Chapter 7
Maintenance Models

7.1 Introduction

It has been commonly known that reliability of any system including telecommunication, computers, aircraft, power plants etc., can be improved by applying the redundancy or maintenance approaches. In general, the failure of such systems is usually costly, if not very dangerous. Maintenance, replacement and inspection problems have been extensively studied in the reliability, maintainability and warranty literature (Pham and Wang, 1996; Wang and Pham, 2006). Maintenance involves corrective (unplanned) and preventive (planned). Corrective maintenance (CM) occurs when the system fails. In other words, CM means all actions performed as a result of failure, to restore an item to a specified condition. Some researchers also refer to CM as repair. Preventive maintenance (PM) occurs when the system is operating. In other words, PM means all actions performed in an attempt to retain an item in specified condition from operation by providing systematic inspection, detection, adjustment, and prevention of failures. Maintenance also can be categorized according to the *degree* to which the operating conditions of an item are restored by maintenance as follows (Wang and Pham, 2006):

a. Replacement policy: A system with no repair is replaced before failure with a new one.
b. Preventive maintenance (pm) policy: A system with repair is maintained preventively before failure.
c. Inspection policy: A system is checked to detect its failure.
d. Perfect repair or perfect maintenance: a maintenance action which restores the system operating condition to 'as good as new', i.e., upon perfect maintenance, a system has the same lifetime distribution and failure rate function as a brand new one. Generally, replacement of a failed system by a new one is a perfect repair.
e. Minimal repair or minimal maintenance: a maintenance action which restores the system to the failure rate it had when it just failed. The operating state of the system under minimal repair is also called 'as bad as old' policy in the literature.

H. Pham, *Statistical Reliability Engineering*, Springer Series in Reliability Engineering,
https://doi.org/10.1007/978-3-030-76904-8_7

f. Imperfect repair or imperfect maintenance: a maintenance action may not make a system 'as good as new' but younger. Usually, it is assumed that imperfect maintenance restores the system operating state.

This chapter discusses a brief introduction in maintenance modeling with various maintenance policies including age replacement, block replacement and multiple failure degradation processes and random shocks. We also discuss the reliability and inspection maintenance modeling for degraded systems with competing failure processes.

Example 7.1 Suppose that a system is restored to "as good as new" periodically at intervals of time T. So the system renews itself at time T, 2 T, ... Define system reliability under preventive maintenance (PM) as.

$$R_M(t) = \Pr\{\text{failure has not occurred by time t}\} \tag{7.1}$$

In other words, the system survives to time t if and only if it survives every PM cycle {1,2,…,k} and a further time (t - kT).

(a) For a system with a failure time probability density function

$$f(t) = \lambda^2 t\, e^{-\lambda t} \quad \text{for } t > 0, \lambda > 0 \tag{7.2}$$

obtain the system reliability under PM and system mean time to first failure.

(b) Calculate the system mean time to first failure if PM is performed every 20 days and $\lambda = 0.005$.

Solution: The system survives to time t if and only if it survives every PM cycle {1,2,…,k} and a further time (t - kT). Thus,

$$R_M(t) = [R(T)]^k R(t - kT) \quad \text{for } kT < t < (k+1)T \text{ and } k = 0, 1, 2, \ldots \tag{7.3}$$

Interval containing t	Formula for $R_M(t)$
$0 < t < T$	$R(t)$
$T < t < 2T$	$R(T)R(t-T)$
$2T < t < 3T$	$[R(T)]^2 R(t-2T)$
...	
$kT < t < (k+1)T$	$[R(T)]^k R(t-kT)$

Thus, system reliability under PM is

$$R_M(t) = [R(T)]^k R(t - kT) \quad \text{for } kT < t < (k+1)T \quad \text{and } k = 0, 1, 2, \ldots \tag{7.4}$$

Given the probability density function as from Eq. (7.1),

$$f(t) = \lambda^2 t e^{-\lambda t} \quad \text{for } t > 0,\ \lambda > 0$$

then

$$R(t) = \int_t^\infty f(x)dx = \int_t^\infty \lambda^2 x\, e^{-\lambda x} dx = (1 + \lambda \text{t})e^{-\lambda t}. \tag{7.5}$$

Thus, from Eq. (7.4), we have

$$\begin{aligned} R_M(t) &= [R(T)]^k R(t - kT) \qquad \text{for } k = 0, 1, 2, \ldots \\ &= \left[(1 + \lambda T)e^{-\lambda T}\right]^k \left[(1 + \lambda(t - kT))e^{-\lambda(t-kT)}\right]. \end{aligned} \tag{7.6}$$

The system mean time to first failure (MTTFF) is

$$\begin{aligned} MTTFF &= \int_0^\infty R_M(t)dt \\ &= \int_0^T R(t)dt + \int_T^{2T} R(T)R(t - T)dt + \int_{2T}^{3T} [R(T)]^2 R(t - 2T)dt + \ldots \\ &= \sum_{k=0}^{\infty} [R(T)]^k \int_0^T R(u)du \\ &= \frac{\int_0^T R(u)du}{1 - R(T)}. \end{aligned}$$

Thus,

$$MTTFF = \frac{\int_0^T R(u)du}{1 - R(T)}. \tag{7.7}$$

Note that, from Eq. (7.5)

$$\begin{aligned} \int_0^T R(x)dx &= \int_0^T (1 + \lambda x)e^{-\lambda x} dx \\ &= \frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}. \end{aligned}$$

From Eq. (7.7), we have

$$MTTFF = \frac{\int_0^T R(u)du}{1 - R(T)} = \frac{\frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}}{1 - (1 + \lambda T)e^{-\lambda T}}.$$

(b) Here $T = 20$ days, and $\lambda = 0.005$ per day.

$$MTTFF = \frac{\frac{2}{0.005}\left(1 - e^{-(0.005)20}\right) - (20)e^{-(0.005)20}}{1 - (1 + (0.005)(20))e^{-(0.005)(20)}} = \frac{19.9683}{0.0047} = 4,267.8 \text{ days}.$$

7.2 Maintenance and Replacement Policies

A failed system is assumed to immediately replace or repair. There is a cost associated with it. One the one hand, designer may want to maintain a system before its failure. On the other hand, it is better not to maintain the system too often because the cost involved each time. Therefore it is important to determine when to perform the maintenance of the system that can minimize the expected total system cost.

Consider a one-unit system where a unit is replaced upon failure. Let.

c_1	the cost of each failed unit which is replaced
$c_2(<c_1)$	the cost of a planned replacement for each non-failed unit
$N_1(t)$	the number of failures with corrective replacements (CM)
$N_2(t)$	the number of replacements of non-failed units during $(0, t]$ interval.

