General Fractional Calculus with Nonsingular Kernels: New Prospective on Viscoelasticity

Xiao-Jun Yang, Feng Gao, and Yang Ju

Abstract In the chapter, the general fractional derivatives in the different kernel functions, such as Mittag-Lefller, Wiman and Prabhakar functions are considered to model the viscoelastic behaviors in the real materials. We investigate the basic formulas of the fractional calculus (FC) in the kernels of the power, Mittag-Lefller, Wiman and Prabhakar functions. We discuss the applications for the general fractional calculus (GFC) in viscoelasticity. As the examples, the Maxwell and Voigt models with the general fractional derivatives (GFD) are considered to represent the complexity of the real materials.

Keywords Mittag-Lefller function · Wiman function · Prabhakar function · General fractional derivative · General fractional integral · General fractional calculus · Viscoelasticity

1 Introduction

Fractional calculus (FC) within the singular power-law kernel in the Riemann–Liouville and Liouville-Caputo types (see [1–9]) has been the increasing interests for scientists and engineers to represent the mathematical models in areas of a great many of the applications in engineering practices, such as the electric circuit [10], control theory [11], physics [12], mechanics [13], heat transfer [14], mathematical economy

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and finance (see [15, 16]), complex population dynamics [17], mathematical biology [18] and many others (see [20] and the cited references therein).

From mathematical and physical point of view, there may exist some of the new prospective of the applications of the operators involving the special functions and power-law functions to linear viscoelasticity (see [21–33]). With the use of the Nutting's observation [28], the laws of deformation with the operators involving the Riemann–Liouville [23, 24], Liouville-Caputo [25] and Caputo-Fabrizio [28] types, local FD [29], general FDs [30], and others [31–33] were reported in detail. The hereditary elastic rheological models, represented as the Volterra integral equation, were reported in [20, 21, 33]. The Maxwell and Voigt models involving the different fractional and fractal operators were proposed in [20, 21, 28–36].

Nowadays, there may exist the new unsolved problems including the Nutting equation [37] and anomalous Nutting equation in the real materials, such as rock and mining rock. Motivated by the above ideas, the brief targets of the chapter are to investigate the general fractional derivatives (GFDs) and the general fractional integrals (GFIs) with the nonsingular power-law kernel to describe the real material with the power-law phenomena by using the general fractional-order Maxwell and Voigt models.

The structure of the present chapter is suggested as follows. In Sect. 2, we introduce the FC and GFC operators with the power-law kernel. In Sect. 3, we investigate the recent applications of the GFDs to the general fractional-order viscoelasticity in the real materials. Finally, the conclusion is given in Sect. 4.

2 Mathematical Tools

In order to discuss the GFC, we introduce the special functions and the FC operator of the Riemann–Liouville and Liouville-Caputo types in this section. Meanwhile, we present the recent results on the GFC operators in the kernels of the special functions. Finally, the Laplace transforms of the FC and GFC operators are considered in detail (see [38–51]).

2.1 The Special Functions with Power Law

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of complex numbers, real numbers, non-negative real numbers, positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, respectively.

The Mittag–Leffler function, introduced by Swedish mathematician Gosta Mittag–Leffler in 1903, is defined as [47]:

$$E_\nu(\eta) = \sum_{\kappa=0}^{\infty} \frac{\eta^\kappa}{\Gamma(\kappa\nu + 1)}, \tag{1}$$

where $\eta, \nu \in \mathbb{C}, \Re(\nu) \in \mathbb{R}_0^+, \kappa \in \mathbb{N}$, and $\Gamma(\cdot)$ is the familiar Gamma function [3].

As first extension of the Mittag–Leffler function, the extended Mittag–Leffler function, structured by Wiman in 1905, is defined as [48]:

$$E_{\nu,\upsilon}(\eta) = \sum_{\kappa=0}^{\infty} \frac{\eta^{\kappa}}{\Gamma(\kappa\nu + \upsilon)}, \tag{2}$$

where $\eta, \nu, \upsilon \in \mathbb{C}, \Re(\nu), \Re(\upsilon) \in \mathbb{R}_0^+$, and $\kappa \in \mathbb{N}$.

As further extension of the Mittag–Leffler function, the extended Mittag–Leffler function, introduced by Prabhakar in 1971, is given as [49]:

$$E_{\nu,\upsilon}^{\phi}(\eta) = \sum_{\kappa=0}^{\infty} \frac{(\phi)_{\kappa}}{\Gamma(\kappa\nu + \upsilon)} \frac{\eta^{\kappa}}{\Gamma(\kappa + 1)}, \tag{3}$$

where $\eta, \nu, \upsilon, \phi \in \mathbb{C}, \Re(\nu), \Re(\upsilon), \Re(\phi) \in \mathbb{R}_0^+, \kappa \in \mathbb{N}$, and the familiar Pochhammer symbol is expressed as [50]:

$$(\phi)_{\kappa} = \begin{cases} 1, \ \kappa = 0, \\ \dfrac{\Gamma(\phi + \kappa)}{\Gamma(\phi)}, \kappa \in \mathbb{N}. \end{cases} \tag{4}$$

For $\lambda \in \mathbb{C}$, the Laplace transforms of the functions with power law are given as [38, 40, 49]:

$$\mathrm{L}\left[\frac{t^{-\nu}}{\Gamma(1-\nu)}\right] = s^{\nu}, \tag{5}$$

$$\mathrm{L}\left[\frac{t^{\nu}}{\Gamma(1+\nu)}\right] = s^{-\nu}, \tag{6}$$

$$\mathrm{L}\left[t^{\upsilon-1} E_{\nu,\upsilon}^{\phi}(\lambda t^{\nu})\right] = \frac{1}{s^{\upsilon}(1 - \lambda s^{-\nu})^{\varphi}}, \tag{7}$$

$$\mathrm{L}\left[t^{\upsilon-1} E_{\nu,\upsilon}(\lambda t^{\nu})\right] = \frac{1}{s^{\upsilon}(1 - \lambda s^{-\nu})}, \tag{8}$$

$$\mathrm{L}\left[E_{\nu}(\lambda t^{\nu})\right] = \frac{1}{1 - \lambda s^{-\nu}}, \tag{9a}$$

$$\mathrm{L}[\delta(t)] = 1, \tag{9b}$$

where $\delta(t)$ is the Dirac delta (see [38]) and the Laplace transform is defined as [38]:

$$L[\Phi(t)] = \Phi(s) := \int_0^\infty e^{-st}\Phi(t)dt. \tag{9c}$$

2.2 GFC in a Kernel Function

The GFD of the Riemann–Liouville type is defined as [38, 40, 41, 45, 46]:

$$\left(D_{(\Xi)}^{RL}\Theta\right)(\tau) = \frac{d}{d\theta}\int_a^\tau \Xi(\tau - t)\Theta(t)dt \left(\tau \in \mathbb{R}_0^+\right), \tag{10}$$

where $\Theta \in AC\left(\mathbb{R}_0^+\right)$, and $\Xi(\tau)$ is the kernel function.

