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Advances in the Anthropological Theory of the Didactic

CRM CENTRE DE RECERCA MATEMÀTICA

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A Word of Introduction

It is difficult to say how grateful I am to the *Centre de Recerca Matemàtica* (CRM) of Catalonia, its director Lluis Alsedà, and all its staff, who made possible what the reader will discover in this volume, and the team of enthusiastic researchers led by Marianna Bosch—Berta Barquero, Ignasi Florensa, Josep Gascón, Pedro Nicolás, and Noemí Ruiz-Munzón—who has so cleverly devised and organised an Intensive Research Programme under the title *Advances in the Anthropological Theory of the Didactic and their Consequences in Curricula and Teacher Education*. The programme took place in Bellaterra (Barcelona) in June and July 2019. Its duration (eight weeks) was exceptional and its achievement remains impressive, to say the least.

This could not have been achieved, of course, without the scientific input of some 90 researchers from 23 countries in Europe, America, and Asia, whose contributions fuelled four courses: (1) *Dialogue between theories*; (2) *Teacher education and the professionalisation of teaching*; (3) *The curriculum problem and the paradigm of questioning the world in mathematics and beyond*, and (4) *Research in didactics at the university level*. Such a wide variety of contributions cannot be summarised in a few words. However, I would like to give the reader some general but firm indications on the matter at hand. To do so, I will begin with the very name of the research field that encompasses all the works presented here: the *anthropological theory of the didactic* (ATD).

Let us start with this rather unusual expression: *the didactic*. In the ATD, we consider *persons* on the one hand and *institutions* on the other hand. As always in the ATD, the word "institution" is understood in a broad sense: a couple is an institution, a family is an institution, a class is an institution, and so are a football team, or a research team. Every institution has various institutional *positions*, such as the position of a mother in a family, a teacher or pupil in a class, a captain in a football team, etc. All persons occupy various positions at the same time and successively. For a more concise presentation, we define the notion of *instance*: an instance is either a person or an institutional position. An *instantial* point of view on a reality is the point of view of a certain person or a certain position within an institution—in the

ATD, institutions and their positions have "points of view", just like persons, and they can also learn, unlearn, etc. All this noted, what is called the didactic is constituted by *any act* (or "gesture") that at least one instance sees as likely to help certain instances to know a certain reality better—to "learn more". Of course, in general, there exist at the same time instances which consider that the act in question will not help to learn more or will even diminish the learning outcomes—in the first case, those instances will consider the act as *isodidactic* and, in the second case, as *antididactic*. The essential fact to underline about the didactic is that research in didactics is too often, and restrictively, focused on acts or gestures considered as didactic by conventional instances—e.g. teachers, teacher trainers, students, curriculum developers and instructional designers, or even the minister of education. In contrast, the researcher working within the framework of the ATD has to be sensitive to the didactic existing in a host of not necessarily "school" institutions, be it the family, the siblings, a group of schoolmates, etc. And, in all these settings, the researcher has to go beyond what is usually labelled as didactic—e.g. a student giving a presentation in class can be didactic for the student as well as for other students who attend the presentation, a pupil reciting his or her lesson to his or her mother or father can be didactic for anyone listening to it, pupils talking about their homework on the phone can be didactic for them but not for their parents (some of whom will be prone to believe that it is mere gossip), etc. However, there is more to it. Research in didactics must certainly focus on elucidating the *economy* of the didactic, i.e. the economy of the potentially didactic acts or gestures *actually performed*. It must also study the *ecology* of the didactic, and in particular, those gestures that might be performed even if some of them will *not* be performed, which make up what can be called the *missing* didactic. What we have to study thus goes beyond what we can actually observe here and now, if only to actually explain the observable.

The search for the possibly didactic and the study of its economy and its ecology must get away from the restricted framework in which most "classical" didactic research has chosen to confine itself. What happens in class cannot be explained *only* by what happens in class. We must remember that the ATD started with the *theory of the didactic transposition of knowledge*, which makes the classroom appear as open to decisions and phenomena that are external to it, proceeding from mechanisms teachers and students can choose to ignore, but which nevertheless determine their didactic destiny. It is at this point that the qualifier *anthropological* used to designate the ATD takes on its meaning and scope. Any act, any "gesture", carried out by anyone in any institutional position, can have didactic effectiveness from some instantial viewpoints with respect to some instances and some specific subject. A classroom is not a sanctuary. What happens there also depends on decisions made, sometimes a very long time ago, at different levels of what is called the scale of levels of didactic co-determinacy—first the school where the class is located, then the society that is their common habitat, then what is called the civilisation (a concept I will not attempt to elucidate here) in which they are immersed, and finally the human species itself. The class, understood in a broad sense like all the other notions here, is the focal place where what we call *didactic systems* are formed, committed to the study of a certain “work” (any possible human creation), in particular, a certain *question* to be “answered”. Didactic systems appear and disappear just to be replaced by new didactic systems. It is in the classroom with its didactic systems that we can begin to identify factors, which in the ATD are called *conditions and constraints*, which are determinative of the ecology and constitutive of the economy of the didactic in this institution called school, where all this takes place. This is the starting point of all research study conducted within the framework of the ATD, regardless of the institutional diversity of the questions selected and the paths taken by the researchers.

One more comment: the ATD, the *anthropological theory of the didactic*, is a *theory*. Even in the social sciences, the term “theory” seems to be polysemous nowadays. Often, the word is used to designate an isolated hypothetical statement, which might be a decontextualised theoretical fragment, but generally appears as an alleged condition, isolated from any explicit theoretical context, as if a theory were made of bits and pieces with little connection between them. Let me remind you that the word *theory* derives from the Greek *theōrós*, which means first of all “spectator”, one who looks at: a theory enables us to “see” the observable world—and, as pointed out earlier, to go beyond the observable—and to model it in order to understand and explain it. Understanding and explaining the didactic (including the “missing” didactic) in any institution is the task, barely begun, of the ATD. What follows is both a testimony to this “will to know” and a timely contribution to our scientific project—the advancing of the anthropological theory of the didactic.

Marseille, France
April 2021

Yves Chevallard

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Dialogue Between Theories in Didactics

Josep Gascón and Pedro Nicolás

The course *Dialogue between theories in didactics* derives from a question we posed to Guy Brousseau in 2013: *Is (or should) didactics be a normative science?* Despite the succinct formulation, the context in which the question was asked contributed to making it clear that we were referring to the fact that knowing and setting a norm are two very different kinds of activities, and that we were questioning the legitimacy of didactics, regarded as a science, to prescribe or outlaw certain teaching techniques. Brousseau's answer went against the one underlying most of the published works in didactics:

> *I do not know what could possibly be a "normative science", really. Is classical mechanics a science? Is it a model? Is it normative? Is biology a normative science? I do not think so. I think that issues concerning scientific truth and problems about cultural, social or economic norms are, and must be, completely separated.*

We thought that such a clear answer gave cause for a promising dialogue between different theories. We therefore reformulated the question as follows:

> *Q_1: To what extent, how, under which conditions, can (or should) didactics make value judgments and write out normative prescriptions in order to provide criteria about how to organise and manage study processes?*

In 2015, we posed this question to several scholars working in different theories in didactics: Brousseau (Theory of Didactic Situations), Michèle Artigue (Comparison between different theories), Ed Dubinsky (Theory of Actions Processes Objects Schemas), María Trigueros (Theory of Actions, Processes, Objects, Schemas), Juan D. Godino (Onto-Semiotic Approach), Koeno Gravemeijer (Realistic Mathematics

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Education), Ricardo Cantoral (Socio-epistemology of Mathematics Education) and Josep Gascón and Pedro Nicolás (Anthropological Theory of the Didactic). The corresponding answers were published, analysed and compared in (Gascón & Nicolás, 2017), and they form what we call *the first step* in the dialogue between theories in didactics.

At the end of the aforementioned article, we explained the need to broaden the dialogue, beyond the discussion about the legitimacy of didactics, to submit value judgements and normative prescriptions., We said that, in order to make progress in the reciprocal understanding of the different theories, we should try to answer a question which could be formulated as follows:

Q_2: Which are the relationships between research in didactics and teaching?

This question sets out an unavoidable key problem for the community of didactics, which still remains open. The answer provided by each theory should specify the *teaching ends* underlying the value judgements and the normative prescription advocated by each approach, as well as bring to light the *postulates* or the *basic assumptions*, and to analyse their influence on the corresponding teaching ends and the kind of statements regarded as *results of the research* by each theory.

Given the crucial importance of this matter, we started *the second step* in our dialogue by suggesting to continue in the direction pointed by these questions. This was reflected in a series of papers published in *For the Learning of Mathematics*: Lerman (2018), Proulx (2018), Bartolini Bussi (2018), Davis (2018), Oktaç et al. (2019), Díaz Godino et al. (2019), Staats and Laster (2019), Gascón and Nicolás (2019).

This stage culminated in a face-to-face advanced course organised by the *Centre de Recerca Matemàtica* (Centre for Mathematical Research) within the framework of the Intensive Research Programme on the ATD. Several researchers such as María Trigueros (Theory of Actions, Processes, Objects, Schemas), Claire Margolinas (Theory of Didactic Situations), Juan D. Godino (Onto-Semiotic Approach), Josep Gascón and Pedro Nicolás (Anthropological Theory of the Didactic), Michèle Artigue (Comparison between different theories) and Yves Chevallard (Anthropological Theory of the Didactic) participated in the course.

Each one delivered a lecture explaining the principles and the methodological and theoretical tools of each approach. Some of them also led a workshop to show those principles and tools in action, with the formulation and the study of a prototypical didactic problem addressed by the corresponding theory. Their contributions constitute the chapters of this first part of the book.

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On the Genesis and Progress of the ATD

Yves Chevallard

1 Theory and Theorization

The word *theory* belongs to the vocabulary of the ATD, where it has a broader meaning than usual: a 3-year-old boy has a theory of dads, a man without education a theory of politics, and so on.

As usual, when we speak of "theory of...", we mobilize a *synecdoche*: a part (the theoretical component of a praxeological complex) refers to the whole (this complex itself).

Correlatively, the work of constructing a theory (in this sense) will be called *theorization*—which is, therefore, the process of constructing the whole complex in question, its *logos* as well as its *praxis*.

What we theorize is the *object* of the theory, that the theory must allow to question, to model, in order to establish the laws of its *economy* (how does this object function?) and of its *ecology* (under what conditions and constraints?).

In the case of the ATD, the object is obviously "the didactic". The object of a theory evolves in time with this theory. We will see this later in the case of the ATD. But we will start with another aspect of the ATD: its anthropological nature.

2 "Anthropological"

The ATD is the "*anthropological* theory of the didactic". The first question we will examine—necessarily in summary form—is the following: why "anthropological"? How and why did this adjective appear?

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What surprised me, from the point of view of research in didactics, when I observed the teaching of mathematics in high school was that the objects taught seemed to be seen by teachers and the noosphere around them as something obvious, or, if you can call it that, natural: *they were taken for granted*.

One of the first questions studied (in a master's thesis) was the following: although they knew the notion of square root, why were eighth graders asked to factor the expression $x^2 - 4$ but not $x^2 - 5$, not to mention $x - 5$?

If we discussed the problem with the teachers concerned, we could get opportunistic answers like "It would be too difficult". In fact, they had not even thought of proposing such expressions, because the then current didactic transposition excluded them.

This type of phenomenon led to a fundamental conclusion: what happens in the classroom can depend on conditions formed *outside* the classroom, sometimes very far away, in social space and historical time, of students and teachers.

The adjective *anthropological* gradually emerged to express, in the first place, that didactic analysis had to take into account multiple conditions, whose location could be outside the classroom, that duly determine all human activity.

Today, we denote by $\mathcal{C}$ the set of all the conditions that prevail at a given time. The set $\mathcal{C}$ can only be known in part, unless the conditions considered are deliberately limited a priori, which we will not do.

3 A Recent Finding

Three economists recently published a study entitled "Social inequalities widen gender gaps in mathematics", subtitled "Equal countries cultivate high-yielding girls" (Breda et al., 2018). They write: "According to the Programme for International Student Assessment (PISA), there are on average only seven girls for ten boys in the top decile of the math performance distribution among the 35 countries belonging to the OECD."

They add that the underrepresentation of girls at high levels of performance is a common feature of all OECD countries and has remained stable since 2000, a result that extends to science and reading (but, in this case, *in favour* of girls).

The authors explain the gender gaps in terms of "general" inequalities: "In more egalitarian countries, [. . .] girls are more represented among high performers. . ." The gender gap in math is a form of social inequality.

In terms of action, instead of persisting on gender-specific measures, this would lead to focus on measures tailored to reduce global social and economic inequalities —which is largely outside the specific sphere of action of teachers.

4 Epistemological Break

The *epistemological break* to be assumed by the didactician consists in making the teacher one of his or her objects of study, whose praxeological determinants (relative to what they do and to what they think) must be brought to light.

The opposite would be to explain what is observed with the means available to the teacher. When researchers are also teachers, they must accept a degree of "schizophrenia" between what they must study and the means they use to do so.

Although they can imaginatively put themselves in the place of what they study, researchers do not have to identify with teachers, who tend to consider as explanatory only the conditions they wish to modify from their position as teachers.

5 Humpty Dumpty

Before I go any further, I want to remind you of what I have called *the Humpty Dumpty principle*, which is essential for building a theoretical language and which is familiar to anyone with a mathematical background.

In *Through the Mirror, and what Alice found there* (1871), Lewis Carroll writes: "When *I* use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."

Here is how the dialogue continues: "The question is, said Alice, whether you *can* make words mean so many different things. The question is, said Humpty Dumpty, which is to be master—that's all."

The above is fiction. Accepting the Humpty Dumpty principle is a requirement to understand what a given theory tells us. The condition of possibility of this principle is the principle of *epoché* or of "suspension of judgment".

Epoché is also translated as *withholding of assent*. Indeed, the epistemologically correct approach to any work also requires the suspension of *disbelief*, which we should expect of every student of the work.

In what follows, the vocabulary used, although as close as possible to common sense, will have a meaning specific to the ATD—this was already the case, in the above, with the words *theory* and *condition* for example.

6 Cognition

The anthropological theory of the didactic contains a *theory of cognition*, which is therefore a subtheory of the ATD. The starting point is the notions of *person x* and *institution I*, or, more precisely, of *institutional position p*.

The basic notion is that of *relation* of x (or p) to an object o, denoted by $R(x, o)$ (or $R_I(p, o)$). We will speak of an *instance* $\hat{\imath}$, either personal ($\hat{\imath} = x$) or positional ($\hat{\imath} = p$), and denote the "instantial" relation to o by $R(\hat{\imath}, o)$.

The relation $R(\hat{\imath}, o)$ is composed of everything that connects the instance $\hat{\imath}$ with the object o—what $\hat{\imath}$ thinks of o, what $\hat{\imath}$ can or cannot do with o, the feelings (of interest, love, indifference, etc.) $\hat{\imath}$ may have towards o, etc.

The relation $R(\hat{\imath}, o)$ evolves throughout the life of $\hat{\imath}$. What a "mentalist" psychology places "in the head" of $\hat{\imath}$ (the Latin word *mens* means "mind", and also "intention", "will"), the ATD places it in an objectifiable reality, $R(\hat{\imath}, o)$.

When we have $R(\hat{\imath}, o) \neq \varnothing$, we say that $\hat{\imath}$ *knows* o, or that o *exists for* $\hat{\imath}$. When $R(\hat{\imath}, o) = \varnothing$, we say that $\hat{\imath}$ *does not know* o, or that o *does not exist for* $\hat{\imath}$. An object o is any entity which exists for at least one instance $\hat{\imath}$. All that exists is object.

Given instances $\hat{\imath}, \hat{\jmath}, k^\wedge$, etc., we can consider $R(\hat{\jmath}, R(\hat{\imath}, o))$, the relation of $\hat{\jmath}$ to the relation of $\hat{\imath}$ to o, and $R(k^\wedge, R(\hat{\jmath}, R(\hat{\imath}, o))$, the relation of $k^\wedge$ to the relation of $\hat{\jmath}$ to $R(\hat{\imath}, o)$. If x_1 and x_2 are pupils and y is the teacher, it may, for instance, be that $R(x_2, R(y, R(x_1, o))) \neq \varnothing$.

The *cognitive universe* of $\hat{\imath}$ is $\Omega(\hat{\imath}) \stackrel{\text{def}}{=} \{o \,/\, R(\hat{\imath}, o) \neq \varnothing\}$: it tells us which objects o are known to $\hat{\imath}$. The *cognitive equipment* of $\hat{\imath}$ is $\Gamma(\hat{\imath}) \stackrel{\text{def}}{=} \{(o, R(\hat{\imath}, o)) \,/\, o \in \Omega(\hat{\imath})\}$: it tells us what is $\hat{\imath}$'s knowledge of o.

In what follows, we take as known the concept of praxeology P. The *praxeological universe* of $\hat{\imath}$ is $\Omega^{\blacklozenge}(\hat{\imath}) \stackrel{\text{def}}{=} \{\mathrm{P} \,/\, R(\hat{\imath}, \mathrm{P}) \neq \varnothing\}$. The *praxeological equipment* of $\hat{\imath}$ is $\Gamma^{\blacklozenge}(\hat{\imath}) \stackrel{\text{def}}{=} \{(\mathrm{P}, R(\hat{\imath}, \mathrm{P})) \,/\, \mathrm{P} \in \Omega^{\blacklozenge}(\hat{\imath})\}$. We have $\Omega^{\blacklozenge}(\hat{\imath}) \subset \Omega(\hat{\imath})$ and $\Gamma^{\blacklozenge}(\hat{\imath}) \subset \Gamma(\hat{\imath})$.

We now assume this key principle: $\Gamma^{\blacklozenge}(\hat{\imath})$ generates $\Gamma(\hat{\imath})$ in the sense that, whatever the object o, $R(\hat{\imath})$ results from all the relations $R(\hat{\imath}, \mathrm{P})$ where $\mathrm{P} \in \Omega^{\blacklozenge}(\hat{\imath})$ involves the object o, whether technically, technologically, or theoretically.

Consequently, in order to analyse in depth the content of a relation $R(\hat{\imath}, o)$ or $R(\hat{\imath}, \mathcal{O})$, where $\mathcal{O}$ is a set of objects o, it is necessary to investigate concretely the praxeologies that generated it, either recently or in a more remote past.

7 Instantial Relativity

There is a key question so far ignored. Which instance judges that, for example, $R(\hat{\imath}, o) \neq \varnothing$? If it is the instance $\hat{\jmath}$, we will write $\hat{\jmath} \vdash R(\hat{\imath}, o) \neq \varnothing$, which can be read as follows: "$\hat{\jmath}$ judges that $\hat{\imath}$ knows o".

There may be another instance $k^\wedge$ such that $k^\wedge \vdash R(\hat{\imath}, o) = \varnothing$. The instances $\hat{\jmath}$ and $k^\wedge$ do not have the same vision of $R(\hat{\imath}, o)$. Note that we have this: $k^\wedge \vdash (\hat{\jmath} \vdash \mathrm{R}(\hat{\imath}, o) \neq \varnothing) \Rightarrow k^\wedge \vdash R(\hat{\jmath}, R(\hat{\imath}, o)) \neq \varnothing$. Similarly, we have: $l^\wedge \vdash k^\wedge \vdash R(\hat{\imath}, o) = \varnothing \Rightarrow l^\wedge \vdash R(k^\wedge, R(\hat{\imath}, o)) \neq \varnothing$.

Let's generalise the above. Let ϑ be any sentence. We can have: $\hat{\jmath} \vdash \vartheta$, $k^\wedge \vdash \neg\vartheta$, $k^\wedge \vdash (\hat{\jmath} \vdash \vartheta)$, etc. If $\hat{\jmath} = \hat{\imath}$, for example, we can have $\hat{\imath} \vdash R(\hat{\imath}, o) \neq \varnothing$ or $k^\wedge \vdash (\hat{\imath} \vdash R(\hat{\imath}, o) = \varnothing)$, or $k^\wedge \vdash (\hat{\imath} \vdash (k^\wedge \vdash R(\hat{\imath}, o) \neq \varnothing)$, or $\hat{\imath} \vdash (k^\wedge \vdash (\hat{\imath} \vdash (R(k^\wedge, o) = \varnothing)$, etc.

The instances $\hat{\imath}, \hat{\jmath}, k^\wedge$, or $l^\wedge$ can be a researcher in didactics ξ, a student x, or a teacher y, so that, depending on the circumstances, we will have for example $\xi \vdash R$

$(y, o) \neq \varnothing$, $\xi \vdash R(x, o) \neq \varnothing$, $y \vdash R(x, o) = \varnothing$, $y \vdash R(\xi, o) = \varnothing$, $\xi \vdash (y \vdash R(\xi, o) = \varnothing)$, etc.

Consider an instance $\hat{\jmath}$ and the cognitive universe $\Omega(\hat{\imath})$ of the instance $\hat{\imath}$. The cognitive universe of $\hat{\imath}$ according to $\hat{\jmath}$ is $\Omega_{\hat{\jmath}}(\hat{\imath}) \overset{\text{def}}{=} \{o \,/\, \hat{\jmath} \vdash R(\hat{\imath}, o) \neq \varnothing\}$. The cognitive equipment of $\hat{\imath}$ according to $\hat{\jmath}$ is then defined by: $\Gamma_{\hat{\jmath}}(\hat{\imath}) \overset{\text{def}}{=} \{(o, R(\hat{\imath}, o)) \,/\, o \in \Omega_{\hat{\jmath}}(\hat{\imath})\}$.

In what follows, we will say that we look at an object o "from the point of view" of $\hat{\imath}$ to say that we consider $R(\hat{\imath}, o)$. But *who* considers $R(\hat{\imath}, o)$? It can only be an instance $\hat{\jmath}$, which is usually the author, ϟ (*koppa*, an archaic Greek letter), of the statement about $R(\hat{\imath}, o)$.

In this description, the researcher ξ does not have a privileged place, even when $\xi =$ ϟ. Why is this the case? There are two main reasons for this. The first refers to any possible instance, including the investigator.

Every instance creates conditions that ξ must take into account. If $y \vdash R(x, o) = \varnothing$, this will have consequences. Si }\hat{z}{ is the education minister and }\hat{z}{ $\vdash \vartheta$, where $\vartheta =$ "the paradigm of visiting works is collapsing", this will also have effects.

The other reason is that researchers should not believe that being researchers gives them automatic access to the truth. They must constantly shape their relation to the objects of their scientific life. Nothing is given to them. Everything has to be conquered.

8 The Didactic Revisited

So far, we have only studied cognitive aspects, whether personal or institutional. Now we move on to the *didactic*. At the starting point we consider an instance $\hat{w}$ and an ordered pair $n^- = (\hat{\imath}, o)$.

The instance $\hat{w}$ is arbitrary, similar to the origin of coordinates in a plane, which can be changed at will: it is the *reference instance*. The couple $n^- = (\hat{\imath}, o)$ is also arbitrary: it is the *cognitive base* considered here.

Let us now consider an instance $\hat{u}$, which will perform a certain "gesture" δ (a "gesture" is a task of a certain type). This gesture changes $\mathcal{C}$, which becomes $\mathcal{C}' = \mathcal{C}^{\swarrow\delta}$ (where $\mathcal{C}^{\swarrow\delta}$ is the set $\mathcal{C}$ "deranged" by the gesture δ).

Can it be said that, at least from the point of view of $\hat{w}$, δ is a "didactic" gesture regarding $n^- = (\hat{\imath}, o)$, in the sense that $R(\hat{\imath}, o)$ will be considered as "cognitively better" after the gesture δ has taken place, i.e., under the conditions $\mathcal{C}^{\swarrow\delta}$?

I used to use a definition of "didactic" inspired by Guy Brousseau's definition of the expression "didactic situation": I said that the gesture δ was didactic if it manifested the intention of $\hat{u}$ to make $\hat{\imath}$ learn more about o.

The current definition generalizes this "old" definition (by the intention lent to $\hat{u}$), insofar as it allows us to dissociate the author $\hat{u}$ of the gesture δ from the observer $\hat{w}$. Of course we can have $\hat{w} = \hat{u}$, but also $\hat{w} = \hat{\imath}$, $\hat{w} = y$, $\hat{w} = \xi$, etc.

To appraise the change in a relation $R(\hat{\imath}, o)$, it is necessary to consider (1) an institutional position $\hat{s} = (I, p)$ that knows o, and (2) an "evaluating instance" v^, able to say whether a relation R' to o is closer to $R^{-} = R(\hat{s}, o)$ than another relation R to the object o.

To decide whether δ is didactic with respect to $n^{-} = (\hat{\imath}, o)$, $\hat{w}$ must imagine a position $\hat{s} = (I, p)$ such that $\hat{w} \vdash R(\hat{s}, o) \neq \varnothing$. The ordered pair $\underline{n} = (\hat{s}, v\wedge)$ is called a *cognitive frame of reference*. The 4-tuple $\tilde{n} = (\hat{\imath}, o, \hat{s}, v\wedge)$ is called a *cognitive nucleus*.

More precisely, if $\hat{w}$ judges that (with obvious notations) we will have $v\wedge \vdash \mathrm{d}(R', R^{-}) < \mathrm{d}(R, R^{-})$, or $v\wedge \vdash \mathrm{d}(R', R^{-}) > \mathrm{d}(R, R^{-})$, or $v\wedge \vdash \mathrm{d}(R', R^{-}) \approx \mathrm{d}(R, R^{-})$, we will say that, for $\hat{w}$, δ is *didactic*, or *antididactic*, or *isodidactic* with respect to $\tilde{n}$ and $\mathcal{C}$.

It is important to note that $\hat{w}$ makes a judgment (of didacticity, antididacticity or isodidacticity) *beforehand*: $\hat{w}$ makes a prediction about the future judgment of $v\wedge$, a prediction based on $\hat{w}$'s knowledge of $\tilde{n} = (\hat{\imath}, o, \hat{s}, v\wedge)$ and $\mathcal{C}$.

All this requires two comments. The first is to note that any gesture δ can be considered didactic, even if there is no "didactic" intention: any gesture is *possibly didactic*, which greatly expands the didactician's universe of interest.

The second comment has to do with the fact that the possible didacticity of a gesture δ is enunciated by $\hat{w}$ a priori, before δ is carried out. Of course, it may be that, in order to judge δ, $\hat{w}$ uses previous observations of δ in like cases.

9 Possibly Didactic Situations

The purpose of didactics, we have said, is to study the didactic. Let us clarify this statement using the notions of cognitive base $n^{-} = (\hat{\imath}, o)$, cognitive frame of reference $\underline{n} = (\hat{s}, v\wedge)$, and cognitive nucleus $\tilde{n} = (\hat{\imath}, o, \hat{s}, v\wedge)$.

To the previous notions, we add another one, which defines the notion of *possibly didactic situation*: $\varsigma = (\tilde{n}, \hat{u}, \delta, \mathcal{C})$. Didactics is the science that studies the *social elaboration* of the didactic, the antididactic, and the isodidactic.

Some of the main questions raised are the following. What instances $\hat{w}$ issue judgments, regarding which gestures δ, which cognitive nucleuses $\tilde{n} = (\hat{\imath}, o, \hat{s}, v\wedge)$, and taking into account which sets of conditions $\mathcal{C}$?

I do not wish to go any further into this matter here, but I want to draw attention to two points. The first point is that the formalization adopted here highlights the fact that *learning is a social fact*, which does not exist outside of a cognitive frame of reference, whatever it is.

The other point refers to $\mathcal{C}$. What does $\hat{w}$ know about $\mathcal{C}$, even if $\hat{w} = \xi$, and what do we know about $\mathcal{C}$ at this point in the history of our discipline? We certainly must not limit ourselves to conditions that can be modified from a given position!

10 By Way of Conclusion

How can a theory $\mathcal{T}$ look at another theory $\mathcal{T}'$? The promoters of $\mathcal{T}$ should ask themselves what objects $\mathcal{T}'$ sees that $\mathcal{T}$ has not yet seen or can hardly see and what concepts it uses to give meaning to what it sees.

I hope that this suggests that a theory is a meticulously built construct, and that this is sufficient to outlaw any opportunistic syncretism. We must each time carefully study $\mathcal{T}'$ and then re-examine $\mathcal{T}$—to improve it, when possible.

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ATD on Relationships Between Research and Teaching. The Case of a Didactic Problem Concerning Real Numbers

Josep Gascón and Pedro Nicolás

1 Introduction

According to what we suggest in the introduction to the course, in order to progress in the dialogue and to move towards the second step in the dialogue between theories in didactics, we start by wondering about how research and teaching are related in the *anthropological theory of the didactic* (ATD):

$\boldsymbol{Q_2}$: *Which are the relationships between research carried out by ATD and different kinds of teaching regarded as valuable by ATD? How are these relationships incarnated in the case of a didactic problem concerning real numbers?*

We will decompose $\boldsymbol{Q_2}$ into two sets of problems, $\boldsymbol{Q_2}$**(i)** and $\boldsymbol{Q_2}$**(ii)**. Section 2 deals with $\boldsymbol{Q_2}$**(i)**, devoted to the analysis of the *ends of research* in ATD. Section 3 deals with $\boldsymbol{Q_2}$**(ii)**, devoted to the analysis of the *teaching ends* which underlay many works in ATD. In this section we introduce the notion of *didactic paradigm*, a key idea in order to explain the links between research and teaching (Gascón & Nicolás, 2019b). Section 4 is devoted to show how the tools of ATD can be applied to a didactic problem concerning real numbers. Finally, in Sect. 5 we propose a third step in the dialogue between theories.

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2 Basic Assumptions and the Object of Study in ATD

In what follows we will present some basic assumptions of the ATD and show their incidence on the constitution of the object of study and research methodology. More precisely, we will answer the following questions:

***Q_2*(i):** *Which are the research ends in ATD? In other words, which are the kind of research problems fostered and the kind of didactic phenomena aimed to be clarified? Which are the kind of results regarded as admissible?*

2.1 ATD as a Theory of Human Activity

Every human activity, and their outputs, can be described in terms of praxeologies. This postulate is at the core of the theory of human activity propose by the ATD. It shapes the formulation of any research problem. Moreover, it plays a crucial role in the formulation of the *teaching ends* and the *means* to achieve those ends. Indeed, both the works to be studied and the way of studying them are to be expressed in terms of praxeologies.

2.2 Research Ends and Teaching Ends Are Postulates

As Weber (1917/2010) would say, the ultimate motivation of human activity, the final ends, are in the "sphere of values", which means that cannot be rationally stablished. This is true for the ends of research activity and the ends of teaching activity. In both cases those ends are chosen by the community involved in a way half conscious and half involuntary.

In the case of the *teaching ends* we distinguish between: (1) those pursued by an institution (for instance, a society or primary education or the Degree in Mathematics); and (2) those pursued by a theory in didactics (for instance, ATD). In both cases, those teaching ends are to be coherent with the underlying postulates, with the assumed facts and values, of the institution or of the theory. Of course, these postulates and the ends may change throughout history.

2.3 Relative Autonomy of Didactics

Didactic phenomena exist, that is to say, there are phenomena which are genuinely didactic, in the sense that they essentially appear in the genesis, development, teaching-learning and diffusion (personal or institutional) of all kind of praxeologies (possibly regardless the split of knowledge into disciplines). Those phenomena *are ruled by didactic laws not reducible to psychological laws, sociological laws, semiotic laws, etc.* The existence of didactic phenomena make didactic a relatively autonomous science (as any other), devoted to the study of didactic phenomena.

2.4 Personal Praxeologies and Institutional Praxeologies

Personal praxeologies result from the history of the personal links to several institutions. Reciprocally, institutional praxeologies result from the interactions of many personal praxeologies coming from different individuals. This interdependence between personal and institutional praxeologies underlies the need to study them together. Due to methodological reasons, at the current state of didactic, ATD prioritise the study of the genesis, development and diffusion of institutional praxeologies, as its generality makes them more accessible to analysis.

2.5 The Praxeological Analysis as an Entrance to Didactic Analysis

ATD places itself in the so-called *Epistemological Programme of Didactic Research* (Gascón, 2003). This means that, to carry out a praxeological analysis, ATD explicitly makes and uses reference models both of the praxeologies related to the knowledge to be studied and of the corresponding didactic praxeologies. In the first case we call them *reference epistemological models* (REM), which account for *what* to study, and in the second case we call them *reference didactic models* (RDM), which account for *how* to study. Both kinds of models are to be included and integrated in the more general notion of *didactic paradigm* (Sect. 3).

2.6 Transpositive Phenomena Are at the Heart of Didactic Problems

Didactic cannot ignore that for a knowledge, coming from an academic institution I_1, *to be taught in a teaching institution* I_2, *must go through a transposition process to be adapted to the epistemological ecology of* I_2. However, the differences between the "academic" and the "taught" knowledge cannot be clearly set out and, moreover, they must be rejected in order to preserve the legitimacy of the taught knowledge. The interpretation of the transposed knowledge creates a strong tension between both institutions (Chevallard, 1985/1991).

The analysis of the transpositive adaptations of knowledge helps to break up with the transparency of the current models of the institutions and, in particular, it enables us to question the so-called *prevailing epistemological models* (PEM) in different institutions. Therefore, this analysis has strong consequences for the formulation of research problems and the adopted methodology in ATD.

2.7 The Minimal Unit of Analysis of Didactic Processes

The ATD states that the minimal unit of analysis of didactic processes includes all the phases involved in the didactic transposition (Fig. 1) (Bosch & Gascón, 2005). *Hence, ATD prioritises to what we could call "macrodidactic" processes.* The unit of analysis determines: (1) the theoretical framework used to formulate research problems; and (2) the empirical field in which we collect, analyse and interpretate data.

2.8 Relativity of Epistemological Roles and Purpose of the REM

There is no privileged epistemological position to be used as a universal reference to guide the praxeological analysis. This postulate leads to the use of provisional and relative REM when dealing with each didactic research problem. The construction of such a REM take into consideration the empirical data coming from all the phases of didactic transposition (Fig. 1). This collection of empirical data is essential to ATD methodology, but it is not sufficient in itself to construct a REM. Indeed, as it is the case in the production of any *scientific hypothesis*, in order to make a REM one needs to make some decisions: the choice and the reject of some didactic facts; the emphasis on some of the chosen facts; and to postulate certain relationships between them. Moreover, one might even consider also some aspects only exceptionally present (or even missing) in the starting collection of data. In conclusion, we could say that a REM is never the result of a process merely inductive (Gascón & Nicolás, 2019a).

A REM is an heuristic tool, with methodological aims, which enables us to draw attention to some phenomena which were previously invisible and unexplained. Thus, it is clear the *phenomenotechnical function* of the REM and their role as *devices for epistemological emancipation* in didactics (Gascón, 2014). Their first

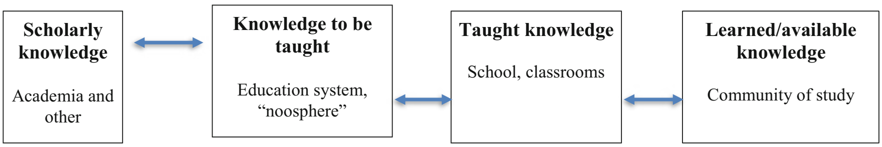

Fig. 1 Phases and institutions involved in didactic transposition

purpose is to provide the required elements to formulate didactic problems, the study of which will allow to enlarge the knowledge of those phenomena revealed by the REM.

2.9 Levels of Conditions Which Affect the Life of Praxeologies

The set of conditions which regulate the institutional life of praxeologies can be stratified according to the so-called scale of didactic codetermination (Fig. 2) (Chevallard, 2002).

For instance, in the case of the discipline *mathematics*, the conditions having an impact on the domains, sectors, themes and questions open to be studied, and on the possible ways of studying them, come from all the upper levels of the scale, including the most generic ones.

2.10 Object of Study and Admissible Research Results

According to ATD, didactics is a science which studies the conditions that govern the genesis, development and diffusion of (personal or institutional) praxeologies concerning the intentional enlargement of knowledge. That is to say, the science that studies the economy and the ecology of these praxeologies. Research results are expected to be didactic laws, *that is, laws describing certain phenomena concerning the economy and ecology of praxeologies.*

Didactic laws can only formulate statements about the rationally suitable *means* to achieve previously fixed *ends*, and about the (desired or not) consequences of certain actions. Research results in didactics *cannot state value judgement or prescribe norms*.

Fig. 2 Scale of levels of didactic codetermination

Humanity
↓↑
Civilisations
↓↑
Societies
↓↑
School
↓↑
Pedagogies
↓↑
Didactic systems
↓↑
Disciplines ↔Domains ↔ Sectors ↔ Themes ↔ Questions

In the study of research problems in didactics we deal in practice with three fundamental dimensions: *epistemological*, *economic* and *ecological* (Gascón, 2011). The construction of a REM is a tentative answer to questions linked to the *epistemological dimension* (the basic dimension of research problems in didactic). Hence, to make explicit a REM is a necessary step in order to formulate real research problems.

Didactic laws concerning the *economic dimension* describe how the praxeologies works at a certain institution, and they also describe which is the official rationale of those praxeologies in that institutions. Didactic laws concerning the *ecological dimension* state: (1) why certain praxeologies behave as they do at a certain institution; (2) what is required in order to modify those praxeologies in a certain way; and (3) the constraints for those modifications.

3 Teaching Ends and Didactic Paradigms Considered by the ATD

In what follows we will deal with the *teaching ends* fostered de facto by the ATD up to now, and with the epistemological model proposed by this theory. Specifically, we want to answer the following questions:

Q_2**(ii):** *Which is the relationship between the teaching ends (implicitly) assumed and advocated by the ATD, the normative prescriptions concerning teaching and the underlying epistemological and didactic praxeological models -that is, the conceptualisations made by the ATD of what and how to study-?*

In order to provide suitable answers we use the key notion of *didactic paradigm*. Before analysing this notion, we would like to point out that a didactic paradigm occurs at different levels: pedagogic, disciplinar and subdisciplinar (Fig. 2).

We will distinguish between: (1) the *current didactic paradigm* at a given institution, shaped by the society in which this institution is immersed and, more especifically, by the pedagogical ideal of that society; and (2) the *reference didactic paradigm* built by a theory to analyse the empirical reality.

3.1 Characterisation of a Didactic Paradigm

A didactic paradigm is made of an *epistemological model* and by certain assumed *educational ends* (formulated in terms of that epistemological model). Typically, a didactic paradigm also contemplates some *means* to achieve those *educational ends*, and hence, unavoidably, it promotes certain normative prescriptions. Frequently, a didactic paradigm appears as a reaction to some undesirable *didactic facts* and, in this sense, the educational ends can be regarded in turn as means to prevent these didactic facts.

When a theory in didactics assumes uncritically the current didactic paradigm at a given institution, this theory implicitly accepts the corresponding pervasive epistemological model of that institution, the educational ends, the means and the associated normative prescriptions. Thus, in order to build in an autonomous way its object of study, a theory in didactics needs to get rid of social impositions as much as possible, and, in particular, to get rid of the didactic paradigm prevailing in the institutions under consideration.

Consequently, to separate from the *paradigm of visiting works* (PVW) which currently in force in didactic institutions, ATD considers, as an alternative, the *paradigm of questioning the world* (PQW). The PVW presents to students certain works in an authoritarian fashion, leaving no room for questions about the corresponding rationale, utility for human beings, alternatives along history, etc. Both paradigms, the PVW and the PQW, live at the pedagogical level and they are *reference didactic paradigms* made by the ATD in order to understand the empirical reality.

3.2 On the Paradigm of Questioning the World (PQW)

The main educational ends assumed by the PQW are: (1) to promote a new cognitive ethos characterised by an open attitude towards knowledge, able to question well-stablished statements and to pose new challenges; and (2) to promote a learning not only based on the study of available knowledge, but also based on the inquiry aimed at the construction of new knowledge.

The *epistemological model* underlying the PQW put the emphasis on the *process of construction of knowledge*, rather than in the *knowledge already constructed*. This process starts with problematic questions concerning a certain system. In order to get answers, one makes a model of that system. Frequently, this model turns out to be provisional and leads to the construction of new models. New knowledge appears based on the successive models and, as it is the case with the models, also that knowledge is often provisional.

In the PQW, the proposed *means* consists of a kind of inquiry, typically analysed in terms of certain *dialectics* (Chevallard, 2007), aimed at the construction of a (temporarily) satisfactory answer to an initial question placed at the starting point of the inquiry.

3.3 On the Paradigm of Mathematical Modelling (PMM)

At the disciplinar-mathematic level, ATD assumes the PMM, provided by: (1) an *epistemological model* that identifies mathematics with *mathematical modelling*

(García et al., 2006); (2) the *educational ends*, which consist in passing on mathematical knowledge while showing the ability of mathematics to understand and master the reality via modelling activity; (3) the *means* to reach those ends, consisting in the so-called *study and research paths* (SRP); and (4) the *didactic phenomena* the PMM tries to fight consists of a vision of mathematics as a rigid activity, ritualised, with compartmentalised knowledge, algorithmic, dogmatic, etc. which prevents the students the practical use of mathematics in order to understand and master the reality (Gascón & Nicolás, 2019b).

The PMM is coherent with the PQW, but it aims to capture the idiosyncratic way in which mathematics help to answer questions about the world.

In conclusion, the relationships between research in ATD and the kind of teaching promoted by the ATD are strongly shaped by the assumed didactic paradigm (at the level of pedagogy, discipline and beyond). Didactic paradigms assumed by a theory are the link between the research made by the community that share this theory and the actions fostered by that community to achieve the educational ends included in those paradigms.

4 The Case of a Didactic Problem Concerning Real Numbers

In the workshop of the advanced course, we showed a problem about the teaching of real numbers in Secondary Education, a REM for that teaching, certain specific *educational ends* (formulated in terms of the REM), and certain *means* to lead the teaching towards the achievement of those educational ends. All this in coherence with the PMM previously explained.

4.1 A Teaching Problem Concerning Real Numbers, and the Institutional Answer

In what follows the fundamental reference is Licera (2017). This work deals with the following *teaching problem*: *What and how is to be taught in relation with real numbers in the last years of Secondary Education?*

After the analysis of the corresponding institutional answer provided by different school systems (French, Argentine, Chilean, Spanish) we verify that: (1) there are *technical problems* in the calculus with numerical approximations; (2) the *unlikeliness of the rationale* for the study of real numbers in Secondary Education; and (3) the only irrational numbers involved in calculations are *radicals* appearing in the realm of *exact measurements*. Remarkably, these empirical data coming from the school mathematics are compatible with the *axiomatic definition of real numbers*, ignoring the usefulness of that kind of numbers.

Concerning academic works in didactics on real numbers, they mostly focus on issues about representation of those numbers and the *understanding* of their basic properties. This leads to a question related to *educational ends*: are the students mainly aimed to *understand* the basic properties of real numbers or, rather, to *use them properly* to solve problems? In other words, should the study of real numbers be *an end by itself* or rather *a means* to solve certain type of tasks, revealing in this way their *functional* role in human beings' activities?

In coherence with the basic assumptions of ATD, before wondering *what to teach* and *how to teach* real numbers, one needs to identify the *institutional constraints* affecting a possible functional use in the last years of Secondary Education. This requires a questioning of the *prevailing epistemological model* (PEM) for real numbers and, consequently, the construction of an alternative *reference epistemological model* (REM). In parallel, there is the need to provide teachers with tools to analyse the (mathematic or para-mathematic) status of real numbers and to manage their teaching. Taking this need into account, the problem on the teaching of real numbers is placed in an institutional scope including not only Secondary Education but the devices devoted to *teachers education*.

4.2 A REM on Real Numbers for Teachers Education

Once the problem has been linked to the realm of teachers education, Licera (2017) detects three problems the REM should deal with:

1. *The problem of a rationale for real numbers*: Which are the types of tasks that can be formulated without mentioning real numbers and that can be optimally solved by using them? Which are the reasons behind the different constructions of real numbers?
2. *The problem of measurement*: Which is the family of numbers that provide the best account for the correspondence between amounts of magnitudes and measurement? Which are the techniques that allow to compare amounts of magnitudes or determine those amounts obtained via the operations of *union*, *subtraction* or *splitting*?
3. *The problem of techniques*: How to develop techniques suitable for the indirect calculation of measurements by using bounded decimal numbers? How to control propagation of errors?

4.3 Economy of Real Numbers in the Step from Secondary to Tertiary Education

The REM deals also with the questions corresponding to the three fundamental dimensions of a didactic problem. Concerning the *economic dimension*:

1. How are real numbers currently taught in Secondary Education?
2. Which are the relationships stablished between real numbers, measurement activities and calculations with numerical approximations?
3. Which is the official rationale?

The answers provided in Licera (2017) can be summarized as follows: (1) real numbers are taught together as an 'incomplete' mathematical praxeology: (a) there are theoretical elements dissociated from the considered types of tasks, and (b) there are no techniques and technological discourses to deal with the problems of numerical approximations and propagation of errors; (2) irrational numbers are identified with a writing (non-repeating unbounded decimal numbers or points of the real line); and (3) there is no rationale for real irrational numbers other than Pi or radical numbers.

Moreover, the REM brings to light two didactic phenomena:

The divorce between numbers and measurement of continuous magnitudes In Secondary Education, measurement activities are missing. The problems about approximation of the measurement of magnitudes and about errors are ignored. The field of numbers used for activities is reduced in practice to rational numbers.

The phenomenon of avoiding irrationals In Secondary Education the problems concerning the use of irrationals is avoided. There are three strategies to avoid irrationals: (1) to identify an irrational number with a rational approximation; (2) to replace it with an arbitrary approximation; and (3) to let the operations indicated but undone.

4.4 Ecology of Real Numbers in the Step from Secondary to Tertiary Education

Concerning the ecology of real numbers the following questions are considered:

- Why the teaching of real numbers is as it currently is in Secondary Education?
- Which are the constraints for a change in the direction suggested by the REM?

Of course, the current teaching of real numbers is as it is because it is immerse in the teaching prescribed by the aforementioned PVW, which is intimately compatible with a finished and static and axiomatic presentation of real numbers, appearing out of the blue, whose basic properties must be studied, and disconnected from any possible source of functionality (such as the scope of activities related to magnitudes and measurements).

5 Towards a Third Step of the Dialogue Between Theories

Many research problems in didactics are concerned about means: *is this a good means for teaching*? Form our view, in this kind of research there is a deep limitation: given that the educational end remains implicit in the formulation of the problem, the question cannot but belong to what Weber called "the sphere of values" (Weber, 1917/2010).

In coherence with the basic assumptions of ATD, the research problem should be formulated as follows: *is M a suitable means to achieve the educational end E?* In agreement with Postman (1995), we think that the possible ends of the education should be openly considered, and the ways of teaching should always be analysed in the light of an explicitly declared educational end. On the hand, as we said before, teaching ends are always shaped by and included in that broader thing called *didactic paradigm*. This is why we propose a third step of the dialogue between theories, in order to deal with the following issues:

- $\boldsymbol{Q_3}$**:** *Are compatible the didactic paradigms assumed by the different theories in didactics? Otherwise, to what extent are the different theories working inside the same discipline?*

In answering these questions we will be forced to make explicit the assumed teaching ends together with the underlying epistemological model. Only in this way we will be ready to move forward in the dialogue between theories.

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From the Networking of Theories to the Discussion of the Educational Implications of Research

Michèle Artigue

1 Introduction

As explained in the introduction of this book, the advanced course on dialogue between theories proposed at the CRM in June 2019 was part of a long process that began with questioning the relationship between the field of mathematics education as a field of research and normativity. The reflection benefitted from the contributions of a panel of researchers and its main results were presented in Gascón and Nicolás (2017). These show that all researchers do not share the same view of this relationship and that the didactic theories they use and to whose development they contribute undeniably influence their responses. When I participated in this first study, I did not position myself as the representative of a particular theory. I preferred to explain how my experience as a researcher grown up in a particular didactic culture and my professional experience more broadly, have conditioned my view of this relationship. In this contribution, too, I do not act as representative of a particular theory. Rather, In this chapter, I try to put the knowledge and experience I have gained by working for about 15 years on what is now called the "networking of theories" at the service of the dialogue between theories and the reflection on their influence on the vision of the educational implications of research. I defend, in particular, the thesis that conceiving theories as components of research praxeologies and taking into account the reality of the work of researchers who, more often than not, combine different theoretical inputs, helps establish a productive dialogue between theories and to work on the issues addressed in the course. In this text, therefore, I begin by introducing the key notions of research praxeology and scale of networking strategies. Then I draw some lessons from my experience of networking, before addressing the relationship between research and educational action, and the

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role of theories in it. I end with a brief report on the contribution to the reflection offered by the seminar associated with the lecture, focused on a particular domain, Algebra, extensively investigated in mathematics education with a variety of theoretical approaches.

2 Theories as Components of Research Praxeologies

In mathematics education, various definitions of theories coexist (see, for instance, the contrast between the structural definition proposed in Niss (2007) and the operational vision proposed in Radford (2008). What is called theory is also highly variable according to researchers and contexts. It goes from very local constructs and distinctions to systems of concepts organized in coherent structures, from constructions mostly 'home-grown' to constructions mostly 'borrowed and adapted' from other fields, from constructions having emerged decades ago to quite recent elaborations. I plaid here for a pragmatic and operational vision of theories. A possible approach is to consider these as components of research praxeologies. Such an approach was first introduced at the third TAD Congress in 2009, then presented at CERME 7 (Artigue et al., 2011). Since that time, it has been refined and its potential for the analysis and comparison of theories made clear (see for instance (Artigue, 2019; Artigue & Bosch, 2014).

As any form of praxeology in ATD, research praxeologies are quadruplets [t, T,θ,Θ], with a praxis block made of the different types of tasks t that the research work asks for and associated techniques T, and a theoretical block made of the technological discourse θ used to describe, justify, interpret both research practices and their outcomes, an a theoretical discourse consisting in "statements of a more general and abstract character, with a generally strong justifying and generating power" (Bosch & Chevallard, 2020). As other praxeologies, research praxeologies are dynamic entities, the technological discourse playing an essential role in the dialectic relationship between their praxis and theoretical blocks which is the source of this dynamics.

Research praxeologies are very diverse as are the types of tasks that the research activity requires. However, there are some emblematic research praxeologies, and especially the following basic one consisting of a research question, a technique or method used to address it, the methodological discourse justifying this technique (note that we recover here the etymological meaning of the world methodology: a logos about a method) with associated constructs, and a theory which serves as a background for this discourse and more globally the whole research work. Of course, this is a very simplified model. Most often researchers in mathematics education do not try to answer one single question, and even in this particular case they generally combine several research techniques. Most often, too, their research theoretical framework is not reduced to a single theory, but combines theoretical constructs coming from different sources. This is understandable considering that, when they address a particular question, researchers cannot emancipate from the state of the art of research in the area at stake, and thus have to take into account pieces of knowledge often obtained through research praxeologies relying on different theories, and whose formulation itself uses theoretical constructs alien to their own theoretical discourse. At least, some conversion work would be necessary to incorporate these in their theoretical discourse. However, this is not an obvious task and the risk of denaturation of the initial constructs is real. To this adds that theoretical approaches in mathematics education can be themselves the result of theoretical combinations.

Most often, research praxeologies emerge at the praxis level, with questions to study; however the vision of what count as a valuable research question and its precise formulation is influenced by the 'theoretical' already there in the environment of the researcher or the research team. Study techniques are generally inspired by those implemented in close research praxeologies, or familiar to the researcher or the research team; however, experience shows that their implementation always requires some adaptation to the particular question at stake, and thus some creativity. These adaptations and associated constructs, for instance in terms of categorizations, contribute to the technological discourse and the praxeological dynamics. This dynamics is also generated by the research results, and the work carried out for their interpretation, through the new questions generated, the didactic phenomena identified or constructed. New constructs enter thus the technological discourse, which, at a later stage, will be incorporated into the theory if their interest becomes 'reasonably' acknowledged and shared. Practically, praxeological development combines horizontal and vertical dimensions, with both the building of new point praxeologies and their progressive organization into local research praxeologies, then regional praxeologies. In fact, a well developed theory always operates at a regional level, unifying in some sense a diversity of research praxeologies to which contribute researchers from different backgrounds, with different research interests, living in different research and educational contexts, and also with different visions of the relationships between research and educational action shaped by their different institutional subjections and personal experiences. I will come back to this point later on, but move now to the introduction of a second conceptual tool: the scale of networking strategies.

3 The Scale of Networking Strategies

This scale was established in the first steps of the networking enterprise. As explained in Bikner-Ahsbahs and Prediger (2008), it aims at showing the diversity of forms that connections between theories can take, and at ordering these between two extremal positions expressing respectively a total absence of relationship and a global unification. As shown in Fig. 1, the scale distinguishes eight intermediate positions according to the degree of integration.

These positions appear in a linear order, structured into pairs such as understanding others and making understandable, comparing and contrasting, etc. The precise

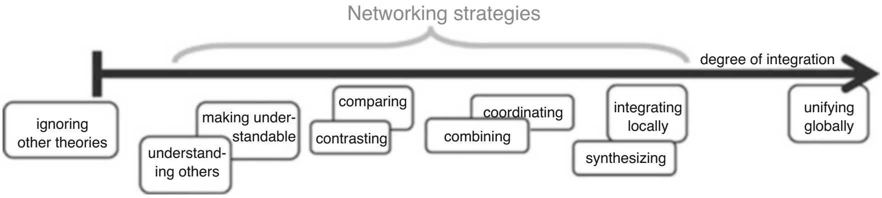

Fig. 1 Networking scale (Bikner-Ahsbahs & Prediger, 2008, p. 492)

meaning of these terms is clarified by the authors, pages 492–497. For instance, it is pointed out that coordinating means that "a conceptual framework is built by well fitting elements from different theories" and supposes the complementarity of the theoretical approaches involved, while combining means that "the theoretical approaches are only juxtaposed according to a specific aspect". The combining strategy can thus involve theories with some conflicting basic assumptions. The scale proposes a linear order, but "it must be emphasized that it is not easy to specify globally their exact topology, since the degree of integration always depends on the concrete realizations and networking methodologies" (ibidem, p. 492). Moreover, researchers who try to connect theories usually combine several strategies. I will add that, for all those involved in the networking enterprise, a unified theory of mathematics education is not the Holy Grail they are pursuing. On the contrary, they are convinced that theoretical diversity is a normal state for this scientific field, and that diversity should not be interpreted as a sign of scientific immaturity.

As stressed in the introduction, for more than one decade, these tools have been effectively used to compare theories, the research practices they nurture and their outcomes. Personally, I have been involved in two major projects carried out in this direction, the ReMath European project (Kynigos & Lagrange, 2014) which followed TELMA (Artigue, 2009), and the Bremen group project (Bikner-Ahsbahs & Prediger, 2014). In the next section, I summarize the lessons that I find most useful to draw from these projects for the discussion.

4 Some Lessons from the Networking Enterprise

The first essential lesson from these projects is the limitation of mutual reading of selected texts and discussions for comparing theories and capitalizing on the outcomes of associated research. This limitation became clear in the collaborative activity carried out in the European research team TELMA of the Kaleidoscope European network of excellence, with the aim of identifying the knowledge accumulated through European research on technology enhanced learning in mathematics. This limitation made clear the necessity of creating specific research practices making it possible the collaborative work on our respective research practices, taken as object of inquiry. From this emerged the idea of methodology of cross-experimentation, later systematized and better conceptualized in the ReMath project whose ambition was to build an integrated vision of the semiotic potential of Dynamic digital artefacts (DDA) (Artigue & Mariotti, 2014). This necessity was also taken into account in the work of the Bremen group, which started from the analysis of a single video by representatives of five different theories (Abstraction in context (AiC), Action, production, communication (APC), ATD, Interest dense situations (IDS) and the theory of didactic situations (TDS)) and their comparison, and continued, over several years, with the progressive development of and collaborative work on a series of research questions leading to explore all strategies identified in the scale of networking, at the exception of the two extreme ones.

The second lesson is the difficulty that many if not most researchers meet at operating the decentration and moves in posture necessary to make sense of other theories without denaturing them; thus, the importance of creating an “antagonist milieu” with the meaning given to this term in TDS, in attempts made to compare and connect theories. This has been achieved in ReMath through the development of specific research techniques such as the design of a common research question then rephrased by each team in each own discourse and complemented by research questions representative of its own interests and approach, or the technique of cross-experimentation making that the same DDA was engaged in two substantial experimentations (one carried out by the team developing the DDA and the other one by another team from another country and with a different theoretical background), and the systematic organization of case studies crossing the analyses of these experimentations. This was also the case in the group of Bremen as attested by the case studies reported in Bikner-Ahsbahs and Prediger (2014). I have experienced how the good functioning of the antagonist milieus so created, favored by the careful design of the tasks themselves and of their conditions of realization, and also the friendly and critical spirit atmosphere of work among researchers that was established in the two projects, helped limit the risks of misunderstanding and denaturation. In the advanced course on theories at CRM, also, we can consider that the substantial group work asked from participants after the conferences has created opportunities for the constitution and exploitation of such an antagonist milieu.

The joint work of researchers with different theoretical backgrounds on the same questions and data also helped overcome the naturalization of constructs which goes along with the development of a theory in a given community. In the work of the Bremen group, a good example is provided by the case study reported in Bikner-Ahsbahs et al. (2014) investigating why, looking at the same short video episode, French, German, and Italian researchers immediately identified respectively a Topaze effect, a Funnel pattern and a Semiotic game. The first two interpretations conveyed a negative vision of the episode from a didactic perspective while the third one conveyed a positive vision. These contrasting interpretations led to a work of denaturalization of these three didactic phenomena, and a process of deconstruction/reconstruction of each of them. It also made clear the complementarity of the views of the episode offered by the Topaze effect and the Funnel pattern, linking this complementarity to the fact that the two phenomena respond to a similar necessity: to give account of the common fact that the fiction of learning has to be maintained in classrooms, but do so within the logic of their respective theoretical inscriptions, TDS for the Topaze effect and Interactionism for the Funnel pattern.

The third lesson I want to mention is the necessity of questioning the exact role played by theories in research praxeologies, beyond the idyllic image presented in research publications for obvious reasons. This was a crucial question in ReMath and it was worked out both regarding the development of the six DDAs that took place in the project, the vision of their didactical functionalities, the associated experimental designs and their a priori and a posteriori analyses. It must be added that the methodological tools designed for this part of the study allowed researchers to distinguish between metaphorical and operational uses of theories. The data collected and their analysis led to relativize the importance of the control by theories of the practical research work, showing the importance, especially in the design of DDAs and scenarios of use, of pragmatic decisions taking into account contextual conditions and constraints, cost issues, familiarity, etc. It was also shown that the use of theories could remain mainly at a metaphorical level, through the reference to principles and ideas.

The last lesson I will mention in this text, is the difference in the conditions to be satisfied for productive networking for analysis and for design. The different case studies developed in ReMath and in the Bremen group showed the potential offered by different types of connections between theories for increasing our understanding of the complexity of learning and teaching processes. Moreover, these projects also showed that the distance between theories is not an obstacle per se if the networking activity is carefully thought and implemented. Combinations and complementarities can be looked for between theories with very different underlying principles, as admitted in the scale of networking strategies. This has been confirmed by further research, for instance the efforts carried out to compare and connect two theories as different as ATD and APOS (Bosch et al., 2017). However, ReMath also showed that, regarding design, building productive connections is not so easy. A clear case was provided by the case study about the DDA Cruislet, a microworld allowing to pilot airplanes above a map of Greece, designed by the Greek team relying on Constructionism. For Cruislet, the alien team was the French team DIDIREM (now LDAR), thus my laboratory, with a vision of design shaped by the concept of didactical engineering and its foundation in TDS, but incorporating also the concerns and tools of the Instrumental approach in research regarding digital environments, as was the case in ReMath. The contrast between the two experiments, the difficulties initially met by the French team at designing a sequence of situations in line with TDS principles and constructs, using Cruislet productively, made visible the fundamental difference between the constructionist and TDS logics of design, despite the shared influence of Piagetian constructivism on their respective vision of learning processes, as explained in Artigue and Mariotti (2014). In fact, despite the epistemological proximity between ATD and TDS, a similar phenomenon can be observed for these theories. The logic underlying design in terms of SRP, now the predominant form of design promoted by ATD, is not the same as the logic underlying design in TDS. These considerations make a natural transition with the next section devoted to the relationships between research, theories and educational action, an issue not directly addressed in the projects just evoked.

5 Research, Theories and Educational Action

As recalled in the introduction of this book and chapter, the research undertaken on normativity issues (Gascón & Nicolás, 2017) has evidenced the diversity of existing 'formal' views regarding the possible normative role of didactic research among researchers, and questioned the possible role of theories in the diversity observed. Most researchers involved in this inquiry were either at the origin of the corresponding theories or in the first circle of their main contributors. Thus it is legitimate to consider that the corresponding theories shape their views of the possible normative role of didactic research. However, my personal reflection and experience, my vision of theories and of their collective development, of research practices, such as expressed above, leads me to consider that the link is not necessarily so straightforward. A priori, normative views of research, if any, should be part of the principles in the background of theories. However these principles, even when made explicit which is not at all always the case, hardly take such a form. They more consist of general assertions positioning the theory regarding the vision of mathematics learning and teaching, or the vision of the mathematics discipline itself; they try to explicit its ambition and scope, or its links with other theories in the field itself or outside it. Moreover, theories are dynamic entities, and, even when attached to the name of a particular researcher, they develop thanks to the collective efforts of communities and the contribution of many different researchers, as stressed above. What shapes most the vision they have of normativity? The theory they mainly use and contribute to and its underlying principles, or other dimensions of their professional and personal life? How does their vision impact their research practices and their outcomes, and the theory itself in return?

In fact, reflecting on these issues, and more globally on the relationship between research and educational action or, formulated differently, on the educational implications of research, I find useful to make a distinction between a normative view and a transformative view of didactic research, certainly more generally shared. As many researchers, I deny a normative aim to didactic research for different reasons linked to my conception of science, to characteristics of this field of research, its nature and state of development (Artigue, 2017), and also because I cannot avoid to see a relation between normative visions of research and the over-valuation of evidence-based research practices considered as the only ones able to provide scientific results and thus a solid foundation to educational action and norms. However, this is not by chance that I have engaged in this field or research and worked in it for nearly five decades. As for most of my colleagues, this is because I am convinced that the knowledge gained through didactic research can and should help improve the current state of mathematics education, while being aware that what is considered an improvement of mathematics education is not just a matter of scientific judgement.

I agree thus with a transformative view of didactic research, not with a normative view, and I also accept that the view I can have, at a given moment, of the educational implications of research outcomes is not just a matter of science but also a matter of values, which should be made more explicit.

However, the distinction introduced above, while useful, does not solve the problem, and, once again, an institutional perspective is helpful. What I have just expressed is a formal position. Practically, didacticians belong to different institutions where they occupy different positions. They are researchers in laboratories, carrying out fundamental and applied research and publishing their results where, quite often, they feel obliged to include implications for teaching, unfortunately often over-generalizing the lessons that can be drawn from the limited study they report. Most of them are professionals engaged in teacher education and professional development, and in the context of these activities, their discourse generally does not fully escape normative assertions, whatever is their formal position. Many of them also play the role of experts in national or international commissions and institutions advising governmental institutions; they are involved in curricular design, in the writing of official curricular resources, of textbooks, in the production of digital educational artefacts, etc. These different subjections and positions result in a diversity of discourses that, at times, take a normative dimension, which can be in contradiction with expressed formal positions. Moreover, normative or not, these discourses influence external and internal didactic transposition processes, often in unpredictably ways.

Such problems have been early identified, especially in relation with the reproduction of didactic engineering, but not so much taken as objects of study by the research community at that time. As I pointed out in Artigue (2017), some evolution is today visible, fostered by unexpected influences of didactic research, by institutional and social pressure and the conditions made to scientific research, and by the theoretical evolution of the field itself providing new conceptual tools and approaches to address the complex issue of relationship between didactic research and educational action. However, very much remains to be done to understand how to develop more productive relations between research and educational action, and how theoretical work can contribute.

In the next and last section, I illustrate how the tools introduced so far were practically used in the seminar to reflect on research practices and their outcomes, their relationship with theories, and potential educational implications.

6 Recovering Research Praxeologies and Relationship to Educational Action from Publications

The seminar associated to the lecture focused on the following question: what access to research praxeologies and the relationship between research and action do research publications allow? As mentioned in the introduction, a mathematical theme, Algebra, was selected and the researchers in charge of presenting the different theories engaged in the course were asked to proposed one or two publications for the seminar work. All positively answered, but the references sent for APOS and ATD were difficult to accommodate within the constraints of such a seminar, and eventually only two publications were discussed in the seminar, involving respectively TDS and OSA. Participants, organized in small groups, were asked to select one article and work on the following questions:

- How to describe the work presented in this article and its results in terms of research praxeology?
- Can I identify resonance, complementarity, incommensurability, between this research praxeology on algebra and those I am the most familiar with? How do these depend on the theories at stake and associated principles?
- Does this article explicitly address the question of didactic action and if so, how? Would I draw similar implications?

Due to limitation of space, I only present below, in a synthetic way, the main outcomes of the work carried out on the TDS text (Barallobres & Bergeron, 2019) presented in a congress on inclusion and diversity held in Granada in April 2019, and focusing on the teaching of algebra to students with learning difficulties. I reproduce below its abstract (p. 1, original version in Spanish):

> The reduction of the level of complexity of mathematical knowledge to be taught is one of the phenomena observed in classes with students who have learning difficulties. This reduction is carried out without any epistemological vigilance regarding the nature of the resulting mathematical activity. One of the forms of this reduction is what we have called the concretization of knowledge (Barallobres, 2016). In this article, we will describe some of the forms that this concretization takes, particularly in the context of algebra teaching, and the impact that it can have on the nature of the mathematical practice to which students are introduced. We will also investigate the way in which the usual conception of abstraction conditions the analysis of difficulties in mathematics and thus contributes to the process of concretization of the knowledge taught. Finally, we will present some examples of situations developed for the teaching of algebra to students with severe learning difficulties, experienced for several years in a school in Montreal.

The text makes explicit three research questions:

- What forms does the phenomenon of concretization take in the teaching of algebra?
- How does the usual conception of abstraction condition the analysis of difficulties in mathematics, with what limitations ?
- Can didactic strategies be developed allowing to overcome these limitations?

The text does not provide information allowing the reader to reconstruct a research praxeology for the two first ones. However, the third question visibly emerges from the results of these, which contribute to the theoretical block of the associated research praxeology. Two hypotheses indeed guides its development. H1: Reducing the level of complexity of the tasks denatures the mathematical object at stake, hindering access to the different levels of abstraction and generalization necessary to the constitution of algebraic thought. H2: Abstract objects can become concrete through the frequentation of varied didactic situations that allow students to perceive their usefulness.

The technique used in this research praxeology is didactic engineering, and it is explicitly connected by the authors with TDS. This didactic engineering is of substantial size (15 sessions of 70 min). The text makes visible classical steps in the implementation of this technique (a priori and a posteriori analyses), and the influence on these of the tools provided by the theory, such as the notion of didactic variable, the organization of the design around families of situations obtained through the controlled variation of didactic variables... However, nothing is said regarding the validation process and how the contrast between a priori and a posteriori analyses is organized, on what data it relies, how these are analysed... The theoretical block of this research praxeology combines different elements: TDS of course, but also an epistemological vision of the field of algebra which is only partially influenced by TDS, and the background provided on the one hand by existing research on the didactic of algebra, and on the other hand by research regarding students with special needs and the critical analysis of it.

We observe thus here an example of research praxeology in which the interaction between the praxis and the theoretical blocks is highly visible, for instance through the move from a vision in terms of general cognitive characteristics of students to characteristics of the learning situations they are exposed to, which guides the formulation of the research question and is paradigmatic of research praxeologies relying on TDS; through the adaptation of the didactic engineering methodology piloted by TDS to the particular context at stake, with the importance attached to the role and organization of the teacher mediation. And, as if often the case for such types of research praxeology, it potentially leads to an ‘existence theorem’ of the type: “Under these conditions, such effect can be obtained” or “Under these conditions, such didactic organization becomes ecologically viable and produces such effects”, positively answering the research question at stake. However, as can be also observed, the limited space allowed for such a substantial design makes that the description is quite incomplete. More would be necessary to appreciate the precise outcomes of this research work.

In this text, there is no doubt that the didactic intentionality is clear and explicit, and permeates the whole research praxeology. However, this didactic intentionality is, by no ways, at the service of some normative aim; rather, this research work can be interpreted as a resource to oppose to the normative consequences of research inspired by influential cognitive perspectives, for a population of students deserving our full attention, because of their fragility. Moreover, we cannot say that this didactic intentionality is aimed at by the use of TDS, which functions more as an instrument to work out the hypotheses made and to show that another mathematical world is possible for these students.

I cannot enter into more details, and just will add that this synthetic description of one single example, however, does not pay justice to the vivid discussions that took place in the seminar. These confirmed the fact that making sense of research activities inspired by other theoretical perspectives than those we are familiar with, without being too much oriented by our own theoretical lens, requires substantial efforts of decentration, but that these are necessary for productively discussing possible connections and complementarities between research works and their outcomes. It confirmed the possibility of establishing connections with research carried out under other theoretical perspectives, but also the fact that the didactic engineering designed obeyed a different logic from those associated to ATD or APOS in the domain of algebra.

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Theory of Didactical Situations in Mathematics: An Epistemological Revolution

Claire Margolinas

1 Scientific Foundations of Theory of Didactical Situations in Mathematics

1.1 Brousseau's Scientific Project

In the late seventies Brousseau played a leading role in the development of '*didactique des mathématiques*' as a scientific discipline. He stated the necessity to consider a "didactical variety" of concepts:

> the study of (didactical) situations must in the end allow the derivation or modification of the necessary concepts currently imported from other scientific fields (Brousseau, 1997, p. 24)

As examples of those fields Brousseau considers for example linguistics and psychology but also mathematics themselves.

His definition of mathematics didactics is thus:

> Didactics of mathematics
>
> Is the science of the specific conditions for the diffusion of mathematical knowledge necessary for human occupations (in a broad sense).
>
> It deals (in a restricted sense) with the conditions under which a 'teaching institution' tries to modify the knowledge of a 'studying institution' when the latter is not in a position to do so independently and does not necessarily feel the need for it. (Brousseau, 1998, p. 1–2, my translation)

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1.2 The Ambition to Understand Teaching and Learning Mathematics

The focus of the French community of research on the understanding of the teaching and learning of mathematics is still an important feature of the aim of our research:

> Didactics, when it was set up as a field of research, was not constituted with normative ambitions, especially in France. [. . .] Research was built on the conviction that priority should be given to understanding the functioning of didactic systems, to clarify the processes of teaching and learning of mathematics which alone could be the basis of reasoned action. (Artigue in Gascón & Nicolás, 2017, p. 4)

Considering the definition of "basic research" by the International Council for Science, didactics of mathematics is considered by Brousseau and Artigue as a basic research:

> Basic scientific research is defined as fundamental theoretical or experimental investigative research to advance knowledge without a specifically envisaged or immediately practical application. It is the quest for new knowledge and the exploration of the unknown. As such, basic science is sometimes naively perceived as an unnecessary luxury that can simply be replaced by applied research to more directly address immediate needs.
>
> However the demarcation between basic research and applied research is not at all clear cut. In reality they are inextricably inter-twined. Most scientific research, whether in the academic world or in industry, is a hybrid of new knowledge generation and subsequent exploitation. Major innovation is rarely possible without prior generation of new knowledge founded on basic research. Strong scientific disciplines and strong collaboration between them are necessary both for the generation of new knowledge and its application. (International Council for Science, 2004, p. 1)

This strong collaboration is considered as very important by Brousseau himself since:

> Only the observation of the singular phenomena that govern the acquisition of knowledge in the conditions specific to them can lead to an understanding, an explanation, and perhaps an improvement of the learning and teaching of mathematics
>
> The theoretical understanding of the way situations work is the aim and not the means to attain a practical goal. (Brousseau, 1975, p. 2 my translation, cited by Perrin-Glorian, 1994, p. 101 my translation)

Brousseau has founded in 1973 the Centre for Observation and Research in Mathematics Education[1] (COREM) that involved an entire school for more than 25 years (until 2000). Some of its operational principles are very important:

> Many people today are inquiring into the relationships among theories, research methods, experiments, results and the practices of teachers. Perhaps my account might be of assistance to them. For example, our observations consisted of watching ordinary classes. But beyond that, the observation school [COREM] made it possible to modify the teaching conditions and observe the result. We learned more about mathematic education from what we had to do in order to observe classes than we did from the observation itself.

[1]COREM : centre d'observation et la recherche sur l'enseignement des mathématiques, see http://guy-brousseau.com/le-corem/

> Another example: in our experimentation, we did not compare the results of the students to determine whether one method was better than another. Instead we restricted ourselves to having the results be equally good, despite the modifications we made, and compared the efforts required by the students and teachers in each case. (Brousseau, 2004, p. 245)

Gascón and Nicolás (2017) has shown that the debate between theories about the very aims of the scientific research has not reach a consensus in mathematics education. For me, a theory is strong and useful if it is coherent, if it is resilient to development of new parts and if it is able to anticipate a significant part of empirical results gathered by researchers that are not claiming this theory as their theoretical framework. For this reason, the understanding of what is at stake in learning and teaching mathematics is an essential part of research and not an 'unnecessary luxury' (in agreement with the International Council for Science, see above).

The role of observations has been in the case of Brousseau's work strongly linked to the existence of the COREM, where different types of observations has taken place over the almost 30 years of its existence. At first, the observations were only related to didactical engineering and was directed mostly toward the understanding of student's procedures and knowledge and their evolution. Gradually, teacher's work has also been observed, using various original experimental settings (in particular in Julia Centeno's doctoral thesis, see Brousseau & Centeno, 1991; Centeno, 1995).

The evolution of TDS due to the work of other researchers has been based on the observation of ordinary teaching (without any intervention of the researcher as proponent of engineering resources, e.g. Hersant & Perrin-Glorian, 2005; Margolinas et al., 2005).

In the following sections, I will develop the TDS concepts I consider as the most important in order to understand learning and teaching mathematics.

2 Epistemological Principles

2.1 Theory of Mathematical Situations (TMS)

In his conference at ICME Copenhagen Felix Klein medallist, Brousseau, (2004) introduces the principles of the Theory of Mathematical Situations (TMS):

> A Theory of Mathematical Situations—Why?
>
> I made the assumption that
>
> – to every piece of mathematical knowledge there corresponds a collection of Situations which can be resolved using this knowledge and reciprocally that
> – in any real environment of a student it is possible to choose elements of one or more Situations that make it possible to identify the knowledge being brought into action by the student's actions. (Brousseau, 2004, p. 250)

Those 'assumptions' or rather principles, are central in the theorization of Theory of Situations, they have many aspects and consequences that I will now examine.

2.2 Situational and Institutional Knowledge

In order to better understand those principles we need the distinction made by between two dual dimensions of knowledge. This distinction is not very easy to understand and I will share here my own ways to explain those two dimensions of knowledge (Margolinas, 2014).

In written societies (Goody, 1986), the transmission of knowledge is based on written texts: in a given institution, knowledge is written, which is particularly true in mathematics, a discipline that has a very strong relationship with a specific form of writing. The term that is associated by Brousseau to this form of knowledge in French is 'savoir' and in Spanish 'saber'.

However, if a knowledge does exist it is because it has been recognised as useful in various situations (Conne, 1992). The usefulness of knowledge and the situations in which this usefulness is revealed most often disappear from the text of knowledge. Moreover, this form of knowledge is often implicit and cannot readily be expressed in situation. The term that is associated by Brousseau to this form of knowledge in French is 'connaissance' and in Spanish 'conocimiento'.

