduce the following notation

$$\ell(p) := \frac{\pi^{\frac{n}{2}} \Gamma(\alpha) \Gamma\left(\frac{n}{2p}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2p} + \alpha\right)},$$

$$k^+(a) := \omega_{n-1} \sup_{0<\varepsilon<p-1} \int_1^\infty t^{-\frac{n}{(p-\varepsilon)'} - 2\alpha} \left(t^2 - 1\right)^{\alpha-1} [a_*(t)]^{\frac{\varepsilon}{p(p-\varepsilon)}} \, dt,$$

$$k^-(a) := \omega_{n-1} \sup_{0<\varepsilon<p-1} \int_0^1 t^{\frac{n}{p-\varepsilon} - 1} \left(1 - t^2\right)^{\alpha-1} [a_*(t)]^{\frac{\varepsilon}{p(p-\varepsilon)}} \, dt,$$

where ω_{n-1} is the area of a unit sphere in $\mathbb{R}^n$, $a_*(\tau) := \sup_{y \in \mathbb{R}^n} \frac{a(\tau|y|)}{a(|y|)}$ is the dilation of function a, see [13].

Theorem 3.1 *Let* $1 < p < \infty$ *and grandizer* a *be integrated by* $\mathbb{R}^n$*. Then, for the operator* $\mathbb{B}^+$ *to be bounded in the grand space* $L_a^{p)}(\mathbb{R}^n)$*, it is necessary that* $\alpha > 0$*. In this case, the lower bound holds:*

$$\ell(p') \leqslant \|\mathbb{B}_+^\alpha\|_{L_a^{p)}(\mathbb{R}^n)}. \tag{3.1}$$

Proof Consider a minimizing family of functions in the form

$$f_\delta(x) = \frac{1}{|x|^{\frac{n}{p}-\delta}(1+|x|)^\delta}, \ \delta > 0.$$

Since $f_\delta \in L^p(\mathbb{R}^n)$ and $a \in L^1(\mathbb{R}^n)$, then $f_\delta \in L_a^{p)}(\mathbb{R}^n)$ by virtue of (2.2). For the operator $\mathbb{B}_+^\alpha$ to be bounded in the grand space $L_a^{p)}(\mathbb{R}^n)$ it follows that it is defined on function f_δ. Here we have

$$\mathbb{B}_+^\alpha f_\delta(x) = \int\limits_{|y|<1} \frac{\left(1-|y|^2\right)^\alpha}{|e_1-y|^n} f_\delta(|x|y)dy.$$

It is plain to check that $f_\delta(xy) \geqslant f_\delta(x) f_\delta(y)$. Then

$$\mathbb{B}_+^\alpha f_\delta(x) \geqslant \ell_\delta f_\delta(x),$$

where $\ell_\delta = \int\limits_{|y|<1} \frac{(1-|y|^2)^\alpha}{|e_1-y|^n} f_\delta(y)dy$. Consequently

$$\|\mathbb{B}_+^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} \geqslant \frac{\|\mathbb{B}_+^\alpha f_\delta\|_{L_a^{p)}(\mathbb{R}^n)}}{\|f_\delta\|_{L_a^{p)}(\mathbb{R}^n)}} \geqslant \ell_\delta.$$

Proceed to the limit at $\delta \to 0$ under the integral sign defining ℓ_δ, which is asserted by Levy's theorem. Here

$$\lim_{\delta\to 0} \ell_\delta = \lim_{\delta\to 0} \int\limits_{|y|<1} \frac{\left(1-|y|^2\right)^\alpha}{|e_1-y|^n} |y|^{-\frac{n}{p}+\delta}(1+|y|)^{-\delta}dy = \ell(p').$$

□

Theorem 3.2 *Let* $1 < p < \infty$ *and grandizer* a *be the radial functions on* $\mathbb{R}^n$*:* $a(x) = a(|x|)$*. If the grandizer mets the following condition* $k^+(a) < \infty$*, then operator* $\mathbb{B}_+^\alpha$ *is bounded in the grand space* $L_a^{p)}(\mathbb{R}^n)$ *and*

$$\|\mathbb{B}_+^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} \leqslant k^+(a). \tag{3.2}$$