In general, the expected total system cost during $(0,T]$, $E_c(T)$, can be defined as follows:

$$E_c[T] = c_1 \mathrm{E}[N_1(T)] + c_2 \mathrm{E}[N_2(T)]. \tag{7.8}$$

We now discuss the optimum policies which minimize the expected costs per unit time of each replacement policy such as age replacement and block replacement.

7.2.1 Age Replacement Policy

A unit is replaced at time T or at failure, whichever occurs first. T is also called a planned replacement policy. Let $\{X_k\}_{k=1}^{\infty}$ be the failure times of successive operating units with a density f and distribution F with finite mean μ. Let $Z_k \equiv \min\{X_k, T\}$ represents the intervals between the replacements caused by either failure or planned replacement for $k = 1,2,\ldots$ The probability of Z_k can be written as follows:

$$\Pr(Z_k \leq t) = \begin{cases} F(t) & t < T \\ 1 & t \geq T. \end{cases} \tag{7.9}$$

The mean time of one cycle is

$$\mathrm{E}(Z_k) = \int_0^T t dF(t) + TR(T) = \int_0^T R(t)dt. \tag{7.10}$$

The expected total system cost per cycle is

$$E_c(T) = c_1 F(T) + c_2 R(T) \tag{7.11}$$

where $R(\mathrm{T}) = 1 - F(\mathrm{T})$.

The expected total cost per unit time for an infinite time span, $C(T)$, is

$$C(T) = \frac{c_1 F(T) + c_2 R(T)}{\int_0^T R(t)dt}. \tag{7.12}$$

Let $r(t) \equiv f(t)/R(t)$ be the failure rate. We wish to find the optimal replacement policy time T^* which minimizes the expected total cost per unit time $C(T)$ in Eq. (7.12).

Theorem 7.1 *Given c_1, c_2, and μ. Assume the failure rate $r(t)$ is a strictly increasing function and $A = \frac{c_1}{\mu(c_1 - c_2)}$. The optimal replacement policy time T^* that minimizes the expected total system cost per unit time $C(T)$ can be obtained as follows:*

If $r(\infty) > A$ then there exists a finite value

$$T^* = G^{-1}\left(\frac{c_2}{c_1 - c_2}\right) \tag{7.13}$$

where $G(T) = r(T)\int_0^T R(t)dt - F(T)$ and the resulting expected total system cost per unit time C(T) is

$$C(T^*) = (c_1 - c_2) r(T^*).$$

(ii) If $r(\infty) \leq A$ then the optimum replacement time T is at: $T^* = \infty$. This implies that a unit should not be replaced unless it fails.

The above results can be obtained by differentiating the expected total cost function per unit time $C(T)$ from Eq. (7.12) with respect to T and setting it equal to 0. We have

$$\frac{\partial C(T)}{\partial T} = (c_1 - c_2)\left(r(T)\int_0^T R(t)dt - F(T)\right) - c_2 \equiv 0 \tag{7.14}$$

or, equivalently, $G(T) = \frac{c_2}{c_1 - c_2}$. Since $r(T)$ is strictly increasing and $G(0) = 0$, we can easily show that the function $G(T)$ is strictly increasing in T.

If $r(\infty) > A$ then $G(\infty) > \frac{c_2}{(c_1 - c_2)}$. This shows that there exists a finite value T^* where T^* is given in Eq. (7.13) and it minimizes $C(T)$.

If $r(\infty) \le A$ then $G(\infty) \le \frac{c_2}{(c_1 - c_2)}$. This shows that the optimum replacement time is $T^* = \infty$. This implies that a unit will not be replaced until it fails.

Example 7.2 Under an age replacement policy, the system is replaced at time T or at failure whichever occurs first. The costs of a failed unit and a planned replacement unit are respectively c_1 and c_2 with $c_1 \ge c_2$. For systems with a failure time probability density function.

$$f(t) = \lambda^2 t e^{-\lambda t} \quad \text{for } t > 0,\ \lambda > 0$$

obtain the optimal planned replacement time T^* that minimizes the expected total cost per cycle per unit time. Given $c_1 = 10$ and $c_2 = 1$, what is the optimal planned replacement time T^* that minimizes the expected total cost per cycle per unit time?

Solution: The pdf is.

$$f(t) = \lambda^2 t e^{-\lambda t} \quad \text{for } t > 0,\ \lambda > 0$$

The reliability function

$$R(t) = \int_t^\infty f(x)dx = \int_t^\infty \lambda^2 x e^{-\lambda x} dx = (1 + \lambda t)e^{-\lambda t}$$

The failure rate

$$r(t) = \frac{f(t)}{R(t)} = \frac{\lambda^2 t e^{-\lambda t}}{(1 + \lambda t)e^{-\lambda t}} = \frac{\lambda^2 t}{(1 + \lambda t)}. \tag{7.15}$$

From Eq. (7.12), the expected total cost per cycle per unit time is

$$E(T) = \frac{c_1 F(T) + c_2 R(T)}{\int_0^T R(t)dt}.$$

The derivative of the function $E(T)$ is given by

$$\frac{\partial E(T)}{\partial T}=\frac{\left[c_1 f(T)-c_2 f(T)\right]\int_0^T R(t)dt-[c_1F(T)+c_2R(T)]R(T)}{\left(\int_0^T R(t)dt\right)^2}.$$

Setting the above equation to 0 we obtain the following:

$$\frac{\partial E(T)}{\partial T}=0 \;\Leftrightarrow\; (c_1-c_2)f(T)\int_0^T R(t)dt-[c_1F(T)+c_2R(T)]R(T)\equiv 0.$$

$$(c_1-c_2)\frac{f(T)}{R(T)}\int_0^T R(t)dt=[c_1F(T)+c_2R(T)]$$

$$(c_1-c_2)r(T)\int_0^T R(t)dt-(c_1-c_2)F(T)=c_2$$

$$(c_1-c_2)\left\{r(T)\int_0^T R(t)dt-F(T)\right\}=c_2$$

or,

$$r(T)\int_0^T R(t)dt-F(T)=\frac{c_2}{(c_1-c_2)}. \tag{7.16}$$

Let

$$G(T)=r(T)\int_0^T R(t)dt-F(T)\quad\text{and}\quad \mathrm{A}_1=\frac{\mathrm{c}_2}{\mathrm{c}_1-c_2}. \tag{7.17}$$

That is, $G(T)=\mathrm{A}_1$. Since $r(T)$ is increasing and $G(0)=0$ we can show that $G(T)$ is increasing in T.