The GFD of the Liouville-Caputo type is defined as [38, 40, 41, 45, 46]:

$$\left(D_{(\Xi)}^{LC}\Theta\right)(\theta) = \int_0^\tau \Xi(\tau - t)\Theta^{(1)}(t)dt \left(\tau \in \mathbb{R}_0^+\right), \tag{11}$$

where $\Theta^{(1)}(\tau) = d\Theta(\tau)/d\tau$, $\Theta^{(1)} \in L_1^{loc}\left(\mathbb{R}_0^+\right)$, and $\Xi(\tau)$ is the kernel function.

The relationship between Eqs. (11) and (10) is given as [40, 41]:

$$\left(D_{(\Xi)}^{C}\Theta\right)(\tau) = \left(D_{(\Xi)}^{RL}\Theta\right)(\tau) - \Xi(\tau)\Theta(0). \tag{12}$$

2.3 FC Within the Singular Power-Law Kernel

With the use of the kernel $\Xi(\tau) = \tau^{-\nu}/\Gamma(1-\nu)$, the Riemann–Liouville FD of the function $\Theta(\tau)$ of order $(0 < \nu < 1)$ is given by [1, 2, 4, 5]:

$$\left({}_0^{RL}D_\tau^{(\nu)}\Theta\right)(\tau) = \frac{1}{\Gamma(1-\nu)}\frac{d}{d\tau}\int_0^\tau \frac{\Theta(t)}{(\tau - t)^\nu}dt \ (\tau > 0). \tag{13}$$

where $\Theta \in AC\left(\mathbb{R}_0^+\right)$, and the Liouville-Caputo FD of the function $\Theta(\tau)$ by [1, 2, 4–9]

$$\left({}_0^{C}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{1}{\Gamma(1-\nu)}\int_0^\tau \frac{\Theta^{(1)}(t)}{(\tau-t)^{\nu}}dt \ (\tau > 0), \tag{14}$$

where $\Theta^{(1)}(\tau) = d\Theta(\tau)/d\tau$ and $\Theta^{(1)} \in L_1^{loc}\left(\mathbb{R}_0^+\right)$.

The relationship between Eqs. (13) and (14) is given as [2]:

$$\left({}_0^{RL}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \left({}_0^{C}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) + \frac{\tau^{-\nu}}{\Gamma(1-\nu)}\Theta(0). \tag{15}$$

Suppose that $\mathbb{N}$ is the set of positive integers, $m \in \mathbb{N}$ and $m-1 < \nu < m$. Equations (13) and (14) yield [2, 4, 7, 8]:

$$\left({}_0^{RL}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{1}{\Gamma(m-\nu)}\frac{d^m}{d\tau^m}\int_0^\tau \frac{\Theta(t)}{(\tau-t)^{m-\nu-1}}dt \ (\tau > 0), \tag{16}$$

$$\left({}_0^{RL}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{1}{\Gamma(m-\nu)}\int_0^\tau \frac{\Theta^{(m)}(t)}{(\tau-t)^{m-\nu-1}}dt \ (\tau > 0), \tag{17}$$

respectively.

The Laplace transforms of the FC operators in the nonsingular power-law kernel are given as [2]:

$$\mathrm{L}\left[\left({}_0^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau)\right] = s^\nu\Theta(s), \tag{18a}$$

$$\left({}_0\mathrm{I}_t^{(\nu)}\Omega\right)(\tau) \ = \ \frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Omega(t)}{(\tau-t)^{1-\nu}}dt. \tag{18b}$$

The inverse operator (the Riemann–Liouville fractional integral) is given as [2]:

$$\left({}_0\mathrm{I}_t^{(\nu)}\Omega\right)(\tau) \ = \ \frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Omega(t)}{(\tau-t)^{1-\nu}}dt. \tag{19}$$

The Laplace transforms of the GFC operators in the nonsingular power-law kernel are given as [2]:

$$\left({}_0^{RL}\mathrm{D}_\tau^{(\nu)}\Theta\right)(t) \ = \ s^\nu\Theta(s),$$

$$\left({}_0^{LC}\mathrm{D}_\tau^{(\nu)}\Omega\right)(t) \ = \ s^{\nu-1}(s\Omega(s) - \Omega(0)),$$

where $\Theta(s)$ is the Laplace transform of the function $\Theta(s)$.

The properties of the GFD in the nonsingular power-law kernel are given as [2]:

$$\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau),$$

$${}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}1 = \frac{(\tau-a)^{-\nu}}{\Gamma(1-\nu)},$$

$${}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}1=0,$$

$$\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau).$$

Remark 1 Liouville derived the fractional derivative formula (see [4]).

$${}_{0}^{C}\mathrm{D}_{\infty}^{(\nu)}\Theta(\tau)=\frac{1}{(-1)^{\nu}\Gamma(\nu)}\int_{0}^{\infty}\Theta^{(m)}(\tau+t)t^{\nu-1}dt,$$

and the formula (see [4])

$$h\int_{0}^{\tau}(\tau-t)^{-\frac{1}{2}}\Theta^{(1)}(t)dt=m(\tau),$$

where $h=1/\sqrt{2\mathrm{g}}$ is the constant, though not quite rigorously from the modern point of view.

So nine introduced the following fractional derivative given as (see [6])

$${}_{a}^{C}\mathrm{D}_{\tau}^{(\nu)}\Theta(\tau)=\frac{1}{\Gamma(p-\nu+1)}\int_{a}^{\tau}(\tau-t)^{p-\nu}\Theta^{(1)}(t)dt,\ \mathrm{Re}(n)<\nu<\mathrm{Re}(n+1).$$

Caputo and Smit and De Vries introduced the fractional derivative in the form (see [7, 8])

$${}_{a}^{C}\mathrm{D}_{\tau}^{(\nu)}\Theta(\theta)=\frac{1}{\Gamma(n-\nu)}\int_{a}^{\tau}\frac{1}{(\tau-t)^{\nu}}\Theta^{(n)}(t)dt.$$

In 1968, Dzhrbashyan and Nersesyan introduced the fractional derivative (see[9])

$${}_{0}^{C}\mathrm{D}_{\infty}^{(\nu)}\Theta(\theta)=\frac{1}{\Gamma(n-\nu)}\int_{0}^{\infty}\frac{1}{(\tau-t)^{n-\nu}}\Theta^{(n)}(t)dt.$$

Theorem 1 (see [49]).