Proof Estimate lower bound $\|\mathbb{B}_+^\alpha f\|_{L^{p-\varepsilon}(\mathbb{R}^n, a^{\frac{\varepsilon}{p}})}$. To do so, $\mathbb{B}_+^\alpha f(x)$ is presented in the form

$$\mathbb{B}_+^\alpha f(x) = \int\limits_{|y|<|x|} \left(|y|^{-\frac{n}{q}} |x|^{-2\alpha} \frac{\left(|x|^2 - |y|^2\right)^\alpha}{|x-y|^n} \left(\frac{a(|x|)^{\frac{\varepsilon}{p}}}{a(|y|)^{\frac{\varepsilon}{p}}} \right)^{\frac{1}{q}} \right)^{\frac{1}{q'}}$$
$$\times \left(|y|^{\frac{n}{q'}} |x|^{-2\alpha} \frac{\left(|x|^2 - |y|^2\right)^\alpha}{|x-y|^n} \left(\frac{a(|y|)^{\frac{\varepsilon}{p}}}{a(|x|)^{\frac{\varepsilon}{p}}} \right)^{\frac{1}{q'}} \right)^{\frac{1}{q}} f(y)dy,$$

where $q > 1$. Applying Hölder's inequality, we obtain

$$|\mathbb{B}_+^\alpha f(x)| \leqslant \left[A_q^+(x)\right]^{\frac{1}{q'}} \left\{ |x|^{-2\alpha} \int\limits_{|y|<|x|} |y|^{\frac{n}{q'}} \frac{\left(|x|^2 - |y|^2\right)^\alpha}{|x-y|^n} \left(\frac{a(|y|)}{a(|x|)} \right)^{\frac{\varepsilon}{pq'}} |f(y)|^q dy \right\}^{\frac{1}{q}}, \tag{3.3}$$

where

$$A^+(x) = |x|^{-2\alpha} \int\limits_{|y|<|x|} |y|^{-\frac{n}{q}} \frac{\left(|x|^2 - |y|^2\right)^\alpha}{|x-y|^n} \left(\frac{a(|x|)}{a(|y|)} \right)^{\frac{\varepsilon}{pq}} dy.$$

Substituting q in expression for $A_+(x)$ by $p-\varepsilon$, $0 < \varepsilon < p-1$, and passing on to polar coordinates $y = \rho y'$, $\rho = |y|$, we get

$$A^+(x) = |x|^{-n} \int\limits_0^{|x|} \rho^{\frac{n}{(p-\varepsilon)'}-1} \left(1 - \frac{\rho^2}{|x|^2}\right)^\alpha \left(\frac{a(|x|)}{a(\rho)} \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} d\rho \int_{S^{n-1}} \frac{dy'}{|x' - \frac{\rho}{|x|} y'|^n}.$$

Now let's use the property $\int\limits_{S^{n-1}} P(x', \xi y')dy' = 1, \forall x' \in S^{n-1}, \xi \in [0, 1)$, Poisson's kernels $P(x', \xi y') = \frac{1}{\omega_{n-1}} \frac{1-\xi^2}{|x'-\xi y'|^n}$. We have

$$A^+(x) = \omega_{n-1} |x|^{-n} \int\limits_0^{|x|} \rho^{\frac{n}{(p-\varepsilon)'}-1} \left(1 - \frac{\rho^2}{|x|^2}\right)^{\alpha-1} \left(\frac{a(|x|)}{a(\rho)} \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} d\rho$$
$$= \omega_{n-1} |x|^{-\frac{n}{p-\varepsilon}} \int\limits_0^1 \rho^{\frac{n}{(p-\varepsilon)'}-1} \left(1 - \rho^2\right)^{\alpha-1} \left(\frac{a(|x|)}{a(\rho|x|)} \right)^{\frac{\varepsilon}{p(p-\varepsilon)}} d\rho.$$