(a) If

$$r(\infty)>A\quad\text{where}\quad \mathrm{A}=\frac{\mathrm{c}_1}{\mu(\mathrm{c}_1-c_2)}\quad\text{and}\quad \mu=\int_0^\infty \mathrm{R(t)dt}. \tag{7.18}$$

then $G(\infty)>A_1$. There exists a finite value T^* where

$$T^* = G^{-1}\left(\frac{c_2}{(c_1 - c_2)}\right). \tag{7.19}$$

(b) If

$$r(\infty) \le A \quad \text{then} \quad G(\infty) \le \frac{c_2}{(c_1 - c_2)} \tag{7.20}$$

then the optimum replacement time is $T^* = \infty$. This implies that a unit will not be replaced until it fails.

We now calculate

$$\int_0^T R(t)dt = \int_0^T (1 + \lambda t)\, e^{-\lambda t} dt = \frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}. \tag{7.21}$$

Then

$$\begin{aligned} G(T) &= r(T) \int_0^T R(t)dt - F(T) \\ &= \frac{\lambda^2 T}{(1 + \lambda T)}\left[\frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}\right] - \left[1 - (1 + \lambda T)e^{-\lambda T}\right] \end{aligned} \tag{7.22}$$

Given $c_1 = 10$, $c_2 = 1$, and $\lambda = 2$, then

$$\frac{c_2}{c_1 - c_2} = \frac{1}{10 - 1} = \frac{1}{9}$$

From Eq. (7.22),

$$\begin{aligned} G(T) &= \frac{\lambda^2 T}{(1 + \lambda T)}\left[\frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}\right] - \left[1 - (1 + \lambda T)e^{-\lambda T}\right] \\ &= \frac{4T}{(1 + 2T)}\left[\frac{2}{2}\left(1 - e^{-2T}\right) - Te^{-2T}\right] - \left[1 - (1 + 2T)e^{-2T}\right] \\ &= \frac{4T}{(1 + 2T)}\left[1 - e^{-2T} - Te^{-2T}\right] - \left[1 - (1 + 2T)e^{-2T}\right]. \end{aligned}$$

Here we can find T^* such as

$$G(T^*) = \frac{1}{9} = 0.1111$$

The expected total cost per cycle per unit time, from Eq. (7.12), is:

$$E(T) = \frac{c_1 F(T) + c_2 R(T)}{\int_0^T R(t)dt}.$$
$$= \frac{c_1\left[1 - (1 + \lambda T)\, e^{-\lambda T}\right] + c_2\left[(1 + \lambda T)\, e^{-\lambda T}\right]}{\left[\frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) - Te^{-\lambda T}\right]}$$
$$= \frac{10\left[1 - (1 + 2T)\, e^{-2T}\right] + \left[(1 + 2T)\, e^{-2T}\right]}{\left[\left(1 - e^{-2T}\right) - Te^{-2T}\right]}$$
$$= \frac{10 - 9(1 + 2T)\, e^{-2T}}{1 - (1 + T)\, e^{-2T}}.$$

T	G(T)	E(T)
0.3	0.0930	7.3186
0.32	0.102	7.2939
0.33	0.1065	7.2883
0.335	0.1088	7.2870
0.34	0.1111	7.2865
0.35	0.1156	7.2881
0.5	0.1839	7.5375

Thus, the optimal planned replacement time T^* that minimizes the expected total cost per cycle per unit time is:

$$T^* = 0.34 \text{ and } E\left(T^*\right) = 7.2865.$$

7.2.2 Block Replacement

Consider that a unit begins to operate at time $t = 0$ and when it fails, it is discovered instantly and replaced immediately by a new one. Under this block policy, a unit is replaced at periodic times $kT(k = 1, 2, \cdots)$ independent of its age. Suppose that each unit has a failure time distribution $F(t)$ with finite mean μ. The expected total system cost per cycle $\mathrm{E_c}(T)$ is given by

$$E_c(T) = c_1 \mathrm{E}[N_1(T)] + c_2 \mathrm{E}[N_2(T)] = c_1 M(T) + c_2 \tag{7.23}$$

where $M(T) = \mathrm{E}(N_1(T))$ is differential and the expected number of failed units per cycle. The expected total system cost per unit time for an infinite time span under block replacement policy is defined as

$$C(T) = \frac{c_1 M(T) + c_2}{T}. \tag{7.24}$$

This indicates that there will be one planned replacement per period at a cost of c_2 and the expected number of failures with corrective replacement per period where each corrective replacement has a cost of c_1.

Theorem 7.2 Given c_1 and c_2. There exists a finite optimum planned replacement time T^* that minimizes the expected total system cost per unit time $C(T)$:

$$T^* = D^{-1}\left(\frac{c_2}{c_1}\right). \tag{7.25}$$

the resulting expected total system cost is $C(T^*)$ where $D(T) = Tm(T) - M(T)$ and $m(t) \equiv dM(t)/dt$.

Similarly from Theorem 7.1, we can obtain the optimum planned replacement time T^* given in Eq. (7.25) which minimizes the expected cost per unit time $C(T)$ by differentiating the function $C(T)$ with respect to T and setting it equal to zero, we obtain

$$Tm(T) - M(T) = \frac{c_2}{c_1}$$

where $m(t) \equiv dM(t)/dt$. The results can immediately follow.

7.2.3 Periodic Replacement Policy

For some systems, we only need to perform minimal repair at each failure, and make the planned replacement or preventive maintenance at periodic times.

Consider a periodic replacement policy as follows: A unit is replaced periodically at periodic times $kT \quad (k = 1, 2, \cdots)$. After each failure, only minimal repair is made so that the failure rate remains undisturbed by any repair of failures between successive replacements (Barlow and Proschan, 1965). This policy is commonly used with computers and airplanes. Specifically, a new unit begins to operate at $t = 0$, and when it fails, only minimal repair is made. That is, the failure rate of a unit remains undisturbed by repair of failures. Further, a unit is replaced at periodic times $kT(k = 1, 2, \cdots)$ independent of its age, and any units are as good as new after replacement. It is assumed that the repair and replacement times are negligible. Suppose that the failure times of each unit are independent, and have a cdf $F(t)$ and the failure rate $r(t) \equiv f(t)/R(t)$ where f is a probability density function density and $R(t)$ is the reliability function. The failures of a unit occur that follow a nonhomogeneous Poisson process with a mean-value function H(t) where $H(t) \equiv \int_0^t r(u)du$ and $R(t) = e^{-H(t)}$.