Let $\tau \in \mathbb{R}_0^+$, $\nu \in (0,1)$, $\Xi \in L\left(\mathbb{R}_0^+\right)$ and $\Omega^{(1)} \in L_1^{loc}\left(\mathbb{R}_0^+\right)$. Then, there is an Abel integral of the form

$$\frac{1}{\Gamma(\nu)}\int_0^\tau \frac{\Xi(t)}{(\tau-t)^{1-\nu}}dt=\Omega(\tau), \tag{20a}$$

with the solution given as

$$\Xi(t)=\frac{1}{\Gamma(1+\nu)}\int_0^\tau (\tau-t)^\nu\Omega^{(1)}(t)dt+\frac{\tau^\nu}{\Gamma(1+\nu)}\Omega(0), \tag{20b}$$

where $\Omega(\tau=0)=\Omega(0)$.

2.4 GFC with the Nonsingular Power-Law Kernel

When the kernel in Eq. (1) is given as $\Xi(\tau)=\tau^\nu/\Gamma(1+\nu)$, the Riemann–Liouville-type GFD of the function $\Theta(\tau)$ of order $(0<\nu<1)$ in the nonsingular power-law kernel is defined as [21, 38]

$$\left({}_0^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(1+\nu)}\frac{d}{d\theta}\int_0^\tau (\tau-t)^\nu\Theta(t)dt\ (\tau>0), \tag{21}$$

where $\Theta \in AC\left(\mathbb{R}_0^+\right)$, and the Liouville-Caputo-type GFD of the function $\Theta(\tau)$ of order $(0<\nu<1)$ in the nonsingular power-law kernel as [21, 38]

$$\left({}_0^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(1+\nu)}\int_0^\tau (\tau-t)^\nu\Theta^{(1)}(t)dt\ (\tau>0), \tag{22}$$

where $\Theta^{(1)}(\tau)=d\Theta(\tau)/d\tau$ and $\Theta(1)\in L_1^{loc}\left(\mathbb{R}_0^+\right)$.

The relationship between Eq. (21) and Eq. (22) is presented as [21, 38]:

$$\left({}_0^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau)=\left({}_0^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau)+\frac{\theta^\nu\Theta(0)}{\Gamma(1+\nu)}. \tag{23}$$

Similarly, for $m-1<\nu<m$, Eqs. (23) and (24) yield:

$$\left({}_{0}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(m+\nu)}\frac{d^{m}}{d\tau^{m}}\int_{0}^{\tau}(\tau-t)^{m-\nu-1}\Theta(t)dt\ (\tau>0), \tag{24}$$

$$\left({}_{0}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(m+\nu)}\int_{0}^{\tau}(\tau-t)^{m-\nu-1}\Theta^{(m)}(t)dt\ (\tau>0). \tag{25}$$

The Laplace transforms of Eqs. (21) and (22) are presented as follows [21, 38]:

$$\mathrm{L}\left[\left({}_{0}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)\right]=\frac{1}{s^{\nu}}\Theta(s), \tag{26}$$

$$\mathrm{L}\left[\left({}_{0}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)\right]=\frac{1}{s^{1+\nu}}(s\Theta(s)-\Theta(0)). \tag{27}$$

Its inverse operator (the general fractional integral) is defined as [21, 38]:

$$\left({}_{a}^{L}\mathrm{I}_{\tau}^{(\nu)}\Omega\right)(\tau)=\frac{1}{\Gamma(-\nu)}\int_{a}^{\tau}\frac{1}{(\tau-t)^{1+\nu}}\Omega(t)dt. \tag{28}$$

The Laplace transforms of the GFC operators in the nonsingular power-law kernel are given as [21, 38]:

$$\left({}_{0}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(t)=s^{\nu}\Theta(s),$$

$$\left({}_{0}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Omega\right)(t)=s^{\nu-1}(s\Omega(s)-\Omega(0)).$$

The properties of the GFD in the nonsingular power-law kernel are given as [21, 38]:

$$\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau),$$

$${}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}1=\frac{(\tau-a)^{-\nu}}{\Gamma(1-\nu)},$$

$${}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}1=0,$$

$$\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{a}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau).$$

Theorem 2 (see [21, 38]).

Let $\tau\in\mathbb{R}_0^+$, $\nu\in(0,1)$, $\Xi\in L\left(\mathbb{R}_0^+\right)$ and $\Omega(1)\in L_1^{loc}\left(\mathbb{R}_0^+\right)$. Then, there is an Abel type integral

$$\frac{1}{\Gamma(-\nu)}\int_0^\tau \frac{\Xi(t)}{(\tau-t)^{1+\nu}}dt = \Omega(\tau), \tag{29a}$$

with the solution given as

$$\Xi(t) = \frac{1}{\Gamma(1-\nu)}\int_0^\tau (\tau-t)^{-\nu}\Omega^{(1)}(t)dt + \frac{\tau^{-\nu}}{\Gamma(1-\nu)}\Omega(0), \tag{29b}$$

where $\Omega(\tau = 0) = \Omega(0)$.

2.5 GFC with the Nonsingular Mittag–Leffler Function Kernel

When the kernel in Eq. (1) is given as: $\Xi(\tau) = E_\nu(-\tau^\nu)$, the GFD of Riemann–Liouville type in the kernel of the Mittag–Leffler function is defined by [21, 38, 41]:

$$\left({}_a^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{d}{d\tau}\int_a^\tau E_\nu(-(\tau-t)^\nu)\Theta(t)dt \ (\tau > a), \tag{30}$$

where $\Theta \in AC\left(\mathbb{R}_0^+\right)$, and the GFD of the Liouville-Caputo type in the kernel of the Mittag–Leffler function by [21, 38, 41]:

$$\left({}_a^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \int_a^\tau E_\nu(-(\tau-t)^\nu)\Theta^{(1)}(t)dt \ (\tau > a), \tag{31}$$

where $\Theta^{(1)}(\tau) = d\Theta(\tau)/d\tau$ and $\Theta^{(1)} \in L_1^{loc}\left(\mathbb{R}_0^+\right)$.