Using the inequality $\sup_{t<1} \frac{a(t)}{a(\xi t)} \leqslant a_*\left(\frac{1}{\xi}\right),\ \xi > 0$, we obtain

$$A^+(x) \leqslant \omega_{n-1}|x|^{-\frac{n}{p-\varepsilon}} \int_0^1 \rho^{\frac{n}{(p-\varepsilon)'}-1}\left(1-\rho^2\right)^{\alpha-1}\left(a_*\left(\frac{1}{\rho}\right)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} d\rho$$
$$\leqslant |x|^{-\frac{n}{p-\varepsilon}} k^+(a).$$

Then, with (3.3) we have

$$\|\mathbb{B}_+^\alpha f\|_{L^{p-\varepsilon}\left(\mathbb{R}^n, a^{\frac{\varepsilon}{p}}\right)} \leqslant [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} \left\{\int_{\mathbb{R}^n} |x|^{-\frac{n}{(p-\varepsilon)'}-2} \int_{|y|<|x|} |y|^{\frac{n}{(p-\varepsilon)'}}\right.$$
$$\left.\times \frac{\left(|x|^2-|y|^2\right)^\alpha}{|x-y|^n}\left(\frac{a(|y|)}{a(|x|)}\right)^{\frac{\varepsilon}{p(p-\varepsilon)'}} |f(y)|^{p-\varepsilon} dy (a(|x|))^{\frac{\varepsilon}{p}} dx\right\}^{\frac{1}{p-\varepsilon}}$$
$$= [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} \left\{\int_{\mathbb{R}^n} |f(y)|^{p-\varepsilon} a(|y|)^{\frac{\varepsilon}{p(p-\varepsilon)'}} A_-(y) dy\right\}^{\frac{1}{p-\varepsilon}},$$

where

$$A^-(y) := |y|^{\frac{n}{q'}} \int_{|x|>|y|} |x|^{-\frac{n}{q'}-2\alpha} \frac{\left(|x|^2-|y|^2\right)^\alpha}{|x-y|^n} a(|x|)^{\frac{\varepsilon}{pq}} dx$$

After substitution $x \to |y|x$ and rotation of the integral in square brackets, we get

$$\|\mathbb{B}_+^\alpha f\|_{L^{p-\varepsilon}\left(\mathbb{R}^n, a^{\frac{\varepsilon}{p}}\right)} \leqslant [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} \left\{\int_{\mathbb{R}^n} |f(y)|^{p-\varepsilon} a(|y|)^{\frac{\varepsilon}{p(p-\varepsilon)'}} dy \times\right.$$
$$\left.\times \int_{|x|>1} \frac{\left(|x|^2-1\right)^\alpha}{|x-e_1|^n} \frac{a(|x||y|)^{\frac{\varepsilon}{p(p-\varepsilon)}}}{|x|^{\frac{n}{(p-\varepsilon)'}}} dx\right\}^{\frac{1}{p-\varepsilon}}$$
$$= [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} \left\{\int_{\mathbb{R}^n} |f(y)|^{p-\varepsilon} a(|y|)^{\frac{\varepsilon}{p}} dy \times\right.$$
$$\left.\times \int_{|x|>1} |x|^{-\frac{n}{(p-\varepsilon)'}} \frac{\left(|x|^2-1\right)^\alpha}{|x-e_1|^n}\left(\frac{a(|x||y|)}{a(|y|)}\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} dx\right\}^{\frac{1}{p-\varepsilon}}$$
$$\leqslant [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} \|f\|_{L^{p-\varepsilon}\left(\mathbb{R}^n, a^{\frac{\varepsilon}{p}}\right)} \times$$

$$\times \left\{ \int\limits_{|x|>1} |x|^{-\frac{n}{(p-\varepsilon)'}} \frac{\left(|x|^2-1\right)^\alpha}{|x-e_1|^n} [a_*(|x|)]^{\frac{\varepsilon}{p(p-\varepsilon)}} \, dx \right\}^{\frac{1}{p-\varepsilon}}$$

$$= [k_\varepsilon^+(a)]^{\frac{1}{(p-\varepsilon)'}} [k_\varepsilon^+(a)]^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}\left(\mathbb{R}^n, a^{\frac{\varepsilon}{p}}\right)}$$

$$= k_\varepsilon^+(a) \|f\|_{L^{p-\varepsilon}\left(\mathbb{R}^n, a^{\frac{\varepsilon}{p}}\right)}.$$

In view of (2.1) we arrive to (3.2). □

Theorem 3.3 *Let* $1 < p < \infty$ *and* $a(x) = \frac{1}{|x|^n}$. *Then*

$$\|\mathbb{B}_+^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} = \ell(p'). \tag{3.4}$$

Proof The bound (3.4) follows from Theorem 3.2, and the lower bound is obtained similar to the proof of Theorem 3.1. □

The following statements are proved similarly to the previous theorems.