In English there is only one term: 'knowledge'. Many attempts of translation has been made (Geiger et al., 2017, 2018) by different researchers (in particular N. Balacheff, M. Cooper, R. Sutherland, and V. Warfield translators of Brousseau (1997). The propositions made in different texts in English that I have read are the following: to leave the French word "savoir" in the English text, to save the term 'knowledge' for 'savoir', to add a letter 's' for '*sapere*' (in Latin) 's-knowledge'; for the French word "connaissance", Balacheff M. Cooper, R. Sutherland, and V. Warfield have coined the noun "knowing" (1997, p.72), and you can also found 'c-knowledge' (from *conoscere*). Those translation's propositions have in common to propose two nouns in English for the nouns 'savoir' and 'connaissance', no translation is yet widely used.

I have been looking into another way to translate these terms: in English, it is not so common to distinguished some linked concepts using different nouns and it is more common to use different adjectives with the same common noun. Since 'savoir' is linked to the formalization of knowledge within an institution (for example mathematics as an institution) I have proposed to translate 'savoir' by '*institutional knowledge*'. Since 'connaissance' is linked to the usefulness of knowledge in situations I have proposed to translate 'connaissance' by '*situational knowledge*'. In English this proposition allows the use of the general term 'knowledge' when it is not necessary to distinguished between institutional and situational knowledge, which is very useful, and difficult in other languages, for example in French or Spanish.

We can now reformulate the principles enunciated above by Brousseau:

– To every piece of mathematical *institutional knowledge* there corresponds a collection of situations which can be resolved using this knowledge *as situational knowledge* and reciprocally that

– In any real environment of a student it is possible to choose elements of one or more situations that make it possible to identify the *institutional knowledge* that correspond to the *situational knowledge* being brought into action by the student's actions.

This reformulation reveals a complexity that was hidden by the use of the single word 'knowledge'. There is always a circulation between those aspects of knowledge: situational knowledge is institutionalized in mathematics institution and thus is transformed into institutional knowledge. Institutional knowledge is supposed to give the power to act in situations and thus is transformed into situational knowledge. Institutional knowledge is easier to determine since it is written, it is organized as a "body of knowledge" (Chevallard, 1989, p. 7) however, situational knowledge is always an observer's interpretation of somebody's actions. Any piece of knowledge involves potentially these two facets, no knowledge is only institutional or only situational. However, in a determined institution, during a certain historical period, a piece of knowledge may present mainly one aspect or the other.

3 Interactions Between Institutional Knowledge and Situational Knowledge: An Example in the Context of Didactic Engineering

3.1 Didactic Engineering

The first principle (see above) is very important for didactic engineering: to every piece of mathematical *institutional knowledge* there corresponds a collection of situations which can be resolved using this knowledge *as situational knowledge*.

In Brousseau's perspective, didactic engineering lays in the *correspondence* between a *sequence* of situations and an institutional piece of knowledge which is the final aim of those situations. Interactions between institutional and situational knowledge work both ways in order to build a didactic sequence of situations.

3.2 From Institutional to Situational Knowledge in Didactic Engineering

In this text I will develop an example about cardinality. The teaching of early number knowledge involves lots of aspects: memorization of oral number sequence, relationships between successive numbers, etc. We will focus here on discrete quantity, that is one of the aspects related to the general concept of cardinality.

From a large mathematics perspective, two sets have the same cardinality if there is a bijection between those two sets. For example, the existence of a bijection between $\mathbb{N}$ and $\mathbb{Q}$ demonstrates the countability of rational numbers set, the existence

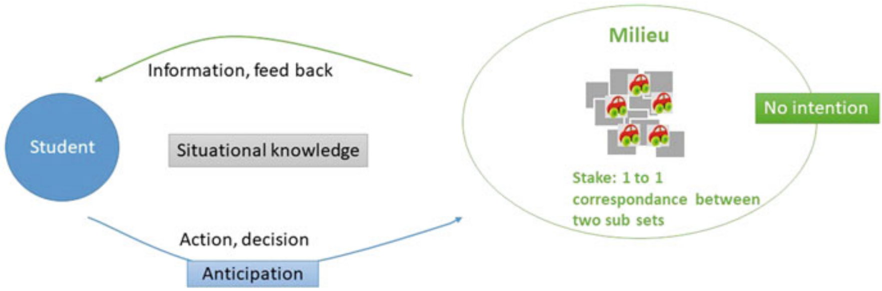

Fig. 1 Schema of action situation

of a bijection between]0;1[and $\mathbb{R}$ demonstrate the uncountality of an open interval of $\mathbb{R}$, etc.

For finite sets, bijection is often named one-to-one correspondence and for finite sets with few elements this one-to-one correspondence can be realized by material association between elements. Thus a sequence of mathematical situations that correspond to discrete quantity will be based on the association between two different collections of objects. For example (Briand et al., 2004) as component of a milieu a collection of garages and a collection of cars and as stake, *given a collection of cars to take exactly what you need* in the collection of garages *to put each car on a garage.*

The one-to-one correspondence allows the possibility of a situational definition of the relation "to have the same quantity" between two collections: If you can put each car on a garage, no car left, no garage left, the collections of cars and garages have *the same quantity.*

3.3 Action Situation

The first and the more general schema of situation is the 'action situation'. The student[2] interacts with a milieu and a stake that have no intention toward this student (Fig. 1).

This figure introduces non only an action and a decision of the student but also an anticipation. In fact if the student has a constant and direct access to both cars and garages then he or she only implements the definition and there is no new situational knowledge involved. For there to be a cognitive stake in the situation, the one-to-one correspondence should be realized only in order to give a feed-back. Thus the immediate action should not be possible: for example stake might be to put the garages on a tray in order to anticipate the realization of the validation, or the car can

[2]The exact term in Theory of situation is "actant", which refers to a hypothetical player who acts rationally and economically, without being subject to the didactic contract.

be put on the student's table but the stock of garages in another part of the room, etc. Thus this schema lead to more than one situation, depending on the possible interactions with the milieu.

There are several useful situational pieces of knowledge in order to realize the stake in those situations, depending on the variables of the situation, in particular: spatial knowledge (to put cars and garages in the same spatial arrangement), and counting knowledge (to count the cars, to memorize the last number, to count the garages up to the same number). The knowledge in this situation is "what" allows you to obtain the desired outcome. It represents a successful adaptation to a particular situation.

Feedback in action situation is not only a win/lose information, it's always more than that. For example, if an student has put the cars on a line and the garages on a line of approximately the same length, the quantity of cars and garages is not necessarily the same, there is thus a feedback about the length as a non always reliable indicator for the quantity.

3.4 During an Action Situation, No Formulation Is **Necessary***, Situational Knowledge Is Mostly Implicit. Formulation Situation*

However, the scientific study of the effects of situational variables on student procedures makes it necessary to experiment with variables that make it necessary to transform knowledge from implicit situational knowledge to explicit formulation of some pieces of knowledge. The gradual transformation of situational knowledge *stricto sensu* into institutional knowledge *stricto sensu* lead to consider the institutionalization as a process of transformation of knowledge.

Situational knowledge encountered during action situation is mostly implicit and is not even identified as useful, there is thus an enormous gap between situational implicit knowledge and institutional written knowledge. TMS studies the major statuses of mathematical knowledge that appear as a bridge over this gap: action, formulation, validation. The method of TMS is to study the specific situations corresponding to those situational knowledge statuses.

As an example of formulation situation in the car-garage setting, I will show some 4–5-year-old students productions in experimental setting (Leterre & Serindat, 2019). This situation has been implemented after the introduction of the car-garage milieu and an action situation (see also Briand et al., 2004, for another experiment of the same situation). The students are told that they have access only to the cars today and they will have access to the garages only another day, the students realize that they will need to remember an information about the cars in order to take the right amount of garages the next day: to write or to draw something that help them to remember. In this experimental setting, no written number line was available. All

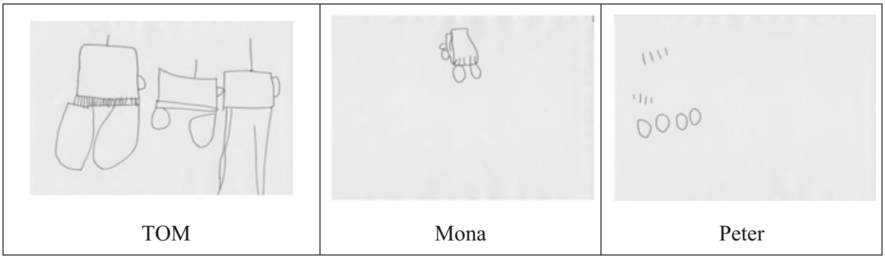

Fig. 2 Three students' written messages

students were given the task individually by a researcher, all students were given 4 cars. You can see in Fig. 2 the messages written by three students.

Tom counted the cars and found four cars. He asked the researcher how to write four, but the researcher did not give him the answer. So he starts drawing. The following day Tom has taken three garages and thus the feedback has been negative.

Mona made a drawing saying: "I draw four lines for the four cars" referring to the four vertical strokes. The following day, Mona as taken a handful of garages and the feedback has been negative.

Peter drew quickly four strokes on the first line and when asked by the researcher it he was finished he add a second line of four stokes and then a third line of four rounds. The following day, Peter has taken four garages and the feedback has been positive.

All students know how to count the four garages, which they may have used as an effective procedure in the action situation. However, in the formulation situation this procedure is not sufficient: the quantity has to be represented.

What the productions show is that new knowledge is invested in the formulation situation: the pupils all try to draw a collection of signs of the same quantity as the collection of cars. Tom draws cars but stops at three, Mona draws a car and invents a coding that she forgets when decoding, Peter is the only one to produce a schema of the car quantity, which he repeats three times. Counting knowledge is no longer sufficient, in particular if the sign "4" is not available. It is thus necessary to produce a collection of signs that have the same quantity as the collection of cars, an 'intermediary' collection (Margolinas & Wozniak, 2012) which is very important for the conception of numbers.

However, in this situation, it is possible to win the stake with a message which is not totally satisfactory (case of Peter's message, see below) or to lose the stake with a message which has some interesting elements (case of Mona's message, see below). In formulation situation, the feedback comes from the action that has been realized thanks to the messages but this does not represent a mathematical validation of those messages.

3.5 Validation Situation

There is thus another situation that represent another step toward mathematical knowledge.

The messages are now objects of study in the validation situation, they are considered as "statements", the question changed the focus from the result of and action informed by a formulation to the discussion of the formulated messages. For example, in the car-garage situation. If the previous message has allowed the student to lose: Is there a reason for this message to fail? Is there a possibility to read this message with another result? If the previous has allowed the student to win: Is there a reason for this message to win? Is there a possibility to read this message with another result?

If we discussed Tom's failed message: Tom has taken three garages, there are three cars drawn in the message. The message itself is false and it is possible to conclude: *Drawing cars is a valid strategy only if the quantity of drawn cars is the same as the quantity of the cars.*

If we discussed Mona's failed message: Mona has taken a handful of garages. There is one car drawn in the message, thus another student may have taken one garage (and fail). There are four vertical lines thus another student may have taken four cars (and win). The message itself is ambiguous: drawing the car is not necessary, four strokes are necessary and sufficient. A general statement can be deduced: *Drawing strokes is a valid strategy only if the quantity of drawn strokes is the same as the quantity of the cars.*

If we discussed Peter's winning message: Peter has taken four garages. However, there are eight vertical lines, another student may have taken eight cars (and fail). There are 12 signs, another student may have taken 12 cars (and fail). The message itself is ambiguous: the repetition of the drawings of four signs is not necessary, four signs are necessary and sufficient. A general statement can be deduced: *Drawing forms is a valid strategy only if the quantity of drawn forms is the same as the quantity of the cars.*

The validation situation led to general statements that are close to a mathematical statement.

3.6 From Situational Knowledge to Institutional Knowledge: Roles of the Different Situations in the Institutionalization Process

The sequence of situations that have been described in the previous paragraph can be implemented with a didactic intention by a teacher, and in this case those situations would be part of bigger situations that include in particular a didactic contract. Those situations have different properties and they do not trigger the same situational knowledge, even if they are built using the same *problem*: to have exactly the right amount of garages in order to put exactly one car on the garages, no car left, no garage left. However, the different situations imply different situational piece of knowledge in the action situations, knowledge is mostly implicit, in formulation situations, some pieces of knowledge are made explicit and written form of the knowledge of quantity may appear, in validation situations, some mathematics statements become useful.

Thus this sequence of situations can play an important part in the institutionalization considered as a process of transformation from situational implicit knowledge toward an explicit and rational knowledge that is closer to the institutional knowledge at stake. Even if the systematic part of institutionalization directed by the teacher was not at the beginning part of the theorization of Brousseau, the process of transformation of knowledge plays a very important role in the theorization of the different situations.

4 Didactical Consequences of Theory of Mathematics Situations on Teaching

4.1 Engineering and "Task Design"

Brousseau has always clearly stated that the experimental process was not designed in order to be generalized for non experimental conditions:

> We have repeatedly insisted that we do not consider this curriculum as a method to be offered to teachers. It should be explained why:
>
> The main reason is the difficulty in communicating all the necessary information [that are very different from those usually followed by teachers, for example relative to assessment and teacher-student relationship]. [...] It would probably be harmful for children to teach them in the traditional way every step of this long genesis and to institutionalize temporary behaviours.
>
> Moreover, the necessities of the epistemological experience have led to choices that it would be at least premature to propose to teachers [...] (Brousseau, 1981, p. 113–114 my translation)

Brousseau's writings can always be [mis]read as readymade didactic proposals. The importance of the sequences of situations that have been experimented at the COREM lay in the properties they have in order to be part of the teaching a given knowledge, with the aim to both situational and institutional knowledge.

4.2 Mathematic Problem, Situations and Observation

In order to interpret the actions of students and teachers, it is necessary to understand the situations that students are investing in, even if they do not correspond to what the teacher is trying to set up. The observer has thus to proceed from the situational

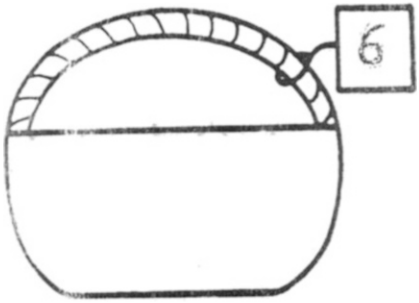

Fig. 3 A written apple-basket task

knowledge required by the situation toward the corresponding institutional knowledge. During this process, some discrepancies often appear between some situations that are considered as 'similar' by teachers, particularly when the same mathematical problem lead to different situations (Clivaz, 2017; Margolinas et al., 2005). This is often the case when teachers give first to students some 'manipulation' tasks and as a final test a written task.

For example, a teacher can ask students to put six apples in a basket. With real objects, students can accomplish this task with the following procedure: take one apple, say 'one' and simultaneously put the apple in the basket, thus take one apple, say 'two' and simultaneously put the apple in the basket, and so on up to 'six'. This procedure can be accomplish fairly quickly and at a steady pace. The oral sequence of numbers 'one, two, three, four, five, six' can thus be pronounced without hesitation. If the student know how to memorize a number as target of counting, and how to count up to six, this is quite easy. In this situation, the 'counting procedure' is sufficient in order to succeed.

If we now consider a written task based on the same problem. (observed in the final year of nursery school, 5–6-year-old students, Fig. 3), with the following instruction: draw the indicated number of apples.

In this situation, the student have not only to understand the meaning of the symbol '6' as 'six' but also to draw apples and to count the drawings. For a student of 5–6 years old, to draw an apple is not an easy task, in particular if this students consider the drawing as a serious task (right colours, right shape, etc.), it is more or less impossible to do this drawing and to count at the same time. But if drawing and counting are disconnected it will frequently lead to a situation where you have a number of drawn apples and you have to compare this number to six. This is a situational knowledge that is never encountered in the apple-basket material setting: is a given number more or less than six?

Often, a student who is not able to succeed in this written task will be submitted again to the material task, he or she will thus succeed and can again failed written the test. Teachers are not always aware that the same mathematical problem can lead to different situations.

4.3 From Situational to Institutional Knowledge: Student's Struggle

Students are always confronted with situations and therefore with situational knowledge, regardless of the teachers's of teaching. The adequacy between situational and institutional knowledge is thus a very important issue of teaching. If students cannot make any link between the situations they have encountered and thus the situational knowledge they have invested and the 'lesson' that is the presentation of institutional knowledge, they may considered the lesson and perhaps mathematics themselves are disconnected from any reality. Conversely, if students consider mathematics as a monumental text of knowledge (chapter "On the Genesis and Progress of the ATD") they will not be able to transform this knowledge into useful situational knowledge.

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The Onto-semiotic Approach in Mathematics Education. Analysing Objects and Meanings in Mathematical Practice

Juan D. Godino, María Burgos, and María M. Gea

1 Introduction

In this seminar a synthesis of the Onto-semiotic Approach (OSA) theoretical system to mathematical knowledge and instruction was presented. We highlighted the problems, principles and research methods that are addressed in this approach and considering the didactics of mathematics as a scientific and technological discipline. In the first part of the seminar we developed the reply to the question posed by Gascón and Nicolás (2017) about the prescriptive nature of didactics of mathematics research from the OSA perspective. This theoretical framework suggests that Didactics should address the epistemological, ontological, semiotic-cognitive, educational-instructional, ecological, and instruction optimization problems (Godino et al., 2019). OSA assumes anthropological, pragmatic and semiotic principles to approach all these types of problems, as well as it embraces sociocultural principles to face the educational-instructional problem.

1.1 Didactics as Science and as Technology

The OSA framework attributes both a scientific and technological character to the knowledge produced by didactic research. On the one hand, it addresses theoretical problems related to the ontological, epistemological and semiotic nature of mathematical knowledge, as far as such problems are related to the teaching and learning processes (the scientific, descriptive, explanatory or predictive component). On the other hand, didactics should intervene in these processes to improve them as much as

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possible (the technological - prescriptive component). While description, explanation and prediction are the main goals of scientific activity, prescription and assessment are the main goals of technological enterprise; however, technological action also includes elements of applied research when solving specific problems. The notion of *didactic suitability* has been introduced as a systemic criterion to address the problem of optimization of mathematical instruction processes.

2 Onto-semiotic Approach: A Modular and Inclusive Theoretical Framework for Mathematics Education

The Onto-semiotic Approach is a modular and inclusive theoretical system for research in mathematics education that provides specific principles and methods to address the:

1. *Epistemological problems:* How does mathematics emerge and develop?
2. *Ontological problems:* What is a mathematical object? What types of objects intervene in mathematical activity?
3. *Semiotic-cognitive problems:* What is knowing a mathematical object? What is the meaning of a mathematical object for a subject given a time and circumstances?
4. *Educational-instructional problems:* What is teaching? What is learning? How do they relate? What types of interactions between people, knowledge and resources are required in the instructional processes to optimize learning?
5. *Ecological problems:* What factors condition and support the development of instructional processes and what norms regulate them?
6. *Instruction optimization problems:* What kind of actions and resources should be implemented in the instructional processes to optimize students' mathematical learning?
7. *Teachers' education problems*: What knowledge and skills should teachers have to manage the teaching and learning processes of mathematics?

These problems, the assumed principles and methods developed to address them are described in Godino et al. (2019). Likewise, a model of teacher's Didactic-Mathematical Knowledge and Competencies based on the OSA (Godino et al., 2017) has been developed. This model considers essential that teachers be trained for the analysis of objects and meanings that intervenes in mathematical practices (onto-semiotic analysis), together with the competences for the analysis of didactical configurations, normative analysis and didactical suitability (Fig. 1).

In the following section, we describe the objectives, methodology and foundations of a workshop for developing the general competence of analysis and didactical intervention.

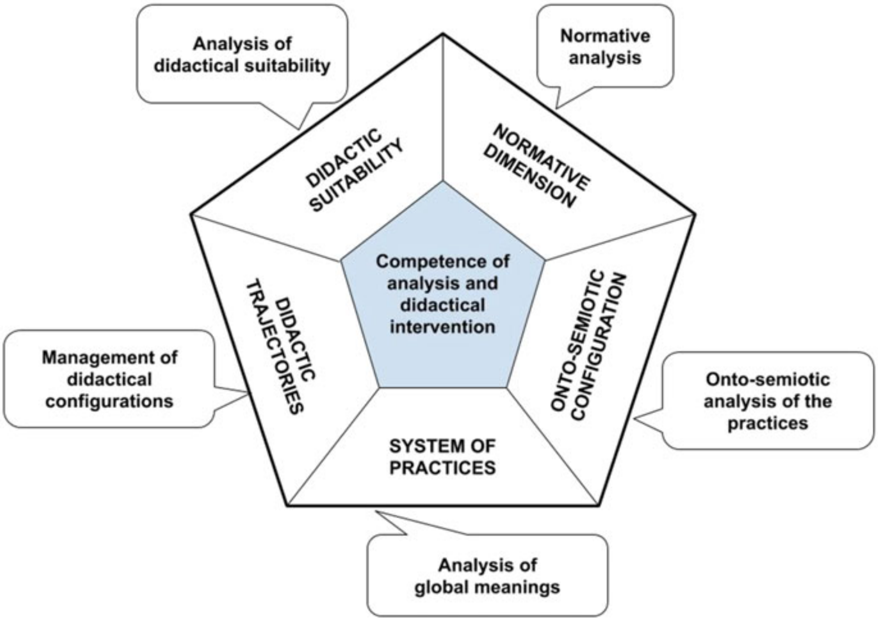

Fig. 1 Components of the analysis and didactic intervention competence (Godino et al., 2017, p. 103)

3 Objects and Meanings Analysis of Mathematical Practices

The notion of *meaning*, frequently used both in research and educational practice, is a central and controversial issue in philosophy, logic, semiotics and other sciences and technologies interested in human cognition. Given the importance of symbolization, communication and understanding processes in mathematics teaching and learning, the question of meaning should occupy a central place in teacher training.

In this workshop we propose to develop the specific analysis competence of the different meanings involved in mathematical practices, applying theoretical tools of the OSA framework (Godino et al., 2007, 2019), which allow micro and macro analysis levels of the communication and interpretation processes in mathematics education.

3.1 Workshop Aims and Method

The workshop main is that the participants:

1. Know various theoretical approaches about the meaning of mathematical objects.
2. Analyze the mathematical practices that are put at stake in problem solving from the point of view of the objects involved and meanings attributed.
3. Reflect on the notion of meaning of mathematical concepts, their relationship with understanding and the design of teaching and learning processes.

The workshop includes a first part in which a text offering a synthesis of various theories of meaning in mathematics education (Godino et al., 2021) is presented and discussed. This reading includes an application example of the analysis method of meanings involved in the solving processes of a mathematical problem (onto-semiotic analysis method, at the micro and macro levels). Next, it is proposed to work in teams of two or three participants to solve a missing value proportionality problem by applying at least two different strategies. Next, the analysis of the objects and meanings put at stake in the practices to solve the problem is carried out. First, the technique described in the reading document, previously discussed, is applied according to the two solutions proposed for the workshop. The micro-level analysis is completed with an exploration of the different meanings of proportionality and its articulation in a global meaning. Then, the unitary and systemic meanings of other solutions different from those proposed in the workshop are analyzed.

3.2 Meaning in the Onto-semiotic Approach

Within the OSA framework the notion of meaning and its relation to the notions of practice and object plays a central role. The fact that certain types of practices are carried out within certain institutions is what determines the progressive emergence of "mathematical objects" and that the "meaning" of these objects is closely linked to the problems and the activity carried out for their resolution, not being pertinent to reduce this object meaning to its mere mathematical definition (Godino & Batanero, 1994).

Although the initial OSA objective was to develop a theoretical model that would answer the question of the meaning of mathematical concepts, in subsequent developments this objective has been extended and applied to any type of object that intervenes in mathematical practices, also proposing a categorization for such objects. It is considered that the epistemological, cognitive and instructional problems that mathematics education has to address should first deal with the ontological problem, that is, clarify the nature and types of mathematical objects whose teaching and learning is intended.

In a first approach, the meaning is that object which is referred by a word, a symbol or any other means of expression, issued by a person in a communicative act with another person or with himself, which takes place in a given context. However, with words and symbols not only things are mentioned or represented, but through them things are also done, that is, they intervene in operative practices. Operations and calculation with the words and symbols are carried out, so that new objects are produced as result of these operative practices.

Therefore, the question arises, what role, besides the representational one, does this word, symbol or expression play in a specific operative practice? This is a central problem that has to be addressed by a holistic theory about meaning, which should takes into account both referential and operational use, responding to the meaning of expressions that refers to concepts (ideal, abstract objects) or any other type of object.

3.3 Pragmatist and Referential Meanings

We explained the use of meaning in the OSA, and its relation to the notions of practice and mathematical object. We contextualized the explanation with the example of a possible demonstration of the elementary arithmetic proposition included in Fig. 2.

In Table 1 (column 2) we summarize the use or operational meaning of the practices required in the demonstration of proposition 2 + 3 = 5 (column 1). Column 3 shows the intervening objects in the practices.

In the realization of each practice, and in the conjunction of all or a part of them, a configuration of objects intervenes whose identification is necessary to understand and manage the teaching and learning processes. The OSA perspective proposes that the problem of signs and their interpretation should not be separated from the ontological problem, understood in terms of inquiring about the nature and types of entities referred to by the signs, as well as the instrumental role played by them in the construction activity and knowledge communication. In addition, the solution of the onto-semiotic problem implies new ways of addressing the epistemological

Proposition: 2 + 3 = 5

Demonstration:

1) The symbols, 2, 3 and 5 represent natural numbers.

2) Natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function $s\colon \mathbb{N} \rightarrow \mathbb{N}$, is defined. In this set, the sum, +, is defined recursively as:

$$n + 1 = s(n); n + s(m) = s(n + m)$$

3) In the sequence, 2 is the successor of 1, $2 = s(1) = 1 + 1$; 3 is the successor of 2, $3 = s(2) = 2 + 1$; and 5 is the successor of 4 which is next of 3, $5 = s(4) = s(s(3))$.

4) The sign = indicates the equivalence of two expressions.

5) The expression 2 + 3 represents the sum of the natural numbers 2 and 3.

6) Taking into account the definition of the sum of natural numbers and successor

$$2 + 3 = 2 + s(2) = s(2 + 2) = s(2 + s(1)) = s(s(2 + 1)) = s(s(3)) = s(4) = 5.$$

7) Therefore, the expressions 2 + 3 and 5 are equivalents.

Fig. 2 Demonstration of an elementary arithmetic proposition (2 + 3 = 5)

Table 1 Use/intentionality and objects in practices to demonstrate 2 + 3 = 5

Sequence of elementary practices	Use/intentionality	Intervening objects
1. The symbols, 2, 3 and 5 represent natural numbers	Attributing meaning to the symbols 2, 3, 5 as natural numbers	Languages: symbolic; natural Concepts: natural number
2. The natural numbers are a set of symbols that satisfy the Peano's axioms, in particular, there is a first element, 1, and a following (successor), injective function s:N→N is defined. In this set, the sum, +, is defined recursively as: n + 1 = s(n); n + s(m) = s (n + m)	Evoking the rules that define natural numbers and their sum, within the framework of a specific axiomatic theory	Language: natural, symbolic Concepts: natural number; set (of symbols); successor, function; first element; sum Propositions: Peano's axioms
3. In the sequence, 2 is the successor of 1, 2 = s (1) = 1 + 1; 3 is the successor of 2, 3 = s(2) = 2 + 1, and 5 is the successor of 4 which is next of 3, 5 = s(4) = s(s(3))	Interpreting the meaning of symbols 2, 3, 5 in Peano's axiomatic theory of natural numbers	Languages: natural; symbolic Concepts: sequence; successor, sum Proposition: 2 is the successor of 1, 3 is the successor of 2, and 5 is the successor of the successor of 3 Arguments: convention based on the properties of the successor function
4. The sign = indicates the equivalence of two expressions	Evoking the meaning of the equality of natural numbers as equivalence of two expressions	Languages: symbolic; naturally Concepts: equivalence of expressions; equality
5. The expression 2 + 3 represents the sum of the natural numbers 2 and 3	Interpreting the meaning of + as the sum of natural numbers	Languages: natural and symbolic Concepts: sum of natural numbers
6. Taking into account the definition of the sum of natural numbers and successor 2 + 3 = 2 + s(2) = s(2 + 2) = s (2 + s(1)) = s(s(2 + 1)) = s(s (3)) = s(4) = 5	Applying the rules that define the following function (successor) and addition of natural numbers	Languages: natural and symbolic Proposition: 2 + 3 = 5 Procedure: addition and successor operations Argument: deductive, based on the definitions of natural numbers, sum and the successor function
7. Therefore, the expressions 2 + 3 and 5 are equivalent	Fixing the new rule of use of the numerical symbols (declare the truth of the proposition)	Languages: natural and symbolic Proposition: statement of practice 7 Argument: deductive sequence of practices 1 to 6

problem about the origin and evolution of knowledge, no doubt essential to address the educational-instructional problem (Godino et al., 2021).

3.4 Workshop Development

Participants were proposed to respond to the instructions given below.

Question 1: Identify the referred objects (meanings) in each of the practices of solution 1 of the problem included in the Annex. Complete Table 2.

Question 2: Identify the referred objects (meanings) in each of the practices of the solution 2 of the problem included in the Annex. Complete Table 3.

In solutions 1 and 2 of the problem, the concept of proportionality intervenes in a decisive way. Taking into account the types of objects and unitary meanings that intervene in the operative and discursive practices that allow solving the problem, we can say that the systemic meaning of the proportionality that is at stake in solution 1 is of arithmetic type, while in solution 2 is of algebraic-functional type (Burgos et al., 2018).

Question 3: Analyze the unitary and systemic meanings of other different solutions by which the problem can be solved.

3.5 Some Conclusions

The identification of the various partial meanings of a mathematical object and its articulation is a phase of the onto-semiotic analysis of mathematical activity. This analysis helps to formulate hypotheses about critical points in the interaction between the various agents in which there may be gaps of meaning or disparity of interpretations that require processes of negotiation of meanings and changes in the instruction process.

Table 2 Object and meanings in solution 1

Sequence of elementary practices to solve the task	Objects referred to in the practices (concepts, propositions, procedures and arguments)	Use and intentionality of practices
…		

Table 3 Objects and meanings in solution 2

Sequence of elementary practices to solve the task	Objects referred to in the practices (concepts, propositions, procedures and arguments)	Use and intentionality of practices
1.		
…		

The theory of meaning that has been elaborated from the OSA perspective is supporting a new field of reflection on what could be called an onto-semiotic analysis, in which the study of signs should be linked to the analysis of the objects referred to by the signs. The OSA attempts to combine realistic and operational theories of meaning, since the problem is approached from the educational context, that is, the setting of construction and dissemination of mathematical knowledge. Although the problem of meaning interests to various disciplines (philosophy, linguistics, psychology, semiotics, etc.), the field of education, and, in particular, mathematics education, provides a rich perspective to address this problem. The onto-semiotic approach proposes not to separate the problem of signs and their interpretation from the ontological problem. This is understood in terms of inquiry about the nature and types of entities referred ~~to~~ by the signs, as well as the instrumental role played by them in the knowledge construction and communication.

The onto-semiotic approach to the meaning of mathematical objects has implications for teachers, since it highlight the complexity of knowledge and, therefore, the challenge of teacher education. In this sense, mathematics teacher should know the different meanings of mathematical objects, as well as the network of objects and processes involved in the mathematical practices, in order to be able to plan the teaching, manage the interactions in the classroom, understand the difficulties and assess the students' learning.

4 Final Reflections

Didactics should provide results that allow effective action on a part of reality: the teaching and learning of mathematics in the different contexts in which it takes place. In addition, it must take into account the four types of problem areas, epistemological, ontological, semiotic-cognitive, educational-instructional, described in Godino et al. (2021) and their interactions. Didactics should offer provisional principles (standards or suitability criteria in OSA framework) agreed by the community interested in mathematics education. These principles and norms are useful in two moments:

1. A priori, the suitability criteria guide the way in which an instruction process should be developed.
2. A posteriori, the criteria serve to assess the teaching and learning process effectively implemented and identify possible aspects to be improved in the redesign.

To generate these principles, researchers in mathematics education should discuss and collaborate with all other sectors interested in improving mathematics teaching (teachers, parents, administration, etc.). This will lead to a consensus that generate principles to guide and value the instruction processes, in order to achieve a suitable teaching of mathematics. It is recognized, however, that the identification of suitability criteria, both general and specific, requires a research agenda that is open to discussion and development in the mathematics education community.

Didactics involves the study of human beings interacting in very diverse contexts, that is complex, dynamic, open, heterogeneous systems engaged in multiple and diverse interactions. These systems have chaotic aspects, where small changes can lead to large deviation. Since minor changes take place at the micro level, they should be studied as possible explanatory factors of the changes observed at the macro level. Consequently, Didactics should contemplate the use of analysis units at the micro level (a task, or a teacher-student interaction of a specific nature), and at the macro level (a field of problems, a long-term didactic trajectory, the sociocultural context). The principles explicitly stated as characteristics of a theory should be interpreted as tools, while the methods are ways of applying these tools to face the solution of specific problems and questions in the field.

Annex. Solutions to a Proportionality Problem

Problem: *A package of 500 g of coffee is sold for 5 euros. At what price should a package of 450 g be sold?*

Solution 1:

1. In everyday life situations of buying and selling, it is usual to assume that, when buying small quantities of coffee, these quantities are of the same type and quality.
2. Consequently, if you buy double, triple, etc. of product, you must pay double, triple, etc. Similarly, if you buy half, the third part, etc. of product, half, the third part, etc. must be paid.
3. If a package of 500 g of coffee is sold for 5 euros, the price of 100 g of coffee (five times less) should be a fifth of 5 euros, that is, 1 euro.
4. The price of 50 g (half of 100 g) must be half the price of 100 g, that is, 0.50 euros.
5. In this way, $4 \times 100 + 50 = 450$ g of coffee should cost, 4×1 € + 0.50 = 4.50 €; that is, 4 euros and 50 cents.

Solution 2:

1. It is assumed that if you buy double, triple, etc. of product, you must pay double, triple, etc. In addition, what we will pay for two different coffee packages will be equal to the price of a package that weighs the same as the two together.
2. Therefore, the correspondence established between the set of the quantities of the product (Q) and the set of the prices paid (P), f: Q → P complies that, the image of the sum of quantities is the sum of the images, $f(a + b) = f(a) + f(b)$, and the product image of an amount a by a real number α is the product of the image quantity by that number, $f(\alpha a) = \alpha f(a)$.
3. That is, the function f: Q → P is linear; then there is k, such that $f(x) = kx$.

4. The coefficient k of the linear function is the coefficient of proportionality, in the case of direct proportionality relations between magnitudes.
5. Applying these properties to the case, we have:
   f (500 g) = 5 €; 500f (1 g) = 5 €; f(1 g) = € 5/500 [One gram of coffee costs 1 cent]
6. 450f(1 g) = 450 × 5/500 €; f (450 g) = 4.5 €
7. Then the price of a package of 450 g should be 4.5 euros.

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APOS Theory and the Role of the Genetic Decomposition

María Trigueros

1 Introduction

APOS Theory is a well-known theoretical framework in mathematics education. In spite of the fact that it has been used by researchers since the 1980s researchers who use this theory in research face in many occasions questions related to the main principles of the theory that seem to be not well understood. These questions probably arise from the fact that subtleties in the theory's communication may be taken for granted when the theory is communicated and need to be clarified. The context of a dialogue among different theories opens a possibility to think on some of those questions and to make an effort to discuss them with some care. In this study the role of the genetic decomposition in the theory, its importance and its use are discussed. Its goal is to respond the following questions: What is a genetic decomposition? How is it developed? How is it used when APOS theory is used in research and in the design of teaching?

This paper presents initially a brief description of APOS theory and its relation to teaching. It follows with the discussion of the role of the genetic decomposition in this theoretical framework, its use in research and in the design of teaching activities. With the goal of providing the readers with a clear explanation, an example is included where the use of the genetic decomposition is presented in the context of a research project and also in the analysis of data obtained from students' work. Finally, the role of the teacher is briefly discussed and some final reflection are included.

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2 APOS Theory and Its Relation to Teaching

APOS theory was developed originally by Dubinsky and advanced and refined by RUMEC (Research in Undergraduate Mathematics Education Community) (Arnon et al., 2014; Asiala et al., 1996). It is based on Piaget's genetic epistemology and can be considered a cognitive framework. As such, the main interest of the theory is to understand how mathematical knowledge is constructed and learnt. Although it has been applied to mathematics learning at different school levels, work with APOS theory has focused mainly at university level.

The description of APOS theory includes an explicit presentation of its basic assumptions and its methodological approach to research are explicitly presented. It emphasizes the construction of knowledge at an individual level, although it does not refer to a specific individual but to a generic student of an institution (Bosch et al., 2017) and takes into account the role of the social component in learning. Its research end can be stated as the identification of the mental constructions developed by students when learning a mathematical concept of a topic and the detailed description of those elements involved in the construction of mathematical knowledge. It also has an educational end that takes into account the specificities of the context where learning takes place (Oktaç et al., 2019).

It is important at this point to make two clarifications linked with common misunderstandings about research using APOS theory and that are closely related to genetic epistemology. First, APOS theory focus on the learning of mathematical concepts has been interpreted as linking its research focus to mathematical definitions; however, in APOS theory concepts are considered as a unit of cognitive meaning about a mathematical object that is constructed by an individual, usually, in a didactical context. Also, genetic epistemology posits that the relation of the cognitive object and the subject is dialectical, so in APOS theory there is an indissoluble unity between the mathematical and cognitive aspects (Bosch et al., 2017). Second, the work of Piaget has been criticized for focusing on individual cognitive processes and leaving aside the social aspects of cognition. However, Piaget did consider these factors in his work. He repeatedly mentioned the identity of intellectual operations and social co-operations and discusses the role of social context in the development of knowledge. He insisted that "progress in logics is equivalent in a non-dissociable way to a process of socialization of thought" (Piaget, 1950/1975, p85). He insisted that the coordination of actions could be the result of cooperation between individuals and that this cooperation, as well as the social context, is important in individual cognition. He also argued that the construction of conservation and reversions needed the cooperation of groups of individuals (i.e. Piaget, 1950/1975, 1967/1971). The importance of the social context and culture in the development of knowledge is also central in Piaget and García Psychogenesis and the history of science (1982).

APOS theory explicitly takes into account the social component of the development of knowledge in its didactic component, the ACE cycle. According to it the teacher should act as a guide of students working collaboratively in small groups on activities (A) that promote the development of knowledge, designed previously in terms of the theory. This work is followed by whole class discussion (C) with the teacher. Reflection is reinforced through this discussion and results obtained are formalized. Students work on their own in the solution of exercises as homework (E). This cycle is repeated during the whole duration of the course (Asiala et al., 1996).

2.1 Mental Structures and Mechanisms

APOS theory proposes that learning happens by the construction of mental structures associated to a concept or a topic of mathematics. These structures are constructed through mental mechanisms. APOS structures are Action, Processes, Objects and Schema.

Actions are needed to start the construction of new knowledge. They are applied to previously constructed structures and are mainly directed by external instructions or stimuli. When students repeat an Action and reflect upon it, it interiorized into a Process. Interiorization is a mechanism related to abstract reflection, when a Process is constructed students are able to apply Actions in a general context, they can skip steps in chains of Actions and can predict the result of an Action. Processes can be reverted. Reflection on Processes can lead to their encapsulation into Objects and Actions can be applied to them. A collection of related structures brought about in the solution of a certain class of problems constitutes a Schema.

Mathematical knowledge is defined in terms of APOS as the individual tendency to respond to mathematical problem situations and their solutions by reflecting on them in a social context and constructing or reconstructing mathematical Actions, Processes and Objects and organizing them in Schemas to use in dealing with the situations (Asiala et al., 1996).

Structures of knowledge are not conceived as levels in APOS theory, they represent stages in learning that are not necessarily constructed as a linear progression. In order to determine a type of conception constructed it is necessary to observe students working with different types of tasks. When an individual, for example, solves problems using mainly Actions, it does not mean that he has failed, Actions are an important part of the construction of knowledge process; and the same can be said of other conceptions, an Object conception does not necessarily mean that the subject will always succeed when working with certain types of tasks (Oktaç et al., 2019).

3 The Genetic Decomposition: Its Role in Research and Teaching

A goal of APOS theory, as a scientific theory, as well as of other theories, is the development of models that can be used to predict and explain different phenomena related to how students learn different mathematical concepts or topics in different contexts using the theory structures. A central component of APOS Theory methodology is the development of an epistemological model describing such construction: a genetic decomposition.

A genetic decomposition consists in a description of the structures and mechanisms that are hypothesized as needed to describe how individuals learn a particular mathematical concept or topic. As a preliminary model, a genetic decomposition is not unique, several genetic decompositions for the same concept can coexist, but, what is important, is that this model needs to be tested experimentally (Dubinsky, 1991). Results from research are used to redesign, refine or to validate the proposed genetic decomposition. After each refinement the genetic decomposition needs to be experimentally tested and this process is repeated until it is validated, which means that research has shown that the model predicts those construction needed in the learning of the concept.

3.1 How Is a Genetic Decomposition Designed?

The development of a preliminary genetic decomposition is not an easy endeavor. It needs a thorough reflection on the role the concept plays in mathematics and its functionality. This reflection is complemented by research on the historical development of the concept, research on results of previous studies in the literature concerning the concept or topic, examination of textbooks, class observations and reflections on the previous experiences of researchers as teachers. Results of this research and reflection results in a description of the previous structures supposed to be needed to start the construction, a description of the Actions needed to be applied on previous structures, how these Actions may be interiorized into Processes, how Processes may be reverted and coordinated to construct new more encompassing Processes, how Processes are encapsulated into Objects. Depending of the level of analysis chosen, the genetic decomposition may describe the structures that are supposed to be elements of the Schema, the relations among those components and how these relations may evolve according to Piaget and Garcia Triad levels (1982), the possibility to assimilate or accommodate other structures to the Schema as it develops and how the Schema can be thematized into an Object. The genetic decomposition if thus an epistemological and cognitive model that describes how an individual can learn and use successfully the mathematical concept in the solution of intra-mathematical or extra-mathematical problems (Arnon et al., 2014).

It is important to point out that although genetic decompositions are presented linearly, as a form of communication, this do not imply that the development of the described concept is linear. Different trajectories may result in the construction of knowledge, there can be discontinuities, loops and other phenomena. What is important is that in the construction of knowledge researchers can find evidence of the need of the proposed structures and mechanisms. It is important to remember that the genetic decomposition is a model, it does not try to describe what happens in a generic student's mind, this is not possible, nor does it offer an exclusive analysis of how the concept is learned. It intends to predict if the student will apply, in one moment or other, the described structures when dealing with mathematical problems related to the concept.

3.2 How Is a Genetic Decomposition Used?

As an epistemological model, the genetic decomposition is used in research as a useful model of cognition for different mathematical concepts and topics. It is an important guide in the design of research instruments and in the analysis of results obtained from the data obtained.

It is also used as a guide in the design of activities where intended structures and mechanisms are reflected. These activities are used in teaching with the ACE cycle (and its variations) where "The role of the instructor is to identify the mental structures that might be needed in learning the concept and to design activities that help students make the proposed mental constructions." (Arnon et al., 2014, p. 179). Its use in research implementation provides data for its refinement and validation and important information about other learning phenomena in the classroom.

Research, teaching and theory development are closely related in APOS theory (Arnon et al., 2014). Through several research cycles it has been found that genetic decompositions reflect adequately how a concept of interest is learned and that it is a useful tool in the design of effective instruction (i.e. Martinez-Planell & Trigueros, 2019; Oktaç, 2018; Weller et al., 2003). Some genetic decompositions have been used by researchers to investigate the teacher's role and the constructions they intend to develop, as inferred from their teaching practices and their relation to students' constructions, or as an approach to their professional knowledge (i.e. Badillo et al., 2011; Gavilán Izquierdo et al., 2007).

4 An Example: The Role of the Genetic Decomposition in the Design of Activities and in Teaching

An example can help to throw light into the role of the genetic decomposition in APOS theory. A genetic decomposition that been used in research and teaching is presented together with the sources used in its design. The teaching and research experience methodologies are summarized and the discussion is focused on how APOS research and methodology, on particular the genetic decomposition, are used to design teaching activities and to identify aspects involved in students' construction of knowledge, together with examples of the role of the teacher in the classroom.

The selected genetic decomposition was designed to construct the concepts of matrix-vector product, matrix-matrix product together with the concept of a matrix transformation (a matrix transformation $T_A(v)$ is a transformation such that $T_A(v) = Av$, where A is a matrix, v is a vector and Av is defined).

4.1 The Genetic Decomposition

The design of the genetic decomposition presented here (Figueroa et al., 2018) included an analysis of the concepts use in mathematics and applications, an analysis of the authors' observations during previous teaching experiences and a review of literature (Larson & Zandieh, 2013; Roa-Fuentes & Parraguez, 2017).

4.1.1 Previous Constructions

The previous constructions needed to initiate the construction of the intended concepts are:

- Table as an Object: students are able to perform Actions on the table, such as identify any element of a given row and column, taking all the entries of a column or a row as a whole.
- Vector as an Object and operations with vectors: sum of vectors, product by a scalar and dot product as a Process, as well as the use of row and column representations of vectors.
- Functions as a Schema, which includes the following Processes: domain and codomain sets, correspondence rule, and composition of functions.

4.1.2 Construction of the Matrix-Vector Product and the Matrix Transformation

A table is used to construct the Process of identifying the element in the i-th row and j-th column (*aij*). Actions are performed using the *aij* elements to construct a matrix *A*. These Actions are interiorized into matrix as a Process when students can construct any matrix given information about its elements, which is encapsulated by doing Actions on the matrix (e.g. comparing, adding and other operations).

The matrix as a Process and the vector Process (obtained by de-encapsulating the vector Object) are coordinated to obtain a new Process that considers the rows and columns of a matrix as vectors and a matrix as an ordered collection of rows and column vectors. Actions are performed to calculate the dot product of a row vector of matrix *A* by any given column vector *x*, and to recognize it as a scalar, denoted by rowi(*A*)· *x*. This Action is repeated for every row of A, and is interiorized into a Process that is coordinated with the Process of a column vector to obtain a new column vector *c* = (row1(*A*) ·*x*, row2(*A*)·*x*, ... rowm(*A*)·*x*). This Process is encapsulated as the product of vector *x* with matrix *A*.

The above Processes are coordinated with the function Process (function as a correspondence rule) to obtain the Process of relating vector c with a function f_A whose correspondence rule is $f_A(x) = c$. This Process allows the calculation of $f_A(x)$ for different vectors *x* to obtain a Process that can be encapsulated as the function Object f_A when students name $f_A(x)$ as *Ax*, and can determine, for example, the relationship with the system of equations *Ax* = *c*, or calculate the domain, codomain, image, and pre-image of the function, among other operations between matrix functions.

4.1.3 Construction of the Matrix Product and the Composition of Matrix Transformations

A given matrix *B* may be desencapsulated into the Process of considering *B* as an ordered collection of column vectors, b_i. This Process is coordinated with the function Process f_A, to obtain a new ordered collection of column vectors ($f_A(b_1)$, $f_A(b_2)$, ..., $f_A(b_k)$).

This new Process allows the writing of the formal definition of matrix product $_{AB}$. This Process is encapsulated into matrix $_{AB}$ through the Action of comparing any ij-th entry of matrix *AB*, *(AB)ij*, with the dot product rowi(*A*) ·*bj*. One may perform Actions on this matrix Object *C* = *AB* to determine its properties (its existence, or non-commutative properties of matrix product, among others).

Additionally, the Processes f_A and f_B are coordinated into a new Process which is the composition $f_A \circ f_B$. This composition Process may be later coordinated with Process f_{AB} through the Action of comparing both Processes and is encapsulated into the composition as an Object $f_A \circ f_B = f_{AB}$ which can be compared to the Object *AB* and considered as equivalent to it.

4.2 Teaching and Research Methodology

The decided teaching strategy was the use of a modeling problem. Although the use of this type of problems is not considered in APOS theory, it is consistent with it. Since working in any type of problem requires students to use their constructed Schemas and they have been proved to be useful in developing the need for new knowledge construction.

The problem designed for this research experience is the following:

Pesticides are sprayed on plants to eliminate harmful insects. Plants absorb some of the pesticides. Parts of these pesticides are absorbed by herbivores when they eat the plants that have been sprayed. How can we determine the amount of pesticide absorbed by an herbivore? If we have carnivores (such a human) that eat the herbivores, can we find out how much of each pesticide has been absorbed by each carnivore?

Two teachers participated in the experience. Each of them used the problem and the worksheets in a classroom with 30 students. Work in the classroom followed the ACE cycle described above. It started with students' work on the problem. Students worked in collaborative teams and after discussing their proposals data for the problem were introduced and as the work progressed the teachers involved in the experience introduced worksheets with activities designed with the genetic decomposition.

A worksheet intended to develop the constructions related with matrix, matrix function and matrix-vector multiplication. A second one had the goal of helping the students to construct the matrix multiplication and its interpretation in terms of the matrix function f_A. The third worksheet intended to foster the construction of the functions f_{AB} and $f_A \circ f_B$ and the relation between the matrix product and the composition of matrix functions. In all the worksheets activities related with the initial problem were included as a context of the intended constructions but opportunities to reflect on the constructions included in the genetic decomposition to learn the proposed mathematical concepts. Table 1 shows some of the designed activities included in the worksheets and their relation to the genetic decomposition.

The experience was developed on three sessions of 2 h each. All the students' work was recovered at the end of each class to be analyzed. Students' work on teams was recorded and responses to a similar problem included in an exam for the students at the end of the course were also analyzed.

4.3 Use of Activities Based on the Genetic Decomposition in Students' Construction of Knowledge

Students strategies when working with the open modeling problem were discussed with the teachers. Two main strategies appeared in both classrooms. Most teams proposed to use tables where they would record the information about the problem

Table 1 Example of activities and their analysis using the GD (Figueroa et al., 2018)

Worksheet-question/activity	Analysis based on the GD
W1-6 / An animal eats 20 plants of type 1, 28 plants of type 2, 30 plants of type 3, and 40 plants of type 4, given this diet what is the animal's consumption of each type of pesticide?	Action of considering and calculating the dot product of a row vector of a matrix by a given vector using the data form the modeling problem
W1-7 / If another animal has the following consumption vector (a, b, c, d) in which the jth entry is the number of plants of type j eaten by the animal, what is the animal's consumption of each type of pesticide?	Interiorization of the above Actions into a Process
W1-8 / Write the above result as a column vector and describe the operation you performed to obtain each entry	Encapsulation of the Process by means of the Action of grouping the resulting scalars into a new column vector
W2-2/ How do you interpret the solution set of system $\mathrm{A}x = b$ in terms of function f_A? If you can, use set notation to write your answer	Coordination of the system of equation Process with that of matrix function into a new Process that identifies the solution set of $\mathrm{A}x = b$ with those vectors in the domain of f_A whose images are b
W2-6/ Apply function f_A to each of the columns of matrix B, how do you interpret these operations in terms of the problem?	Coordination of the row-column dot product Process with that of matrix function into the Process of finding the images of the columns of B that constitute the columns of a new matrix (AB)
W2-7/ If AB is the matrix whose ij-entry is c_{ij}, the amount of pesticide i absorbed by herbivore j, calculate every entry of AB and describe each column of AB with respect to function f_A	Encapsulation of the matrix product Process, as an ordered collection of images of f_A
W2-8 / Describe entry c_{ij} as an operation that relates a row of A with a column of B	Encapsulation of the matrix product as an operation between matrices
W3-3/ Calculate and interpret in the problem the following: (a) $f_{\mathrm{B}}(x) =$ _____ if x = (4,5,1); (b) $f_{\mathrm{AB}}(x) =$ _____ if x = (4,5,1); (c) $f_{\mathrm{A}}(f_{\mathrm{B}}(x)) =$ _____ if x = (4,5,1); (d) $\mathrm{AB}x =$ _____ if $x = (4,5,1)$	Action of calculating and comparing the results of applying a matrix function and the composition of matrix functions
W3-6/ How much of pesticide i is absorbed by carnivore j?	Interiorization of the Actions of W3-3 into the Processes of matrix product and composition of matrix function. Coordination of both Processes into their equivalence Process
W3-7/ Calculate: (a) (AB)C ; (b) A(BC); (c) Compare this last two calculations and interpret them in terms of composition of functions	Encapsulation of the product matrix by performing Actions to find its properties and encapsulation of the function composition by comparing it with the matrix product

and suggested to use table rows and columns as vectors and to use linear combinations to determine a mathematical model to work with the problem. The second strategy proposed consisted on using functions to model the absorption of pesticides by the plants and then consider new functions for the herbivores and the carnivores. These ideas were taken by the teachers, one of them discussed with students the possibility to operate with tables and then introduced the first worksheet while the other decided to introduce it after the discussion of the strategies.

The analysis of the data obtained while students worked in class with the designed activities can be give information about students' possible constructions. Some examples of the types of students' responses that give evidence of different types of the constructed structures follow.

During work with the activity W1-6 the following students' assertions were recorded:

> S15: We have the table. We can first calculate the products of the number of plants and the units of pesticide each of them has. The herbivore 1 eats 20 28 30 40 of plants 1, 2, 3 and 4 respectively. And plants have 2,3,4,3 units of pesticide 1. Then we add them. . .it consumes 364 units of pesticide 1. We do the same for each of the plants.
>
> S19: We can think of it, the consumption of an animal herbivore a of plant 1, b plant 2, and so on. We can look at the table, as a matrix as we did before and there is four types of plants, to find the consumption if it eats the 3 pesticides, then we have to add we have a consumption matrix it must have 4 types of plants and if we have 3 pesticides, we do the scalar product of the vectors, one row and one column so it consumes ai1a plus ai2b plus ai3c plus ai4d. Here it is 364 units. We can do that for all the herbivores and the pesticides.

Student 15 shows that she can respond to the question by applying Actions to the information given; as the question can be responded by doing Actions her response is correct. Student 19 describes a general procedure to find the consumption of any herbivore. Her response is also correct, but she shows she has constructed the Process of calculating the dot product of a row vector and a column vector. She then applies this Process to the particular information given. She also shows the construction of the matrix-vector product as a Process. Working with other activities S22 commented to his team members:

> S22: What I can see is the multiplication of a matrix A by a vector x, is the same as the function, a function with vectors. This function f A has a rule and it gives the units of pesticide in a herbivore when it ate certain quantities of plants. With the dot product of each row of A as a vector and the consumption vector.

This assertion shows that this student has coordinated the Process of vector and that of function through the dot product. He gives evidence of the construction of the Process where the product Ax can be considered the rule of correspondence of a function f_A. This evidence was also found in his work on other similar problems.

In worksheet 2 a new matrix associated to the problem situation was presented. Students were all able to do Actions to calculate the i-th pesticide absorbed by the j-herbivore. They were all able to use the dot product of a row of matrix A and a column of matrix B and could interpret this operation in the context of the problem:

> S42: It is the same as before, but now matrix a represents the plants and matrix B represents each herbivore, what each of them consumes. Now the dot product of this row and a column gives what this herbivore has eaten of each plant, when we do all the rows and the columns, we get how much pesticide is absorbed by each herbivore. With each of this numbers obtained. . . as elements, we can write a new matrix with all the information of the pesticide each herbivore has absorbed.

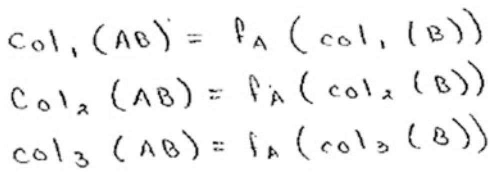

Fig. 1 Example of coordination or processes by student 48

Other students gave evidence that they had interiorized the product matrix *AB*. They deduced a general rule to calculate each of its elements. For example;

S08: We have to calculate the dot product of each row of A by each column of B. We thus obtain c11, c12 etcetera. We repeat it so we can write like a formula cij = rowi(A) · colj (B) to obtain all the elements of the new matrix C.

S09 Yes, this can also be written as a sum, as we did before with the teacher (she wrote cij $= \sum_{k=1}^{n} a_{ik} b_{kj}$).

Student S47 evidenced she had encapsulated the matrix product as an Object. She deduced the previous sum with her colleagues and then she realized that the product is not commutative, demonstrating show could do Actions on the product to find one of its properties:

S47 These matrices have different numbers of rows and columns. We can only do the dot product if the elements of each row of A are the same in number as those of the columns of B, then it is clear that the product of matrices is not commutative, you see that the product AB can be defined, for example but the product of BA may not be defined, for example if A has 2 rows and 4 columns and B has 4 rows and 6 columns we can do AB but BA is not defined. . . the product is not commutative.

Some students showed they coordinated the matrix function with the matrix product Processes by writing the product procedure in terms of functions, as shown in Fig. 1.

Only a few students demonstrated encapsulation of the matrix transformation. They were able to coordinate the Process of product of matrices and the Process of function into a Process that where they can use them indistinctly and to encapsultate this later Process into an Object where both representations are considered as equivalent. Descriptions such as the following can be considered as evidence of this construction:

S47: . . . but here we have fB of x, it is a vector, and we also know that fA of x is another vector v. When we do the product, we can calculate fB and then fA of the vector obtained and we get a new vector. We can verify this now with these matrices, we do fA (fB)(x)) and then AB. . .they are the same, so we can also think of AB as fAB, this is nice!

The previous examples illustrate how researchers can find evidence of construction described in the genetic decomposition. It may be clear, however, that one instance where a student shows a construction is not enough to determine his or her constructed structures. To be able to determine them it is necessary to take into account student's responses and work while facing different problem situations related with the same concepts. This needs to follow students' work through a course and to complement the information obtained through semi-structured interviews or open questionnaires.

4.4 The Role of the Teacher

The methodology used in the class was the ACE cycle. As we have focused on examples showing students' constructions, it may seem that the teacher passively uses the worksheets but it is not so. During each class the teachers talked to students while they worked on the activities to observe what students were doing and to pose them some questions when needed (A). They also interrupted students' work when they considered it pertinent to discuss results obtained with the whole class, compare different strategies followed by teams, to ask new questions and to formalize when considered pertinent (D) and gave students other tasks to work individually or collectively in the classroom or at home (E), depending on what they considered students needed. This behavior is typical in classes using ACE cycle.

Teachers in the experience described above collaborated as researchers in the design of the worksheet used, they were familiar with the genetic decomposition and with the ACE cycle. However, the experience has been repeated with three teachers unfamiliar with the design of the activities included in the worksheets but used the ACE cycle in the class. Results obtained are similar to the described in Figueroa et al. (2018).

5 Final Reflections

The close connection between a genetic decomposition, the design of activities and problem situations is illustrated through the description of the examples presented above. Each task focuses on details involved in the learning of a concept or group of concepts as is the case in this study. Some activities may be similar to those that can be found in textbooks, but most of the times, the need to design an activity that is intended to help students do a very specific construction results in original tasks that are not usually considered without the guide of a model such as the genetic decomposition. Sequences of activities or worksheets, as those described above, provide students with several opportunities to reflect on fine-grained details related to the construction of concepts and to look at them from different angles. Discussion among students and with the teacher involves the need to share ideas, strategies and arguments to justify points of view and opportunities to develop strategies and deepen their understanding of mathematics.

In terms of research, the use of APOS theory provides detailed information about students' constructions. The need to test the genetic decomposition experimentally provides opportunities to critically analyze the constructions predicted and to use new information obtained from research to either validate it, refine it or discard it.

APOS theory aims are also related to its limitations. It is clear that it cannot be used to find answers to questions that are not comprised within its chosen level of analysis. The possibility to establish a dialog with other theories, as has been the case with the Anthropological theory of didactics (Bosch et al., 2017) can help to broaden the use of APOS theory, as changed by the results of such dialogue.

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Introduction to Part II Mathematics Teacher Education and the Professionalisation of Teaching

Berta Barquero and Marianna Bosch

Formulating the Problem of Teacher Education within the ATD

The ATD approach of teacher education takes as its starting point the attempt to convert the craft of teaching into a full-fledged profession. The core of this line of research is the study of teaching "semi-professions" at school, university or vocational school. It is based on two main concepts, first that of a profession (as distinguished from the notion of "semi-profession") and second, the concept of problems of a profession, which refers to the difficulties to face in the practice of this profession. The main idea here is that, regardless of how subjective it may seem to others, any observed difficulty must be taken seriously and looked at as a problem to be solved within the framework of the ATD.

According to Bosch and Gascón (2009), following Cirade (2006), we can propose an initial formulation of the problem of teacher education in the following terms:

> What knowledge and competences are necessary (or, at least, useful) for mathematics teachers to act in an effective and appropriate way in the students' mathematical training? And, what can be done to help teachers become aware, build and/ or acquire said knowledge and competences?

As in any other human activity, the notion of *praxeology* appears as a key tool to approach *teachers' praxis*—what they do to organise teaching processes—and *logos*—why they do it the way they do—as inseparable dimensions. The developments proposed by Cirade (2006), Sierra (2006) and Ruíz-Olarría (2015) introduced

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a distinction among different *types of praxeologies* to better express the complexity of the praxeological equipment involved in the practice of the teaching profession: the "mathematical praxeologies" (to be taught), which are part of the "mathematical praxeologies for teaching", included in the broader set of "teachers' didactic praxeologies", that can finally be understood from the perspective of the didactic research praxeologies. The problem of teacher education can then be reformulated as follows:

> What is the praxeological equipment necessary (or, at least, useful) for mathematics teachers to be able to act in an effective and appropriate way in students' education? And what can be done to help teachers integrate it into their own personal praxeological equipment?

Addressing this set of questions needs to include the *analysis of didactic transposition processes* (Chevallard, 1985; Chevallard & Bosch, 2020) to a certain extent. In other words, the problem of teacher education cannot be separated from the problem of the curriculum that is presented in part 3 of this volume and inherits the main assumptions that are made in this regard. These assumptions mainly correspond to the type of didactic paradigm (Chevallard, 2006, 2015) that prevails in the considered didactic institutions, including schools and teacher education institutes or faculties. Teacher education then appears as a privileged environment to analyse the conditions and constraints produced by the prevailing pedagogical "paradigm of visiting works" and to examine the possible ways to make it evolve toward the one of "questioning the world" (Barquero et al., 2018).

In a first approximation, the second formulation of the teacher education problem can be considered as belonging to the paradigm of visiting works if one assigns a set of supposedly pertinent works to the project of enriching the teacher's praxeological equipment. If we move to the paradigm of questioning the world, instead of focusing on the teacher's praxeological equipment, it is important to place the questions, difficulties or problems to which teachers must provide answers through their professional activity, at the heart of teacher education. Thus, the problem of describing the praxeologies that can increase the teacher's praxeological equipment becomes the problem of determining the questions that are at the origin of these praxeologies. The problem of teacher education then admits a new formulation, which we can consider as dual to the previous one:

> What are the crucial questions that teachers have to face in their teaching practice? And, what can be done to help them construct satisfactory answers to these questions?

The dual character of the two formulations lies in the fact that the answers to the crucial questions are precisely the basic ingredients of the teacher's praxeological equipment. It has the advantage of suggesting that the collective construction of answers does not necessarily follow the visit of some pre-established bodies of knowledge and know-how. Moreover, it has the virtue of keeping the problem of the description of this praxeological equipment open and, therefore, of its construction and dissemination in educational processes. It must therefore be made clear that these questions are not teachers' personal difficulties due to, for instance, an alleged lack of vocation, interest or dedication. On the contrary, they refer to problems that the teaching profession must face and to which it must provide a collective response, i.e. by developing (or making available) appropriate technical and theoretical—praxeological—resources. Didactics can find its function as basic instrumental knowledge for teachers in both the detection and formulation of these questions and the elaboration and dissemination of answers. In the words of Chevallard (2009, our translation):

> A teacher education project approaching the questions raised by the practice of mathematics teaching in a real and effective way [. . .] needs tight cooperation between the school system which constitutes the "field" for the teaching practice; research in didactics, which acts as a source of questioning and production of praxeological resources to the renewal and improvement of this activity; and the teaching profession itself, which is the one who should, in the end identify the—always evolving—needs that all its members should face.

An interesting example of a teacher education proposal founded on the ATD can be found in the pioneer study of Cirade (2006) and the instructional device of the *questions of the week*, based on the questions formulated by pre-service teachers during their internship in schools while following a course in didactics of mathematics. Another approach to educational devices for teachers and, more particularly, to the *lesson studies* that are spreading from Japan, is formulated in terms of *paradidactic systems and infrastructures*. Miyakawa and Winsløw (2013 and 2019) propose characterising teacher knowledge and practices at stake in terms of *didactic and paradidactic praxeologies*. This approach helps identify different types of professional tasks and techniques, as well as professional technologies and theories (García et al., 2019), including questioning the paradidactic ecology, that is, the conditions and constraints acting on paradidactic systems and, more generally, on the teaching profession. Finally, Sierra (2006) and Ruiz-Olarría (2015) proposed *study and research paths for teacher education* (SRP-TE) as an instructional format that places the study of professional questions at the core of teacher education programmes.

Collective Advances During the Lectures and Workshops in Course 2

In this second course, there was a total of 45 participants from 14 different countries. For two intensive two weeks, we all participated in discussing the advances regarding the topic of mathematics teacher education and the professionalisation of teaching. Lectures were followed by workshops, which were, in most cases, closely related to the content of the lectures. We invite readers to have a look at the contributions of the researchers participating in the course that are accessible in (Barquero et al., 2021).

With respect to the lectures, we start with the chapter written by Yves Chevallard (Aix-Marseille Université), titled "Challenges and advances in teacher education within the ATD", which aims to outline the main aspects of the modelling, in the framework of the ATD, of the teacher's position. The author underlines the fragility of the teacher position, which has slowly emerged from its menial origins to stabilise itself as a mere semi-professional occupation, to stress that we find ourselves facing a praxeological revolution which calls for the coordinated use of all the means of thought and action of didactics.

In chapter 2, T. Miyakawa's "Paradidactic infrastructure for mathematics teachers' collective work" addresses the questions related to the mathematics teachers' collective work around classroom teaching. The author investigates how and to what extent different theoretical constructs of the ATD can facilitate the analysis of teachers' collective work inside and outside the classroom. Some examples from Japan and France are considered to show how to characterise mathematics teachers' knowledge and practices.

For their part, F. J. García, E. M. Lendínez and A. M. Lerma present their work "On the problem between devices and infrastructures in teacher education within the paradigm of questioning the world" (chapter 3). They start by facing the challenge of placing teacher education within the paradigm of questioning the world. The authors connect this challenge with two basic problems: the problem of the devices and the problem of the infrastructures in teacher education. They then focus on the lesson study devices to reformulate them in terms of professional praxeologies that take place in different paradidactic systems. This reformulation contributes to elucidating how lesson study can work coherently within the paradigm of questioning the world and to advance in the problem of the infrastructures needed to support it.

The fourth lecture was given by Floriane Wozniak and Claire Margolinas. It concerned the "Introduction of ordinal numbers at the beginning of the French curriculum: a study of a professional teaching problem". In chapter 4, the authors focus on the study of ordinal numbers in pre-primary school that has allowed them to build a didactic engineering approach to observe the evolution of the students' knowledge (as ordinal numbers are not part of ordinary teaching, at least not in France). They used the data collected in an implementation that was put into practice in a class of two teachers with pre-elementary school students (5–6 years old). The aim was to better understand the knowledge that students could build in a sequence of adidactic situations and to better understand the praxeological needs of the teacher and the means for observing them.

The fifth and last lecture was delivered by Avenilde Romo (Instituto Politécnico Nacional, CICATA) and Berta Barquero (Universitat de Barcelona). It was about the "Study and research paths for teacher education (SRP-TE): some advances on teacher education in the paradigm of questioning the world". Chapter 5 focuses on the proposal of the SRP-TE, as an inquiry-based process combining practical and theoretical questioning of school mathematical activities, to place teacher education in the new paradigm of questioning the world. Two case studies of SRP-TE for pre-service and in-service mathematics teachers are presented, for primary and secondary education. Both case studies are used to show the transposition of research tools to teaching practice and how they help teachers to progress in the critical issue of identifying institutional constraints hindering the integration of mathematical modelling.

Chapter 6 presents a workshop guided by Michèle Artaud (Aix-Marseille Université) and Jean-Pierre Bourgade (Université de Toulouse - Jean Jaurès) about the "Transpositive Phenomena of Didactics in Teacher Training". The authors describe the process of institutional transposition of knowledge in an institution that is not a school, as a process called *archididactic transposition*. In light of the didactic transposition theory, and of the latest developments in the anthropological theory of the didactic, the authors examine the process of transposition of mathematical didactics in the particular case of mathematics teacher education.

A workshop led by Vicenç Font, Alicia Sánchez and Gemma Sala (Universitat de Barcelona) about "Prospective teachers' narrative analysis using the didactic-mathematical knowledge and competences model" is described in chapter 7. The authors present the use of tools proposed by the onto-semiotic approach based on a system of categories of knowledge and competences of the mathematics teacher. These categories belong to what the authors have called the didactic-mathematical knowledge and competences model (DMKC). The model is applied to the analysis of a narrative of a classroom session, in order to identify the knowledge and competences of the pre-service teacher.

Chapter 8 is devoted to a workshop guided by Koji Otaki (Hokkaido University of Education) and Yukiko Asami-Johansson (University of Gävle). It deals with "Exploring the paradidactic ecosystem: conditions and constraints on the teaching profession". This chapter aims to show the work using different theoretical resources for investigating the teachers' design and analysis of didactic situations. By introducing notions such as the nested triptych of *paradidactic systems* and the *scale of paradidactic determinacy*, the authors show the kind of levels and institutions that might be considered for the study of the conditions related to the paradidactic work of teachers.

Last but not least, chapter 9 "Teacher learning in collaborative settings: analysis of an open lesson" presents a workshop guided by T. Miyakawa and F. J. García. It concerns the work developed for the analysis of an open lesson organised in a Japanese primary school. Some of the questions the participants addressed were about analysing the mathematical and didactic organisation in the lesson plan and the textbook developed in the classroom, and the paradidactic praxeologies that could be identified in the open lesson.

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Challenges and Advances in Teacher Education Within the ATD

Yves Chevallard

1 By Way of Warning

This presentation focuses on an institutional reality, to wit, the teacher *position* (and some other positions), in the sense of "position" that the anthropological theory of the didactic gives to this word.

A person x occupying a position p, for example that of teacher, is subjected to that position: x is a *subject* of p. A person x is the subject of a great many institutional positions.

The caveat I wish to make here is the following: while the analysis presented is about a *position* p, it happens quite often that the listener or reader misunderstands it and imagines that it really applies to the *persons* in position p.

If one asserts that the teacher "profession" is still a "semiprofession", it does *not* mean that teachers do not have a high personal potential, only that, generally, this potential will not be fully realized in that position such as it is now.

Many institutional positions require from those who occupy them much *less* than what they could give. Projecting on these people the image of the *position* they occupy is often a caricature that ignores and denies their full personality. Beware!

2 A Typical Example

I use the word *teacher* to mean the person y in a didactic system $\mathbb{S}$ (formed in a school σ) denoted by $\mathbb{S} = S(X; y; ♥)$, where X is the set of students and ♥ is the didactic stake.

An English wit and Anglican cleric, Sydney Smith (1771–1845), writes:

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> It made me a very poor man for many years, but I never repented it. I turned schoolmaster to educate my son, as I could not afford to send him to school. Mrs. Sydney turned schoolmistress, to educate my girls, as I could not afford a governess. I turned farmer, as I could not let my land. A manservant was too expensive; so, I caught up a little garden-girl, made like a milestone, christened her Bunch, put a napkin in her hand, and made her my butler. The girls taught her to read, Mrs. Sydney to wait, and I undertook her morals; Bunch became the best butler in the county. (Holland, 1855, p. 159)

Sydney Smith, his wife and his daughters are amateur teachers. This, it seems, has become the exception rather than the rule. But, in fact, a "professional" teacher is an amateur teacher turned professional. A teacher always has to assume an ounce of amateurism: When "certified teachers" have to teach something that they never studied at the university, they must accept to start by being amateur teachers.

This is valid for professions in which the work content may evolves suddenly. What matters is that the profession clearly shows that the ongoing change, considered vital, is under the *profession*'s control, a point to which I shall return.

Sydney Smith's life shows another essential aspect of the teaching trades: just as you can do the cleaning yourself if you can't pay someone to do it, you can make yourself a teacher if you can't afford to pay someone for that. The conclusion to be drawn from this example is unpleasant but little questionable: the teaching trades are generally regarded as low-level occupations. This is, until today, an invariant in the history of teachers.

Sydney Smith shows us another petty craft, that of butler. An improvised teacher, Mrs. Sydney, trains a young girl, Bunch, to become a butler, an occupation that does not rely on a science and has low technological requirements. When an occupation is founded on the personal qualities of its practitioners rather than on a solid body of knowledge, it is bound to be a low-level occupation, particularly if its practitioners adhere to this self-satisfied self-definition.

3 Starting from Scratch

Vannes is a town in Morbihan, Brittany (France). In "Le collège de Vannes en 1830" (1886), Jules Simon (1814-1896), a former Minister of Public Instruction in the provisional government of 1870, wrote the following:

> Our teachers, who were almost all priests, knew Latin perfectly well. Perhaps they also knew, as best they could, a little bit of theology. I can attest that they knew nothing else. In 1829 we were given a physics teacher. We hadn't heard of this kind of study at the college of Vannes since 1789. Mr. Merpaut, who was in charge of this teaching, was like the college: he had never heard of it. He bought an old copy of Father Nollet's *Physics*. "I don't understand it," he said, "but we will read it together, and perhaps by helping each other, we can find out what it means." We couldn't do it. We looted two cabinets containing some outdated physics instruments and many miscellaneous substances. We were very keen to mix the vials with each other under Mr. Merpaut's watchful eye ... We ended up playing pucks during class with the discs from a Voltaic pile. I must say, to pay tribute to the truth, that Mr. Merpaut had a very noisy game. The rhetorics teacher, our neighbour, complained about the noise. Mr. Merpaut was magnificent: "Go tell your master that we are here to study

the laws of nature and that we give him full freedom to do whatever he wants with the laws of rhetoric." (Simon, 1886, pp. 149–150).

The author reports the birth of a didactic system *S*(*X*; *Y*; ♥), with Mr. Merpaut as a physics teacher *y* who, at this stage, knows almost nothing about physics, even if he has heard of Jean-Antoine Nollet (1700–1770), a renowned physicist.

However, the real birth of Mr. Merpaut as a physics teacher *y* occurs when he haughtily declares to his colleague's student: "Go tell your master that we are here to study the laws of nature . . .". Mr. Merpaut certainly knows only that the Greek word *phýsis* (φύσις) means "nature". In this case, the study of nature starts almost from zero. This phenomenon continues to this day, even with "certified" teachers.

In fact, it occurs whenever they have to teach a field of "their" discipline that they hardly know—which was the case in France a few years ago when some mathematics teachers had to teach, for the first time ever, the basics of graph theory.

4 The Teacher as Servant

As is well known, in Ancient Greece, a *pedagogue*—from Greek παιδίον (*paidíon*) "child" and ἀγωγός (*agōgós*) "guide"—was a slave who led the master's children to school and more or less supervised their schoolwork. Given the evolution of European societies, the pedagogues have ceased to be slaves. However, they have become members of the domestic staff: they became servants.

If we look at things from the point of view of the teacher himself (in the case of men), the teaching profession has long been what, given his education, one could do if he could not do "better" (for example, become a civil engineer).

For parents of affluent social origin, the idea has been preserved that teachers are people who, like servants in a household, work in the school so that society outside it can live fully, without worrying that much for the basic social preparation of the rising generations.

5 A Semiprofession

The teaching trade is nowadays considered as a *semiprofession*. To clarify this point, I borrow from the article "Semiprofession" in Wikipedia, in which we read the following (Semiprofession (n.d.)):

> A semiprofession is an occupation that requires advanced knowledge and skills but is not widely regarded as a true profession. Traditional examples of semiprofessions include social work, journalism, librarianship, teaching and nursing.

The article adds that semiprofessions often have "less clear-cut barriers to entry" than professions, while their practitioners "often lack the degree of control over their own work" associated with doctors and lawyers.

