The Onto-semiotic Approach in Mathematics Education. Analysing Objects and Meanings in Mathematical Practice

Juan D. Godino, María Burgos, and María M. Gea

1 Introduction

In this seminar a synthesis of the Onto-semiotic Approach (OSA) theoretical system to mathematical knowledge and instruction was presented. We highlighted the problems, principles and research methods that are addressed in this approach and considering the didactics of mathematics as a scientific and technological discipline. In the first part of the seminar we developed the reply to the question posed by Gascón and Nicolás (2017) about the prescriptive nature of didactics of mathematics research from the OSA perspective. This theoretical framework suggests that Didactics should address the epistemological, ontological, semiotic-cognitive, educational-instructional, ecological, and instruction optimization problems (Godino et al., 2019). OSA assumes anthropological, pragmatic and semiotic principles to approach all these types of problems, as well as it embraces sociocultural principles to face the educational-instructional problem.

1.1 Didactics as Science and as Technology

The OSA framework attributes both a scientific and technological character to the knowledge produced by didactic research. On the one hand, it addresses theoretical problems related to the ontological, epistemological and semiotic nature of mathematical knowledge, as far as such problems are related to the teaching and learning processes (the scientific, descriptive, explanatory or predictive component). On the other hand, didactics should intervene in these processes to improve them as much as

J. D. Godino (✉) · M. Burgos · M. M. Gea
Universidad de Granada, Granada, Spain
e-mail: mariaburgos@ugr.es; mmgea@ugr.es

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possible (the technological - prescriptive component). While description, explanation and prediction are the main goals of scientific activity, prescription and assessment are the main goals of technological enterprise; however, technological action also includes elements of applied research when solving specific problems. The notion of *didactic suitability* has been introduced as a systemic criterion to address the problem of optimization of mathematical instruction processes.

2 Onto-semiotic Approach: A Modular and Inclusive Theoretical Framework for Mathematics Education

The Onto-semiotic Approach is a modular and inclusive theoretical system for research in mathematics education that provides specific principles and methods to address the:

1. *Epistemological problems:* How does mathematics emerge and develop?
2. *Ontological problems:* What is a mathematical object? What types of objects intervene in mathematical activity?
3. *Semiotic-cognitive problems:* What is knowing a mathematical object? What is the meaning of a mathematical object for a subject given a time and circumstances?
4. *Educational-instructional problems:* What is teaching? What is learning? How do they relate? What types of interactions between people, knowledge and resources are required in the instructional processes to optimize learning?
5. *Ecological problems:* What factors condition and support the development of instructional processes and what norms regulate them?
6. *Instruction optimization problems:* What kind of actions and resources should be implemented in the instructional processes to optimize students' mathematical learning?
7. *Teachers' education problems*: What knowledge and skills should teachers have to manage the teaching and learning processes of mathematics?

These problems, the assumed principles and methods developed to address them are described in Godino et al. (2019). Likewise, a model of teacher's Didactic-Mathematical Knowledge and Competencies based on the OSA (Godino et al., 2017) has been developed. This model considers essential that teachers be trained for the analysis of objects and meanings that intervenes in mathematical practices (onto-semiotic analysis), together with the competences for the analysis of didactical configurations, normative analysis and didactical suitability (Fig. 1).

In the following section, we describe the objectives, methodology and foundations of a workshop for developing the general competence of analysis and didactical intervention.