Theorem 3.4 *Let* $1 < p < \infty$.

1. *If grandizer a is integrated on* $\mathbb{R}^n$, *then for the operator* $\mathbb{B}_-^\alpha$ *to be bounded in the grand space* $L_a^{p)}(\mathbb{R}^n)$ *it is necessary that* $\alpha > 0$. *Here the lower bound holds:*

$$\ell(p) \leqslant \|\mathbb{B}_-^\alpha\|_{L_a^{p)}(\mathbb{R}^n)}.$$

2. *Let grandizer a be the radial function on* $\mathbb{R}^n$, *meeting the condition* $k^-(a) < \infty$. *Then operator* $\mathbb{B}_-^\alpha$ *is bounded in the grand space* $L_a^{p)}(\mathbb{R}^n)$ *and*

$$\|\mathbb{B}_-^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} \leqslant k^-(a).$$

3. *If* $a(x) = \frac{1}{|x|^n}$, *then*

$$\|\mathbb{B}_-^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} = \ell(p).$$

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Lebesgue Means of Trigonometric Integrals and Their Rate of Convergence

S. S. Volosivets

Abstract In 2012 F. Móricz obtained two sufficient conditions for Lebesgue summabilty of trigonometric integrals. In this note we give quantitative estimates for rate of convergence of classical and generalized Lebesgue means of such integrals.

Keywords Trigonometric integral · Lebesgue summability · Equiconvergence

1 Introduction

Let $\{c_m\}_{m\in\mathbb{Z}}$ be a sequence of complex numbers. We consider the trigonometric series

$$\sum_{m\in\mathbb{Z}} c_m e^{imx} \tag{1.1}$$

and its symmetric partial sums $S_n(x) = \sum_{|m|\le n} c_m e^{imx}$, $n \in \mathbb{Z}_+ = \{0, 1, \dots\}$. Formal integration of (1.1) gives

$$l(x) := c_0 x + \sum_{|m|=1}^{\infty} \frac{c_m e^{imx}}{im}, \tag{1.2}$$

provided that the series on the right-hand side of (1.2) converges. It is clear that the condition $\sum_{|m|=1}^{\infty} |c_m/m| < \infty$ guarantees absolute and uniform convergence of series in (1.2).

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If the function $l(x)$ exists in some neighborhood of a point $x_0 \in \mathbb{R}$ and if

$$\frac{\Delta_{2h} l(x_0)}{2h} := \frac{l(x_0 + h) - l(x_0 - h)}{2h} \to s, \quad h \to 0,$$

then the series (1.1) is said to be summable at x_0 to s by the Lebesgue method. It is easy to see that in this case

$$\frac{\Delta_{2h} l(x_0)}{2h} = c_0 + \sum_{|m| \geq 1} c_m e^{imx_0} \frac{\sin mh}{mh},$$

for sufficiently small $h > 0$. About this method, see more in [6, Ch.9, §2].

Lebesgue summability of integrals was introduced by O. Szász [3]. For a locally integrable function $f : \mathbb{R} \to \mathbb{C}$, we consider the (formal) trigonometric integral

$$I(f)(x) = \int_{\mathbb{R}} f(t) e^{itx}\, dt, \quad x \in \mathbb{R}, \tag{1.3}$$

with symmetric partial integrals $I_s(f)(x) = \int_{|t| \leq s} f(t) e^{itx}\, dt$, $s \in (0, \infty)$. We say that the integral (1.3) converges to the limit I at a point $x \in \mathbb{R}$, if there exists $\lim_{s \to +\infty} I_s(f)(x) = I$.