Consider one cycle with a constant time T from the planned replacement to the next one. Then, since the expected number of failures during one cycle is $E(N_1(T)) = M(T)$, the expected total system cost per cycle is

$$E_c(T) = c_1 E(N_1(T)) + c_2 E(N_2(T)) = c_1 H(T) + c_2, \tag{7.26}$$

where c_1 is the cost of each minimal repair.

Therefore the expected total system cost per unit of time for an infinite time span is

$$C(T) \equiv \frac{1}{T}[c_1 H(T) + c_2]. \tag{7.27}$$

If a unit is not replaced forever, *i.e.*, $T = \infty$, then $\lim_{T\to\infty} R(T)/T = r(\infty)$, which may be possibly infinite, and $C(\infty) = c_1 r(\infty)$.

Given c_1 and c_2. There exists a finite optimum replacement time T^* such that

$$Tr(T) - H(T) = \frac{c_2}{c_1} \tag{7.28}$$

that minimizes the expected total system cost per unit time $C(T)$ as given in Eq. (7.27).

Differentiating the function $C(T)$ in Eq. (7.27) with respect to T and setting it equal to zero, we have.

$$Tr(T) - H(T) = \frac{c_2}{c_1}$$

If the cost of minimal repair depends on the age x of a unit and is given by $c_1(x)$, the expected total system cost per unit time can be defined as

$$C(T) = \frac{1}{T}[\int_0^T c_1(x) r(x) dx + c_2]. \tag{7.29}$$

One can obtain the optimum replacement policy T that minimizes the expected total system cost $C(T)$ by taking a derivative of the function $C(T)$ with respect to T given the function $c_1(x)$.

7.2.4 Replacement Models with Two Types of Units

In practice, many systems are consisted of vital and non-vital parts or essential and non-essential components. If vital parts fail then a system becomes dangerous or suffers a high cost. It would be wise to make the planned replacement or overhaul at suitable times. We may classify into two types of failures; partial and total failures, slight and serious failures, or simply faults and failures.

Consider a system consists of unit 1 and unit 2 which operate independently, where unit 1 corresponds to non-vital parts and unit 2 to vital parts. It is assumed that unit 1 is replaced always together with unit 2. Unit i has a failure time distribution $F_i(t)$, failure rate $r_i(t)$ and cumulative hazard $H_i(t)(i = 1, 2)$, *i.e.*, $R_i(t) = \exp[-H_i(t)]$ and $H_i(t) = \int_0^t r_i(u)du$. Then, we consider the following four replacement policies which combine age, block and periodic replacements:

(a) Unit 2 is replaced at failure or time T, whichever occurs first, and when unit 1 fails between replacements, it is replaced by a new unit. Then, the expected total system cost per unit time is

$$C(T) = \frac{c_1 \int_0^T f_1(t)R_2(t)dt + c_2F_2(T) + c_3}{\int_0^T R_2(t)dt} \tag{7.30}$$

where c_1 is a cost of replacement for a failed unit 1, c_2 is an additional replacement for a failed unit 2, and c_3 is a cost of replacement for units 1 and 2.

(b) In case (a), when unit 1 fails between replacements, it undergoes only minimal repair. Then, the expected total system cost per unit time is

$$C(T) = \frac{c_1 \int_0^T r_1(t)R_2(t)dt + c_2F_2(T) + c_3}{\int_0^T R_2(t)dt}, \tag{7.31}$$

where c_1 is a cost of minimal repair for failed unit 1, and c_2 and c_3 are the same costs as case (a).

(c) Unit 2 is replaced at periodic times $kT(k = 1, 2, \cdots)$ and undergoes only minimal repair at failures between planned replacements, and when unit 1 fails between replacements, it is replaced by a new unit. Then, the expected total system cost per unit time is

$$C(T) = \frac{1}{T}[c_1H_1(T) + c_2H_2(T) + c_3], \tag{7.32}$$

where c_2 is a cost of minimal repair for a failed unit 2, and c_1 and c_3 are the same costs as case (a).

(d) In case (c), when unit 1 fails between replacements, it also undergoes minimal repair. Then, the expected total system cost per unit time is

$$C(T) = \frac{1}{T}[c_1H_1(T) + c_2H_2(T) + c_3], \tag{7.33}$$

where c_1 is a cost of minimal repair for a failed unit 1, and c_2 and c_3 are the same costs as case (c).

7.3 Non-repairable Degraded System Modeling

Maintenance has evolved from simple model that deals with machinery breakdowns, to time-based preventive maintenance, to today's condition-based maintenance. It is of great importance to avoid the failure of a system during its actual operating; especially, when such failure is dangerous and costly. This section discusses a reliability model and examines the problem of developing maintenance cost models for determining the optimal maintenance policies of non-repairable degraded systems with competing failure processes. The material in this section are based on Li and Pham (2005a).

Notation

C_c	Cost per CM action
C_p	Cost per PM action
C_m	Loss per unit idle time
C_i	Cost per inspection
$Y(t)$	Degradation process
$Y_i(t)$	Degradation process i, i = 1, 2
$D(t)$	Cumulative shock damage value up to time t
S	Critical value for shock damage
$C(t)$	Cumulative maintenance cost up to time t.
$E[C_1]$	Average total maintenance cost during a cycle
$E[W_1]$	Mean cycle length
$E[N_1]$	Mean number of inspections during a cycle
$E[\xi]$	Mean idle time during a cycle
$\{I_i\}_{i\in N}$	Inspection sequence
$\{U_i\}_{i\in N}$	Inter-inspection sequence
P_{i+1}	Probability that there are a total of $(i + 1)$ inspections in a renewal cycle
P_p	Probability that a renewal cycle ends by a PM action
P_c	Probability that a renewal cycle ends by a CM action ($P_c = 1- P_p$)

Consider that:

- The system has the state space $\Omega_U = \{M, \ldots, 1, 0, F\}$ and it starts at state M at time $t = 0$;
- System fails either due to degradation ($Y(t) > G$) or catastrophic failure $\left(D(t) = \sum_{i=1}^{N_2(t)} X_i > S\right)$. System may either goes from state i to the next degraded state i-1 or directly goes to catastrophic failure state F, $i = M,.0.1;$
- No repair or maintenance is performed on the system; and
- The two processes $Y(t)$ and $D(t)$ are independent.