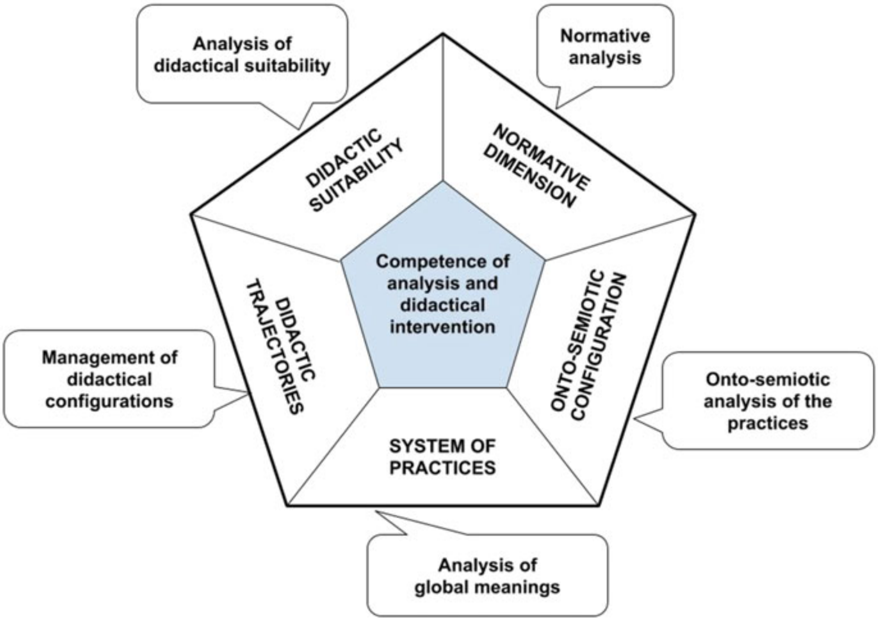

Fig. 1 Components of the analysis and didactic intervention competence (Godino et al., 2017, p. 103)

3 Objects and Meanings Analysis of Mathematical Practices

The notion of *meaning*, frequently used both in research and educational practice, is a central and controversial issue in philosophy, logic, semiotics and other sciences and technologies interested in human cognition. Given the importance of symbolization, communication and understanding processes in mathematics teaching and learning, the question of meaning should occupy a central place in teacher training.

In this workshop we propose to develop the specific analysis competence of the different meanings involved in mathematical practices, applying theoretical tools of the OSA framework (Godino et al., 2007, 2019), which allow micro and macro analysis levels of the communication and interpretation processes in mathematics education.

3.1 Workshop Aims and Method

The workshop main is that the participants:

1. Know various theoretical approaches about the meaning of mathematical objects.
2. Analyze the mathematical practices that are put at stake in problem solving from the point of view of the objects involved and meanings attributed.
3. Reflect on the notion of meaning of mathematical concepts, their relationship with understanding and the design of teaching and learning processes.

The workshop includes a first part in which a text offering a synthesis of various theories of meaning in mathematics education (Godino et al., 2021) is presented and discussed. This reading includes an application example of the analysis method of meanings involved in the solving processes of a mathematical problem (onto-semiotic analysis method, at the micro and macro levels). Next, it is proposed to work in teams of two or three participants to solve a missing value proportionality problem by applying at least two different strategies. Next, the analysis of the objects and meanings put at stake in the practices to solve the problem is carried out. First, the technique described in the reading document, previously discussed, is applied according to the two solutions proposed for the workshop. The micro-level analysis is completed with an exploration of the different meanings of proportionality and its articulation in a global meaning. Then, the unitary and systemic meanings of other solutions different from those proposed in the workshop are analyzed.

3.2 Meaning in the Onto-semiotic Approach

Within the OSA framework the notion of meaning and its relation to the notions of practice and object plays a central role. The fact that certain types of practices are carried out within certain institutions is what determines the progressive emergence of "mathematical objects" and that the "meaning" of these objects is closely linked to the problems and the activity carried out for their resolution, not being pertinent to reduce this object meaning to its mere mathematical definition (Godino & Batanero, 1994).

Although the initial OSA objective was to develop a theoretical model that would answer the question of the meaning of mathematical concepts, in subsequent developments this objective has been extended and applied to any type of object that intervenes in mathematical practices, also proposing a categorization for such objects. It is considered that the epistemological, cognitive and instructional problems that mathematics education has to address should first deal with the ontological problem, that is, clarify the nature and types of mathematical objects whose teaching and learning is intended.