The American Association of Colleges for Teacher Education (AACTE) has published a list of 12 criteria that help delineate a semiprofession:

1. Lower in occupational status
2. Shorter training periods
3. Lack of societal acceptance that the nature of the service and/or the level of expertise justifies the autonomy that is granted to the professions
4. A less specialized and less highly developed body of knowledge and skills
5. Markedly less emphasis on theoretical and conceptual bases for practice
6. A tendency for the individual to identify with the employment institution more and with the profession less
7. More subject to administrative and supervisory surveillance and control
8. Less autonomy in professional decision making, with accountability to superiors rather than to the profession
9. Management by persons who have themselves been prepared and served in that semiprofession
10. A preponderance of women
11. Absence of the right of privileged communication between client and professional
12. Little or no involvement in matters of life and death

One crucial observation needs to be mentioned: "In most semiprofessional fields, efforts at professionalization are ongoing." (The opposite process also occurs; professions are subject to processes of deprofessionalization.)

The status of semiprofession expresses itself in a number of "symptoms", the disappearance of which would mean that this semiprofession is professionalizing itself—while, correlatively, their persistence expresses that this status continues unchanged.

One of the most remarkable and, it seems, one of the most obstructive obstacles in moving towards the professionalization of the teaching trades is the fact that teachers regularly fall prey to pedagogical fads. In an article entitled "25 Years of Teaching Fads and Bad Educational Science", a British noospherian, Ross Morrison McGill, has proposed (see Fig. 1) a table of pedagogical fads that overwhelmed teachers over the past two decades (McGill (2016)).

In contrast, the didacticians' dream would be to stop these cycles of waiting/pedagogical agitation/disappointment, and to substitute them with the linear time of the progress of knowledge—whose advent, however, is still to come . . .

6 Towards a Profession?

The path to the status of profession is arduous. The *praxis* and *logos* of any occupation with a long history have a strong inertia, so that, finally, they collapse rather than transform themselves. In the case at hand here, what are these "products" of history? The brief answer that I will give here describes a teacher position which

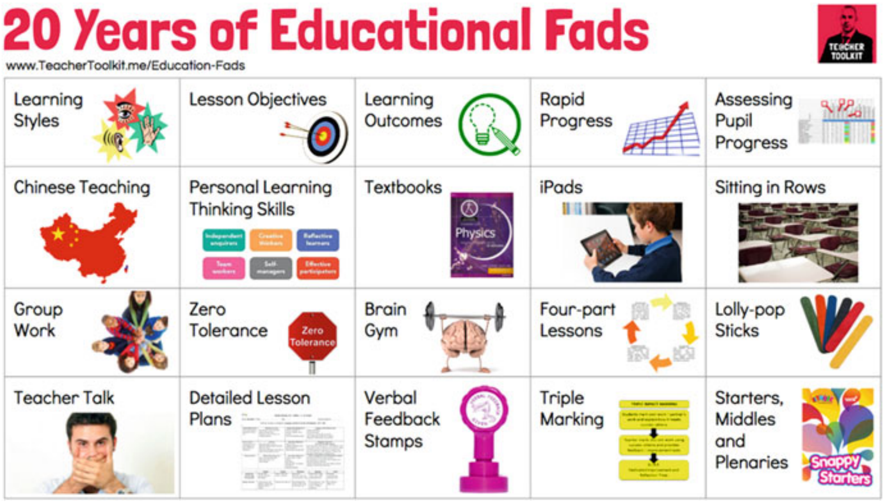

Fig. 1 Unending educational fads

seems to have been so far occupied by a great many teachers, who, therefore, were subjected to it.

The main constraint imposed on teachers by this position is to conceive of oneself and act as an autonomous and relatively solitary "system"—the teacher is similar in this respect to a self-employed worker and a sole contractor. What is often referred to as "the teacher's pedagogical freedom" is the obverse of a coin of which the reverse is the teacher's practical obligation to fend for himself, an obligation whose respect is also a point of honour for him or her.

During the golden age, when teachers played a role, not only as guides, but also as experts of the works to visit, their key achievement (and the essential proof of their adequacy to the teacher position) was the lessons they designed and taught. For a long time, the constraints imposed by the position of teacher pushed teachers *not* to share "their" course with anyone but their students. This is an attitude that, with the emergence of the Internet, has begun to fade away.

7 A Few Reminders

How can things evolve without deteriorating? First of all, a few reminders. At the basis of any form of education there are *didactic systems* $\mathcal{S} = S(X; Y; \heartsuit)$, within a school σ in a school system Σ.

Here, the set X is the set of "students" (we should have $\text{card}(X) \geq 1$) and Y is the set of "teachers" (we can have $Y = \varnothing$ but often have $\text{card}(Y) = \text{card}(\{y\}) = 1$). The symbol ♥ refers to the "didactic stake", the work to study and learn.

Let us denote by p_s the student position, to which the students $x \in X$ are subjected, and by p_t the teacher position in $\mathcal{S}$. A teacher $y \in Y$ first and foremost considers not so much the students $x \in X$ taken one by one as the student *position* p_s.

The teacher y refers to a *cognitive nucleus* $\tilde{n} = (\hat{\imath}, o, \hat{s}, v^\wedge)$, with $\hat{\imath} = p_s$ and $o = ♥$: the *cognitive base* is $n^- = (\hat{\imath}, o) = (p_s, ♥)$. The *standard instance* $\hat{s} = (I, p)$ may be of y's own choosing, with the *evaluating instance* $v^\wedge$ being y: $\underline{n} = (\hat{s}, v^\wedge) = (\hat{s}_y, y)$.

However, the *cognitive frame of reference* $\underline{n} = (\hat{s}, v^\wedge)$ may be quite diverse. For example, in σ or Σ, it may be that, to all classes of the same grade, there corresponds a common *cognitive frame of reference* $\underline{n} = (\hat{s}, v^\wedge)$, specific to σ or Σ.

We will assume hereafter that the principle of teachers' activity is the following: from their own point of view, teachers must act in order to make the relation $R(p_s, o)$ evolve to be judged by $v^\wedge$ *closer* to $R(\hat{s}, o)$.

The expression $\varsigma = (\tilde{n}, \hat{u}, \delta, \mathcal{C})$ denotes a *possibly didactic situation*: $\tilde{n}$ is the cognitive nucleus $(p_s, ♥, \hat{s}, v^\wedge)$, $\hat{u}$ is any instance making a *gesture* (i.e., any task of any type) δ, and $\mathcal{C}$ is the set of conditions prevailing *before* δ is performed.

We focus here on the case where $\hat{u} = y$, so that $\varsigma = (\tilde{n}, y, \delta, \mathcal{C})$, with $\tilde{n} = (p_s, ♥, \hat{s}, v^\wedge)$. The gesture δ that y will perform depends on y's personal relations to p_s, ♥, $\underline{n} = (\hat{s}, v^\wedge)$, and $\mathcal{C}$: it will therefore depend on y's *praxeological equipment*.

The gesture δ changes the set $\mathcal{C}$ into $\mathcal{C}' = \mathcal{C}^{/\delta}$, (read "$\mathcal{C}$ deranged by δ"), i.e., δ changes the *ecology* of $\mathcal{S} = S(X; Y; ♥)$. We thus have a function $\delta^\wedge{}_y$ from the set of ordered pairs $(\tilde{n}, \mathcal{C})$ to a set of gestures Δ: $\delta^\wedge{}_y: (\tilde{n}, \mathcal{C}) \mapsto \delta^\wedge{}_y(\tilde{n}, \mathcal{C}) = \delta \in \Delta$.

It is assumed that Δ contains the "do nothing" gesture $\delta_\varnothing$: when $\delta^\wedge{}_y(\tilde{n}, \mathcal{C}) = \delta_\varnothing$, y does nothing. The study of *the ecology and economy* of *possible didactic functions* $\delta^\wedge{}_.$ and their learning yields is a major research type of tasks of didactics.

If it is true that a didactician is free to consider other standard instances than the one y has in mind, the learning yields mentioned above are always relative to some standard instance $\underline{n} = (\hat{s}, v^\wedge)$: *learning is a socially defined process*.

8 Some Questions

In the foregoing, we mentioned a teacher y, i.e., a *person*. In fact, we are interested in the teacher *position* p_t: we are concerned with the *ideal subject* $\bar{y}$ of p_t, subjected to p_t only, unlike a real person.

Teacher training is based on didactic systems denoted as $\mathcal{S} = S(Y; \check{Z}; \check{o})$, with obvious notations. This requires the creation of a position p_{wt} for "*w*ould-be" teachers $\check{y} \in Y$ and a position p_{tt} for teacher trainers $\check{z} \in \check{Z}$.

It thus requires to define relations $R(p_{wt}, \check{o})$ and $R(p_{tt}, \check{o})$, which raises the first big question for educational institutions: What should the position p_{wt}'s praxeological equipment be made of? Put naively: *What should student teachers study?*

A key issue for $\bar{y}$ is: if $\mathcal{C}^*$ is a set of conditions that $\bar{y}$ would like to see fulfilled, do we have $\Delta^* = \{\delta / \delta \in \Delta \wedge \mathcal{C}^{/\delta} \supset \mathcal{C}^*\} \neq \varnothing$? In a host of cases, Δ^* *is* empty: for example, $\bar{y}$ cannot by himself reduce economic inequalities in society.

This is a type of problems that student teachers must unswervingly address. For any "difficulty" observed in $\mathcal{S} = S(X; Y; \heartsuit)$, student teachers will have to study which sets $\mathcal{C}^*$ eliminate it and which gestures $\delta \in \Delta$ are such that $\mathcal{C}^{\prime\delta} \supset \mathcal{C}^*$.

Behind all this, there lurks another question, which is still today an open question, or rather an *unasked* question: What should the position p_{tt}'s praxeological equipment be made of? *What should teacher trainers study?*

This question must be highlighted here. Even today, many academics start to train professors as Mr. Merpaut taught physics. But what was meritorious in 1830 at the college of Vannes seems today somewhat incongruous.

9 The Paradigm of Visiting Works and Its Limitations

What a teacher must know how to do depends on many parameters. The most important of them is the *school paradigm* in which the teacher acts, and which specifies "what they do in class"—do they *visit works* or do they *question the world*?

A didactic system $\mathcal{S} = S(X; y; \heartsuit)$ can now be written $\mathcal{S} = S(X; y; W)$, where W is a "work", i.e., any reality made by humans. The difference between paradigms appears first with the level of *pedagogy*.

$$\begin{gathered} \ldots \\ \downarrow\uparrow \\ \text{Schools} \\ \downarrow\uparrow \\ \textit{Pedagogies} \\ \downarrow\uparrow \\ \text{Didactic systems} \end{gathered}$$

In the ATD, pedagogy is the activity of leading students $x \in X$ to the work W to study. In a school σ, grouping (or not) students into *classes* (where didactic systems will form) is an important part of the school's pedagogy.

Another important aspect is that, generally, access to a work W is not random: the class follows a path laid out in σ (or Σ), which leads to W and comprises a succession of nested fields: Discipline $\supset$ Domain $\supset$ Sector $\supset$ Theme $\supset$ Subject.

Let us consider a given work W. In the paradigm of visiting works, a class will study W if, and only if, W appears in one of the nested successions mentioned above. In what follows, W is a "subject", i.e., here, a type of tasks.

Let's take the case of a mathematical work long known as the "rule of three", a name now used (anachronistically) to refer to the technique of solving proportions by cross-multiplying. Suppose there exists in Σ this pedagogic path: *Mathematics* $\supset$ *Arithmetic* $\supset$ *Fractions and decimals* $\supset$ *Special uses* $\supset$ *Rule of three*. No such path is given by nature: it is a human creation—hence, here, its perhaps unusual aspect.

Suppose that *y* wants to convey to the students a technique t^{-} more user-friendly than cross-multiplication, by generalizing the more elementary technique *t* illustrated by this example: "if 3 books cost 6*s.* 7*d.* what will 12 cost?"

The technique *t* works thus: 12 is 4 times 3, and therefore 12 books will cost 4 times the price of 3, i.e., 4 × 6 *s.* 7*d.* = £1 6 s. 4d. Now if we want to know the price of 17 books or 329 books, since 17 and 329 are *not* multiples of 3, what will we do?

The example used is on page 52 of *A Treatise on the first principles of Arithmetic, after the method of Pestalozzi* (London, 1847) by Thomas Tate, under the title "Rule of three (where a knowledge of Fractions is not required)".

The scope of the technique *t* astoundingly increases if we accept the concept of a *fractional* number of times: 17 is 17/3 times 3, so that if 3 books cost, say, 34.5 €, 17 books will cost 17/3 times 34.5 €, or (17 × 34.5 €)/3, i.e., 195.5 €.

Because the list of works to visit is pre-established, the paradigm of visiting works, when strictly implemented, does not allow replacing the currently used technique (by cross-multiplication) with the technique t^{-}.

10 The Paradigm of Questioning the World

Indeed, in the paradigm of questioning the world, the "teacher" *y* and the class are free to consider the following question: "Is it possible to extend the scope of the elementary rule of three *t*, where one of the numbers of items must be a multiple of the other?"

The historical transition to the new paradigm has a high price for *y*, especially (but not only) in pedagogic terms: starting from a question *Q*, it is now up to the students led by *y* to identify the fields of knowledge useful for their inquiry on *Q*.

Suppose a class of eighth graders (13–14 years) studies the following question: "What is this thing called *apparent temperature*?"

Let's skip the first steps of the inquiry. The class has arrived at this passage from the article "Wind Chill" in Wikipedia (Wind chill (n.d.)):

> The standard formula for Environment Canada is: $T_{wc} = 13.12 + 0.6215T_a - 11.37v^{+0.16} + 0.3965T_a v^{+0.16}$ where T_{wc} is the wind chill index, based on the Celsius temperature scale; T_a is the air temperature in degrees Celsius; and *v* is the wind speed at 10 m (33 ft) standard anemometer height, in kilometres per hour".

A part of the formula giving the index T_{wc} is clear to the students. But the expression $v^{+0.16}$ is unknown to them, although it looks like v^1, v^2 or v^3. They conclude that it is known to at least *one* position in *one* institution, that of meteorology.

The same article specifies that, if $T_a = -20$ (°C) and $v = 5$ (km/h), then $T_{wc} = -24$; and that, if $T_a = -20$ and $v = 30$, then $T_{wc} = -33$. The students decide to use their calculator and to interpret the expression $v^{+0.16}$ as meaning *v* ^0.16. They get the following results:

13.12 + (0.6215 * (−**20**)) − (11.37 * (**5**^0.16)) + (0.3965 * (−**20**) * (**5**^0.16)) = −**24.27**85032833
13.12 + (0.6215 * (−**20**)) − (11.37 * (**30**^0.16)) + (0.3965 * (−**20**) * (**30**^0.16)) = −**32.56**80444763

They conclude that they now know *another* institution that also knows decimal exponents such as +0.16: the "institution of calculators". Such are the very first steps in their inquiry on "apparent temperature".

11 A Foreseeable Future

Many types of tasks that y will have to perform in the new paradigm are still to explore, such as working with the students to identify essential or useful fields of knowledge (here, mathematics, but also physics, meteorology, etc.).

The case of apparent temperature highlights a real difficulty of the new paradigm: When a class answers a question Q, what can be the *standard instance* $\hat{s}$ against which this answer will be appraised by some *evaluating instance* v^?

There is thus a whole history to build collectively through the work of didactic systems $S(p_s; p_t; Q)$ and of teacher training didactic systems $S(p_{wt}, p_{tt}, Q)$, where all the questions raised by the new paradigm will have to be studied.

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Analyzing Mathematics Teachers' Collective Work in Terms of the Inquiry

Takeshi Miyakawa

1 Introduction

Mathematics teachers working and learning through collaboration are gaining particular attention today in terms of a form of teacher professional development and an object of mathematics education research. The Japanese lesson study (Fernandez & Yoshida, 2004; Isoda et al., 2007) is one of the typical examples often referred to in such a context. Apart from it, one may find a wide range of teachers' collective work inside and outside school across and within educational systems in different parts of the world (Robutti et al., 2016). The teachers may officially or unofficially collaborate face to face or online for different purposes: preparing ordinary mathematics lessons, developing teaching materials or textbooks, collaborating in a research project of teachers and researchers, and so forth.

In this lecture, focusing on the teachers' work around the classroom teaching and the system of *conditions* and *constraints* that shapes them, we investigate how and to what extent different theoretical constructs of *the Anthropological Theory of the Didactic* (ATD, hereafter) allow us to analyze and better understand teachers' collective work inside and outside the classroom. The questions we address in this lecture are the following:

- How to characterize mathematics teachers' collective work inside or outside school?
- How to analyze the teachers' knowledge and practices in such work?
- How to analyze the dynamic process of teacher collaboration?

This lecture is organized mainly with two examples of teachers' collective work from the following two perspectives. We investigate, on the one hand, Japanese

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teachers' work inside and outside mathematics class in terms of *paradidactic praxeologies*, which characterize teacher knowledge and practices at stake (e.g. Miyakawa & Winsløw, 2013, 2019). On the other hand, teachers' collective preparation of mathematics lessons in France in terms of *inquiry*, which characterizes the dynamic process of the development of lessons (e.g. Trouche et al., 2019).

2 Teacher's Work Inside and Outside Mathematics Classroom

Teacher's individual work and/or teachers' collective work cannot be reduced to the teaching practices in the classroom. Mathematics teachers work individually or collectively around these teaching practices for developing and implementing mathematics tasks, lessons, or resources (textbooks, worksheets, etc.), as well as for their professional growth. The mode of teacher's work varies across and even within countries depending on the contexts, such as professional development and everyday teaching.

2.1 Paradidactic Praxeologies

One of the issues we often encounter in the international community of research on mathematics education, and in particular teacher education, is how to characterize teachers' knowledge devoted to such practices and/or developed during these practices. It is well known that Shulman (1986) proposes some categories of teacher's knowledge (subject matter content knowledge, pedagogical content knowledge, and curricular knowledge), and Ball and her colleagues specify, following Shulman's idea, the domains of mathematical knowledge for teaching (Ball et al., 2008).

Within the ATD, the notion of *praxeology* is a critical idea to model the practices and knowledge inside and outside the mathematics classroom. In addition to the mathematical and didactical praxeologies (MP and DP respectively hereafter) that are to be implemented or that are implemented in the classroom, this notion allows us to describe teacher's practices and knowledge outside classroom, as a *paradidactic praxeology*, which is a praxeology associated with the praxeologies at stake in the classroom (Winsløw, 2011; Miyakawa & Winsløw, 2013). Further, while the previous research work—the ones mentioned above, for instance—tends to deal with teachers' practices and knowledge separately, the notion of praxeology brings them together and clarifies how a specific knowledge is invested in a teacher's specific action.

As the variety of teachers' practices can be found in the educational fields of different countries, there is a variety of the types of tasks required for the teachers and, as a consequence, the variety of teacher's knowledge associated with and developed with it. As Chevallard (2002, p. 6) mentions, "the system $(T_{\pi}^{(k)})_{k\in \mathrm{I}}$ of professional tasks is neither fixed once for all, even at the level of a career nor totally predefined". The main issue for our research work is to identify the paradidactic praxeologies with different instances of teacher's individual or collective work in different countries and reveal how they are related to the praxeologies in the classroom (MP and DP). Besides, it would also be important to identify the evolutions of paradidactic praxeologies when considering professional development or teacher learning.

In the previous work, some examples of praxeological analysis of teachers' paradidactic work have been proposed with the data of Japanese open lesson, including an actual lesson with a detailed lesson plan and a post-lesson discussion, and show how to characterize teachers' collective work (Miyakawa & Winsløw, 2013). In this lecture, I further propose another way to analyze teachers' collective work in terms of *inquiry*, which is a principal object of study within the ATD.

2.2 Paradidactic Infrastructure

Teachers' paradidactic praxeology does not exist "in vacuo", like any other praxeology. It is shaped by the *conditions* that support such a praxeology to exist and the *constraints* that hinder it. In particular, the basic system of these conditions and constraints that is indispensable to make a paradidactic praxeology viable has been called *the paradidactic infrastructure* (Winsløw, 2011; Miyakawa & Winsløw, 2013), following the notion of *didactic and mathematical infrastructure* introduced by Chevallard (2009, 2019). The notion of paradidactic infrastructure is critical to characterize the different settings or contexts of teachers' work outside the classroom and to understand better teachers' didactic or paradidactic practices that vary according to the didactic institutions.

Due to the diversity of teachers' work as mentioned above, the ordinary practices in a given institution seems, from time to time, extraordinary from the perspective of another institution. This was a case for the Japanese paradidactic practices that a number of teachers observe a lesson given by a colleague in the classroom and discuss it right after the lesson. Such practices are carried out within the context of lesson study or open lesson, which constitute a paradidactic infrastructure that supports them. However, while its characteristics are more or less known today outside Japan, it is not evident yet, even for Japanese people, what conditions such practices and hinders them. This is because these practices are very natural for Japanese teachers and researchers, and so they are involved in these practices as participants without asking about their viability and sustainability. It is necessary, therefore, for the researcher to investigate the paradidactic infrastructure encompassing these practices in a specific case of paradidactic practices like the study of Miyakawa & Winsløw (2019), in addition to describing the setting of teachers' practices. In this paper, there is not enough room to discuss this issue, but it is certainly a critical issue to tackle in the mathematics education research on teacher education.

3 Analysis of Teachers' Collective Work in Terms of the Inquiry

In addition to identifying the key praxeologies in teachers' work for the classroom teaching, some theoretical constructs of ATD allow us to describe the dynamic process of how some specific MP and DP could be realized in the classroom through the teachers' different work outside the classroom and how they acquire or learn the professional knowledge related to the mathematics teaching.

3.1 Teachers' Work as an Inquiry

The key idea is to regard teachers' work related to mathematics teaching inside and outside the classroom as an *inquiry*. In order to design and implement tasks and/or a teaching sequence in the classroom, the teachers investigate curricular materials such as national curricula, mathematics textbooks, and related literature, and understand or clarify the goals of teaching, as well as for deciding or inventing which tasks they are going to use in the classroom and how. This process is similar to the process of *inquiry*. In fact, Japanese teachers' individual and/or collective work on their teaching practices in the context of *lesson study* (Fernandez & Yoshida, 2004) and *practice research* (Miyakawa & Winsløw, 2019; Miyakawa & Xu, 2019) is regarded as a kind of research which investigates the effective teaching (Takahashi & McDougal, 2016). This is why in Japan, the term *kenkyu* (study or research) is very often used in teachers' community, such as *jugyo-kenkyu* (lesson study), *kyozai-kenkyu* (study of teaching materials), and *jissen-kenkyu* (practice research). Further, even in the community of mathematics education research, this idea of regarding teachers' work as a process of inquiry is not something odd. Some previous research work analyzes teachers' work from the perspective of *documentational inquiry* (Margolinas & Wozniak, 2010; Assude, 2010).

In the ATD, *the inquiry* is an important object of study in the context of the paradigm of *questioning the world*, and several theoretical tools have been developed to characterize it. The notion of inquiry is broadly defined as "the action taken to provide an answer A to a question Q" (Chevallard & Bosch, 2019, p. xxiv). The teacher's work, such as designing lessons or developing resources, can be naturally considered as a kind of inquiry. The rich analytical tools in the ATD allow us to describe the dynamic process of teachers' work as well as for their learning. Let us analyze two example cases: lesson study in Japan and teachers' collective designing of lessons in France.

3.2 The Case of Lesson Study

The lesson study in Japan is the teachers' practices, including the designing and implementation of lessons, often carried out in the context of school-based professional development called *konai-kenshu* (see Fernandez & Yoshida, 2004 for more details). It is not usually easy to identify the critical aspects in terms of the professional development to characterize or conceptualize it because this is a teachers' spontaneous practice in the field, and there is a diversity in its form in Japan. For example, the lesson study is often described as a cyclic process (Lewis et al., 2006), but it is not often the case in Japan that the long process of lesson study is iterated several times because the focus is not a single lesson, but the sequence of lessons for a unit or chapter and one lesson study does not usually continue to the next school year. Further, while teacher collaboration is often emphasized in the description of lesson study outside Japan, it is scarce in Japan that a group of teachers collaborates all through the process of lesson study. Usually, a single teacher is designated as a principal teacher who designs and implements the lessons. It is a professional development mainly for that teacher. If another lesson study is organized in the same school, another teacher will be designated as a principal teacher and design lessons for another grade and unit.

When describing a practice, the researcher implicitly adopts a specific viewpoint that seems important to remark on. It is, therefore, crucial to set up explicitly a framework or viewpoint to characterize the practice in scientific research. In this paper, we regard teachers' work in the lesson study as an inquiry and the theoretical constructs related to the inquiry within ATD are adopted. From this perspective, our claim is that the structure of the lesson study is characterized as in Fig. 1, while there should be some variations according to the context.

As mentioned earlier, teachers' practices in the lesson study could be characterized in terms of paradidactic praxeologies. What describes teachers' practices from

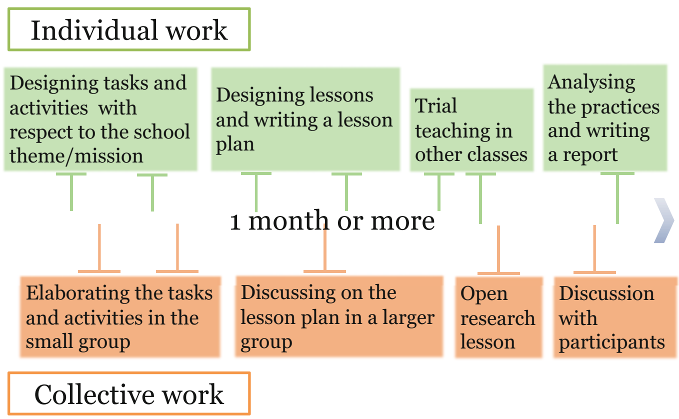

Fig. 1 The dialectic of the individual and the group and the types of associated tasks

this perspective is, first of all, the *types of tasks*, that is, paradidactic tasks in our case. Figure 1 describes in the chronological order paradidactic types of tasks the teachers are working on in the process of lesson study. I also take into consideration *the dialectic of the individual and the group*, which is a dialectic that can be found in the inquiry (Chevallard & Bosch, 2019, pp. xxiii–xxiv). This aspect is critical in the context of *Study and Research Paths* (SRP) or inquiry-based teaching as well as for in the lesson study because, on the one hand, there is always individual work in some parts of lesson study and inquiry and, on the other hand, this aspect describes the *topogenesis* (Chevallard, 1991) of the teachers' overall practice and knowledge (i.e. paradidactic praxeology) as well as the mathematical and didactical praxeologies to be implemented in the classroom. The notion of *topogenesis* describes in the ATD the roles of different participants (usually teacher and learners; principal teacher and other teachers or experts in the case of lesson study) in terms of the emergence of knowledge. In contrast, the horizontal bar (timeline) shows the *chronogenesis* (Chevallard, 1991) that describes the timings of the emergence and development of a paradidactic praxeology as well as the mathematical and didactical praxeologies. Let me briefly further explain this diagram (Fig. 1).

In the lesson study within a school, an individual teacher is designated to carry out a lesson study. He or she first designs mathematical activities (MP) to be implemented in the classroom in relation to the theme of the school. Then, the teacher brings them into the discussion with the colleagues in a small group and develops these activities. After the discussion, there is still individual work, and repeat this discussion two or three times. After designing the activities, the teacher writes a formal lesson plan which details the goal of lessons, the structure of the sequence of lessons, the ideas of lesson design, and the progression of a lesson that will be demonstrated to the colleagues and visitors [see the examples given by Fernandez and Yoshida (2004) and Miyakawa and Winsløw (2013)]. Then, this principal teacher takes it to another meeting in a larger group in the school and discusses the detail of the lesson plan. After the discussion, this teacher revises the lesson plan. Then, before the open research lesson, the teacher makes essays in other classes, if possible. Then the teacher carries out the open research lesson to the colleagues within the school. If this is a big open lesson, there are teachers from other schools. Then, the post-lesson discussion follows. The participants discuss the reasons for teacher's instruction in the classroom, or what was good, what was not good enough, how it can be improved, and so forth. After the lessons, the teacher analyses his/her teaching practices during these lessons (no single lesson) and writes a *practice report* [see for the examples Miyakawa and Winsløw (2019) and Miyakawa and Xu (2019)]. Usually, the lesson study is finished at this step. But sometimes, the teacher is asked to present these teaching practices in the meeting or congress of teachers' association. And a very few times, these practices are published in a book for sale at bookstores.

The characterization of teachers' practices in the lesson study, as shown in Fig. 1, is a first step of the analysis of the inquiry. While it shows an overall process of teachers' work, it does not show how the teacher develops MP and DP to be implemented in the classroom with supports from his/her colleagues. In particular, another aspect of the development of praxeology, which is often called *mesogenesis* that describes the evolution of *milieu* (Chevallard, 1992), has not been taken into account. This aspect is going to be discussed with the next example in the micro analysis of teachers' work.

3.3 The Case of Collaborative Design of Lessons

The teachers' work of designing and implementing lessons in the classroom happens in the process of lesson study. The theoretical development of ATD related to the inquiry allows us to go further to investigate this dynamic process of teachers' work. In what follows, we provide an example of the analysis of teachers' practices, in the case of the collaborative design of lessons in France (Trouche et al., 2019). The French case, not the Japanese case, is discussed here due to the methodological difficulty of data collection as lesson design is often carried out as individual work. While the context of teachers' practice is different from the lesson study in Japan, I consider that the same analytical method could be applied in the case of lesson study.

Two teachers of a French public lower secondary school were working together for preparing lessons on a topic, *algorithmic*, which was newly introduced in the French mathematics curriculum. The data was collected from a one-hour session wherein the teachers explore several textbooks and discuss what activities they will use and how to teach this topic. The process of collaborative design in terms of the inquiry within the ATD is summarized in Fig. 2.

This diagram accounts for, first of all, the *dialectic of questions and answers* (also called *the dialectic of inquiry*; Chevallard & Bosch, 2019, p. xxiv), which is a main characteristic of inquiry from the perspective of ATD. The timeline is segmented into stages according to the *questions* (given below the timeline) the teachers dealt with during the collective work. The *answers* to these questions are given above the timeline. These answers were also classified according to the two-kinds of praxeologies: algorithmic praxeology in blue and didactic praxeologies in orange. The diagram shows that two-thirds of the session was mainly used to explore mathematical activities for students that were in line with the target of algorithmic praxeology; one third was used for the discussion on the didactic praxeology with respect to the activities they found. There are also some interactions between them, which are called in the ATD the *didactic codetermination* (Chevallard, 2002), the dynamic process wherein the mathematical praxeology and didactical praxeology have determined each other.

Figure 2 provides further details about the dynamic process at a micro-level how the teachers delimit the praxeologies to be implemented in the classroom through the discussion with the colleague and the investigation of related materials and documentation. This process is called in the ATD *the dialectic of media and milieus* (Chevallard & Bosch, 2019, p. xxii), which is another main characteristic of inquiry. The *media* is defined as "a medium being any system that issues messages", and the *milieu* is "a system deemed to be devoid of any intention to prove or disprove σ

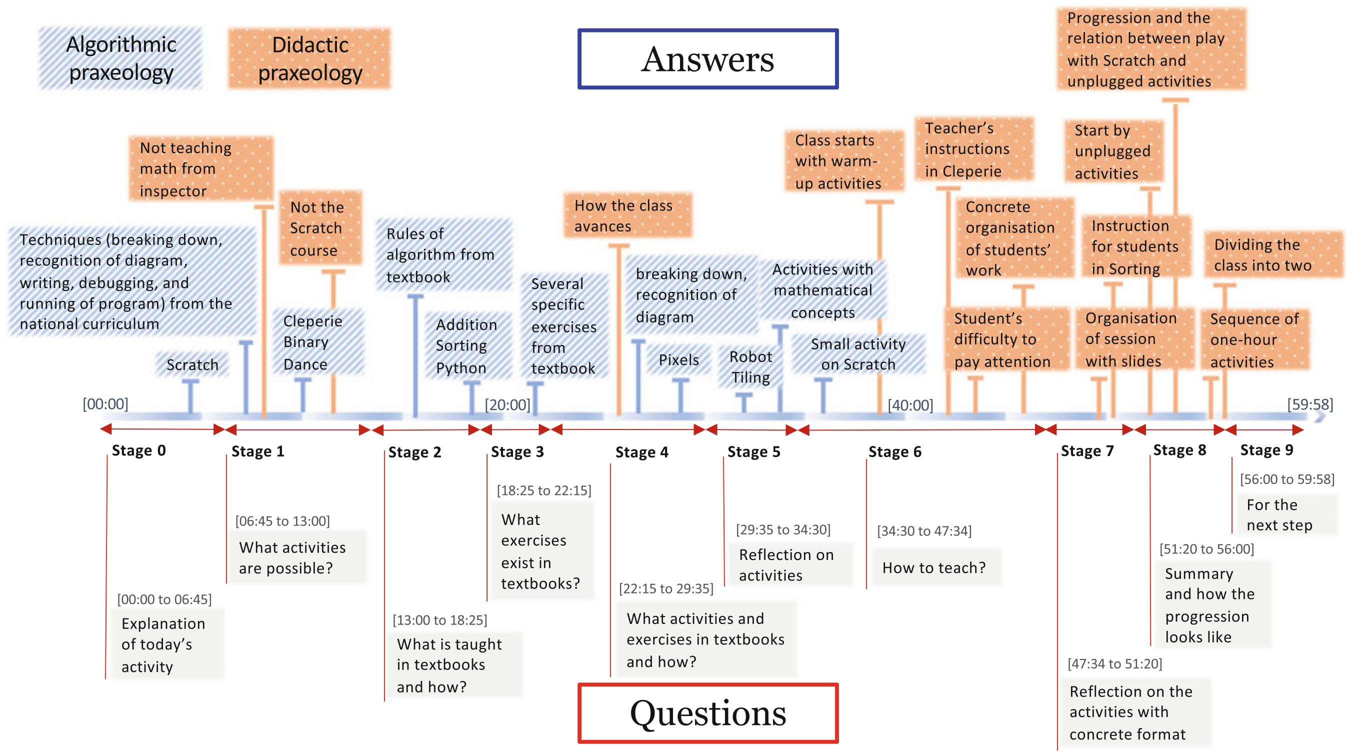

Fig. 2 Questions, answers, and praxeologies on a timeline (Trouche et al., 2019)

[a statement], much like a part of the inanimate world" (idem), which is able to produce feedbacks to the inquirer (the teacher in this case). In the process of designing the lessons, the teacher obtained the answers or information (given above the timeline) not only by themselves but also from different resources (media): textbooks, internet, curricular documents official or not, colleagues, and so forth. These answers constitute a part of the *milieu* with which the teacher interacts during the lesson design. Therefore, the answers above the timeline in Fig. 2 shows the evolution of the *milieu* for the teachers, that is to say, the *mesogenesis* of the paradidactic praxeology.

In addition, the timeline implies, as in the analysis given in Fig. 1, the *chronogenesis* of teachers' paradidactic praxeology as well as the mathematical (or algorithmic) and didactical praxeologies to be implemented in the classroom.

I briefly further explain the detail of this diagram (Fig. 2). For the collaborative work, the different kinds of resources (*media*) had been prepared so that the teachers would be able to consult during the session, such as the national curriculum, several series of mathematics textbooks, the documents related to the algorithmic teaching, and so forth. The two teachers worked on the different questions for designing lessons to be realized in the classroom. As mentioned above, many of them relate to the mathematical activities (MP) and some to the teacher's instruction on how to teach (DP). At the beginning of the session (Stage 1), the teachers consulted the *media* and proposed some tentative answers to the initial question on the students' activities to be implemented. They referred to the national curriculum as well as the direction given by an inspector in the professional development program one of the teachers had taken prior to this collaborative work and then proposed some activities they preferred (Creperie, binary activity, and dance). From Stage 2, the teachers started investigating different textbooks (media) and found different activities and exercises as well as the teaching methods. In these stages (mainly from Stage 2 to Stage 5), the main concern of the one teacher was to find 'unplugged' activities which are carried out without using the computers; and the other teacher was working on the activities or exercises they found in the textbooks, in addition to browsing the different textbooks. One can find here more and more the interaction with the *milieu*. From Stage 6, after finding preferred activities, the teachers began working on the didactic questions on how to organize teaching activities in the classroom. In particular, they decided to start the lessons with unplugged activities such as Creperies (a kind of sorting activity) and sorting.

The process of designing lessons in the lesson study in Japan would be different in terms of the way of collaboration between teachers. In the French case, the two teachers were really working together with the same goal and with the same responsibility, while in the Japanese lesson study, the roles and the responsibilities (*topos*) attributed to the participants are often different according to the kinds of participates. In terms of the ATD, the *topogenesis* of mathematical and didactical praxeologies to be implemented in the classroom is very different. However, it is our further question to what extent these characteristics of paradidactic practices are different between France and Japan. The process of developing the mathematical activities and designing the lessons might be similar to each other.

4 Discussion and Conclusion

This lecture aims to show how we can analyze teachers' collective work inside and outside the classroom in terms of the ATD. Its theoretical constructs related to the inquiry allow us to elucidate teachers' work involved in the process of lesson study and the lesson design. In particular, the dialectics related to the inquiry (questions and answers, media and milieus, and the individual and the group) play crucial roles to characterize the three principal aspects of the genesis of teacher's paradidactic and didactic knowledge: *topogenenesis*, *chronogenesis*, and *mesogenesis*.

Based on these results, two main issues seem crucial to be addressed in our further study. The first one is the issue mentioned earlier on the paradidactic infrastructure that makes teacher collaboration viable in different countries. In this paper, I discussed the two example cases from the two different countries. Both of them actually exist as a possible form of teacher collaboration. However, it is not evident at all what cultural elements allow such existences. Some previous studies claim, in the case of lesson study, that teachers' associations, regional or national congress, journals or bulletins for publication would be the critical conditions for teachers' collaborative activities like lesson study (Miyakawa & Winsløw, 2019; Miyakawa & Xu, 2019). Further detailed analysis is necessary.

The second issue is related to the theoretical constructs and the characterization of teacher collaboration. The frameworks used to characterize teachers' work in this paper are the ones developed in the general context of teaching and learning within ATD. Another important issue for our future study is to develop analytical tools to characterize the aspects specific to teacher collaboration and to the design of mathematics lessons. There are some attempts to tackle this issue. A Franco-Italian group, for instance, tries to characterize teachers' collaborative work with researchers in terms of the *Meta-Didactical Transposition*, which describes the mechanism of how the researchers' praxeologies and teachers' praxeologies are shared and developed to the new ones (Aldon et al., 2013; Arzarello et al., 2014). Since the teachers' collaborative project often include different kinds of participants, such as teachers, teacher educators, researchers, and politicians, the different praxeologies according to them play important roles in the collaborative work. Another recent theoretical development by Otaki et al. (2020) would also be an important contribution in this regard. They propose six dialectics that characterize the constraints specific to the teachers' work around mathematics classroom as well as other types of teachers' paradidactic work: the dialectics of *stakes and gestures*, *period and study program*, *milieu and infrastructure*, *the predidactic and the postdidactic*, *school and noosphere*, and *the designer and the analyzer*.

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On the Problem Between Devices and Infrastructures in Teacher Education Within the Paradigm of Questioning the World

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1 Introduction

Schools, in a broad sense, are social institutions created to facilitate that certain groups of people get access to certain kind of knowledge. With any social institution, the way they work is affected by a complex set of conditions and restrictions. These determine, very often implicitly, what could be studied within a given institution and how. That is what Chevallard (2015), in the framework of the Anthropological Theory of the Didactic (ATD), called a didactic paradigm. Initially, he distinguished between two ideal cases, named the *paradigm of visiting works* (PVW) and the *paradigm of questioning the world* (PQW). In short, in the former paradigm, the study is organized as if mathematical works had already been created elsewhere and the students are just visiting them. Mathematical works are presented as finalized works, giving students little opportunities to understand why these works were created and what they serve for (it is usually said that their *raison d'être* is absent). In the latter, the study is organized around the exploration of crucial and live questions. Thus, it is fostered a functional approach to mathematical works as means worth knowing to deal with such questions (that is, their raison d'être is put at the front of the study process). The PQW is generally represented with the developed Herbartian scheme (Fig. 1), describing a didactic system S in which: X are the students, Y are the 'study helpers' (teacher(s), in a broad sense), Q the question students are exploring, already existing answers that they could consider when exploring Q, W_j already existing works useful to study and deconstruct such answers, Q_k derived questions that could emerge along the study process (whose exploration could need considering new answers and new works W_j), and $A^{\heartsuit}$ the answer X could get to, as a consequence of the whole study process, that could be

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$$[\, S(X; Y; Q) \rightarrow M=\{A_1^{\lozenge}, \ldots, A_m^{\lozenge}; W_{m+1}, \ldots, W_n; Q_{n+1}, \ldots, Q_p\} \,] \hookrightarrow A^{\heartsuit}$$

Fig. 1 Developed Herbartian scheme

offered as a possible answer to Q (for a more detailed explanation, see the ATD glossary at Chevallard et al., 2020).

The notion of didactic paradigm has been extensively used in research within the ATD, normally applied to institutions in which mathematics or other disciplines are studied. Our aim here is to expand the use of this notion to institutions focused on teacher education, where the didactic stake is not just mathematics works, but also didactic ones, as the generating question is connected with the teaching and learning of mathematics. Following the Herbartian scheme above, now S is interpreted as a teacher education system in which: X are teachers, Y are teacher educator(s), (eventually $Y = \varnothing$ if the study process is self-regulated), Q is a professional question (in the sense of a question that addresses issues that matters to the profession). Already existing answers and works W_j could be diverse, but would include, at least, those connected with mathematics (more related to the epistemological dimension of teacher education) and those connected with the teaching and learning of mathematics (more related to the pedagogical dimension of teacher education). Likewise, derived questions Q_k could be more related to epistemological issues and/or pedagogical ones. Finally, the tentative answer $A^{\heartsuit}$ might be interpreted as a professional answer, including both mathematical and didactic praxeologies, although the balance between them could differ depending on the nature of the professional question Q they are exploring (more epistemology oriented or more pedagogy oriented).

Organising teacher education under the *paradigm of questioning the world of the teaching profession*, which we will represent as PQW_{TP}, might be desirable since it would favour a functional and meaningful access to professional knowledge, in contrast with teacher education programmes structured more in the sense of visiting mathematical and pedagogical works. It leads to interesting questions like: is it possible to organise teacher education under the PQW_{TP}? What does a teacher education process under the PQW_{TP} look like? What are the conditions needed to realise the PQW_{TP} (in initial or in-service teacher education)?

Answering these questions goes beyond the scope of this chapter. However, our aim is to initiate the exploration of them. In our understanding, any possible answer to these questions should be connected, to some extent, with the issue of teacher education devices and infrastructures.

2 Devices and Infrastructures: an ATD Perspective

Chevallard et al. (1997) introduced the notion of devices that facilitate the study of a mathematical work. They defined a *didactic device*, in a general sense, as "any kind of mechanism arranged to produce some educational aims" (p. 277, our translation). They offered some examples: a "mathematics lesson", "a textbook", "the school library", "exam papers", or "the questions that a teacher uses in his/her lesson" could be seen as didactic devices. In a first approach, the notion might be seen as too broad and undetermined. Later on, they distinguished between pedagogical and didactic devices, depending on whether they are general ones (like "textbooks") or specific to the teaching of a discipline (like, for instance, the "mathematical practices workshops"). This distinction offers a hint to clarify and sharpen the notion: devices could be conceptualised in relation to the different levels of didactic codeterminacy.

These levels are explained in Chevallard and Bosch (2020) as 'places' from which conditions that affect study processes emanate (Fig. 2), considering that conditions originated in one level might appear as restrictions to what is possible in the levels below it.

From these levels, it is possible now to interpret pedagogical devices as those connected with the so-called higher levels (above 'didactic systems': pedagogy, school, society . . .) while didactic devices could be those related to the so-called lower levels (below 'didactic systems': discipline, domain, sector, theme, subject). For instance, a "lesson" could be seen as a device related to the school level, a "textbook" as one related to the pedagogy level, while "mathematics textbooks" and "calculators" could be interpreted as devices at the discipline level, the "(general) number line" a device at the domain level (arithmetic), "natural/integer numbers lines" as devices at sector level (natural numbers and integer numbers, respectively). From this perspective, it is important to notice that a device in one level could include devices from levels below or could be part of devices related to levels above.

Devices are essential in any study process. Indeed, it is hard to imagine a study process without them. However, study processes do not occur isolated but within institutions that are subject to conditions and restrictions arising from the different levels of didactic codetermination. Devices need supporting conditions to exist and produce the educational aims they are supposed to. The notion of infrastructure arises then.

According to Chevallard and Bosch (2020), an infrastructure is a general concept within the ATD that refers to "the underlying base needed to develop any determined, superstructural activity" (p. xxix). The notion appeared previously in Chevallard (2009), in the context of the conditions for the implementation of the pedagogy of study and research activities and paths. Chevallard (2009) argued that,

Fig. 2 Scale of levels of didactic codeterminacy

Humankind ⇆ Civilisation ⇆ Society ⇆ School ⇆ Pedagogy ⇆ Didactic System ⇆
⇆ Discipline(s) ⇆ Domain ⇆ Sector ⇆ Theme ⇆ Subject

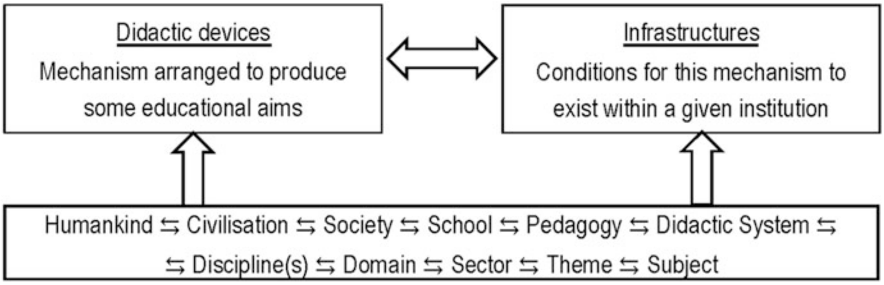

Fig. 3 Didactic devices and infrastructures

for such kind of devices to exist, a mathematical-didactic infrastructure is needed. In the same vein, Miyakawa and Winsløw (2013) consider an infrastructure as the "set of conditions for the teachers' work in the classroom—with a concrete mathematical organisation and didactic organisation—[. . .] and refers ultimately to the levels of co-determination of those organisations" (p. 189). Later, they state that "the infrastructure of the classroom [. . .] involves both elements which are generic to all teachers of a given school and some which are specific to the teaching of mathematics and thus to the development of students' mathematical praxeologies" (Miyakawa & Winsløw, 2019, p. 284). They admit that, while the notion is abstract, what matters is the systemic point of view it offers, since it brings to the front "a multitude of apparently unrelated factors as a coherent whole which conditions and constrains a particular set of praxeologies, without determining them entirely" (Ibid., p. 284).

Considering both notions together offer an interesting perspective that we formulate here as a working hypothesis: any study process entails the activation of a set of devices, which is possible only if an underlying infrastructure exists. Both devices and infrastructure are notions that rely on each other and relate to the different levels of didactic codeterminacy (Fig. 3).

3 Devices and Infrastructures for Teacher Education in the Paradigm of Questioning the World of the Teaching Profession

In the introduction of this chapter, we opened the question related to if it is possible to organise teacher education under the PQW_{TP}. If so, new questions arise: how would these study processes look like? And, what conditions are needed to support them? Now, we can reformulate these questions more precisely, as the interaction of two didactic problems:

- **The problem of the devices**: what kind of teacher education devices could be used to realise the PQW_{TP}?

- **The problem of the infrastructures**: what kind of infrastructures are needed to support such devices so that they can produce the intended educational aims?

Within the ATD, different kinds of teacher education devices have been designed, implemented and analysed, as well as the infrastructure needed to support them. For instance, Cirade (2006) proposed the ‘question of the week’ device, Ruiz-Olarría (2015) and Barquero et al. (2020) proposed the ‘study and research paths for teacher education[1]’, or García et al. (2020) who proposed a teacher education device based on a ‘structured exploration of professional questions’.

Explaining the affordances and limitations of these devices, as well as the infrastructure needed to support them, would go beyond the scope of this chapter. However, it is important to point out that (1) these devices have been developed with a focus on the exploration of meaningful questions of the teaching profession (according to the Herbartian schema applied to teacher education), and (2) their implementation in the initial and/or in-service education of teachers would need the creation of infrastructural elements that are currently missing in the institutions in charge.

The aim of our research is to find out whether lesson study, in an appropriate form, could be interpreted as a teacher education device within the PQW_{TP}. And, from this, to advance in the problem of the infrastructures related to this device.

4 Lesson Study as a Teacher Education Device Within the Paradigm of Questioning the World: An ATD Perspective

In brief, a lesson study could be described as a device that allows the teacher to develop their professional knowledge through the collaborative and careful design of a lesson, its implementation and direct observation in the classroom, and a joint analysis in a post-lesson discussion (Doig & Groves, 2011; Fernández & Yoshida, 2004). Usually, it is structured as a cyclical process (Fig. 4). Based on Fujii (2014, 2016), its main features are:

1. Lesson study starts from some teachers’ concern about students’ learning, which leads to the formulation of a ‘research question’.
2. The group of teachers engages in collaborative research activity around the topic at stake, curricular documents, existing resources, etc. that leads to the design of a lesson. This lesson is detailed in a document (called ‘lesson plan’), which usually includes: the objective of the lesson, its connections to the research question, relation with the curriculum, a detailed analysis of students’ mathematical activity within the lesson, and a description of teacher’s actions within the classroom.

[1]With a modular structure based on existing study and research paths designed for primary, secondary or tertiary levels.

Fig. 4 Lesson study cycle (Fujii, 2014, 2016)

3. The implementation of the lesson (called the 'research lesson') by a teacher of the lesson study group, while the others observe it.
4. A post-lesson collective discussion focused on students' strategies and difficulties, and connected to the research question. This discussion could be enriched with some input made by an external expert.
5. A final reflection, to consolidate and carry forward learning, and to identify new questions for the next cycles. It could include the writing and publication of a report.

The lesson study has already been problematised within the ATD. Winsløw (2011) uses the notion of didactic system *S(X, Y, O)*: formed by a group of 'students' *X*, supported by a 'teacher' *Y,* and with the aim that *X* study (and learn) *O*. He considers that, in lesson study, there are two kinds of didactic systems: (1) the one in school (captured in the lesson plan and happening in the research lesson, where *O* is a mathematical work), (2) the one in which *X* are teachers, *Y* could be empty, another teacher and/or an expert, and *O* is a work about the teaching and learning of mathematics. These systems correspond to teachers' collective work during the lesson planning, the research lesson, and the post-lesson discussion. To differentiate between them, Winsløw proposes using the word 'paradidactic systems', considering that, in fact, these are systems in which teachers work and learn about a didactic system.

Based on Winsløw (2011), García et al. (2019) consider that teachers' activity within each paradidactic system could be described in terms of praxeologies. The benefits of this approach are that teachers' activity within these systems can now be analysed in terms of types of tasks, techniques, technologies and theories. These would contribute to a better understanding of lesson study as a teacher education device, as well as to how it contributes to developing teacher knowledge. Figure 5 synthetises this approach, making visible some of the type of tasks teachers carry out when they engage in lesson study.

We argue that lesson study could be interpreted as a teacher education device within the PQW_{TP}. First of all, because it is a study process that starts from a

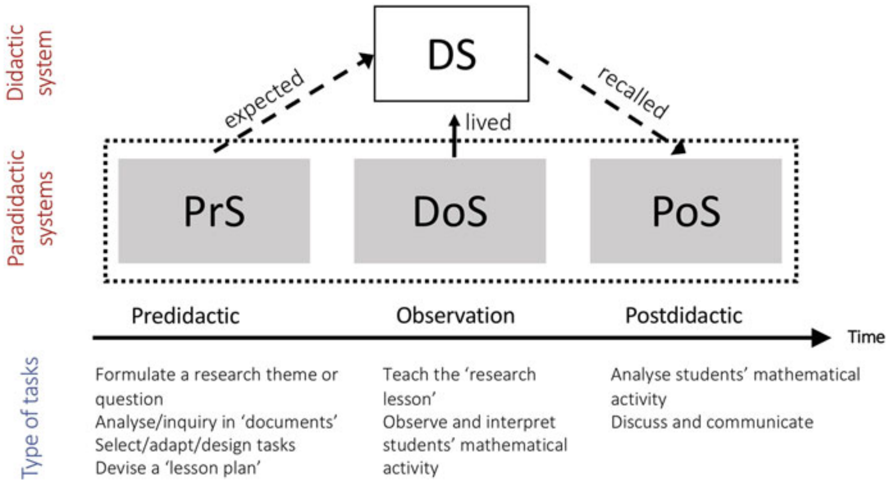

Fig. 5 Lesson study from the ATD perspective (García et al., 2019, based on Winsløw, 2011)

professional question. Second, because, to develop the lesson plan, teachers have to engage in a research process in which 'answers' of different nature might be considered and analysed (like curricular documents, textbook, online resources, professional books, research papers . . .). Finally, because it leads to the building of an 'own answer', in the form of a lesson plan and a research lesson, besides the device allows the teacher to reflect on their answer, to discuss and to develop it further. Thus, teachers engage in collective work throughout different paradidactic systems, carrying out certain types of tasks (Fig. 5) that potentially would contribute to an enlargement of their professional knowledge.

5 Lesson Study and the Problem of the Infrastructure

Supporting lesson study is challenging. Research has extensively reported about conditions that favour or hinder lesson study in each country, particularly in Japan, where the practice originated and seems to be established. Usually, researchers rely on a general notion of 'culture' to find out what are the conditions that favour lesson study, which includes a mixture of conditions. For instance, in Japan, Lewis and Tsuchida (1998) mention a frugal curriculum, a culture of collaboration among teachers and critical self-reflection, and stability in educational policies as key factors, while Shimizu (2014) identifies the formulation of clear students' learning goals in the national curriculum and teachers' voluntary efforts with the support of administrators. Doig et al. (2011) add the existence of classroom culture in which each student is willing to engage in the tasks and contribute together with a teacher professional culture opened to other perspectives of teaching and seeing commentaries as positive contributions.

Without denying the relevance of these and other ‘cultural’ factors already identified in the lesson study literature, our hypothesis is that the notion of infrastructure, and its relation to the levels of didactic codetermination, could contribute to our understanding of what might make a lesson study sustainable and successful.

A first consequence of this approach (García et al., 2019) is that most of the ‘factors’ (infrastructural components) already identified are more related to the upper levels of didactic codeterminacy (above discipline), while ‘factors’ more related to the lower levels (discipline and below) seems to be neglected. In fact, García et al. (2019) analysed three cases of lesson study in three different contexts (Japan, England and Spain) which were organised under three different epistemological and didactic models: in Japan under the ‘structured problem-solving approach’ model, in England under the ‘teaching problem-solving’ approach, and in Spain under the Theory of Didactic Situations (TDS). This analysis showed that the professional praxeologies within each paradidactic system and, particularly, the professional tasks teachers face, how they tackle them, and the technological-theoretical discourses that explain and justify teachers’ activity in lesson study, are critically affected by the epistemological models about mathematics and its teaching that the lesson study community assumes. As a consequence, it would be stated that a crucial dimension of lesson study infrastructure is the epistemological and didactic models assumed, very often implicitly, by the lesson study community. For instance, in the case of Japan, the success of lesson study in elementary and middle school could largely be explained by a shared understanding and adoption of the ‘structured problem-solving approach’ as an epistemological and didactic model of reference. In the Spanish case analysed, carried out with prospective teachers, researchers found out that a sufficient understanding of the TDS was crucial in the implementation of the lesson study cycles.

6 Conclusions

Implementing teacher education under the paradigm of questioning the world is a challenge that we have connected here with two basic problems: the problem of the devices and the problem of the infrastructures in teacher education. This conceptualisation seems to be useful to advance our understanding of how organising teacher education and how best to support it. Different kinds of teacher education devices have already been experimented with in the ATD, like the ‘question of the week’, ‘study and research paths for teacher education’ or the ‘structured exploration of professional questions’. In this chapter, we focus on another device, which has attracted interest from researchers worldwide: lesson study. Specifically, in the chapter, we have argued that (1) lesson study can be reformulated as an articulated set of paradidactic systems, (2) activity within these systems can be modelled in terms of professional praxeologies (thus, as types of professional tasks and techniques, as well as technological and theoretical discourses), (3) supporting these systems need the existence of adequate infrastructure,

(4) an essential component of this infrastructure is the epistemological and didactic models adopted by the lesson study community. Thus, we consider that this approach could contribute to a better understanding and a theorisation of lesson study, as well as of other teacher education devices.

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Introduction of Ordinal Number at the Beginning of the French Curriculum: A Study of Professional Teaching Problem

Floriane Wozniak and Claire Margolinas

1 Introduction

In mathematics, cardinal and ordinal are two aspects of integers. However, cardinal aspects are usually dominant at the beginning of the mathematics curriculum (pre-elementary and beginning of elementary curriculum: 3–7 year-old).

We will interpret this teaching problem using different theoretical approaches in a complementary perspective. Within ATD, we can consider this teaching problem as a question of didactic transposition (Chevallard, 1985) and of the use the praxeological analysis in order to understand its different constraints (Chevallard, 2011). Within TDS (Brousseau, 1997), we can consider this teaching problem in a didactic engineering design that begins with the determination of a fundamental situation and includes experimental observations (Bessot, 2011).

A recent French curriculum (Ministère de l'éducation nationale, 2015) emphasizes the importance of ordinal number and more generally of numbers as means of remembering a position (Margolinas & Wozniak, 2014). This new aspect of numbers thus poses a teaching problem: how teachers are supposed to interpret this piece of curriculum?

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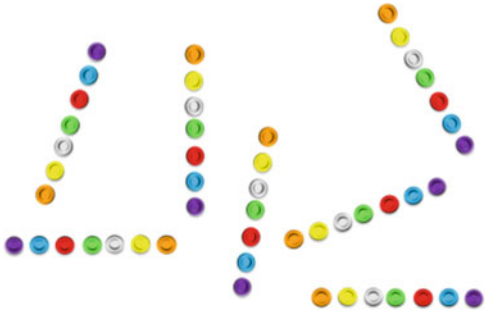

Fig. 1 Some rows of tokens

2 A Praxeological Analysis of the Mathematical Coordinate System's Knowledge

An elementary way to distinguish ordinal number and cardinal number is to examine two situations. Let us consider two bags both containing 10 beads: 9 blue and 1 red. These bags have the same content and same quantity of beads. There is no way to distinguish their content. Let us consider now, two bits of thread knotted at one end. We thread the beads from the two bags separately: an order is created and the red bead's position among the blue ones is different on the two threads (9 chances out of ten). A cardinal number is a number used to remember the quantity of a set of objects and an ordinal number is a number used to remember the position of an object on a list. List and set in one hand, position and quantity in the other hand are the two main elements to distinguish these different functions of numbers.

Let us consider now some praxeologies to locate an object on a one-dimension space (a line) in order to better characterize an ordinal number. A young girl calls her friends: Lara, Luca, Lila . . . It is easy to determinate that Lila is the third on the list. The enunciation of the names implicitly gives the starting point (the first name enunciated) and the enunciation of the sounds is ordered by the time order (one name *after* one name). It is not the same situation if we are looking for the second token from the green token in identical rows of tokens in various dispositions (Fig. 1). Is it the yellow or blue token?

In this case, there is no implicit linear coordinate system (usually called "number line" in English and "repère" in French) with an origin, an orientation, and a unit. Such a coordinate system is needed in order to locate an object on a one-dimensional space. It can be more or less implicit in situation or explicitly created. For example, in Fig. 1, you can say "starting from the first orange token, the green token is the fourth": you do not need to explicit the direction of the number line (from the orange token there is only one direction) or the unit (each token represent a unit), but you need to explicit the origin or the first unit.

If all objects on a list are different, it is possible to locate one without using any ordinal number. For instance, let us suppose the origin is the purple token in Fig. 1 (the direction is then given towards the other tokens). By enumerating the colours of tokens, it is possible to order each of them: purple, blue, red, etc. A few words are

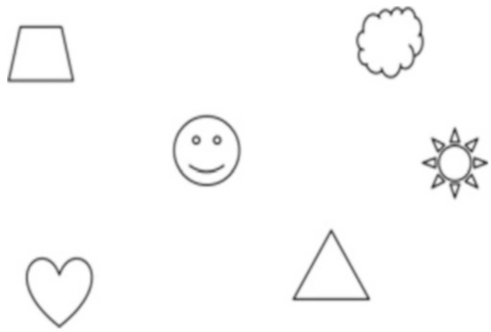

Fig. 2 A collection of different forms

then enough to locate a particular token: for instance, the grey token is between the green token and the yellow one, or the blue token is after the purple one. So if the objects are different from each other, a coordinate system is not necessary, even if the objects are arranged in a line. The use of a line or a list is not a sufficient condition to need an ordinal number.

Another aspect must be studied now: the relationship between cardinal number and ordinal number. Let us consider a collection of different forms (Fig. 2). To count them transforms the collection into a list by successively pointing at them: trapezium, heart, head, triangle, cloud, sun.

There are 6 drawings and the last drawing pointed is the sun, thus the sun is in sixth position on the oral list. The cardinal number witch expresses the quantity of elements in a set corresponds to the ordinal number witch expresses the position of the last element in the list of enumerated elements. Couturat (1896, p. 305) has shown the duality between cardinal number and ordinal by proposing a way to define cardinal numbers from ordinal numbers [our translation]:

> "Let us imagine a simple infinite sequence of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, . . . A set of signs such that one of them is the first (1) and for each of them corresponds to another determined one that follows it immediately: we say that we have built a system of ordinal numbers. It should be noted that these signs have no other meaning than that resulting from the previous definition; in other words, each of them, 5 for example, has no other property than to immediately follow a given sign (4) and to be immediately followed by another given sign (6). [. . .] A collection of distinct objects is given, they are in one by one correspondence to consecutive numbers, starting from 1; in other words, each of them is assigned successive order numbers: 1, 2, 3, 4, 5. . . without forgetting or repeating one of them. The last ordinal number thus used, *n*, is called the *cardinal number* of the given collection, or the number of given objects; and we said that there are *n* objects in the collection. This operation is called *numbering/counting*; it consists in *counting* the objects [. . .] The set of these signs arranged in this order is called the natural sequence of whole numbers.

Thus, counting numbers (one, two, three . . .) arises from the need to use an ordered and stable list of words to count or to locate.

Finally, the mathematical study of locating praxeologies makes possible to define a reference praxeological model of the ordinal number at elementary school in France. The "raison d'être" of ordinal number is to identify a position by using the oral number sequence. The question to be studied is how to locate a distinct object in a list of identical objects. It is the place that has to be located, and not a particular object among singular objects. The successful technique consists in choosing a coordinate system for the list: an origin and an orientation. Numbering the elements from the first after the origin until the targeted object gives the position (last pronounced number).

3 A Didactic Engineering for Research

Two dimensions must be considered to design a didactic engineering: a mathematical organization as an ecological condition (definition of a reference praxeological model); a didactical organization as an economical condition.

How can the main characteristics of the number as position concept be transposed into situations? Using Brousseau's terms, we are looking for a fundamental situation:

> "[. . .] a situation's schema capable of generating, by the interplay of the didactic variables that determine it, all the situations corresponding to a given institutional knowledge.[1] Such a situation, when it can be identified, offers teaching opportunities but above all a representation of institutional knowledge through the problems in which it intervenes, making it possible to restore the meaning of the knowledge to be taught". (Brousseau, 1998, p. 3, our translation).

The search of a suitable fundamental situation leads to consider:

Milieu

- Objects are disposed on a line
- All objects are identical but at least one is different
- It is possible to determine an origin on the line

Situation's stake

- To position a distinguished object on a list of neutral objects in order to have a one to one correspondence with a model

Definition in act of "same position" through one to one correspondence

Goal of the fundamental situation

- Necessity to build a system of coordinates

Epistemological bases

- Make explicit the cardinal/ordinal duality
- Depart from the dominant model which consists in masking ordinal number

Theory of didactical situations (Brousseau, 1998; Brousseau et al., 2014) considers an evolution of knowledge statuses that lead to consider different situations

[1]The terms "institutional knowledge" has been introduced by Margolinas in this volume.

Fig. 3 Two identical necklaces

based on the same fundamental situation: action situations (the model is close to the material or far away) with anticipation of the feedback, formulation situations (the model is given some day and the material is given another day or the model is given to a student and the material to another one).[2]

Our experimental choices, developed in 2012, is based on a *milieu* of a knotted thread, nine identical neutral beads and one coloured bead (Fig. 3). Our choices are coherent with our praxeological analysis.

In particular, by proposing a knotted thread, we offer the possibility of considering an origin (the knot) which:

- is not one of the objects whose position we can locate,
- corresponds both to the position from the origin and to the succession of the manipulation of the beads that we thread.

Thus, the first bead from the origin is also the first bead threaded on the knotted thread. Moreover, there is no ambiguity between the origin (which must mathematically have the abscissa 0) and the first pearl (which must mathematically have the abscissa 1).

The conclusion from our engineering for research about students' situational knowledge (see Margolinas & Wozniak, 2014) confirms that number is only known to students (5–6 years old) as a designation of quantity. During the formulation situation, students were able to combine this mathematical knowledge with their linguistic knowledge of the order of the written words (Goody, 1977). They have coined an 'oriented quantity'. For example, in order to remember the necklace of Fig. 3, they would have written 3 1 6, that means starting from the knot, there are 3 neutral beads, 1 coloured bead and 6 neutral beads. This is a new situational knowledge, which take into account the origin and a succession using the successive words read. However, the principle of the economy of the ordinal number is only rarely encountered in these situations.

[2]We do not refer here to the validation situation (discussion of the messages produced during the formulation situation, see Margolinas in the same volume).

4 Towards Training: Teacher's Mathematical and Didactical Knowledge Needs

In France, before 2015, ordinal number was mentioned in the national curriculum but was missing from the list of required skills at the end of pre-elementary school (3–6 year-old students). Thus, teachers taught only counting and some number songs. In primary school (6–11 year-old students), the order relation was mentioned with the type of task « placing a number/a fraction on the graduated numbers line » but not the ordinal number. Since 2015, the ordinal number is explicitly mentioned in the text and into the list of required skills [our translation]

> Use the number to designate a rank, a position: The number also allows memorising the rank of an element in an organized collection. To memorise the rank and position of the objects (third bead, fifth hoop), children must define a reading direction, a direction of way, i.e. give an order. (Ministère de l'éducation nationale, 2015, p. 14)

> Locate a position in a row or in a list. Make the link between the rank in a list and the number of items before it. (Minitère de l'éducation nationale, 2018, p. 24)

Why this change? In France, it is usual for a new Minister of education to change curriculum. In 2015, the committee in charge of writing the new curriculum consulted some researchers about mathematics education in primary school. Claire Margolinas was part of them and she has presented our works and some elements have been then introduced. This is an illustration of a didactic transposition phenomenon: There is regularly the need to renew the works of the curriculum, even if no one knows how to teach the new introduced piece of knowledge, in this case, the institution "research in mathematics education" has legitimized this change. This lead to a problematic of research in didactics that Chevallard (2011) considers as a *primordial problematic*:

> Given a project of activity in which a particular institution or person is contemplating engaging, what is, for that institution or person, the praxeological equipment that may be considered essential or merely useful in the design and execution of that project? (Chevallard, 2011, p. 98, our translation)

This new curriculum thus trigger two questions: (1) How teachers are supposed to interpret this piece of curriculum? (2) How teachers' educators can support them in this change on the curriculum?

To support a change on the curriculum, teacher educators must work on the conditions and constraints affecting teaching systems and practices. Teacher educators have to determinate which can be the useful praxeological equipment for teaching ordinal number and what training content might be proposed. This requires identifying teachers' missing praxeologies to be acquired, and the relationships between existing praxeologies and future praxeologies to be developed.

Concerning mathematical and didactical organizations, the questions to be considered are: How to characterize a given piece of knowledge? What is its "raison d'être"? How can a given piece of knowledge lives in the didactical system? How is it possible to interpret students' praxeologies?

In the case considered here, the introduction of ordinal number as a new piece of knowledge, taken into account our previous work, the specific question is: What does didactic engineering for research teach us about mathematical and didactical praxeologies useful for the teacher?

In our studies (Margolinas & Wozniak, 2014, 2015), we organised three types of observations related to research engineering: (1) Post-experimentation: the experimentation as a *milieu*; (2) Free implementation as an engineering-based research: presentation of didactical purposes and research results; and (3) Free implementation: the engineering as a resource (not related to research).

Three main indicators of praxeological needs can be identified by observing teachers' practices. First, what teacher takes or modifies (ostensives, tasks, techniques, technologies) compared to their usual practices through the didactic engineering. Second, the part of *logos* in teachers' praxeologies and its effects on the mathematical praxeologies (mute, weak, strong, see Wozniak, 2012). Third, the distance from the reference praxeological model used as the didactic engineering foundation [degree of conformity, as introduced by Chevallard (2020)].

However, the direct observation of teachers' practices is not the only way to study the relationships between existing praxeologies and future praxeologies to be acquired. The study of existing responses in teaching resources (e.g. textbooks[3]) can reveal the ecological or economical constraints of the didactical system. Let us consider a textbook (first year of elementary school) for illustrating the ecological or economical constraints on the didactical system (Figs. 4, 5 and 6).

In the situation evoked in this textbook, Rose has 10 stones and has placed a stone in each card. The arrow "Departure" represents the origin and the direction of the line. The first question is "*On which card did she arrived*?". In a second time, the cards are turned face down and two cards are marked with coloured tokens. The type of question is "*Which are the drawings on those cards*?". The mascot gives the indication of the expected answer: the green token is in sixth position on the line.

This problem is in conformity with the curriculum ("*Locate a position in a line or un a row*", see above) and the ordinal number is introduced via the relationship with cardinal number. Since the arrow is an origin outside the line of cards, there are six cards from the first card up to the sixth card. The cardinal number remains the reference for a reason (origin outside the line of objects) that may not be explicit. After the introductory situation, some exercises are proposed (Fig. 5). The questions are "*Which is the card in 5^th^ and in 16^th^ position*?"

Thus, after the introduction of the relationship between quantity and ordinal number, new words to express ordinal numbers are given and used. Ordinal number appears, therefore, only as a simple vocabulary matter.

Let us consider now an example of pre-service teacher training provided by an experienced pre-elementary school teacher who is also a part-time teacher educator. Our hypothesis is that the specific status of this teacher educator reveals what the

[3]In France, teachers are free to choose textbooks for theirs students. Textbooks are freely published without any control from the Ministry of education.

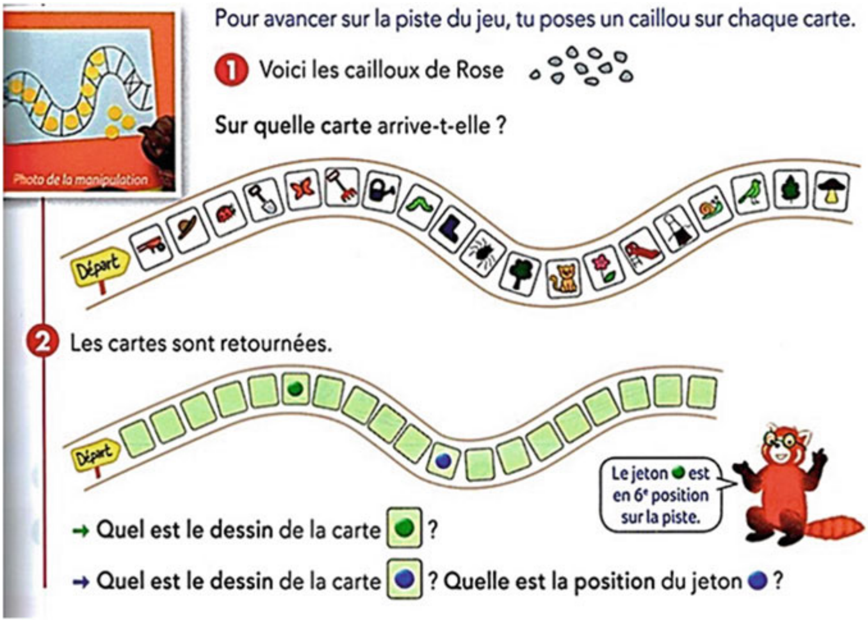

Fig. 4 An introductory situation

Fig. 5 Exercise

profession understands about its institutional demand. The content of the courses is organised in three parts: presentation of the curriculum, list of the vocabulary needed for locating oneself, other persons or things (In front of, behind, before, after, back, in the middle, between, next to, at the beginning, at the end, first, last), and study of an example of teaching situation: “the train” (Fig. 6).

We find again, as in the textbook, the question of vocabulary. But the most important is the inherent difficulty in this situation: orientation of the list of numbers for numbering and movement of the train are in two opposite directions: The train moves from right to left, and the list numbers for expressing the place from left to right (Fig. 7).

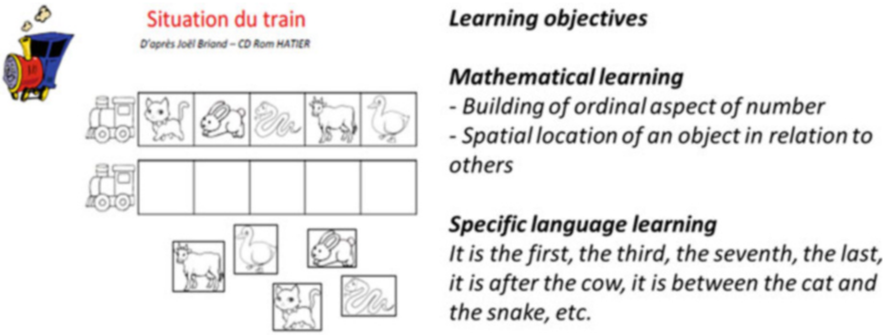

Fig. 6 An example of teaching situation: "the train" [our translation]

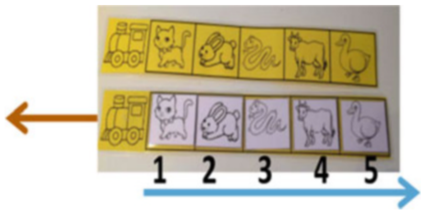

Fig. 7 An inherent difficulty, the opposite direction

The coordinate system is thus define by an origin which is not a wagon (the locomotive) and an orientation which is from left to right, thus identical to the written words direction. Thus even if there is no awareness from the students (and the teacher) of the coordinate system, the knowledge of the orientation of writing is sufficient to answer all questions. In all the textbooks that we have consulted, this is always the case, and that leads to make the hypothesis that the inherent difficulty of a conflict of orientation may never been encountered and neither any new knowledge about coordinate system.

Nevertheless, the significant element of the situation is about the learning objectives announced: the ordinal number must be used for the spatial location of an object in relation to others, but students don't need to use numbers in a coordinate system to succeed. Two didactic variables are essential: the number of compartments and the characteristic of objects (animals) in each of them. With 5 compartments, students can duplicate the model by using words such as: the first, after the first, the last, before the last, the middle. With all different animals, students can also duplicate the model by remembering the list of animals in a certain order: "cat, rabbit, snake, cow, duck" or "duck, cow, snake, rabbit, cat". In this situation there is a confusion between spatial location (which is an important part of pre-elementary school curriculum) and ordinal number. The ordinal number is a tool for spatial location, but it is possible to locate without numbers, and some words in the list of specific words to be learned have no relationship with numbers.

In summary, the study of these « works » shows that cardinal number dominates ordinal number. In our previous study (Margolinas & Wozniak, 2014), we showed how students' productions in didactical engineering for research invent a new kind of number we called "oriented number". In the other hand, the study reveals that the conditions for locating with a technique based on the use of ordinal numbers instead of spatial techniques are unknown. The previous example of situations in pre-elementary school textbook and the pre-service teacher training provided by an experienced teacher illustrate it quite well. Thus, our answer about teacher's required praxeological equipment follows three directions: epistemological needs, mathematical needs and didactical needs.