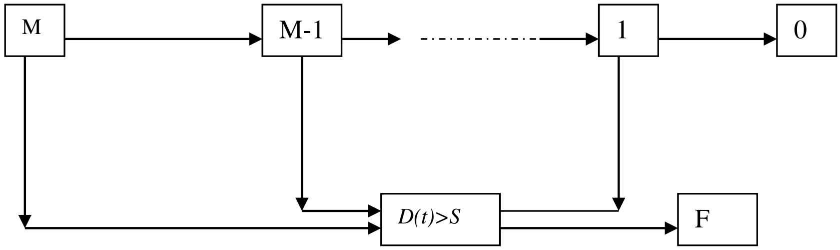

Fig. 7.1 Flow diagram of the system with two competing failure processes (Li and Pham, 2005a, b)

Figure 7.1 illustrates the case where systems are subject to two failure competing processes: degradation process *Y(t)* and the random shocks process *D(t)* and whichever process occurred first would cause the system to failure.

Suppose that the operating condition of the system at any time point could be classified into one of a finite number of the states, say $\Omega_U = \{M, \ldots, 1, 0, F\}$. A one-to-one relationship between the element of $\Omega = \{M, \ldots, 1, 0\}$ and its corresponding interval is defined as follows:

$$\begin{array}{ll} \text{State M if} & Y(t) \in [0, W_M] \\ \text{State M} - 1 \text{ if} & Y(t) \in (W_M, W_{M-1}] \\ \vdots & \\ \text{State i} & Y(t) \in (W_{i+1}, W_i] \\ \text{State 1} & Y(t) \in (W_2, W_1] \\ \text{State 0} & Y(t) > W_1 \end{array}$$

Let $P_i(t)$ be a probability that the value of *Y(t)* will fall within a pre-defined interval corresponding to state *i* and *D(t)* $\leq$ *S*. From state *i*, the system will make a direct transition to state (*i-1*) due to gradual degradation or to state *F* due to a random shock (Fig. 7.1). The reliability function is defined as:

$$R_M(t) = \sum_{i=1}^{M} P_i(t) = P(Y(t) \leq G, D(t) \leq S) \tag{7.34}$$

where $P_i(t)$ is the probability of being in state *i*. Let *T* be the time to failure of the system. Then *T* can be defined as: $T = \inf\{t > 0 : Y(t) > G \text{ or } D(t) > S\}$. The mean time to failure is given by:

$$\mathrm{E[T]} = \int_0^{\infty} P(Y(t) \leq G, D(t) \leq S)dt$$

$$= \int_0^{\infty} P(Y(t) \le G) \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j e^{-\lambda_2 t}}{j!} F_X^{(j)}(S)dt \tag{7.35}$$

or, equivalently,

$$E[T] = \sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \int_0^{\infty} P(Y(t) \le G)(\lambda_2 t)^j e^{-\lambda_2 t} dt \tag{7.36}$$

Let $F_G(t) = P\{Y(t) \le G\}$, then $f_G(t) = \frac{d}{dt} F_G(t)$. The pdf of the time to failure, $f_T(t)$ can be easily obtained:

$$\begin{aligned} f_T(t) &= -\frac{d}{dt}[P(Y(t) \le G)P(D(t) \le S)] \\ &= -\sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \frac{d}{dt}\left[P(Y(t) \le G)(\lambda_2 t)^j e^{-\lambda_2 t}\right] \end{aligned}$$

After simplifications, we have

$$\begin{aligned} f_T(t) = -\sum_{j=1}^{\infty} \frac{F_X^{(j)}(S)}{j!} \\ \left[f_G(t)(\lambda_2 t)^j e^{-\lambda_2 t} + F_G(t) j \lambda_2 (\lambda_2 t)^{j-1} e^{-\lambda_2 t} - \lambda_2 F_G(t)(\lambda_2 t)^j e^{-\lambda_2 t}\right] \end{aligned} \tag{7.37}$$

Assume that the degradation process is described as the function $Y(t) = W \frac{e^{Bt}}{A+e^{Bt}}$ where the two random variables A and B are independent, and that A follows a uniform distribution with parameter interval [0,a] and B follows exponential distribution with parameter $\beta > 0$. In short, $A \sim U[0, a]$, $a > 0$ and $B \sim Exp(\beta)$, $\beta > 0$.

The probability for the system of being in state M is as follows:

$$\begin{aligned} P_M(t) &= P(Y(t) \le W_M, D(t) \le S) \\ &= \left\{ \int_{\forall A} P\left(B < \frac{1}{t} \ln \frac{u_1 A}{1-u_1} | A = x\right) f_A(x) dx \right\} P(D(t) \le S) \\ &= \left\{ 1 - \frac{1}{a} \left(\frac{1-u_1}{u_1}\right)^{\frac{\beta}{t}} \left(\frac{t}{t-\beta}\right) \left(a^{1-\frac{\beta}{t}} - 1\right) \right\} e^{-\lambda_2 t} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S) \end{aligned} \tag{7.38}$$

Then the probability for the system of being in state i can be calculated as follows:

$$P_i(t) = P(W_{i+1} < W \frac{e^{Bt}}{A + e^{Bt}} \le W_{i,} D(t) \le S)$$

$$= \left\{ \int_0^a P\left(\frac{1}{t} \ln \frac{u_{i-1}A}{1-u_{i-1}} < B \le \frac{1}{t} \ln \frac{u_i A}{1-u_i} | A = x \right) f_A(x) dx \right\} e^{-\lambda_2 t} \sum_{j=1}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S)$$

$$= \left\{ \frac{1}{a} \left(\frac{t}{t-\beta} \right) \left(a^{1-\frac{\beta}{t}} \right) \left[\left(\frac{1-u_i}{u_i} \right)^{\frac{\beta}{t}} - \left(\frac{1-u_{i-1}}{u_{i-1}} \right)^{\frac{\beta}{t}} \right] \right\} e^{-\lambda_2 t} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S) \tag{7.39}$$

where $\mu_i = \frac{W_i}{W}$, $i = M\text{-}1,..,1$.

Similarly, the probability for the system of being in state *0* is as follows:

$$P_0(t) = P\left(Y(t) = W \frac{e^{Bt}}{A + e^{Bt}} > G, D(t) \le S\right)$$
$$= \left\{ \frac{1}{a} \left(\frac{1-u_M}{u_M} \right)^{\frac{\beta}{t}} \left(\frac{t}{t-\beta} \right) \left(a^{1-\frac{\beta}{t}} \right) \right\} e^{-\lambda_2 t} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S)$$

The probability for a catastrophic failure state F is given by:

$$P_F(t) = P\left(Y(t) = W \frac{e^{Bt}}{A + e^{Bt}} \le G, D(t) > S\right)$$
$$= \left\{ 1 - \frac{1}{a} \left(\frac{1-u_1}{u_1} \right)^{\frac{\beta}{t}} \left(\frac{t}{t-\beta} \right) \left(a^{1-\frac{\beta}{t}} \right) \right\} \left\{ 1 - e^{-\lambda_2 t} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S) \right\}$$

Hence, the reliability $R_M(t)$ is given by:

$$R_M(t) = \sum_{i=1}^{M} P_i(t)$$
$$= \left\{ 1 - \frac{1}{a} \left(\frac{1-u_M}{u_M a} \right)^{\frac{\beta}{t}} \left(\frac{t}{t-\beta} \right) \left(a^{1-\frac{\beta}{t}} \right) \right\} \left\{ e^{-\lambda_2 t} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j}{j!} F_X^{(j)}(S) \right\} \tag{7.40}$$

Example 7.3 Assume.