In a first approach, the meaning is that object which is referred by a word, a symbol or any other means of expression, issued by a person in a communicative act with another person or with himself, which takes place in a given context. However, with words and symbols not only things are mentioned or represented, but through them things are also done, that is, they intervene in operative practices. Operations and calculation with the words and symbols are carried out, so that new objects are produced as result of these operative practices.

Therefore, the question arises, what role, besides the representational one, does this word, symbol or expression play in a specific operative practice? This is a central problem that has to be addressed by a holistic theory about meaning, which should takes into account both referential and operational use, responding to the meaning of expressions that refers to concepts (ideal, abstract objects) or any other type of object.

3.3 Pragmatist and Referential Meanings

We explained the use of meaning in the OSA, and its relation to the notions of practice and mathematical object. We contextualized the explanation with the example of a possible demonstration of the elementary arithmetic proposition included in Fig. 2.

In Table 1 (column 2) we summarize the use or operational meaning of the practices required in the demonstration of proposition 2 + 3 = 5 (column 1). Column 3 shows the intervening objects in the practices.

In the realization of each practice, and in the conjunction of all or a part of them, a configuration of objects intervenes whose identification is necessary to understand and manage the teaching and learning processes. The OSA perspective proposes that the problem of signs and their interpretation should not be separated from the ontological problem, understood in terms of inquiring about the nature and types of entities referred to by the signs, as well as the instrumental role played by them in the construction activity and knowledge communication. In addition, the solution of the onto-semiotic problem implies new ways of addressing the epistemological

Proposition: 2 + 3 = 5

Demonstration:

1) The symbols, 2, 3 and 5 represent natural numbers.
2) Natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function $s: \mathbb{N} \to \mathbb{N}$, is defined. In this set, the sum, +, is defined recursively as:

$$n + 1 = s(n); n + s(m) = s(n + m)$$

3) In the sequence, 2 is the successor of 1, $2 = s(1) = 1 + 1$; 3 is the successor of 2, $3 = s(2) = 2 + 1$; and 5 is the successor of 4 which is next of 3, $5 = s(4) = s(s(3))$.
4) The sign = indicates the equivalence of two expressions.
5) The expression 2 + 3 represents the sum of the natural numbers 2 and 3.
6) Taking into account the definition of the sum of natural numbers and successor

$$2 + 3 = 2 + s(2) = s(2 + 2) = s(2 + s(1)) = s(s(2 + 1)) = s(s(3)) = s(4) = 5.$$

7) Therefore, the expressions 2 + 3 and 5 are equivalents.

Fig. 2 Demonstration of an elementary arithmetic proposition (2 + 3 = 5)