The result of our analysis is that the type of task T: *To reproduce the position of an object in a line composed of indistinguishable objects*" is the "raison d'être" of ordinal number.

Different situations have to be associated with this type of task (see Margolinas in this volume about the difference between problem—or type of task—and situation): action situation and formulation situation, in particular time delay is important in order to experience the situational pieces of knowledge needed in order to memorize a position.

From a mathematical point of view, it is necessary to know that for locating an object on a line composed by indistinguishable objects, the line must be transformed into a list. In other words, we must create a coordinate system: origin, orientation, and unit. Thus, the one-to-one comparison in sequence is the technique to assess the success of the achievement of T. At last, concerning didactical needs, teachers have to understand that the construction of a coordinate system by students is a criterion of understanding what ordinal number is.

5 Conclusion

Reflecting on the teaching of the ordinal number leads to questions on both "number" and "order". "Number" is implicitly understood by its cardinal aspect, i.e. to account for a quantity. As soon as a collection of objects is present, the quantity is always available, but in order to be able to recognasi and talk about position, this collection must be constituted as a list, i.e. it must be ordered. There is therefore no situation in which the quantity is excluded for the sole benefit of the position, the cardinal and the ordinal always appear together.

On the other hand, "counting", i.e. pronouncing the oral sequence of word-numbers by enumerating objects, is an useful technique both for determining the cardinal of a collection of these objects and for numbering these objects and thus saying the position of each of these objects in an ordered list.

However the technologies that explain the use of the technique of "counting" in order to obtain the cardinal or the ordinal are different.

When one "counts" a collection of objects in order to compare the quantity of this collection with the quantity of another collection, one makes a one-to-one correspondence between objects and word-numbers. The order of the word-numbers is not essential: "one, two, three, four" is a collection of words that has the same quantity as "four, three, two, one" or "one, three, four, two". Of course if one always uses the words in the same order, it is possible to remember only the last one (the cardinal) but also one can compare two collections using any intermediary collection of objects or words (Margolinas & Wozniak, 2012).

When one "counts" a list of objects in order to compare the place of a distinguished object on the list with the place of a distinguished object on another list, the order of the word-numbers is one of the keys of the success of the technique, the other is the designation of the origin, and sometimes also the orientation of the list. For example, if one says on the phone that you have to open the second drawer of a desk, this is not sufficient to know which drawer it is (the second from the top or the bottom). Thus teachers are confronted with a problem which is not frequent: to make a distinction between two praxeologies where the techniques are the same but the technologies (and of course the theories) are different. Furthermore, one of the praxeology is always available and the other is not. Last but not least, "order", when it deals with a spatially ordered list, inherits implicitly the order of writing, which, in mathematics, is not relevant and plays the role as a hidden technique.

Our observations, both the engineering for research and their subsequent outcomes on teachers' practices (Margolinas & Wozniak, 2015), lead to consider that the designed sequence cannot resist to the tendency to consider only the cardinal praxeology. There is thus a challenge not only to share some carefully planned tasks and situations to the teacher but most of all to share the "raison d'être" of those situations.

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Study and Research for Teacher Education: Some Advances on Teacher Education in the Paradigm of Questioning the World

Berta Barquero and Avenilde Romo-Vázquez

1 Introduction: Study and Research Paths for Teacher Education

Considering the general problem of moving toward the paradigm of questioning the world (PQW) (Chevallard, 2015) in current educational systems, this chapter focuses on the inevitable step of the professional development of teachers. In the framework of the ATD, when thinking about how to plan teacher education in the PQW, various assumptions raise about what teacher education may deal with and how to plan teacher education programmes. On the one hand, previous research on the ATD on teacher education (Cirade, 2006; Bosch & Gascón, 2009) stated that teacher education programmes should include the questions affecting the development of teachers' practice. That is, these *professional questions* may be at the core of teacher education programmes. These questions of the teachers' profession are of different levels of generality since the more specific ones concerning a theme to the ones about the school organisation or pedagogical decisions. However, it is important to consider that many of them have an essential mathematical component. In other words, some of the problems that teachers should face in their daily professional life are related to mathematics and, particularly, to the didactic transposition process of this knowledge at stake.

On the other hand, another aim of teacher education might be to facilitate the link between new knowledge resulting from educational research with the reality of the

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classroom and present these tools to deal with and start proposing answers to these professional questions, rather than presenting basing teacher education on as a set of more or less dogmatic knowledge. Simultaneously, this didactic knowledge can be used to question both curricular content and forms of teaching, thus allowing teachers in training to formulate new problematic issues that implicitly hinder teaching, acting transparently for all actors in the educational process.

These assumptions materialize in the proposal of *study and research paths for teacher education* (SRP-TE), initially experimented by Sierra (2006) in the case of preschool teacher education and developed by Ruiz-Olarría (2015) for the initial training of secondary mathematics teachers. In this chapter, two case studies of SRP-TE with pre-service and in-service mathematics teachers are discussed, which were implemented in two different university contexts for teacher education. The common aspect is that both courses start from a similar initial professional question about how to analyse, adapt and integrate a learning process related to mathematical modelling in school.

In the following sections, we present the general structure of the SRP-TE that, far from being a close structure, is then adapted for each particular case study. We are particularly interested in several aspects of both experiences. The first consist of looking at what are the main derived *professional questions* that are at the core of the different steps of the SRP-TE, guiding its development, and which answers are expected to be built collectively between students-teachers and educators. The second relies on the analysis of the evolution of the *milieu* that each experience with the SRP-TE has been able to create between students-teacher and educators to enable teachers' epistemological and didactic questioning. The third concerns the individual and collective work developed to jointly elaborate new mathematical and didactic infrastructure or, at least, jointly acknowledge its necessity to deal with the prevailing institutional constraints hindering the integration of the PQW in current school systems.

2 General Structure of the Study and Research Paths for Teacher Education

Five general modules organise an SRPs-TE, which appear as an inquiry-based process combining practical and theoretical questioning of school mathematical activities. For more details, Barquero et al. (2020) presents an overview of the different SRP-TE implemented with different modalities of development in teacher education.

- *Module 0* starts by considering a professional question (i.e., how to teach proportionality, algebra, integers or linear regression? How to integrate inquiry in mathematics teaching?) in front of which student-teachers are invited to search for available answers among the different accessible media (books, textbooks, curricula guidelines, etc.), which eventually include some instructional proposals coming from educational research.

- *Module 1* consist of proposing student-teachers experience a teaching activity (based on the design of an *study and research path* (SRP)) that could, to a certain extent, exist in a regular classroom closely linked to what could exist in school. Student-teachers are asked to act as students within the SRP under the guidance of the educators.
- *Module 2* involves the analysis of the experienced SRP and the collective work on the design of a lesson plan. Student-teachers might design an adapted version of the previously experienced mathematical activity for a specific group of students (theirs, if possible). This design takes the form of a lesson plan as close as possible to teachers' practice, including an *a priori* design of the activity. During this adaptation, it usually happens that teachers "reduce" the potential of the proposed instructional activity to face their school institutional constraints.
- *Module 3* entails the implementation and *in vivo* analysis of the lesson plan. Teachers are asked to implement their adapted teaching proposal in a real classroom or with students in an out-of-class activity. In this module, teachers are supposed to use the *a priori* design as a tool for managing the implementation of the activity and for developing its *in vivo* analysis.
- *Module 4* consist of the *a posteriori* analysis of the lessons. This last module is devoted to sharing the teaching experiences, looking at what has happened (compared to the *a priori* designs) and reflect on the conditions created and constraints faced in the implementation(s). Teachers are asked to share and compare the institutional constraints found and the level at which they manifest themselves. Teachers finally can present a new adaptation of the instructional proposal and a detailed analysis of the entire process. At the very end of the SRP-TE, the results of the experimentation can be partially used as an answer to the first initial question ('How to teach . . .?') that was at the origin of the whole process.

The specific activities proposed depend on the initial professional questions, on the particular context for teacher education and the possibilities of developing all the modules, or only some of them.

3 The Herbartian Schema as a Didactic Model of Reference for the Analysis of the SRP-TE

With the aim to analyse what exists (and what could exist) in the transition from the paradigm of visiting works toward the paradigm of questioning the world, the Herbartian schema appears as a useful didactic model of reference for this analysis. As explained in Bosch (2018), the Herbartian schema indicates the main elements of the *inquiry* process.

$$[S(X, Y, Q_0) \curvearrowright \mathrm{M}\ ] \hookrightarrow \mathrm{A}^{\heartsuit}$$

We start from a didactic system S in which X are the students, Y are the guides of the study (teachers, in a broad sense), and Q_0 is the generating question to which X, with the help of Y, has to provide an answer $A^{\heartsuit}$. The study of Q_0 generates an inquiry process involving a *didactic milieu M* made up of different types of objects or tools for the inquiry.

$$[S(X, Y, Q_0) \curvearrowright \mathrm{M} = \{\mathrm{A}_1^{\diamond}, \mathrm{A}_2^{\diamond}, \ldots, \mathrm{A}_{\mathrm{m}}^{\diamond}, \mathrm{W}_{\mathrm{m+1}}, \mathrm{W}_{\mathrm{m+2}}, \ldots, \mathrm{W}_{\mathrm{n},} \mathrm{Q}_{\mathrm{n+1}}, \mathrm{Q}_{\mathrm{n+2}}, \ldots, Q_p, D_{p+1}, D_{p+2}, \ldots, D_q\}] \hookrightarrow \mathrm{A}^{\heartsuit}$$

The $A_m^{\diamond}$ are already existing answers that seem helpful to address Q_0 (or its derived questions Q_p) that the X and Y have discovered in the institutions around them. The W_n are works drawn upon to make sense of the $A_m^{\diamond}$, analyse and reconstruct them to build up $A^{\heartsuit}$. The Q_p are the questions derived from the study and inquiry into the Q_0, the $A_m^{\diamond}$ and W_n, raised by the construction of $A^{\heartsuit}$. Finally, the D_q are sets of data gathered in the course of the inquiry.

When we use the Herbartian scheme to analyse teacher education, the didactic system S is composed of X students-teachers in training, Y are teacher-educator(s), and Q_0 is a professional question that generates the teacher education process. We designate it as the *generating question for teacher education*: $Q_{0\text{-TE}}$. The particular SRP-TEs, we here considered, have the same initial generating question about:

> How to analyse, adapt and develop a learning process related to mathematical modelling in our teaching practice? How to institutionally sustain long-term learning processes based on modelling? What constraints might be faced, and how overcome them?

The *milieu M* includes a diversity of elements (with the derived questions Q_p, the existing answers $A_m^{\diamond}$, works W_n and data gathered D_q) of different nature, as they can come from questioning different dimensions of teacher education. These elements can be more connected to the epistemological dimension, that is, to the questioning of the mathematical knowledge at stake (in our case, about how to conceptualise modelling and how to describe and analyse it); or, to economic dimension (asking about what exist in secondary or primary school institutions concerning modelling, or about which particular conditions facilitate its integration); or, to the ecological dimensions (asking about the conditions and constraints that facilitate or hinder the dissemination of modelling practices in a broader sense). To characterise these elements, we have used the following main dialectics (following the previous works, Chevallard, 2011; Barquero et al., 2019).

First, we use the *dialectics of the questions and answers* to describe the main professional questions and answers structuring the SRP-TE. As the structure, in term of modules and on main professional questions addressed, are common for both case study, Fig. 1 shows the main questions addressed in each module and the expected answers to be built. We retake this figure in the conclusions to highlight the main commonalities of both experiences.

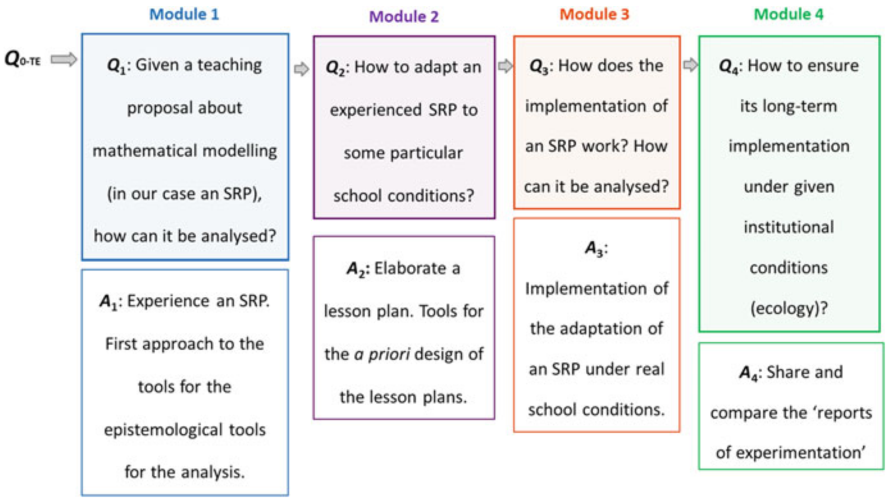

Fig. 1 Structure of the SRP-TE in term of the main Q-A in each module

Second, we analyse each particular case study with respect to the *dialectics of media and milieus* established in the teacher education process, focusing on what is made available for teachers and which means are provided. And third, we focus on the organisation of the individual and the collective work to make the *didactic milieu* evolve as expected.

4 Analysis of Two Case Studies with SRP-TE

First, we present the experience with the SRP-TE organized by the CICATA (Instituto Politécnico Nacional) that has been implemented during the last six academic years, from 2014/15 to 2019/20. Participants are in-service secondary and university teachers, mostly from Mexico and other Latin American countries (Argentina, Chile, Colombia, Guatemala, Paraguay and Uruguay). The course ran over 4–5 weeks with an expected work from participants of about 80 h. This case study is particularly interesting due to its adaptation to the online and in distance modality, considering the multimedia tools, forums, videos, Moodle platform, as well as the asynchronous working conditions. Moreover, the optimal conditions of the number of hours, the reduced number of participants (about fifteen in each edition of the course, with about five educators), and the fact that participants are in-service teachers made possible to develop all the modules of an SRP-TE. The second case study has been implemented with pre-service primary school teachers at the University of Barcelona (Spain). It has been implemented since the academic year 2011/12. The SRP-TE ran over a compulsory course called "Didactics of

Mathematics II", with fourth-year students, which aims to introduce tools from didactics of mathematics to address the professional tasks to analyse, design and evaluate mathematical practices in primary school. All the classroom sessions have been developed face-to-face over about 2 months within 2-h session twice a week.

4.1 Case Study 1: The SRP-TE About Modelling the Evolution of Facebook Social Network

We focus on the particular SRP-TE developed in 2018/19. For 5 weeks, we approached the $Q_{0\text{-TE}}$ presented, with an adaptation of the general structure of the SRP-TE into four main activities that covered from module 1 to module 4. They mainly consisted of (Act1) Carry out the SRP (role of a student); (Act2) Preparing an adaptation of the SRP for their students (role of a designer); (Act3) Implementing an SRP (role of a teacher); and (Act4) Analysing the implemented SRP (role of an analyst-didactician).

More concretely, the first activity—corresponding to Module 1—proposed the resolution and analysis of a modelling activity about 'Modelling the forecast of Facebook user' with the main aim to let participants experiment an SRP close to what could exist in the classrooms [a more detailed description of an implementation of this SRP at university level can be found in Barquero et al. (2018b)]. Participants were asked to 'live' it as mathematical learners or students. They were asked to become a team of mathematical consultants and provide an answer to a request from "Publicity" (an invented name for a consultant firm specialized in social networks) which wanted to have a deep study on the initial question about "How to predict the evolution of users Facebook: growth of the number of users by geographical area, existing predictions, variables at stake, open and unused accounts, etc.?". To develop this activity, five groups of teachers were formed. Each group was guided by one educator. They interacted in a forum to prepare the first prediction, and they used common forums to compare their work with the rest of the working groups. In the end, they had to prepare a report with their final answer to the request of "Publicity". In this first activity, looking at the dialectics of the media and milieus, many new elements started to be included in the *milieu*, such as dataset about Facebook users' evolution, derived questions and temporary answers generated by the students about the role of models, mathematical models used to fit data, elements for justification and validation, etc. But it also brought to light some important constraints about, for instance, the absence of mathematical notions, concepts and discourses to talk about the modelling work developed, as well as the transparency of the use of some media (such as GeoGebra or Excel). In particular about the pertinence of the functions used to fit data, the meaning of the coefficient to compare the different models (such as R^2), etc.

When these first reports were delivered, the educators asked them to analyse the activity carried out. Concerning the analysis of this modelling activity, the main tool

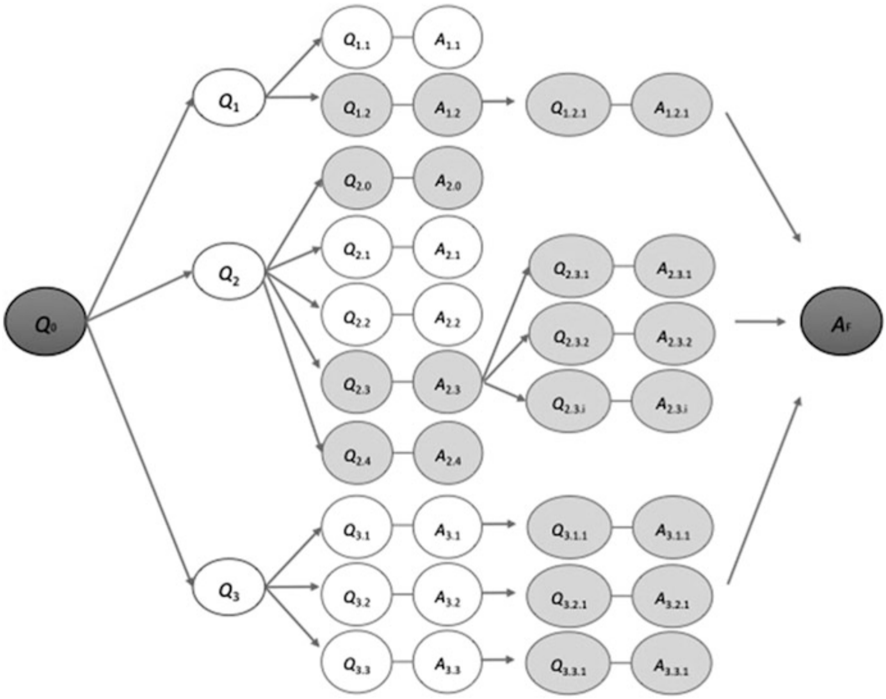

Fig. 2 Example of *Q-A* map from a group of participants (Barquero et al., 2018a, p. 38)

provided to the participants took the form of a *questions-answers map* (Winsløw et al., 2013; Barquero et al., 2018a). Students were asked to sketch the modelling process they had followed, underlying the dialectics established between the questions addressed and the answers obtained. The educators helped the students by providing them with a summary of some of the questions and answers (*Q–A*) that appeared in their discussions or reports (see Fig. 2). As explained in Barquero et al. (2018b), the aim of this analysis was to break with the usual way of describing school mathematical contents—which priorities concepts, notions and techniques to the detriment of questions and problems—without using complex terminology.

> It enabled the participants to change the order of priorities, highlighting the dialectic between questions and answers in the complete modelling process they had followed, and using the arborescence of Q–A to analyse the different stages of the modelling process. (Ibid., p. 37)

In Activity 2—corresponding to Module 2—participants were asked to prepare a "lesson plan" as an adaptation of the SRP that they come to experience to be then transposed to secondary school level. First, participants prepared an individual version of the lesson plan, which was later shared with her/his working group and with the instructor. The work led to the preparation of a final common lesson plan to be adapted and implemented to their particular conditions in the following activity. Generally speaking, at this stage, the tendency was to "close" the activity with respect to the different elements they included. More concretely, some decisions that could be read in the individual lesson plans were about: (a) describing the objectives mostly in terms of the notions to be introduced (function graphs, linear regression, etc.); (b) structuring the activities as a set of small tasks "to be solved" by students; (c) assigning to the teacher most of the responsibilities related to the introduction of the models to be used, later applied by the students; (d) describing teachers' work, but independently from students' work; (f) assigning the teacher the responsibility to validate the models and answers provided by the students, among other decisions. This can be interpreted as the difficulty of the participants to detach from the dominant paradigm of visiting work, dominant in the educational institutions where most of us are developing our professional practices. But these initial lesson plans constituted a rich *milieu* to question many epistemological and didactic decisions that had been made. The educators intervened more actively to question how to plan a more inquiry-based activity with a more active role of students in this process. In the following, there is a comment of one of the educators guiding the work from Group B in the discussion forum:

> [*Educator, following Team B in the forum*] I read your three individual lesson plans, and I found that they share many commonalities, [. . .]. I found that the discussion about which elements to include in the lesson plan is interesting and a good starting point. But we may go one step further and compare the underlying pedagogical and didactic proposals. For example, to what extent can we leave the activity open, or should we necessarily establish the mathematical tools to be used in advance? What means of information (besides the teacher's explanations) can students' access? What difficulties do we contemplate from students, and what kind of answer can we provide? How much autonomy will students' have to assume during the modelling activity?

Participants finished activity 2 with a new version of the lesson plan, common to all the members of the working group. Then, each of them had to adapt this common lesson plan to the particular condition for implementation. Activity 3 consisted of carrying out this implementation and collecting all the empirical data from its observation in the classroom (which acted as main *media* in this module). Participants acted now as "teacher" in their classrooms and guided the activity as close as they could, according to their *a priori* designs. At the end, they had to prepare an "experimentation report", with a prefixed structure provided by the educators. These reports constituted part of the *milieu* to be then shared with the rest of the participants. The reports allowed, first, to reflect on their own practice in an objective and common way and, second, to know about how the designs had been worked in different school contexts. This rich *milieu* facilitated the *a posteriori* analysis of the designs and to open the discussion about the conditions and constraints detected.

Finally, Activity 4—linked to Module 4—was devoted to elaborating a final revision of the lesson plan to propose a new version of it. This had to take into account their initial design, the own experience and those of the team partners. The *milieu* was then enlarged by detailed feedback from the instructors to the experimental reports and some reading (Chevallard, 2015). Educators also provided a clearer structure of the final form of the lesson plan, which asked to include: (a) a mathematical analysis of the teaching activity, (b) a didactic analysis of the teaching

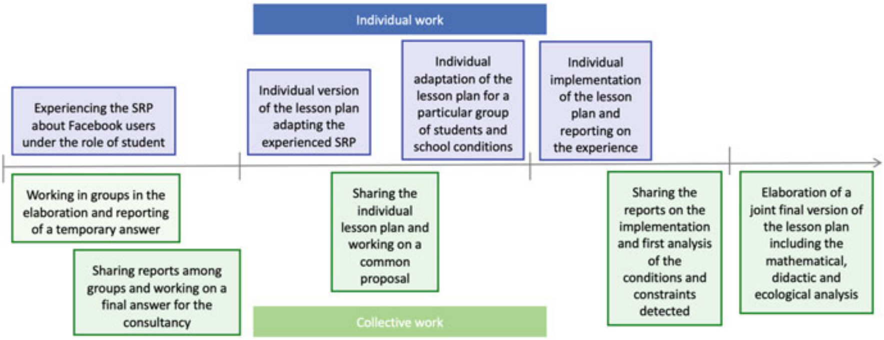

Fig. 3 The dialectic of the individual and collective work in the first case study

activity in the form of lesson plan—as re-elaboration of the previously produced in activity 2—, and (c) a description of the conditions and constraints detected in the implementation.

A summary of the development of this SRP-TE, we take the proposal of Miyakawa (2021) to show the *dialectics between the individual and the collective work*, to synthesise the chronological evolution of the modules and of the most important instrument facilitating the evolution of the *milieu* shared in this training process (Fig. 3).

4.2 *Case study 2: the SRP-TE about modelling the cake box*

We do not develop the entire SRP-TE in this chapter. More details about its implementation can be read in Barquero et al. (2020). We explain here its main particularities with respect to the adaptation for pre-service primary school teachers and the main differences in comparison to the previous case study. In particular, we want to mention two main differences: the first is about the way how Module 0 was planned in the SRP-TE. The second concerns module 1, which can be considered as the more important module in this case. Participants were pre-service primary school teachers, and that they did not have hours in the course for implementing any proposal in the school, either easy access to primary school classrooms. That is why there were no easy conditions to develop module 3 and 4. This SRP-TE was more planned to work on the epistemological dimension, in other words, to question what the role of modelling can be in the teaching and learning of mathematics, how mathematics can be interpreted as a modelling tool and how to describe modelling. It also addressed the need of introducing new epistemological tools to analyse and design mathematical activity in an approach closer to the PQW. We thus focus on briefly describing the work developed in relation to modules 0 and 1.

About module 0, the first activity started by presenting $Q_{0\text{-TE}}$ about “How can modelling be introduced in Primary school education? What kind of modelling activities could be used?". Participants were asked to make a first analysis of the role of modelling in mathematics education using different resources, such as local curriculum, textbooks or some teaching proposals. This activity is two-fold: firstly, to describe the conditions under which modelling is proposed to be introduced in primary school education and, secondly, to establish some shared terms to refer to modelling (such as system, model, variables of the system, mathematical results, interpretation, model validation, etc.). In comparison to the first case study, this way to start Module 0 facilitated the first encounter to $Q_{0\text{-TE}}$, helping to set up a shared terminology to talk about modelling.

With respect to the second activity—the starting of Module 1—, educators proposed a modelling activity, the *cake box*, which consisted of helping a pastry chef to find the most appropriate measures of the material to buy to build boxes with lids to pack the cakes (and other sweets) she sells. This modelling activity, which leads to inquiry into the relation of measure of the material to build the boxes with the corresponding measures of the boxes and the lids, corresponds to an adaptation of the one proposed by Ruiz-Higueras (2008) and Chappaz and Michon (2003). Participants were asked to assume the role of primary school students and experience the different steps of this activity. The activity of the cake box was easily accessible to participants, and it has always generated a rich modelling activity involving different types of models, from the ones more numerical and geometrical to the more advanced ones involving pre-algebraic or algebraic models.

The third activity was about the analysis of the mathematical activity developed—ending of Module 1—. Participants had to analyse the modelling process as it was experienced by themselves and by other groups in the class. They used the session reports delivered by each group and the class debates as the main media considered in this task. The educator proposed to do this work using the proposal of the *questions-answers maps*. She proposed an initial sketch of the Q-A map, with the main steps or phases of the activity with the initial questions she had presented during the implementation. Participants had to complete it with a description of their work. Each group worked on producing their Q-A map of the cake box activity, which was used later to analyse the path followed by another group in the class. This new task helped them to enrich the Q-A maps by including new questions, answers, strategies, etc., showing the potentialities of using this tool for future implementation of the activity.

This way of describing the modelling process not only provided the participants with new terminology, but it also appeared as an alternative way to talk about doing mathematics, breaking with the usual "static" way of describing school activities, more focused on concepts, notions and techniques to the detriment of questions, models and provisional answers. Furthermore, these Q-A maps were later used when participants started working on the design of a teaching proposal. And, in the case they could implement, participants used them as tools for the *in vivo* and *a posteriori* analysis of the implementations.

5 Conclusions and Discussion

In this chapter, we have described two experiences with SRP-TEs implemented under different institutional conditions. The first with in-service secondary and university teachers in the CICATA centre within online and in distance modality. The second with pre-service primary school teacher at the University of Barcelona. They both start from a similar generating question about how to analyse, adapt and develop a learning process related to mathematical modelling in our teaching practice? Although their evident differences in the institutional context for implementation and the school level teachers are trained, some common aspects and regularities appeared in their analysis.

Both experiences were designed by the same team of researchers with a similar structure supporting the design of the SRP-TE. We have used Fig. 1 to sketch the main derived professional questions (and the expected answers from the participants) that were central in each of the modules. The first common question was about how to analyse a teaching proposal, and in particular, an SRP. The SRP proposed in each case study was different: "Forecasting Facebook users" for the first case and "The cake box" for the second one. Their selection was made according to the school level teacher were trained and the experiences we had with the SRP. This first question was addressed similarly in both cases by making teachers experience an SRP under the role of students (as it could exist in a school institution) and then proposing to analyse it. The tools provided to do so were also a common aspect of the case studies examined. First, teachers use the *questions and answers maps* as the main epistemological tool to analyse knowledge developed during the SRP. Furthermore, this tool played a crucial role when the teachers adapted the experienced SRP in a specific school institution. At this moment, teachers use Q-A maps not only as an epistemological tool for the *a posteriori* analysis of teaching proposals. They also use them during the *a priori* analysis—when they design the lesson plans and use Q-A maps to anticipate and evaluate possible paths to be followed by students—and in the *in vivo* analysis—when they used them as a tool to institutionalise knowledge.

The second common questions were about how to adapt the experienced SRP to some particular school conditions. The strategy followed was to asked teachers to elaborate a *lesson plan* (which were retaken and re-elaborated in different parts of the modules). The work with respect to the (re)elaboration of the lesson plans with in-service teachers (the first case study) was richer than in the second case. This was, in part, thanks to the professional experience that participants had in secondary school and the fact that they could change of position easily. In particular, when they were asked to prepare a first proposal of the lesson plan, they easily adopted a new the position and assumed the dominant conditions in their school institutions, most of them closer to the paradigm of visiting work. This allowed a rich discussion on the implicit assumptions that were considered, and the related constraints derived from the paradigm of visiting works. In the second case study, pre-service primary school teachers are less subjected to the primary school conditionings. That can be a reason why their work on the lesson plan did not make emerged as many constraints as detected in the first case, and the discussion on the ecological analysis was highly limited, as teachers could not feel by themselves these limitations.

Modules 3 and 4 were only developed in the first case study, but it is important to mention the efforts of creating the appropriate devices (such as the lesson plan "in evolution", the reports of the implementation, the last version (with fixed) structure of the lesson plans) to enrich the *milieu* progressively. Our option consisted of elaborating a rich enough *milieu* between the educators and the participants to be used as a confrontation device for the teaching proposals and the theoretical tools introduced in the course.

In any case, the role-play, which was also a common strategy in both cases, appeared to be a successful strategy to make participants place themselves in different positions and make different institutional constraints emerge. It allowed to make visible some constraints that lead to the introduction of new tools for the epistemological and didactic analysis and to open a more general discussion on the constraints derived from the paradigm of visiting works and the collective awareness their scope.

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Transpositive Phenomena of Didactics in Teacher Training

Michèle Artaud and Jean-Pierre Bourgade

1 Introduction

The process of didactic transposition of knowledge in educational institutions has given rise to many works equipped by the theory of didactic transposition. It has also been shown that the process of institutional transposition of knowledge in an institution that is not a school could, under certain conditions at least, be read as a process of didactic transposition: we have called it a process of *archididactic transposition*. In light of these two aspects of the didactic transposition theory (ATD) and the latest developments in the anthropological theory of the didactic, we examine in this workshop some aspects of the process of transposition of mathematical didactics into the training of mathematics teachers.

We have considered the following question that Yves Chevallard has presented in his lecture: "What should student-teachers study?" or, more precisely, "What should the position p_{wt}'s praxeological equipment be made of?" where p_{wt} is the student-teacher position. We assume that this praxeological equipment must include some didactics. Therefore, the above question can be considered as the question of the didactic transposition process of the didactics in teacher training.

The difference with several works presented in the workshops, if we have correctly understood them, is that we are not going to model an existing didactic praxeology but instead *produce a model* of a praxeology that could, or even should, exist. Such a praxeology could be *part of a didactic infrastructure for the position of teacher, p_t,* and then something that should be studied in the p_{wt} position.

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2 A Didactic Organisation Produced with the ATD: An Example of Infrastructure

As we have written in some previous papers (Artaud, 2010, 2011, 2016), the model of the *didactic moments* (Chevallard, 2002), modelling *six functions of the study*, is a productive tool to analyse the didactic within a situation, because it allows seeing a lot of didactic phenomena that could be hindered by mathematical considerations. However, it is also useful to develop or to construct some didactic praxeologies because the development is based on the functions of the study—the *moments*—, which is a condition favouring the avoidance of a structural approach to the construction of praxeologies.

In this workshop, we have considered the following didactic type of tasks: "to *realise an exploratory moment*", that is a moment (which can take a long time, even several sessions) during which, in the process of study of a praxeology, the technique is produced, elaborated, and it generally necessitates the exploration of several tasks of the same type. The exploratory moment could be realised in several episodes. We have worked on *elaborating a didactic organisation (DO) around* this type of tasks. This DO should be *produced*, justified and made intelligible *by ATD*, and we have tried to place ourselves as much closer as we can in the paradigm of questioning the world.

To make things easier and save time, we have based our work on the DO that appears in the study plan presented during the workshop described in Miyakawa and Garcia (this volume). We have found there an analysis of a technique for the realisation of the exploratory moment relative to the type of tasks, "to solve an addition or subtraction situation" and, more precisely, to the subtype of tasks, "to make a drawing of the situation". The task studied in this lesson is the following:

> There were 16 people in a bus. Later, some people got on this bus. Now, as a whole, the number of people is 34. How many people got on later on?

Some theoretical technological elements (part of the *logos*) also appear. And we have assumed that the type of tasks has been encountered in the process of studying a certain question *Q*. *The aim of the first part of the workshop was to develop this technique and this technological-theoretical environment (the* logos*) into a praxeology for the realisation of the exploratory moment, a praxeology that would, at least, be partially produced and justified by ATD.*

The work performed by the four groups who took part in the workshop provided good support to progress. The first point that we can highlight is that the four groups began by writing the technique. It is a good way to start when analysing an existing didactic organisation—that is the first step. Nevertheless, when one wants to modify some DO, it is better to think about the technological or theoretical elements first or "dialectically" with the analysis of the technique. If we consider the techniques, the first group analysed the technique for the realisation of the exploratory moment by restricting the study to the exploration of the mathematical task at stake. This first group proposed:

> Encourage the students to draw. (*sic*)
> Tell pupils not to use formulae. (*sic*)
> Firstly, letting them draw individually.
> For pupils who get stuck, encourage them to draw a concrete situation.
> Remind of their previous works (drawing).
> Then, dividing them into groups and letting them draw together.

One more time, it is the first step, and then we can generalise the analysis to see to what extent the (didactic) technique depends on the mathematics at stake. For instance, here, we could write: to encourage the students to use an experimental technique and not a deductive one (a graphical one in most cases); first, let them work individually before constituting groups and asking them to share their techniques; etc.

The second technique proposed by the second group is a little more general technique but not a very operational one.

> Start with the *praxis*, not with *logos*.
> Consider diagrams (informal ostensives) as important, which are made by students.
> Use a realistic problem for making the problem lively for students (modelling problem).
> Allow the students to share different diagrams.
> Combine individual work and group work.

It is more some subtype of tasks for which we need a technique, and the formulation includes at least a technological statement as it is important to "make the problem lively for students". That is also a difficulty encountered when analysing techniques: what must and can be "put into words", and what can be considered as "well-known"? what is a part of the technique and what is justifying it?

In the techniques proposed by the third and fourth groups, there were proposals to modify the mathematical task in order to improve the technique of realisation of the exploratory moment, as well as the technique of the direction of the study. For instance, the third group explained that they would propose to:

> Divide the class into 2; let us say student type A & student type B. Both groups get different tasks in which the particular numbers are different. [. . .] They both get subtraction situations with the same context for the "task" (the bus context).

And, the fourth group wrote:

> We would propose the location of the bus with numbers between 5 and 20. [. . .] We would change the size of the numbers up to 50, and in each case, we would propose that they represent and look for answers to the different questions that have been proposed.

It is interesting to see that these proposals boil down to have different tasks in the class that change the parameters of the situation. Another time, this is expressed in the context of the mathematical task at stake.

If we consider the technology and the theory, we can notice that the second group gave an extensive theoretical assessment (as it is usual for theory) provided from ATD: "Preparing pupils to live in human society (questioning the world)" but no technological ones. The other groups give technological assessments, some of which are related to the choice of the mathematical task but not explicitly related to ATD. For instance:

> Teacher's understanding about the MO: drawing at the same time as reading the statement allows pupils to organise numerical values, to reconstruct the problem situation by means of their own drawings or words, and therefore to understand it. Working in small groups makes it easy to start searching for answers to the situation.

To develop an infrastructure for the realisation of the exploratory moment, we first consider that the *dialectics of the study* (Kim, 2015) are relevant to analyse the realisation of the didactic moments (this is a theoretical assessment). We also assume that the *dialectics of media and milieus* is one of the main relevant dialectics for the realisation of the exploratory moment (technological assessment). Finally, we include the definition of the dialectics of media and milieus (technological assessment): the dialectics of media and milieus enables a renewed relationship with the tools used to obtain and check information. Assessments obtained from a media are checked by the confrontation to some milieus.

We also take into account the following definition of the exploratory moment: it is the moment when the type(s) of tasks at stake is(are) explored and when at least an embryo of technique is produced. We are now able to produce a technique, or at least an infrastructure of a technique, for the realisation of the exploratory moment. For instance:

> To realise an exploratory moment, the type of tasks at stake must be explored; this exploration is performed by the students, who are given several tasks of the type at stake; for each task, the techniques that students will have implemented must be presented then tested by some milieus (that have to be available to the students) in order to keep the elements that stand the test and if necessary to develop them in order to build a technique or part of a technique.

Obviously, this infrastructure must be developed to integrate some characteristics of the mathematics at stake. For instance, we could add to take into account types of tasks related to addition or subtraction:

> If the type of tasks explored belongs to addition or subtraction types of tasks, vary the size of the numbers between the tasks at stake and ensure that a graphical milieu is available, as well as a calculator.

We turn now into the question of the didactic transposition by studying the conditions under which a teacher, that is, someone who occupies the p_t position, who would like to improve or develop his or her didactic praxeologies, could encounter, or even learn, the DO that we have just developed. When studying this question, we study some aspects of the *archididactic transposition process* of didactics.

3 Archididactic Transposition Process

The notion of *archidactic transposition* was introduced to analyse the transposition of mathematics into a knowledge-producing institution, economics (Artaud, 1993, 1994, 1995). In particular, we have highlighted that mathematics is a *fundamental* *knowledge* for economics, in the sense that mathematics makes it possible to produce economics or to use it as a system of knowledge production. When the production of knowledge K uses mathematics or any other K′ knowledge that plays a fundamental role for it, one may wonder through which channels the fundamental knowledge arrives in the sphere P_K of the production of the knowledge K under consideration.

Two answers must *a priori* be considered. First, either the elements of the fundamental knowledge necessary to P_K are already developed and may therefore be borrowed, accompanied by possible transpositive changes. In this case, the knowledge K′ "manipulated" in P_K is clearly exogenous. Second, the elements of knowledge K′ have been developed in P_K, by specialized actors, or actors who specialize for the occasion. In this case, the knowledge K′ used in P_K is endogenous.

From this basic schema, we observe the implementation of a more complex configuration, whose development is part of the historical time frame. On the one hand, actors who specialize in borrowing, adapting and developing useful K′ knowledge elements are emerging at the P_K border. With regard to economics and mathematics, we can look at these actors, permanent or occasional, as mathematical economists.

On the other hand, the use of knowledge K′ in P_K is recorded *in the very training* of future P_K actors. At a certain historical stage of development, in particular, a satellite institution of P_K, which we call the school associated with P_K (generically noted as E_K) emerges. The work carried out on K′ must then take into account new constraints: not only those linked to its import into a given region of P_K with a view to its use, but also the–strictly didactic–constraints generated by the desire to introduce K′ into the training of future P_K actors and, more generally, of all those who, in their activity, claim to know K because they recognise that this knowledge is relevant to their practice.[1]

Thus, in each period of its history, P_K will "learn" the part of K′ useful to P_K through its actors, either directly from $P_{K'}$, or indirectly, through the associated school, E_K.

Let us consider K = economics (E) and K′ = mathematics (M). The study of the modalities and conditions for integrating mathematics (M) into economists' training courses—i.e. in E_E—is traditionally part of the field of analysis of didactic transposition processes. The transposition processes that transport mathematical matter from P_M to the production institution of economics, P_E, are processes of institutional transposition of mathematical knowledge.

The elaboration of all the analyses carried out in the study of mathematisation in economics has gradually but irresistibly led to the thinking—that is, to the modelling—of observable phenomena in the genuine terms of the theory of didactic transposition. What is obvious is that, with regard to mathematical knowledge, the P_E institution functions like a school and its *noosphere*. This is an institution that can be seen as a school without being mainly a school since the "institutional problem"

[1]For instance, the profession of commercial recognizes that economics (K) is necessary to its activity and therefore incorporates mathematics (K′) into its members' training.

of P_E remains the production of economics knowledge. Nevertheless, it is an institution that also functions in the same way as a school (of mathematics).

Thus, within the *noosphere*—where the controversy over the relevance of mathematics takes place—a whole series of debates is developed, articulated around the question of the economist's mathematical needs, and aimed at clarifying and regularly modifying what can be regarded as a *curriculum*. This curriculum determines the subject be learned by attaching comments to its presentation, which are all "official instructions". Real didactic strategies are developed, generally unsophisticated ones and, to put it bluntly, somewhat rudimentary ones. Didactic systems appear and then disappear in an erratic but recurrent way. Finally, an evolving corpus of mathematical knowledge is built up in which, in each historical period, archaisms and innovations coexist.

This school, since this is how we look at the P_E institution, when compared to what we have called the school associated with P_E, E_E, appears as a primitive and primordial school, of which the associated school E_E is basically only the "offshoot".

For this reason, we can say that the P_E institution, which is seen as a school of mathematics –from which, in fact, the actors learn mathematics—is an *archischool*. From this point of view, the institutional transposition mentioned above—from P_M to P_E—can then be considered as an *archidactic transposition*.

As explained at the first edition of the ATD congress, "ATD is a very concrete thing, a machine for producing praxeologies, in particular, teaching and training praxeologies" (Artaud, 2007, pp. 256–257). In other words, didactics and, in particular the ATD, is a *fundamental knowledge* for the profession of teacher. Therefore, there must be a process of archidactic transposition, which transposes the didactics from the institution producing this knowledge to the profession of teacher. It will be noted as I_{MP} in the following, which is not a didactic institution, but which function as such with regard to didactics, by constituting an archischool. This archischool is the matrix of another institution, strictly didactic for its part: a school that trains I_{MP} actors, where the position p_{wt} exist.

Therefore, we consider it important to study the transposition processes and to examine archididactic processes too, as both processes can create some conditions and constraints that partially explain didactic transposition phenomena.

4 Conditions and Constraints of Didactic Transposition Phenomena of Didactics in Teacher Training

Let us go back now to the question at stake, namely whether a teacher who would want to improve or develop his or her didactic praxeologies could encounter the DO that we have just developed. That is, in other words, is the archididactic transposition process effective (or even possible?) in regards to the didactic knowledge involved in this DO?

In order to answer this question, we consider an ideal subject of p_t, and we try to formulate and test the conditions or constraints which allow for or prevent from developing his or her praxeological equipment to include this didactic organisation. We did not have time to fully accomplish and discuss this task in the workshop. We intended to test, in a dialectics of media and milieus, some of the following assertions (or to produce others).

A1. Unlike what is happening today for the couple (Mathematics, Economics), the process of archidactic transposition of didactics in I_{MP} is still embryonic, and this is caused by different conditions (see A2, and A4, for instance).

A2. Didactics is not yet seen by society as a *scholarly knowledge*, following the characterization of knowledge due to Y. Chevallard (1985), and it hinders its dissemination, including in "professional" institutions.

A3. Assertion A2 is reinforced in the case of teacher training by the constraint of the denial of the didactics (Chevallard, 2010). This leads, first, to discuss what is going to be taught, and it prevents the identification of a didactic relation to the issues of the study—or an institutional relation in the position of a teacher suitable for the study direction. Second, in a non-independent way, the denial of the didactics pushes the emphasis on the pedagogical aspects of professional gestures to the detriment of the more specific aspects that, nevertheless, largely determine the effectiveness of the study organisations.

A4. The teaching profession is a *semi-profession*, as developed by Y. Chevallard and G. Cirade (2010). This means that the *profession* (Chevallard & Cirade, 2010) makes little claim to provide scientific knowledge for the profession and, when it does, this knowledge does not appear fundamental to the core of the profession—namely the direction of the study of questions and the setting up, on this occasion, of praxeological organisations. In other words, the profession does not express needs for didactics as knowledge and, as a consequence, there are no "didactician professors" similar to "mathematician economists". Indeed, if some professors are interested in didactics to improve their professional praxeologies, this remains within the framework of the paradigm of the "small independent producer", and it is, in fact, almost invisible to the profession. It can be noticed that whereas the strategy of demathematization of the economy is second, the "dedidactisation" of professorial praxeologies seems to be first in the profession. In addition, the debates or controversies surrounding the relevance of didactics, which any didactician dealing with vocational training encounters, are institutionally invisible, which hinders a clear questioning of the teacher's didactic needs, not to mention that of the constitution of a study program.

To conclude, these conditions and constraints affect the didactic transposition process of didactics in teachers' training because training systems are not isolated from the profession. It is not easy to set up an institutional relation to didactics and, especially, to praxeologies produced by ATD, which would be functional when, in the profession, praxeologies around the same types of tasks exist whose *logos* are not based on didactics. The fact that these praxeologies are not well justified, or explainable, or even effective is not sufficient to disqualify them. For this reason, we think that it is of importance to develop in teachers' training analysis and evaluation of observed praxeologies, mathematical as well as didactic ones (Cirade & Crumière, 2019). It gives some *milieu* to elaborate professional praxeologies whose *logos* are based on didactics. But the development of observed praxeologies is also important, as it is, therefore, necessary for the researcher to build teaching praxeologies that could or should exist or, at least, infrastructures of such praxeologies to support teachers' training and that help on the diffusion of the didactics in the teacher position.

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Prospective Teachers' Narrative Analysis Using the Didactic-Mathematical Knowledge and Competences Model (DMKC)

Vicenç Font, Alicia Sánchez, and Gemma Sala

1 Introduction

In the design of Mathematics teacher training programs, it seems necessary to use a model of categories of knowledge and competences that are considered useful for the development of the professional activity. This is the reason why, in the literature on research about teachers' education, we find different proposals of such systems of categories. The analysis of the teachers' (or pre-service teachers') practicum reports is used, in many cases, to infer their competences and knowledge, according to one of such models of categories of knowledge and competences.

Based on the theoretical notions of the onto-semiotic approach (OSA) (Godino et al., 2019) and its several contributions to the field of teacher education, it has been developed a model (the DMKC model) that intends to assemble several categories of didactic-mathematical knowledge and professional competences of the Mathematics teacher required for a suitable teaching of mathematics (Godino et al., 2017; Breda et al., 2017; Pino-Fan et al., 2018). The theoretical tools of the DMKC model make possible to answer the following research question: *Which knowledge and teacher competences are involved when teachers describe, explain and assess their teaching practice?*

In the development of the workshop, we present the practicum report of a pre-service teacher, with the observations of some mathematics classes. This is a narrative made by a pre-service teacher, coming from the observation of a class of an in-service teacher and a guide with the categories of the DMKC model. Attendees are asked to address the following questions: Which competences (and at which degree of development) can be inferred from the report? Which type of knowledge (and which knowledge) can be inferred from the report? Which aspects of the guide

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used for the observation could be improved in order to better answer the first two questions?

In the following sections, we present a brief explanation of the Didactic-Mathematical Knowledge and Competences model (DMKC model), the methodology used in the workshop and an example of the analysis performed by the participants, and, in the end, some final considerations.

2 Didactic-Mathematical Knowledge and Competences Model

A theoretical model of the Didactic-Mathematical Knowledge (DMK) of the Mathematics teacher has been developed within the framework of the OSA (Godino, 2009; Pino-Fan et al., 2015, 2018). One of the perspectives of development of this model is the fitting of the notion of knowledge and the notion of competence. In addition, within the framework of OSA, there have been other studies regarding Mathematics teachers' competences (Font et al., 2015; Giacomone et al., 2018; Pochulu et al., 2016; Seckel & Font, 2015) that have also exposed the need of having a model of teachers' knowledge to evaluate and develop their competences. These two research agendas have joined, thus generating the Mathematics teachers' Didactic-Mathematical Knowledge and Competences model (DMKC model) (Breda et al., 2017; Godino et al., 2017; Pino-Fan et al., 2017).

2.1 The Notion of Competence

The Mathematics teacher should be able to address didactic problems related to the teaching of this subject, for which they need some specific competences. Therefore, two important questions to develop the DMKC model appear: (1) How is the notion of competence understood? and (2) Which are the key competences that the mathematics teacher should have? The competence in the DMKC model is understood from the action competence perspective, considering it as a combination of knowledge, skills, affective dispositions for action, tools for reflection, etc., that allows an effective performance, within typical contexts of the profession, of the actions aforementioned in its formulation. It consists in a potentiality that is updated in the performance of effective (competent) actions.

This formulation of the competence needs a characterization of its development (definition, levels of development and descriptors), in order to be operational. According to Seckel and Font (2015), the resolution of a task is the starting point for the development and evaluation of a teacher competence, since the task generates the perception of a professional problem that needs to be solved, and for this purpose, the teacher mobilizes skills, knowledge and attitudes, in order to develop a practice that intends to solve the problem. Furthermore, we can expect that such practice is performed with more or less achievement. It can be considered as an evidence that the person can perform practices that are similar to the ones described by some descriptors of the competence, which is at the same time associated to a certain level of development of the competence.

In the DMKC model, the mathematical competence and the competence in analysis and didactic intervention are considered the two key competences of the mathematics teacher. The core of the second one (Breda et al., 2017) is designing, applying and assessing sequences of one's own learning and others', through techniques of didactic analysis and quality criteria to establish cycles of planification, implementation, assessment and outline suggestions for improvements. In this work, we mainly focus on the competence in analysis and didactic intervention. This general competence consists of different subcompetences (Breda et al., 2017): (1) subcompetence in the analysis of the mathematical activity—as described in Godino et al. (2017), this subcompetence is decomposed into two more (the competence in analysis of global meanings and the competence in ontosemiotic analysis of mathematical practices)—; (2) subcompetence in the analysis and management of the interaction and its effect on students' learning; (3) subcompetence in the analysis of norms and meta-norms; and (4) subcompetence in the assessment of the didactic suitability of the process of instruction.

2.2 The Notion of Knowledge of the Mathematics Teacher

There are several models regarding the knowledge that a mathematics teacher should have to properly manage their students' learning. Pino-Fan et al. (2015) propose a model to characterize the didactic-mathematical knowledge (DMK) of the teachers, which considers, among other aspects, the contributions and development of several models of the mathematics teacher's knowledge, and the theoretical and methodological developments of the OSA. Thus, the DMK model (a part of the DMKC model) suggests that teachers' knowledge is organized into three big dimensions: mathematical, didactical and meta didactic-mathematical.

The first dimension, the mathematical one, refers to the knowledge that enables teachers to solve mathematical problems or tasks that are typical of the educational level in which they teach (common knowledge), and link the mathematical objects of that level to mathematical objects that are studied at higher levels (extended knowledge). Researchers who propose different models of the mathematics teacher's knowledge agree that, apart from the mathematical content, the teacher should have knowledge about several factors that influence the teaching of that mathematical content. The third dimension of the DMK, the meta didactic-mathematical dimension, refers to the knowledge needed to reflect on the own practice, that enables the teacher to assess the instructional process and improve it in a redesign for future implementations.

2.3 Competence in the Analysis and Didactic Intervention and Its Relation to the Model of Analysis of Instructional Processes Proposed by the OSA

The OSA (Font et al., 2010) considers five type of analysis of the instructional processes: (1) identification of mathematical practices; (2) elaboration of configurations of mathematical objects and processes; (3) analysis of trajectories and didactic interactions; (4) identification of the system of norms and metanorms; and (5) assessment of the didactic suitability of the instructional process. The first type of analysis explores the mathematical practices performed in a mathematical instruction process. The second one focuses on the mathematical objects and processes that intervene in the performance of the practices, as well as those that emerge from them. The third type of analysis is oriented to the description of patterns of interaction, didactic configurations and the articulation between them in didactic trajectories; the configurations and trajectories are conditioned and supported by a weave of norms and metanorms. The fourth type of analysis studies this weave. The fifth type is based on the four previous analysis and is oriented to the identification of potential improvements of the instructional process in new implementations.

The development of the competence in analysis and didactic intervention enables the teachers to do these types of didactic analysis proposed by the OSA and, at the same time, the training programs for the teaching and learning of these types of didactic analysis contribute to the development of this competence and the acquisition of teachers' knowledge of the DMKC model. They are training cycles (workshops) designed as powerful learning environments where: (1) attendees have an active participation starting with the analysis of classroom episodes; and (2) the types of analysis proposed by the model of analysis emerge from the interaction with the whole group.

In the different implemented workshops, with the just mentioned aim, we have observed the following regularities: (1) Teachers and pre-service teachers express comments in which aspects of description and/or explication and/or assessment can be found, when they have to give their opinion (without a previous given guide) about a classroom episode implemented by other teacher; (2) These teachers' opinions can be considered as evidences of some of the six facets (epistemic, cognitive, ecologic, interactional, mediational and emotional) of the didactic-mathematical knowledge model (DMK) of the Mathematics teacher (a part of the DMKC model); (3) When opinions are clearly appraising, they are implicitly or explicitly organized with some descriptors of the components of the didactic suitability criteria (another component of the DMKC model) proposed by the OSA (epistemic, mediational, ecologic, emotional, interactional and cognitive suitability); (4) The positive assessment of these descriptors is based on the implicit or explicit assumption that there are certain trends in mathematics teaching that indicate how a mathematics teaching of quality should be. These trends are related to the DMKC model, since some of them are the base to propose some of the didactic suitability criteria; and (5) The levels of depth of the analysis performed by the teachers vary from superficial analysis to expert, based on the theoretical tools used to perform them.

3 Methodology

We use a practicum report and a document with a brief explanation of some constructs of the DMKC model and some tasks that have to be answered in group. Finally, we have a discussion with the whole group to answer the three tasks previously mentioned.

Participants received a narrative, made with a guide of observation by a preservice teacher that attended a class of a teacher in-service, and a series of tasks that the group should answer, as well as an explanation about the theoretical tools presented in the previous sections. Finally, we had a discussion with the whole group to answer the following questions related to the narrative: Which competences (and at which degree of development) can be inferred from the report? Which type of knowledge (and which knowledge) can be inferred from the report? Which aspects of the guide used for the observation could be improved in order to better answer the first two questions?

4 Results

The practicum report was elaborated by a preservice teacher during the teaching practice in a group of primary education (six-years-old). In this report, she had to identify and describe a teaching and learning situation in the classroom and she received a guide to do it, which she answers in her narrative. The questions in the guideline are grouped into three blocks that require describing, interpreting and completing (in the sense of designing new and better situations). There are five questions about describing, three about interpreting and two about completing.

Participants had a guide with the categories of the DMKC model that they use to answer the three questions previously mentioned. In particular, each of these questions of the guide used by the preservice teacher to elaborate her practicum report, were analysed a priori systematically identifying the theoretical categories of the DMKC, with the aim to answer the following questions: Does the guide include the different categories of knowledge and competences of the mathematics teacher? How could the proposed questions be refined? In the following lines, we show an example of the work of analysis for one of the questions considered constructive.

Example of constructive question: *10. Modify the original task proposed by the teacher so that the student that has had difficulties to achieve the intended objective of learning, could achieve it. Justify your change.*

(1) Type of analysis: appraising (justify the change: it is the previous assessment of the didactic design that enables to make decisions for the redesign of new didactic

sequences); (2) Depth of the analysis: Level 1 (analysis of some tasks, required practices and mathematical objects used by the students with learning difficulties); (3) Phase of the study process: assessment and redesign; (4) Knowledge dimension: (a) mathematic: component of common knowledge, (b) didactic-mathematical: epistemic facet (necessary knowledge for the design of didactic sequences), (c) cognitive facet (noticing of learning difficulties and curricular adaptations). The other facets of the knowledge can also intervene, depending on the answer given, (d) meta-didactic-mathematical: didactic suitability criteria (the other criteria can also intervene, depending on the answer given); (5) Competence: subcompetence in assessment of the mathematical suitability (cognitive suitability).

5 Considerations

Conclusions of this course are that the participants infer knowledge and competences of a preservice teacher, when they use some of the characteristic of the DMKC model. Participants were able to deduce about the categories of the DMKC model that appear in the answers of the preservice teacher, through a qualitative analysis, in particular, they infer the level of development of the competence in analysis and didactic intervention and the different types of the teacher's knowledge.

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Exploring the Paradidactic Ecosystem: Conditions and Constraints on the Teaching Profession

Koji Otaki and Yukiko Asami-Johansson

1 Introduction: An Epistemological Obstacle in the Research on Teachers' Work

In this workshop, we aim to give participants some theoretical resources for investigating teachers' design and analysis of didactic situations, i.e. teachers' work outside their lessons. Teachers' activities in this kind more recently have attracted increasing research attention in didactics (e.g., ICMI-study 25 on "Teachers of mathematics working and learning in collaborative groups", 2020). However, in our view, researching such teachers' work—or *paradidactic* work (we explain it in detail later)—holds an epistemological obstacle that has to be overcome for the *scientific* inquiry. We tend to regard paradidactic reality as an "accessory" of (purely) didactic reality—remember the meaning of the prefix *para-*. This can be illustrated through looking at some challenges for transferring Japanese *lesson study*—a type of paradidactic work—into other countries. Such transplantation usually aims to adopt not only the methodology and institutional system of lesson study itself but also Japanese teachers' didactic and pedagogical views and methods of teaching. Because of this "second-class" status of the paradidactic reality, the teachers' work seems not to be regarded as a full-fledged object of study in didactics yet comparing to the phenomena in didactic situations. In our view, this obstacle is deeply rooted in two properties of the paradidactic. The first property is the intimate relationship between didactic activities and paradidactic activities. Didactic actions are planned and reflected in the paradidactic context, and paradidactic actions are organised

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around didactic situations. They are inseparable superficially, even though they are different entities in didactic research themes. The second property is that teachers' paradidactic work is rather closer to our own research activities than their activities for teaching, which used to be the central objects of our didactic research. In other words, both teachers out of lessons and researchers usually ask questions about didactic situations in their investigation, even if their ways of questioning the didactic reality (that is, the problematics) are essentially different. This means that the paradidactic is easy to become "transparent" for researchers—but of course, this is an illusion. This fate of the paradidactic as objects of the study in didactics implies that research on the paradidactic work needs well-constructed theories for detaching ourselves from familiar objects, much more than the case of usual research on didactic situations. This workshop especially focuses on a tool for studying the conditions on teachers' paradidactic work.

2 Didactic Systems Involved in Complex Institutional Ecosystems

In this section, we briefly introduce some basic ideas for studying *didactic* phenomena within ATD before inviting you to investigate *paradidactic* phenomena. It is because most notions and methods in paradidactic research are based on them in didactic research.

2.1 A Triptych of Didactic Systems

Any didactic situation—situations where someone teaches something to others—emerges in a system consisted of three elements: a work at stake, studying people and teaching people. Within ATD, such a system is called a *didactic system*, which is denoted by $S(X, Y, ♥)$, where X is a set of *students*, Y is a set of *teachers*, and ♥ is called a *didactic stake* (cf. Chevallard & Bosch, 2019). It can function as a simple model of an ordinary classroom, which have a group of students as X, a teacher as Y, and some piece of knowledge-to-be-taught as ♥. However, this is only an example of applying the triptych. In the sense of ATD, the meaning of the term *didactic system* is quite broad. For example, you can imagine a check-in procedure at a hotel. It usually activates the functioning of a small didactic system for teaching how to use the service and equipment of the hotel.

2.2 The Scale of Levels of Didactic Co-determinacy

Each didactic system *S*(*X*, *Y*, ♥) has its own right. However, this does *not imply that the system is closed*. All the elements of *S*(*X*, *Y*, ♥) can be affected by the factors out of the system. This fact is obvious; imagine a didactic system of a typical classroom. Undoubtedly, numbers of different factors outside the didactic system are involved in there: international educational movements, national schooling systems, school equipments, teachers' educational perspectives, etc. Within ATD, such factors are called *conditions* and *constraints* on didactic systems. According to the terminology of ATD, the word *constraint* means the conditions that are unadjustable for the players of a given system (more properly speaking, for a given *position*).

S(*X*, *Y*, ♥) is opened to the conditioning from its *super-systems* at different levels. ATD distinguishes six levels of conditions starting from the level of *didactic systems* (Fig. 1; cf. Chevallard, 2019). The level of *pedagogies* means a category for all pieces of knowledge and know-how of teaching something, which is independent of the specificity of didactic stakes ♥: for example, the so-called inquiry-based teaching. And then, a certain didactic system *S*(*X*, *Y*, ♥) within a pedagogy is consecutively subsumed into a *school*, a *society*, a *civilisation*, even a *humankind*, all of which bring about their own conditions for *S*(*X*, *Y*, ♥). In addition, didactic systems can be identified to different sub-levels depending on sizes or granularities of ♥: *Disciplines* ⇄ *Domains* ⇄ *Sectors* ⇄ *Themes* ⇄ *Subjects*.

Let us emphasise here that this kind of classification is *not a realistic picture of the system of conditions but a pragmatic toolkit for research on it*. The reality of the system looks like a spectrum rather than a bundle of discontinuous levels. An important research task is not identifying the levels of well-known conditions but finding out unknown conditions on a given phenomenon in terms of the scale.

2.3 The Ecology and Economy of Didactic Systems

Any didactic system functions in a unique way for handling various conditions and constraints, i.e. a given *institutional ecology*. Such functioning of a didactic system brings about its *behaviours* and *properties*. The set of such behaviours and properties

Fig. 1 The scale of didactic co-determinacy levels

Humankind
↓↑
Civilisations
↓↑
Societies
↓↑
Schools
↓↑
Pedagogies
↓↑
Didactic systems

of a certain didactic system $S(X, Y, \heartsuit)$ is called the *economy* of $S(X, Y, \heartsuit)$ (see also Gascón & Nicolás, 2019). In general, didactic research is supposed to include some analysis of the economy of didactic systems. For example, the theory of didactic situations (TDS) studies different states of didactic systems, i.e. didactic situations of devolution, action, communication, validation, and institutionalisation (cf. Brousseau, 1997). And, most of didactic research aims to create new "desirable" didactic economies through introducing new conditions.

3 Towards Paradidactic Research

3.1 A Nested Triptych of Paradidactic Systems

Let us define here teachers' work within ATD. We start from the fact that such a work deals with didactic systems $\mathcal{S} = S(X, Y, \heartsuit)$. Teachers design and analyse—simply speaking, *study*—didactic systems $\mathcal{S}$. We call the systems of teachers' working on $\mathcal{S}$ the *paradidactic systems* denoted by $\mathfrak{S}$. The adjective *paradidactic* has been introduced by Winsløw (2012) for studying teachers' work *about* the didactic situations. The fact that $\mathfrak{S}$ study $\mathcal{S}$—let us represent it by $\mathfrak{S} \succ \mathcal{S}$—suggests the existence of a nesting relationship between $\mathfrak{S}$ and $\mathcal{S}$. In our research project, we define the paradidactic system $\mathfrak{S}$ by $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$. It is composed of a didactic system $S(X, Y, \heartsuit)$ as a *paradidactic stake* $\spadesuit$, a set of *didactic engineers* $\mathfrak{X}$, and a set of *didactic mentors* $\mathfrak{Y}$: $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, \spadesuit) = \mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, S(X, Y, \heartsuit))$. In short, any paradidactic system $\mathfrak{S}$ can be modelled by a nested notation based on the triptych of didactic systems. Let us illustrate the usage of it. This workshop began with your analysis of a video data-collection of an elementary-school mathematics lesson $\mathcal{S}_{\mathrm{JP}}$ in Japan, which is simply defined by $S(X, Y, \heartsuit)$. The point is that we set up a paradidactic system $\mathfrak{S}_{\mathrm{WS1}}$ where $\mathfrak{S}_{\mathrm{WS1}} \succ \mathcal{S}_{\mathrm{JP}}$. We can describe $\mathfrak{S}_{\mathrm{WS1}}$ as $\mathfrak{S}(\mathfrak{P}, \mathfrak{W}, \mathcal{S}_{\mathrm{JP}})$ where participants $\mathfrak{P}$ as $\mathfrak{X}$; "workshopers" $\mathfrak{W}$ (i.e. KO & YAJ) as $\mathfrak{Y}$; and the lesson $\mathcal{S}_{\mathrm{JP}}$ as $\spadesuit$:

$$\mathfrak{S}_{\mathrm{WS1}} = \mathfrak{S}(\mathfrak{P}, \mathfrak{W}, S(X, Y, \heartsuit)).$$

After that, you will analyse your analysing process about the lesson: a new "transcendental" system $\boldsymbol{\Sigma}_{\mathrm{WS2}}$ will emerges where $\boldsymbol{\Sigma}_{\mathrm{WS2}} \succ [\mathfrak{S}_{\mathrm{WS1}} \succ \mathcal{S}_{\mathrm{JP}}]$. In short, $\mathfrak{P}$ will be not only players of this paradidactic system, but also its analysers. Thus, precisely speaking, the system $\boldsymbol{\Sigma}_{\mathrm{WS2}}$ will have a triple nested structure:

$$\boldsymbol{\Sigma}_{\mathrm{WS2}} = \Sigma(\mathfrak{P}, \mathfrak{W}, \mathfrak{S}(\mathfrak{P}, \mathfrak{W}, S(X, Y, \heartsuit))).$$

In our view, this triality plainly symbolises the difficulty of the paradidactic research.

3.2 Economic Usage of the Scale of Levels of Didactic Co-determinacy

The scale of levels of didactic co-determinacy is usually used for explaining the *ecology of didactic systems*. In addition, within the paradidactic research, it can also play a new role in describing the *economy of paradidactic systems*. For example, Otaki, Asami-Johansson, and Bahn (2020) clarify a propensity of teachers' post-lesson discussion in which teachers talk solely about specific mathematical topics and general teaching methods. They take for granted the conditions at the levels of sector, domain, and discipline—we call it the *paradidactic bipolarisation*. This kind of usage of the scale of didactic co-determinacy levels is not our original. Some researchers have consciously, or not, used this tool for the purpose of conducting economic analysis. To begin with, our identification of the bipolarisation phenomenon is based on the Spanish researchers' description of teachers' professional tendency in terms of the scale: the *thematic confinement* and the *pedagogical generalism* (e.g., Barbé et al., 2005; Florensa et al., 2018). In another case, Artigue and Winsløw (2010) apply the scale for identifying the foci of international comparative analyses, e.g. in PISA and TIMSS (This imply that the international surveys of this type are paradidactic activities of a special genre!). Let us name this usage of the scale the *economic usage* in contradistinction to the ecological usage.

3.3 The Scale of Levels of Paradidactic Determinacy

Any paradidactic system $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$ has an intimate relationship with its didactic system $S(X, Y, \heartsuit) = \spadesuit$, that is to say, $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$ brings about $S(X, Y, \heartsuit)$, and vice versa. However, this does not mean that each system lives under the same ecological conditioning. $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$ is born upon by a given ecology through a unique route and influence, which are different from those of $S(X, Y, \heartsuit)$. For example, a current teachers' paradidactic system and its didactic system can be conditioned by an identical condition of inquiry-based teaching movement. This movement might be easy to affect the paradidactic system; that is to say, teachers study inquiry-based teaching and consider its possibility in their didactic systems. By contrast, the didactic systems themselves resist such radical changes. This fact suggests a need for a specialised scale for investigating the ecology of paradidactic systems (Fig. 2)—we call it the *scale of levels of paradidactic determinacy* (Otaki et al., 2020). In this scale, the level of *professions* means the *didactic* profession, that is, the category of possible professions involved in paradidactic activities: schoolteacher, governmental official, textbook-designer, teacher-educator, mathematician, didactic researcher, etc. The *noospheres* of the next level indicates the *didactic* noospheres, which are fuzzy and extensive institutions thinking about school systems with didactic systems at stake (see also, Chevallard, 1992a, 1992b).

Fig. 2 The scale of paradidactic determinacy levels

Humankind
↓↑
Civilisations
↓↑
Societies
↓↑
Noospheres
↓↑
Professions
↓↑
Paradidactic systems

Fig. 3 A possible (but too much) integration of the two scales

Humankind
↓↑
Civilisations
↓↑
Societies
↓↑
Noospheres ⇄ Professions ⇄ Paradidactic systems
↓↑
Schools
↓↑
Pedagogies
↓↑
Didactic systems

To explain the case of the paradidactic bipolarisation, let us briefly introduce one of the constraints: the *lack of didactic theories*—e.g., the absence of the notions of *didactic contract* and *didactic transposition*—, which is located at the levels of societies and noospheres. Our noospheres around the world do not have enough terminology for recognising the didactic phenomena, especially those that are involved in the levels of disciplines, domains, and sectors. By contrast, schoolteacher can use pedagogical notions and very specific didactic notions. As a consequence, they tend to focus on general pedagogic matters and microscopic contents at stake in their paradidactic systems, with an illusion that the articulation of disciplinary knowledge-to-be-taught described in curricular documents is universal.

4 Relation of the Paradidactic Theory to the Fundamental Theory in ATD

The "A" of ATD encapsulates its fundamental assumption: *every human action is didactic in some sense*. According to this anthropological principle, any paradidactic entity is also didactic. We have no intention to resist it. In fact, we model the paradidactic systems as the didactic systems of a special type. Moreover, it is very likely able to integrate the paradidactic scale to the didactic scale, if we want to do so (Fig. 3). This integrated scale clarifies that the paradidactic research focuses on the interface—or "ecotone" (cf. Chevallard, 1992a)—between the levels of societies and schools (see also Chevallard & Bosch, 2019). However, our interest is *not* in such over-sophistication of the general theory (we consider that the integrated scale is specifically useful for justifying that our paradidactic theory is not independent of the fundamental theory of ATD). Praxeologically speaking, using the vocabulary of ATD, the paradidactic approach is *a domain of research praxeologies* based on the theoretical infrastructure of ATD. Based on the praxeology model, a typical sequence of four research actions can be described as following four components: having a problem about certain didactic facts (e.g., didactic facts around geometry); gathering data about the facts; constructing a spontaneous solution based on the data; and deconstructing and reconstructing the solution (and problem) within a theoretical framework (see also Artigue & Bosch, 2014). By using mathematical simile, we can make a following statement about such a didactic research activity: the fundamental part of ATD seems like a kind of set theory in didactics. Every (possible) domain of didactics—e.g., the didactics of mathematics, the didactics of Catalan, and the didactics of didactics—can be based on the fundamental theory of ATD.

At this point, we remind you of the epistemological obstacle with paradidactic research mentioned in the first section. Our slightly complex theorisation of paradidactic reality is not motivated by a purely theoretical concern, but deeply rooted in research practice with sensitivity to the transparency illusion. In our view, almost nothing is as "invisible" as the paradidactic in what we study within didactics. We need *scientific* theories constructed with relevant epistemological vigilance in social science, when we study the paradidactic reality.

5 Final Remarks: What Could Be the *raison d'être* of Paradidactic Research?

As we have already mentioned in the introduction, teachers' paradidactic work is on the way to gaining acceptance as a legitimised theme of study in didactics. Then, where could its epistemological legitimacy come from? Let us give here our tentative answer. A main aim of science is to understand the conditions under which its target phenomenon emerges (see also Chevallard et al., 2015). In the case of didactics, such conditions seem to be categorised into two major interrelated types: the *noospheric* and the *school*. Every didactic phenomenon is more or less affected by the institutional ecologies of both types, where a certain piece or body of knowledge related to the phenomenon lives. In other words, any didactic stake emerges under different conditions not only of the home-institution of a school of it (we have to remember that the term *school* has quite broad meaning according to ATD), but of also the transit-institution of a noosphere in its didactic transposition history. Within ATD, the existence of the school type of conditions is emphasised by the scale of levels of didactic co-determinacy. By contrast, the noospheric type does not fully attract research concern, even though we have the technical term of *noosphere*, which allows us to recognise the complex process of dissemination of knowledge. The noospheres are implicit but influential institutions between the schools and the societies in the didactic co-determinacy levels. In our view, the information about conditions of the noospheric type is crucial, especially when we study the possibility, viability, and reproducibility of radical didactic proposals like the *study and research path* (cf. Bosch, 2019). The importance of research on paradidactic phenomena will probably continue to grow more and more in the future.

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Teacher Learning in Collaborative Settings: Analysis of an Open Lesson

Takeshi Miyakawa and Francisco J. García

1 Teacher Knowledge Within the Anthropological Theory of the Didactic

The notion of praxeology is central within the Anthropological Theory of the Didactic (ATD). Coined more than 20 years ago, it offers a theoretical tool to describe, analyse, and question any human activity.

> [. . .] the notion of praxeology is at the heart of the ATD. This notion generalises different common cultural notions—those of knowledge [savoir] and know-how [savoir-faire], a word that generically refers to "*an ability that has been acquired by training*". It should allow designating, without epistemological-cultural implications (this is knowledge, this is not, this is "just" a skill, etc.), and without judging an a priori or a posteriori value, any possible structure of knowledge (Chevallard, 2009, p. 4, our translation).

When the focus is on the teaching and learning of mathematics, we usually talk about mathematical praxeologies or organisations (MP or MO) and didactic praxeologies or organisations (DP or DO). To make a long story short, a basic assumption within ATD is that any problem about mathematics teaching and learning should be considered in a wider sense as a problem about doing mathematics (Chevallard et al., 1997). Thus, *learning* is about carrying out a certain mathematical activity within an institution that can be described in terms of MP. And when this *study process* is supported by someone else (normally a teacher, in regular school institutions), the activity he or she makes to support learners' mathematical activity can be described in terms of DP. In this case, a *didactic system* is formed, designated by $S(X; Y; O)$,

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being *X* the people studying *O* (and that should be engaged in certain mathematical activity about *O*), being *Y* those that will help *X* studying *O* (normally, *Y* is a single teacher *y*) and *O* the mathematical organisations at stake (made up of praxeologies, or containing some praxeological elements at least). As a conclusion, we can state that a teaching and learning process could be described as a joint activation of MP and DP in mutual dependency (called *didactic codetermination*).

If we put our lenses in the teacher, as the study helper or facilitator, a crucial question arises: what kind of knowledge should he or she have in order to facilitate a fruitful and meaningful encounter of *X* with mathematical organisations MO? This is an old and nuclear question within research in mathematics education. Several theoretical frameworks have been developed to try to inquire about the nature of this knowledge (see Venkat & Adler, 2014). A possible answer from the ATD to this issue is the notion of *praxeological equipment of the teaching profession*, interpreted as the set of praxeologies teachers need to intervene effectively and pertinently in the mathematical education of their students (see Cirade, 2006; Bosch & Gascón, 2009; Chevallard, 2009; Ruiz-Olarría, 2015). Some important remarks about this notion are:

- It offers a unifying model of teacher knowledge (*logos*) and teacher action (*praxis*).
- This equipment is essentially made of mathematical and didactic praxeologies.
- It hypothesized its collective and institutional nature beyond personal *idiosyncrasies*.

Cirade (2006) pointed out that this equipment should include not only the mathematical praxeologies teachers would have to teach, but also praxeologies teacher would need to (a) delimit, interpret, connect and make explicit the raison d'être of these praxeologies, (b) conceive and construct the didactic praxeologies associated to the mathematical praxeologies to teach. From this perspective, an important research program within ATD has to do with the characterisation of the praxeological equipment needed, or at least useful, for teachers to intervene effectively and pertinently in their students' mathematical education in a given institution.

2 Teacher Learning: Devices and Infrastructures

The notion of *praxeological equipment* of the teaching profession as a model of teacher knowledge is one side of the coin. The other side is how teachers build and develop their knowledge. This relates to another important domain in the research in mathematics education: teacher education and professional development.