$Y(t) = W \frac{e^{Bt}}{A+e^{Bt}}$ where $A \sim U[0,5]$ and $B \sim Exp(10)$; and critical values for the degradation and the shock damage are: G = 500 and $S = 200$, respectively. The random shocks function:$D(t) = \sum_{i=1}^{N_2(t)} X_i$ where $X_i \sim Exp(0.3)$ and $X_i' s$ are i.i.d. Figure 7.2 shows the reliability of the system using Eq. (7.40) for $\lambda_2 = .12$ and $\lambda_2 = .20$.

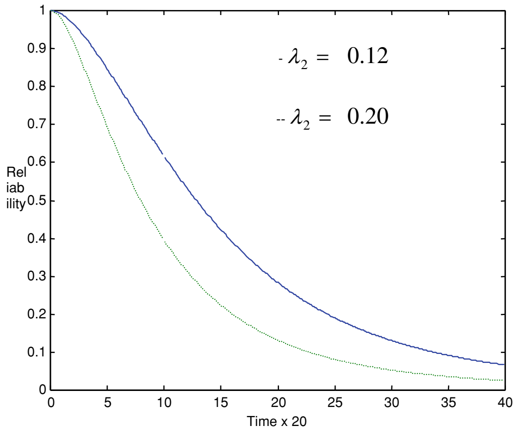

Fig. 7.2 Reliability $R_M(t)$ versus time t (Li and Pham, 2005a, b)

7.4 Inspection-Maintenance Repairable Degraded System Modeling

The system is assumed to be periodically inspected at times $\{I, 2I, \ldots, nI, \ldots\}$ and that the state of the system can only be detected by inspection. After a PM or CM action the system will store it back to as-good-as-new state. Assuming that the degradation $\{Y(t)\}_{t\geq 0}$and random shock $\{D(t)\}_{t\geq 0}$ are independent, and a CM action is more costly than a PM and a PM costs much more than an inspection. In other words, $C_c > C_p > C_i$.

From Sect. 7.3, T is defined as the time-to-failure $T = \inf\{t > 0 : Y(t) > G \text{ or } D(t) > S\}$ where G is the critical value for $\{Y(t)\}_{t\geq 0}$ and S is the threshold level for $\{D(t)\}_{t\geq 0}$ (Li and Pham, 2005a, b).

The two threshold values L and G (G is fixed) effectively divide the system state into three zones as shown in Fig. 7.3. They are: Doing nothing zone when $Y(t) \leq L$ and $D(t) \leq S$; PM zone when $L < Y(t) \leq G$ and $D(t) \leq S$; and CM zone $Y(t) > G$ or $D(t) > S$. The maintenance action will be performed when either of the following situations occurs:

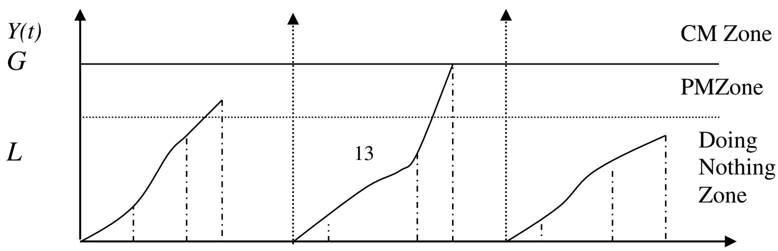

Fig. 7.3 The evolution of the system (Li and Pham, 2005a, b)

The current inspection reveals that the system condition falls into PM zone, however this state is not found at previous inspection. At the inspection time iI, the system falls into PM zone which means $\{Y((i-1)I) \leq L, D((i-1)I) \leq S\}$ $\cap\{L < Y(iI) \leq G, D(iI) \leq S\}$. Then PM action is performed and it will take a random time R_1. When the system fails at T, a CM action is taken immediately and would take a random of time R_2. Note that after a PM or a CM action is performed, the system is renewed and end of the cycle.

The average long-run maintenance cost per unit time can be defined as follows:

$$EC(L, I) = \frac{E[C_1]}{E[W_1]}. \tag{7.41}$$

The expected total maintenance cost during a cycle $E[C_1]$ is defined as:

$$E[C_1] = C_i E[N_I] + C_p E[R_1] P_p + C_c E[R_2] P_c \tag{7.42}$$

Note that there is a probability P_p that the cycle will end by a PM action and it will take on the average $E[R_1]$ amount of times to complete a PM action with a corresponding cost $C_p E[R_1] P_p$. Similarly, if a cycle ends by a CM action with probability P_c, it will take on the average $E[R_2]$ amount of times to complete a CM action with corresponding cost $C_c E[R_2] P_c$. We next discuss the analytical analysis of $E[C_1]$.

Calculate $E[N_I]$.

Let $E[N_I]$ denote the expected number of inspections during a cycle. Then

$$E[N_I] = \sum_{i=1}^{\infty} (i) P\{N_I = i\}$$

where $P\{N_I = i\}$ is the probability that there are a total of i inspections occurred in a renewal cycle. It can be shown that

$$\begin{aligned} P(N_I = i) = {} & P(Y((i-1)I) \leq L, D((i-1)I) \leq S)\, P(L < Y(iI) \leq G, D(iI) \leq S) \\ & + P\{Y(iI) \leq L, D(iI) \leq S\} P\{iI < T \leq (i+1)I) \end{aligned} \tag{7.43}$$

Hence,

$$\begin{aligned} E[N_I] = \sum_{i=1}^{\infty} i (& P(Y((i-1)I) \leq L, D((i-1)I) \leq S) P(L < Y(iI) \leq G, D(iI) \leq S) \\ & + P(Y(iI) \leq L, D(iI) \leq S) P(iI < T \leq (i+1)I)) \end{aligned} \tag{7.44}$$

Assume $Y(t) = A + Bg(t)$ where $A \sim N(\mu_A, \sigma_A^2), B \sim N(\mu_B, \sigma_B^2)$, and A and B are independent. We now calculate the probabilities $P(Y((i-1)I) \leq L, D((i-1)I) \leq S)$ and $P(L < Y(iI) \leq G, D(iI) \leq S)$. Given $g(t) = t$.