Table 1 Use/intentionality and objects in practices to demonstrate 2 + 3 = 5

Sequence of elementary practices	Use/intentionality	Intervening objects
1. The symbols, 2, 3 and 5 represent natural numbers	Attributing meaning to the symbols 2, 3, 5 as natural numbers	Languages: symbolic; natural Concepts: natural number
2. The natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function $s: N \to N$ is defined. In this set, the sum, +, is defined recursively as: $n + 1 = s(n)$; $n + s(m) = s(n + m)$	Evoking the rules that define natural numbers and their sum, within the framework of a specific axiomatic theory	Language: natural, symbolic Concepts: natural number; set (of symbols); successor, function; first element; sum Propositions: Peano's axioms
3. In the sequence, 2 is the successor of 1, $2 = s(1) = 1 + 1$; 3 is the successor of 2, $3 = s(2) = 2 + 1$, and 5 is the successor of 4 which is next of 3, $5 = s(4) = s(s(3))$	Interpreting the meaning of symbols 2, 3, 5 in Peano's axiomatic theory of natural numbers	Languages: natural; symbolic Concepts: sequence; successor, sum Proposition: 2 is the successor of 1, 3 is the successor of 2, and 5 is the successor of the successor of 3 Arguments: convention based on the properties of the successor function
4. The sign = indicates the equivalence of two expressions	Evoking the meaning of the equality of natural numbers as equivalence of two expressions	Languages: symbolic; naturally Concepts: equivalence of expressions; equality
5. The expression 2 + 3 represents the sum of the natural numbers 2 and 3	Interpreting the meaning of + as the sum of natural numbers	Languages: natural and symbolic Concepts: sum of natural numbers
6. Taking into account the definition of the sum of natural numbers and successor $2 + 3 = 2 + s(2) = s(2 + 2) = s(2 + s(1)) = s(s(2 + 1)) = s(s(3)) = s(4) = 5$	Applying the rules that define the following function (successor) and addition of natural numbers	Languages: natural and symbolic Proposition: 2 + 3 = 5 Procedure: addition and successor operations Argument: deductive, based on the definitions of natural numbers, sum and the successor function
7. Therefore, the expressions 2 + 3 and 5 are equivalent	Fixing the new rule of use of the numerical symbols (declare the truth of the proposition)	Languages: natural and symbolic Proposition: statement of practice 7 Argument: deductive sequence of practices 1 to 6

problem about the origin and evolution of knowledge, no doubt essential to address the educational-instructional problem (Godino et al., 2021).

3.4 Workshop Development

Participants were proposed to respond to the instructions given below.

Question 1: Identify the referred objects (meanings) in each of the practices of solution 1 of the problem included in the Annex. Complete Table 2.

Question 2: Identify the referred objects (meanings) in each of the practices of the solution 2 of the problem included in the Annex. Complete Table 3.

In solutions 1 and 2 of the problem, the concept of proportionality intervenes in a decisive way. Taking into account the types of objects and unitary meanings that intervene in the operative and discursive practices that allow solving the problem, we can say that the systemic meaning of the proportionality that is at stake in solution 1 is of arithmetic type, while in solution 2 is of algebraic-functional type (Burgos et al., 2018).

Question 3: Analyze the unitary and systemic meanings of other different solutions by which the problem can be solved.

3.5 Some Conclusions

The identification of the various partial meanings of a mathematical object and its articulation is a phase of the onto-semiotic analysis of mathematical activity. This analysis helps to formulate hypotheses about critical points in the interaction between the various agents in which there may be gaps of meaning or disparity of interpretations that require processes of negotiation of meanings and changes in the instruction process.

Table 2 Object and meanings in solution 1

Sequence of elementary practices to solve the task	Objects referred to in the practices (concepts, propositions, procedures and arguments)	Use and intentionality of practices
…		

Table 3 Objects and meanings in solution 2

Sequence of elementary practices to solve the task	Objects referred to in the practices (concepts, propositions, procedures and arguments)	Use and intentionality of practices
1.		
…		

The theory of meaning that has been elaborated from the OSA perspective is supporting a new field of reflection on what could be called an onto-semiotic analysis, in which the study of signs should be linked to the analysis of the objects referred to by the signs. The OSA attempts to combine realistic and operational theories of meaning, since the problem is approached from the educational context, that is, the setting of construction and dissemination of mathematical knowledge. Although the problem of meaning interests to various disciplines (philosophy, linguistics, psychology, semiotics, etc.), the field of education, and, in particular, mathematics education, provides a rich perspective to address this problem. The onto-semiotic approach proposes not to separate the problem of signs and their interpretation from the ontological problem. This is understood in terms of inquiry about the nature and types of entities referred ~~to~~ by the signs, as well as the instrumental role played by them in the knowledge construction and communication.

The onto-semiotic approach to the meaning of mathematical objects has implications for teachers, since it highlight the complexity of knowledge and, therefore, the challenge of teacher education. In this sense, mathematics teacher should know the different meanings of mathematical objects, as well as the network of objects and processes involved in the mathematical practices, in order to be able to plan the teaching, manage the interactions in the classroom, understand the difficulties and assess the students' learning.