To tackle this domain of research within the ATD, two important notions could be borrowed and adapted from the ATD. The first one is the notion of *educational devices*, considered as any *mechanism* (in a general sense) arranged to produce some educational aims (Chevallard et al., 1997). The second one is the notion of *mathematical-didactic infrastructure*, based on Chevallard (2009), which could be considered as the set of conditions and constraints that affect and determine what is studied (MO) and how (DO) in a given institution. These two notions have been introduced mainly connected with mathematical study processes in school (from kindergarten to higher education). However, they could be considered in a wider sense, as far as any teacher education process is nothing but a kind of study process (not only about mathematics, but also about teaching them) in a given institution (normally, schools of teacher education in the teachers' initial education, but teacher centres, schools, or whatever, in teacher professional development). Thus, the problem of teacher education could be tackled by both questioning: (a) the *teacher education devices* that could be useful to build and develop teachers' praxeological equipment, and (b) the *infrastructures* necessary for such devices to exist and produce their intended outcomes.

To differentiate these *infrastructures* that affect teacher education from those related to the teaching of mathematics, the term *paradidactic infrastructure* has been introduced (Winsløw, 2011; Miyakawa & Winsløw, 2013, 2019). A *paradidactic infrastructure* has been defined as "the set of conditions for the work outside the classroom, related to a given MO and DO" (Miyakawa & Winsløw, 2019, p. 189). Here, it could be considered even in a more general sense, as the conditions and constraints that affect the existence and functioning of teacher education devices.

3 The Teacher Education Device "Open Lessons" and the Paradidactic Infrastructure in Japan

In general, an open lesson (in Japanese, *kokai-jugyo*) in Japan means any lesson wherein the colleagues inside or outside school are invited to observe the classroom teaching and provide comments mainly for the purpose of professional development. Its form is very simple: a teacher prepares a lesson attentively and writes a lesson plan for the observers; this teacher teaches in the classroom, and the others observe the lesson; the post-lesson discussion is organized after observing the lesson. It occurs at any educational level today, even in the university for the faculty development. Because the terms used in the educational field in Japan are not clear-cut, an open lesson could be considered as a kind of lesson study (Shimizu, 2014) or as a part of lesson study (synonymous with *research lesson*). This teacher education device forms a part of paradidactic infrastructure in Japan that allows mathematics teachers to work together.

Open lessons, and in general lesson study, have been acknowledged as a powerful professional development device. It has attracted worldwide attention mainly since Stigler and Hierbert's (1999) seminal work. Thus, lesson study has become a study object for many didacticians from different theoretical perspectives, with different research aims. Particularly, researchers from the ATD have focused on open lessons and lesson study using the ATD framework, with the aim to better understand the affordances and limitations of this practice, as well as its ecology within teacher education institutions. Among others, it is worth considering publications such as Winsløw (2011), Miyakawa and Winsløw (2013, 2019), Rasmussen (2016), or García et al. (2019).

4 Workshop's Rationale and Guiding Questions

It is the aim of this workshop to invite participants to engage in a research process about teacher learning in the context of an open lesson using ATD tools. The *open lesson* to be analysed in this workshop was taught in 2009, in a 2nd grade class (pupil age ≈ 7 years) of the elementary school attached to Joetsu University of Education, about word problems of subtraction (Fig. 1). This open lesson formed part of a 2-day "study meeting" (*kenkyu-kai*), held annually at this school, and attended by hundreds of teachers from all over Japan. During these 2 days, the school showcases lessons from all school disciplines, as well as other aspects of the school's life, such as after school musical and sports activities. Each activity contributes to give a holistic impression of the school's life, governed by the general aim of "preparing students to live in human society" (an approximate translation of the school's motto). The mathematics lesson considered in this paper was observed by about 70 teachers.

The workshop was structured as follows: first of all, we showed short video clips of the lesson and the post-lesson discussion to get an overall image of the open lesson, then, we proposed the participants analyse the data, considering what they had seen in the videos plus the detailed lesson plan, excerpt from textbook connected with the topic addressed in the lesson, the transcript of open lesson, and the transcript of the discussion after the lesson. The mathematical problem used in this lesson was a world problem of subtraction, which requires to use the segment diagram: "There were 27 passengers in a bus. Later, some passengers got on this bus. Now as a whole, the number of passengers is 34. How many people got on later on?" (see Hitotsumatsu, 2005, p. 72).

Participants' work was guided by the following questions:

- What MO and DO are planned to be implemented in the lesson plan and the textbook?
- What MO and DO are implemented in the classroom?

Fig. 1 An *open lesson* about word problems of subtraction in second grade class in Japan

- What paradidactic praxeologies could be identified in the open lesson?
- What teacher learnings happen in the open lesson?

5 Discussion and Perspectives

The detailed results of analysis of this open lesson can be found in Miyakawa and Winsløw (2013). We do not refer to them here. Instead, we discuss an aspect that has not been discussed in that paper, but they were in the workshop.

An important question that has been addressed in the workshop is how the choices of didactic organisations realized in the Japanese lesson can be justified. One answer proposed by the authors is the epistemological and didactic model adopted in the mathematics lessons of Japanese primary and middle schools, that is the teaching approach *mondai-kaiketsu-gata-jugyo* (literally translated into English as a *lesson of the form of problem-solving*, and also called in English *structured problem-solving approach* by Stigler & Hierbert, 1999). The lesson based on this approach is "designed for students to acquire knowledge and skills through creative mathematical activity by presenting challenging problems to students" (Takahashi, 2008). In addition to the development of problem-solving strategies and skills, which is the target of traditional problem-solving approach, this Japanese approach also aims at developing the conceptual knowledge on the specific mathematical contents. The teaching progression in a lesson with this approach is usually organised into four or five phases: posing a problem (*hatsumon*), students' work on the problem individually or in groups (*kikan-shido*), whole-class discussions of various solutions (*neriage*), summing up of the lesson (*matome*), and exercises or extension (*hatten*; optional). The lesson analysed in this workshop also followed this model. In addition to this model, the notion of *mathematical thinking* (*sugakuteki-kangaekata*) is also often referred to when discussing the structured problem-solving lessons in Japan (Hino, 2007; Isoda, 2012). We do not go into the detail of this notion, but it constitutes, with the structured problem-solving approach, an epistemological and didactic model of Japanese lessons.

Coming back to the ecological perspective of teacher collaboration, we consider that such an epistemological and didactic model is a critical condition that supports the teachers' collaborative work in the lesson study, that is to say, it is a part of the didactic as well as paradidactic infrastructure. In order to design and discuss the mathematics teaching in the group of teachers, it is necessary to have such a model of reference that justifies or criticises the didactic organisation proposed by a teacher. It can be the structured problem-solving approach and the mathematical thinking like in the case of Japanese lessons, or it could be any other, like the ATD or the Theory of Didactical Situations (Brousseau, 2002) the frameworks that allow us to design and analyse MO and DO to be implemented or have been implemented in the classroom. However, what is important to consider is that using an approach or another, will it be implicitly or explicitly, will have a direct impact on how the teacher education device is organised and what kind of knowledge will teachers build as a consequence of being engaged in such devices (García et al., 2019). The role of these epistemological and didactic models as essential components of the paradidactic infrastructure has been overlooked in existing research about lesson study/open lessons as professional development devices. Since we do not have enough empirical studies that investigate the relationship between the epistemological and didactic model and the teacher education devices such as lesson study, we think it is an important issue to be addressed in the future.

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Introduction to Part III The Curriculum Problem and the Paradigm of Questioning the World

Marianna Bosch and Noemí Ruiz-Munzón

The Curriculum and the ATD

The study of the curriculum has always been at the core of the anthropological theory of the didactic (ATD) since its first developments with the theory of the didactic transposition in 1980 (Chevallard, 1985, Chevallard & Bosch, 2020). This theory clearly expanded the object of study of didactics, which was mainly focused on teaching and learning processes taking place in the classroom. It introduced new analytical methods to start questioning different subject matter constructions that are at the core of school education. It also requires distinguishing the knowledge that is actually taught and learnt at school from what is designated as knowledge to be taught and from its original forms in the scholarly institutions that produce and use knowledge for purposes other than teaching it. The initial question for researchers is not how a given piece of knowledge can be better taught and learnt, but the different possible conceptions of this piece of knowledge and how it has initially been elaborated or produced and then transformed into something to be learnt at school. Analysing didactic transposition processes needs approaching knowledge as a social construction and also as structured activities taking place in social institutions.

Didactic transposition processes take place over different periods of time, and include several breaks in the form of curriculum reforms. They are carried out by

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Fig. 1 Scale of didactic codeterminacy in the paradigm of visiting works

Humanity
↓↑
Civilisations
↓↑
Societies
↓↑
Schools
↓↑
Pedagogies
↓↑
Disciplines ⇆ Domains ⇆ Sectors ⇆ Themes ⇆ Questions
↓↑
Didactic system $S(X, Y; \wp)$

different types of agents (scholars and producers of knowledge in different disciplines, but also the users of knowledge or educators and policy-makers) and in different institutions surrounding the school. The result is not always a coherent construction, and objects of knowledge from different historical periods can coexist. The problem of the ecology of knowledge emerges. The objects of knowledge are seen as alive entities that live in certain environments—*niches*—with certain raisons d'être that can be modelled in terms of *trophic levels* (Chevallard, 2007, 2017).

New notions and methods were needed to approach this universe of knowledge objects that appear in didactic transposition processes, their interactions and evolution. The development led to the emergence of the anthropological theory of the didactic in the 1990s, with a description of the social universe in terms of institutions, persons, objects, positions, relations, etc. and the modelling of human activities in terms of praxeologies. In Chevallard's words (2007, our translation), didactics as a field of research aims to study "the conditions and constraints under which praxeologies start to live, migrate, change, operate, perish, disappear, be reborn, etc. within human groups".

Approaching the conditions and constraints affecting the praxeologies that are to be taught at school will produce a new extension of the research focus that is represented by the scale of didactic codeterminacy (Fig. 1). While didactic transposition addresses the problem of how contents are selected, elaborated and organised to be transformed into themes, sectors, domains and disciplines, the study of the ecology of school praxeologies also questions how the upper levels of the scale affect their existence and possible evolution (including their disappearance).

In our societies, teaching and learning processes are mainly organised according to what Chevallard (2015) calls the paradigm of visiting works. In this paradigm, instructional processes are determined by the selection of a set of works or praxeological organisations—a curriculum—that students are asked to "visit" under the guidance of the teacher. The visit includes learning what those works are made of, which their main elements are and how they can be used, for instance, to solve some given sets of problems—usually called "applications". It not only comprises

Fig. 2 Scale of didactic codeterminacy in the paradigm of questioning the world

Humanity
⇅
Civilisations
⇅
Societies
⇅
Schools
⇅
Pedagogies
⇅
Didactic system $S(X, Y; Q)$
⇅
Disciplines ⇆ Domains ⇆ Sectors ⇆ Themes ⇆ Questions

becoming aware of their existence, but also acknowledging their importance as historical productions. The raisons d'être of these praxeologies, not only the reasons for learning them but also their reasons for existence, can remain in the shadow or simply be delayed, presented as something that will appear later on—if it does.

To avoid assuming the current state of things as if it were the only possible one, the paradigm of visiting works is subsumed into a larger pedagogical paradigm, the paradigm of questioning the world, which can also appear as a counter-paradigm because of the important changes it requires in the scale of didactic codeterminacy. The main element to define the paradigm of questioning the world is the notion of study and research path (SRP) based on the so-called Herbartian schema:

$$[S(X;Y;Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$$

where M is the "didactic milieu", i.e., the set of resources potentially used by the class $[X, Y]$ to construct the answer $A^{\heartsuit}$ defined to be the class's answer to question Q. In this case, we say that the class $[X, Y]$ studies Q or inquires about it. The Herbartian schema indicates the main elements of the inquiry process. Its dynamics is captured in terms of some dialectics (question-answer, media-milieu, individual-collective, among others) that describe the production, validation and dissemination of $A^{\heartsuit}$.

Adopting the perspective of the paradigm of questioning the world represents a crucial change. The pre-established organisations of knowledge do not precede the educational process generated by the didactic system $S(X, Y; Q)$. It is during the inquiry about Q that the potential knowledge tools are searched, selected, studied, tested and adopted as new elements of the milieu or they are simply rejected. The scale of didactic codeterminacy has to be modified accordingly (Fig. 2).

The latest advances in the ATD related to the curriculum problem are located at the interface between both paradigms: trying to determine the conditions needed for didactic systems to evolve towards the paradigm of questioning the world while identifying the constraints imposed by the current paradigm of visiting works that hinders, and sometimes even impedes, such an evolution.

Collective Advances During the Lecture and Workshops in Course 3

Chapter 3.1 contains the notes of Yves Chevallard's four-lecture course. They take stock of the current state of certain areas of the ATD and, on this basis, address the crucial question of the didactic paradigm shift that is becoming increasingly clear throughout the world today. Chevallard plays special attention to the higher levels of the scale of didactic codeterminacy (Humanity, Civilisations, Societies, etc.) and highlights the notions of pedagogy—in particular, the pedagogy of inquiry. He shows how to model classical study paradigms and the paradigm of questioning the world that is developing today.

Chevallard's notes are followed by four short chapters derived from the workshops that completed Course 3, illustrating some recent research works related to the curriculum problem. In chapter 3.2, Hamid Chaachhoua and Annie Bessot, from the Université Grenoble Alpes, and Julia Pilet, from the Université *Paris-Est-Créteil*, in France, study the case of quadratic equations through the analysis of dominant praxeological models. They discuss methodological issues regarding a reference praxeological model designed to conduct a comparative study of two institutions using different curricula for quadratic equations.

The didactic analysis of a school piece of knowledge, such as quadratic equations proposed by the French authors, corresponds to the didactic transposition methodology and its developments in the praxeological analysis. This line of research is carried out within the paradigm of visiting works. The three chapters that end Part 3 of this book address the design, implementation and analysis of study and research paths, an instructional format that can be located in the transition towards the paradigm of questioning the world.

Britta Eyrich Jessen, from the University of Copenhagen in Denmark, presents experimentations of study and research paths (SRPs) in chapter 3.3. Some correspond to implementations carried out in the researcher's classrooms; others come from her engagement in in-service teachers education courses about the design, development and analysis of SRP-based teaching. She emphasises the methodological choices concerning both the implementation and the analysis of the outcomes.

In chapter 3.4, Koji Otaki, from Hokkaido University of Education in Japan, develops the analysis of SRPs through the dialectic of questions and answers. He studies the dynamics of an SRP about calculating cube roots using simple pocket calculators, and uses it to illustrate the relationship between question-answer maps and the notion of praxeology.

Chapter 3.5 follows a similar direction. Verónica Parra and María Rita Otero, from the Universidad Nacional del Centro de la Provincia de Buenos Aires in Argentina, consider didactic-mathematical indicators of the dialectics that nourish the dynamic of SRPs and provide tools to pilot research-based teaching. They introduce a set of indicators of the inquiry dialectics and analyse the data obtained when implementing an SRP in the last year of the Argentinian secondary education level.

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Toward a Scientific Understanding of a Possibly Upcoming Civilizational Revolution

Yves Chevallard

1 Prefatory Remarks

The ATD is a complex theory, which in many ways breaks with the common relation to teaching and learning facts. Of course, everyone is free to study a given problem—the question of teacher education or the question of curriculums—using the tools and through the channels they want. However, we are here to address the problem of questioning the world in the framework of the ATD. And this presupposes that our relation to the ATD has a solidity that, first of all, I would like to examine and strengthen. I will therefore take the liberty of recalling basic elements of the ATD, insisting in particular on the key points that seem to me to be the least well identified in general.

For newcomers to the ATD, let me recall here what I have called the "Humpty Dumpty principle", which I borrow from Lewis Carroll's *Through the Mirror, and what Alice found there* (1871), and which applies to all sciences: "When *I* use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less." This principle says in direct terms what Blaise Pascal, around 1658, in *The Geometric Spirit and the Art of Persuasion*, stated as follows:

> Hence it appears that definitions are very arbitrary, and that they are never subject to contradiction; for nothing is more permissible than to give to a thing which has been clearly designated, whatever name we choose. It is only necessary to take care not to abuse the liberty that we possess of imposing names, by giving the same to two different things.

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2 Revisiting Some Basics of the ATD

2.1 Persons, Institutions, and Positions

The Macmillan Dictionary online defines "basics" as meaning "the most important aspects or principles of something that you should learn, think about, or deal with first". The point of departure of the ATD as a modeling tool of the didactic is the joint notions of a *person* and an *institution*, which, as everyone knows, are to be understood in a broad sense—newborns, infants, and toddlers are persons, for example; and families, classes, couples, clubs, and circles, among a countless number of entities, are full-fledged institutions—they are all "instituted" by human activity.

I must pause here to make explicit something very important to the ATD, i.e., *the use of letters and other symbols* to name the entities we are interested in. As a rule, *persons* are denoted by the small letter x (with a subscript if needed) and institutions by the capital letter I (same remark).

Let me then introduce the *third* key notion, that of *institutional position* p in an institution I. We shall denote such a position by the letter p if the institution I is clear from the context and otherwise by the ordered pair (I, p). An institution has at least one, and generally many, positions. In a school class, there are two main positions: let us call them the *student position* and the *teacher position*. In a family, there are generally the parent position and the child position. A given position (I, p) may be unoccupied briefly or for a long time: the class's teacher may be absent, for example.

A person x who is a "member" of an institution I occupies a position p in I. We say that x is a member of the institution I in the position p. We also say that the person x is a *subject* of p or is subjected to p. Persons are always subjected to *many* institutional positions—that of a native speaker of their mother tongue, of son or daughter, of husband or wife, of lover, of student or teacher (or both), etc. We shall also say that persons are the *resultants* of the (evolving) sets of (personal) subjections. Institutions are continually built up by persons, and, conversely, persons are shaped by institutions: this *dialectic of persons and institutions* is the beating heart of society as seen from the ATD.

2.2 Relations to Objects

We can now take a crucial step forward. Given any entity or, as we shall say, any *object* o, for any person x or institutional position (I, p), we define the *relation* of that person or position to the object o as the set made up of all the links relating x or p to o. These sets are denoted by $R(x, o)$ and $R_I(p, o)$, respectively. (Of course, one can also denote $R_I(p, o)$ by $R((I, p), o)$.)

In what follows, I shall subsume the notions of person and institutional position under a common notion, that of *instance* (think of Humpty Dumpty's principle), often denoted by a letter with a circumflex accent. An instance $\hat{\imath}$ can, therefore, be either a *personal* instance x or an *institutional* (or *positional*) instance (I, p). With this in mind, we shall denote the relation of the instance $\hat{\imath}$ to the object o by $R(\hat{\imath}, o)$. In the former case, we shall have $R(\hat{\imath}, o) = R(x, o)$, in the latter $R(\hat{\imath}, o) = R_I(p, o)$.

This subsumption may seem, at first, somewhat counterintuitive. However, we consider that, just like a person, an institution or, more exactly, an institutional position, can know, learn, forget, like, dislike, obsess over any "object". (Beware! An "object" o may as well be a person x.) When a person says "Here, we don't like dogs!", this person refers to a position that she occupies and at which dogs are not welcome. When people say "In here, they know too little mathematics!", they refer to a position to which—it seems—they are themselves subjected.

The notation used here allows us to denote, for example, the relation of an instance $\hat{\jmath}$ to an object $\tilde{o}$ which is the relation of the instance $\hat{\imath}$ to the object o, i.e., $R(\hat{\jmath}, \tilde{o}) = R(\hat{\jmath}, R(\hat{\imath}, o))$. In particular, $R(y, R(x, o))$ is the relation of the teacher y to the relation of the student x to the object o. In what follows, the Greek letter ξ will generically denote a researcher in the field of didactics, while $\widehat{\rho}$ will be a *position* of researcher in didactics. Of course, what will be of primary interest to us is the case of a didactician ξ doing research in the framework of the ATD. In other words, we shall be concerned with contributing to the definition of a researcher's position $\widehat{\rho}_{\mathrm{ATD}}$ within the framework of the ATD.

When the relation of an instance $\hat{\imath}$ to an object o is *not* empty, i.e., when $R(\hat{\imath}, o) \neq \varnothing$, we say that $\hat{\imath}$ *knows* o—this is, of course, a deliberately minimalist definition!—or that the object o *exists for* $\hat{\imath}$. When $R(\hat{\imath}, o) = \varnothing$, we say that $\hat{\imath}$ *does not* know o—or that o *does not exist for* $\hat{\imath}$. As a result, we have: $\hat{\imath}$ knows $o \Leftrightarrow R(\hat{\imath}, o) \neq \varnothing$. Now a *crucial* problem arises. What instance asserts that "we have $R(\hat{\imath}, o) \neq \varnothing$"? More generally, what instance $\hat{\jmath}$ professes that it is so? If such an instance $\hat{\jmath}$ exists, we shall write: $\hat{\jmath} \vdash R(\hat{\imath}, o) \neq \varnothing$, to be read: "the instance $\hat{\jmath}$ judges (or opines, or asserts) that $R(\hat{\imath}, o)$ is not empty" (or: "according to $\hat{\jmath}$, $\hat{\imath}$ knows o"). There may exist another instance $\hat{k}$ such that $\hat{k} \vdash R(\hat{\imath}, o) = \varnothing$ or $\hat{k} \vdash \neg(R(\hat{\imath}, o) \neq \varnothing)$. In other words, the instances $\hat{\jmath}$ and $\hat{k}$ have different views of $R(\hat{\imath}, o)$. Note that we have this: $\hat{k} \vdash (\hat{\jmath} \vdash R(\hat{\imath}, o) \neq \varnothing) \Rightarrow \hat{k} \vdash R(\hat{\jmath}, R(\hat{\imath}, o)) \neq \varnothing$. Similarly, we have: $l^\wedge \vdash \hat{k} \vdash R(\hat{\imath}, o) = \varnothing \Rightarrow \hat{l} \vdash R(\hat{k}, R(\hat{\imath}, o)) \neq \varnothing$. This noted, it may be the case that we also have: $\hat{k} \vdash (\hat{\jmath} \vdash R(\hat{\imath}, o) \neq \varnothing)$. More generally, let ϑ be any statement; we can have $\hat{\jmath} \vdash \vartheta$, $\hat{k} \vdash \neg\vartheta$, $\hat{k} \vdash (\hat{\jmath} \vdash \vartheta)$, etc. Of course, it can be the case that $\hat{\jmath} = \hat{\imath}$ or $\hat{k} = \hat{\imath}$ (so that, for example, $\hat{\imath} \vdash R(\hat{\imath}, o) \neq \varnothing$ or $\hat{\imath} \vdash R(\hat{\imath}, o) = \varnothing$). Some of the more important cases for us are when one or more of the instances $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are a *researcher in didactics* ξ, a *student* x, or a *teacher* y. In particular, according to the circumstances, we can have $\xi \vdash R(\hat{\imath}, o) \neq \varnothing$, $\xi \vdash R(x, o) \neq \varnothing$, $y \vdash R(x, o) \neq \varnothing$, $y \vdash R(\xi, o) = \varnothing$, $\xi \vdash R(\xi, o) = \varnothing$, etc.

However, a question is still open at this stage. When I write that, for example, $\hat{\jmath} \vdash \vartheta$, who then asserts that $\hat{\jmath} \vdash \vartheta$? Of course, it is the present author, the only person who can write (or say) "I" in the present context. In the following, I shall denote this "authorial instance" by the (archaic) Greek letter ϟ (koppa), so that if I write "ϟ $\vdash (\hat{\jmath} \vdash \vartheta)$", I arrive at a redundant formulation ("I assert that the author, i.e., myself, asserts that $\hat{\jmath} \vdash \vartheta$"), which I'll try to avoid.

2.3 Who's Speaking?

Before going further, it is important to stress three more points briefly. First, we define an *object* (in a human group or society) as anything that exists for at least one person or one institution (in that group or society). Let me add that, in consonance with the general ethos of the *anthropological* theory of the didactic, there is no object o that, at one time or another, does not deserve the interest of researching didacticians—all objects are *anthropologically* interesting.

Second, the functions $\hat{\imath} \vdash \ldots$, where $\hat{\imath}$ can be *any* instance, allow a multifocal approach to the didactic, in which every instantial "worldview" *may* count, depending on the research questions the didactician ξ is dealing with. Thus, the ATD stands at a distance from theorisations in didactics that centre on the relations to objects of the researcher ξ considered or, even more flippantly, of the researcher position $\widehat{\rho}$, and disregard most other instances' viewpoints. There is indeed a crucial advantage to the ATD's open stance towards otherness: it allows us to develop unconfused analyses of research situations. Here is a recently observed example.

2.4 The Case of Peer Review

A researcher ξ submits a paper to a research journal. The proposed study is about a class session where the students had to carry out a certain (mathematical) task t. Following the referees' reports, the paper is not considered acceptable by that journal's editors. In fact, it is critically required of ξ to explain further why, as ξ views it, the task t has to do with the learning of *algebra*. Now the author ϟ of the paper, i.e., ϟ $= \xi$, only asserts that it is some of the instances *observed by* ξ that view the task t as being likely to make the students aware of their need for "algebraic" tools. It is these instances, notably the teacher who designed the didactic scenario involving t and the teachers who implement it, who let it be known that they view t as likely to efficiently push the students into the world of (elementary) algebra. If we denote the latter statement ("The task t is likely to push the students…") by ϑ, we can say that $\xi \vdash (y \vdash \vartheta)$, where y is the designer or any one of the teachers involved, *not* that $\xi \vdash \vartheta$: in fact, it might be as well that $\xi \vdash \vartheta$ or that $\xi \vdash \neg\vartheta$. Therefore ξ should be required to justify that $\xi \vdash (y \vdash \vartheta)$, not that $\xi \vdash \vartheta$! It is true that ξ can be required to analyze the *reasons why* the designer and the teachers hold the belief that ϑ, which certainly has to do with both their relation to t, i.e., $R(y, t)$, and their relation to the mathematical field of algebra, i.e., $R(y, \mathcal{A})$, where the letter $\mathcal{A}$ stands for "algebra".

Note that, in complying with this requirement, ξ would have to draw on his relations $R(\xi, R(y, t))$ and $R(\xi, R(y, \mathcal{A}))$, which, in turn, will *partially* depend on his relations $R(\xi, t)$ and $R(\xi, \mathcal{A})$. It is not unreasonable to presume that what motivated the referees was twofold. The first incentive has a systemic source: referees are supposed (including by themselves) to know at least one of the fields of expertise to which the question tackled by the author $\daleth = \xi$ in the paper they have to review can be related more or less explicitly. At the same time, however, they are generally ignorant of the nontrivial aspects of the question under study because of the peculiarity of the material made available by the author and, of course, because the question under consideration is supposed to be on the front lines of science. As a result, referees tend to insist on unspecific aspects of the paper submitted to them.

One of these aspects consists in making (often bitter) remarks on methodological matters, which seem liberally open to comments and suggestions—the referee, so he or she usually suggests, would have done otherwise! The second "solution" consists in criticizing the author's supposedly inadequate command of some scientific field implicated in the proposed study. In the episode I refer to here, it will be for example "algebra", denoted here by $\mathcal{A}$, with referees implicitly boasting about their own relation to $\mathcal{A}$ and, at one go, querying the author's one. It boils down to saying that $R(\daleth, \mathcal{A})$ will not be validated by the referee ω unless it resembles closely enough—according to the referee's discernment—the referee's own relation $R(\omega, \mathcal{A})$, held by ω to be the "right" relation to $\mathcal{A}$ (humility is usually not a referee's forte).

From the point of view of the ATD, this is epistemological wrongdoing—bragging is rarely a good thing on a researcher's part. From the vantage point of $\widehat{\rho}_{\mathrm{ATD}}$, there is no such thing as a good-in-itself relation to *any* object. The important point is not about the relation to $\mathcal{A}$ of the researcher $\xi = \daleth$ but about ξ's ability to adequately *model* such relations as $R(x, t)$, $R(y, t)$, $R(x, \mathcal{A})$, and $R(y, \mathcal{A})$, which depends only *in part* on $R(\xi, \mathcal{A})$. In this respect, it turns out that, quite often today, innovative and committed teachers fall in with the age-old but enduring view that algebra should be construed as "generalized arithmetic". This fact, of course, should be part of $R(\xi, R(\widehat{\mu}, \mathcal{A}))$, where $\widehat{\mu}$ is the mathematics teacher position. Researchers always have to alter and enrich their relations to many objects to free themselves from the current mainstream conception. Another view, and indeed a competing view of (elementary) algebra, is that $\mathcal{A}$ is a set of tools to *model* ("algebraically") number phenomena. The key word here is *model*.

2.5 The Case of Algebra

Let me indulge in a quick example. Consider the numbers $a = 2$ and $b = 8$. Their arithmetic mean is $m = \frac{2+8}{2} = \frac{10}{2} = 5$. Here is an arithmetic phenomenon: in chained notation, we have $2 < \frac{2+8}{2} < 8$. We can check on other ordered pairs that the mean m seems to always lie *between* a and b. This leads to the conjecture that if $a < b$, then $a < \frac{a+b}{2} < b$. This is an algebraic model of the presumed arithmetic phenomenon we are interested in. How can we generate this model?

We can do that in a manner that is *not* the generalization of any "natural", basically arithmetic procedure. Indeed we have: $a < b \Rightarrow a = \frac{2a}{2} = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = \frac{2b}{2} = b$. As anyone can see here, the amount of "algebraic" calculations is kept to a minimum: at every step of the way, we slightly alter the form of the (algebraic) expression obtained so far. But there is more to it than that. By looking at numerical examples, we can also convince ourselves that m lies right in the middle of the interval $[a, b]$. Once again, the "algebraic" work to perform to check this conjecture is tantamount to *seeing* the forms of the algebraic expressions that are displayed before our eyes. When we look at our "proof" that $a < m < b$, we first see that in going from $a = \frac{a+a}{2}$ to $m = \frac{a+b}{2}$ we replace an a by a b, which produces an increase of $\frac{b-a}{2}$; then in going from $m = \frac{a+b}{2}$ to $b = \frac{b+b}{2}$, where once again an a is replaced by a b, we observe another increase by the same amount, i.e., $\frac{b-a}{2}$. We thus arrive at the equalities $m - a = b - m = \frac{b-a}{2}$, which prove our guess.

The glossary of a book on mathematics (Morrison & Hamshaw, 2015, p. 616) defines algebra as "the use of letters and other symbols to write mathematical information". This is a good, i.e., minimalist, "definition". Now a key question is left blank: *which uses* will it serve? This is exactly the question I have tried to consider here. The onus of addressing it is *indefinitely* on researchers in didactics, for all answers to this question are necessarily provisional—from the point of view of the practicing researcher at least.

2.6 The Praxeological Analysis of Action

Let us consider an instance $\hat{w}$ that we shall call the *reference instance* (in truth, $\hat{w}$ can be *any* instance). Given an instance $\hat{\imath}$, the *universe of objects* or *cognitive universe* of $\hat{\imath}$ according to $\hat{w}$ is defined by: $\Omega_{\hat{w}}(\hat{\imath}) \stackrel{\text{def}}{=} \{o \mathbin{/} \hat{w} \vdash R(\hat{\imath}, o) \neq \varnothing\}$. The set $\Omega_{\hat{w}}(\hat{\imath})$ tells us *which* objects exist for $\hat{\imath}$ according to $\hat{w}$. The *cognitive equipment* of $\hat{\imath}$ according to $\hat{w}$ is then defined by: $\Gamma_{\hat{w}}(\hat{\imath}) \stackrel{\text{def}}{=} \{(o, R(\hat{\imath}, o)) \mathbin{/} o \in \Omega_{\hat{w}}(\hat{\imath})\}$. The set $\Gamma_{\hat{w}}(\hat{\imath})$ specifies *how* $\hat{\imath}$ knows the objects that $\hat{\imath}$ knows according to $\hat{w}$, i.e., the objects $o \in \Omega_{\hat{w}}(\hat{\imath})$.

What causes the relations to objects to exist in persons and institutions? The answer hinges on the notion of *praxeology*, the key notion of the theory of human action propounded by the ATD. Let us consider an institution I and a position p in I. The subjects of the instance $\hat{o} = (I, p)$ engage in an activity $\tilde{a}$ which is seen by an instance $\hat{w}$ as the implementation of a praxeology $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w})$. The symbol $\overset{A}{p}$ denotes the *praxeological functor*, which supplies a praxeological analysis of the activity $\tilde{a}$, that is to say, essentially, specific answers to three questions: (a) *What* do people in p do?, (b) *How* do they do it?, and (c) *Why* do they do it that way?

From the point of view of the ATD, the answer to the first question can be formulated in terms of *tasks* of some *type*. The ATD posits that all action splits into a sequence of *tasks* t_i of different *type* T_i. As is usual in the ATD, the notion of task is a very wide notion: many human activities that seem to some of us "natural", such as the task of conversing, for example, or of drinking a glass of water, must be looked at as *tasks*. As is usual, too, the notion of task is invariant under changes of "size" (in terms of the amount of work required to perform the task): "calculating the difference of two integers", "writing a poem", "opening a mustard jar", "buying a new car", "getting married", are all types of tasks.

The answer to the second question is that, to carry out some task t of type T, one needs a *technique* τ (from Greek τέχνη), i.e., a determined way of doing tasks t of the given type T. It should be noted that a technique τ succeeds only on a set of tasks $T_\tau \subsetneq T$ which is called the *scope* of τ. It must be emphasized that the preceding requirement governs every type of action: a specific technique is therefore required to speak, sing, walk, squat, blow one's nose, brush one's teeth, swallow a pill, or prove that the number e is transcendental. The ordered pair made up of a type of tasks T and a technique τ relative to T is denoted by [T / τ] and is called the *praxis block* (or block of "know-how").

The answer to the third question is that any technique requires a *justifying* comment called its *technology*, denoted by the Greek letter θ. This technology θ is itself coupled with a supporting discourse at a higher level, that of the *theory* Θ of the technique τ. The ordered pair made up of the technology θ and the theory Θ is denoted by [θ / Θ]: it is the *logos block* (or block of "knowledge"). It can be observed that the quality of the justification of the *praxis* block [T / τ] provided by the *logos* block [θ / Θ] is variable according to the institution I. But one point is not debatable: all *praxis* blocks tend to couple with a *logos* block, thereby constituting a fully-fledged, though sometimes questionable, praxeology $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w}) = [T / \tau / \theta / \Theta]$. Let us denote the blocks [T / τ] and [θ / Θ] by the Greek capital letters Π and Λ, respectively. We thus have: $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w}) = [T / \tau / \theta / \Theta] = [T / \tau] \bigoplus [\theta / \Theta] = \Pi \bigoplus \Lambda$.

A fourth question must be raised to complete the praxeological analysis of an activity $\tilde{a}$: *Why* do people in p do what they do? In other words, what is the reason why they do what they do? What is the *raison d'être*—the reason for being—of the task t carried on? What motivates it? The kind of answer which is relevant here is based on the following phenomenon: a task t of some type T is performed because performing it is part and parcel of a technique τ_1 relative to a type of tasks T_1: to perform a task $t_1 \in T_1$, one has at some time or other to perform a task $t \in T$. If you want to cook ravioli (type of tasks T_1), you have to boil water (type of tasks T). If you want to sum two fractions (T_1), you may need—according to the technique τ_1 used—to calculate the least common multiple of two numbers (T). Etc.

2.7 Praxeological Obsolescence

What is sometimes the case is that some type of tasks T, which used to be imposed by a widespread technique τ_1, remains in a curriculum, while the technique τ_1, which justified its presence therein, has long vanished from the curricular scene. In such a case, if the performing of tasks of type T is not required by any other technique τ_2, T becomes "unmotivated", deprived of any raison d'être and useless, and the performing of tasks t ends up resembling a mysterious rite of passage—"Don't ask why, you have to do it!" In some lower high school curriculums—until recently, this was the case in France, for example—, there remained the formula $\frac{a}{b} = a \times \frac{1}{b}$ (with

$b \neq 0$). This is certainly mathematically correct. But what is it for? What praxeology is it a part of? A century ago, it was a key technological component of a taught praxeology whose purpose was to facilitate the division of two numbers. Suppose you have to calculate $20 \div 8$ and you know in advance that $\frac{1}{8} = 0.125$; you have: $20 \div 8 = \frac{20}{8} = 20 \times 0.125 = 2 \times 1.25 = 2.5$. Of course, you have to know in advance, so to speak by heart, the value of the reciprocal of the divisor, as, for example, $\frac{1}{5} = 0.2$. In many cases, the reciprocal is *not* a decimal number, and you have to settle for an approximate value: $\frac{1}{3} \approx 0.33$; $\frac{1}{7} \approx 0.142$; etc.

One important case was the division by π: if you want to calculate the radius of a circle of length 10 m, you have to calculate $\frac{10}{2\pi} = \frac{5}{\pi} = 5 \times \frac{1}{\pi}$. Just as everyone knew that $\pi \approx 3.14$, everyone had to know that $\frac{1}{\pi} \approx 0.318$ or 0.32. So that you had: $5 \times \frac{1}{\pi} \approx 5 \times 0.318 = 1.59$ or $5 \times \frac{1}{\pi} \approx 5 \times 0.32 = 1.6$. Today, a calculator gives at one click: $\frac{10}{2\pi} = 1.59154943\ldots$

It may happen that a praxeological element $\mathfrak{p}$ remains lastingly in the curriculum even though it is no longer praxeologically motivated—it is precisely the case with the equality just considered. Two remarks, however, are in order. The first is that, when teaching an unmotivated praxeological element $\mathfrak{p}$, a teacher generally concocts a "didactic" justification, no matter how vague it may be. The second remark is that a praxeological element long unmotivated in a given institution can recover a role in another institution and even can reassume the function it played in its "traditional" role.

This is typically the case in modern computing with the method called "Division by Invariant Multiplication", whose pivot is the equality $\frac{a}{b} = a \times \frac{1}{b}$, where the reciprocal $\frac{1}{b}$ is *precomputed*. Suppose $b = 7$. If we have $\frac{10^{20}}{7} = 14285714285714285714\ldots$ we arrive at, say,

$$106 \div 7 \approx 106 \times 14285714285714285714 \times 10^{-20} = 1514285714285714285 26 \times 10^{-20}$$
$$= 15.142857142857142856 84.$$

Note that, because $\frac{106}{7}$ is a repeating decimal, its true value is $15.142857142857142857\underline{14}\ldots = 15.\overline{14287}$.

2.8 Praxeologies and Relations

Where do personal and institutional relations come from? For any instance $\hat{\imath}$ and for every praxeology $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w}) = [T / \tau / \theta / \Theta]$, we consider the relation $R(\hat{\imath}, \mathrm{P})$. We define the *praxeological universe* of $\hat{\imath}$ according to $\hat{w}$ by $\Omega_{\hat{w}}^{\bigstar}(\hat{\imath}) \stackrel{\text{def}}{=} \{\mathrm{P} / R(\hat{\imath}, \mathrm{P}) \neq \varnothing\}$ and $\hat{\imath}$'s *praxeological equipment* according to $\hat{w}$ by $\Gamma_{\hat{w}}^{\bigstar}(\hat{\imath}) \stackrel{\text{def}}{=} \{(\mathrm{P}, R(\hat{\imath}, \mathrm{P})) / \mathrm{P} \in \Omega_{\hat{w}}^{\bigstar}(\hat{\imath})\}$. We have $\Omega_{\hat{w}}^{\bigstar}(\hat{\imath}) \subset \Omega_{\hat{w}}(\hat{\imath})$, where $\Omega_{\hat{w}}(\hat{\imath}) = \{o / \hat{w} \vdash R(\hat{\imath}, o) \neq \varnothing\}$ is the *cognitive* universe of $\hat{\imath}$ according to $\hat{w}$, and $\Gamma_{\hat{w}}^{\bigstar}(\hat{\imath}) \subset \Gamma_{\hat{w}}(\hat{\imath})$, where $\Gamma_{\hat{w}}(\hat{\imath}) = \{(o, R(\hat{\imath},$

$o)) \,/\, o \in \Omega_{\hat{w}}(\hat{\imath})\}$ is the *cognitive* equipment of $\hat{\imath}$ according to $\hat{w}$. We now posit that, conversely, $\underset{\widehat{w}}{\cup}\Gamma_{\hat{w}}^{\star}(\hat{\imath})$ generates $\underset{\widehat{w}}{\cup}\Gamma_{\hat{w}}(\hat{\imath})$ in the following sense: whatever the object o, the relation $R(\hat{\imath}, o)$ results from all the relations $R(\hat{\imath}, \mathrm{P})$ where $\mathrm{P} \in \underset{\widehat{w}}{\cup}\Omega_{\hat{w}}^{\star}(\hat{\imath})$ involves the object o, whether technically, technologically or theoretically.

This generating principle applies to any object o. Thus the relation $R(x, x')$ of the person x to a person x', for example, to x's mother (in which case $x' \neq x$), or to x themself ($x' = x$), arises from all the praxeologies to which x has a non-empty relation and which involve x'. The same applies to all instances $\hat{\imath}$ and $\hat{\imath}'$. To analyze in depth the contents of a relation $R(\hat{\imath}, o)$ or $R(\hat{\imath}, \mathcal{O})$, where $\mathcal{O}$ is a set of objects o, one has to concretely inquire about the praxeologies that generated it, either recently or in a more remote past.

Let us consider an activity $\tilde{a}$ carried out in a positional instance $\hat{o} = (I, p)$ and which an instance $\hat{w}$ regards as the implementation of a certain praxeology $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w}) = [T \,/\, \tau \,/\, \theta \,/\, \Theta]$. This equality signifies that $\hat{w}$ interprets the activity $\tilde{a}$ as the performing in the position $\hat{o}$ of tasks t of type T by means of the technique τ, according to the technological-theoretical block $\Lambda = [\theta \,/\, \Theta]$. Now let $\tilde{A}$ be the set of praxeologies $\mathrm{P} = \overset{A}{p}(\hat{o}, \tilde{a}, \hat{w})$ for all $\hat{o}$ and $\hat{w}$. Given an instance $\hat{\imath}$, let us consider the relation $R(\hat{\imath}, \tilde{A})$. If $\hat{\imath}$ establishes in an instance $\hat{o}'$ an activity $\tilde{a}'$ regarded by $\hat{\imath}$ as corresponding to $\tilde{a}$, i.e., such that $\hat{\imath} \vdash \tilde{a}' \mathrel{\widehat{=}} \tilde{a}$ (where $\widehat{=}$ means "corresponds to"), we say that the praxeology $\mathrm{P}' = \overset{A}{p}(\hat{o}', \tilde{a}', \hat{\imath}) = [T' \,/\, \tau' \,/\, \theta' \,/\, \Theta']$ is a (personal or institutional) *transpose* of P according to $\hat{\imath}$, which we denote by: $\mathrm{P}' = \mathrm{P}^{\mathrm{tr}}$. (The *transposition functor* is a multivalued or plurivocal functor.) Of course, if $\hat{w}'$ is another instance, we can have $\hat{w}' \vdash \neg(\tilde{a}' \mathrel{\widehat{=}} \tilde{a})$. Generally speaking, $\tilde{a}'$ and P' will depend on $R(\hat{\imath}, \tilde{A})$ and on the *conditions* that prevail in the position $\hat{o}'$. It is the core notion of condition, central to the theory of the didactic developed in the ATD, that we will now examine.

2.9 Conditions and Constraints

What causes the relations to objects to exist in persons and institutions? The general answer is: the *conditions* that prevail around them. The changes that affect these conditions explain the changes that the relations $R(\hat{\imath}, o)$ may undergo. Now, what is a condition? This question can receive a formal answer: given any system $\mathcal{S}^*$, a *variable* (i.e., a variable quantity) $\boldsymbol{v}^*$ defined on the states $\boldsymbol{s}^*$ of $\mathcal{S}^*$, and a set V of possible values v of $\boldsymbol{v}^*$, the condition c is defined by: $c \overset{\mathrm{def}}{\equiv} \boldsymbol{v}^*(\boldsymbol{s}^*) = v$, where $v \in V$. (Note that $\boldsymbol{v}^*$ can be a dichotomous or a polytomous variable as well as a continuous variable.) If $\mathcal{S}^*$ is a classroom, we can consider the condition "the room's temperature is between 21 °C and 23 °C".

More generally, we can define a condition by a combination of variables defined on $\mathcal{S}^*$. If $\mathcal{S}^*$ is a class of s students comprising f female students and m male students, we can consider the condition $f > \frac{1}{3}m$ and $m > \frac{1}{3}f$, which is the same as $f > \frac{1}{4}s$ and

$m > \frac{1}{4}s$. (If $s = 24$, for this condition to be satisfied, we must have $f \geq 7$ and $m \geq 7$.) Other conditions can be "Each student has a calculator", "Each student has a workable Internet connection", "Students have had enough to eat before coming to school", "This student does not know what the derivative of e^x is", etc.

Let us denote by $\mathcal{C}(t)$ the set of all prevailing conditions relative to *all* systems in the universe. This definition has the advantage of emphasising that, in the ATD, we theoretically take into account *all* possible conditions, relative to *all* possible systems—we do not restrict our interest to a limited number of them that tradition has brought to our attention. But this definition must be criticised on one count: the "set of prevailing conditions $\mathcal{C}(t)$" depends on the instance $\hat{w}$ that considers it: there exists no all-encompassing vantage point to look at the set $\mathcal{C}(t)$. For this reason, instead of writing $\mathcal{C}(t)$, we shall henceforth write $\mathcal{C}_{\hat{w}}(t)$ to denote the set of prevailing conditions *known to* $\hat{w}$. Some conditions may be absent from $\mathcal{C}_{\hat{w}}(t)$ simply because of t—until recently in the history of humanity, no instance $\hat{w}$ could take into account the level of ionising radiation (which is detectable with a Geiger counter but not by the human senses).

This can be generalized. Very often, $\mathcal{C}_{\hat{w}}(t)$ excludes conditions on which $\hat{w}$ has no control, i.e., that $\hat{w}$ is not allowed to alter. Such conditions are called *constraints for* $\hat{w}$. However, any instance $\hat{w}$ is fully aware of *many* conditions that are constraints for $\hat{w}$ but which are not constraints for some other instances $\hat{w}'$. For example, students know that they cannot freely set the date of their next test: they know that it is a prerogative of the teacher or the school administration. More generally, given two instances $\hat{w}$ and $\hat{w}'$, we may have $\hat{\imath} \vdash \mathcal{C}_{\hat{w}}(t) \neq \mathcal{C}_{\hat{w}'}(t)$. It is of the utmost importance for ξ (and $\widehat{\rho}$) to learn to distinguish between $\mathcal{C}_{\hat{w}'}(t)$ and $\mathcal{C}_{\hat{w}}(t)$ because each of them is generally characteristic of the type of instance concerned. For example, the set of conditions that exist for a "classical" didactician seems at times to be almost disjoint from the set of conditions contemplated by a classical pedagogue or, for that matter, by a sociologist of education. In this respect, the ambition of the ATD is to theoretically and practically overcome the limitations that tend to characterise each of these instances.

2.10 "Unknown Unknowns"

All this notwithstanding, we cannot brush aside a fundamental limitation, which was—surprisingly enough—expressed in 2002 by the then US Secretary of State for Defense, Donald Rumsfeld, at a Defense Department briefing, when he declared ("There are known knowns" (n.d.); "*Known and Unknown: A Memoir*" (n.d.)):

> . . . as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns—the ones we don't know we don't know.

In $\mathcal{C}_{\hat{w}}(t)$, the "known knowns" are those conditions c defined by a set of variables $\boldsymbol{v}^*_1, \boldsymbol{v}^*_2, \ldots, \boldsymbol{v}^*_n$ of which $\hat{w}$ knows the values $v_1, v_2, \ldots, v_n$, at least approximately.

The "known unknowns" are types of conditions c for which $\hat{w}$ knows $\boldsymbol{v}^*$ but not, even approximately, its current value v. Lastly the "unknown unknowns" are the possible types of conditions c defined by variables $\boldsymbol{v}^*$ that $\hat{w}$ doesn't even imagine to exist.

In truth, these notions had been used much earlier. The Canadian American engineer Elio D'Appolonia (1918–2015) wrote in 1979 (Jackson, 2019, 98):

> The risk posed by unknowns is somewhat dependent on the nature of the unknown relative to past experience. This has led me to classify unknowns into one of the following two types: (i) *known unknowns* (i.e., expected or foreseeable conditions), which can be reasonably anticipated but not quantified based on past experience as exemplified by case histories, and (ii) *unknown unknowns* (i.e., unexpected or unforeseeable conditions), which pose a potentially greater risk because they cannot be anticipated based on past experience or investigation. Known unknowns result from phenomena which are recognized, but poorly understood. On the other hand, unknown unknowns are phenomena which cannot be expected because there has been no prior experience or theoretical basis for expecting the phenomena.

Once an "unknown unknown" has been discovered, it is easy to convince oneself that the existence of the type of conditions in question is crystal-clear and that it is at worst a *known* unknown. This is a fact that we must be firmly aware of, at the risk of diminished epistemological alertness. I will first mention three examples of discovery of such sensitive conditions, in cases more or less foreign to our domain of endeavour.

The first example relates to "puerperal fever" or childbed fever, i.e., "bacterial infections of the female reproductive tract following childbirth or miscarriage" ("Postpartum infections", (n.d.)). In that case, the "unknown" variable—poor hygiene on the part of midwives, nurses, and doctors—was discovered several times since the end of the eighteenth century—by, among others, Alexander Gordon (1752–1799), in Aberdeen (Scotland), Oliver Wendell Holmes (1809–1894), in Boston (USA), or Ignaz Semmelweis (1818–1865), first in Vienna (Austria), then in Pest (Hungary). But the discovery was not taken seriously, and the discoverers were blamed and ridiculed for uttering statements regarded as offensive to doctors. Referring to one of his colleagues, Dr. D. Rutter, an opponent of Holmes and the "contagionists", Charles Delucena Meigs (1792–1869), typically wrote (Meigs, 1854):

> Still, those of you who are contagionists will say that he [Dr D. Eutter] carried the poison from house to house; and if so, then you ought to give some rationale of the fact. Did he carry it in his hands? *But a gentleman's hands are clean* [emphasis added]. Did he carry a nebula or halo about him? Then why not I also? If the nebula adhered to his clothing, it might as well have adhered to mine. (p. 104)

What was lacking—besides the lack of humility of the medical profession!—was an explanation of the alleged mechanism of contagion—the germ theory of disease had not yet been developed or was not fully accepted.

My second example concerns physics and, more particularly, the experimental work conducted by Heinrich Hertz (1857–1894) in the years 1886–1888 on what came to be called "Hertzian waves", i.e., radio waves. Let us first do a little physics with Alan Chalmers (1999):

> Hertz was able to use his apparatus to generate standing waves, which enabled him to measure their wavelength, from which he could deduce their velocity. His results indicated that the waves of longer wavelength travelled at a greater speed in air than along wires, and faster than light, whereas Maxwell's theory predicted that they would travel at the speed of light both in air and along the wires of Hertz's apparatus. The results were inadequate for reasons that Hertz already suspected. Waves reflected back onto the apparatus and the walls of the laboratory were causing unwanted interference. (pp. 33–34)

The "hidden" variable was thus... the size of the room in which Hertz's experiments were taking place. Hertz (1962, p. 14, quoted by Chalmers, 1999, p. 34) wrote:

> The reader may perhaps ask why I have not endeavored to settle the doubtful point myself by repeating the experiments. I have indeed repeated the experiments, but have only found, as might be expected, that a simple repetition under the same conditions cannot remove the doubt, but rather increases it. A definite decision can only be arrived at by experiments carried out under more favorable conditions. More favorable conditions here mean larger rooms, and such were not at my disposal. I again emphasize the statement that care in making the observations cannot make up for want of space. If the long waves cannot develop, they clearly cannot be observed. (p. 14)

In this case, the "unknown" condition is presented by Hertz as a (temporarily insuperable) constraint ("larger rooms... were not at my disposal").

My third example is far more recent. In 2014, specialists of "pain research" published a study entitled "Olfactory exposure to males, including men, causes stress and related analgesia in rodents" (Sorge et al., 2014). Their introduction speaks for itself:

> We found that exposure of mice and rats to male but not female experimenters produces pain inhibition. Male-related stimuli induced a robust physiological stress response that results in stress-induced analgesia. This effect could be replicated with T-shirts worn by men, bedding material from gonadally intact and unfamiliar male mammals, and presentation of compounds secreted from the human axilla. Experimenter sex can thus affect apparent baseline responses in behavioral testing.

It seems that this discovery—the influence of the experimenters' *sex* (rather than *gender*) on the mice's responses—was utterly unexpected. Of course, this kind of influence was known when the "subjects" are human beings. But it came as a surprise in the case of rodents.

The issue of "unknown unknowns" does not spare didactics. As far as I know, an "unknown" set of conditions, essentially revealed by Guy Brousseau, was first and foremost the set of conditions grouped under the name of *didactic contract* (Brousseau, 2002). This key notion is a pivotal example of a very generic case. Let us consider a person x in the process of being subjected to some institutional position (I, p), for example, to the student position in some school class. Experience shows that this process may be more or less difficult and challenging, which can be explained by the following mechanism: the person x is already subjected to other institutional positions $(\check{I}_k, \check{p}_k)$. Given this situation, one or more of these subjections may conflict with the subjection of x to the position (I, p), and even may impede this process. As an example, I will quote from an interview of the British sociologist Peter Woods (2012) with two female students, Yvonne and Dianne, as part of "a discussion on reports, and reference therein to 'ladylike behavior'":

Yvonne: I don't think they're . . . Well, it seems stupid to me . . . We're women . . . I don't care what anyone says.
PW: What do you think they mean by 'ladylike'?
Yvonne: Someone that goes around stinking of perfume, 'aving 'er 'air up, an' wearin' little earrings.
Dianne: Rather like Miss Sparkes. (*Deputy head*)
Yvonne: Yeah, spittin' image of Miss Sparkes.
Dianne: That's what she's trying to get us to be like you know, trying to get us to be like her. But that's one thing I could never do, because ever since I've been five I've been climbing trees, climbing on top of garages at the back 'ere – you can climb up trees and swing over on the back of the garages. I don't think I could ever adjust to the way Miss Sparkes... Oh no! (pp. 197–198)

These students refuse to look like their teacher, at least as far as how they behave in everyday life. If some form of ladylike composure is indeed a didactic stake o in Miss Sparkes's teaching, they reject the very aim of the education imposed on them, which is a crippling condition with respect to the targeted education. If, however, the object o—the way one behaves, walks or dresses—is *not* considered a goal of this education, these students fear more or less unconsciously that, in accepting to resemble their teacher as regards an object $\hat{o}$ (say English grammar, or arithmetic) of which the mastery is a "true" educational objective, they will automatically look like her on many other points, including o. It is true that, when students learn from their teachers, they "look" more like them. There is no question that, for example, if a student learns that the derivative of cosine is negative sine, this student will look a little more like the teacher, at least in "mathematical behaviour".

2.11 Within Reach or out of Reach?

An important issue must be highlighted here. It may happen that teachers seek to do something to neutralize the students' subjections which would hinder their commitments to school work. In other words, teachers may believe that the hostile conditions that determine the student's "negative" attitude towards school and school work are not *constraints* for teachers, that teachers can modify these conditions to reduce their "bad" influence. We shall see that this hypothesis is part of a broader scheme on which the management of education is globally based. However, in many cases, this hypothesis proves wrong: if some solution exists, it is often the case that it is not within the reach of the teacher position, while, at the same time, *other* institutional positions can act efficiently to solve the riddle.

Here is a typical example. In a study titled “Societal inequalities amplify gender gaps in math” and subtitled “Egalitarian countries cultivate high-performing girls”, the authors (Breda et al., 2018) write:

> According to the Programme for International Student Assessment (PISA), there are on average only seven girls for ten boys in the top decile of the math performance distribution among the 35 countries belonging to the Organisation for Economic Co-operation and Development (OECD). Underrepresentation of girls at high levels of performance is a common feature of all OECD countries [. . .] and has remained remarkably stable since 2000 [. . .]. Gender gaps of the same magnitude are also observed in science and reading, the latter in favor of girls [. . .]. (p. 1219)

As the title and subtitle suggest, the authors explain the gender gaps in terms of “general” inequalities existing in the societies which are being compared:

> It is striking that general indicators of inequalities can explain so well the patterns of gender differences in math, science, and reading performance across countries (whereas other indicators directly related to gender stratification have limited explanatory power). In more egalitarian countries, differences in initial status seem less likely to translate into differences in performance, and girls are more represented among high performers as are, for example, students from a low socioeconomic and cultural background. This suggests that the gender gap in math is a form of social inequality like many others. (p. 1220)

In terms of action, instead of persisting on gender-specific measures, this would lead to focus on measures tailored to reduce global social and economic inequalities—which is largely outside the specific sphere of action of teachers as such.

2.12 The Possibly Didactic

In order to make the link with the foregoing, let us first consider an instance $\hat{\imath}$. For simplicity’s sake, we shall leave implicit the reference instance $\hat{w}$. As a general rule, $\hat{\imath}$’s cognitive universe $\Omega(\hat{\imath})$ and cognitive equipment $\Gamma(\hat{\imath})$ change in time. If a person x is subjected to a position (I, p), in order for x to be a “good subject” of I in the position p, we must have $R(x, o) \cong R_I(p, o)$ for every object $o \in \Omega_I(p)$, where the symbol $\cong$ means that the relation $R(x, o)$ “conforms” with the relation $R_I(p, o)$—as ever, from the point of view of the reference instance $\hat{w}$ kept implicit.

It may happen that this conformity is achieved at some time t, which can be denoted by $R(x, o) \cong_t R_I(p, o)$ (or, more exactly, $\hat{w} \vdash R(x, o) \cong_t R_I(p, o)$), and turns out to be lost at a later time $t' > t$, which will be written $R(x, o) \ncong_{t'} R_I(p, o)$ (or, more exactly, $\hat{w} \vdash R(x, o) \ncong_{t'} R_I(p, o)$). In such a case, we can assume that some exogenous institutional subjection, maybe unnoticed until then, manifests itself as a “counter-subjection”—at least from the standpoint of $\hat{w}$. This is a vicissitude of the cognitive life of persons and institutions. We will now look at what can, more generally, determine the cognitive evolution of persons and institutions.

The ATD defines didactics as *the science of the didactic*, i.e., of “didactic facts”. The “traditional” word used in the ATD is *gesture*, taken in a metaphorical sense, to designate any act or action (and not only a “bodily movement”). The use of “gesture” was adapted to the original definition of "the didactic", for it conveys the idea of an intention on the part of the "gesturer", which can be here any instance $\hat{u}$: In short, I used it to define—in the wake of Guy Brousseau's definition of a "didactic situation"—a *didactic* gesture δ as an act that manifests an intention, on the part of $\hat{u}$, to help some instance $\hat{\imath}$ to "learn" some object o. Now this definition has appeared over time to be fragile, uncertain, and too narrow.

A difficulty lies in the notion of intention itself. How can one know that $\hat{u}$ had the intention to perform a gesture of a "didactic" nature, except maybe $\hat{u}$ themself? Paradoxically enough, it seems that observers are more prone to see an intention in $\hat{u}$'s action when they regard the effects of this action as "bad", and to deny it when they regard the results of this action as "good" (Knobe, 2006; and also Cova, 2016). It is not unreasonable to assume that this attitude has deep-rooted, religious origins. According to Gil Brodie (n.d.), "it is axiomatic in scholastic theology that the intention to perform a sinful act, even though not executed, is a sin in itself. (This is sometimes referred to as the 'Priority of the Intention.')."

This is not the sole obstacle to using the criterion of intention. Any gesture, of any "kind" whatever, matters to the didactician, even when the gesturer had no "didactic" intention in mind, for it can change the prevailing conditions in ways that modify the ecology of learning, as we are going to see now.

To move forward, we must first give more development to the theme of *evaluation*. We start from what we will call a *cognitive base*, i.e., the ordered pair $n^{-} = (\hat{\imath}, o)$ of an instance $\hat{\imath}$ and an object o. How can the "quality" of $R(\hat{\imath}, o)$—i.e., how "good" or "bad" it is—be assessed? We must first assume that the relation to o of some institutional instance $\hat{s} = (I, p)$, i.e., $R(\hat{s}, o) = R_I(p, o)$, is regarded (more or less explicitly) as the "right", *standard* relation to o, to which $R(\hat{\imath}, o)$ must be compared. We then assume an *evaluating instance* $\hat{v}$, capable of making judgments about the quality of relations $R(\hat{\imath}, o)$, for at least some objects o and some instances $\hat{\imath}$. Of course, we suppose that $\hat{v}$ can make rough judgments such as $\hat{v} \vdash R(\hat{\imath}, o) \cong R(\hat{s}, o)$ and $\hat{v} \vdash R(\hat{\imath}, o) \ncong R(\hat{s}, o)$. But more is needed of $\hat{v}$. We assume that $\hat{v}$ is able to estimate a "degree of conformity" $\varphi(R, R^{-})$ between $R = R(\hat{\imath}, o)$ and $R^{-} = R(\hat{s}, o)$. More precisely, given relations R and R' to o, $\hat{v}$ can issue one of the following verdicts: $\hat{v} \vdash \varphi(R, R^{-}) < \varphi(R', R^{-})$; $\hat{v} \vdash \varphi(R, R^{-}) > \varphi(R', R^{-})$; $\hat{v} \vdash \varphi(R, R^{-}) \approx \varphi(R', R^{-})$. The ordered pair $\underline{n} = (\hat{s}, \hat{v})$ is then called a *cognitive frame of reference*. The quadruple $\tilde{n} = n^{-}\frown\underline{n} = (\hat{\imath}, o, \hat{s}, \hat{v})$ is a *cognitive nucleus*. The notion of cognitive nucleus will now allow us to define the key concept of a *possibly didactic situation*.

Let us consider a reference instance $\hat{w}$ and some instance $\hat{u}$ that "makes a gesture" δ. When shall we say that the gesture δ is a *didactic gesture from the point of view of* $\hat{w}$ (or is a $\hat{w}$*-didactic gesture*)? To do so, we first consider a cognitive base $n^{-} = (\hat{\imath}, o)$ which is likely to "profit by" the gesture δ once performed, in the sense that $\hat{\imath}$ will know o "better" than before δ was made. We also consider a cognitive frame of reference $\underline{n} = (\hat{s}, \hat{v})$ likely to appraise the change in $R(\hat{\imath}, o)$ *after* the gesture will be made. And finally, we have to take into account the set $\mathcal{C}$ of conditions that prevail *before* the gesture δ is made. (Note that, of course, we can have, for example, $\hat{v} = \hat{w}$ or $\hat{v} = \hat{\imath}$.)

For notational convenience, we denote by R the relation $R(\hat{\imath}, o)$ before δ takes place, and R' the "same" relation *after* the accomplishment of δ. That being so, we say that the gesture δ is *didactic from the point of view of* $\hat{w}$ (or $\hat{w}$*-didactic*) *with respect to* $\tilde{n}$ *and* $\mathcal{C}$ if $\hat{w}$ conjectures that $\hat{v}$ will consider R' "closer" to $R^{-} = R_I(p, o)$ than was the case of R, i.e., that $\hat{v} \vdash \varphi(R, R^{-}) < \varphi(R', R^{-})$. If $\hat{w}$ conjectures that $\hat{v}$ will consider it further away from $R^{-} = R_I(p, o)$ than was the case of R, i.e., that $\hat{v} \vdash \varphi(R, R^{-}) > \varphi(R', R^{-})$, we say that the gesture δ is *antididactic from the point of view of* $\hat{w}$ (or $\hat{w}$*-antididactic*) *with respect to* $\tilde{n}$ *and* $\mathcal{C}$. The gesture δ is said to be *isodidactic from the point of view of* $\hat{w}$ (or $\hat{w}$*-isodidactic*) with respect to $\tilde{n}$ and $\mathcal{C}$ if $\hat{w}$ conjectures that $\hat{v}$ will find R and R' almost equally compliant with R^{-}, i.e., that $\hat{v} \vdash \varphi(R, R^{-}) \approx \varphi(R', R^{-})$.

The quadruple $\varsigma = (\tilde{n}, \hat{u}, \delta, \mathcal{C})$ is called a *possibly didactic situation* and is said to be a didactic, antididactic, or isodidactic situation from the point of view of $\hat{w}$, respectively, according to whether the gesture δ is didactic, antididactic, or isodidactic from the point of view of $\hat{w}$ with respect to $\tilde{n}$ and $\mathcal{C}$.

Let me note here that if we represent these notions on a number line by assigning them, respectively, the values 1 (didactic), −1 (antididactic), and 0 (isodidactic), then we can think of enriching much the gamut of judgments possibly issued by $\hat{w}$ about δ: $\hat{w}$ might assert, for example, that δ is *most probably* didactic, or *in all likelihood* antididactic, or *very surely* isodidactic. We could also distinguish between what $\hat{w}$ "thinks" or "believes", and what $\hat{w}$ may *wish* for—in which case δ might be held to be "hopefully" didactic, or "optimistically" isodidactic, etc. However, in what follows, we will limit ourselves to our initial three-valued model, without regard to the "degree of belief" that might affect $\hat{w}$'s judgment (Kyburg, 2003).

To *conjecture* that the gesture δ performed by $\hat{u}$ is didactic or isodidactic or antididactic with respect to $\tilde{n}$ and $\mathcal{C}$, $\hat{w}$ relies in particular on $\hat{w}$'s relations to $R(\hat{\imath}, o)$, $\hat{s}$, $\hat{v}$ and $\mathcal{C}$, i.e., on $R(\hat{w}, R(\hat{\imath}, o))$, $R(\hat{w}, \hat{s})$, $R(\hat{w}, v^\wedge)$, and $R(\hat{w}, \mathcal{C})$. The set of triples $(\tilde{n}, \hat{u}, \mathcal{C})$ with respect to which the gesture δ is didactic (or antididactic, or isodidactic) from the point of view of $\hat{w}$ is called the didactic (or antididactic, or isodidactic) *scope of* δ from the point of view of $\hat{w}$. It must then be underlined that the object o and the instances $\hat{u}$, $\hat{s}$, $\hat{v}$, and even $\hat{\imath}$, are often merely *imagined* by $\hat{w}$, in which case we shall denote them, respectively, by $*o$, $*\hat{u}$, $*\hat{s}$, $*\hat{v}$, or $*\hat{\imath}$. The corresponding cognitive nucleus, $*\tilde{n}$, is written as the case may be, $(\hat{\imath}, o, *\hat{s}, *\hat{v})$, $(\hat{\imath}, o, *\hat{s}, \hat{v})$, $(\hat{\imath}, *o, \hat{s}, *\hat{v})$, etc. The set $\mathcal{C}$ of prevailing conditions is itself partially unknown to $\hat{w}$ and largely imagined by this instance so that we should denote it by $*\mathcal{C}$. Cognitive nuclei are therefore frequently *underdefined*. This seemingly invasive phenomenon is at the root of a major tendency, the tendency towards the surreptitious generalization of a gesture's didactic (or antididactic, or isodidactic) *scope* from the point of view of $\hat{w}$.

The gesture δ changes the prevailing conditions $\mathcal{C}$, in particular if it modifies $R(\hat{\imath}, o)$. Let us denote by $\mathcal{C}'$ the *new* set of conditions after δ has taken place. We call $\mathcal{C}'$ a *derangement* of $\mathcal{C}$ and write $\mathcal{C}' = \mathcal{C}^{\wedge\delta}$, which can be read "$\mathcal{C}$ deranged by δ", where the symbol used (‸) is the "caret insertion point"—a symbol familiar to any copyeditor or proofreader. We thus have $\mathcal{C}' = \mathcal{C}_0 \cup \mathfrak{D}_\delta$, with $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathfrak{D}_\delta \cap \mathcal{C} = \emptyset$. This may lead to rewriting the situation ς as $\hat{\varsigma} = (\tilde{n}, \hat{u}, \mathfrak{D}, \mathcal{C})$, where the gesture δ is replaced by the set $\mathfrak{D}$ of *deranging conditions*. Let us emphasize that the instance $\hat{w}$'s judgment is formed *before* the gesture δ takes place: it is, therefore, a prediction about the judgment that the evaluating instance $\hat{v}$ will issue *after* δ has been performed. When ς and δ are (roughly speaking) reproducible, $\hat{w}$ may have integrated into their relation to ς results observed in past occurrences of the situation ς. But $\hat{w}$ will nevertheless issue an *a priori*, conjectural judgment relating to the a posteriori judgment of $\hat{v}$.

One of the major difficulties of forecasting in general (and not only in didactics) is our lack of knowledge about the set $\mathcal{C}$ of conditions and their effects on $R(\hat{\imath}, o)$—which relates to the existence of "unknown unknowns" commented above. The commendable effort to neutralize *some* of these conditions does not eliminate the fact that we are unaware of even more of them. Although the possibly didactic is always unsure, it is a vital necessity to human societies, which nonetheless tend to repress it as if it were an insuperable flaw. They, therefore, hide it in selected, isolated places, e.g., schools and classrooms. Didacticians should, however, look for *the possibly didactic* wherever it occurs in society—not only where society pretends to maintain it.

3 Humanity, Civilizations, and Societies

3.1 The Assertedly (or Allegedly) Didactic

Any gesture δ made by any instance $\hat{u}$ may be held to be didactic (or antididactic, or isodidactic) for at least some instances $\hat{w}$. One case must be highlighted. It often happens that an instance $\hat{u}$ considers making some gesture δ that, from the point of view of $\hat{u}$, is a didactic gesture, essentially justified as such in $\hat{u}$'s perspective. We then call δ an *assertedly* (or *allegedly*) *didactic* gesture—a gesture that, of course, can be seen as antididactic or isodidactic by instances $\hat{w} \neq \hat{u}$.

A possibly didactic gesture is any possible gesture and can therefore be performed by any instance in any institution. By contrast, the coming into being of an *assertedly* didactic gesture δ supposes that $\hat{u}$ is authorized to do so by some institution endowed with the (more or less disputed) legitimacy and authority to host such gestures. We shall call institutions of this kind *declaredly* didactic institutions.

One such institution is, in many societies, the Ministry of Education, about which Wikipedia notes the following ("List of education ministries", n.d.):

> An education ministry is a national or subnational government agency politically responsible for education. Various other names are commonly used to identify such agencies, such as Ministry of Education, Department of Education, Ministry of Public Education, and so forth, and the head of such an agency may be a Minister of Education or Secretary of Education. Such agencies typically address educational concerns such as the quality of schools or standardization of curriculum. The first such ministry ever is considered to be the Commission of National Education (Polish: *Komisja Edukacji Narodowej*, Lithuanian: *Edukacinė komisija*) founded in 1773 in the Polish-Lithuanian Commonwealth.

In a more general way, all declaredly didactic institutions must seek and obtain some form of *political* legitimacy that gives them momentarily some acknowledged place as "didactic" institutions in the society where they exist. Like all other aspects of society, the allegedly didactic borders on the political.

3.2 The Infrastructure/Superstructure Dialectic

The above considerations may serve to illustrate, however briefly, *the dialectic between ecology and economy*, in the sense given to these words in the ATD. In the word "ecology", coined in German by Ernst Haeckel (1834–1919) as *Ökologie*, the suffix *-logy* relates to the set $\mathcal{C}$ of prevailing conditions as far as it can be objectively known, regardless of any changes to be made to it. In the word "economy", which derives from Greek *oikonomia* "household management", the suffix -*nomy* comes from *nemein* "manage".

In order that some gesture δ becomes ecologically possible, the set $\mathcal{C}$ of prevailing conditions, i.e., the prevailing ecology, often has to be "deranged" by some preliminary gesture δ′, which by definition partakes of the economy, so that the ensuing ecology, i.e., $\mathcal{C}' = \mathcal{C}^{\measuredangle\delta'}$, makes δ possible. The ecology ($\mathcal{C}$) determines the economically possible (δ′), which in turn reshapes the ecology ($\mathcal{C}^{\measuredangle\delta'}$). This dialectic takes the concrete form, at all levels, of the interplay between two functions: that of *infrastructure*, i.e., basic "facilities", and that of *superstructure*, i.e., the gestures which the existing infrastructure makes possible.

To give an elementary mathematical example, let us suppose we wish to prove that the three medians of a triangle *ABC* intersect each other at the triangle's centroid *G*. Such is, therefore, the superstructural mathematical work that we wish to do. To achieve this, we have to use some mathematical infrastructure. Suppose we have at our disposal the mathematical infrastructure of barycentric coordinates. Let *O* be any point in the triangle's plane—in fact, *O* can be anywhere in space. For any point *M*, the vector $\overrightarrow{OM}$ is denoted by the letter $\boldsymbol{m}$ corresponding to the letter *M*. The points *I* and *J* being the midpoints of segments *BC* and *CA*, respectively, we have

$$\boldsymbol{i} = \frac{\boldsymbol{b}+\boldsymbol{c}}{2} \text{ and } \boldsymbol{j} = \frac{\boldsymbol{c}+\boldsymbol{a}}{2}, \text{ i.e., } 2\boldsymbol{i} = \boldsymbol{b}+\boldsymbol{c} \text{ and } 2\boldsymbol{j} = \boldsymbol{c}+\boldsymbol{a}.$$

By subtracting the second equality from the first, we arrive at $2\boldsymbol{i} - 2\boldsymbol{j} = \boldsymbol{b} - \boldsymbol{a}$ and therefore at $2\boldsymbol{i} + \boldsymbol{a} = 2\boldsymbol{j} + \boldsymbol{b}$. Dividing by 3 we get $\frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{a} = \frac{2}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{b}$. This equality shows that the point *G* defined by $\boldsymbol{g} = \frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{a} = \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c}) + \frac{1}{3}\boldsymbol{a} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ belongs to both line segments *AI* and *BJ* (which proves that these segments intersect each other) and also to segment *CK*, where $\boldsymbol{k} = \frac{\boldsymbol{a}+\boldsymbol{b}}{2}$, since we have: $\boldsymbol{g} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) = \frac{2}{3}\frac{\boldsymbol{a}+\boldsymbol{b}}{2} + \frac{1}{3}\boldsymbol{c} = \frac{2}{3}\boldsymbol{k} + \frac{1}{3}\boldsymbol{c}$.∎

As should be clear from this example, the work done depends heavily on the available infrastructure. Here we have used a simple but powerful mathematical infrastructure, which allows us to convert a proof into an easy algebraic calculation, repeating after Gottfried Leibniz (1646–1716): *Calculemus*—let's calculate. It is known that the creation of this infrastructure was, however, the end result of a great scientific saga, which required an incredible amount of superstructural work and imagination based on already existing mathematical and physical infrastructures.

Despite its high specificity, the above example illustrates a fundamental problem of all human societies and civilizations: the building up of praxeological infrastructures (of all possible natures: mathematical, juridical, industrial, political, medical and social, etc.), so as to make possible the praxeological superstructural activities people want to do before they die (it's humour, folks). The dialectic of infrastructures and superstructures throughout the prehistory and history of humankind is the material with which civilizations are formed, deformed and formed again. In this global perspective, the possibly didactic (and the assertedly didactic) play a key role: persons and institutions learn, unlearn, and relearn. Their evolution is a complex, didactically driven process. In what follows, we will address, however briefly, a major issue about the root of didactics: Why on earth are there didactic gestures at all?