Formal integration of (1.3) with respect to x implies

$$\int_{\mathbb{R}} f(t)(it)^{-1} e^{itx}\, dt =: L(x), \quad x \in \mathbb{R}. \tag{1.4}$$

It is clear that the condition $f(t)/t \in L^1(\mathbb{R})$ is sufficient for the existence of (1.4) and its uniform convergence with respect to x. We say that the integral (1.3) is Lebesgue summable at a point $x_0 \in \mathbb{R}$ to the limit I if the function $L(x)$ exists in $(x_0 - \delta, x_0 + \delta)$, for some $\delta > 0$, and there exists $\lim_{h \to 0} \Delta_{2h} L(x_0)/(2h) = I$. It is easy to see that in this case, for sufficiently small h,

$$L_h(f)(x) = \frac{\Delta_{2h} L(x)}{2h} = \int_{\mathbb{R}} f(t) e^{itx} \frac{\sin th}{th}\, dt.$$

F. Móricz [2] proved two theorems.

Theorem A *If $f(t) \in L^1(-T, T)$ for all $T > 0$ and*

$$\lim_{s \to +\infty} s^{-1} \int_{|t| \leq s} |tf(t)|\, dt = 0, \tag{1.5}$$

then $L(x)$ in (1.4) exists everywhere on $\mathbb{R}$, and we have uniformly in x

$$\lim_{h \to 0+0} \left(\frac{\Delta_{2h} L(x)}{2h} - I_{1/h}(x) \right) = 0.$$

Theorem B *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *is such that* $f \in L^1(-T, T)$ *for all* $T > 0$ *and there is* $s_1 > 0$ *with property*

$$s^{-1} \int_{|t|\le s} |tf(t)|\, dt \le B, \quad s > s_1. \tag{1.6}$$

If the integral (1.3) *converges to* I *at some point* $x \in \mathbb{R}$, *then it is Lebesgue summable at* x *to* I.

Let us consider the family of operators

$$L_h^{(\alpha)}(f)(x) = \int_{\mathbb{R}} f(t)e^{itx} \left(1 - \left(1 - \frac{\sin th}{th}\right)^{\alpha}\right) dt, \quad \alpha \ge 1.$$

It is clear that $L_h^{(1)}(f)(x) = L_h(f)(x)$. The aim of the present note is to obtain a quantitative extension of Theorem A for $L_h^{(\alpha)}(f)(x)$, $\alpha \ge 1$, and of Theorem B for $L_h^{(1)}(f)(x)$.

2 Auxiliary Propositions

Lemma 1 *For all* $|t| \in (0, 1)$, *we have* $0 < 1 - \sin t/t < t^2/6$.

Proof The left inequality is well known. By the Taylor formula with remainder in Lagrange's form, we obtain, for $t \in (0, 1)$,

$$\frac{t - \sin t}{t} = \frac{t^3 \cos z}{6t} \in (0, t^2/6),$$

where $z \in (0, t) \subset (0, 1)$. □

Lemma 2 *For* $t > 0$ *and* $H(t) = \sin t/t$, *the inequality* $|H'(t)| \le t/2$ *holds.*

Proof Let us consider the function $\varphi_1(t) = t^3/2 + t\cos t - \sin t$. Then $\varphi_1''(t) = 3t - \sin t - t\cos t > 0$ on $(0, +\infty)$ and $\varphi_1'(0) = 0$, whence $\varphi_1'(t) > 0$ on $(0, +\infty)$. Since $\varphi_1(0) = 0$, we deduce that $\varphi_1(t) > 0$ on $(0, +\infty)$. Thus, $\sin t - t\cos t \le t^3/2$, $t \ge 0$. In a similar way we obtain the inequalities $-\sin t + t\cos t \le t^3/2$ and $|\sin t - t\cos t| \le t^3/2$ for $t \ge 0$. Using the formula $H'(t) = t^{-2}(t\cos t - \sin t)$, we prove the inequality of Lemma 2. □