The relationship between Eqs. (30) and (31) becomes [21, 38, 41]:

$$\left({}_0^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \left({}_0^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) - E_\nu(\tau^\nu)\Theta(0). \tag{32}$$

Similarly, for $m-1 < \nu < m$, Eqs. (13) and (14) yield:

$$\left({}_a^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{d^m}{d\tau^m}\int_a^\tau E_\nu(-(\tau-t)^\nu)\Theta(t)dt \ (\tau > 0), \tag{33}$$

$$\left({}_{a}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau) = \int\limits_{a}^{\tau} E_{\nu}(-(\tau - t)^{\nu})\Theta^{(m)}(t)dt \ (\tau > 0). \tag{34}$$

Its inverse operator (the general fractional integral) is defined as

$$\left({}_{0}\mathrm{I}_{t}^{(\nu)}\Omega\right)(\tau) = \Omega(\tau) - \frac{1}{\Gamma(\nu)}\int\limits_{0}^{\tau} \frac{\Omega(t)}{(\tau - t)^{1-\nu}}dt$$

Remark 2 Hille and Tamarlcin proposed the Abel type integral equation of the second kind (see [50]).

$$\Omega(\tau) - \frac{\tau}{\Gamma(\nu)}\int\limits_{0}^{x} \frac{\Omega(t)}{(\tau - t)^{1-\nu}}dt = \Theta(\tau), 0 < \alpha < 1,$$

with the solution given as

$$\Omega(\tau) = \frac{d}{d\tau}\int\limits_{0}^{\tau} E_{\nu}\left[\lambda(\tau - t)^{\nu}\right]\Theta(t)dt.$$

Hille introduced the following fractional differential operator (see [51])

$$\left({}_{a}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau) = \lambda\frac{d}{d\tau}\int\limits_{0}^{\tau} E_{\nu}\left[\lambda(t - x)^{\nu}\right]f(t)dt.$$

Atangana and Baleanu introduced the general fractional derivative with the Mittag–Leffler function involving the normalization parameter (see [43])

$$\left({}_{a}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau) = \frac{\Im(\nu)}{1-\nu}\int\limits_{a}^{\tau} \mathrm{E}_{\nu}\left(-\frac{\nu}{1-\nu}(\tau - t)^{\nu}\right)\Theta^{(1)}(t)dt,$$

where $\Im(\nu)$ is the normalization parameter.

The Laplace transforms of the GFC operators in the nonsingular Mittag–Leffler kernel are given as [21, 38, 41]:

$$\left({}_{0}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(t) = \left(1 - s^{-\nu}\right)^{-1}\Theta(s),$$

$$\left({}_{0}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Omega\right)(t) = s^{-1}\left(1 - s^{-\nu}\right)^{-1}(s\Omega(s) - \Omega(0)).$$

The properties of the GFD in the nonsingular Mittag–Leffler kernel are given as [21, 38, 41]:,

$$\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau),$$

$${}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}1=E_{\nu}(-\tau^{\nu}),$$

$${}_{\alpha}^{LC}D_{\tau}^{(\nu)}1=0,$$

$$\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau).$$

2.6 GFC with the Nonsingular Wiman Kernel

When the kernel in Eq. (1) is given as: $\Xi(\tau)=\tau^{\upsilon-1}E_{\nu,\upsilon}(-\tau^{\nu})$, the GFD of Riemann–Liouville type in the kernel of the Wiman function is defined by [21, 38, 41]:

$$\left({}_{a}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{d}{d\tau}\int_{a}^{\tau}(\tau-t)^{\upsilon-1}E_{\nu,\upsilon}(-(\tau-t)^{\nu})\Omega(t)dt\ (\tau>a),\qquad(35)$$

where $\Theta\in AC\left(\mathbb{R}_0^+\right)$, and the GFD of the Liouville-Caputo type in the kernel of the Wiman function by:

$$\left({}_{E_{\nu,\upsilon}(-)}^{C}\mathrm{D}_{a}^{(\nu)}\Omega\right)(\tau)=\int_{a}^{\tau}(\tau-t)^{\upsilon-1}E_{\nu,\upsilon}(-(\tau-t)^{\nu})\Omega^{(1)}(t)dt\ (\tau>a),\qquad(36)$$

where $\Theta^{(1)}(\tau)=d\Theta(\tau)\big/d\tau$ and $\Theta^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$.

The relationship between Eqs. (35) and (36) is [21, 38, 41]:

$$\left({}_{a}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\left({}_{a}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)-\tau^{\upsilon-1}E_{\nu,\upsilon}(-\tau^{\nu})\Omega(0).\qquad(37)$$

Similarly, for $m-1<\nu<m$, Eqs. (13) and (14) yield:

$$\left({}_{a}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{d^m}{d\tau^m}\int_{a}^{\tau}(\tau-t)^{\upsilon-1}E_{\nu,\upsilon}(-(\tau-t)^{\nu})\Omega(t)dt\ (\tau>0),\qquad(38)$$

$$\left({}_{a}^{CT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\int_{a}^{\tau}(\tau-t)^{\upsilon-1}E_{\nu,\upsilon}(-(\tau-t)^{\nu})\Theta^{(m)}(t)dt\ (\tau>0). \quad (39)$$

Its inverse operator (the general fractional integral) is defined as [21, 38, 41]

$$\left({}_{0}\mathrm{I}_{t}^{(\nu)}\Omega\right)(\tau)=\int_{0}^{\tau}(\tau-t)^{-\upsilon}E_{\nu,1-\upsilon}^{-1}(-(\tau-t)^{\nu})\Omega(t)dt$$

The Laplace transforms of the GFC operators in the nonsingular Wiman kernel are given as [21, 38, 41]:

$$\left({}_{0}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(t)=s^{1-\upsilon}\left(1+s^{-\nu}\right)^{-1}\Theta(s),$$

$$\left({}_{0}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Omega\right)(t)=s^{-\upsilon}\left(1+s^{-\nu}\right)^{-1}(s\Omega(s)-\Omega(0)).$$

The properties of the GFD in the nonsingular Wiman kernel are given as [21, 38, 41]:

$$\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}(\Theta_{1}+\Theta_{2})\right)(\tau)=\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}\Theta_{1}\right)(\tau)+\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}\Theta_{2}\right)(\tau).$$

$${}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}1=\tau^{\upsilon-1}E_{\nu,\upsilon}(-\tau^{\nu}),$$

$${}_{\alpha}^{LC}D_{\tau}^{(\nu)}1=0,$$

$$\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}(\Theta_{1}+\Theta_{2})\right)(\tau)=\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}\Theta_{1}\right)(\tau)+\left({}_{\alpha}^{LC}D_{\tau}^{(\nu)}\Theta_{2}\right)(\tau).$$