3.3 Homo sapiens *and the Didactic*

Our species has given an essential place to the didactic—a place with a similar extension in no other animal species. Many reasons have been given for this all-pervasive phenomenon in human societies. Certainly, one of the oldest considerations in this regard relates to the "indeterminacy" of the human being, looked upon as intrinsically incomplete. Here are two quotes that testify to our age-long awareness of our native incompleteness. The first is from the *Oration on the Dignity of Man* (*De hominis dignitate*) of Giovanni Pico della Mirandola (1463–1494), sometimes called the "Manifesto of the Renaissance" (Pico della Mirandola, 1496/1956):

> At last, the Supreme Maker decreed that this creature, to whom He could give nothing wholly his own, should have a share in the particular endowment of every other creature. Taking man, therefore, this creature of indeterminate image, He set him in the middle of the world and thus spoke to him: "We have given you, Oh Adam, no visage proper to yourself, nor any endowment properly your own, in order that whatever place, whatever form, whatever gifts you may, with premeditation, select, these same you may have and possess through your own judgment and decision. The nature of all other creatures is defined and restricted within laws which We have laid down; you, by contrast, impeded by no such restrictions, may, by your own free will, to whose custody We have assigned you, trace for yourself the lineaments of your own nature. I have placed you at the very center of the world, so that from that vantage point you may with greater ease glance round about you on all that the world contains. We have made you a creature neither of heaven nor of earth, neither mortal nor immortal, in order that you may, as the free and proud shaper of your own being, fashion yourself in the form you may prefer. It will be in your power to descend to the lower, brutish forms of life; you will be able, through your own decision, to rise again to the superior orders whose life is divine." (pp. 6–8)

The main conclusion of Pico's *Oratio* is present three centuries later under the pen of Johann Heinrich Fichte (1762–1814) in his *Science of Rights* (*Grundlage des Naturrechts*):

> In short, all animals are perfect and complete; man, however, is merely suggested. [. . .]. Every animal *is* what it is; man alone is originally nothing at all. What man is to be, he must become; and as he is to be a being for himself, must become through himself. Nature completed all her works; only from man did she withdraw her hands, and precisely thereby gave him over to himself. (pp. 118–119)

From a biological perspective, the indeterminacy of human beings is linked to a key fact: the phenomenon of *neoteny*. The word "neotenia" was coined in German (*Neotenie* or *Neotänie*) around 1885 by the anatomist, zoologist and anthropologist Julius Kollman (1834–1918), who composed it from the Greek *neos* "young" and the verb *teinein* "to extend". Neoteny, i.e., the prematurity inherent in the human species, results in the continued infancy and helplessness of the young *Homo sapiens* and the ensuing vital necessity of a very long period of care and education.

In this perspective, one should also mention the more recently formulated ("Grandmother hypothesis", n.d.), which relies on the fact that "long postmenopausal lifespans" distinguish humans from all other primates. This hypothesis completes the above picture. Rachel Caspari and Sang-Hee Lee (2004) write:

> The human life history pattern differs from that observed in the great apes in its delayed maturation, slower growth, higher fertility, and increased longevity, which is associated with menopause in women [. . .]. These are evolutionary changes that have implications for the development of human culture. Longevity, in particular, may be necessary for the transgenerational accumulation and transfer of information that allows for complex kinship systems and other social networks that are uniquely human. It is also a focal point of the grandmother hypothesis, which posits that increased longevity is important in enhancing the inclusive fitness of grandmothers who, perhaps as early as the first *Homo* populations, invested in their reproductive-age daughters and their offspring.

More generally, human societies, through their institutions and their subjects, are endlessly coping with the problem of an undetermined future—what will they become? This is true both at the levels of societies and individuals. The anthropologist Marshall Sahlins (2008) notes:

> We have the equipment to live a thousand different lives, as Clifford Geertz observed, although we end up living only one. This is only possible on the condition that biological needs and drives do not specify the particular means of their realization. Biology becomes a determined determinant. (p. 107)

In this perspective, Sahlins highlights the fact that, since the start of the human adventure, "Culture" took the lead over "Nature":

> Culture is older than *Homo sapiens*, many times older, and culture was a fundamental condition of the species' biological development. Evidence of culture in the human line goes back about three million years; whereas the current human form is but a few hundred thousand years old. (p. 104)

To sum up, *Homo sapiens*, Knowing man, had to become *Homo discens*, Learning man, and *Homo docens*, Teaching man (the Latin verb *docere* means "to show, teach, cause to know", and *discere* means "to learn, acquire or attempt to acquire knowledge, study").

Before we move forward, let me posit a key tenet of the ATD. Whatever the possibly didactic gestures made, a necessary visible outcome of a possible didactic complex should be the formation, flourishing, and passing away, in some institution, of a type of structures called *didactic systems*. A didactic system $\mathcal{S}$, usually denoted by $\mathcal{S} = S(X; Y; \heartsuit)$, brings together a "class of students" X, a "team of teachers" Y, and a "didactic stake" ♥. All these words must be taken *cum grano salis*—not too literally. In fact, X may be just a singleton, with one "student" only, x, who can be a two-year-old child, a doctoral student or a seasoned researcher, for example. The set Y may be the empty set $\varnothing$ (the system is then said to be an *autodidactic* system), or $y \in Y$ may be the mother—or, as we have seen, the grandmother!—of the child x, or the supervisor of the doctoral student x. As for the didactic stake ♥, it is the entity X has to "study" and "learn", with the possible help of Y.

A didactic system $\mathcal{S}$ can also be denoted by $\mathcal{S} = \dot{S}(p_s, p_t, \heartsuit)$, where p_s is the student position, and p_t the teacher position, to highlight the centrality of these positions. The possibly didactic gestures that are made within a didactic system $\mathcal{S}$ are said to be possibly didactic *in the strict sense*—all other gestures, performed outside any didactic system, being possibly didactic *in the broader sense*.

Here again, we must take care not to suggest that the existence of a possible didactic system is a glaringly conspicuous fact: it may exist from the point of view of some instance $\hat{w}$ while another instance $\hat{w}'$ will ignore it altogether. Among the first questions that should be addressed by researchers in didactics is the question of what possible didactic systems $\mathcal{S}$ are recognized as didactic systems, by what instances $\hat{w}$. Once again, we will refer to *declared* didactic systems, notably those so labelled by the institutions that host them. Of course, other questions of fundamental relevance to the didactician are: Which kinds of $x \in X$ and $y \in Y$ can appear in $\mathcal{S}$, and about which didactic stakes ♥? A large part of the history of education could be told simply by answering just these questions. All this boils down to the study of *the ecology and economy of possible didactic systems*.

3.4 Humankind: We Versus They, and Way Beyond

In the ATD, the prevailing conditions $\mathcal{C} = \mathcal{C}(t)$ are, at every time t, distributed across the levels of the *scale of didactic codeterminacy*, which, in its standard form, can be represented as follows:

Humanity
⥯
Civilizations
⥯
Societies
⥯
Schools
⥯
Pedagogies
⥯
Didactic systems

On reading the tags from top to bottom, we observe they are all nouns in plural, except the first, *Humanity*, because there is today only one humankind.

We begin with this first level of conditions. The generic human being carries a host of conditions and constraints, which are determinants of all human endeavours and, in particular, of learning processes. We have already noted that neoteny and prolonged infancy in humans are factors favouring learning. This is only one of many key factors, of which the most prominent is the existence of *language*, in both its oral and written forms, which set *Homo sapiens* apart from all other animals.

Regarding mathematics, particularly, language in the usual sense of the word should be seen as a part of the wealth of *ostensives* that make up the noetic tools of humans. We remind the reader that the noun *ostensive* used here derives from the Latin verb *ostendere* "to show, expose to view": an ostensive is any entity that can be grasped by the senses, as for example, a sound, a glyph, a gesture (in the restricted sense of the word). The adjective *noetic* derives from the Greek verb *noein* "to think": noetic tools are tools that allow us to think.

I will abstain here from conducting an in-depth study of the main characteristics of humans relevant to didactics. By way of example, we shall consider two different aspects of *Homo sapiens*'s deep-rooted behaviour. The first concerns humans as members of collectives. Generally speaking, every human group tends to fight against, or to distinguish oneself from, every other human group: it is "We vs they". In his booklet *Race and History* (1952), the illustrious anthropologist Claude Levi-Strauss (1908–2009) has noted, not without a touch of humour, that, paradoxically, excluding others and feeling superior to other peoples is the touchstone of humanity:

> This attitude of mind, which excludes "savages" (or any people one may choose to regard as savages) from human kind, is precisely the attitude most strikingly characteristic of those same savages. We know, in fact, that the concept of humanity as covering all forms of the human species, irrespective of race or civilization, came into being very late in history and is by no means widespread. Even where it seems strongest, there is no certainty—as recent history proves—that it is safe from the dangers of misunderstanding or retrogression. So far as great sections of the human species have been concerned, however, and for tens of thousands of years, there seems to have been no hint of any such idea. Humanity is confined to the borders of the tribe, the linguistic group, or even, in some instances, to the village, so

> that many so-called primitive peoples describe themselves as "the men" (or sometimes—though hardly more discreetly—as "the good", "the excellent", "the well-achieved"), thus implying that the other tribes, groups or villages have no part in the human virtues or even in human nature, but that their members are, at best, "bad", "wicked", "ground-monkeys", or "lousy eggs". (pp. 11–12)

This enduring attitude, which shows through racism and xenophobia, sexism and gender discrimination, ageism, and all types of collective or individual egocentrism, must have had an adaptive advantage in the human adventure—but I will shun the issue here.

In this respect, however, two aspects must be brought to researchers' attention in didactics. First, while it is true that the *logos* block of the praxeological equipment of many instances may contain stereotypes, prejudices, and implicit biases, researchers ξ have to consider them, first and foremost, as *objects of study* that have to be problematized, *not* taken for granted. The key tool to master in that conjuncture is the practice of *epoché*, a Greek word (ἐποχή) usually rendered as "suspension of judgment" or sometimes "withholding of assent". Researchers need not condemn nor endorse any behavioural, i.e., praxeological, element on a moral or political basis (for example), which would be at variance with the very idea of *epoché*. Their commitment is to analyze the *conditions* that maintain the corresponding praxeological elements alive and the changes in conditions that might lead to their extinction.

The deprecation of other groups, which seems connatural to the human species, is the reverse of a coin whose obverse is also worthy of some comments. Two words will enlighten us in that regard. The first is *out-group*, defined by the Merriam-Webster Dictionary as "a group that is distinct from one's own and so usually an object of hostility or dislike". The second is *in-group*, defined as "a group with which one feels a sense of solidarity or community of interests". The in-group is, so to speak, a "fostering group" of its members, with which some of them may develop an almost exclusive relationship—which always carries the dangers of parochialism.

However, throughout their lifetime, persons are bound to become members of *several* in-groups, each of them contributing a part of those persons' empowerment. In particular, for persons to escape the "tyranny" of a given in-group, they will become members of some group new to them, which was, until then, an out-group, but which will soon become an in-group—unless the induction process fails.

In this *dialectic between in-groups and out-groups*, the support provided by an in-group is crucial. In terms of learning outcomes, in particular, an in-group can be compared to a peloton in a road bicycle race, to wit, "a densely packed group of riders who stay together *for their mutual advantage*" (*Wordnet*). It should be remembered here that learning processes are, first and foremost, *collective* processes, in which each learner benefits from *common*—rather than "personalized"—instruction and the de facto support of fellow in-group members. In fact, learning is a "tribal" process: persons learn because the institutional group to which they are subjected itself learns.

The preceding remarks apply notably to two contexts of interest to us. The first one is that of in-groups of students, whatever their age, i.e., "classes". A class [X, y]

is an in-group which should help its members to learn, especially when the subjection of a student to this "new" institutional position mitigates the possibly detrimental effects of exogenous or previous institutional subjections—think for example of Dianne and Yvonne's attitude towards Miss Sparkes's world. Ideally speaking, the class composition should beget a new "species" of students—a "counter-species"—to counteract and neutralize the supposedly undesirable conditions—i.e., $\hat{w}$-antididactic conditions, for some instances $\hat{w}$—brought into the classroom, implicitly or explicitly, by its members. In any case, a new reality has to emerge dialectically—which can take time.

Let me emphasize that the first help the teacher y as such can get comes *from the class itself*—which, it is true, can, at times, be the main hindrance to the teacher's projects. In a commonplace classroom, the students follow the teacher who fosters the dynamics of the collective study process. Conversely, the teacher uses the class's "kinetic energy" to move forward—at the risk of gliding off the track. It is here time to signal that the verb *to teach* originally meant "to show, point to" (the "teacher" was formerly the... index finger); while *to learn* referred to the idea of following a track—the learner follows the track that the teacher is pointing to.

Let us now turn to the second context announced, which is the case of researchers in didactics and, more generally, of all scientific communities. The forming of a scientific community typically follows the "laws" highlighted above, in the wake of the so-called "Invisible college" (n.d.) dear to Robert Boyle (1627–1691): there is little need to say much more about its utility. However, there's a hidden side to this auspicious view, which is the splintering of the field of didactics into an indefinite series of "subfields", usually (though not always) defined by restricting the knowledge domain they focus on.

This specialization strategy can never be taken at face value, as if it were practically imposed by the knowledge researchers have to come to grips with. Not only can it be disputed on epistemological grounds (because the "splintering" of a domain of study may surreptitiously exclude objects that are key to understanding the phenomena under study), but also it leads to the exclusive appropriation of the "remaining" objects of study by an in-group of researchers and, so to speak, to the expropriation of all the researchers who do not wish to pledge allegiance to this in-group.

Many consequences follow from this confiscation that, more often than not, transforms a research domain open to all into the stronghold of a few. The most notable side effect, it seems, is the lack of scientific debate in research communities so well protected against interventions from the outside, at the risk of surrendering to dogmatic convictions. The *dialectic of media and milieus*, to which we will return, then remains sourly underdeveloped.

3.5 Humankind: An Answer to Every Question?

The level of Humanity is the seat of the conditions and constraints carried by "modern humans" as such. Many observations have been made about the trials and tribulations of *Homo sapiens* in their quest for truth. Be they mere students or seasoned researchers, our fellow humans are prone to yield to the so-called *confirmation bias* that Francis Bacon (1561–1626) already diagnosed in *The New Organon or true directions concerning the interpretation of Nature* (1620):

> The human understanding when it has once adopted an opinion (either as being the received opinion or as being agreeable to itself) draws all things else to support and agree with it. And though there be a greater number and weight of instances to be found on the other side, yet these it either neglects and despises, or else by some distinction sets aside and rejects, in order that by this great and pernicious predetermination the authority of its former conclusions may remain inviolate. (XLVI)

Before and after Bacon's own comments, a great many observations have been made on the human propensity to favour what we already believe. In his book *The Seven Deadly Sins of Psychology* (2017), Chris Chambers writes:

> Since the mid-1950s, a convergence of studies has suggested that when people are faced with a set of observations (data) and a possible explanation (hypothesis), they favor tests of the hypothesis that seek to confirm it rather than falsify it. Formally, what this means is that people are biased toward estimating the probability of data if a particular hypothesis is true, p (data|hypothesis) rather than the opposite probability of it being false, p(data|~hypothesis). In other words, people prefer to ask questions to which the answer is "yes," ignoring the maxim of philosopher Georg Henrik von Wright that "no confirming instance of a law is a verifying instance, but... any disconfirming instance is a falsifying instance." (p. 5)

A lot of research in psychology, in fact, focuses on the conditions and constraints of which human beings are the seat. Some of these findings have epistemological and methodological implications, as we shall see now.

Some people seem to have an answer to every question, even if the question has never been raised since the beginning of time! Many decades ago, working with "split-brain" patients, whose hemispheres didn't communicate anymore because their corpus callosum had been severed for medical reasons, the psychologist and neuroscience expert Michael Gazzaniga showed what the cue to this mystery might be ("Left-brain interpreter", n.d.):

> The *left-brain interpreter* is a neuropsychological concept developed by the psychologist Michael S. Gazzaniga and the neuroscientist Joseph E. LeDoux. It refers to the construction of explanations by the left brain hemisphere in order to make sense of the world by reconciling new information with what was known before. The left-brain interpreter attempts to rationalize, reason and generalize new information it receives in order to relate the past to the present.

This conclusion was prompted by famous experiments with split-brain patients, the principle of which can be summarized as follows (Carrier & Mittelstrass, 1991):

> One of these patients is asked to perform a certain action and this request is made using optical information provided only to the left side of the patient's field of vision and, hence, to

> the right (mute) cerebral hemisphere. When the patient is then asked why he performed this action, the left hemisphere of the brain (where the linguistic capability resides) knows nothing of the request the right side obtained; the patient gives other, entirely sensible reasons. (p. 227)

In fact, "although the concept of the left-brain interpreter was initially based on experiments on patients with split-brains, it has since been shown to apply to the everyday behaviour of people at large" ("Left-brain interpreter", n.d.). The conclusion follows:

> The drive to seek explanations and provide interpretations is a general human trait, and the left-brain interpreter can be seen as the glue that attempts to hold the story together, in order to provide a sense of coherence to the mind. In reconciling the past and the present, the left-brain interpreter may confer a sense of comfort to a person, by providing a feeling of consistency and continuity in the world. This may in turn produce feelings of security that the person knows how "things will turn out" in the future.

This, it seems, is how we humans "connect the dots". Here is a more concrete example given by Gazzaniga (2011):

> We showed a split-brain patient two pictures: A chicken claw was shown to his right visual field, so the left hemisphere only saw the claw picture, and a snow scene was shown to the left visual field, so the right hemisphere saw only that. He was then asked to choose a picture from an array of pictures placed in full view in front of him, which both hemispheres could see.
>
> The left hand pointed to a shovel (which was the most appropriate answer for the snow scene) and the right hand pointed to a chicken (the most appropriate answer for the chicken claw). Then we asked why he chose those items. His left-hemisphere speech center replied, "Oh, that's simple. The chicken claw goes with the chicken," easily explaining what it knew. It had seen the chicken claw.
>
> Then, looking down at his left hand pointing to the shovel, without missing a beat, he said, "And you need a shovel to clean out the chicken shed." Immediately, the left brain, observing the left hand's response without the knowledge of why it had picked that item, put into a context that would explain it. It interpreted the response in a context consistent with what it knew, and all it knew was: Chicken claw. It knew nothing about the snow scene, but it had to explain the shovel in his left hand. Well, chickens do make a mess, and you have to clean it up. Ah, that's it! Makes sense.
>
> What was interesting was that the left hemisphere did not say, "I don't know," which truly was the correct answer. It made up a post hoc answer that fit the situation. It confabulated, taking cues from what it knew and putting them together in an answer that made sense. We called this left-hemisphere process *the interpreter*. ("Why do we feel unified")

That said, we are led to conclude that *Homo sapiens* seems to be primarily engrossed in "making sense" of the world, more than in the quest for *truth*. It will be the role of the *dialectic of media and milieus* to go beyond this outcome of evolution.

3.6 Societies and Civilizations

The sense of the word *society* involved in the ATD is, up to now, not discordant with the common meaning of the term. Broadly speaking, we can refer to this dictionary definition of "society" (*Macmillan Dictionary*): "people in general living together in organized communities, with laws and traditions controlling the way that they behave toward one another." A particular case of interest to us is when a society is governed by a *state*—is a "statist society", as are today the French, the Spanish, or the German societies, for example.

By contrast, any human community is not necessarily a (fully-fledged) society: to be considered so, it has to provide infrastructures addressing the basic human needs regarding shelter, food, safety, family life (making and raising children), education, health, communication and information, transportation, ageing, dying, etc. But there is more to it than that: one crucial observation is that the way a society ministers to the needs of the persons and institutions it comprises depends heavily on time and space. Healthcare in Greece in Plato's day, for example, was not the same as healthcare in Barcelona nowadays. And the same holds for education.

The reverse exists too. Some aspects of the Spanish society resemble much their French counterparts. The same certainly applies to other societies, notwithstanding the propensity of *Homo sapiens* to believe in their unrivalled "singularity". This is where the notion of *civilization* comes up. Let us first remember the upper levels of the scale of didactic codeterminacy:

Humanity
⇃↾
Civilisations
⇃↾
Societies
⇃↾
…

A warning must be issued before we go any further. Here, "civilisation" is defined on a strictly *local* and *comparative* basis. Given a reference instance $\hat{w}$, the societies $\mathcal{S}_1$ and $\mathcal{S}_2$ and institutions I_1 in $\mathcal{S}_1$ and I_2 in $\mathcal{S}_2$, we say that, from the point of view of $\hat{w}$, the institutional positions (I_1, p_1) and (I_2, p_2) belong to the *same* civilisation with respect to a set $\mathcal{O}$ of objects, if, for all $o \in \mathcal{O}$, we have: $\hat{w} \vdash R_{I_2}(p_2, o) \approx R_{I_1}(p_1, o)$.

A case of much interest is when the two societies are the "same" society $\mathcal{S}(t)$ at different points in time (for example, the French society now and a century ago). If $t_2 > t_1$, for some position p in I that exists at both times, it may be the case that $\hat{w} \vdash R_{I_2}(p_2, o) \not\approx R_{I_1}(p_1, o)$, with obvious notations. In such a case, we shall say that, from the point of view of $\hat{w}$, a *civilisational change* has taken place in p with respect to o between times t_1 and t_2.

The impact of such changes cannot be overestimated. They often generate passions for and against, with horrified haters, who cannot abjure the "mores" of yesterday and accept those of today, and unqualified supporters of the change, who dogmatically hate the haters. Any commitment to a reform regarded by many as a civilisational change is sure to come across these opposing passions. Of course, here, we can think of the historical transition to the paradigm of questioning the world!

4 Questioning and Studying: The Human Art of Learning

4.1 The Notion of School

The word *school* appears at the fourth level (counting from the top) in the scale of didactic codeterminacy:

Humanity
⇃↾
Civilisations
⇃↾
Societies
⇃↾
Schools
⇃↾
…

The notion of school used in the ATD is, once again, a very extensive notion: a school is *any institution that can host—more or less legitimately—didactic systems of a certain type at least*. Although schools in this sense exist in every society, the formulation of its founding principle becomes clearly explicit in ancient Greece. The *Online Etymology Dictionary* reminds us that the word *school* comes from Latin *schola,* which means "intermission of work, leisure for learning; learned conversation, debate, lecture; meeting place for teachers and students, place of instruction; disciples of a teacher, body of followers, sect." Latin *schola* comes in turn from Greek *skhole*, "spare time, leisure, rest, ease; idleness; that in which leisure is employed; learned discussion" and also "a place for lectures, school". In fact, the original meaning of Greek *skhole* was "leisure", which, circa 300 BCE, passed to mean "otiose discussion", "in Athens or Rome the favorite or proper use for free time", and then a place for such discussion. The time for "studious discussion", for school in the new sense, was therefore a special time when *all other activities stopped* to make room for *study* time. The Roman grammarian Festus, who flourished in the second century (CE), fittingly wrote that schools had been named after a Greek word (σχολή, *skholḗ*) meaning originally "leisure" but which came to refer to a time when, “all other activities left aside” (*ceteris rebus omissis*), “children devote themselves to liberal studies” (*vacare liberalibus studiis pueri debent*).

It cannot be overemphasized that “school” means—by definition—a time apart (which usually requires a particular place: the school) from the busy and sometimes chaotic worldly life outside school. The paradox of school is that, while schools keep “mundane” affairs outside, they are, or should be, the very place where *questions* about worldly matters of *any* kind can be raised and addressed with equanimity. As is well known, Greek *skhole* and Latin *schola* opened the way to many other European languages, a fact that the author of the *Dictionary of Word Origins* (Ayto, 1990) sums up as follows:

> [School] was borrowed [. . .] from medieval Latin *scōla*, and has since evolved into German *schule*, Dutch *school*, Swedish *skola*, and Danish *skole*, as well as English *school.* The medieval Latin word itself goes back via classical Latin *schola* to Greek *skholḗ*. This originally denoted ‘leisure,’ and only gradually developed through ‘leisure used for intellectual argument or education’ and ‘lecture’ to ‘school’ (in the sense ‘educational assembly’) and finally ‘school’ the building. The Latin word has spread throughout Europe, not just in the Romance languages (French *école,* Italian *scuola,* Spanish *escuela*), but also into Welsh *ysgol*, Irish *scoil*, Latvian *skuola*, Russian *shkola*, Polish *szkola*, etc.

It is a postulate of the ATD that, behind any didactic system, there exists a “school” that accommodates this didactic system, in which it can legitimately thrive, whatever its lifespan. In any analysis conducted according to the ATD, a didactic system being given, one has first to identify the school that enshrines it. The next step will consist of the analysis of the school’s *pedagogy* relative to the didactic system in question. As we will see, the existence and functioning of didactic systems crucially depend on their pedagogies.

4.2 The Level of Pedagogies

Let us look again at the lowest levels of the scale of didactic codeterminacy:

…
⇂↾
Schools
⇂↾
Pedagogies
⇂↾
Didactic systems

The words *pedagogy* and *pedagogue* have acquired in the ATD meanings faithful to their origins but which, however, have to be carefully delineated. The word *pedagogue* derives (through Latin and French) from the Greek *paidagōgós* (παιδαγωγός), from *paîs* (παῖς), “child” and *agōgós* (ἀγωγός), “guide”—from *ágō* (ἄγω), "lead". A pedagogue was at first a "slave who escorts boys to school and generally supervises them", and later "a teacher". The *Online Etymology Dictionary*, from which we borrow this information, adds that "hostile implications in the word are from at least the time of Pepys (1650s)"—for example, the *MacMillan Dictionary* defines a pedagogue as "a teacher who uses strict old-fashioned method". Of course, the uses we will do of the word in the ATD will be rid of these pejorative nuances. Nonetheless, as didacticians, we would have to explain—among other things—the emergence and spread of this disparaging meaning.

What is pedagogy? To answer this question, we have to introduce the notion of a *work*. According to the *Online Etymology Dictionary*, the expression "work of art" is attested by 1774, but earlier (1728) this expression was simply taken to mean an "artifice, production of humans (as opposed to nature)." It is in this sense that we will use the word *work* on its own, as well as in expressions like "work of mathematics", "work of linguistics", "work of chemistry", or… "work of art". Note also that any human notion for study, that of a rational number, mountain, sea, lake, or pond, are "productions of humans" and therefore works. Consequently, any didactic system $\mathcal{S}$ can be denoted by $\mathcal{S} = S(X; Y; W)$, where W is the work to study and "learn".

To study W, students $x \in X$ have to come "in contact" with W, be it the resolution of quadratic equations or horse riding. Pedagogy is, if one may say so, the "art" of putting the students $x \in X$ in contact with the didactic stake—the work for study, W. The first contact can be of various natures: it can be achieved through mere storytelling or, to the contrary, through direct contact—by boldly attempting, for the first time in one's life, to solve a quadratic equation or to ride a horse. It must be emphasized that the notion of pedagogy as used in the ATD is *not* independent of the contents studied: leading students towards a given work W (or a set of works $\mathcal{W}$) obviously depends on the nature of W (or $\mathcal{W}$).

A way of putting students in contact with W is through a "field trip", which the lexical database *WordNet* defines as "a group excursion (to a museum or the woods or some historic place) for firsthand examination". But there are other possibly didactic structures that should be highlighted as well. A first pedagogic gesture consists of grouping students in *classes* according to some criteria: a class $[X, Y]$ is a structure in which didactic systems $S(X; Y; W)$ form. A second major pedagogic gesture consists of giving a structure to the set of works W to study—a gesture that results in a "course of study" or "curriculum". At this point, a pedagogical bifurcation occurs according to the *study paradigm* being implemented. In the still current paradigm of *visiting works*, founded on *disciplines* as a jumping-off point, a study programme is most often structured into five successive levels, as shown below:

Disciplines
↓↑
Domains
↓↑
Sectors
↓↑
Themes
↓↑
Subjects

If, for example, we consider the mathematical discipline, we may have *domain* = "Geometry", *sector* = "Plane transformations", *theme* = "Isometries", *subject* = "Isometries of a rectangle". Beware! Such a structuration of the works to study *is always an artificial construction* that changes in time and across institutions: one cannot expect to meet a "context-free" structuring—a fact which has to do with the didactic transposition process. We will now slowly move towards the other branch of the bifurcation, i.e., the *paradigm of questioning the world*.

4.3 The Pedagogies of Inquiry

The difference between the paradigm of visiting works and the paradigm *of questioning the world* cannot be reduced to a single aspect. Formally speaking, the expression $\mathcal{S} = S(X;\ Y;\ W)$ should be replaced, in the case of the paradigm of questioning the world, by the formula $\mathcal{S} = S(X;\ Y;\ Q)$ where Q is a question. Note that a question is indisputably a work, i.e., a human creation. Note also that, according to the *Online Etymology Dictionary*, Latin *quaestionem* meant "a seeking, a questioning, inquiry, examining" and *quaerere* "ask, seek".

Without further delay, I introduce here the *Herbartian schema* in its semi-developed form:

$$[S(X;Y;Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$$

where M is the "didactic milieu", i.e., the set of resources potentially used by the class $[X, Y]$ in constructing the answer $A^{\heartsuit}$ defined to be the class's answer to Q. In this case, we shall say that the class $[X, Y]$ *studies Q* or *inquires* into it. While, just like *question*, *inquire* has to do with Latin *quaerere* "ask, seek", the verb *to study* has something special about it. According to the *Dictionary of Word origins* (Ayto, 1990), the underlying notion is that of "application of extreme effort", which is neither in *learn* nor in *teach*, and was already in Latin *studium* "eagerness, intense application", hence "application to learning".

The notion of inquiry used in the ATD *is a general modelling concept*. It applies to whatever didactic system $\mathcal{S} = S(X; Y; ♥)$ in which the didactic stake ♥ is a question Q. In fact, it applies indirectly to *all* didactic systems. If ♥ is a work W that is *not* a question, the study of W takes the form of the study of questions $Q_1, Q_2, \ldots, Q_n$ related to W: What is W? What is its praxeological structure? How does W "function"? What are its raisons d'être? Where does it come from? Who made it? Etc.

Let us go back to the Herbartian schema. Note first that Johann Friedrich Herbart (1776–1841) was a German philosopher, often regarded as the founder of pedagogy as an academic discipline. Although the Herbartian schema is not formally due to Herbart, it retains something of Herbart's pedagogical views. Moreover, according to a law of the history of mathematics formulated by historian Carl B. Boyer (1906–1976), "mathematical formulas and theorems are usually not named after their original discoverers". This extends easily to other fields of endeavour.

We suppose that every member of $X \cup Y$, i.e., of the class $[X, Y]$, strives to come up with their own answer: if $x \in X$ (respectively, $y \in Y$), we denote this answer by $A_x^♥$ (resp., $A_y^♥$). It may be that the students work in groups X', X'', X''', …, such that $X = X' \cup X'' \cup X''' \ldots$ If $x \in X'$, then $A_x^♥ = A_{X'}^♥$, etc. (The same applies to Y.) In the most general case, $A^♥$, i.e., $A_{[X,Y]}^♥$, results *in some way* from the different answers $A_x^♥$ ($x \in X$) and $A_y^♥$ ($y \in Y$).

When $Y = \{y\}$, it may happen that the teacher imposes his own answer $A_y^♥$ on his class so that $A^♥ = A_y^♥$. To describe such a case and the other potential cases, we use the notion of *topos*—the Greek word *topos* means "place, region, space". The *topos* of an institutional position $\hat{\imath}$ is the set of types of tasks that a person occupying the position $\hat{\imath}$ is institutionally allowed to perform on their own, without interaction with any other person. In the case in point, the authorizing institution is the school or the class itself. The teacher position's *topos* is maximal, and the student position's *topos* is minimal—students are only required (and allowed) to study and learn the teacher's answer $A_y^♥$, i.e., we have $A^♥ = A_y^♥$. We shall speak of y as a "maximal teacher" and the student x as a "minimal student"—of course, what is maximal (respectively, minimal) is *not* the teacher (resp., the student), but the teacher *position* (resp. the student *position*). The latter case, which can be regarded as a limiting case, suggests that any study format can be analyzed in terms of inquiry. At this point, the notion of paradigm will enable us to put some order in the profuse possibilities offered by human didactic creativity.

4.4 Study Paradigms

The notion of *study paradigm* gives an answer to the generic question: “What do they do at school?” According to the *Online Etymology Dictionary*, the word *paradigm* derives from Greek *paradeigma* “pattern, model; precedent, example”, from *paradeiknynai* “exhibit, represent,” literally “show side by side” (from *para-* “beside” and *deiknynai* “to show”). In what follows, I will present four main paradigms designated, respectively, by the symbols ϖ_1, ϖ_2, ϖ_3, and ϖ_4. Although these models seem to follow one another historically, they should be considered as contemporary: they exist in synchrony, even though it is suggestive to study them diachronically.

The first paradigm is the paradigm *of reading the Book*. In a given civilization, the Book (seen as unique) is a text B that both raises *questions* Q and gives *answers* A. Saying that a class $[X, y]$ studies B means that, under the direction of y, X studies the questions Q raised and the answers A given in B. The students do not choose the questions for study and, of course, do not officially take part in the construction of the answers A. In this study format, the teacher y will be called the *reader* (the word derives from Old English *rædere* “person who reads aloud to others; lector; scholar; diviner, interpreter”).

The supposed author of B may be an alleged divinity, a legendary figure, or a person who once existed but whose book has been at least partially anonymized by the passing of time, changing its author into a legend (the word *legend*, from Latin *legere* “to read”, originally meant “things to be read”). The paradigm of reading the Book has a long past. In some civilizations, there was a Book supposed to cover all the needs of an accomplished life, a book I will call a *total encyclopedia*. For a long time, the Bible was the total encyclopedia par excellence in the Christian world, as the Koran is in the Muslim world. I will take here a non-religious example that may be unexpected for some readers, that of Homer, the legendary author of the *Iliad* and the *Odyssey*. (You certainly know the logician’s joke that these poems “were not written by Homer, but by another Greek man of the same name”.) According to some specialists, Homer’s works were *the* Book, i.e., the total encyclopedia of the Hellenic world, in which generations of young Hellenes studied “their” world. I will draw here on Eric Alfred Havelock (1903–1988)’s book *Preface to Plato* (1963), where the author writes:

> To approach Homer in the first instance as a didactic author is asking a good deal from any reader and is not likely to win his early sympathy. The very overtones of the word ‘epic’, implying as they do the grandiose sweep of large conceptions, vivid action, and lively portraiture, seem to preclude such an estimate of Europe’s first poet. Surely for Homer the tale is the thing. Didactic or encyclopedic elements that may be there—one thinks for example of the famous Catalogue of the Ships—are incidental to the epic purpose and likely to weigh as a drag on the narrative. However, we are going to explore the argument that the precise opposite may be the case; that the warp and woof of Homer is didactic, and that the tale is made subservient to the task of accommodating the weight of educational materials which lie within it. (p. 61)

In a footnote, the author adds a comment on the adjective "didactic" (p. 84, note 2): "This adjective may mislead, if it suggests an emphatically conscious purpose on the part of the oral poet, yet it is difficult to choose a better. He is didactic by necessity, but also in large part unconsciously."

The paradigm ϖ_1 of reading the Book has descendants to this day, which we will continue to call the paradigm of reading the Book. But here, the total encyclopedia is replaced by a library of *partial* encyclopedias β_1, β_2, ... β_l, where β_i is *the* book on some part of the lived world. Such was, for example, Edward Cocker's *Arithmetick*, if we are to believe the article about it in Wikipedia:

> *Cocker's Arithmetick*, also known by its full title "Cocker's Arithmetick: Being a Plain and Familiar Method Suitable to the Meanest Capacity for the Full Understanding of That Incomparable Art, As It Is Now Taught by the Ablest School-Masters in City and Country", is a grammar school mathematics textbook written by Edward Cocker (1631–1676) and published posthumously by John Hawkins in 1677. *Arithmetick* along with companion volume, *Decimal Arithmetick* published in 1684, were used to teach mathematics in schools in the United Kingdom for more than 150 years.

In a generation before mine, mathematics students perused John L. Kelly's *General Topology* (1955) or William Feller's *An Introduction to Probability Theory and Its Applications* (1957). In my generation, we studied Serge Lang's *Algebra* (1965) and Walter Rudin's *Real and Complex Analysis* (1966).

In a sense now archaic, an "author" is an authority, a source of authoritative information to be found in the author's book. In the study format long associated with the paradigm of reading the Book, a book about some specific topic—*the* Book—is chosen, and the students "study" it under the direction of the "reader", *who is not always a specialist* of the subject matter treated in β, but who rules and directs the study of β. In France, these "underteachers" were long called *régents*, from Latin *regere* "to rule, direct", a verb which, according to the *Online Etymology Dictionary*, derives "from PIE root **reg-* 'move in a straight line,' with derivatives meaning 'to direct in a straight line,' thus 'to lead, rule'."

The famous French writer Stendhal (1783–1842), in his posthumously published *The Life of Henry Brulard,* described his mathematics class at the "École centrale" of Grenoble, i.e., Grenoble high school, with the teacher, Dupuy, sitting in an armchair and questioning the pupil at the blackboard on some lesson of *the* book—which, in that case, was Étienne Bézout's *Cours de mathématiques* (1798) (Fig. 1).

In France, this study format culminated in the practice of *colles*, i.e., weekly oral examinations, which were central to the French Higher School Preparatory Classes (*Classes préparatoires aux grandes écoles*), and is at the origin of the "flipped classroom" (see, e.g., Rickey & Shell-Gellasch, 2010).

The most crucial point here is that the paradigm of reading the Book is not incompatible with any other paradigm, and, in particular, with the paradigm of questioning the world. We may place ourselves in this paradigm when, during an inquiry, we want to know if a certain work W can be useful to our inquiry. The difference, as always, is that we ask what W, and therefore the book β on W, can bring to the inquiry.

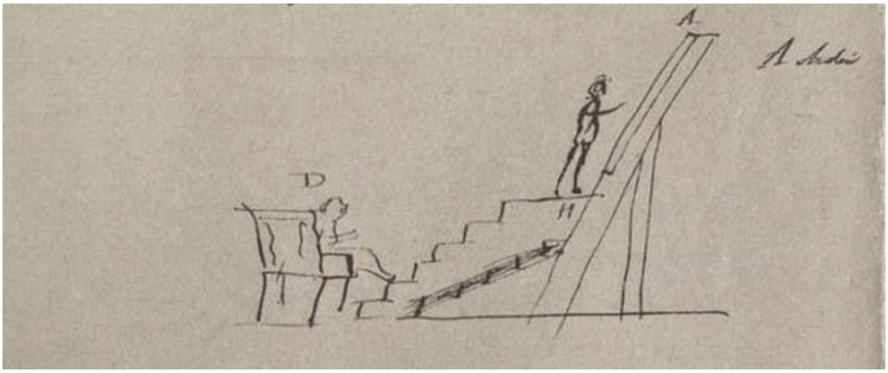

Fig. 1 The young Beyle at the blackboard (https://www.gutenberg.org/files/53779/53779-h/53779-h.htm)

The paradigm ϖ_2 is the *paradigm of celebrating great authors*. In it, the focus is less on a work than on the work's *author*. We do not study political science but Aristotle's *Politics*; we do not study arithmetic or geometry but Euclid's *Elements*, etc. Of course, we do so because these supposed masterpieces are (or were, at some point in history) *the* Books on political science and on arithmetic and geometry, respectively. While the implementations of the paradigm of reading the Book is often low-level, the paradigm of celebrating great authors is often high-level. One consequence of this fact is that the study of a great author's works not only seeks to know the *questions* raised and the *answers* given by this author, but also the *didactic milieus* drawn upon and the *dialectic of media and milieus* used.

This paradigm ϖ_2 survives and even flourishes to this day, generally in areas that are now part of history: history of literature, history of philosophy, and even history of science. This paradigm is the bearer of a learned tradition from which we can learn a lot. The paradigm ϖ_2 is, of course, compatible with the paradigm of questioning the world—in many inquiries, we are led to study some author's work whose understanding seems to require an analysis of the conditions and constraints under which the author worked.

The paradigm ϖ_2 was part of an intellectual universe that paved the way for the third study paradigm, ϖ_3, the paradigm *of visiting works*. Indeed, it is in medieval universities that the standard curriculum made up of the "seven liberal arts" was developed. Its lower division was called the "trivium", i.e., the place where three roads (*via*) meet, to wit, *grammar*, *logic*, and *rhetoric*. According to John Ayto's *Dictionary of Word Origins* (1990), "the notion of 'less important subjects' led in the 16th century to the use of the derived adjective *trivial* for 'commonplace, of little importance'".

The upper division, the *quadrivium*, was made up of four "arts": *arithmetic*, *geometry*, *music*, and *astronomy*. Although flourishing during the Middle Ages, the whole "liberal" curriculum had been established in ancient Greece (the names *trivium* and *quadrivium* came later): it appears as a transition between the paradigm ϖ_2 of celebrating great authors and the paradigm ϖ_3 of visiting works.

4.5 The Disciplines and the Professor

The paradigm ϖ_3 is well known to all of us, and I will therefore not dwell much on it. This paradigm is the paradise of academic disciplines, which Wikipedia characterizes in the following way ("Discipline (academia)", n.d.):

> An *academic discipline* or *academic field*, also known as a *field of study*, *field of inquiry*, *research field* and *branch of knowledge*, is a subdivision of knowledge that is taught and researched at the college or university level. Disciplines are defined (in part), and recognized by the academic journals in which research is published, and the learned societies and academic departments or faculties to which their practitioners belong. It includes scientific disciplines.

How does the paradigm ϖ_3 differ from ϖ_1 and ϖ_2? In the paradigm of reading the Book, the "teacher" y is a mere *reader*. A reader does not claim to be a master of the discipline(s) to which the Book belongs: the one who claims to be an expert is the more or less distant and more or less anonymous author of the Book studied.

By contrast, in the paradigm ϖ_3 of visiting works, the teacher is a *professor*, i.e., someone who *professes* the discipline taught. According to Ayto's *Dictionary of Word Origins* (1990), a *professor* "is etymologically someone who 'makes a public claim' to knowledge in a particular field", and, incidentally, someone's *profession* "is the area of activity in which they 'profess' a skill or competence". Let us note in passing that something of the distinction made here between *reader* and *professor* survives in British universities, where, according to the *Macmillan Dictionary*, "someone begins as a lecturer, then becomes a senior lecturer, then sometimes a reader, and finally a professor".

The somewhat surreptitious change from reader to professor brings about an irreversible change in the paradigm of visiting works: the *monumentalisation* of the works studied, with the correlative deletion of their raisons d'être. That is where we are today. As a teacher educator, I used to ask student teachers endowed with the best training in mathematics, not *What is a straight line?* but *What are the raisons d'être of straight lines?* And the same with the notions of *ray*, *angle*, or *parallelogram*. Why the devil, in geometry, are there rays, angles, and parallelograms rather than nothing? Not, therefore, *What is this thing?* But *What is this thing* for*?* The fact that they could not seriously answer this type of questions was a symptom of the increasing monumentalisation of mathematics teaching. There were questions to which the "text of knowledge" they would have to teach brought an answer. Still, in this text, the interplay between questions and answers remained generally implicit, and the teachers themselves were unaware of it. This is where the research work on the paradigm of questioning the world started.

4.6 Steps Towards Questioning the World

The fundamental impetus was given by Guy Brousseau's *theory of didactic situations* (2002), which brought about a decisive epistemological break. Given some (mathematical) work *W*, the problem is to find a question *Q* whose study, under certain conditions $\mathcal{C}$ (of which only a few are explicitly considered), will lead the students to encounter *W* and establish thereby a more epistemologically genuine relation to *W*. In this way, the use made of *W* in the study of *Q* reveals at least one of *W*'s raisons d'être in the institution where the study process takes place—a raison d'être is always relative to some institutional position, at some point in the history of the institution.

During a first period of time, the incipient ATD followed close in the wake of the TDS, with some adjustments to take into account: (1) the anthropological concern that already prevailed in the theory of didactic transposition, and (2) the adaptation to the ecology of (French) *secondary* mathematics teaching. In this perspective, at least some of the TDS's tenets gave birth to the notion of a *study and research activity* (SRA). In an SRA, for example, one starts with a certain task of a culturally well-known type, known as the tick task (✓), whose completion bumped into a problematic task *t* of a certain type *T*, in such a way that, under the supposedly prevailing conditions $\mathcal{C}$, the building-up of a *technique* for *T* and of a surrounding *logos* block would have to draw critically on the work to study *W*.

It is under these conditions that the model of *didactic moments* (Chevallard, 2020) was developed as a counterpart to Guy Brousseau's "dialectics" (of action, formulation, etc.): but I shall not go into it here. The design and implementation of SRAs proved expensive in terms of mathematical and didactic imagination and work: for every notion to be "taught", the (student) teacher had to devise a task ✓ with the appropriate properties and, above all, the pupils had to discover each time a *new* type of tasks *T*, which, in addition, was generally thought of as a pure means, and not as a valued end of the teaching process.

This led me to introduce the notion of a study and research *path* (SRP), in which the study of a large type of tasks *T* with a great number of subtypes of tasks T_1, T_2, … T_n, leads to encounters with works W_1, W_2, … W_n pertaining to the official mathematics curriculum. In the curriculum that had long prevailed since the nineteenth century, one of these supertypes of tasks *T* consisted in "calculating the distance between two *inaccessible* points". Here is a basic example (Beck, 2011; see Fig. 1):

> Suppose that you can measure the distance between A and B and the angles from A and B to two inaccessible points C and D. For example, A and B might be on a straight road in a valley and C and D might be two visible mountain peaks. Then you can calculate the distance between C and D using the law of sines and the law of cosines.

Let us note in passing that this problematic task exemplifies a fundamental (though elementary) raison d'être of angles.

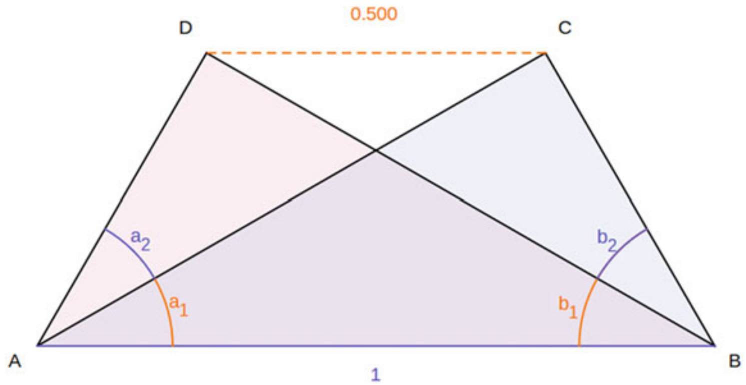

Fig. 2 Distance between "inaccessible" points

Another classic example of an SRP "with large reach" is associated with the supertype of tasks T' consisting in calculating a numeric value using *graphical methods*. Here is an example borrowed from Carl Runge (1856–1927)'s book *Graphical Methods* (1912; see Fig. 2a, b):

> In order to multiply a given quantity *c* by a given number, let the number be given as the ratio of the lengths of two straight lines *a*/*b*. If the quantity *c* is also represented by a straight line, all we have to do is to find a straight line *x* whose length is to the length of *c* as *a* to *b*. This can be done in many ways by constructing any triangle with two sides equal to *a* and *b* and drawing a similar triangle with the side that corresponds to *b* made equal to *c*. As a rule it is convenient to draw *a* and *b* at right angles and the similar triangle either with its hypotenuse parallel [Fig. 3a] or at right angles [Fig. 3b] to the hypotenuse of the first triangle. Division by a given number is effected by the same construction; for the multiplication by the ratio *a*/*b* is equivalent to the divisions by the ratio *b*/*a*. (p. 4)

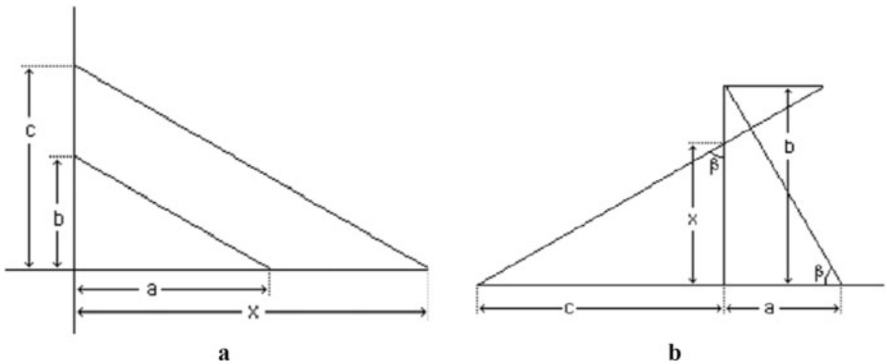

Fig. 3 (a) and (b) Graphical multiplications

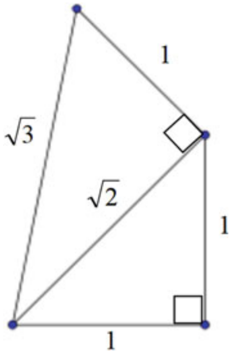

Fig. 4 Calculating square roots

The graphical calculations above use crucial elementary knowledge in geometry. It can be shown that the supertype T' has quite a wide reach in terms of mathematical works. In the language of the *ecology of knowledge*, i.e., of the *ecology of praxeologies*, we say that, given a ("standard") instance $\hat{s} = (I, p)$, a praxeological entity P "eats" the praxeology P* according to $\hat{s}$ if, according to $\hat{s}$, the construction of P draws on P*, be it technically, technologically, or theoretically. For example, using Pythagoras' theorem P_Σ (= P*), it is easy to calculate graphically $\sqrt{5}$: it all boils down to measuring the hypotenuse of a right triangle of which the other two sides have lengths 1 and 2. In the case of, say, $\sqrt{7}$, the same technical gesture cannot be used directly, but we can rely on the recursive technique suggested by Fig. 4.

We can conclude that, according to some $\hat{s}$, T' "eats" Pythagoras' theorem P_Σ, a fact we can denote by: $\hat{s} \vdash T' \,\square\, P_\Sigma$ or $\hat{s} \vdash P_\Sigma \hookrightarrow T'$. The transitive relation $\rho_{\hat{s}}$ that holds between a type of tasks T and a praxeology P when, according to $\hat{s}$, T eats (either directly or indirectly) the praxeological entity P is called the *trophic* relation—the adjective "trophic" derives (1845) from Greek τροφή (*trophḗ*), "food". Given T, we define the *trophic span* of T according to $\hat{s}$ by $\rho_{\hat{s}}(T) = \{P \,/\, \hat{s} \vdash T \,\square\, P\}$ and the trophic span by $\rho_{\hat{S}}(T) = \bigcup\{\rho_{\hat{s}}(T) \,/\, \hat{s} \in \hat{S}\}$ for any set $\hat{S}$ of instances.

How can square roots be calculated graphically? A technique used the notion of *power of a point*, based on the following theorem (see Fig. 5).

For any chord PN, we have $\overline{PM} \times \overline{PN} = \overline{PA} \times \overline{PB} = \overline{PT}^2$. If PM = 1 and PN = 7, then PT = $\sqrt{7}$.

This technique was formerly used in some secondary mathematics curriculums, so that it used to be within the span of T' according to some quite decent instances. This is an example of a crucial phenomenon, which will allow us to make the transition from "finalized" SRPs, i.e., SRPs with an assigned finality (in terms of "bumping" into a prescribed set of works), to "unfinalized" SRPs. In the first case, an SRP is supposed to lead the students to encounter works W_1, W_2, etc., designated in advance. In the second case, the SRP only aims to answer a question without seeking to arrive at any predesignated work. In the first case, the question Q to which the didactic system $\mathcal{S} = S(X; Y; Q)$ seeks to answer appears as a means to an end, as a stratagem, a manoeuvre, a trick to come across some predetermined work. In the second case, question Q appears self-justifying, with a value in itself, without considering what it might be conducive to (Fig. 5).

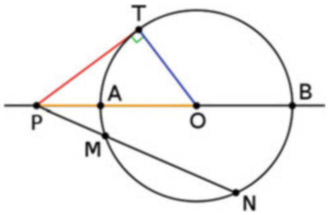

Fig. 5 Using the power of a point

4.7 *Questioning the World? Really?*

As far as I am concerned, the real breaking point in the research on the paradigm of questioning the world was the introduction, during the 2000–2001 school year, in grade 11 of French high schools, of a new study format known by the French acronym of TPE (*Travail personnel encadré*): the *supervised personal work*. Eleventh graders, generally working in teams of three, had to study on their own (although a teacher of their class tutored them) a question Q of their choosing and to submit a written report narrating the whole study process and giving the answer A they had arrived at. The question Q was freely chosen, provided it appeared very likely that its study would draw on resources from at least two of the subjects taught to the students—mathematics and chemistry, biology and physics, etc.

Despite such a limitation, this represented a remarkable break with the dominant paradigm of visiting works. In fact, it gave rise to the idea of an *unfinalised* SRP, in which the question tackled was not a hidden means to compel the students to study some *prescribed* work. The first question that I decided to study as if I were an 11th grader, and which I borrowed, through his teacher, from a true 11th grader, was: "Why has plague disappeared from European countries?" During an in-service training course, a team of science teachers (some of them taught physics and chemistry, others biology) chose the following question: "Why do roses smell good?" Like their own students, these teachers had to ask me (I was their "supervisor" for a while) to validate their proposal. To do this, I asked them if they could assure me that they had no a priori idea of the answer. Their answer was positive, and I agreed with their choice.

Let us remember here the four study paradigms mentioned above, to wit, the paradigm *of reading the Book* (ϖ_1), the paradigm *of celebrating great authors* (ϖ_2), the paradigm *of visiting works* (ϖ_3), and the paradigm *of questioning the world* (ϖ_4). Now the (relatively) free choice of the question Q in the supervised personal works established an essential break with the previous paradigms ϖ_1, ϖ_2, and ϖ_3. It is indeed fundamental to note that, in these paradigms, both the choice of *questions* and the construction of *answers* are the prerogative of tradition, i.e., of more or less legendary, heroized, or deified authors (ϖ_1), of supposedly superlative authors (ϖ_2), or fully certified teachers (ϖ_3).

Let us also remember the Herbartian schema in its semi-developed form: $[\mathcal{S} \curvearrowright M] \hookrightarrow A^{\heartsuit}$, where $\mathcal{S} = S(X; Y; Q)$. The Herbartian schema applies to all possible study paradigms. For example, in the paradigm ϖ_3, given a question Q (determined implicitly by the teacher), it is generally the case that teachers y have inquired on Q, under the supervision of expert academics z, when they were university students (in which case we had $\mathcal{S} = S(y; z; Q)$), and they resume this inquiry when preparing their teaching notes (in which case we generally have $\mathcal{S} = S(y; \varnothing; Q)$).

In such a case, we'll almost always have $A^{\heartsuit} = A_y^{\heartsuit}$. The teacher y presents to the students the report of his or her inquiry into Q. The students are, at this point, mere *spectators*—certainly committed spectators, but spectators. The only *actor* of the inquiry is the teacher, in front of a captive audience. Students will become (secondary) actors in the inquiry by studying the teacher's report, i.e., the lesson content, in the framework of personal autodidactic systems $S(x; \varnothing; A_y^{\heartsuit})$.

In many respects, the creation of the TPEs (or other similar study formats) caused an *epistemological and didactic break*. The change affected the relation to the object "knowledge", designated here by the letter $\mathcal{K}$, of any instance $\hat{\imath}$, i.e., $R(\hat{\imath}, \mathcal{K})$. A situation of inquiry tends to promote a relation to knowledge in which what is already known is a priori less important—since it was learnt in previous inquiries and has, therefore, a low probability of being once again relevant—than what remains to be studied next, to meet the needs of the inquirers engaged in a new inquiry. *Retrocognition* (or "knowing backward"), which was the old school habitus, has to give way to an attitude of *procognition* (or "knowing forward"). What the instance $\hat{\imath}$ will have to learn in the near future is likely to be more decisive in the ongoing inquiry than what $\hat{\imath}$ has already learned.

In fact, what became suddenly obvious was that the student position's cognitive equipment, as seen by myself ($\lightning$), i.e., $\Gamma_{\lightning}(p_s) \stackrel{\text{def}}{=} \{(o, R(p_s, o)) \,/\, o \in \Omega_{\lightning}(p_s)\}$, had to deeply change. The same conclusion applied to the teacher position p_t. The change in question was described—at least partially—in terms of *dialectics*. We define a dialectic to be any *praxeology* that allows some instance to overcome two opposed, contrary types of *constraints* by turning them into a new kind of *conditions* that supersede them. These new conditions are said to be the outcome of an operation of *supersession* (in German, *Aufhebung*, in French, *dépassement*, in Spanish, *superación*).

The work done from the very beginning (in 2000) clarified the *contract changes* required by the situation of inquiry with respect to the traditional class situation. These changes were identified with the mastery of a (finite) number of *dialectics* (in the sense just indicated). I shall now present, in no particular order, these dialectics by borrowing from a recent glossary (Chevallard, 2020).

Let us begin with the so-called dialectic *of the individual and the group*, or dialectic *of idionomy and synnomy*:

> In a school class $[X; Y]$ where $Y = \{y\}$, it is usually supposed that every student $x \in X$ inquires on the question Q on his or her own to produce an answer $A_x^{\heartsuit}$. In general, the answer $A_x^{\heartsuit}$ supplied by x will cease to have any relevance from the very moment the teacher discloses his or her own answer $A_y^{\heartsuit}$, which will displace all answers $A_x^{\heartsuit}$, $x \in X$, in accordance with the *degenerate* Herbartian schema $S(X; Y; Q) \hookrightarrow A_Y^{\heartsuit}$. This leads students to develop an individualistic relation to knowledge (and to ignorance), caught as they are between their *idionomy* (from Greek *idios* "one's own" and *nomos* "law") and the *heteronomy* imposed by the teacher. By contrast, in an inquiry as modeled by the (non degenerate) Herbartian

> schema, answers $A_x^\heartsuit$ are no more *but no less* than answers A° that will constitute part of the milieu M from which $A^\heartsuit$ is to be produced according to the semi-developed Herbartian schema, $[S(X; Y; Q) \rightarrow M] \hookrightarrow A^\heartsuit$, where $M = \{A_1^\circ, A_2^\circ, \ldots, A_n^\circ, W_{n+1}, \ldots, W_m, \ldots\}$. In such a perspective, a student is no longer accountable *only* for his or her own answer $A_x^\heartsuit$: *all* students are collectively accountable for the answer $A^\heartsuit$ and its construction. Their main need is therefore to establish in the class a common law, determined and applied collectively, to which they will be accountable. Such a *synnomy* (from Greek *syn* "together") must counterbalance the idionomy that remains indispensable for each student in his or her personal effort to investigate the question Q and bring his or her share to the advancement of the inquiry.

Knowing how to navigate between the individual (myself) and the group (me with the others) in order to contribute to the creation of a collective answer $A^\heartsuit$, is obviously fundamental if we do not want that, as in traditional classes, the answers given to the questions studied are only those of the teacher. But the class has to set to work. An important "gesture" is to look for existing answers A° coined by various institutions. This is, in fact, a new breach of contract: in a traditional class, that would be "copying" and therefore "cheating"—while in scientific research, it does appear as a *duty* you cannot evade! The situation faced by the students x as inquirers (and the teacher y as chief inquirer) is new and raises many problems. One of them is taken care of by a dialectic with a strange name, the dialectic *of the parachutist and the truffle hound*:

> When looking for information in the course of some inquiry, one has to sweep vast areas, thus acting as a (military) parachutist, while knowing that the information searched for will be found (in the way a truffle hound—or hog, or pig—does) only in some sporadic, unexpected places. The capacity to do so is identical with the mastery of the dialectic of the parachutist and the truffle hound.

When the class's resources cease to be strictly organized by the teacher, as is usual, when the information given to the students ceases to be neither insufficient nor in excess, new challenges arise, that the dialectic of the parachutist and the truffle hound helps to overcome.

A difficulty created by the sudden expansion of the areas to be explored and the disappearance of the teacher's control over access to information has to do with the uncertainty associated with the exploration to be carried out: is the document under review relevant to the ongoing inquiry or is it likely to lead us astray? This is what the dialectic *of on-topic and off-topic* allows us to do:

> At school, the course followed by an inquiry is traditionally supposed to remain on-topic all the time, without wandering off-topic even for a short detour that would seem promising in terms of unexpected but hopefully relevant encounters. Proper mastery of the dialectic of on-topic and off-topic makes it possible to overcome this institutional limitation and go away at times from the apparent right path, in search of the unforeseen.

The study format of the inquiry as promoted by the paradigm of questioning the world thus requires less pusillanimous, more daring "inquirers" than the traditional strictly guided work.

Another difficulty results from the breach of contract related to a "free" documentary search characteristic of the type of inquiry in question. In the answers A°

and the other works W that this search brings to light, there are grey areas comprising entities that the inquirer is unfamiliar with or does not know at all. This, again, requires an attitude towards the unknown or unfamiliar that is not that required in an ordinary classroom, where retrocognition is the rule and procognition is a fault. This is what the praxeology called the dialectic *of black boxes and clear boxes* should enable us to achieve:

> [The dialectic of black boxes and clear boxes is a] praxeology that allows one, when confronted to some praxeological element, to manage one's way between full ignorance (black box) and supposedly complete knowledge (clear or white box) of that element. To cast it in formulaic style: this dialectic helps one determine *the right shade of grey* to work with.

From the traditional school perspective, the dialectic of black boxes and clear boxes may seem to weaken the conclusions that will lead to the answer $A^{\heartsuit}$, because they may hinder the correct understanding of the answers A° and the other works W. On the other hand, we do not always have to elucidate all the aspects of an answer A° or a work W: a supposedly "in-depth" study is rarely necessary on the way to $A^{\heartsuit}$—but a *relevant* study is. With this remark, which is essential to an epistemologically sound understanding of the so-called dialectic *of questions and answers* (we have to question answers A° and works W, and answer—*up to a point*—the corresponding questions), we come to the dialectic that we must look at as the alpha and omega of inquiry work: the dialectic *of conjecture and proof*.

The dialectic of conjecture and proof is best known under the name of dialectic *of media and milieus*, and can be described as follows:

> In the course of an inquiry on a question Q by a didactic system $S(X; Y; Q)$, X is confronted with statements expressed by what is generically called *media*, a *medium* being any system that issues messages—a textbook, a teacher, a newspaper, the Internet are all media. Of course this list should also include X insofar as this group utters statements regarding the question Q. Notwithstanding their plausibility, mostly all the statements "received" by X (including those coming from Y) should be regarded as *conjectures*, i.e., as statements based on incomplete evidence. Looking for evidence is thus the sinews of inquiry. Proof of statement ϑ should be looked for by questioning media which, with respect to ϑ, behave like "adidactic" *milieus*. Such an adidactic milieu—or simply milieu, if no ambiguity is to be feared—is a system deemed to be devoid of any intention to prove or disprove ϑ, much like a part of the inanimate world. The dialectic of media and milieus enables the pursuit of truth—even in cases where there is no decisive test.

In all areas of human activity, this dialectic is central to the search for truth and obliges us to look for ever more media and milieus. Contrary to the usual pedagogic contract, the teacher as a chief inquirer does not have to validate the answer $A^{\heartsuit}$ arrived at by the class: this validation is the responsibility of the class, by means of the dialectic of conjecture and proof. It is the evidence gathered by the class that counts, not the possible verdict of the teacher—who, of course, may oppose a conclusion proposed by the students, not as a traditional teacher, perhaps, would do, but with the support of clear antagonistic evidence.

All this is particularly true in mathematics, where, unlike in ordinary school practice, a multiplicity of proofs of a statement will be sought—these proofs will, in particular, replace the teacher's veridictive interventions. Here is a small, deliberately humorous dialogue between a dimwit (β) from the old world-that-is-not-to-be-questioned and an average person trained in the paradigm of questioning the world (ω):

β: "7 times 9?"
ω: "7 times 9? Damn, I don't know about that anymore! Well, 7 times 10 = 70. . ."
β: "No, no, no! Answer me right now!"
ω: "Allow me. . . So 7 times 9, that's 70 minus 7, or 63."
β: "That's it! That's it!"
ω: "Or it's 7 times 3 times 3 (because 3 times 3 = 9), that is 21 times 3 = 63. Or, since I think I remember that 7 times 8 = 56, 7 times 9 = 56 plus 7, that is 56 plus 6, 62, plus one, 63. Or it is also equal to $(8 - 1)(8 + 1)$, or $8^2 - 1$, or 64 minus 1; so 63. Or... Okay. It is also 9 times 9 = 81, minus 2 times 9 = 18, so 81 minus 20 plus 2, or 61 plus 2, or 63. Yes, that's my answer: 63. At least I think so!"
β: "That's right, that's right!"
ω: "But how do you know it is true?"
β: "I know that 7 times 9 = 63."
ω "Are you sure about that? Aren't we both wrong? When I was a child I liked to count in base 3."
β: "?"
ω "Let's count in base 3, my dear! In base 3, the number 7 is written... 21 and 9 is written... 100. Their product is therefore 2100, i.e., $0 + 0 + 3^2 + 2 \times 3^3$, i.e., 9 plus 2 times 27, or 9 plus 54, i.e., 63. We can't get out of it!"
β: "Hey, what's taking you so long?"
ω "It's better than making a mistake, isn't it?"
β: "That's not untrue..."
ω "Mathematics, my dear, deserves a little respect; and people deserve the same."
β: "What do you mean? What do you mean?"
ω "Well, people deserve a little respect. Especially as concerns their relation to mathematics. Don't you think so?"
β: "Maybe. . ."

Other dialectics of inquiry have been identified: we will examine them below.

4.8 The Question of Questions

The question of the choice of questions to study is crucial. However, it seems that today, this question has been little studied. To begin with, I list some questions studied in a workshop entitled "Inquiries on the Internet" which I had created in a French "collège" (junior high school). The workshop took place during four successive school years, from 2008 to 2012:

Q_1. A billion (dollars) is a thousand million (dollars), but what is a trillion (dollars)?
Q_2. Why do insects rush to light sources at night?
Q_3. Why does the onion make you cry?
Q_4. It is sometimes said that the great battles of the past (before the Second World War) were much more deadly than those of today. Is that true?
Q_5. What is the 500th decimal place of π?
Q_6. When you copy a URL into a browser's address box and press the "Enter" key (for example), you usually see a web page more or less quickly. Where does this page come from? How does it get to the computer screen?
Q_7. What is the 500th decimal place of $\sqrt{2}$?
Q_8. It is said that the use of mobile phones can endanger the health of users. What exactly are the dangers involved?
Q_9. How long does it take for a plastic bottle to be destroyed? 5 years? 50 years? 500 years? 5000 years? And how do we know that?
Q_{10}. Coal and oil are said to be unsustainable sources of energy (one day, coal or oil reserves will be depleted). But what about nuclear power? Would this source of energy be inexhaustible?
Q_{11}. A traveller goes from Paris (France) to São Paulo (Brazil). The flight lasts about 12 hours. He dreams that the Earth is half as small as it is, to shorten the journey. What would happen to humans if the Earth were as he dreams? Would life on Earth be the same? How would it change?
Q_{12}. What is the 100th decimal place of 31/19? The thousandth? The ten thousandth? The hundred thousandth? The millionth?
Q_{13}. Ice cubes are said to float on the water because the ice would be "lighter than water". Is that true? It is also said that we do not know why ice is lighter than water. Is that true?
Q_{14}. It seems that there are sites on the Internet where you can ask which day of the week was a given date. Is that true? Can we also do it ourselves, without using such calculators, and even mentally?

I cannot consider these questions in detail here. But I would like to make a few simple remarks about them. A first remark is that they are formulated in a somewhat "naïve" way. There is a logic behind this: if persons address a question, we can think that they do not know the answer that can be given to it, and more broadly, they do not know the domain to which the question belongs.

Second remark: when a question contains one or more assertions that are not explicitly questioned (they are "presuppositions" of the question), these assertions have to be examined in the inquiry on the question: this is part of the contract that regulates the "new" world.

Thirdly, several questions are about rumours and hearsays: the expected inquiries are a privileged tool for deconstructing erroneous beliefs.

The questions above generate *unfinalised* inquiries, whose "facilitators" did not seek to have the students meet this or that work of this or that nature. However, this is a rare material in our community where, for good reasons in general, many of us are led to consider (and design) *finalized* SRPs. If we could consult the inquiry reports (they are in French) relating to the questions given as examples above, it would also be an opportunity to observe how incredibly important, and often unexpected, the *trophic span* of the questions studied (i.e., of the task of answering them) can be.

Newcomers to the paradigm of questioning the world may fear, in some students, the copy-paste syndrome. I suppose that many questions—the most non-standard ones notably—in the list above will curb that fear. Even when the "best" answer, i.e., the optimal answer given the constraints under which the inquirers work, is quite similar to an answer A°, i.e., $A^{\heartsuit} \approx A^{\circ}$, we must remember that the "moment of the evaluation", which is part and parcel of the "model of didactic moments" at the heart of the ATD, does not only concern the answer $A^{\heartsuit}$ but also the *relation* of the student x to $A^{\heartsuit}$, particularly x's capacity to explain, justify, and comment on the answer $A^{\heartsuit}$, its genesis and justification.

How should we choose the questions for study? If it is the class's or the teacher's choice, we can think of "mysteries" recently encountered in our daily life. For example, during this series of lectures, I have mentioned the "grandmother hypothesis", or the "Out of Africa" theory. Anyone here could record these questions in their personal "question book", in order to inquire about them later on. In fact, this happens rarely: It is the fate of questions to be forgotten, left aside for a later time that, too often, never comes. Our questioning curiosity sometimes verges on the sexual zeal of giant pandas. Of course, when we engage in the paradigm of questioning the world, as teachers, as teacher trainers, or as researchers, this attention to the questions that come to us in one form or another must be developed and systematized.

Like any gesture, the choice of a question requires a technique. What requirements can be considered as determinants of this technique? The choice can, of course, be random—and in some cases, this is a relevant or an even optimal technique. A mitigated form of this randomness is to accept a significant place for the *unexpected*. As in genuine research, we have to handle the dialectic *of the planned and the unexpected*. This dialectic has a companion, which is the dialectic *of a priori and in vivo analyses*. I will say nothing here about the notion of a priori analysis since, I believe, it is well-known to all. The *in vivo* analysis is linked to the unexpected in any inquiry: decisions have to be made that cannot be planned and anticipated so that the inquirers and the "chief inquirer" have to analyse "in vivo" the current situation and discuss the suggestions or decisions that result from it. Once again, it is clear that both the student craft and the teacher craft change.

We now arrive at a big issue! Suppose a "statist society" with a ministry of education (or any similar governmental agency), that determines and publicizes teaching programs. In the paradigm of visiting works, such a program $\dot{\mathrm{P}}$ is formulated in terms of works to be visited: $\dot{\mathrm{P}} = \{W_1, W_2, \ldots, W_n\}$. In the paradigm of questioning the world, $\dot{\mathrm{P}}$ will be formulated in terms of "great questions" to be inquired into: $\dot{\mathrm{P}} = \{\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m\}$. Note that the great questions $\mathcal{Q}$ should be more or less generic (e.g., "Being human, being social: Forms of sociality in human societies", or "Knowledge for the citizen: What to learn?") whereas the questions studied in classes (such as the questions $Q_1, \ldots, Q_{15}$ above) will be more specific (e.g., "Is the distribution of single-parent families by income group uniform, unimodal, or bimodal?" or "What are quadratic equations? Are they useful to the ordinary citizen? If so, how?") and should be contributions—however small!—to the study of some "great question" Q.

The big challenge we have to face is to determine how—by what societal technique—this program of questions could be created. I'll deliberately keep this major problem out of this series of lectures: it will be on our research agenda for the years to come. I'll simply make a few remarks concerning the principles that might guide the building-up of such a program.

A first guiding principle is that school is an institution that must enable the younger generations to enter society by "entering" into the questions that arise there and the answers that can be given to them.

A second guiding principle is that the ability to produce, under reasonable conditions of supervision and assistance, an answer A to a question Q can be considered essential in training citizens to exercise their public and private responsibilities.

Against the habitus of immediacy that tends to be generated by the old school world, whose virtue of patience is not the strongest, a third guiding principle is that the construction of an answer A to a question Q cannot be almost "instantaneous": it presupposes a series of draft answers A', A'', ..., several times "dismantled", enriched, and elaborated anew. In the best of cases, this sequence will seem to converge towards a "provisionally definitive", even if hypothetical, answer A.

4.9 Conducting an Inquiry

I would like now to consider what I believe to be the three structuring principles of any SRP in the paradigm of questioning the world. The first one is quite simply the fact that a question is chosen. The second structuring principles lies in the implementation of *the five basic gestures of the study of any question Q*:

1. Observe the answers $A^{\diamond}$ in the various institutions.
2. Analyze, at both experimental and theoretical levels, these answers $A^{\diamond}$.
3. Evaluate these answers $A^{\diamond}$.
4. Develop a specific answer $A^{\heartsuit}$.
5. Defend and illustrate the answer $A^{\heartsuit}$ thus produced.

All these gestures are related to the dialectics of inquiry and, in particular, to the dialectics not yet examined.

The first of these dialectics is the *dialectic of reading (or "excribing") and writing (or inscribing)*:

> Most information comes to us in texts, as happens with the answers $A^{\diamond}$ appearing in the developed Herbartian schema. Texts are made of assertions that both follow from and manifest praxeologies which, usually, remain hidden to the casual reader. These praxeologies have been "inscribed" (and thus concealed) in the text, so to speak; conversely, the serious reader, who feels concerned with the praxeologies put to use to produce the assertions he reads, will have to "undo" the inscribing by—to use a neologism—"excribing" them, i.e., by questioning the text about its hidden content, so as to bring to the fore normally

> latent praxeologies. It follows from all this that, reciprocally, in producing $A^♥$, X (and therefore Y) has to devote much effort to “inscribing” it into the text that will preserve it from oblivion and make it known more widely. Altogether, all this necessitates *much writing* and, above all, *different kinds* of writing (such as in a notebook, a progress report, a draft, etc.).

The second dialectic is the dialectic *of diffusion and reception*:

> Whatever the answer $A^♥$ to some question Q, be sure that it will diffuse outside of $S(X; Y; Q)$. For example, if $[X; Y]$ is a school class, $A^♥$ will be known, in essence, to other teachers, to parents, etc. Bringing an answer to a question is a social act, the product of which cannot be cooped up in a single place—“leaks” are sure to happen. The diffusion that takes place alters the ecology of $A^♥$ and may therefore diminish its viability, even within $[X; Y]$. How $A^♥$ will be received is thus a crucial concern for its producers and potential users. The dialectic of diffusion and reception is therefore a key tool of inquiry.

The third structuring principle is, of course, to put to use the seven dialectics of inquiry, to wit, the dialectics *of the individual and the group*, also called dialectic *of idionomy and synnomy*, *of the parachutist and the truffle hound*, *of on-topic and off-topic*, *of black boxes and clear boxes*, *of conjecture and proof*, also called dialectic *of media and milieus*, *of reading (or “excribing”) and writing (or inscribing)*, and *of diffusion and reception*.

4.10 An Endless Analysis

Before I stop, I would like to highlight a point that has to do with the dialectics of diffusion and reception. Answering a question is a social act that must be recognized and known, and therefore evaluated, outside of the class. The paradigm of questioning the world is, undoubtedly, more than the old paradigms, a paradigm in which inquiry work ceases to be confined within a classroom. In a school, for example, it can be imagined that the inquiries conducted in a class $[X_1, Y_1]$ could be evaluated by another class $[X_2, Y_2]$.

This openness must also concern the works used in the inquiry: if it draws upon work $W_1, W_2, \ldots, W_m$, its supervisor or its evaluators can ask the question: What about the work W^* that has not been considered? Would it significantly change the inquiry’s outcome? How? Is the answer given stable, “resilient”, with respect to other works not taken into consideration? And, last but not least, what about mathematics?

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The Analysis of Dominant Praxeological Models with a Reference Praxeological Model: A Case Study on Quadratic Equations

Hamid Chaachoua, Julia Pilet, and Annie Bessot

1 Introduction

Depending on the country, the choices in the curricula differ, the mathematical objects to be studied are introduced in different orders and according to ecological issues (Chevallard, 1997) specific to each institution. These choices lead to different institutional relationships to the objects of knowledge, embedding the development of the personal relationships of the institution's subjects. How to access the dominant praxeological model of an institution? At what level of granularity should it be described? We already assume that this level of granularity depends on research questions. How to compare the dominant praxeological models of each institution?

Several studies (Chevallard, 1984; Gascón, 1994; Bosch & Gascón, 2001) show that there is an implicit model of the taught mathematical knowledge in any institution. Didacticians must necessarily take a step back from the educational system they are studying by constructing a "frame of reference" based on an epistemological approach to the knowledge considered (Gascón, 1994, p. 44). In Bosch and Gascón (2005), this frame of reference is expressed as an epistemological model, which can be formalised in terms of a reference praxeological model (*RPM*).