Lemma 3 *For* $|t| > 1$ *and* $\alpha \ge 1$, *we have the inequality*

$$|1 - (1 - \sin t/t)^{\alpha}| \le Ct^{-1}.$$

Proof By the Lagrange mean value theorem

$$|1^\alpha - (1-\sin t/t)^\alpha| = |\alpha z^{\alpha-1}\sin t/t| \le \alpha t^{-1},$$

since $z \in (1-\sin t/t, 1) \subset (0,1)$ and $x^{\alpha-1}$ increases in x. □

Denote by Φ the set of continuous and increasing on $\mathbb{R}_+ = [0,\infty)$ functions ω such that $\omega(0) = 0$ and $\omega(2t) \le C\omega(t)$, $t \in \mathbb{R}_+$. If $\omega \in \Phi$ and $\int_0^\delta t^{-1}\omega(t)\,dt = O(\omega(\delta))$, $\delta \in (0,+\infty)$, then ω belongs to the Bary class B; if $m > 0$, $\omega \in \Phi$ and $\delta^m \int_\delta^\infty t^{-m-1}\omega(t)\,dt = O(\omega(\delta))$, $\delta \in (0,+\infty)$, then ω belongs to the Bary-Stechkin class B_m (see [1]). The following lemma is proved in [5] for $m \in \mathbb{N}$ but its proof is appropriate for any $m > 0$.

Lemma 4

(i) *If $m > 0$, $\omega \in B$, $g(t)$ is a non-negative measurable function such that $y^m g(t) \in L^1_{loc}(\mathbb{R}_+)$ and*

$$\int_0^y t^m g(t)\,dt = O\left(y^m \omega(1/y)\right), \quad y > 0, \tag{2.1}$$

then

$$\int_y^\infty g(t)dt = O\left(\omega(1/y)\right), \quad y > 0. \tag{2.2}$$

(ii) *If $m > 0$, $\omega \in B_m$, $g(t)$ is a non-negative locally integrable function, then* (2.2) *implies* (2.1).

3 Main Results

Theorem 1 *Let $\alpha \ge 1$, $\omega \in \Phi$ and $f : \mathbb{R} \to \mathbb{C}$ be such that $f \in L^1_{loc}(\mathbb{R})$ and*

$$\int_{|t|\le y} |t|^{2\alpha}|f(t)|\,dt = O(y^{2\alpha}\omega(y^{-1})), \quad y > 0. \tag{3.1}$$

Then $L_h^{(\alpha)}(f)(x)$ exists as a Lebesgue integral at every $x \in \mathbb{R}$, and we have, uniformly in x,

$$L_h^{(\alpha)}(f)(x) - I_{1/h}(f)(x) = O(\omega(h)), \quad h > 0. \tag{3.2}$$

Proof By definitions of $L_h^{(\alpha)}(f)(x)$ and $I_{1/h}(f)(x)$, we have

$$I_{1/h}(f)(x) - L_h^{(\alpha)}(f)(x) = \int_{|t|\le 1/h} f(t)e^{itx}(1 - \sin th/(th))^{\alpha}\, dt$$

$$+ \int_{|t|\ge 1/h} f(t)e^{itx}(1 - (1 - \sin th/(th))^{\alpha})\, dt = J_1(x) + J_2(x). \tag{3.3}$$

By Lemma 1 and condition (3.1), we find that

$$|J_1(x)| \le \int_{|t|\le 1/h} |f(t)|\frac{|th|^{2\alpha}}{6^{\alpha}}\, dt \le C_1\omega(h). \tag{3.4}$$

Let us note that for $\omega_1(t) = t\omega(t)$, one has

$$\int_0^{\delta} t^{-1}\omega_1(t)\, dt = \int_0^{\delta} \omega(t)\, dt \le \delta\omega(\delta) = \omega_1(\delta),$$

i.e. $\omega_1 \in B$. Using this fact, Lemma 4 and (3.1) in the form

$$\int_{|t|\le y} |t|^{2\alpha+1}\frac{|f(t)|}{|t|}\, dt = O(y^{2\alpha+1}\omega_1(1/y)), \quad y > 0,$$

we obtain

$$\int_{|t|\ge y} \frac{|f(t)|}{|t|}\, dt = O(\omega_1(y)), \quad y > 0. \tag{3.5}$$

Hence, by Lemma 3 and (3.5), we deduce that

$$|J_2(x)| \le \int_{|t|\ge 1/h} |f(t)|\frac{C_2}{|t|h}\, dt \le C_3\frac{\omega_1(h)}{h} = C_3\omega(h). \tag{3.6}$$