$D(t) = \sum_{i=0}^{N(t)} X_i$ where $X_i's$ are i.i.d. and $N(t) \sim Possion(\lambda)$.

Then

$$P(Y((i-1)I) \leq L, D((i-1)I) \leq S)$$

$$= P(A + B(i-1)I \leq L)P(D((i-1)I) = \sum^{N(i-1)I)} X_i \leq S)$$

$$= \Phi\left(\frac{L - (\mu_A + \mu_B(i-1)I)}{\sqrt{\sigma_A^2 + \sigma_B^2((i-1)I)^2}}\right) e^{-\lambda(i-1)I} \sum_{j=0}^{\infty} \frac{(\lambda(i-1)I)^j}{j!} F_X^{(j)}(S)$$

and

$$P(L < Y(iI) \leq G, D(iI) \leq S)$$

$$= \left\{\Phi\left(\frac{G - (\mu_A + \mu_B iI)}{\sqrt{\sigma_A^2 + \sigma_B^2(iI)^2}}\right) - \Phi\left(\frac{L - (\mu_A + \mu_B iI)}{\sqrt{\sigma_A^2 + \sigma_B^2(iI)^2}}\right)\right\} e^{-\lambda iI} \sum_{j=0}^{\infty} \frac{(\lambda iI)^j}{j!} F_X^{(j)}(S)$$

Since T is $T = \inf\{t > 0 : Y(t) > G \text{ or } D(t) > S\}$, we have:

$$\begin{aligned} P(iI < T \leq (i+1)I) = P(Y(iI) \leq L, Y((i+1)I) > G)P(D((i+1)I) \leq S) \\ + P(Y((i+1)I) \leq L)\, P(D(iI) \leq S, D((i+1)I) > S) \end{aligned} \tag{7.45}$$

In Eq. (7.45), since $Y(iI)$ and $Y((i+1)I)$ are not independent, we need to obtain the joint p.d.f $f_{Y(iI),Y((i+1)I)}(y_1, y_2)$ in order to compute $P(Y(iI) \leq L, Y((i+1)I) > G)$.

Assume that $Y(t) = A + Bg(t)$ where $A > 0$ and $B > 0$ are two independent random variables, $g(t)$ is an increasing function of time t and $A \sim f_A(a)$,$B \sim f_B(b)$. Let

$$\begin{cases} y_1 = a + bg(iI) \\ y_2 = a + bg((i+1)I) \end{cases} \tag{7.46}$$

After simultaneously solving the above equations in terms of y_1 and y_2, we obtain:

$$a = \frac{y_1 g((i+1)I) - y_2 g(iI)}{g((i+1)I) - g(iI)} = h_1(y_1, y_2)$$

$$b = \frac{y_2 - y_1}{g((i+1)I) - g(iI)} = h_2(y_1, y_2)$$

Then the random vector $(Y(iI), Y((i+1)I))$ has a joint continuous p.d.f as follows

$$f_{Y(iI),Y((i+1)I)}(y_1, y_2) = |J|f_A(h_1(y_1, y_2))f_B(h_2(y_1, y_2)) \tag{7.47}$$

where the *Jacobian* J is given by

$$J = \begin{vmatrix} \dfrac{\partial h_1}{\partial y_1} & \dfrac{\partial h_1}{\partial y_2} \\ \dfrac{\partial h_2}{\partial y_1} & \dfrac{\partial h_2}{\partial y_2} \end{vmatrix} = \left| \frac{1}{g(iI) - g((i+1)I)} \right|. \tag{7.48}$$

Note that $D(iI)$ and $D(I_{i+1})$ are independent (Li and Pham, 2005a, b), therefore,

$$P(D(iI) \le S, D((i+1)I) > S) = P(D(iI) \le S)P(D((i+1)I) > S). \tag{7.49}$$

Calculate P_p

Note that either a PM or CM action will end a renewal cycle. In other words, PM and CM these two events are mutually exclusive at renewal time point. As a consequence, $P_p + P_c = 1$. The probability P_p can be obtained as follows:

$$\begin{aligned} P_p &= P(\text{PM ending a cycle}) \\ &= \sum_{i=1}^{\infty} P(Y(i-1)I) \le L, L < Y(iI) \le G)P(D(iI) \le S) \end{aligned} \tag{7.50}$$

Expected Cycle Length Analysis

Since the renewal cycle ends either by a PM action with probability P_p or a CM action with probability P_c, the mean cycle length $E[W_1]$ is calculated as follows:

$$\begin{aligned} E[W_1] &= \sum_{i=1}^{\infty} E[(iI + R_1)I_{PM \text{ occur in } ((i-1)I, iI]}] + E[(T + R_2)1_{CM \text{ occur}}] \\ &= \left\{ \sum_{i=1}^{\infty} iIP(Y((i-1)I) \le L, D((i-1)I) \le S) \right. \\ &\quad P(L < Y(iI) \le G, D(iI) \le S)\} + E[R_1]P_p \\ &\quad + (E[T] + E[R_2])P_c \end{aligned} \tag{7.51}$$

where $I_{PM \text{ occurs in}((i-1)I,iI]}$ and $I_{CM \text{ occurs}}$ are the indicator functions. The mean time to failure, $E[T]$ is given by:

$$
\begin{aligned}
E[T] &= \int_0^{\infty} P\{T > t\}dt \\
&= \int_0^{\infty} P\{Y(t) \le G, D(t) \le S\}dt \\
&= \int_0^{\infty} P\{Y(t) \le G\} \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j e^{-\lambda_2 t}}{j!} F_X^{(j)}(S)dt
\end{aligned} \tag{7.52}
$$

or, equivalently, that

$$
E[T] = \sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \int_0^{\infty} P\{Y(t) \le G\}(\lambda_2 t)^j e^{-\lambda_2 t}dt \tag{7.53}
$$

The expression $E[T]$ would depend on the probability $P\{Y(t) \le G\}$ and sometimes it cannot easy obtain a closed-form.