In this study, we discuss methodological questions in order to build such a reference praxeological model. To what extent is this model independent of the institution considered by the researcher for the conduct of his or her research problem? How does the analysis of dominant praxeological models in different institutions enrich the reference model and at what level: global, regional, local,

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punctual? How can praxeologies be structured to reflect the dynamics in their study? Are new theoretical tools needed? These questions are by nature open-ended and will be addressed here.

To approach this problem, we have chosen the field of elementary algebra which is widely studied in the context of the anthropological theory of the didactic and which has already been the subject of several reference epistemological models. We deal more specifically with the case of the solving of quadratic equations in French and Brazilian secondary education.

2 A Reference Epistemological Model of Elementary Algebra

Researches in didactics of elementary algebra (secondary education) converge to the findings on the prevalence of non-functional manipulation of algebraic expressions (Chevallard, 1989; Schneider, 2012) and on the difficulties encountered by students (Kieran, 2007) in this field, which is essential for further scientific education. For Chevallard (1989) the epistemological *raison d 'être* (reason of being in English) of elementary algebra is to be a tool for modelling other mathematical praxeologies.

Some researchers such as (Chaachoua, 2010; Grugeon-Allys et al., 2018; Ruiz-Munzón, 2010; Pilet, 2015; Sirejacob, 2016; Jolivet, 2018) have attempted to construct didactic engineering that takes into account the epistemological *raison d'être* of elementary algebra. They have grown rich around a common core presented in Fig. 1. The reference praxeological model of algebra reflects a hierarchical structure according to the scale of the codetermination levels. The field of algebra is at the global level of mathematical organisation. It is divided into three

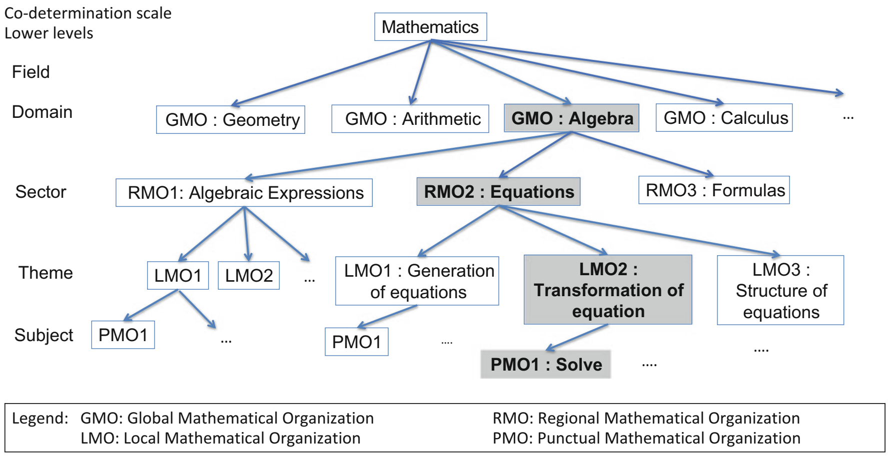

Fig. 1 A partial representation of an RPM of elementary algebra

Table 1 Three techniques of algebraic resolution of a quadratic equation

Technique	Example
$\tau_{\text{null\_product}}$: • Grouping all of the terms in the left member • Factorizing • $T_{\text{null\_product}}$: Solve an equation of form $P \cdot Q = 0$	$4x = 4x^2$ $-4x^2 + 4x = 0$ $4x(-x + 1) = 0$ $4x = 0 \; or - x + 1 = 0$ $x = 0 \; or \; x = 1$
$\tau_{\text{square\_root}}$: • Grouping all of the non-constant monomes in the left member and constants in the right • Factorizing the left member • $T_{\text{square\_root}}$: Solve an equation of form $P(x)^2 = k$	$x^2 - 3 = 0$ $x^2 = 3$ $x = \sqrt{3} \; or \; x = -\sqrt{3}$
$\tau_{\text{discriminant}}$: • Grouping all of the terms in the left member • Developing and reducing • $T_{\text{trinomial}}$: Solving a trinomial equation by using the discriminant formula	$x^2 + 5x + 4 = 0$ $\Delta = 5^2 - 4 \cdot 1 \cdot 4\Delta = 9$ $x = \frac{-5+3}{2} \; or \quad x = \frac{-5-3}{2}$ $x = -1 \; or \; x = -4$

regional mathematical organisations: algebraic expressions (Pilet, 2015), equations ((Sirejacob, 2016) for the first degree) and formulas. Each is divided into several local mathematical organisations.

This study deals with the regional mathematical organisation "Equations" (RMO2) and, at the local level, with the transformation and solving of algebraic equations. At the punctual level, we focus on the type of tasks T: Solve a one-variable quadratic equation.

We consider three techniques of algebraic solving of a quadratic equation (Table 1), each with a specific technological-theoretical environment.

The null product factorisation technique, noted $\tau_{\text{null\_product}}$, consists in factorising to obtain an equation of the form $PQ = 0$. The technological-theoretical level is based on the property of the null product. $\tau_{\text{square\_root}}$ technique consists of factorising to obtain an equation of the form $(ax + b)^2 = k$ (where a, b and k are real). At the technological-theoretical level, it is based on the definition of the square root. Finally, the discriminant technique, noted $\tau_{\text{discriminant}}$, consists in rewriting the equation in the form of a trinomial and applying the discriminant formula that gives the roots of the equation. It is based on the discriminant formula.

There are other techniques for solving quadratic equations such as arithmetic, calculus or geometric techniques, but we restrict the study to the three algebraic techniques presented above, given the school levels considered and what is expected in the institutions analysed.

3 A Comparative Study of the Teaching of Quadratic Equations in Two Curricula

We aim to describe and compare dominant models inside two different institutions, Brazil and France, in relation to the solving of quadratic equations. The methodology exemplified consists of a set of round trips between modelling and confrontation with contingency thanks to different empirical surveys. The different steps are as follows: description of a reference epistemological model as presented in the previous section, empirical survey of students from both institutions to describe their personal relationship to solving quadratic equation, analysis of curricula and textbooks from both institutions to describe their dominant models, confrontation between the dominant model and the epistemological model. To do this, we will seek to enrich the reference praxeological model by describing the punctual level.

4 Students' Personal Relation with Solving Quadratic Equations in both Institutions

We designed a survey on the solving of quadratic equations (Table 2) to find out to which solving techniques French and Brazilian students gave priority. This information allows them to make assumptions about the dominant models in France and Brazil. As shown in the a priori analysis (Table 2), the structure of the equations and coefficients are chosen so that among the three solving techniques, one is more optimal than the others.

The equations are presented to the students on a sheet of paper with the following instructions: "Solve the following equations, taking care to make your solving process visible". We have not given instructions about the calculator.

This survey was conducted among French students in 2005 (Nguyễn, 2006) and 2019 and among Brazilian students in 2019. All of them had studied what is expected by their institution on solving quadratic equations. We present only a

Table 2 Equations used in the survey and a priori analysis on the optimal solving technique

Equations used in the survey	$\tau_{\text{null\_product}}$	$\tau_{\text{square\_root}}$	$\tau_{\text{discriminant}}$
a. $(3-x)(x+2)+x+2=0$	X		
b. $x^2+8x+16=0$	X		
c. $x^2-7=0$		X	
d. $x^2-\left(1+\sqrt{3}\right)x+\sqrt{3}=0$	X		
e. $2x^2+5x+7=0$			X
f. $25x^2-90x+81=0$	X		
g. $(3x-4)^2-(-5x+1)^2=0$	X		
h. $x^2-11x+24=0$			X
i. $4x=x^2$	X		

part of the experimental results in Table 3. About equations (a) and (b) from Table 2, between 2005 and 2019, French students did not use the same techniques. In 2019, they exclusively used the discriminant technique, whereas in 2005 more than half of the students used the optimal factorising technique. Like them, Brazilian students turned to the discriminant technique. This trend is confirmed for the other equations. The "other" category corresponds to the absence of answer or to student productions that do not fall into the other three techniques analysed a priori.

The massive use by French students (2019) of the $\tau_{\text{discriminant}}$ technique, even if a priori calculations are expensive as in equation d, seems linked to the presence of programmable calculators. The excerpt from the copy of a French student in Fig. 2 illustrates this. This student transforms the equations to identify coefficients of the trinomial and then writes the value of the discriminant as well as those of the roots without writing intermediate calculations. His or her calculator seems to provide exact and approximate roots values. The presence of the calculator modifies a priori analysis presented in Fig. 3. In fact, with this tool, $\tau_{\text{discriminant}}$ technique becomes optimal, the only cost being to identify coefficients and to enter them in the calculator.

The French institutional context provides interpretative elements on evolutions of techniques used by French students between 2005 and 2019: curricular changes have led to a reduction in the time devoted to technical work, in particular factorisation, and introduction of the algorithmic domain has encouraged a frequent use of programmable calculators.

The productions analysed are observable of the institutional relation to quadratic equations and therefore of the dominant model in each institution. They are not sufficient to characterise and understand the dominant model. It is necessary to analyse other observables such as curricula and textbooks from both countries. We only present the analysis of the textbooks here.

5 Textbooks Analysis: Which Are the Dominant Praxeological Models in France and Brazil for Solving Quadratic Equations?

To explain the differences observed in the techniques used by the students, we propose to question the dominant models of the two institutions. This analysis requires refining the level of granularity in the description of the RPM. Indeed, the teaching of a notion is done at the level of the subject in the scale of levels of codetermination, and therefore of punctual praxeologies. The identification of these punctual praxeologies is based on a specific methodology that interrogates several study materials. In our case, we limit the analysis to that of some textbooks by considering them as a good observable of the institutional relation (Assude, 1996; Chaachoua & Comiti, 2007) and therefore of the dominant model. This analysis

Table 3 Some experimental results about equations (a) and (b)

	a: $(3 - x)(x + 2) + x + 2 = 0$			**b: $(3x - 4)^2 - (-5x + 1)^2 = 0$**		
(%)	FR 2005 (62 students)	FR 2019 (26 students)	BR 2019 (114 students)	FR 2005 (62 students)	FR 2019 (26 students)	BR 2019 (114 students)
$\tau_{null\_product}$	52	0	0	61	0	0
τ_{square_root}	0	0	0	0	0	0
$\tau_{discriminant}$	48	73	64	39	59	32
Other	0	17	36	0	41	68

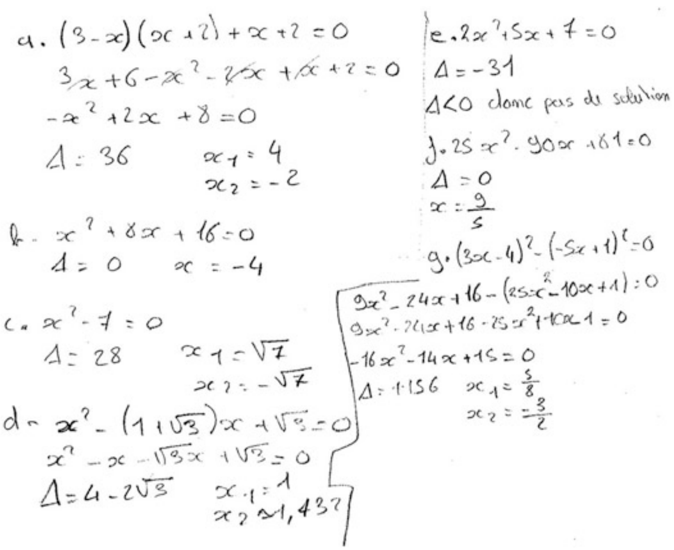

Fig. 2 Copy of a French student who appears to be using a programmable calculator

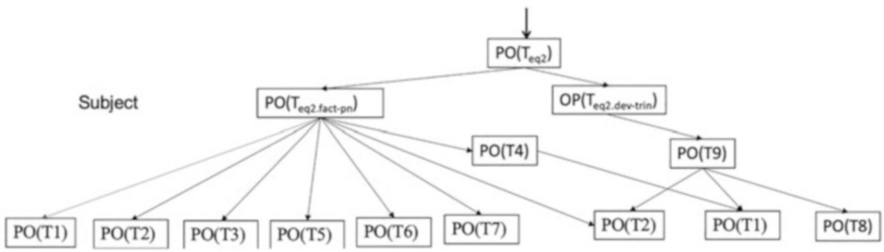

Fig. 3 Enrichment of the RPM by analysing the praxeologies of the two institutions BR and FR

seeks to identify the technological environment and praxis based on the textbook lesson and exercises.

First of all, it should be noted that the official institutional time devoted to the study of quadratic equations is very different between the two institutions studied: 1 year in Brazil (BR) and 3 years in France (FR). The praxeological analysis of textbooks revealed different study dynamics around 11 types of tasks, as shown in Table 4.

This analysis led us to enrich the RPM on solving quadratic equations, as presented in Fig. 3.

This enriched model allows us to produce and understand the dominant praxeological models of the two institutions that we describe below.

For Brazil, the study is motivated by situations in the field of geometry (GMO: Geometry) whose role is to produce algebraic expressions and equations to be solved (Theme level). The resolution of equations is restricted to the resolution of the trinomial and its variants: PMO(T1), PMO(T2), PMO(T8) and PMO(T9). For task

Table 4 The 11 types of tasks in the order in which they appear in each textbook

Types of tasks	BR grade9	FR grade9	FR grade 10	FR1 grade 11
T1 (Solve an equation of the form $ax^2 + c = 0$, with $a \neq 0$)	2	6	6	3
T2 (Solve an equation of the form $ax^2 + bx = 0$, with $a \neq 0$)	3	4	5	4
T3 (Solve an equation of the form $(ax + b)^2 - (cx + d)^2 = 0$, with $a \neq 0$ et $c \neq 0$)			8	
T4 (Solve an equation of the form $(ax + b)^2 - c = 0$, with $a \neq 0$, $c > 0$)		1	3	
T5 (Solve an equation of the form $(ax + b)^2 = 0$, with $a \neq 0$)		5	7	
T6 (Solve an equation of the form $[a^2]x^2$ +/− $[2ab]x + [b^2] = 0$, with $a \neq 0$, $b \neq 0$) where the expressions in brackets are total values. Eg. $4x^2 - 12x + 9$		2		8
T7 (Solve an equation of the form $(ax + b)(cx + d) = 0$, with $a \neq 0$, $c \neq 0$)		3	4	9
T8 (Solve an equation of the form $x^2 + bx + c = 0$, with $b \neq 0$ and $c \neq 0$)	4			
T9 (Solve an equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$)	5			2
T10 (Solve a quadratic equation or reduce it to $P(x) = Q(x)$)			1	10
T11 (Solve a third-degree equation of the form $P(x) = Q(x)$)			10	

type T1, the $\tau_{\text{square\_root}}$ technique is expected. This technique will become part of the technological environment to produce the $\tau_{\text{discriminant}}$ technique, a unique technique for solving the T9 type of tasks. Although factorisation has been studied before, it rarely produces a technique for solving second-degree equations and, even then, it is only in very simple cases. On the other hand, the $\tau_{\text{discriminant}}$ technique produces a new factorisation technique for second-degree polynomials.

For France, the panorama is very different because there is a valorisation of the $\tau_{\text{null\_product}}$ technique during the "troisième" (14–15 years) and "seconde" (15–16 years) where the only equations proposed are factorisable by the technological elements that are distributivity and remarkable identities. It is only at the "première" school level (16–17 years old) that the trinomial form appears, which cannot be factorised by previous technological elements. The $\tau_{\text{null\_product}}$ technique will become part of the technological environment to produce the $\tau_{\text{discriminant}}$ technique to solve T9. Moreover, the study of quadratic equations and functions feed into each other. At present, in France, the habitat of equations migrates from the field of algebra to that of calculus during secondary education.

The dominant models in France and Brazil and their dynamics are therefore highly contrasted. How can we then explain the similar answers of the students observed previously between FR2019 and BR2019? As analysed above, the introduction of calculators in French classrooms and recent curricular developments may provide some answers. More in-depth studies would be necessary to provide solid results, but it is not the purpose of this article that aims to show a possible methodological path.

6 Conclusion

The methodology we have tried to present here is based on the following steps.

First of all, a first step is to build an RPM on the basis of an epistemological study-oriented by a research question. Then, the RPM and the analysis of a contingent (in this case textbooks and productions of students from an institution) enrich the RPM by possible breakdowns into punctual praxeologies according to a more or less fine level of granularity according to the requirements of the research. Finally, comparison with one or more other institutions (fictitious, distant in time or geographically distant) is a relevant contribution to identifying dominant models, continuing to enrich the RPM and questioning the possibilities.

This methodology is a way of describing not only the composition of the dominant model but also its praxeological dynamics.

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Study and Research Paths, Ecology and In-service Teachers

Britta Eyrich Jessen

1 Introduction

In Denmark, we have experimented with study and research paths (SRP) in upper secondary education and in pre-service teacher education for lower secondary school (Jessen, 2014, 2017; Rasmussen, 2016). Still, SRP based teaching has not become part of teachers' practices in classrooms detached from research activities. In this paper, we will discuss conditions and constraints for engaging upper secondary mathematics teachers in SRP based teaching.

We here refer to the SRP-based teaching as situations where the teacher poses a generating question, Q_0. Q_0 is an open question the students understand enough to inquire but are unable to answer. For this, they need to learn something new, through the combination of known methods and notions potentially supported by the study of media. When preparing the generating question, the teacher must analyse the potential paths students might take when studying the question. This study includes the derived questions they might raise and the potential media they might consult when developing answers for the derived question.

The construction of an answer covers the construction of new knowledge developed in an iterative process where students continuously go from studying media to use the new insights in research processes building new answers (see further in Chevallard, 2004; Winsløw et al., 2013; Jessen, 2017). Thus, the dialectic between media and milieu or between study and research is considered the venue where learning emerges. As part of the design and a priori analysis of Q_0, we create question-answer maps, which are tree-like mind maps indicating what derived questions students might pose, see Fig. 1. In the beginning, they might have an open nature, such as, e.g., "if I am to answer this. . . I need to know more about how

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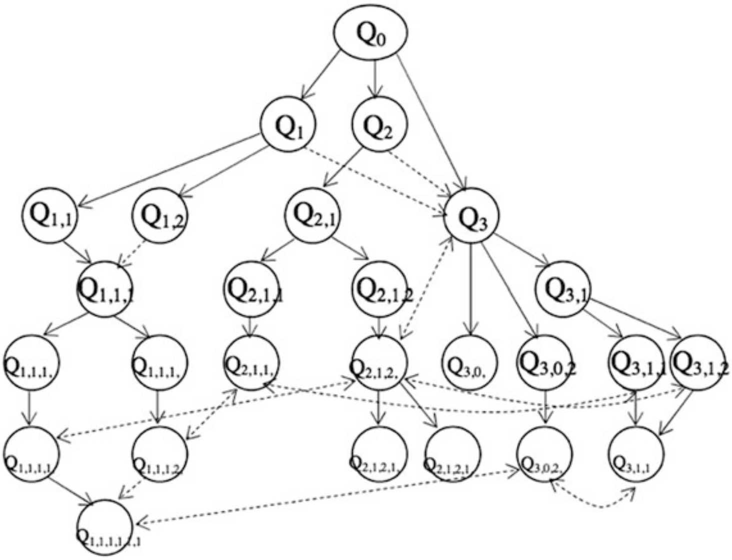

Fig. 1 A tree diagram of the a priori analysis (Jessen, 2014, p. 207)

Civilizations ↔ Societies ↔ Schools ↔ Pedagogies ↔
Disciplines ↔ Domains ↔ Sectors ↔ Themes ↔ Questions

Fig. 2 The scale of levels of codetermination affecting teaching (Jessen, 2019, p. 128)

we define a differential equation?". Further in the process, the questions might take the form of a type of task and address a point praxeology (see Chevallard, 2002; Barbé et al., 2005).

During the last decades, initial experimentations with SRP have been conducted at all levels of the educational system (e.g. García et al., 2006; Barquero, 2009). They indicate successful learning outcomes in terms of the development of more coherent praxeological organisations among students when teaching is based on a generating question engaging students in study and research processes. Similarly, the Danish experiments with SRP have been studying the potentials of introducing SRP based teaching under the constraints and conditions of ordinary teaching. To study the constraints and conditions, we employ the scale of levels of codetermination, as shown in Fig. 2. Below we refer to this scale when analysing the Danish situation.

2 The Context and In-service Teachers

Constraints and conditions for implementing SRP-based teaching arrive from several parts of the educational system. Jessen (2019) discusses the backwash effect of high-stake written exit examinations as a constraint for implementing SRP based teaching. In Denmark, the exit examinations for upper secondary school function as the entrance to the higher education as well. Most students attend a written exam in mathematics, and therefore, the written exam is a high stake for the students and teachers. Jessen (2019) has reported how this leads to a practice where students and teachers preproduce templates for solving typical types of tasks in the written exam, which students use to practice rather than learning the underlying substance of the techniques required to solve the task. This focus on students being able to identify and solve standard problems further support the teachers' emphasis on the contents from the curriculum, such as how to determine the tangent line to a differentiable function $f(x)$ in a given point x_0, which can be reduced to fill in this formula: $y = f^{'}(x_0)(x - x_0) + f(x_0)$. This seems to further the teaching paradigm of transmission of knowledge.

The less often used oral examination offers potentially good conditions for SRP based teaching. The case study presented by Jessen (2014) on interdisciplinary SRP based teaching is an example of another high stake examination, where students are supposed to demonstrate that they can address a question or problem from two different disciplines and connect those in more complete answers in a written project report. From 2017, this report must be defended in an oral exam (Danish Ministry of Education, 2017c). Jessen (2014) argues how SRP is a strong tool for designing and realising this kind of project exam since the generating questions often address problems beyond one school discipline. Furthermore, the tree diagrams indicate the interdisciplinary potential of the problem along with explicit indications of how and when the study of different media might be relevant to the students. Thus we can argue that SRPs meet the societal demand of the school system to teach students to be able to apply knowledge and competences beyond the school discipline and in more real-world contexts preparing the students for higher education (Danish Ministry of Education, 2017a). This is part of the justification for upper secondary education why it becomes a condition at school level at the scale of levels of codeterminacy. Constraints, arriving from the teachers' lack of experience with interdisciplinary teaching and the schools classical distinction between school disciplines, becomes a challenge at the level of the discipline and below, if we wish to base the teaching on generating questions addressing problems that go beyond the boundaries of school disciplines.

The dominant pedagogy is still based on the transmission of knowledge, although different initiatives, like in-service courses and curricula changes, support a more inquiry-based approach to teaching in all disciplines. The ministerial guidelines describe in their 'didactical principles' how: "Reasoning must become explicit when working with pure mathematical theories, in modelling and through an inquiry approach to the content knowledge, where students autonomously discover "new" mathematical theorems [. . .]" (Danish Ministry of Education, 2017b, p. 21). The document continues by stating: "Problem solving must be organised in ways where students learn individually to formulate mathematical questions (and tasks) to pose a problem with relevant questions for modelling purposes. Problem posing can be the goal itself or the questions can, e.g. be answered (solved) by other students." (Danish Ministry of Education, 2017b, p. 22). This surely resonates with the idea of SRP, where students engage in the dialectic of questions and answers to solve the generating question or problem. On the other hand, the Danish curriculum is divided into mathematical goals (in terms of competences, see Niss, 2003) and core goals, which is a monumentalistic description of the content knowledge. This affects the levels from pedagogy and below.

In order to accommodate the SRP based teaching to the constraints and conditions for upper secondary mathematics formed by the high-stake exit examination, we have experimented with the design of sequences of study and research activities (SRA). These can be considered a branch of an SRP or an SRP with a more delimited Q_0, aiming at specific elements or monuments of the curriculum. When the SRAs are directed towards specific goals in the curriculum they carry the risk of being reduced to transmission of knowledge, meaning that students do not develop the full rationale behind the techniques developed during the SRA (Chevallard, 2004, p. 6, 2006, p. 18). Barquero et al. (2016) provide a categorisation of SRAs linked to the degree to which the SRA supports the development of coherent notions or concepts. Jessen (2017) reports on an SRA concerning the doubling time for exponential functions being part of a sequence of SRAs addressing the notion of functions in grade 10. Jessen indicates how the students develop many more details on the topic compared to the goals of curriculum including more theoretical arguments. The students performed above average at their oral exam.

The teachers' education represents a mixture of constraints and conditions for the implementation of SRP based teaching. To become an upper secondary school teacher in Denmark, one needs a master degree with a minor in one discipline and a major in another. To gain 'teaching competency' students need to take university courses fulfilling a number of requirements: 60 ECTS[1] core mathematics (calculus, analysis, linear algebra, algebra, probability theory and discrete mathematics), 30 ECTS advanced courses (advanced courses in the previously mentioned fields), 20 ECTS courses providing a broader perspective (history of mathematics, programming in relation to core courses and mathematics in a broader context as other natural sciences) and 10 ECTS in didactics and philosophy of mathematics (Danish Ministry of Higher Education and Science, 2018). Universities organise study programmes according to the above, but with rather different realisation—especially in the last requirement. This means that students can have had next to no didactical knowledge before entering the classrooms as teachers. Most Danish universities provide courses where didactics is intertwined with or mainly concerns learning theory and more

[1]European Credit Transfer and Accumulation System, where 60 ECTS points correspond to 1 year fulltime university studies.

generic approaches to teaching. At University of Copenhagen the required course for pre-service teachers introduces the students to learning theories (e.g. Piaget and Vigotsky), the theory of didactic situations (Brousseau, 1997), inquiry-based science education through the 5E-model (Bass et al., 2009, p. 91), interdisciplinary teaching (as Jantsch, 1972) and assessment (e.g. Black et al., 2004). Those who take a major in mathematics can choose to take the advanced course on didactics of mathematics, which covers elements of ATD including SRP. Students from this course can choose to write their master thesis in didactics of mathematics, which has led to more than 40 projects, where 6 of them has studied the challenges of developing, implementing or analysing SRPs (e.g. see Hansen & Winsløw, 2011; Christensen, 2018; Uglebjerg, 2019). Thus some pre-service teachers enter the profession with detailed knowledge about SRP-based teaching or at least prerequisites for engaging in this rather swiftly. However, the vast majority has next to no knowledge about didactical theories such as ATD. We, therefore, consider the teachers' initial education both representing constraints and conditions for implementing SRP based teaching and this affects the level of pedagogy in the scale of levels of codeterminacy in Fig. 2. Jessen et al. (2019) present a complete analysis of conditions and constraints for implementing SRP based teaching in Denmark compared to Japan.

The above-mentioned experiences on implementing SRP, SRA and analyses of the ecology of doing so, were the outset for the course design, Math in Change, which has been offered to all secondary mathematics teachers in Denmark (see more detailed description in Jessen, 2020). The suggestion from Barquero and Bosch (2015) using didactical engineering as a research methodology for the experimentations with ATD has been the starting piont for the course design. However, we suggest considering a slightly transposed version for the teachers, when engaging in the development of their own practice through the development of SRP based teaching. The four activities of didactical engineering are: preliminary analysis, design and a priori analysis, experimentation and 'in vivo' analysis and finally a posteriori analysis, validation and further development of the design. These components resonate with components of the paradidactical infrastructures existing in Japanese Lesson Study. Below we will explain how these were combined and related in the course design of Math in Change.

3 Reflections Upon Course Design

The above-mentioned constraints and conditions are part of what conditions the creation of an in-service course on how to design and implement SRP based teaching for ordinary teachers. The teachers are not familiar with the idea of questioning the content knowledge, creating a priori analysis or work more theoretically with the planning of their teaching nor the shared reflections about the outcomes of their teaching.

We have therefore considered how we can introduce the teachers to SRP and ATD without using the full theoretical framework of ATD and how to teach them an idea as learning from media-milieu dialectics without Herbartian schema and the notion of praxeology? We have looked at the experiences with SRP in teacher education (SRP-TE), which indicate that tree-diagrams function well when engaging teachers in considering how to design SRPs. Still, Barquero et al. (2015) have encountered difficulties when teachers are to implement their designs in their classrooms, where they tend to turn to transmission of knowledge. As one participant in an online SRP-TE course formulates it: "As I said, the implementation is complicated in my case, and I can only use one session, not more. [. . .] I would apply it to a group of students to observe if they are able to graph a scatterplot and fit different functions to it, but I cannot go further." (Barquero et al., 2015, p. 40). This indicates a transfer problem issue from rich designs in a course to the classroom reality. This is a general problem with more open and inquiry-based teaching formats. García (2013) argues that when we engage in-service teachers in inquiry-based mathematics education, we need more than good designs, the teachers need further support for implementation of the activities produced during professional development courses and he suggests Lesson Study as a way to organise this support. For the course Math in Change we strived to achieve elements of this by encouraging the teachers to participate in pairs or small groups from each school to support the implementation of their designs, to share responsibilities of the design and to observe and reflect upon the outcomes of the designs developed by the teachers during the in-service course.

We chose to teach the course 'hands-on' where SRP was introduced through an exercise, where a generating question was shared with the participants who created the first attempt of an a priori analysis answering the question: "What strategies, questions and answers can you imagine students follow, when posed this generating question?". We do not consider the outcome of the teachers work a full mapping of the potential praxeological organisation, which could be derived from the generating question. We named their diagrams 'knowledge maps', since they map the content knowledge potentially addressed by students and guide or support the teachers in the classroom when validating the contributions of the students, for instance by posing new questions or prompting them for further study and research. Afterwards, the teachers were presented with the Danish case studies mentioned above, introducing the participants to how we can engage students in the study process using all sorts of media. The teachers were encouraged to explicitly incorporate this when they started to design SRPs.

Initially, teachers were introduced to the elements of Japanese Lesson Study using Danish research on how to engage teachers in this working mode to lower secondary teachers. This is further described by Jessen and Rasmussen (2018), also addressing the choices made with respect to how ATD theory was taught to teachers not knowing much about didactics of mathematics.

The course Math in Change was organised as 7 times 4 hours sessions. The first session focused on SRP, inquiry-based education and elements of lesson study, including lesson plans. The second session focused on the teachers' design of SRPs on vector algebra and geometry. Probability theory was also a new topic in curriculum, which was unfolded from a mathematical perspective. For the next session, the participants developed SRPs covering parts of probability theory together with sharing their experiences from teaching their SRP on vectors. In this way each session represented both a shared preparation of a lesson for the teachers together with a shared reflection upon their previously realised SRPs. For every design teachers presented a lesson plan and a knowledge map which was discussed with the participants leading to adjustments of the design. In the groups of teachers coming from the same or nearby schools, the designs were implemented and different testimonials were collected in order to discuss the outcomes in relation to the lesson plan during the following course session. The data took very different forms: observation notes, students' assignments, pictures or video recording of classrooms and students—and in one case we managed to move the course session to one of the participants' classroom, first observing a lesson together before reflecting upon the experience.

The experiences from the course Math in Change were promising with respect to engaging in-service teachers in inquiry-based teaching in terms of SRP. They saw the potential of using 'knowledge maps' and a priori analyses as tools for keeping the student inquiry open and guiding the students to validate their hypotheses and strategies by sharing and comparing. The components of shared preparation and reflections seemed to further their engagement, and several teachers expressed the excitement of working more in detail with the mathematics to be taught, than they had done previously. Still, the reflections were more plenum discussions and questions to the responsible groups than explicit a posteriori analyses of potentials realised by the students. More details about why certain potentials were unfolded where others ignored might have furthered the teachers' knowledge about their own practice and the content knowledge at stake. Last but not least, it is doubtful if the teachers kept using and designing SRPs after the completion of the course, when the structured time for preparation, observation and reflections ended with the course. Thus more research is needed regarding the constraints and condition of creating more sustainable changes or development of the participants teaching practices.

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Analysing the Dialectic of Questions and Answers in Study and Research Paths

Koji Otaki

1 Introduction: A Dialectical Generalisation of the Notion of "Study"

The purpose of this workshop makes participants familiar with the analysis of *inquiry* characterised by the notion of SRP, i.e. *study and research path* in the anthropological theory of the didactic, ATD (Chevallard, 2006). The so-called inquiry-based teaching attracts increasing interests of mathematics education. Such a pedagogical tenet emphasises students' "autonomous" inquiry which might resemble researchers' one. However, the name of "inquiry-based teaching" seems to be an oxymoron. On the one hand, the word *inquiry* highlights an unfettered action according to students' own interests; on the other hand, the term *teaching* implies that there is something-to-be-taught which is forced to students. If you can see here some double-bind situation, it is because you are probably possessed with the *study paradigm of visiting works* (Chevallard, 2015). This paradigm regards the didactic situation as it of predetermined knowledge of some kind. Most national curricular projects are typical hypostatisations of it. The visiting-works paradigm is dominant not only in mathematics education action but also within the research on it. For example, the theory of didactic situations, TDS (cf. Brousseau, 1997) aims to investigate fundamental situations for different bodies of knowledge, which assume the existence of targeted knowledge in some sense. The different cases of research on the inquiry-based teaching are probably similar. However, if we recall PhD courses as didactic situations of a genre, then we can notice that this view is relatively confined to school common-sense. PhD didactic situations must be "inquiry-based". This case leads us to the possibility of another paradigm called the *questioning the world* (Chevallard, 2015). This new paradigm underlines not only studying

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pre-established answers, but also inquiring students' own questions. In short, the school realisation of didactic situations like PhD courses is insisted by this "counter-paradigm" (ibid.). Let me put brief comments here in the prefix *counter-*. It usually means "against" and "opposite direction": counterblow, counterculture, counterexample, and so on. However, we have to understand here the counter-paradigm as including the paradigm of visiting works, although it actually denies the traditional teaching view in some sense. To use an expression by the French epistemologist and philosopher Gaston Bachelard, the questioning-the-world is a *dialectically generalised* perspective of the classic paradigm.

2 Didactic Systems and Their Study-and-Research Functioning

2.1 The Name of "Study and Research Path" and Its Two Referents

The study and research path (SRP) is a central idea of a new didactic proposal which is conducted by the questioning-the-world paradigm. The SRP is a relatively metonymical term for emphasising that any inquiry process has two fundamental and complementary poles: the study pole for acquiring existing knowledge and the research pole for producing new knowledge. Precisely speaking, the name of SRP means two things. The first and main meaning is the students' activity for solving their own problems. The idea of SRP highlights that the inquiry is the inquiry *into questions*. An SRP as a students' activity starts from some question which can be fruitful sources of students' inquiry, that is, have the *generating power*: for example, "how to calculate the cube root of a given number by using a simple pocket calculator?" (Otaki et al., 2016). Such questions are alive, that is to say, the questions are legitimised by some needs, demands, or values: cultural, disciplinary, and functional (García et al., 2006). The second and applied meaning of SRP is the teachers' activities which is a complex of knowledge and know-how for supporting students' inquiry. Such an SRP as the teachers' activity and sequence of didactic situations is also named the *knowing-through-inquiry didactic organisation* (Chevallard, 2019). In this paper, I will use the name of SRP in the first meaning.

2.2 A Generic Model of Didactic Systems

Let me introduce a fundamental model for describing the course of SRP, which is a general model of *didactic systems*. Didactic systems—which are denoted by $S(X; Y; ♥)$—consist of a *set of students X*, a *set of teachers Y*, and a "target" ♥ called a *didactic stake* (cf. Chevallard & Bosch, 2019). It should be obvious that ordinary mathematics classes and lessons can be described by this model: $S(X; y; \heartsuit)$. In addition, the model can also represent didactic systems like those of homework situations: $S(x; x; \heartsuit)$ or $S(x; \varnothing; \heartsuit)$. Let me add a small comment here that, within the ATD's terminology, the adjective *didactic* has quite broad meaning, that is to say, any fact related to dissemination of knowledge is a didactic fact: teaching how to add numbers, to come to Barcelona from Japan, to politely eat full-course meal of French cuisine, and so on.

2.3 The Herbartian Schema as a Model of Inquiry Processes

In ordinary classrooms, didactic systems $S(X; Y; \heartsuit)$ are constructed around predetermined pieces of knowledge ķ:—in the case of mathematics, for example, ķ can be different theorems, notions, or techniques. Such $S(X; Y; \heartsuit)$ can be denoted by $S(X; Y;$ ķ$)$. By contrast, in SRPs, $S(X; Y; \heartsuit)$ function around questions q without forced works to be studied: $S(X; Y; q)$. This characterisation implies that $S(X; Y; q)$ bring their answers a into being: $S(X; Y; q) \rightarrow a$. This formula is called the *Herbartian schema* (cf. Bosch, 2019; Chevallard, 2019), precisely speaking the *reduced* Herbartian schema.

2.4 The Three Genetic Facets of SRP

The Herbartian schema implies that any course or history of inquiry begins with some question and ends with some answer. Generally speaking, the history of a certain object has different facets—in the case of a person, there could be many historical aspects: biological, intellectual, family-relational, economical, educational, and so on. Within the framework of the ATD, when considering the history of SRP, one focuses on three genetic facets: *chronogenesis*, *mesogenesis*, and *topogenesis* (cf. Bosch, 2019; Chevallard & Ladage, 2008). The chronogenesis of inquiry is the process of (re)producing and disseminating knowledge in a given didactic system $S(X; Y; q)$, which is promoted by the *dialectic of questions and answers*. The prefix *chrono-* underlines that the reality of *knowing* evolves over time in didactic situations. Precisely speaking in the ATD—especially in the theory of personal and institutional relations—, any *relation* $R(\hat{\imath}, o)$ of an *instance* $\hat{\imath}$, that is a *person* x or an *institutional position* (I, p), to an *object* o is at a given time t, that is, $R(t, \hat{\imath}, o)$ (see also Chevallard, 2019). The mesogenesis of inquiry is related to the growing and evolving of "toolkit" of $S(X; Y; q)$. About this dimension, the paradigm of questioning the world insists on the importance of the *dialectic of media and milieus*, which emphasises that any course of inquiry includes not only production of new knowledge but also the acquisition of pre-established knowledge in previous research. The topogenesis of inquiry refers to the division of roles and responsibilities in an SRP. In general, an answer a is more or less a collective product, which is handled by individuals of different “places” or *topoi*. For a metaphorical illustration, let me use a football game, which has a main product “getting-points” by the teamwork of players in different places of a given field (it is *not* an individual product by Lionel A. Messi!).

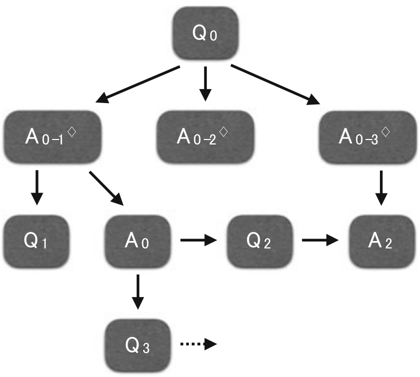
3 Investigation into the Chronogenesis of SRP

This workshop focuses on the chronogenesis of SRP, that is to say, we concentrate on how to analyse the evolution of questions q and answers a on it. In the following, I will introduce three points of view for analysing dynamics of the questions/answers dialectic. Its introducing order progresses in a zoom-in way from a landscape photography of SRP to a close-up photography of a.

3.1 Questions-and-Answers Map

A first approximation of the chronogenesis of SRP is a tree diagram of questions q_1, q_2, . . ., q_i, and answers a_1, a_2, . . ., a_j (Fig. 1), which is called the *Q-A map* (e.g., Winsløw et al., 2013). Let me highlight here that the Q-A map is regulated not only by chronological order expressed by subscript numbers, but also by epistemological order implied by bi- or multi-furcating structure. The relationship between two kinds of orders are quite complicated, so we still do not have clear and explicit methods for drawing Q-A maps. However, it is mere theoretical problem related to the Q-A map, and drawing it seems to be actually not so difficult. For example, in my personal experience of SRPs for teacher education as a lecturer, pre- and in-service school-teachers could easily write their own Q-A maps.

Fig. 1 An example of Q-A maps (Otaki et al., 2016, p. 17)



3.2 Two Dimensions of Answers

Every process of SRP starts from some *initial question* q_0, which brings about *derived questions* $q_1, q_2, \ldots, q_i$. Such questions more or less belong to either two types of questions: *how-type* $\overleftarrow{Q}$ or *why-type* $\overrightarrow{Q}$ (cf. Chevallard & Bosch, 2019). The question "*how to calculate the cube root of a given number by using a simple pocket calculator?*" belongs to $\overleftarrow{Q}$. And, the inquiry of this question naturally produces a why-question in $\overrightarrow{Q}$: "*why does such a method allow the calculation of the cube root?*" (Otaki et al., 2016). Of course, this why-question comes after answering the initial how-question (Fig. 2). Let me name here such type of answers to the how-type questions $\overleftarrow{Q}$, the *practical type of answers* denoted by $\overleftarrow{\mathscr{A}}$. By contrast, the type of answers to the why-type $\overrightarrow{Q}$ is named the *theoretical type of answers* denoted by $\overrightarrow{\mathscr{A}}$. In this case, the convergence of the exponent part is an answer in $\overrightarrow{\mathscr{A}}$ (Fig. 3). Let me explain how to handle a major type of q, that is, *what-type question*. About this, an important fact is that what-type question can be translated to how-type question (cf. Chevallard & Bosch, 2019): how to define it? This is even the same in the case of why-type question (Ibid.): how to explain it? These facts remind us that distinction between the practical type and the theoretical type of answers is not absolute but relative. In other words, they are not the natures of answers but the functions of them. We will return to this matter at the following section about the *praxeology model*.

Comparing with the first dimension of answers—practical and theoretical—, the second dimension of answers seems to be more familiar with us: *ready-made type* $\mathscr{A}^{\circ}$ and *home-made type* $\mathscr{A}^{♥}$. $\mathscr{A}^{\circ}$ means that answers in it have been produced by other persons or institutions, that is to say, they are findings of previous research. By contrast, if a certain inquiry originally creates answers, then they belong to $\mathscr{A}^{♥}$. In short, there can exist four (ideal) types of answers in SRP: $\overleftarrow{\mathscr{A}}^{\circ}$, $\overleftarrow{\mathscr{A}}^{♥}$, $\overrightarrow{\mathscr{A}}^{\circ}$, and $\overrightarrow{\mathscr{A}}^{♥}$.

Fig. 2 Two ways for calculating cube-roots using simple pocket calculators (Otaki et al., 2016, p. 14)

$A_{0\text{-}1}$	$A_{0\text{-}2}$
1. [*a*] [√] [√]	1. [*a*] [√] [√] [×]
2. [×] [*a*] [=] [√] [√]	2. [*a*] [√] [√] [√] [√] [×]
3. [×] [*a*] [=] [√] [√]	3. [*a*] [√] [√] [√] [√] [√] [√] [×]
4. ...	4. ...

Fig. 3 A justification of the calculating method (Otaki et al., 2016, p. 14)

$$((((((\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}\cdots$$

$$=\frac{1}{4}+(\frac{1}{4})^2+(\frac{1}{4})^3+(\frac{1}{4})^4+\cdots+(\frac{1}{4})^n+\cdots$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}=\frac{1}{4}\cdot\frac{4}{3}\cdot 1=\frac{1}{3}$$

3.3 Praxeological Functions of Answers

When we use the word *answer*, it usually means some statement or sequence of statements to a given question. However, we sometimes answer a given question by gestures: for example, when I go to a restaurant, I can answer a question "how many people?" through hand-gesture. This naive example reminds us that the answer is not the statement but the activity. An ATD's research assumption is that any human activity can be described as a *praxeology* or a system of praxeologies (cf. Chevallard, 2019). Within the framework of the ATD, the word *praxeology* is a technical term for representing activities, indicating and underlining that any activity consists of four components: *type-of-tasks* T, *technique* τ, *technology* θ, and *theory* Θ. A type-of-tasks T is any origin or motivation of a given how-question in $\overleftarrow{Q}$. In the abovementioned case about the pocket calculator, the type-of-tasks T_C is "to calculate the cube root of a given number by using a simple pocket calculator", which includes *tasks* t_1, t_2, . . ., t_k as in cases where given numbers are 2, 3, 100, and so on. A technique τ is any specific way for doing a certain T. In our case, calculating methods in Fig. 2 can be τ for T_C denoted by τ_{TC}. Let me emphasise that T and τ are not natures of objects but their functions in a given human action. For example, T_C can work as a technique in a situation where T is "approximately identifying cube root values of certain natural numbers". The term *technology* has unusual meaning in the ATD. It is the discourse for explaining τ denoted by θ, according to the etymological composition of the word, "technique" + "logy". In our case, the proof of Fig. 3 is an essential component of θ. Finally, the theory Θ is the discourse explaining θ. The ε-N definition of the notion of limit of a sequence can be a crucial ingredient of Θ in our case. In the same way as distinction of T and τ, it is relational and relative matter whether a given argument in $\overrightarrow{\mathcal{A}}$ is technological or theoretical.

Let me remind you of an ATD's fundamental assumption: any human activity $\tilde{A}$ can be divided into a sequence of finite tasks t_1, t_2, . . ., t_k: $\tilde{A} = t_1 \wedge t_2 \wedge \ldots \wedge t_k$. A task t_1 is accomplished by some technique τ_1, and τ_1 generates some new task t_2, and so forth. This assumption implies that any praxeological analysis has to make many tasks in its targeted praxeology undescribed. For example, our praxeological analysis abovementioned does not explicitly describe tasks of a genre like "to prove", which is no less present in the praxeology but implicit or absent in the product of the analysis. Types-of-tasks clarified in any praxeological analysis are selected for reasonably reconstructing observed activities. There is the intentional omitting of different tasks for making research results simple and clear. However, it does *not* mean that there are no other tasks in the realised praxeologies. As with other models in different scientific fields, any praxeological model is an approximation of the system studied.

4 Final Remarks: How to Analyse the Question on Inquiry?

I have just introduced some tools for analysing dialectical interplay between questions and answers on SRP. However, as you notice, such tools are partial to the answer. This seems to be an epistemological obstacle in our current didactic research:

> Our mathematical epistemology is much richer when it comes to designate results, properties and objects as defined in answers than to describe and develop the questions which did or could lead to the answers, or be posed based on them (Bosch & Winsløw, 2015, p. 392).

The entity of question is more invisible than the answer, and taken for granted. Indeed, we have only very naive tools for analysing questions: how/why, yes-no/what, and so on. In my view, there is a situational similarity between such a state of our research and the dawn of Gérard Vergnaud's theory of conceptual fields (Vergnaud, 2009), which theorises the notion of answer as a triplet model of concept. He problematised the notion of answer at that time. We have to problematise the notion of question now.

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Experimentation of a Study and Research Path: Didactic-Mathematical Indicators of Dialectics

Verónica Parra and María Rita Otero

1 Study and Research Path's Generative Question and Its Hypotheses

Study and research path's generative question is the following Q_0: *How to determine the set of values of the variables that satisfy the equilibrium condition in a given offer and demand market model?* This question is formulated based on the hypothesis (H) of the existence of an equilibrium state:

H_0: The market balance exists, and it is possible to obtain.
H_1: The market balance is produced when the offered quantity is equal to the demanded quantity, for a certain price.
H_2: The offers and demands functions are linear and both depend on the price of the unique product.

For example, if nothing is supposed about the existence of an obtainable equilibrium state, the generative question could be asked in terms of *How to determine whether or not the equilibrium point is obtainable?*

We consider that a generative question is related not only to the initial hypotheses, but also to the institutions where the SRP experimentation is carried out. That is to say, depending on the contexts of implementation, the question can be specified or extended just like the set of hypotheses. For example, we can incorporate a hypothesis about the nature of offer and demand functions, of the variables that affect their formation, whether or not to assume that the equilibrium point is obtainable, etc. In

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Fig. 1, the scheme of a possible praxeological model is presented that contemplates possible routes according to the questions and hypotheses.

2 SRP's Implementation Contexts

SRP was implemented in 4 groups of students with different characteristics which we grouped in two institutional contexts. On the one hand, a school context composed of students of the last year of the Argentinian secondary education, within the curricular space destined to the classes of "Mathematics". On the other hand, a context of mathematics teachers in training, within the curriculum space called "Didactics of Mathematics". SRP implemented in the institution "secondary school" begins from the Q_0, assumes the hypotheses H_0, H_1 and H_2. In this way, the search for answers to Q_0 permits to study several topics of the Mathematics' study program of the last year of the secondary education. In return, SRP implemented in the "university" institution begins from Q_0, also assuming the equilibrium hypothesis (H_0), but nothing assumes about the nature of offer and demand functions. So, teachers in training go through three stages: experimentation of SRP, analysis of SRP in terms of derived questions, involved praxeologies, and proposal of possible extensions or restrictions to the questions and hypotheses.

In the "secondary school" context, SRP was implemented in two different grades of the last year, with students between 17 and 18 years old. In the first of them, the implementation lasted 36 class sessions of two hours each and the group was made up of 32 students. On the first day of class the first question of SRP was asked. This decision was taken during the stage of designing the SRP, so there was not a previous training in the praxeologies that would allow answers to the questions, so, the students would not know which mathematical notions would allow them to provide answers. The study process was completely managed by the researcher. In the second grade, the implementation lasted 8 class sessions of two hours each and the group consisted of 35 students. The generative question was presented to the class group after 5 months of the school year and the study process was not managed by the researcher. In this case, the researcher was present in the classroom as an observer.

From the analysis of data obtained in these implementations, a possible set of didactic-mathematical indicators of each of the dialectics was generated (Parra & Otero, 2017, 2018).

In the "Argentinian university level" context, SRP was implemented in two groups of mathematics teachers in training (students between 22 and 35 years) in a Didactics of Mathematics course.

This course is part of the third year of the mathematics teacher training curriculum. The course programme proposes, among other units, to study the anthropological approach in mathematics teaching. In this case, both implementations were carried out by the researcher. The first lasted 7 sessions of classes of three hours each and the group consisted of 12 students. The second implementation lasted

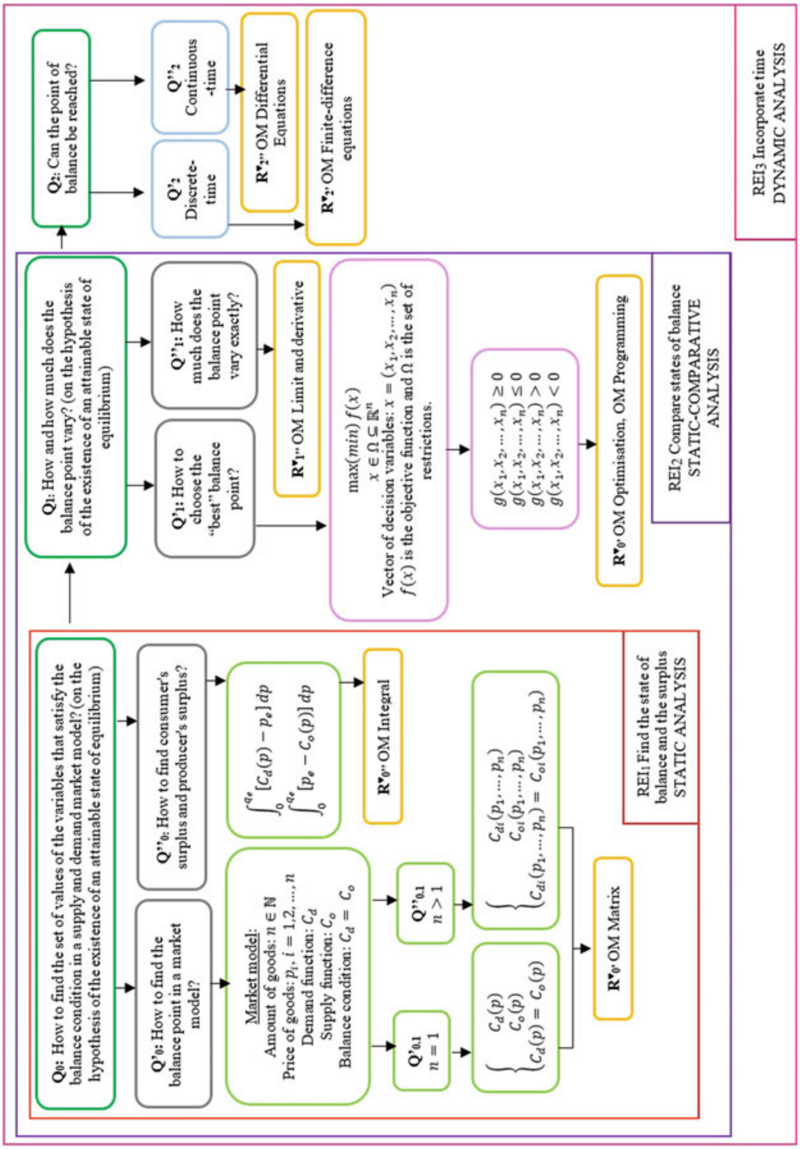

Fig. 1 Scheme of a possible praxeological reference model

Table 1 Information obtained in a test on previous sales

Price per unit (in $ARG)	Amount of demand	Amount offered
10	300	
11		174
13	270	
14		231
23		402
24	160	
25		440
26	140	

8 class sessions of three hours each and the group consisted of 11 students. In this context, no data were presented and only the equilibrium hypothesis was formulated.

The questions given to the secondary level students were the following:

Q_0: Let's suppose that we will make a product and that our aim is to sell and collect money. The following information corresponds to information obtained in a previous test of sales (Table 1):

What model would allow to study the behavior of the offer and demand on this market? How to determine the price and the quantity so that the demand could satisfy at the same time that the offer does not have excess?

Q_1: How to study the behavior of the law of offer and demand for any couple of linear functions? How to determine the point of balance in this case?
Q_2: If the parameter "ordinate" of the model is modified: How to describe the variation of the point of balance?
Q_3: If the parameter "gradient of a line" of the model is modified: How to describe the variation of the point of balance?
Q_4: How much does the point of balance change exactly in each case?

The questions Q3, Q4 and Q5 correspond to the study of the variations in the point of balance after modifying the parameters of the model. The parameters were modified one by one because the official curriculum of the Argentina secondary school prescribes only the study of functions of an independent variable. The curriculum does not propose the study of functions of two or more independent variables. The students answered the questions to the variations of a parameter and at the same time built different modes calculating the point of balance (in an analytical way or using the software GeoGebra®) in each case. They described the variations and answered questions of the type: if it increases or diminishes one of the parameters "How does it change the balance?" The answer to this question was qualitative (increase or decrease). Then, the teacher proposed the following question: "Q_5: How much does the point of balance change exactly in each case?"

Some derived questions were the following ones: What is a model of market? What is the point of balance? What is "the demand and the offer"? "How does the offer and demand behave?" These questions were answered by the students through internet search, books and asking the teacher of economy at the institution. Several groups of students not only worked on the characterisation of the economic model, but also researched, among others, about the factors that cause an increase or decrease of the demand; factors that cause an increase or decrease of the offer, and of course, the question: "How to build a model?" Here, there is a way out to the scope of microeconomics, but also to the area of mathematics. In order to build this model, it is necessary to study how to build the equation of a line with two or more known points, how to solve a system of two linear equations with two unknown quantities and to represent that model or situation in a system of these components.

To answer the questions derived from Mathematics, the professor acted, in some cases, as a source of information. For example, he reminded the students that this model, being linear, could behave like the linear functions they have studied in previous school years. Here, an "output" to the linear functions and to the resolution of systems of linear equations with two unknowns were necessary. Once done the research and the study, it was necessary to go back to the initial question and build an acceptable answer, at least, by the study community (students and teacher).

The study of the questions referred to the variations that were developed during several classes, until arriving to the concept of derivative functions as a useful tool to describe reason of change between two variables. This required the study of the limits of the functions to define the derivative of a function, a new exit or way out of the theme. The questions were the following: "What is the "intuitive idea" of a limit? Does the limit of a function exist always? Can function have two different limits? Which are the properties of the limit? Which are the infinite limits? Which are the limits in the infinity? How many indeterminations can we find? How can they be "saved"?"

3 Didactic-Mathematical Indicators of the Dialectics

1. $\boldsymbol{D}_{\textbf{E-I}}$. **Dialectic of research and study:** we identified this dialectic when it appears at any moment during the class:

$\boldsymbol{I}_{\textbf{1DE-I}}$: A search on the internet, in books of different disciplines, consulting the teachers of different disciplines, consulting different professionals and any other search in different medias who are not the teacher. For example, in this case, the search on internet or in math books and microeconomics.

$\boldsymbol{I}_{\textbf{2DE-I}}$: A study of answers A_i°, such as, the study of available answers, the works O_j that are useful in the building of the answer to the general question or its derivates. For example:

- A_1°: *O*Mat on linear function.
- A_2°: *O*Mat on parallel straight lines and perpendicular straight lines.
- A_3°: *O*Mat on two linear equation systems with two unknown quantity.

- A_4°: *O*Micro on the models of offer and demand.
- A_5°: *O*Micro on the displacement of the offer and demand curves.

$I_{3DE\text{-}I}$: The formulation of the derived questions in the different groups and search for answers. For example:

- Q_{ME1}: What is a model of offer and demand?
- Q_{ME2}: What is the function of the offer?
- Q_{ME3}: How does the function of demand behave?
- Q_{ME4}: What is the point of balance in microeconomics?
- Q_{ME5}: How do we represent a group of data in a cartesian coordinate system?

2. **DI-C. Dialectic of the individual and collective**: we identify this dialectic when the following actions are identified.

$I_{1DI\text{-}C}$: A group decision taken by the students, for example: to agree in a model (if the amounts offered and demanded depend on the price or if the price depends on the amount offered and demanded).

$I_{2DI\text{-}C}$: A member mentions that the production made is not his but from the group and vice versa.

$I_{3DI\text{-}C}$: Each group pact how to expose and defend the answer knowing that is it a production of the group collective, not individual, assigning tasks and individual responsibilities in this spread out of information.

$I_{4DI\text{-}C}$: The teacher and the students decide what subject to study.

$I_{5DI\text{-}C}$: The teacher prepares the common settings regarding the need to move forward in the study process.

$I_{6DI\text{-}C}$: The students incorporate questions during the common settings to redirect the study process according to the production of each group.

3. **DASP-ASD. Dialectics of the praxeological analysis-synthesis/didactic synthesis-analysis**: we identify this dialectic when we observe an action of the following type:

$I_{1DASP\text{-}ASD}$: An analysis of the different answers $A_i^{\diamond}$ that requiers a decision on what of this work to study in order to build the answer $A^{\heartsuit}$. For example: what and how to study the system of two lineal equations with two unknown quantity? what and how to study the displacement of the functions? How to study the models of offer and demand? What and how to study the relationships between variables?

$I_{2DASP\text{-}ASD}$: An analysis of the information obtained by different information systems: internet, books, microeconomics books, teachers, economists, merchants, etc.

$I_{3DASP\text{-}ASD}$: An analysis of the questions asked in each study group.

$I_{4DASP\text{-}ASD}$: A synthesis of techniques, technology and theories that make up different $A_i^{\diamond}$.

$I_{5DASP\text{-}ASD}$: A synthesis of the information obtained by the different media prioritising what is necessary and adequate to give answers to the different questions.

$I_{6DASP\text{-}ASD}$: A synthesis of the answers to the derived questions.

4. $D_{\text{T-FDT}}$. **Dialectic of subject and out-of-subject**: The separation between mathematics and microeconomics is done in terms of exploring different environments that apparently do not have any direct relation with the issue considered. For example, the study of limits of the functions was produced when there were questions asked about the variations of price and amount of balance. This exploration was not obvious when considering question Q4. That is how we identified this dialectic for the following actions:

$I_{\text{1DT-FDT}}$: Students go to different disciplines of mathematics. For example, to microeconomics. The decision over the domain of validity of the parameters of the model implies to study lays of offer and demand and adjust them.

$I_{\text{2DT-FDT}}$: In mathematics, a solution to the same discipline. For example:

- The study of limits of the functions in order to enter the study of the derivate of functions as limits of the incremental quotient.
- The study of equation systems to enter the calculation of point of balance.

5. $D_{\text{P-T}}$. **Dialectic of the parachutist and the truffle hound**: This dialectic starts working when it is introduced for the first time to a new question, a derived question, a $A^{\diamond}$ and or any other work, that when doing a search in different media and without a strict analysis, it seems to be useful to the construction of answer $A^{\heartsuit}$. We identify the functioning of the dialectics when at some point of the class we observe:

$I_{\text{1DP-T}}$: The group of students cannot determine how to start answering the question and the productions delivered do not give a partial answer to the questions.

$I_{\text{2DP-T}}$: The search on the internet is wide and starts to focus on what can be useful.

$I_{\text{3DP-T}}$: The search in books leads us to rule out different chapters that were not useful for answering the questions.

6. $D_{\text{CN-CC}}$. **Dialectic of black boxes and clear boxes:** We identify this dialectic when at some point in the class there is a partial study of fragments or parts of some work. So, when a study is produced in a *grey level*. For example: actions belonging to this level of *grey* are the following:

$I_{\text{1DCN-CC}}$: To study only one way to solve a system of equations.

$I_{\text{2DCN-CC}}$: To build the equation of the line that goes through two points without doing the mechanical study of the formula.

$I_{\text{3DCN-CC}}$: To study straight lines without studying perpendiculars.

$I_{\text{4DCN-CC}}$: To study the derivatives of functions as a limit of the incremental quotient.

7. $D_{\text{M-M}}$: **Dialectic of media-milieu:** We identify this dialectic when at some point in the class:

$I_{\text{1DM-M}}$: Questions are asked in terms of "why?" and the results obtained or proposal of a media (source of information) are questioned. For example: "Which of the two models obtained are correct? Why are both models of offer and demand suitable?"

$\boldsymbol{I}_{\mathbf{2DM\text{-}M}}$: A different answer is studied in any media (that is not the teacher).

$\boldsymbol{I}_{\mathbf{3DM\text{-}M}}$: Questions in terms of "how?", that is, questioning: "How to prove that the model chosen is the correct one? How do we prove that the point of balance varies? How do we prove that the point of balance and the parameters are related? etc." This implies the need to look for new information.

8. $\boldsymbol{D}_{\mathbf{L\text{-}E}}$. **Dialectic of reading and writing:** We identify this dialectic when at some point of the class the students:

$\boldsymbol{I}_{\mathbf{1DL\text{-}E}}$: The students underline or highlight what they consider important from the internet researches, or when they copy to their folders what can be useful in this search and the use of books, or asking for information from economy and math teachers.

$\boldsymbol{I}_{\mathbf{2DL\text{-}E}}$: They prepare the synthesis of their own work or from the information obtained in a media.

9. $\boldsymbol{D}_{\mathbf{D\text{-}R}}$. **Dialectics of diffusion and reception**: We identify this dialectic when at some point in the class study groups communicate and defend their answers, that is, when they share the productions in each common setting.

4 Final Thoughts

From the set of indicators of each and every one of the dialectics constructed based on the data of the SRP experimentation at the secondary level, we conclude that the most frequent dialectics was from the individual and collective, which is due to the group working in the class. The dialectic of subject and out-of-subject is another frequent one as well as praxeological analysis-synthesis/didactic synthesis-analysis and black boxes and clear boxes. The dialectic of the praxeological analysis-synthesis/didactic synthesis-analysis and the black boxes and clear boxes were used in class immediately. Both of them are strongly linked since the realisation of the analysis to a synthesis requires determining a level of grey useful to the study of the works.

Question number 4 (Q_4) presents higher occurrence of indicators of all of the dialectics, especially the dialectic of research and study, subject and out-of-subject and of the individual and collective. This indicates a higher occurrence in the search of information and development of the investigations, more agreements from each group and higher entering and coming out from different subjects (such as mathematics and microeconomics). There is a significant difference with the remaining questions, possibly because Q_4 was allowed to generate more derived questions (characteristics not determined beforehand) and a study more sustained in time. More sessions were devoted to the construction of answers to Q_4 and its derived questions. This particularity was maybe due to the fact that Q_4 allowed to address aspects of the curriculum that had not been studied before by the class.

The search for answers to Q_3 did not present indicators of the dialectic of research and study nor the dialectic of reading and writing possibly it is so because Q_3 did not generate any derived questions, and in consequence it was not necessary to look or research in different sources of information. Therefore, there were no lectures with subsequent rewriting and interpretations from the students. In Q_2 there were no indicators of the dialectics of reading and writing but there were of the dialectic of research and study, possibly because Q_2 generates some questions related to straight parallels, to the use of *GeoGebra*®, and the intersection of two straight lines: mathematics that the students knew and used. Generally, the occurrence of indicators was similar for Q_1 and Q_0 detecting an inferior number in the corresponding to Q_4, but higher to Q_2 and Q_3.

However, the indicators introduced here correspond to this particular implementation and have been developed under specific conditions and limitations of SRP. We conclude that the more generative a question is, the more indicators of dialectics we can find. This work expects to move forward in the construction of a set of indicators extending them to future implementations and other researches. The work of Salgado et al. (2017) and Gazzola (2018) go in this direction.

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Introduction for Part IV Research in Didactics of Mathematics at the University Level

Ignasi Florensa and Pedro Nicolás

Research in didactics of mathematics at university level can be considered as a relatively novel field. In fact, the first research works date from the decade of the 1970s. These initial works were framed within cognitive approaches and studied phenomena related to student conceptions and difficulties, learning processes, and modes of thinking among others. However, in the early 1980s, and due to the limitations of such approaches, researchers progressively adopted the epistemological approach that was flourishing in France. The works of Artigue (1988), Artigue and Rogalski (1990) and Gascón and Bosch (1995) pioneered this transition. The adoption of the epistemological approach accelerated in the 1990s with the development of the Anthropological Theory of the Didactic (ATD), which has proven to be extremely fruitful in this domain as stated in the summary of the TWG14 of the CERME11 (Gonzalez-Martin et al., 2019).

Nowadays, the field of didactics of mathematics at the university level has spread widely: the creation of a specific group in the CERME7 in 2011 and the creation of the INDRUM network in 2016 are illustrative facts. Some of the open questions that the research community faces can be found in these two documents: *Background: the birth of INDRUM2016* (Nardi & Winsløw, 2016) and the call for papers of the TWG14 of CERME11 conference. We summarise them here:

- Transitions between secondary and tertiary institutions seem to produce problematic phenomena related to high failure rates.
- Implementation and analysis of new teaching formats. Diverse teaching formats such as *Study and Research Paths* have been implemented in Mathematics

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degrees (or in mathematics courses in other degrees). What challenges does this implementation face at research level?
- Teaching of advanced mathematics topics that tend to be highly theorised, leaving their raisons d'être in the shade.
- Initial training of lecturers. Teacher training is based mainly on research achievements and the didactic training of teachers is absent or based on general pedagogic approaches. Should didactic devices be incorporated as training tools?

These questions cannot be detached from two phenomena that we consider specific of the teaching of mathematics at tertiary level. First, the illusion of closeness to scholarly knowledge: the fact that the lecturer is a researcher in the taught field may generate the idea of teaching pure and unique scholarly knowledge. However, this closeness, even if it may have existed in the past, remains an illusion, and we consider that didactics may help making this illusion explicit. Second, the ambivalent nature of the profession of lecturer: combining teaching mathematics with research in didactics. This twofold character generates important and unavoidable tensions between the practice, the implementation of empirical experiences, and research.

Considering the open questions in the field, these phenomena and the ATD developments, we planned the course *Research in didactics of mathematics at the university level* to debate and elaborate on the following themes:

- How to deal with knowledge at university level and question, analyse and describe it? How to do that as researchers? What is the role of the praxeological analysis, reference praxeological models, question-answer maps and other theoretical constructs?
- How to be sure that the theoretical developments and empirical experiences enrich from each other? How to guarantee fruitful feedback from both sides? In other words: how to avoid empty theoretical advances and naïve empirical experiences?
- How to deal with the tension between a prevailing epistemology with highly crystalised works in the domain and the need of living questions when implementing an inquiry study process?

The first week of the course included four lectures and four workshops revolving around these open issues. In the following chapters, the papers summarising the content of each lecture can be found. The first lecture, entitled *Institutional transitions in university mathematics education,* was given by Michèle Artigue, emeritus professor at the Université Paris Diderot. The paper addresses an exhaustive revision on how the ATD analyses the problem of institution transitions as well as the associated design of alternative practices trying to smooth them. Taking the doctoral dissertation of Frédéric Praslon as a starting point, the lecture presents the progression of the theoretical and empirical research within the ATD framework.

The second lecture was given by Reinhard Hochmuth, full professor at the Leibniz Universität Hannover. This conference was entitled *About the use of mathematics in other sciences*. Hochmuth, who signs his work together with Jana Peters, analyses the consequences of the fact that applying mathematics to real-world phenomena requires empirical sciences like physics, engineering, sociology, psychology, etc. In consequence, this interaction among disciplines make epistemological issues emerge. In particular, two aspects of mathematical practices in empirical sciences are discussed: first, the identification of mathematical formal quantities with measurable quantities and, second, the analysis of mathematical discourses inherent in mathematical practices.

The third lecture was delivered by Carl Winsløw, full professor at the University of Copenhagen, and was entitled *Mathematical analysis at university.* Taking the works of Bergé (2008) as a starting point, Winsløw provided an overview of ATD research on the teaching and learning of analysis at the university level. Then, this overview is illustrated by some examples of recent research leading to the outline of related problems for future research.

The final lecture was given by Chris Rasmussen, professor and associate chair in the department of mathematics and statistics at San Diego State University, and an associated workshop was also proposed. Rasmussen's lecture and workshop were entitled *Examining individual and collective level mathematical progress.* Rasmussen's lecture and its associated workshop contribute to coordinating different analyses to develop a more comprehensive account of teaching and learning. In particular, Rasmussen proposes to expand the constructs in Cobb and Yackel's (1996) interpretive framework that allow for coordinating social and individual perspectives.

In addition to the workshop associated to Rasmussen's lecture, another three were held. The first one was conducted by Alejandro González-Martín, full professor at the Université de Montreal. The workshop was *Using tools from the ATD to analyse the use of mathematics in engineering tasks. Some cases involving integrals.* The workshop proposed the analysis of certain tasks related to engineering in which integrals are used. The paper summarises the main results of our analysis as well as the key ideas raised in the workshop discussion; and ends by discussing implications regarding the role of calculus courses in engineering, as well as some perspectives on its teaching.

The second workshop was entitled *Describing mathematical activity: dynamic and static aspects* and was organised by Catarina Lucas, researcher at the Instituto Da Saúde Pública Da Universidade Do Porto, Ignasi Florensa, tenure track lecturer at the Escola Universitària Salesiana de Sarrià in Barcelona. The authors asked the participants to mobilise two different ATD tools to describe different aspects of mathematical activity and their complementarity with the praxeological analysis. The ATD tools proposed were question-answer maps as a tool to describe the genesis and evolution of knowledge involved in inquiry processes and the Herbartian schema used to describe different aspects of inquiry processes.

The final workshop was led by Thomas Hausberger, tenure track lecturer at the Université de Montpellier and was entitled *Mathematical structuralism: A didactic invention*? The workshop was dedicated to a discussion of epistemological and didactic aspects of mathematical structuralism with a focus on Group Theory. The participants worked on a corpus of documents comprising excerpts of the Bourbaki

Manifesto "the architecture of mathematics" and the transcript of a discussion thread from a mathematical forum online.

The second week of the course was devoted to scientific interactions and the participants' contributions about ongoing research works. The interested reader can access them in (Barquero et al., 2021).

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Institutional Transitions in University Mathematics Education

Michèle Artigue

1 Introduction

Institutional transitions in education are inherently problematic. Students must adapt to changes in institutional relationships that may be abrupt, and often are not explicitly discussed, or in terms too much general to be useful. Teachers, who are often only superficially informed about the institutional relationships that prevail in the institutions from which their students come, are ill-equipped to effectively support their acculturation. This is by no means a recent phenomenon. Already at the beginning of the twentieth century, the mathematician Felix Klein denounced, in the introduction to the famous book associated with his lectures for Gymnasium professors in Germany, a double discontinuity directly linked to such institutional transitions. He wrote:

> The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally, he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching (Klein, 2016, p. 1).

It is therefore not surprising that the discontinuities of the secondary/university transition have been an object of attention since the emergence of research on university mathematics education. This attention has increased with the massification of university education, and also with a growing demand for accountability making unacceptable the high failure rates engendered by these

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discontinuities, insufficiently considered. The first ICMI study devoted to the teaching and learning of mathematics at the university level (Holton, 2001) particularly underlined this point. Nevertheless, there is no doubt that research in this area has developed considerably in the last decades, partially renewing its approaches, as shown by the following succession of syntheses:

- The ICMI Study 11 (Holton, 2001)
- Gueudet (2008) resulting from a course at the 2007 summer school of didactics
- The survey piloted by Mike Thomas for ICME12 (Thomas et al., 2015)
- The entry (Gueudet & Thomas, 2020) in the *Encyclopedia of Mathematics Education*.

These syntheses clearly show both the quantitative increase of research within this field and the enrichment of its perspectives. Research has primarily focused on identifying and understanding students' difficulties, mobilizing epistemological and cognitive perspectives. The work carried out around the idea of "Advanced Mathematical Thinking" within the international group PME (Tall, 1991) is a good example of this. This research also pointed out new requirements in terms of connections between domains of knowledge, flexibility between modes of thinking and semiotic representations, relationship with proof and with formal language, forms of discourse (see for instance the reference book (Dorier, 2000) for research on linear algebra).

However, progressively, we can observe an increasing awareness of the role played by university teaching practices and assessment modes in the difficulties experienced by students, going along with an increasing influence of socio-cultural perspectives in mathematics education research at large, which leads to question the dominant visions, and provides new frameworks and conceptual tools to address transition issues. The anthropological theory of the didactic (ATD) has played an important role in this evolution, and in this text, I focus on the contribution of ATD research to the study of transition issues in university mathematics education (UME).

2 The ATD Perspective on Transition Issues

ATD provides a global approach of institutional transitions, not reserved to transitions at stake in UME. In fact, to my knowledge the first research using ATD to address transition issues was the thesis by Brigitte Grugeon, a doctorate student of mine (Grugeon, 1995), and the transition at stake was the transition between vocational high school and technological high school in France.

ATD obliges its users to a radical move of lens, from the students to the institutions shaping their relationship with mathematics knowledge, from the analysis of students' difficulties to the analysis of mathematical and didactical praxeologies in the institutions at stake. Doing so, students' difficulties become the sign of some institutional dysfunction. When adopting such a perspective, a basic concept is the one of praxeology used in the theory to model any type of human practice. A praxeology is defined as a quadruplet [t,T,θ,Θ], with a praxis block made of the different types of tasks t and associated techniques T, and a theoretical block made of the technological discourse θ used to describe, justify, produce these techniques, and a theoretical discourse consisting in "statements of a more general and abstract character, with a generally strong justifying and generating power" (Bosch & Chevallard, 2020). For instance, a classical type of task in Calculus is to prove that an equation $f(x) = c$, f being a real function of one real variable, has a unique solution on an interval $I = [a,b]$. To solve it, different techniques exist. The French high school curricula favours the following one: to show that f is continuous and strictly monotonic on I and that c belongs to the interval $[f(a),f(b)]$ in the increasing case (resp. to $[f(b),f(a)]$ in the decreasing case). The use of this technique can mobilize new tasks and techniques, for instance if the derivative is used to prove that f is strictly monotonic. In this praxeology, the intermediate value theorem, which is part of the theory of real functions of one real variable, is an essential ingredient. It may be a more or less explicit part of the technological discourse, according to the level of schooling and institution.

Praxeologies structured around a type of task, also called "punctual praxeologies", coalesce into "local praxeologies" sharing a common technological discourse, for instance the intermediate value theorem or the fundamental theorem of Calculus, and local praxeologies themselves coalesce into regional praxeologies sharing some common theoretical ground, for instance the theory of real functions of real variables, which leads to complex mathematical organizations. As has been shown by research on the secondary/tertiary transition, this progressive structuration of praxeologies needs to be carefully examined (see for instance (Bosch et al., 2004)).