Combining (3.3), (3.4) and (3.6) yields

$$|L_h^{(\alpha)}(f)(x) - I_{1/h}(f)(x)| \le (C_1 + C_3)\omega(h)$$

uniformly in x. Theorem is proved. □

Corollary 1 *Suppose that f and α are as in Theorem 1, $I(f)(x)$ exists for all $x \in \mathbb{R}$, $\omega \in \Phi$ and $|I(f)(x) - I_s(f)(x)| = O(\omega(1/s))$, $s \in (0, +\infty)$. Then the inequality*

$$L_h^{(\alpha)}(f)(x) - I(f)(x) = O(\omega(h)), \quad h > 0. \tag{3.7}$$

holds.

Corollary 2 *Let $\alpha \geq 1$, $\omega \in B_{2\alpha}$ and $f : \mathbb{R} \to \mathbb{C}$ be such that (3.5) holds. If $I(f)(x)$ exists for all $x \in \mathbb{R}$ and $I(f)(x) - I_s(f)(x) = O(\omega(1/s))$, $s \in (0, +\infty)$, then (3.7) holds.*

Proof First, we note that the conditions $\omega_1(t) = t\omega(t) \in B_{2\alpha+1}$ and $\omega(t) \in B_{2\alpha}$ are equivalent. By Lemma 4 (ii), relation (3.5) implies (3.1), and due to Corollary 1 we finish the proof. □

Theorem 2 *Let $\omega \in \Phi$, $\eta \in B_2$, $f : \mathbb{R} \to \mathbb{C}$ be such that $f \in L^1_{loc}(\mathbb{R})$ and*

$$\int_{|t|\leq s} t^2 |f(t)|\, dt = O(s^2\omega(1/s)), \quad s > 0. \tag{3.8}$$

If $x \in \mathbb{R}$ is fixed, $I(f)(x)$ exists and

$$|I(f)(x) - I_{1/h}(f)(x)| = O(\eta(h)), \quad h \in (0, +\infty),$$

then

$$L_h(f)(x) - I(f)(x) = O(\omega(h) + \eta(h)), \quad h > 0. \tag{3.9}$$

Proof As in the proof of Theorem 1, we have

$$I_{1/h}(f)(x) - L_h(f)(x) = \int_{|t|\leq 1/h} f(t)e^{itx}(1 - \sin th/(th))\, dt$$

$$+ \int_{|t|\geq 1/h} f(t)e^{itx}(\sin th/(th))\, dt = J_1(x) + J_2(x).$$

Since

$$F(u) = \int_0^u (f(t)e^{itx} + f(-t)e^{-itx})\, dt = I_u(f)(x),$$

using integration by parts and extending $H(x)$ in Lemma 2 by $H(0) = 1$, we see that

$$J_1(x) = \int_0^{1/h} (1 - H(ht))dF(t) = I_{1/h}(f)(x)(1 - H(1))$$

$$+ \int_0^{1/h} I_t(f)(x)hH'(th)\, dt = (I_{1/h}(f)(x) - I(f)(x))(1 - H(1))$$

$$+ \int_0^{1/h} (I_t(f)(x) - I(f)(x))hH'(th)\, dt = J_{11}(x) + J_{12}(x).$$

It is clear that $|J_{11}(x)| \leq C_1\eta(h)$. By Lemma 2 and the condition $\eta \in B_2$, we obtain

$$|J_{12}(x)| \leq \int_0^{1/h} C_1\eta(1/t)|hH'(th)|\,dt = C_1\int_h^\infty \eta(u)h|H'(h/u)|u^{-2}\,du$$

$$\leq C_1\int_h^\infty \eta(u)\frac{h^2}{2u^3}\,du \leq C_2\eta(h), \quad h > 0$$

As in the proof of Theorem 1, we have (3.5) and (3.6), in other words, $J_2(x) = O(\omega(h))$, $h > 0$, under condition (3.8). Combining estimates of $J_{11}(x)$, $J_{12}(x)$ and $J_2(x)$, we obtain (3.9). Theorem 2 is proved. □

Remark Note that (3.8) is weaker than estimate (1.5). Indeed, if (1.5) holds, then

$$\int_{|t|\leq s} |t^2 f(t)|\,dt \leq s\int_{|t|\leq s} |tf(t)|\,dt = O(s^2\omega(1/s)), \quad s > 0.$$

Thus, Theorem A follows from Theorem 1.