2.7 GFC with the Nonsingular Prabhakar Kernel

When the kernel in Eq. (1) is given as: $\Xi(\tau)=\tau^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-\tau^{\nu})$, the GFD of Riemann–Liouville type in the kernel of the Prabhakar function is defined as [21, 38, 41]:

$$\left({}_{a}^{RLT}\mathrm{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{d}{d\tau}\int_{a}^{\tau}(\tau-t)^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-(\tau-t)^{\nu})\Omega(t)dt\ (\tau>a), \quad (40)$$

where $\Theta\in AC\left(\mathbb{R}_{0}^{+}\right)$, and the GFD of the Liouville-Caputo type in the kernel of the Prabhakar function as [21, 38, 41]:

$$\left({}_{E_{\nu,\upsilon}(-)}^{C}\mathrm{D}_a^{(\nu)}\Omega\right)(\tau) = \int_a^\tau (\tau-t)^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-(\tau-t)^{\nu})\Omega^{(1)}(t)dt \ (\tau > a), \tag{41}$$

where $\Theta^{(1)}(\tau) = d\Theta(\tau)/d\tau$ and $\Theta^{(1)} \in L_1^{loc}(\mathbb{R}_0^+)$.

The relationship between Eqs. (40) and (41) is [21, 38, 41]:

$$\left({}_a^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \left({}_a^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) - \tau^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-\tau^{\nu})\Omega(0). \tag{42}$$

Similarly, for $m-1 < \nu < m$, Eqs. (13) and (14) yield:

$$\left({}_a^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{d^m}{d\tau^m}\int_a^\tau (\tau-t)^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-(\tau-t)^{\nu})\Omega(t)dt \ (\tau > 0), \tag{43}$$

$$\left({}_a^{CT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \int_a^\tau (\tau-t)^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-(\tau-t)^{\nu})\Theta^{(m)}(t)dt \ (\tau > 0). \tag{44}$$

Its inverse operator (the general fractional integral) is defined as [[21, 38, 41]]

$$\left({}_0\mathrm{I}_t^{(\nu)}\Omega\right)(\tau) = \int_0^\tau (\tau-t)^{-\upsilon}E_{\nu,1-\upsilon}^{-\phi}(-(\tau-t)^{\nu})\Omega(t)dt$$

Remark 3 Kilbas, Saigo and Saxena introduced the following general fractional derivative (see [42]).

$$\left({}_a^{RLT}\mathrm{D}_\tau^{(\nu)}\Theta\right)(\tau) = \frac{d^m}{d\tau^m}\int_a^\tau (\tau-t)^{\mu+m-\beta-1}E_{\nu,\mu+m-\beta}^{\phi}(\lambda(\tau-t)^{\nu})\Theta(t)dt,$$

which is called the Kilbas-Saigo-Saxena GFD.

The Laplace transforms of the GFC operators in the nonsingular Prabhakar kernel are given as [21, 38, 41]:

$$\left({}_0^{RL}\mathrm{D}_\tau^{(\nu)}\Theta\right)(t) = s^{1-\upsilon}\left(1+s^{-\nu}\right)^{-\phi}\Theta(s),$$

$$\left({}_0^{LC}\mathrm{D}_\tau^{(\nu)}\Omega\right)(t) = s^{-\upsilon}\left(1+s^{-\nu}\right)^{-\phi}(s\Omega(s)-\Omega(0)).$$

The properties of the GFD in the nonsingular Prabhakar kernel are given as [21, 38, 41]:

$$\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{\alpha}^{RL}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau),$$

$${}_{a}^{RL}\mathrm{D}_{\tau}^{(\nu)}1=\tau^{\upsilon-1}E_{\nu,\upsilon}^{\phi}(-\tau^{\nu}),$$

$$\left|{}_{\alpha}^{LC}D_{\tau}^{(\nu)}1=0,\right.$$

$$\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}(\Theta_1+\Theta_2)\right)(\tau)=\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_1\right)(\tau)+\left({}_{\alpha}^{LC}\mathrm{D}_{\tau}^{(\nu)}\Theta_2\right)(\tau).$$

3 The Rheological Models with GFCs Involving the Nonsingular Kernels

3.1 *Complex Phenomena in Viscoelasticity*

The stress–strain-time relation with the positive-parametric Mittag–Leffler function can be written as

$$\sigma_{\nu}(\tau)=\mathrm{M}E_{\nu}(-\tau^{\nu})\varepsilon_{\nu}(0). \tag{45}$$

where $\sigma_{\nu}(\tau)$ is stress, $\varepsilon_{\nu}(0)$ is the initial strain, τ is time and M is the material constant.

There are

$$E_{\nu}(-\tau^{\nu})\propto\tau^{-\nu}, \tag{46}$$

$$E_{\nu,1}(-\tau^{\nu})\propto\tau^{-\nu}, \tag{47}$$

$$E_{\nu,1}^{1}(-\tau^{\nu})\propto\tau^{-\nu}, \tag{48}$$

which, after taking the Laplace transform, leads to

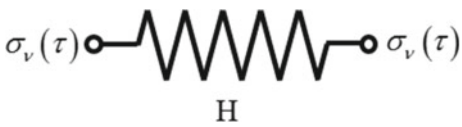
$$\sigma_{\nu}(s)=\mathrm{M}\varepsilon_{\nu}(0)s^{-1}\left(1+s^{-\nu}\right)^{-1}\propto\mathrm{M}\varepsilon_{\nu}(0)\Gamma(1-\nu)s^{\nu}. \tag{49}$$

The phenomena in rheological behaviors are called as the Nutting behaviors in the real materials.

In another hand, there may exist the stress–strain-time relation with the positive-parametric Mittag–Leffler function can be written as

$$\sigma_{\nu}(\tau)=\mathrm{M}E_{\nu}\left(-\tau^{-\nu}\right)\varepsilon_{\nu}(0). \tag{50}$$

Fig. 1 The spring element



where $\sigma_\nu(\tau)$ is stress, $\varepsilon_\nu(0)$ is the initial strain, τ is time and M is the material constant.

There are

$$E_\nu\left(-\tau^{-\nu}\right) \propto \tau^\nu, \tag{51}$$

$$E_{\nu,1}\left(-\tau^{-\nu}\right) \propto \tau^\nu, \tag{52}$$

$$E^1_{\nu,1}\left(-\tau^{-\nu}\right) \propto \tau^\nu, \tag{53}$$

The phenomena in rheological behaviors are called as the anomalous Nutting behaviors in the real materials.