4 Final Reflections

Didactics should provide results that allow effective action on a part of reality: the teaching and learning of mathematics in the different contexts in which it takes place. In addition, it must take into account the four types of problem areas, epistemological, ontological, semiotic-cognitive, educational-instructional, described in Godino et al. (2021) and their interactions. Didactics should offer provisional principles (standards or suitability criteria in OSA framework) agreed by the community interested in mathematics education. These principles and norms are useful in two moments:

1. A priori, the suitability criteria guide the way in which an instruction process should be developed.
2. A posteriori, the criteria serve to assess the teaching and learning process effectively implemented and identify possible aspects to be improved in the redesign.

To generate these principles, researchers in mathematics education should discuss and collaborate with all other sectors interested in improving mathematics teaching (teachers, parents, administration, etc.). This will lead to a consensus that generate principles to guide and value the instruction processes, in order to achieve a suitable teaching of mathematics. It is recognized, however, that the identification of suitability criteria, both general and specific, requires a research agenda that is open to discussion and development in the mathematics education community.

Didactics involves the study of human beings interacting in very diverse contexts, that is complex, dynamic, open, heterogeneous systems engaged in multiple and diverse interactions. These systems have chaotic aspects, where small changes can lead to large deviation. Since minor changes take place at the micro level, they should be studied as possible explanatory factors of the changes observed at the macro level. Consequently, Didactics should contemplate the use of analysis units at the micro level (a task, or a teacher-student interaction of a specific nature), and at the macro level (a field of problems, a long-term didactic trajectory, the sociocultural context). The principles explicitly stated as characteristics of a theory should be interpreted as tools, while the methods are ways of applying these tools to face the solution of specific problems and questions in the field.

Annex. Solutions to a Proportionality Problem

Problem: *A package of 500 g of coffee is sold for 5 euros. At what price should a package of 450 g be sold?*

Solution 1:

1. In everyday life situations of buying and selling, it is usual to assume that, when buying small quantities of coffee, these quantities are of the same type and quality.
2. Consequently, if you buy double, triple, etc. of product, you must pay double, triple, etc. Similarly, if you buy half, the third part, etc. of product, half, the third part, etc. must be paid.
3. If a package of 500 g of coffee is sold for 5 euros, the price of 100 g of coffee (five times less) should be a fifth of 5 euros, that is, 1 euro.
4. The price of 50 g (half of 100 g) must be half the price of 100 g, that is, 0.50 euros.
5. In this way, $4 \times 100 + 50 = 450$ g of coffee should cost, 4×1 € $+ 0.50 = 4.50$ €; that is, 4 euros and 50 cents.

Solution 2:

1. It is assumed that if you buy double, triple, etc. of product, you must pay double, triple, etc. In addition, what we will pay for two different coffee packages will be equal to the price of a package that weighs the same as the two together.
2. Therefore, the correspondence established between the set of the quantities of the product (Q) and the set of the prices paid (P), f: Q → P complies that, the image of the sum of quantities is the sum of the images, $f(a + b) = f(a) + f(b)$, and the product image of an amount a by a real number α is the product of the image quantity by that number, $f(\alpha a) = \alpha f(a)$.
3. That is, the function f: Q → P is linear; then there is k, such that $f(x) = kx$.

4. The coefficient k of the linear function is the coefficient of proportionality, in the case of direct proportionality relations between magnitudes.
5. Applying these properties to the case, we have:
   $f\,(500\text{ g}) = 5$ €; $500f\,(1\text{ g}) = 5$ €; $f(1\text{ g}) =$ € 5/500 [One gram of coffee costs 1 cent]
6. $450f(1\text{ g}) = 450 \times 5/500$ €; $f\,(450\text{ g}) = 4.5$ €
7. Then the price of a package of 450 g should be 4.5 euros.

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