Optimization Maintenance Cost Rate Policy

We determine the optimal inspection time I and PM threshold L such that the long-run average maintenance cost rate $EC(L, I)$ is minimized. In other words, we wish to minimize the following objective function (Li and Pham, 2005a):

$$
\begin{aligned}
&EC(L, I) \\
&= \frac{\sum_{i=1}^{\infty} iP\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\}}{\left\{\sum_{i=1}^{\infty} I_i P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\right\} + E[R_1]P_p + E[R_2]P_c} \\
&+ \frac{\sum_{i=1}^{\infty} iV_i\{P\{Y(I_i) \le L, Y(I_{i+1}) > G\}P\{D(I_{i+1}) \le S\} + P\{Y(I_{i+1}) \le L\}P\{D(I_i) \le S, D(I_{i+1}) > S\}\}}{\left\{\sum_{i=1}^{\infty} I_i P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\right\} + E[R_1]P_p + E[R_2]P_c} \\
&+ \frac{C_p E[R_1] \sum_{i=1}^{\infty} P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\}}{\left\{\sum_{i=1}^{\infty} I_i P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\right\} + E[R_1]P_p + E[R_2]P_c} \\
&+ \frac{C_c E[R_2]\left\{1 - \sum_{i=1}^{\infty} P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\}\right\}}{\left\{\sum_{i=1}^{\infty} I_i P\{Y(I_{i-1}) \le L, D(I_{i-1}) \le S\}P\{L < Y(I_i) \le G, D(I_i) \le S\right\} + E[R_1]P_p + E[R_2]P_c}
\end{aligned} \tag{7.54}
$$

where $I_{i-1} = (i-1)I, I_i = iI, I_{i+1} = (i+1)I$ and $V_i = P\{Y(iI) \le L, D(iI) \le S\}$.

The above complex objective function is a nonlinear optimization problem. Li and Pham (2005a, b) discussed a step-by-step algorithm based on the Nelder-Mead downhill simplex method.

Example 7.4 Assume that the degradation process is described by $Y(t) = A + Bg(t)$ where A and B are independent, A~U(0, 4), B~Exp(−0.3t), respectively, and

$g(t) = \sqrt{t}e^{0.005t}$. Assume that the random shock damage is described by $D(t) = \sum_{i=1}^{N(t)} X_i$ where X_i follows the exponential distribution, i.e., X_i~$Exp(-0.04t)$ and N(t)~$Poisson(0.1)$. Given $G = 50$, $S = 100$, $C_i = 900$/inspection, $C_c = 5600/CM$, $C_p = 3000/PM$, R_1~Exp$(-.1t)$, and R_2~Exp$(-.04t)$. We now determine both the values of I and L that minimizes the average total system cost per unit time $EC(I,L)$.

From Eq. (7.54), the optimal values are $I^* = 37.5, L^* = 38$ and the corresponding cost value is $EC^*(I, L) = 440.7$. See Li and Pham (2005a, b) for the detailed calculations.

7.5 Warranty Concepts

A warranty is a contract under which the manufacturers of a product and/or service agree to repair, replace, or provide service when a product fails or the service does not meet intended requirements (Park and Pham, 2010a, 2010b, 2012a, 2012b, 2012c, 2016). These agreements exist because of the uncertainty present in the delivery of products or services, especially in a competitive environment. Warranties are important factors in both the consumers and manufacturers' decision making. A warranty can be the deciding factor on which item a consumer chooses to purchase when different products have similar functions and prices. The length and type of warranty is often thought of as a reflection of the reliability of a product as well as the company's reputation.

Warranty types are dependent on the kind of product that it protects. For larger or more expensive products with many components, it may be cheaper to repair the product rather than replacing it. These items are called repairable products. Other warranties simply replace an entire product because the cost to repair it is either close to or exceeds its original price. These products are considered non-repairable. The following are the most common types used in warranties:

Ordinary Free Replacement - Under this policy, when an item fails before a warranty expires it is replaced at no cost to the consumer. The new item is then covered for the remainder of the warranty length. This is the most common type of a warranty and often applies to cars and kitchen appliances.

Unlimited Free Replacement - This policy is the same as the ordinary free replacement policy but each replacement item carries a new identical warranty. This type of warranty is often used for electronic appliances with high early failure rates and usually has a shorter length because of it.

Pro-rata Warranty - The third most common policy takes into account how much an item is used. If the product fails before the end of the warranty length, then it is replaced at a cost that is discounted proportional to its use. Items that experience wear or aging, such as tires, are often covered under these warranties.

Different warranty models may include a combination of these three types as well as offering other incentives such as rebates, maintenance, or other services that can satisfy a customer and extend the life of their product. Bai and Pham (2006a,2006b, 2004, 2005), Wang and Pham (2006), Pham (2003), Wang and Pham (2010, 2011, 2012), and Murthy and Blischke (2006) can be served as good references of papers and books for further studies on maintenance and warranty topics.

7.6 Problems

1. Suppose that a system is restored to "as good as new" periodically at intervals of time T. So the system renews itself at time T, 2 T, ... Define system reliability under preventive maintenance (PM) as

$$R_M(t) = \Pr\{\text{failure has not occurred by time t}\}$$

In other words, the system survives to time t if and only if it survives every PM cycle {1,2,...,k} and a further time (t - kT).

(a) For a system with a failure time probability density function

$$f(t) = \frac{1}{\beta^2} t e^{-\frac{t}{\beta}} \quad \text{for } t > 0,\ \beta > 0$$

obtain the system reliability under PM and system mean time to first failure.

(b) Calculate the system mean time to first failure if PM is performed every 25 days and $\beta = 2$.

2. Suppose that a system is restored to "as good as new" periodically at intervals of time T. So the system renews itself at time T, 2 T, ... Define system reliability under preventive maintenance (PM) as

$$R_M(t) = \Pr\{\text{failure has not occurred by time t}\}$$

In other words, the system survives to time t if and only if it survives every PM cycle {1,2,...,k} and a further time (t - kT).

(a) For a system with a failure time probability density function

$$f(t) = \frac{1}{6}\lambda^3 t^2 e^{-\lambda t} \quad \text{for } t > 0,\ \lambda > 0$$

obtain the system reliability under PM and system mean time to first failure.

(b) Calculate the system mean time to first failure if PM is performed every 50 days and $\lambda = 0.035$. (Hints: See Example 7.1)

3. Under an age replacement policy, the system is replaced at time T or at failure whichever occurs first. The costs are respectively c_p and c_f with $c_f > c_p$. For systems with a failure time probability density function

$$f(t) = \frac{1}{6}\lambda^3 t^2 e^{-\lambda t} \quad \text{for } t > 0,\ \lambda > 0$$

obtain the optimal planned replacement time T^* that minimizes the expected total cost per cycle per unit time. Given $c_p = 5$ and $c_f = 25$, what is the optimal planned replacement time T^* that minimizes the expected total cost per cycle per unit time?

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