3.2 The Viscoelastic Elements with GFDs

3.2.1 The Spring Element

Model 1

As shown in Fig. 1, the spring element follows the Hooke's law given as [20, 21]

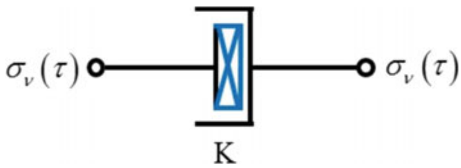
$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(\tau), \tag{54}$$

where H is the Young's modulus of the material.

3.2.2 The Viscoelastic Elements

As shown in Fig. 2, the viscoelastic elements with the FD and GFDs were presented to describe the viscoelastic behaviors in the real materials.

Fig. 2 The viscoelastic element



Model 2

The viscoelastic element with the FD in the singular power-law kernel is given as [20, 21]:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}_{0}^{LC}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(\tau). \tag{55}$$

where K is the coefficient of viscosity.

Model 3

The viscoelastic element with the GFD in the singular power-law kernel is represented in the form:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}_{0}^{LC}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(\tau). \tag{56}$$

where K is the coefficient of viscosity.

Model 4

The viscoelastic element with the GFD in the kernel of the Mittag–Leffler function is can be expressed as:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}_{0}^{LC}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(\tau). \tag{57}$$

where K is the coefficient of viscosity.

Model 5

The viscoelastic element with the GFD in the kernel of the Wiman functions is represented as:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}_{0}^{LC}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(\tau). \tag{58}$$

where K is the coefficient of viscosity.

Model 6

The viscoelastic element with the GFD in the kernel of the Prabhakar functions is represented in the form:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}_{0}^{LC}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(\tau). \tag{59}$$

where K is the coefficient of viscosity.

The creep and relaxation representations are given through the equations of the Volterra type:

$$\varepsilon_\nu(\tau) = \sigma_\nu(0) J_\nu(\tau) + \int_0^\tau J_\nu(\tau - t)\left({}_0^{CT}\mathrm{D}_t^{(\nu)}\sigma_\nu\right)(t)dt \tag{60}$$

and

$$\sigma_\nu(\tau) = \varepsilon_\nu(0) G_\nu(\tau) + \int_0^\tau G_\nu(\tau - t)\left({}_0^{CT}\mathrm{D}_t^{(\nu)}\varepsilon_\nu\right)(t)dt, \tag{61}$$

where the creep compliance and relaxation modulus are given by: $J_\nu(\tau) = \varepsilon_\nu(\tau)/\sigma_\nu(0)$ and $G_\nu(\tau) = \sigma_\nu(\tau)/\varepsilon_\nu(0)$, respectively.

3.3 The Maxwell Models with GFDs

As shown in Fig. 3, the Maxwell models with the GFDs and FD consists of a Hookean element and a general fractional-order Newtonian element in series.

The constitutive equation of the Maxwell model with GFDs can be written as

$$\left({}_0^{CT}\mathrm{D}_\tau^{(\nu)}\varepsilon_\nu\right)(\tau) = \frac{\sigma_\nu(\tau)}{\mathrm{K}} + \frac{1}{\mathrm{H}}\left({}_0^{CT}\mathrm{D}_\tau^{(\nu)}\sigma_\nu\right)(\tau).$$

Case 1

The creep compliance of the Maxwell model with the FD in the singular power-law kernel can be written as [20, 21]

$$J_\nu(\tau) = \frac{1}{\mathrm{K}}\frac{\tau^\nu}{\Gamma(1+\nu)} + \frac{2}{\mathrm{H}},$$

and the relaxation modulus of the Maxwell model with the FD in the singular power-law kernel is given as [20, 21]

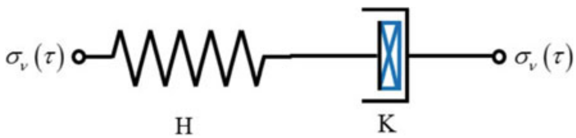


Fig. 3 The Maxwell model via FD and GFDs

$$G_\nu(\tau) = 2\mathrm{K}\tau^{-\nu} E_\nu\left(-\frac{\mathrm{K}}{\mathrm{H}} t^\nu\right).$$

Case 2

The creep compliance of the Maxwell model with the GFD in the nonsingular power-law kernel is

$$J_\nu(\tau) = \frac{1}{\mathrm{K}} \frac{\tau^{-\nu}}{\Gamma(1-\nu)} + \frac{2}{\mathrm{H}},$$

and the relaxation modulus of the Maxwell model with the GFD in the nonsingular power-law kernel can be given as

$$G_\nu(\tau) = 2\mathrm{K}\tau^{\nu} E_{\nu,1+\nu}\left(-\frac{\mathrm{K}}{\mathrm{H}} t^\nu\right).$$

Case 3

The creep compliance of the Maxwell model with GFD in the kernel of the Mittag–Leffler function can be written as

$$J_\nu(\tau) = \frac{1}{\mathrm{K}} + \frac{2}{\mathrm{H}} + \frac{1}{\mathrm{K}} \frac{\tau^{\nu}}{\Gamma(1+\nu)},$$

and the relaxation modulus of the Maxwell model with general fractional derivative in the kernel of the Mittag–Leffler function becomes

$$G_\nu(\tau) = \frac{\mathrm{K}}{\mathrm{H}+\mathrm{K}} E_\nu\left(-\frac{\mathrm{H}}{\mathrm{H}+\mathrm{K}} \tau^\nu\right).$$

Case 4

The creep compliance of the Maxwell model with general fractional derivative in the kernel of the Wiman function can be represented in the form:

$$J_\nu(\tau) = \frac{1}{\mathrm{K}}\left(\frac{\tau^{1-\upsilon}}{\Gamma(2-\upsilon)} + \frac{\tau^{1-\upsilon+\nu}}{\Gamma(2-\upsilon+\nu)} + \frac{\mathrm{K}}{\mathrm{H}}\right),$$

and the relaxation modulus of the Maxwell model with the GFD in the kernel of the Wiman function is

$$G_\nu(\tau) = \mathrm{K} E_{\nu,\nu}\left(-\left(\frac{\mathrm{K}}{\mathrm{H}} + 1\right)\tau^\nu\right).$$

Case 5

The creep compliance of the Maxwell model with the GFD in the kernel of the Prabhakar function can be expressed as

$$J_\nu(\tau) = \frac{1}{\mathrm{K}}\left(\tau^{1-\upsilon} E_{\nu,\upsilon}^{-\phi}(-\tau^\nu) + \frac{2}{\mathrm{H}}\right),$$

and the relaxation modulus of the Maxwell model with the GFD in the kernel of the Prabhakar function is

$$G_\nu(\tau) = 2\mathrm{K} \sum_{n=0}^{\infty} \left(-\frac{\mathrm{K}}{\mathrm{H}}\right)^n \tau^{(n-1)(2-\upsilon)} E_{\nu,(n-1)(2-\upsilon)+1}^{(1-n)\phi}(-\tau^\nu).$$

3.4 *The Voigt Models with GFDs*

As shown in Fig. 4, the Voigt models with the GFDs and FD consists of a Hookean element and a general fractional-order Newtonian element in parallel.