On the other hand, (1.5) is stronger than (1.6). We give a quantitative estimate of convergence also in the case where (1.6) is valid.

Theorem 3 *Let $\eta \in B \cap B_2$ and $f : \mathbb{R} \to \mathbb{C}$ be such that $f \in L^1_{loc}(\mathbb{R})$ and (1.6) holds. If $x \in \mathbb{R}$ is fixed, $I(f)(x)$ exists and $|I(f)(x) - I_{1/h}(f)(x)| = O(\eta(h))$, $h > 0$, then for some $\delta > 0$, we have*

$$L_h(f)(x) - I(f)(x) = O(\eta^{1/3}(h)), \quad 0 < h < \delta.$$

Proof Let $\mu > 1$ be a real number the value of which will be specified later. Then

$$L_h(f)(x) - I_{\mu/h}(f)(x) = \int_{|t|\leq\mu/h} f(t)e^{itx}(\sin th/(th) - 1)\,dt$$

$$+\int_{|t|\geq\mu/h} f(t)e^{itx}(\sin th/(th))\,dt = J_1(x) + J_2(x).$$

By Lemma 4(i), for $g(t) = f(t)/t$, $m = 2$ and $\omega(t) = t$, condition (1.6) implies $s\int_{|t|\geq s} |f(t)/t|\,dt = O(1)$, $s > 0$, and we have

$$|J_2(x)| \leq h^{-1}\int_{|t|\geq\mu/h} \frac{|f(t)|}{t}\,dt \leq C_1\mu^{-1}. \tag{3.10}$$

On the other hand, as in the proof of Theorem 2, we obtain

$$
\begin{aligned}
J_1(x) &= \Big|(I_{\mu/h}(f)(x) - I(f)(x))(1 - H(\mu)) \\
&\quad + \int_0^{\mu/h} (I_t(f)(x) - I(f)(x))hH'(th)\,dt\Big| \\
&\le C_2\eta(h/\mu) + \frac{C_2h^2}{2}\int_0^{\mu/h} \eta(t^{-1})t\,dt
\end{aligned}
$$

$$
= C_2\eta(h/\mu) + \frac{C_2\mu^2}{2}\frac{h^2}{\mu^2}\int_{h/\mu}^{\infty} \frac{\eta(u)}{u^3}\,du \le C_3\mu^2\eta(h/\mu). \tag{3.11}
$$

If $\eta \in B$, then $\eta(h/\mu) = O((\ln\mu)^{-1}\eta(h))$, $0 < h < 1/2$ (see [4, Ch.7, Sect.7.1.21]). For a sufficiently small h, we can find $\mu > e$ such that $\ln\mu/\mu^3 = \eta(h)$ (the function $t^{-3}\ln t$ decreases for $t > e^{1/3}$). Therefore, $\mu^{-1} \le \eta^{1/3}(h)$, and by (3.10), one has $|J_2(x)| \le C_1\eta^{1/3}(h)$. In turn, by (3.11), we derive

$$
|J_1(x)| \le C_4\frac{\mu^3}{\ln\mu}\eta(h)\mu^{-1} = C_4\mu^{-1} \le C_4\eta^{1/3}(h).
$$

From these estimates we deduce that

$$
L_h(f)(x) - I_{\mu/h}(f)(x) = O(\eta^{1/3}(h)), \quad 0 < h < h_0.
$$

On the other hand, $\lim_{h\to 0+0}\eta(h) = 0$ and

$$
I_{\mu/h}(f)(x) - I(f)(x) = O(\eta(h/\mu)) = O(\eta(h)(\ln\mu)^{-1})) = O(\eta^{1/3}(h)),
$$

for small h. Hence, we obtain $L_h(f)(x) - I(f)(x) = O(\eta^{1/3}(h))$, $0 < h < \delta$, which completes the proof. □

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