The constitutive equation of the Voigt model can be written as

$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(\tau) + \mathrm{K}\left({}_0^{RLT}\mathrm{D}_\tau^{(\nu)}\varepsilon_\nu\right)(\tau).$$

Case 1

The creep compliance of the Voigt model with the FD in the singular power-law kernel can be written as [20, 21]

$$J_\nu(\tau) = \frac{1}{\mathrm{H}}\left(1 - E_\nu\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^\nu\right)\right),$$

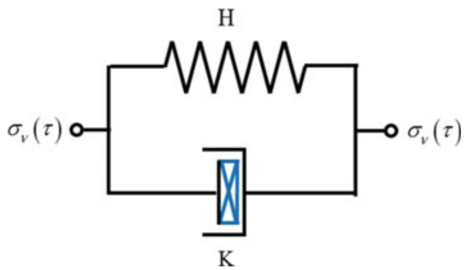


Fig. 4 The Voigt model via FD and GFDs

and the relaxation modulus of the Voigt model with the FD in the singular power-law kernel is given as [20, 21]

$$G_\nu(\tau) = \mathrm{H} + \mathrm{K}\frac{\tau^{-\nu}}{\Gamma(1-\nu)}.$$

Case 2

The creep compliance of the Voigt model with the GFD in the nonsingular power-law kernel is represented as

$$J_\nu(\tau) = \frac{1}{\mathrm{H}} E_\nu\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^\nu\right),$$

and the relaxation modulus of the Voigt model with the GFD in the nonsingular power-law kernel can be given as

$$G_\nu(\tau) = \mathrm{H} + \mathrm{K}\frac{\tau^{\nu}}{\Gamma(1+\nu)}.$$

Case 3

The creep compliance of the Voigt model with the GFD in the kernel of the Mittag–Leffler function is

$$J_\nu(\tau) = \frac{1}{\mathrm{H}+\mathrm{K}}\left(E_\nu\left(-\frac{\mathrm{H}}{\mathrm{H}+\mathrm{K}}\tau^\nu\right) + E_{\nu,\nu+1}\left(-\frac{\mathrm{H}}{\mathrm{H}+\mathrm{K}}\tau^\nu\right)\right),$$

and the relaxation modulus of the Voigt model with the GFD in the kernel of the Mittag–Leffler function is given as

$$G_\nu(\tau) = \mathrm{H} + \mathrm{K}E_\nu(-\tau^\nu).$$

Case 4

The creep compliance of the Voigt model with the GFD in the kernel of the Wiman function is expressed by

$$J_\nu(\tau) = \frac{\tau^{1-\upsilon}}{\Gamma(2-\upsilon)} + \frac{\tau^{1+\nu-\upsilon}}{\Gamma(2+\nu-\upsilon)} + \frac{2\mathrm{K}}{\mathrm{H}},$$

and the relaxation modulus of the Voigt model with the GFD in the kernel of the Wiman function can be written as

$$G_\nu(\tau) = \frac{2\mathrm{H}}{\mathrm{K}} E_{\nu,\nu}\left(-\left(\frac{\mathrm{K}}{\mathrm{H}}+1\right)\tau^\nu\right).$$

Case 5

The creep compliance of the Voigt model with the GFD in the kernel of the Prabhakar function is

$$J_\nu(\tau) = \frac{1}{\mathrm{H}}\left(\sum_{n=0}^{\infty}\left(-\frac{\mathrm{K}}{\mathrm{H}}\right)^{-n}\tau^{n(1-\upsilon)}E^{-n\phi}_{\nu,n(1-\upsilon)+1}(-\tau^\nu)\right.$$
$$\left.+\frac{\mathrm{K}}{\mathrm{H}}\sum_{n=0}^{\infty}\left(-\frac{\mathrm{K}}{\mathrm{H}}\right)^{n}\tau^{(n-1)(2-\upsilon)}E^{(1-n)\phi}_{\nu,(n-1)(2-\upsilon)+1}(-\tau^\nu)\right),$$

and the relaxation modulus of the Voigt model with the GFD in the kernel of the Prabhakar function is given as

$$G_\nu(\tau) = \mathrm{H} + \mathrm{K}\tau^{\upsilon-1}E^{\phi}_{\nu,\upsilon}(-\tau^\nu).$$

For more details of the applications of the GFC operators to the viscoelastic behaviors, see [20, 21].

4 Conclusion

In the present work, we investigated the basic formulations of the FC and GFC operators with the special functions with the power law. The Laplace transforms of the GFDs and GFIs formulations were discussed in detail. The anomalous Nutting behaviors in the real materials can be proposed for the first time. The applications of the GFC operators to the viscoelastic behaviors can be represented in the use of the complexity of the real materials. The Maxwell and Voigt models with the GFDs in the nonsingular kernels were obtained with the help of the Laplace transforms of the special functions. The results can be explained the complex phenomenon in the mining-rock materials.

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References

1. Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Amsterdam (1993)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Academic (2006)
3. Machado, J.T., Kiryakova, V., Mainardi, F.: Recent history of fractional calculus. Commun. Nonlinear Sci. Numer. Simul. **16**(3), 1140–1153 (2011)
4. Liouville, J.: Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions. J. de l'Ecole Polytechnique **13**, 1–69 (1832)
5. Riemann, B.: Versuch einer allgemeinen Auffassung der Integration und Differentiation, 14 Janvier. Bernhard Riemann's Gesammelte Mathematische Werke **1892**, 353–362 (1847)
6. Sonine, N.: Sur la différentiation à indice quelconque.Matematicheskii Sbornik **6**(1), 1–38 (1872)
7. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent—II. Geophys. J. Int. **13**(5), 529–539 (1967)
8. Smit, W., De Vries, H.: Rheological models containing fractional derivatives. Rheol. Acta **9**(4), 525–534 (1970)
9. Dzhrbashyan, M., Nersesyan, A.: Drobnye proizvodnye i zadacha cauchy dlya di_erencial.nykh uravneniy drobnogo poryadka. Izvestiya Akademii Nauk Armyanskoj SSR, ser. Matematika **3**(1), 3–29 (1968)
10. Kaczorek, T., Rogowski, K.: Fractional Linear Systems and Electrical Circuits. Springer International Publishing, Switzerland (2015)
11. Vinagre, B.M., Podlubny, I., Hernandez, A., Feliu, V.: Some approximations of fractional order operators used in control theory and applications. Fractional Calculus Appl. Anal. **3**(3), 231–248 (2000)
12. Richard, H.: Fractional Calculus: An Introduction for Physicists. World Scientific (2014)
13. Drapaca, C.S., Sivaloganathan, S.: A fractional model of continuum mechanics. J. Elast. **107**(2), 105–123 (2012)
14. Blasiak, S.: Time-fractional heat transfer equations in modeling of the non-contacting face seals. Int. J. Heat Mass Transf. **100**, 79–88 (2016)
15. Machado, J.T., Mata, M.E.: Pseudo phase plane and fractional calculus modeling of western global economic downturn. Commun. Nonlinear Sci. Numer. Simul. **22**(1), 396–406 (2015)
16. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Phys. A **284**(1), 376–384 (2000)
17. Rivero, M., Trujillo, J.J., Vázquez, L., Velasco, M.P.: Fractional dynamics of populations. Appl. Math. Comput. **218**(3), 1089–1095 (2011)
18. Magin, R.L.: Fractional calculus models of complex dynamics in biological tissues. Comput. Math. Appl. **59**(5), 1586–1593 (2010)
19. Podlubny, I.: Fractional Differential Equations. Academic, San Diego (1998)
20. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific (2010)
21. Yang, X.J., Gao, F., Ju, Y.: General Fractional Derivatives with Applications in Viscoelasticity. Academic (2020)
22. Gemant, A.: A method of analyzing experimental results obtained from elastiviscous bodies. Physics **7**, 311–317 (1936)
23. Blair, G.S.: The role of psychophysics in rheology. J. Colloid Sci. **2**(1), 21–32 (1947)
24. Gerasimov, A.N.: A generalization of linear laws of deformation and its application to problems of internal friction. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 251–260 (1948). (Russian)
25. Caputo, M., Mainardi, F.: Linear models of dissipation in anelastic solids. La Rivista del Nuovo Cimento **1**(2), 161–198 (1971)
26. Rossikhin, Y.A., Shitikova, M.V.:. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. Appl. Mech. Rev. **63**(1), 010801 (2010)

27. Bagley, R.: On the equivalence of the Riemann-Liouville and the Caputo fractional order derivatives in modeling of linear viscoelastic materials. Fractional Calculus and Applied Analysis **10**(2), 123–126 (2007)
28. Gao, F., Yang, X.J.: Fractional Maxwell fluid with fractional derivative without singular kernel. Therm. Sci. **20**, 871–877 (2016)
29. Yang, X.J., Gao, F., Srivastava, H.M.: New rheological models within local fractional derivative. Romanian Rep. Phys. **69**(3), 113 (2017)
30. Yang, X.J.: New general fractional-order rheological models within kernels of Mittag-Leffler functions. Rom. J. Phys. **69**(4), 115 (2017)
31. Rabotnov, Y.: Equilibrium of an elastic medium with after effect (in Russian). Prikladnaya Matematika i Mekhanika **12**(1), 53–62 (1948)
32. Stiassnie, M.: On the application of fractional calculus for the formulation of viscoelastic models. Appl. Math. Model. **3**(4), 300–302 (1979)
33. Koeller, R.C.: A theory relating creep and relaxation for linear materials with memory. J. Appl. Mech. **77**(3), 031008 (2010)
34. Bagley, R.L., Torvik, P.J.: On the fractional calculus model of viscoelastic behavior. J. Rheol. **30**(1), 133–155 (1986)
35. Heymans, N., Bauwens, J.C.: Fractal rheological models and fractional differential equations for viscoelastic behavior. Rheologica Acta **33**(3), 210–219 (1994)
36. Schiessel, H., Metzler, R., Blumen, A., Nonnenmacher, T.F.: Generalized viscoelastic models: their fractional equations with solutions. J. Phys. A **28**(23), 6567 (1995)
37. Nutting, P.G.: A new general law of deformation. J. Franklin Inst. **191**(5), 679–685 (1921)
38. Yang, X.-J.: Theoretical studies on general fractional-order viscoelasticity, Ph.D. Thesis, China University of Mining and Technology, Xuzhou, China (2017)
39. Yang, X.-J.: General fractional calculus: a new prospective on viscoelasticity. In: International Workshop on Theory and Applications of Fractional Partial Differential Equations. Qingdao, China (2018)
40. Yang, X.J.: General Fractional Derivatives: Theory, Methods and Applications. CRC Press (2019)
41. Yang, X.J., Machado, J.T., Baleanu, D.: Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler type functions. Romanian Rep. Phys **69**(4), 115 (2017)
42. Kilbas, A.A., Saigo, M., Saxena, R.K.: Generalized Mittag-Leffler function and generalized fractional calculus operators. Integr. Transf. Special Funct. **15**(1), 31-49 (2004)
43. Atangana, A., Baleanu, D.: New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. Thermal Sci **20**(2), 763–769 (2016)
44. Kiryakova, V.S.: Generalized Fractional Calculus and Applications. CRC Press (1993)
45. Kochubei, A.N.: General fractional calculus, evolution equations, and renewal processes. Integr. Eqn. Oper. Theory **71**(4), 583–600 (2011)
46. Mittag-Leffler, G.M., Sur La Nouvelle Fonction $E_\alpha(x)$. Comptes Rendus de l'Acad′emie des Sciences, **137**, 554–558 (1903)
47. Wiman, A.: über Den Fundamental Satz in Der Theorie Der Funktionen $E_\alpha(x)$. Acta Math. **29**, 217–234 (1905)
48. Prabhakar, T.R.: A singular integral equation with a generalized mittag leffler function in the kernel. Yokohama Math. J. **19**(1), 7–15 (1971)
49. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions, Related Topics and Applications, vol. 2. Springer, Berlin (2014)
50. Hille, E., Tamarldn, J.D.: On the theory of linear integral equations. Ann. Math. **31**, 479–528 (1930)
51. Hille, E.: Notes on linear transformations. II. Analyticity of semi-groups. Ann. Math. **40**(1), 1–47 (1939)