A second ATD essential construct for approaching institutional transitions is the hierarch of scales of codeterminacy, helping us better understand the complex system of conditions and constraints that condition the ecology of mathematical and didactical praxeologies. In its current presentation (Chevallard, 2019), ten different levels in interaction are distinguished,[1] from the level of conditions and constraints resulting from the fact that we are human beings (humanity level) to the level of topics and questions. Considering the field of functions, for instance, the teaching of a topic such as the variation of exponential functions is shaped by a diversity of conditions and constraints which go beyond those associated with its inscription in a particular theme (exponential functions), sector (transcendent functions of one real variable) and domain (Calculus or Analysis) of the mathematics discipline. It is also shaped by more global conditions and constraints for instance regarding the role given to digital tools (level of pedagogy), the curricular choices which may more or less emphasize connections between scientific disciplines and shape assessment practices (level of school). These choices, in turn are constrained

[1]These ten levels are, in decreasing hierarchical order: humanity, civilization, society, school, pedagogy, which constitute the supra-didactic levels, and discipline, domain, sector, theme, topic/question, which constitute the didactic level.

by society expectations, habits and values (level of society), which, for many of them, transcend a particular society (level of civilization or more in our globalized world).

3 Investigating the Secondary/University Transition with ATD: Pioneering Theses

The doctoral thesis by Frédéric Praslon (2000) was to my knowledge the first research using this ATD perspective to approach the secondary/university transition. As was the case in Grugeon's thesis, Praslon uses ATD to question dominant visions, in his case, the dominant vision of the secondary-university transition as a radical move from the proceptual world to the formal world of mathematics, from intuitive to rigorous practices. His theoretical framework combines ATD with constructs such as the distinction between the tool and object dimensions of mathematical objects due to Douady (1986), the idea of semiotic register due to Duval (1995), the idea of procept and the three mathematical worlds of Tall (2004). This combination helps him build on already established knowledge regarding the teaching and learning of Calculus and Analysis, which, at the time, is often expressed through the use of such constructs.

The analysis of mathematical praxeologies follows a standard methodology based on the analysis of curricular documents and resources, textbooks and teaching material, assessment texts. This analysis makes clear that a substantial praxeological universe around the notion of derivative already exists at the end of high school, but that a dramatic increase of the praxeological landscape takes place in the first six months at university. It also makes clear that the transition does not correspond to the radical move described above, rather to an accumulation of smaller and less visible breaches, that are not appropriately taken in charge. The main breaches that Praslon identifies are the followings:

- An acceleration in the introduction of new objects.
- A greater diversity of tasks preventing routinization.
- Much more autonomy given in the solving process, and in the selection and management of semiotic registers of representations or mathematical settings.
- A new balance between the particular and the general, between the tool and object dimensions of mathematical concepts.
- Objects more controlled by definitions, results more systematically proved, and proofs which are no longer "the cherry on the cake" but take the status of mathematical methods.

The conjunction of these breaches creates a substantial gap, but university teachers tend to under-estimate the cognitive charge induced for their students. As pointed out in Artigue (2017), in the expression of the results of this pioneering research, we can observe the effect of the theoretical combination at stake in its theoretical framework. To make university teachers and students sensitive to these changes, Praslon designed a set of tasks in the gap between the two cultures: a priori compatible with high school knowledge but fully exotic in high schools, and at the same time not really university tasks. These tasks were proposed to students at their inscription at the university and discussed with them and the staff in the first week of teaching. Here is an example of such a task.

Let us consider the periodic function f with period 1 defined by $f(x) = x \cdot (1-x)$ on [0, 1[(see the corresponding graph).

Q1: Is this function continuous? Differentiable?

Q2: The symmetric derivative of f at a is defined as the following limit: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a-h)}{2h.}$

Compute the derivatives and symmetric derivatives of f, if they exist, at points ½, ¼ and 0, and compare these.

Q3: Are the following three conjectures true or false? Justify your answers:

C1: Every even function defined on $\mathbb{R}$ has a symmetric derivative at 0.

C2: Every even function defined on $\mathbb{R}$ has a derivative at 0.

C3: If a function defined on $\mathbb{R}$ has a derivative at a, it has also a symmetric derivative at a, and the two are equal.

A French student entering the university after a scientific baccalauréat (Bac S) in the late nineties had been taught the mathematics required for solving this task. Nevertheless, it was not part of the high school culture. For instance, f was defined by pieces and students had to understand that the given expression could only be used on [0,1[. They had already met functions defined by pieces, but these remained marginal objects and, generally, those used were defined by two or at most three different algebraic expressions given to the students. Question Q1 was not a new question and a graphical representation was offered making visible the existence of acute points, but the techniques students had routinized, based on the conservation of continuity and differentiability through algebraic operations and composition of functions, were not sufficient. In question Q2, a new notion was introduced directly through a formal definition and the students were asked to use it; this was not usual, but the definition was close to the familiar definition of the derivative. Question Q3 proposing three general conjectures was also rather unusual, but Q1 and Q2 prepared it.

The students' answers showed that students recognized these questions as questions they could address and tried to answer them but that, in their great majority, they were deprived of the mathematical means and experience necessary for their solution, except for the calculation of derivatives and symmetric derivatives in non-problematic cases. Moreover, Praslon observed the lack of connection between graphical and algebraic perspectives, attested for instance by the fact that most of those who had identified points of no-differentiability in Q1, using the graphical representation, fell down in the algebraic trap when calculating the derivative at 0 in Q2.

Praslon's thesis was a source of inspiration for two other doctorate students of mine interested in transition issues. The first one was Analia Bergé, a university teacher at the University of Buenos Aires who studied the evolution of the students' conceptualization of the completeness of $\mathbb{R}$ and continuity of the real line, along their university mathematics studies where the theme was addressed in four successive courses (Bergé, 2004, 2008). The transition at stake was not the secondary/university transition but transitions internal to the university system. Her thesis showed that the four courses functioned as disconnected institutions, and that the connection between the perspectives they offered on completeness was let to the students' private work. As a consequence, most of the students who successfully passed these courses did not distinguish the completeness of $\mathbb{R}$ from the density of its order, these two properties being associated for them, indiscriminately, with the idea of set without holes. Also, in their great majority, they neither understood the foundational role of this property for the field of Analysis, nor were able to explain where it was used in the demonstration of key theorems such as the Intermediate Value Theorem mentioned above. Visibly, the situation only improved when, in the last course, the completeness of $\mathbb{R}$ became a particular case in the study of complete metric topological spaces.

The second thesis was that of Ridha Najar, prepared in cotutelle with the University El Manar in Tunis (Najar, 2010). This time, as in Praslon's thesis, the transition at stake was the secondary/university transition, and the domain that of functions. The student population considered was a privileged population, made of top level and highly motivated students, those entering the selective program of CPS (Scientific Preparatory Classes) preparing to engineering schools. Najar focused on another main source of discontinuity in institutional relationships regarding functions in the secondary/tertiary transition, linked to the extension of the function habitat, with the move from praxeologies mobilizing essentially real functions of one real variable for solving Calculus tasks to praxeologies involving functions conceived as set theoretical objects or homomorphisms between algebraic structures. Najar developed a detailed study of these functional praxeologies, paying specific attention to the respective topos of teachers and students, to the use of semiotic resources and reasoning modes, mobilizing once again different constructs for that purpose.

He showed the praxeological discontinuity resulting from the new inscription of functions (as applications) in the domain of set theory and algebraic structures at university, despite the introduction of some set theory perspectives and discourse in the teaching of geometric transformations in high school, and once again the poor sensitivity of the institution to this discontinuity. The type of task "Proving that a function is a bijective mapping" illustrates this difference, when one compares the techniques favoured in high school mentioned above and those used in set theory or abstract algebra, coming back to the definition or using specific characteristics of homomorphisms in abstract or linear algebra. The example below, a typical example of task often proposed to students in the first worksheet on set theory and functions, is analysed by Najar in a very detailed way for illustrating this difference.

E, F, G and *H* are sets and *H* has two elements at least, *f* is an element of *A(F,G)*, the set of applications from *F* to *G*; prove the following equivalences:

$$f\text{-}surjective \Leftrightarrow [\forall g, h \in A(G, H), (gof = hof \Rightarrow g = h)]$$

$$f\text{-}injective \Leftrightarrow [\forall g, h \in A(E, F), (fog = foh \Rightarrow g = h)]$$

The thesis shows the limitations of high school praxeologies around geometric transformations that could prepare the transition. These are limited to a few punctual praxeologies, isolated and rigid, and the technological and theoretical discourse is reserved to the teacher's topos. The contrast is clear with the CPS praxeologies centered around their theoretical block, paying limited attention to the needs of the technical work and to the practical dimension of the technological discourse (Castela & Romo Vázquez, 2011) and with a much more reduced gap between the teachers' and students' topos. Taking into account these results, Najar designed a didactic intervention addressing under-estimated breakdowns that resulted reasonably effective despite the strong constraints imposed by the specific context of CPS.

As commented in Artigue (2017), these doctoral theses contributed to show the potential of ATD for understanding transition issues in their institutional and systemic dimensions and the difficulties they generate, in diverse contexts. One can observe an evolution in their use of ATD, as far as the theory develops. For instance, Najar refers to the idea of completeness of local praxeologies introduced in (Bosch et al., 2004) and uses associated indicators (see below). However, there is no doubt that in the light of current ATD research, their results could be expressed differently. This naturally leads me to the next section devoted to the evolution of research since these pioneering theses, still focusing on research inspired by ATD perspectives.

4 The Evolution of ATD Research Praxeologies on Transition Issues

This evolution has impacted both the praxis block and the theoretical block of research praxeologies on transitions, as could be expected considering the dialectic relationship between these two blocs (Artigue et al., 2011). In this text, I can only discuss a few of them.

4.1 The Evolution of the Praxis Block

The three theses I have evoked approach mathematics from an internal perspective. They do not consider mathematics in their relationships with other scientific fields, and the courses they study are not service courses. This is not surprising knowing the long-term predominance of such internal perspectives in research carried out at university level. One important evolution in the last decade is the increasing attention paid to the teaching of mathematics for non-specialists, attested by the existence of a specific entry in the second version of the *Encyclopedia of Mathematics Education* (Hochmuth, 2020). There is no doubt that ATD contributes to this evolution in an original way, as shown for instance by the pioneering theses by Barquero (2009) investigating the possible ecology of modelling practices in courses for economic majors and by Romo Vázquez (2009) investigating how engineering students respond to the mathematical needs they face in engineering projects, and the circulation of mathematical knowledge between the different institutions involved. These pioneering theses make clear that in university education, mathematics are engaged in a diversity of transitions beyond the sole secondary/tertiary transition, and that many of these engage a diversity of institutions whose mathematical praxeologies remain for the most part *terra incognita*, even for those who teach mathematics in service courses. I will not enter into more details, as several contributions to the CRM advanced course and thus to this book show, in a detailed way, how these questions are today addressed in ATD research.

The evolution of the praxis block of praxeologies has also resulted from the distinction introduced by Chevallard between two paradigms for mathematics education: the paradigm of visiting works and the paradigm of questioning the world, with the associated introduction of the concept of SRP (Study and Research Path), and the resulting evolution of the conceptualization of didactic engineering (Barquero & Bosch, 2014). This evolution has clearly influenced the methodology of ATD research on transitions. In university mathematics education, once again, Barquero's thesis was a pioneering example, with the essential role given to an SRP on the evolution of populations implemented in a workshop parallel to the main course. This methodological construction obeys another logic than the one at the base of Praslon's and Najar's experimental designs mentioned above that rely on the traditional vision of didactic engineering inspired by the theory of didactic situations.

Regarding transitions in university mathematics education, a recent example is Catarina Lucas' doctoral thesis (Oliveira Lucas, 2015). This work searches for a possible rationale for the domain she calls Elementary Differential Calculus (EDC) at stake in the secondary/tertiary transition. Capitalizing on existing ATD research, she hypothesizes that such a rationale can be found in terms of functional modelling. This thesis very well illustrates the crucial role given in ATD design to the elaboration of what is called a Reference Epistemological Model (REM) allowing the researcher to question official institutional views and rationales, build alternatives, and then study their possible ecology. This REM permeates the whole research process from the formulation of precise research questions, to the analysis of institutional practices, the identification of possible mathematical paths (MP) incarnating it, and the design and experimentation of SRPs. It helps establish conditions addressing identified obstacles to the development of functional modelling such as those linked to the dominant applicationist view of mathematics, or the disassembling of functional relationships in secondary education. SRP based on this REM are experimented with first year university students in a course of nuclear medicine, and their ecology carefully studied.

4.2 The Evolution of the Theoretical Block

As could be expected, this evolution is tightly intertwined with the evolution of the praxis block. In this part of the text, I point out some important evolutions especially due to research on transitions in university mathematics education, and more precisely three of them.

The first one is an outcome of the research carried out by Bosch et al. (2004) on limits in the secondary/tertiary transition, contrasting the high school and university praxeologies. This research has led to the idea of "completeness" of local mathematical praxeologies associated with a list of seven indicators of degree of completeness (integration of the different types of tasks, existence of different techniques and criteria for choosing between them, non-rigidity of the ostensives associated with the techniques, existence of inverse tasks and techniques, existence of a technological discourse for the interpretation of techniques and the results they produce, existence of open mathematical tasks, generative influence of technological elements on the types of tasks and techniques). These have been widely used since then, and not only by researchers working on transitions in university mathematics education.

The second one is the distinction introduced by Winsløw (2008) in the context of Calculus and Analysis. It differentiates two types of praxeological changes: the first one from praxeologies nearly limited to their praxis block to full praxeologies, the second one when tasks and techniques themselves become of theoretical nature, that is to say when the theoretical block of existing praxeologies becomes the source of new types of tasks and techniques. His research also makes clear that this second change does not necessarily occur in the first university courses. Once again, this theoretical distinction has been used since then by different researchers (see for instance Hausberger (2018)). It could also be used to incorporate results already obtained and expressed in other terms in the current ATD discourse, facilitating thus the capitalization of knowledge. There is no doubt for me that this would be possible, for instance, for part of the results obtained by Najar and Bergé.

The last theoretical development I would like to mention is due to Castela and Romo Vázquez (2011) and motivated by the specific terrain of engineering studies. To give account of the circulation of knowledge between the different institutions involved and of characteristics of the technological discourse in engineering courses and professional work, these researchers felt the need to extend the praxeological model. On the one hand, this model refines the usual description of the technological discourse as a discourse developed for justifying, explaining and producing techniques, by distinguishing six functions of the technological discourse: describing the type of task and technique, validating the technique, explaining the technique, facilitating the implementation of the technique, motivating the technique and associated gestures, evaluating the technique. On the other hand, it distinguishes between the scientific institutions in charge of producing the reference mathematics knowledge P(M) and the associated technological discourse θ^{th} made of proofs based on existing theories, and institutions I_u where this mathematical knowledge is used, and where validation obeys to processes proper to I_u, leading to original forms of technological discourse, practical and empirical, noted θ^p. This extended model was first used in Romo Vázquez's thesis to compare different courses on Laplace transform. Its need is still debated in some quarters; however, it is productively used today in different contexts (see for instance González-Martín & Hernandes Gomes, 2017; Peters et al., 2017)).

5 Final Reflections

In this text on transition issues in university mathematics education, I have tried to make clear the original and substantial contribution of research developed under the ATD theoretical umbrella, for our understanding of the complex nature of these transitions that do not limit to the secondary/tertiary transition, and for supporting educational action. Coming back to some pioneering doctoral theses, I have tried to show the change in perspectives on transition issues induced by ATD, and I have also tried to give the reader some flavour of the progressive evolution of ATD research in this area, pointing out some important evolutions of associated research praxeologies, dialectically involving their praxis and theoretical blocks. However, this contribution on transition issues remains schematic and partial, partial regarding the research work on transitions developed within ATD itself, partial also because it focuses on ATD research, which only represents a small part of the research on transition issues involving university institutions, as made clear for instance by the corresponding entry in the revised version of the *Encyclopedia of Mathematics Education* (Gueudet & Thomas, 2020). I have no doubt that the reader will find in this volume many complements to this partial presentation.

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Examining Individual and Collective Level Mathematical Progress

Chris Rasmussen

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahs & Prediger, 2014; Hershkowitz et al., 2014). An early effort at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper we expand the interpretive framework for coordinating social and individual perspectives by offering a set of constructs to examine the mathematical progress of both the collective and the individual. Building off the work of Rasmussen et al. (2015), we illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses.

1 Theoretical and Methodological Background

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The interpretive framework, shown in Table 1, lays out the constructs in the emergent perspective. The significance of accounting for both individual and collective activity is highlighted by Saxe (2002), who points out that, "individual and collective activities are reciprocally related. Individual activities are constitutive of

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Table 1 The interpretive framework

Social perspective	Individual perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities" (p. 276–277).

Our prior work with the interpretative framework (e.g., Rasmussen et al., 2009; Yackel & Rasmussen, 2002; Yackel et al., 2000) has raised our awareness of the opportunity (and need) to go beyond the constructs in the interpretative framework. In particular, we expand the ways we can analyze individual and collective mathematical progress. We use the phrase "mathematical progress" as an umbrella term that admits analyses of collective practices and individual conceptions and activity.

On the bottom left-hand side of the interpretive framework (Table 1), the construct of classroom mathematical practices is a way to conceptualize the collective mathematical progress of the local classroom community. In particular, such an analysis answers the question: What are the normative ways of reasoning that emerge in a particular classroom? Such normative ways of reasoning are said to be reflexively related to individual students' mathematical conceptions and activity. In prior work that has used the interpretive framework, individual conceptions and activity has been treated as a single construct that frames the ways that individual students participate in classroom mathematical practices (e.g., Stephan et al., 2003).

In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, we split the bottom right hand cell into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. With these two constructs for individual progress we now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? And what conceptions do individual students bring to bear in their mathematical work?

Our prior work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices. In order to more fully account for what often occurs at the undergraduate level, we therefore expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices. Thus we add an additional cell to the bottom left row of the interpretive framework, disciplinary practices. We can now answer two different questions about collective mathematical progress, one related to disciplinary practices (What is the

Table 2 Expanded interpretive framework

Social perspective		Individual perspective	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions

Table 3 Four constructs for analyzing mathematical progress and respective research questions

Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions
What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to mathematical progress that occurs across small group and whole class settings?	What conceptions do individual students bring to bear in their mathematical work?

mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?) and one for classroom mathematical practices (What are the normative ways of reasoning that emerge in a particular classroom?).

To summarize, Table 2 shows our expansion of the bottom row of the interpretive framework, which now entails four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions. The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individual students. The contribution that this expansion makes is in providing researchers with a more comprehensive means of bringing together analyses from social and individual perspectives. In particular, the expanded interpretive framework enables a researcher to answer the questions listed in Table 3.

1.1 Classroom Mathematical Practices

Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, etc. This means that particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist. The empirical approach (see Rasmussen & Stephan, 2008) makes use of Toulmin's (1958) argumentation scheme, the core of which consists of: data, claim, and warrant. In an argument, a speaker or speakers makes a claim and presents evidence or data to support that claim. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant, or a connector between the two. Finally, the argumentation may also include a backing, which demonstrates why the warrant has authority to support the data-claim pair. Toulmin's model also includes qualifiers and rebuttals. All argumentations are then analyzed to identified ways of reasoning that function as if shared using three well-defined criteria.

1.2 Disciplinary Practices

Disciplinary practices refer to the ways in which mathematicians typically go about their profession. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen et al., 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices. In this workshop we focus on algorithmatizing, which encompasses both creating and using algorithms. The method for documenting algorithmatizing makes use of a grounded approach, with an eye toward the disciplinary nature of students' mathematical activity. For example, an important algorithm in differential equations is Euler's method, which is a numerical technique for obtaining an approximate solution to an initial value problem. A common instructional approach is to simply tell students what this algorithm is and then to have them practice the method. This kind of approach to teaching Euler's method does not offer students an opportunity to engage in doing mathematics like mathematicians do. In contrast, in the example analyzed in this workshop we see students engage in the authentic practice of creating and using algorithms, which involved engaging in the goal directed activity of creating predictions, isolating attributes, forming quantities, creating relationships between quantities, and expressing relationships symbolically.

1.3 Mathematical Conceptions

As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various conceptions of the ideas being discussed and potentially modify their conceptions. Our analysis of individual student conceptions follows a grounded approach, while at the same time making use of relevant analyses from prior work that have characterized different ways that students think about mathematical ideas.

1.4 Participation in Mathematical Activity

This analysis draws on recent work by Krummheuer (2011), who characterizes individual learning as participation within a mathematics classroom using the constructs of production design and recipient design. In production design, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of *author* is given when a speaker is responsible for both the content and formulation of an utterance. The title of *relayer* is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance. A *ghostee* takes part of the content of a previous utterance and attempts to express a new idea, and a *spokesman* is one who attempts to express the content of a previous utterance in his/her own words. Within the recipient design of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A *conversation partner* is the listener to whom the speaker seems to allocate the subsequent talking turn. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called *co-hearers*. Those who seem tolerated by the speaker but do not participate in the conversation are *over-hearers*, and listeners deliberately excluded by the speaker from conversation are *eavesdroppers.*

2 Background and Setting

I illustrate the four constructs and address the respective research questions from Table 3 using data from a semester-long classroom teaching experiment (Cobb, 2000) in differential equations conducted at a medium sized public university in the Midwestern United States. I selected a 10-minute small group episode from the second day of class based on its potential to illustrate all four constructs. There were four students in this group, Liz, Deb, Jeff, and Joe (all names are pseudonyms).

There were 29 students in the class. Class met four days per week for 50-minute class sessions for a total of 15 weeks. The classroom had movable small desks that allowed for both lecture and small group work. The classroom teaching experiment was part of a larger design based research project that explored ways of building on students' current ways of reasoning to develop more formal and conventional ways of reasoning (Rasmussen & Kwon, 2007). A goal of the project was to explore the adaptation of the instructional design theory of Realistic Mathematics Education (RME) to the undergraduate level. Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level (Freudenthal, 1991). In this process, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided reinvention (e.g., Rasmussen & Blumenfeld, 2007; Rasmussen & Marrongelle, 2006; Rasmussen et al., 2005).

As previously stated, the analysis comes from video recorded work of a small group of four students, Liz, Deb, Jeff, and Joe, on the second day of class, and they have received no instruction on any analytic, numerical, or graphical techniques. Just prior to the small group work students completed the following task: The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions: There is only one species, the species have been in the lake for some time before what we are calling time $t = 0$, the resources (food, land, water, etc.) are unlimited, and the species reproduces continuously. Given these assumptions for a certain lake with fish, sketch three different population versus time graphs (one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$).

This task was relatively straightforward for students and brought forth an imagery of exponential growth and the graphs they sketched were consistent with this imagery. The instructor then used their graphs as an opportunity to introduce the rate of change equation $dP/dt = 3P$ as a differential equation that was consistent with their graphs. In particular, as P values increase, so does the slope of the graph of P vs. t.

The follow up task, which students worked on for approximately 10 minutes, however, was much more cognitively demanding for students.

> Consider the following rate of change equation, where P(t) is the number of rabbits at time t (in years): $dP/dt = 3P(t)$ or in shorthand notation $dP/dt = 3P$. Suppose that at time $t = 0$ we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits.
>
> At $t = 0.5$ and $t = 1$.
>
> At $t = 0.25$, $t = 0.5$, $t = 0.75$, and $t = 1$.

The following is a priori analysis of the reasoning that we expect students to engage in when solving this task: Establishing connection between P and dP/dt (if you know P you can find dP/dt); Given P and dP/dt at a moment in time allows one to find P at a later time; Applying the connection between P and dP/dt at that later time one can find the corresponding dP/dt; The previous can be combined into a repeating loop.

3 Sample Analysis

The 10-minute transcript was split into four segments. As a sample of the theoretical and methodological approach, I analyze only the first segment in which students get starting on the problem.

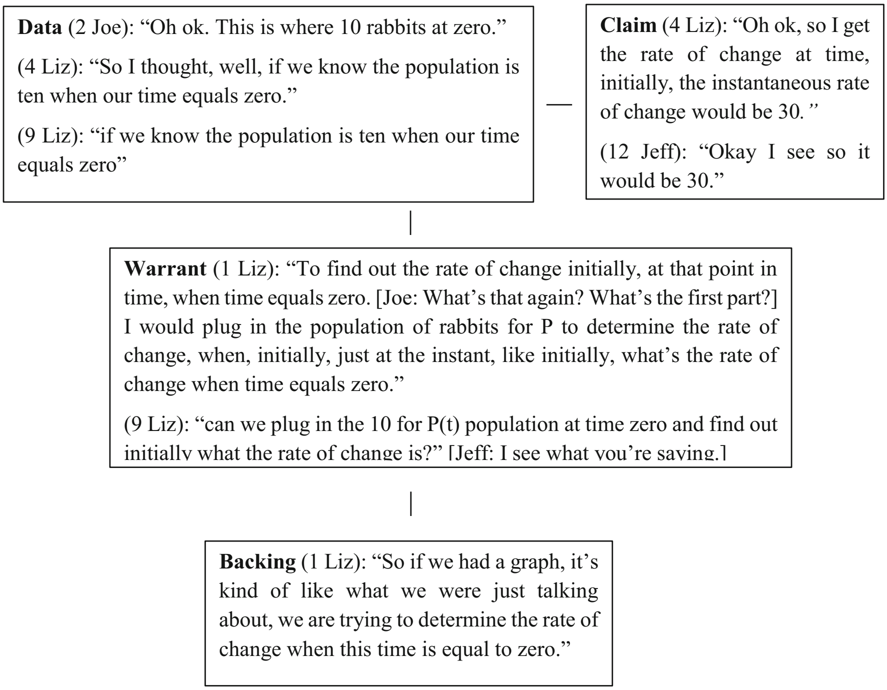

Fig. 1 Argument 1—Determining the initial rate of change

The first four minutes of small group work consisted of 19 talk turns and the production of four arguments (a la Toulmin). We start by detailing the first of the four arguments, as this argument constituted the basis for their subsequent work and hence is the lynchpin for what follows. Through this argument, we are able to highlight various participation roles, meanings that emerge, and the start of students’ algorithmatizing. The totality of Argument 1, shown in Fig. 1, was generated over the first 12 talk turns and involved contributions from three of the four students. Because people often explicate the motivation or inspiration before stating their claims, it is not unusual for arguments to unfold in a manner such that a warrant or backing comes before the claim and data. Moreover, parts of an argument are often repeated and a complete argument may develop over multiple turns. All of this is the case in Argument 1. In Fig. 1 talk turns and speakers (e.g., 4 Liz) are provided so that a reader is able to glean the order in which utterances were made and by whom.

In Argument 1 we see three of the students take on different *production* roles. Liz is the *author* for three components of the argument (Claim, Warrant, and Backing) because she is the first one to articulate the respective ideas. Jeff and Joe are both *conversation partners* and Jeff plays the role of *relayer* for the Claim (which is that the initial rate of change is 30). Recall that a relayer is someone who restates a previously articulated idea. This is significant in this case because it provides evidence that Jeff shares and agrees with the claim that Liz made. Jeff gave similar confirmation on what we coded as Liz's warrant (see (9 Liz) in Fig. 1). Joe, for his part, is also an author because he is the first to articulate the Data, although as we see that in Argument 3 his interpretation of what the 10 is is incorrect and inconsistent with the interpretation of his groupmates. Deb, who has seen this problem before, intentionally pulls herself out of the discussion and hence functions as an *eavesdropper* in this entire segment. Eavesdroppers are those that are intentionally excluded from conversations, which often has negative connotations, but here we see this as a positive participation role because Deb allows space for her groupmates to develop their own ideas while she privately (re)works the problem. Deb rejoins the discussion, however, in the next segment.

In turns 1–12 in which Argument 1 was developed, we also see articulated two different meanings for dP/dt. In particular, in the Claim for Argument 1 Liz plugs in the initial population value of 10 into the rate of change equation and gets 30, which she interprets to be the "instantaneous rate of change." She says this while sketching an exponential graph and pointing her pen to where the graph intersects the vertical axis. In calculus, the notion instantaneous rate of change is often associated with the slope of the tangent line at a point, and this *might* be how Liz interprets the 30, however we do not have strong evidence for this graphical interpretation of instantaneous rate of change. A second meaning that emerges for dP/dt is "the change in the population over the change in time." This discrete, ratio-based interpretation was authored by Liz and is the Claim for Argument 2, which occurred on turn 7 and was provided by Liz in response to a question from Joe, "Are we trying to figure out what P is?"

The development of Argument 1 is also illustrative of the group's start toward the creation of an algorithm to approximate future population values. That is, their participation in the disciplinary practice of algorithmatizing. In particular, we see here students engaged in two aspects of creating an algorithm: engaging in goal-directed activity and isolating attributes. The first of which was partnered with the participation role of *focuser* and second of which was partnered with articulating different meanings for dP/dt. Recall that a focuser is one of four new facilitator roles that we found necessary to add to Krummheuer's (2011) set of production and recipient roles in order to capture newly identified participation roles. We define a *focuser* as someone who directs others attention toward a specific goal or activity. Liz takes on this role in turn 1 when she directs her and her groupmate's attention "To find out the rate of change initially, at that point in time, when time equals zero." With this statement Liz sets out a specific activity of finding the initial value of dP/dt, which an expert will see as a specific case of a component of the more general Euler algorithm. As we saw in Fig. 1, this goal was realized with Liz and Jeff agreeing that the initial value of dP/dt is 30. Joe also acted as a focuser when in turn 6 he posed the following question to his group: "Are we trying to figure out what P is?" With this question, Joe focuses his group on what attribute of the problem situation are they trying to determine, P or dP/dt? Joe's question promoted Liz to create we coded as Argument 2 where she articulated a new meaning for dP/dt, the "change in population over the change in time." This is a powerful meaning for this group that enables them to make use of the isolated attribute of dP/dt to figure out a general approach to

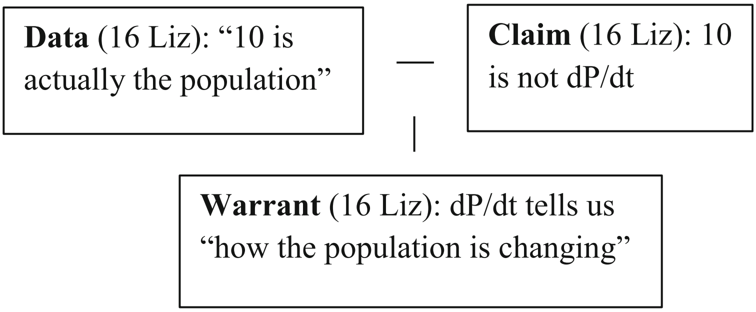

Fig. 2 Another meaning for dP/dt

compute the change in population, which is a necessary component of the Euler method algorithm. Moreover, both focusing acts led to an argument's claim, which was mathematically productive as it resulted actually finding the initial rate of change and interpreting this value as a ratio of two discretely changing quantities.

The last turn that makes up Argument 1 is line 12, which is followed by Liz in turn 13 asking her groupmates, "30, I mean does that make sense?" This is the first time we see anyone in this group "checking in" with their peers about their thinking and thus gave rise to our identifying another new Facilitator role, that of *checker*. Acting as a checker can serve multiple functions in a group. For example, it can lead to coherence and shared understanding and build confidence in their ideas, it can open a space for someone to ask a clarifying question, and it can be an opportunity for someone to disagree. In this case, Liz's check in gave rise to Jeff indicating agreement, "Yeah, that makes sense," and for Joe to offer a counter argument, which we coded at Argument 3. In this argument Joe asserts (incorrectly) that 10 is actually equal to 3P(t). This incorrect claim by Joe turns out to be productive for it gives rise to Liz putting forth what we coded as Argument 4, show in Fig. 2.

As a whole Argument 4 relates again to the algorithmatizing aspect of isolating attributes because it reasserts the meaning of 10 as the initial population and it puts forth a new meaning for the quantity dP/dt, namely "how the population is changing." Joe does not object with Liz's argument and appears from the video to think quietly about what has been said. Thus far, the meanings for dP/dt include instantaneous rate of change, change in population over change in time, and how the population is changing. Developing meaning(s) for attributes that figure prominently in their mathematical work will serve to ground their reinvention of Euler's method as a product of their own reasoning and sense making.

Next, Liz and Jeff act as focusers in that they each ponder what now to do with the 30. In particular, Liz says, "So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after, say, a half year has elapsed?" and Jeff says, "how would we work time into the equation to get the next, uh, population or change in population?" As before, this particular facilitator role of focuser promotes students' goal directed activity toward creating an algorithm. At this point Deb finally joins the discussion and says to the group, "That is exactly what I did."

4 Conclusion

There exist multiple ways in which coordination across the four lenses, some of which will be explored in the workshop. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin schemes that comprise the normative way of reasoning. How do those contributions coordinate with those students' production design roles within the individual participation lens?

An instructional implication that this analysis raises is the how to help promote productive interactions between small group members. In this particular class the small group analysed worked extremely well together, even on the second day of class. This was largely good fortune. So then what might an instructor do to facilitate more productive interactions in small groups that do not function as well as Liz, Deb, Joe, and Jeff?

Acknowledgement The work presented here was done in collaboration with Megan Wawro and Michelle Zandieh.

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Mathematical Analysis at University

Carl Winsløw

1 What Is Analysis and Why Is It Important?

Mathematical Analysis allows us to model physical phenomena such as motion and volume with as much precision as we want. Already in Antiquity, the "method of exhaustion" was applied by Archimedes to evaluate the volume of solids, while Aristotle suggested that paradoxes about motion (such as the famous race between Achilles and the tortoise) could not be solved without considering time as infinitely divisible. To some extent, the development of Analysis is inseparable from that of Arithmetic—that is, of our concept of number; and of course, modern Analysis is impossible without the notion of completeness, a property held by the real number field, but not by the field of rational numbers (despite the fact that these include "arbitrarily small" numbers). However, the most useful results of Analysis did not have to wait until the modern theory of the real number field was developed; in fact, the Calculus of the Newton and Leibniz (eighteenth century, with several precursors also in the seventeenth century) was mainly based on bold applications of symbol manipulation, including Cartesian models of curves. Brave calculations with infinite sums and with "infinitesimal" quantities allowed the formulation of the first, rudimentary versions of the "Fundamental theorem", which relates the calculation of instantaneous growth (derivatives) with the calculations of accumulated growth (integrals). In classical mechanics, this corresponds to the fundamental relations between speed and travel distance during motion in Euclidean space, such as $s = s_0 + \int_0^t v$. These early developments also led to spectacular advances in the new field of "Analytic Geometry", where, for instance, the algebraic description of curves and other geometric objects could now be applied to solve a large number of problems related to their numerical properties, such as curvature or area. Later on, these basic

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techniques to handle growth and accumulation have also led to spectacular advances in many other fields, including Biology, Engineering and even social sciences such as Economy and Demography. As concerns present-day mathematical sciences, the importance of Analysis as an area of research can be roughly estimated as about 25%: among the 63 "main categories" in the Mathematics Subject Classification (https://mathscinet.ams.org/msc/msc2010.html), 17 categories (numbered from 26 to 49) pertain to Analysis. The categories range from more classical ones (such as "30: Functions of a complex variable") to more recent ones (such as "37: Dynamical systems and ergodic theory").

The wealth and breadth of "applications" of the mathematical techniques of the Calculus, initially developed in the context of Physics, explains why a large number of university students today are required to learn those techniques, even if the appearance of Computer Algebra Systems has also led to new developments as to how these techniques are practiced and learnt.

The first explosive period of technical progress, with its daring use of intuition and computation, also led to a number of new paradoxes, which showed the necessity of more precise theoretical control of the meaning and validity of computations. Only towards the end of the nineteenth century did the field fully mature at the level of theory, as is it taught in introductory or semi-advanced university courses today. These developments have not profoundly affected the teaching of the Calculus praxis—although, at the level of technology, it is assorted with more precise descriptions related to the permissibility of certain computations (such as convergence criteria for sums). The modern theory of the Calculus—based on the completeness property of the real numbers, definition and distinction of various kinds of integral, and so on—is mainly taught to students of mathematics and closely related fields. In fact, even such students rarely meet and almost never acquire the full scope of the theoretical advances in the field which professional mathematicians consider, today, the state of the art. It is therefore increasingly apparent that what most students meet at university is a transposition of the Calculus of the eighteenth century, with mainly superficial theoretical decorations drawing on later developments. The visibility of *didactical transposition* phenomena in secondary school mathematics motivated the beginnings of the anthropological theory of the didactic. But also at university level, the mathematical sciences—and not least Mathematical Analysis—do, in fact, offer some of the most striking cases of an increasing *distance* between scholarly knowledge (including the knowledge held and developed by teachers) and the knowledge actually taught and learnt. This leads to an increasing need of epistemological and institutional vigilance for didacticians who endeavour to understand the constraints and obstacles faced by the teaching of Calculus and Analysis, and to design and experiment innovations with specific goals that are important to institutions or are considered desirable for other reasons.

2 The Transition from Calculus into Analysis

One can increasingly distinguish two kinds of "Analysis" in the didactical transpositions found in universities:

(1) *praxis* focused courses which focus on developing students' skills with Calculus as a computational and modelling tool, without any significant treatment of theory based on the topology of the real number field;
(2) courses in which such praxis continues to be of some importance, but where a strong focus lies also on studying the *theory* which explains and justifies the praxis, at least at was done towards the end of the nineteenth century.

In short, these are often referred to as Calculus (1) and Analysis (2). First courses on the latter are often called "Real Analysis" or the like, where "Real" refers to the real numbers or $\mathbb{R}^n$, as opposed to the somewhat different field of Complex Analysis that is usually studied later on. In both cases, the main objects of the courses are *functions* and their properties, and the role of the number field is to appear as domain and range of functions. Modern Analysis as a whole is certainly not limited to the study of functions but involves also more abstract sectors such as Operator Theory, Distribution Theory and so on.

The didactical research literature is surprisingly scarce when it comes to anything beyond Calculus; the investigations we did to prepare the recent Encyclopedia article (Winsløw, 2018) suggests that there are less than 30 papers in major international journals that have such a focus. This can, to some extent, be explained by the relatively larger student populations that study Calculus at university, but not by an absence of institutional stakes and didactical challenges in the case of the smaller populations that study Real Analysis. We shall return to the principal relevance of these stakes and challenges for Didactics of Mathematics in the section on the Teaching Profession.

The arithmetical foundations of university level Analysis have been the object of some studies in ATD, beginning with Bergé (2008) who, in a series of papers, investigated tasks which university students are given on this subject, and well as their (mis-)understandings of the completeness axiom and its consequences. These studies do generally not go beyond the arithmetical foundations, let alone question alternatives such as hyperreal numbers; but this is done, for instance, by Tall and Katz (2014). We notice here that although the latter paper is presented as a "cognitive" study and does not involve an explicit institutional analysis, it questions the mathematical contents from a number of perspectives relevant to ATD.

Concerning the Calculus-Analysis transition, a main result in ATD based research on the teaching of post-Calculus Analysis is that at least two major transitions take place (Winsløw, 2008; Winsløw et al., 2014): the first (type I) in which "Calculus" praxeologies are refined and formalized by the study of theory that involves definitions, theorems and proofs based on the basic topological properties of the real numbers; and the second (type II) when tasks and techniques are introduced which take, as objects, elements of the theory blocks previously introduced. For instance, integrals of abstract functions appear as theoretical objects in most introductory real analysis courses; they become simple examples of functionals in the study of function spaces. It is apparent that a type II transition may, again, be followed by a type I transition, as the new praxis is supplied with theoretical superstructure (in the example, topological vector spaces etc.); in principle, one can in fact assume that the continued study of Analysis involves an ongoing accumulation and dialectics of such transitions.

The role of definitions in first Analysis courses, and in particular in type I transitions, was further studied by Winsløw (2019). It identifies one characteristic feature of how such courses define their primary objects (such as integrals): conditions for *existence* of an object are given simultaneously with its value, as in the following example of a classical εδ-definition (where the condition for existence of the number I also show what it is in case it exists):

A function f: [a, b]→$\mathbb{R}$ is integrable if there is a number $I \in \mathbb{R}$ with the following property: for all $\varepsilon > 0$ there is a $\delta > 0$ such that for any partition $a = x_1 < x_2 < \ldots < x_n = b$ of [a, b] with $|x_{k+1} - x_k| < \delta$ for all k, and for any choice of middle points $t_k \in]x_k, x_{k+1}[$, one has

$$\left| I - \sum_{k=1}^{n-1} f(t_k)(x_{k+1} - x_k) \right| < \varepsilon$$

In this case one says that I is the integral of f over [a, b].

It is clear that definitions of this type are quite different from the kind of definition typically given in a Calculus course, where the working definition of the integral is simply a formula (corresponding to the Fundamental Theorem of Calculus). This kind of definition is found throughout Analysis courses, also when defining notions that do not appear in Calculus (such as the norm of an operator on a Hilbert space). They are usually formulated as quantified statements (with the verbal form of quantifiers being something like "for all", "there is") involving inequalities that relate the variables of the quantifiers. In typical first courses on Real Analysis, definitions of this type are presented in lectures with only few requirements for students to actively engage with their meaning. Students' failure to complete the first transition can, more broadly, be explained by the new theory blocks being purely related with relevant praxis blocks, both those known from Calculus (Kondratieva & Winsløw, 2018), and new ones. These failures therefore lead naturally to the didactic problem of task design.

3 Task Design in Analysis

What we have considered so far amounts mainly studies based on descriptive models using key notions from ATD, in particular the modelling of mathematics itself based on praxeologies. A very large body of the broader field of research on university mathematics education remains, in fact, at the level of ethnographic studies, trying to capture significant features (especially problematic ones) of the spontaneous or "normal" didactical practice in university mathematics. But the expectations which the institutions (and society at large) have to research on university mathematics are typically that such research produces concrete and documented interventions which can lead to improved didactical practice and solve some of the problems. These expectations are often driven by unsatisfactory learning outcomes that typically appear through high failure rates, not least in undergraduate courses in Analysis. Institutions, and not least university mathematics teachers, have a tendency to expect that problems could be solved with broad, universal remedies, such as new lecturing styles ("flipped classroom" is currently a popular one) or the addition of support measures such as exercise clinics, remedial courses etc. These expectations, of course, appeal to those who wish to maintain a clear division of labour between mathematics teachers (who deliver the contents) and pedagogues (who advise on the form of delivery). The substantial advantage of the ATD approach is to show how didactical and mathematical praxeologies are intrinsically intertwined. Didactical tasks involve, among other things, to construct mathematical tasks for the students to work on. As mathematical tasks are at the basis of the mathematical praxeologies which can be developed, it is apparent that central features of the contents which can be delivered are constituted by the specific didactical praxis and theoretical assumptions taken by those who deliver (the teachers), and not just by the external forms in which delivery takes place.

The design of mathematical tasks for didactical use is thus a crucial didactical task that is frequently undervalued by university mathematics teachers who tend to select tasks for students from end-of-chapter collections, while they concentrate on the details of exposing theory and technology related to these types of tasks. In Analysis courses beyond Calculus, textbook tasks tend to be strictly theoretical, with little or no link to the praxis blocks learned in Calculus (and which much of the theory actually serves to describe and justify)—and, at the same time, fairly trivial (like verifying that a definition is satisfied by some example or applying a theorem to a simple special case). It is therefore a possible strategy for improving students' learning to design and experiment new tasks for students—i.e., exercises—that have more subtle goals than verifying the students' comprehension of a few lines in the text. A number of recent research studies in ATD are based on doing so, both for initial Analysis courses to facilitate transitions of type I (e.g. Gyöngyösi et al., 2011; Gravesen et al., 2017), and for more advanced courses where the main challenges relate to transitions of type II (e.g. Grønbæk & Winsløw, 2007). In all of these studies, explicit reference models for the specific mathematical contents—usually in terms of praxeological organisations—are crucial to the systematic identification of learning gaps, and subsequent design and experimentations of new student tasks.

4 Analysis and the Teaching Profession

Undergraduate courses in Analysis are not only taken by future mathematics researchers. In particular, the initial education of secondary level mathematics teachers often includes one or more such courses. More than 100 years ago, Felix Klein (1908) noted that the relevance of such an experience to secondary school teaching might not be evident to students as they have it, or when they become teachers. Indeed, there is necessarily many practical and theoretical components of such courses that do not appear in secondary level teaching. Klein's solution to this problem was to propose what we would now call "capstone lectures", in which the university mathematics contents were applied to endow students with what he called *an advanced perspective* on secondary level items such as numbers, functions, derivatives and so on. Analysis certainly offers important contributions here, particularly when it comes to exhibit and elucidate subtle features of the real number field (often represented informally as points on a "number line" at secondary level) as well as its roles in defining the most common functions and operations studied at secondary level (cf. Grønbæk & Winsløw, 2014). Designing and experimenting such "bridges" between undergraduate Analysis courses, and the mathematics to be taught in secondary school, is certainly an interesting and important aspect of Mathematical Analysis at university, with potential bearings and links to wider areas of research in Didactics of Mathematics—including the currently very active field related to the study and questioning of the knowledge held by and required for mathematics teachers. At the same time, the undergraduate courses in question need to be investigated and possibly redesigned with the needs of the teaching profession in mind.

Let us consider a concrete example, partially based on Kondratieva and Winsløw (2018). Students have encountered the notion of *angle* already in primary school, where it has two different meaning that are not always strictly separated: the "opening" between two-line segments, and a "measure" of the size of this region, usually strongly associated with material measurement using a protractor. The former is purely informal and linked to visual ostensives (circle segments drawn between the line segments). The latter comes, as all other measures, with a unit—at primary level, angles are invariably measured in "degrees", with 90° being a "right" angle and 360 making up a full circle (corresponding to the "outer" angle between two superposed line segments). This suffices to introduce, for instance, the basics of triangle geometry, including trigonometry based on computations with side lengths.

However, the relation between the measures of length (usually in metric units) and measures of angles (in the archaic degree system) remains somewhat mysterious. Then, usually in upper secondary school, a link appears with "radians", a new unit to measure angles. It is related to the (usually postulated) formula from primary school, which allows one to compute the length of a full circle in terms of the length of its diameter d, namely πd. The number π remains, in particular a complete mystery; defining it as the length of a circle with diameter 1 (in whatever unit one may like, as long as the same unit is applied to both diameter and circle) does not help much, as long as nothing is said about what the length of the circle itself could be. A possible informal explanation, occasionally given at secondary level (cf. e.g. Loeng, 2019), is linked to the thought experiment of "winding" a soft ruler (or "measurement band") around the circle, in "positive" direction ("against the clock"). Doing so with a circle of diameter 1 (units) would thus yield the number π. The new unit for angles is then based on doing this "winding" with a circle of radius 1 (diameter 2) and omitting the unit for length: the full circle—called the "unit circle"—then measures 2π. Curiously, the number π was already met in primary school, as the area of circle of diameter 2; in uses of the "area formula" πr^2, students have also become used to a postulated value of π, namely some "approximation" like 3.14. At the same time, to prepare the introduction of sine and cosine as functions defined on all of $\mathbb{R}$, one extends the possibility of "winding" to allow for "winding" around the circle an arbitrary number of times, and also to "wind" in "negative" direction, so that the new "angle measure" may in fact assume any "real" number value.

But even without these supplementary extensions, remains the question: how can one measure the length of a circle in the same units as a straight line? And the area of the circle in terms of the usual area units introduced for rectangles (in primary school), given that the circle cannot be divided into rectangles? It seems clear that teachers, whether in primary or in secondary school, need to know at least plausible answers to these questions, if they are teaching mathematics as more than natural history, i.e. as more than inexplicable facts of nature based on observation and more or less approximate measurement. It also explains why teachers at this level need to know something about analysis, and in particular about integrals in the sense of "infinite sums". Here the most basic idea is that of *length*: we associate length to any curve in the plane if the sum of lengths of cords obtained by joining any collection of points on the curve has an upper bound, or more informally, if the sum of cord lengths "converge" when adding still more points on the curve and joining them by cords. To *prove* that this is actually the case for a circle—in particular a unit circle—certainly goes beyond the normal secondary mathematics curriculum. In a rigorous course on vector calculus, it is the theorem that allows concluding that a circle is *rectifiable*, usually in terms of the differentiability of a parametrization. But the "approximation" of the circle by inscribed regular polygons is intuitively convincing. Then, the link with the area can also be established by intuitive means, since regular polygons are made up of disjoint triangles, with base on the polygon and height approximately the radius r of the circle; the sum of these areas, then, is approximately half the sum of the length of the bases (= the length of the circle) times the radius of the triangles (approximately, r), so that the area becomes half the length of the full circle times r. Of course, the "convergence" of the area of the inscribed polygon to the "area" of the full circles is also rather informal, and its formalization requires yet another piece of Analysis, namely a definition for the existence and value of area, and a criterion that permits to conclude in the case of a circle. Teachers should not only know the informal argument—along with ways it could be shared with students, and that need to be carefully designed and experimented. They need also to know what makes it different from a rigorous

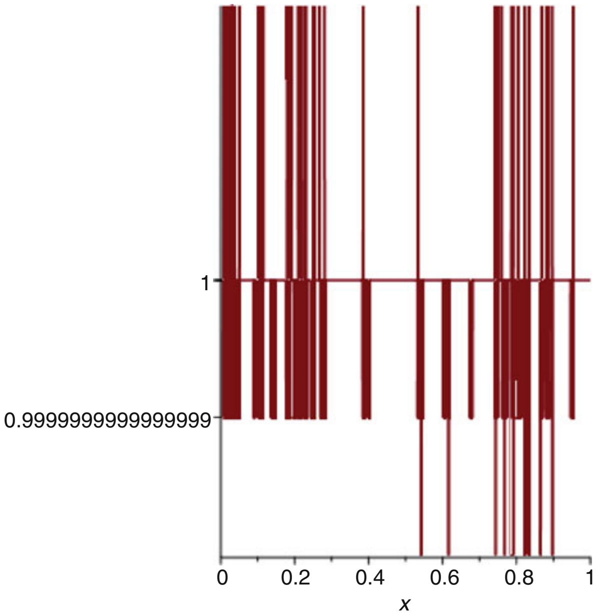

Fig. 1 Plot of the function $\cos^2 x + \sin^2 x$ produced in Maple 2017

mathematical proof—hence, they need to be exposed not only to relevant proofs, but also to activities that enable them to master the technicalities (concerning the real numbers, including completeness, and the ways these are built into vector analysis) which the informal argument hides as "evidences". To encounter the existence of simple non-rectifiable curves, such as the Koch curve, is probably a necessary step, and transpositions to secondary mathematics might even be imagined with full use of the digital tools that students at this level are usually only using on tame cases where intuition and tool always coincide.

As this example shows, Analysis does not only appear as a (more or less advanced) superstructure to the more frequently taught Calculus. Certain elements of Analysis—especially infinite series—appear as a source of modern mathematical foundations whenever the real numbers are involved, which include all contexts involving continuous measures, such as those found in primary school geometry. We currently know little about the actual or potential implications for teaching and mathematics teacher education. In schools, digital tools are increasingly used to carry out computations and visualisations related to functions, which are almost invariably assumed to be defined on the real number line (or part of it). This leads to surprises such as Fig. 1 (obtained with *Maple*, one of the most sophisticated CAS tools currently available). Secondary school teachers may need a specialized education on power series in relation to how such digital tools represent real numbers and functions, in order to be prepared to manage students' use of such tools.

5 Problems for Future Research

Based on the previous outline of existing research studies, we now come to more or less open problems which ATD researchers could turn their attention to in the future.

The first one concerns the *scale* of the studies undertaken. For material and institutional reasons that are quite general, studies so far conducted—for instance, to make visible and to theoretically delimit transitions of type I and II—seem to be solely based on case studies, usually done within a single course unit at the institution of the authors. Studies that encompass larger populations of students, as well as several institutions, need to be organized (and, in particular, funded) in order to make sense of these phenomena as more than institutional incidences, and to identify also the variations that exist among institutions, and their causes. This goes obviously not only to test the tools that have been developed to identify transitions but also for didactical designs—in particular, task designs—that have so far only been experimented in single institutions, often only in one occurrence of a single course. Certainly, successful adaptation of designs always requires careful attention to specific conditions in the institution, but on the other hand, carefully crafted task design—developed with explicit reference epistemological models, in view of overcoming specific transition problems—could also be expected to have some general form of “adaptability” to a variety of conditions. One could even say that if they do not, they have in principle no interest beyond development purposes in a single course context.

One specific obstacle to going beyond course units and more evidently single institutions is the almost total lack of comparative studies of curricula. While we often—also in this paper—postulate similarity or even a kind of isomorphism between course units (such as “first analysis courses”) or between larger chunks of course units (such as “Calculus courses”, “Real Analysis courses”), we need explicit reference models to declare and investigate the mathematical praxeologies which they actually expose students to—whether we use these models to analyse the official aims of course units, the way they are assessed, or the praxeologies actually observed to be developed by students. Such models are usually only developed at a very local scale and they are only applied to one institutional context. An ongoing research programme aims to carry out comparative analyses of the contents and construction of undergraduate mathematics programmes in Europe (Bosch et al., 2019). This should be followed by comparisons not only of individual Analysis courses in such programmes, but of the whole sequences of undergraduate courses on Calculus and Analysis. It would also be especially important, for such courses, to go beyond the context and rationales of undergraduate programmes in pure mathematics, especially to study the more elementary courses which are also taken by students in other programmes. Is Calculus at university level mainly maintained for service purposes? What are the effective role of Calculus courses in these non-mathematics study programmes? What mathematical needs—in particular, within Analysis—exist in other parts of these programmes, and are they satisfied by the courses offered?

Another largely unexplored challenge for the teaching of post-Calculus Analysis is the potential—but apparent rare use—of Computer Algebra Systems and other powerful digital tools, which have increasingly found their way into Calculus teaching at both secondary and tertiary levels (see e.g. Varsavsky, 2012). We have some small-scale experiments of didactical designs aiming at facilitating transition of type I, by integrating the use of simple instrumented techniques into students work with theoretical tasks (e.g., Gyöngyösi et al., 2011). To go beyond initial courses closely related to praxis blocks from Calculus, designs would likely need to be based on more thorough investigation of mathematicians' usage of tools to support theoretical research, where studies such as Bunt et al. (2013) point out gaps between both existing software and current uses in education on the one hand, and the practices of "expert problem solving". This field, indeed, represents a fascinating instance of the teaching-research nexus problem (Madsen & Winsløw, 2009; Winsløw, 2015).

The interaction between physics and mathematics continues to be strong and fruitful in scholarly institutions but seems to be more difficult and less well understood when it comes to both secondary and university level education (see e.g., Karam et al., 2019). The borderland between advanced Analysis and specific domains of physics (e.g., quantum mechanics and quantum field theory) currently begins to be investigated within ATD (ongoing thesis work by N. Lombard at the University of Montpellier). The investigation and crafting of semi-advanced courses on Analysis that cater to specific future professional needs, for instance in teaching and engineering, is only an emergent topic in mathematics education research (see e.g., Grønbæk & Winsløw, 2014; Wasserman et al., 2017). We have already touched upon the case of teacher education, which is probably closer the current concerns of ATD based researchers, while the links established with Analysis courses beyond Calculus are still few and timid. Again, an anthropological perspective would need to connect praxeological needs in professional practices with what is currently or potentially offered in such courses. In order to identify the extent to which Analysis courses contribute, or could contribute, to professional education outside the strict domain of mathematics or mathematics teaching, it is also relevant to investigate how more or less advanced mathematical ideas appear in the teaching of engineers, such as the recent studies of Hochmuth and Peters (2018) focusing on mathematical praxeologies in Signal Theory.

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Using Tools from ATD to Analyse the Use of Mathematics in Engineering Tasks: Some Cases Involving Integrals

Alejandro S. González-Martín

1 Introduction

In recent years, research on mathematics education has become increasingly focused on the teaching and learning of university-level mathematics by non-specialists (González-Martín et al., 2021). Given the large number of students worldwide who take university mathematics courses as a programme prerequisite, as compared with the number of students enrolled in specialist mathematics programmes, it is worthwhile questioning whether these courses could better prepare students for the professional workplace. This issue becomes even more important if we consider the dropout rates of students who fail calculus (Rasmussen & Ellis, 2013), which are particularly high in engineering programmes (Faulkner et al., 2019). Literature examining the teaching and learning of mathematics in engineering programmes reports that students encounter many difficulties with mathematics in their first years of study; moreover, it has been reported that "poor mathematics skills are a major obstacle to completing [. . .] engineering programs" (Fadali et al., 2000, p. S2D-19). One specific challenge for these students relates to their difficulty connecting previously learned mathematical content with the content of the professional engineering courses (González-Martín & Hernandes-Gomes, 2018), which may lead them to view mathematics courses as irrelevant to their professional needs.[1] Faced with this scenario, researchers and educators alike seem to agree that traditional

A longer paper with details of the results presented here is available in González-Martín (2021).

[1] In the words of Flegg et al. (2011), "without an explicit connection between theory and practice, the mathematical content of engineering programs may not be seen by students as relevant" (p. 718).

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calculus content and teaching methods are not meeting current professional needs and do not allow students to adequately develop the mathematical skills they require for the workplace (González-Martín & Hernandes-Gomes, 2019). Loch and Lamborn (2016, p. 30) stated that "mathematics is often taught in a 'mathematical' way, with a focus on mathematical concepts and understanding rather than applications. The applications are covered in later engineering studies."

Noss (2002) and Kent and Noss (2003) conducted pioneering work on the mathematical needs of engineers. Using interviews and a questionnaire survey, they concluded that, in general, structural engineers do not use any advanced mathematics in their workplace (Noss, 2002) and that the majority of civil engineers employ basic arithmetic on a daily basis (Kent & Noss, 2003). Their study reveals that the content of university mathematics is "transformed into something else" (Kent & Noss, 2003, p. 54) and that only traces of it can be detected in engineers' actual professional activity. More recently, the anthropological theory of the didactic (ATD) has been used by researchers to provide further insight into this phenomenon. Romo-Vázquez (2009), after having analysed three engineering capstone projects, found "a weak presence of the Teaching of Mathematics institution: mathematics teachers do not propose projects and are not used as resource persons in developing these [capstone projects]" (p. 288, [translation]). Moreover, it is evident that mathematics is used in different ways in these projects: "for a mathematician whose research is not already related to the type of issues tackled in these projects, the investment is no doubt heavy." (p. 288 [translation]) These observations led Romo-Vasquez to conclude that "the invisibility of mathematics is, therefore, a result that is broadly confirmed" (p. 289, [translation]). With that in mind, she identified two types of mathematical needs:

- "Elementary" needs, which in general call for secondary mathematics, at least in spirit: working on formulae, analysing and using functional dependences, finding orders of magnitude, performing calculations, assessing intervals of possible values for given measurements, calculating simple integrals, solving simple linear differential equations, and using trigonometry.
- More advanced mathematics: Laplace transform, dimensional analysis, and finite elements. (p. 289, [translation])

Romo-Vázquez (2009) adds that elementary needs lead to mathematical techniques being adapted to a given engineering task, which requires engineers to interpret the meaning of the objects at hand. As this meaning is never strictly mathematical in this context, mathematics and engineering notions become intertwined. Regarding advanced mathematics, the use of software facilitates mathematical work but also modifies it (e.g., allowing for explorations), making the interpretation of results essential.

More recently, Quéré (2019) conducted an online survey which was completed by 261 professionally active French engineers. Of this large sample, only 24% stated that their university mathematical training was adequate for their current professional requirements, whereas 52% reported that this training was ill adapted to their needs. Regarding their real need for university mathematics in their workplace, only

129 (49.4%) of the participants answered positively, with only 43% of them (21.24% of the entire sample) stating that they required knowledge of content from calculus/ analysis courses.

In order to contribute to this area of research, we previously analysed the use of mathematics in teaching praxeologies employed by teachers with different academic and professional backgrounds who teach in the same engineering programme (González-Martín & Hernandes-Gomes, 2020). Our results indicate that the propensity for rigour can be stronger in teachers with a mathematical background, while teachers who have worked as engineers may be more likely to incorporate this professional experience into their teaching practices. Our current research program investigates how calculus notions are used in engineering courses; we seek to identify possible ruptures between how notions are introduced in calculus courses and how they are later used in professional engineering courses. This type of research can help pinpoint the actual needs of engineers and reveal how they use mathematics both in the workplace and in their academic programmes, sparking a debate over the content of mathematics courses for engineers.

In the first stage of our research programme, we are interested in better understanding how single-valued integrals are used in engineering courses and whether this represents a disconnection with their use and content in calculus courses. At this stage we are analysing engineering textbooks, working under the principle that most tertiary instructors use textbooks as a major resource in planning their curriculum (e.g., Mesa & Griffiths, 2012). At this time, our analyses focus on a Strength of Materials course in a civil engineering programme and a General and Experimental Physics course in an electrical engineering programme (González-Martín & Hernandes-Gomes, 2019). We have developed detailed analyses examining the way the books introduce bending moments (González-Martín & Hernandes Gomes, 2017) and first moments of an area (González-Martín & Hernandes-Gomes, 2018). We have also provided an overview of both textbooks' use of integrals (González-Martín & Hernandes-Gomes, 2019). We believe this type of research is necessary, since, as Castela (2016) puts it:

> . . .mathematicians need to take some distance with their own culture [. . .]. They have to reconsider the following questions: which mathematical praxeologies are useful for such engineering or professional domains? What needs would be satisfied? Which discourse makes the mathematical technique intelligible? This is actually an epistemological investigation that we consider as a prerequisite to the design of mathematics syllabi for professional training programs. (pp. 424–425)

2 Theoretical Framework

We will now provide a brief description of the theoretical tools used in our research (for more details, see González-Martín & Hernandes Gomes, 2017, 2018, 2019). We use tools from ATD (Chevallard, 1999), which considers human activities as institutionally situated. In this sense, knowledge about these activities and their *raison d'être* are also institutionally situated. In particular, ATD offers a general epistemological model of mathematical knowledge, where mathematics is seen as a human activity through which various types of problems are studied.

A key element in ATD is the notion of praxeology (or, in our case, mathematical organisation or mathematical praxeology—MO hereinafter), which is formed by a quadruplet $[T/\tau/\theta/\Theta]$ consisting of a type of task T to perform, a technique τ which allows the task to be completed, a rationale (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. The first two elements $[T/\tau]$ are the practical block (or know-how), whereas the knowledge block $[\theta/\Theta]$ describes, explains, and justifies what is done. These two blocks are important elements of the ATD model of mathematical activity that can be used to describe mathematical knowledge. Furthermore, ATD distinguishes different types of MO: punctual, which are associated with a specific type of task; local, which integrate multiple punctual MOs that can be explained using the same technological discourse; and regional, which integrate local MOs that accept the same theoretical discourse.

Praxeologies, like knowledge in general, may move from the institution in which they originate to other institutions where they are used in different ways. This is the case, for example, with mathematical notions that are used to solve engineering problems. In this instance, the concerned praxeologies are subject to *transposition* effects (Castela, 2016; Castela & Romo Vázquez, 2011; Chevallard, 1999): this means that when a piece of knowledge that is produced within one institution moves to and is used in another institution, it may undergo certain transformations. In this boundary-crossing process, some (or all) elements of the original praxeology may evolve. In the case of a praxeology crossing the boundary from one research institution to another in order to be taught or used, Castela (2016) proposes the following model (Fig. 1).

In this model, I_r represents a research institution, while the arrow between the praxeologies represents the institutional processes, which have an epistemological and a social dimension. The tasks of type T can be addressed by an institution that has only a pragmatic relationship to the praxeology (I_p). The asterisks indicate that every component of the original praxeology may evolve. I^*_r, which is created and controlled by I_r and I_p, operates these evolutions or transformations. Finally, θ^p represents a practical technology that is developed and acknowledged by I_p on specific empirical bases, "possibly sustained by a [rationale] of second level" (p. 423).

Although our work does not examine *the same* praxeologies in different institutions, this model informs our work, in that we are examining objects (integrals) that originate in one institution (calculus courses) which are then used to solve tasks in

Fig. 1 Transposition model proposed by Castela (2016, p. 422)

another institution (professional engineering courses). Therefore, it is important to analyse the types of tasks and techniques as well as the rationales and theories employed. To that end, our research identifies specific local MOs present in professional engineering courses; we analyse how calculus notions are used (practical block) and whether this use relates to the way these notions are usually introduced in calculus courses (knowledge block).

3 Methodology

We first analyse calculus textbooks to identify how the content on integrals is structured and to pinpoint the tasks where integrals are used along with the rationales given. We then analyse engineering textbooks to identify how integrals are used in introducing specific engineering notions. For the case of bending moments, we analysed the book by Beer et al. (2012). With respect to the content presented in this workshop, we analysed the praxeologies employed in introducing techniques for sketching bending-moment diagrams. We analysed the theoretical sections to identify (a) how notions are defined and how properties are justified, (b) the tasks proposed to justify and use the new content, and (c) the techniques that are employed to solve these tasks. For more details, see González-Martín and Hernandes Gomes (2017) and González-Martín (2021).

4 Some Results

Our analysis of the Mechanics of Materials book used in a Strength of Materials course show that, with regard to first moments of an area (Q), moments of inertia (I), polar moments of inertia (J), bending moments (M) and centroids (C), integrals are mostly used in the theoretical sections to introduce and define notions proper to engineering, as well as to deduce certain properties (González-Martín & Hernandes-Gomes, 2019). That said, students can turn to the tables and formulae provided to find values and solve most of the tasks. The actual technique does not rely on integrals, and it is only in the explanation of the technique (technology) that integrals appear. In many cases, the explicit justifications rely on a professional discourse that is not (at least for the student) explicitly related to explanations and properties that would appear in a calculus course.

For example, our analysis of the way integrals are used to define bending moments for beams in a Strength of Materials textbook reveals different uses of the "integral" object (González-Martín & Hernandes Gomes, 2017). Although bending moments are defined as an integral, the tasks, techniques and justifications used in calculus courses are very different from the ones presented in this professional engineering course; this may result in students not recognising "the same" object in two different courses, which means they may question the relevance of

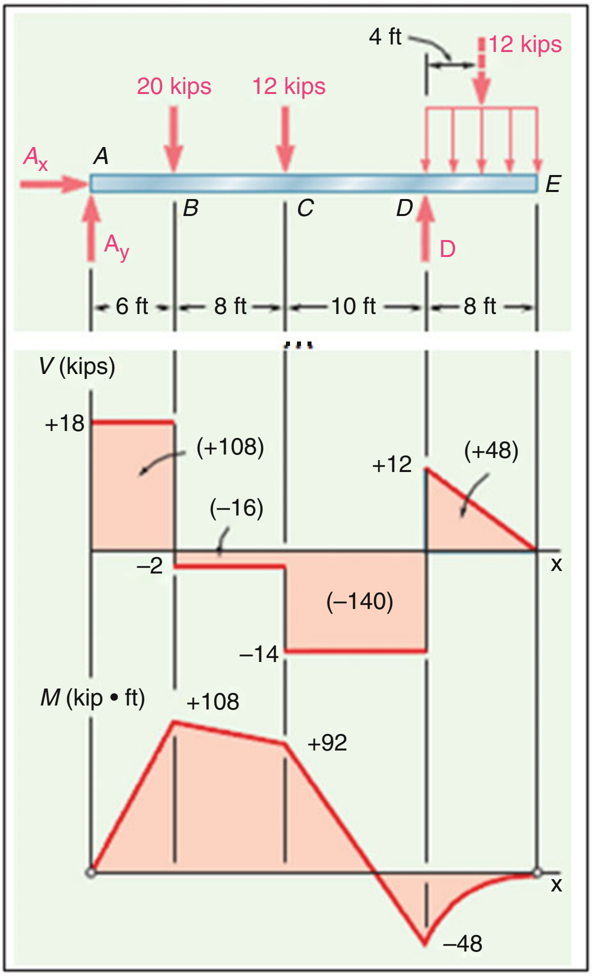

Fig. 2 Example of task involving bending moments (González-Martín & Hernandes Gomes, 2017, p. 2078, adapted from Beer et al., 2012)

integration techniques that are not used in tasks concerning bending moments. For instance, Fig. 2 shows a typical task: sketching the bending moment diagram (lower graph, M) referring to the distribution forces on a beam (upper diagram). The technique to solve the task can be summarised as follows:

1. Using the diagram showing the distribution of loads (upper diagram in Fig. 2), identify the key points (A, B, C, D, E) where loads are applied or distributed.
2. Using these points and some arithmetic formulae (based on the principle that the addition of forces equals zero), calculate the values of the shear force at these points.
3. Using these values, sketch the graph of the shear force (middle graph in Fig. 2).

4. By calculating areas in that graph and using arithmetic formulae, find the values of M at A, B, C, D, and E.
5. Sketch the graph of M (lower graph in Fig. 2).

The rationale for Step 5 is given as follows:

> Note that [in a solved example] the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve a cubic (third degree). The shear and bending-moment curves are always one and two degrees higher than the load curve, respectively. With this in mind, the shear and bending-moment diagrams can be drawn without actually determining the functions $V(x)$ and $M(x)$. (Beer et al., 2012, p. 362)

We can therefore see that the technique to solve this task relies on basic formulae and on geometric considerations. As Kent and Noss (2003) put it, only traces of the calculus content can be detected in the actual technique. Our results concerning first moments of an area (González-Martín & Hernandes-Gomes, 2018) are similar: the tasks proposed to students require mostly the use of geometric considerations and of ready-to-use formulae.

Our analysis of the General and Experimental Physics textbook used in an electrical engineering course (Halliday et al., 2014) focused on the content related to electromagnetism, (Chaps. 21–24, for a total of 108 pages; see González-Martín & Hernandes-Gomes, 2019). Once again, our results echo previous findings. Although we note the use of more challenging functions, in this book integrals are also used mostly to introduce and define notions proper to electrical engineering, and it is rather the interpretation of an integral that allows for a proper analysis of the phenomena under study. As with the previous book, most techniques call for the use of given properties or tables, which means that tasks can be solved without students being aware of the use of integrals. In most cases where students need to calculate an integral, immediate integration techniques are sufficient.

5 Final Comments

During the workshop, the discussion centred around the following main questions:

- How are integrals used to introduce or define bending moments? What type of rationale is used to justify certain properties or techniques?
- What tasks are presented that use integrals, and what are the techniques available to solve them? Are integrals explicitly used, or they are implicitly involved in the rationale that justifies the techniques?
- What parts of calculus course content are (and are not) used? Can recommendations be made regarding certain topics that should (or should not) be taught in calculus courses?
- Is it possible to suggest some tasks that could be used in a calculus course for engineers to better motivate the introduction of integrals?

The discussion led the participants to agree with the points presented earlier in this chapter. They also questioned the amount of time traditionally spent in calculus courses on techniques used for calculating the integral of very complex functions.

The analyses described in this chapter are intended to illuminate how mathematics is actually used in professional engineering practices. This in turn may help determine which content should be emphasised in calculus courses (such as the interpretation of integrals) and which content may be less important (such as the integration of very complex functions).

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A Workshop on the Epistemology and Didactics of Mathematical Structuralism

Thomas Hausberger

1 Epistemology of Mathematical Structuralism

Mathematical structuralism takes its roots in the re-foundation of algebra operated by German algebraists at the beginning of the twentieth century. Workshop participants were thus given a few historical landmarks on the emergence of the group, ring and field structures, following Wussing (2007) and Corry (1996). Synthetic accounts may be found in (Hausberger, 2018b).

The idea of algebraic structures as a unifying principle is due to Noether: in the 1920s, she got Abstract Algebra away from thinking about operations on elements (such as addition or multiplication in groups or rings) but described structures in terms of selected subsets (such as normal subgroups of groups or ideals in Ring Theory) and homomorphisms. Noether and her school thus changed the way theorems were proved in algebra, focusing on general proofs that limit the calculations and put to the fore the "most general and fundamental" (hence simpler, according to structuralist views) concepts. This mathematical re-foundation of algebra also paved the way for unprecedented mathematical constructs, such as noetherian rings.

The structuralist method is well described in the Bourbaki (1950) Manifesto, entitled "the architecture of mathematics", written by a group of French mathematicians who were the great promoters of structuralist thinking. Indeed, Bourbaki set out to apply the method developed by the German algebraists to all fields of mathematics. The first part of the workshop was dedicated to the discussion of excerpts of the Manifesto. Bourbaki describes the structuralist use of the axiomatic method both as a method of exposition of mathematical theories and a method of discovery of new results (thus a heuristic). He draws conclusions on the impact of

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structuralism on mathematical activity and the organization of mathematical theories.

The starting point of the workshop was a couple of remarks that guided the reading of the Manifesto. Firstly, Bourbaki is aiming at the “profound intelligibility” of mathematics through the axiomatic method which “teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light”. This suggests that *didactic principles* are governing Bourbaki’s reconstruction of mathematical knowledge. Secondly, Bourbaki sees structures as “tools for the mathematician”; he emphasizes the “standardization of mathematical techniques” by means of the axiomatic method which is “nothing but the Taylor system for mathematics”. This connects to the *praxeological point of view* of the Anthropological Theory of Didactic (ATD) which sees mathematical activity as the development of combinations of praxis and logos (praxeologies, Chevallard, 2006). The following questions were thus discussed:

1. Which statements of the Bourbaki discourse express didactic concerns? To what extent does mathematical structuralism rely (or not) on didactic principles as a method of exposition and a method of discovery? How to phrase these principles in terms of didactics of mathematics?
2. Using the vocabulary of ATD, how to describe the impact of structuralism on mathematical praxeologies, in particular the praxis and logos blocks, the interrelations of both, on mathematical organizations in general?

Participants underlined that the writing of the Manifesto is itself a didactic gesture: in order to disseminate mathematics (Bourbaki wrote a treatise in several volumes: *Elements de mathématiques*), one must have a clear vision of what mathematics is. As pointed out in the Manifesto, “[it is] out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves cannot conceive in its totality”. This quote refers to a stage of development of mathematics in which diversity hindered the production, communication, and dissemination of mathematics. This is where the structuralist method comes into play as a didactic method to promote understanding and sense-making through “separating out the principal mainsprings of the arguments; then taking each of them separately and formulating it in abstract forms, to develop the consequences which follow from it alone”. Bourbaki also refers to an “economy of thought”, which may be related to the economy of didactic memory. Nevertheless, setting out abstraction and generality as principles was also debated among workshop participants (as among mathematicians, for instance Mandelbrot who discussed relationships between explanation, generality, and abstraction). At first sight, a few educators—perhaps those who least share the Bourbaki culture—exclaimed: “Don’t do that!” (the use of such principles in classrooms).

In fact, Bourbaki describes and justifies in the Manifesto techniques to solve problems and to write and communicate mathematical theories. In other words, all the elements of mathematical research praxeologies may be found. Readers may draw a list of structuralist techniques to solve problems: “to recognize among the elements relations which satisfy axioms of known types", "to apply the arsenal of general theorems which belong to the structure of that type", "to orient the intuitive course of one's thought [according to the structural insights]"; and structuralist techniques for the exposition of mathematics: "to look for deep-lying reasons", "to find common ideas of theories and bring them forward", "to separate out the mainsprings of its arguments", "to inquire how these different components influence each-other", "to set up the axiomatic theory of a given structure".

The phenomena of unification of punctual praxeologies into regional and global praxeologies was related by several participants to the Bourbaki discourse of unification of mathematics under the axiomatic method. The holistic vision of Bourbaki was questioned as an epistemological point of view that may be relativised (it applies to algebra but maybe not with the same pertinence to every mathematical domain) and as a socio-cultural norm. Participants pointed out that distinctions should be made between mathematical praxeologies in teaching institutions (including universities), in research institutions (also including universities), and in different fields or domains, considering also variations between societies and civilisations (in reference to the scale of levels of didactic codeterminacy). Nevertheless, it was argued that some common points (consistent with the Bourbaki discourse) may be found privileging the "most general formulations" (of theorems, definitions, etc.); the role played by Theory to unify mathematical sectors and domains (regional praxeologies); a further Metatheory to unify the discipline in a global praxeology. Other participants underlined that the structuralist method induced changes at the discipline level, for instance the creation of new domains and sectors (general set-theoretic topology, algebraic topology, etc.). It was accompanied by important modifications in the praxis (and of course logos), as new questions appeared and new ways of proving results (new techniques). At this stage of the workshop, the need of new tools emerged to make such statements more precise.

2 Modeling Mathematical Structuralism Within ATD

The notion of *structuralist praxeology* (Hausberger, 2018a) aims at modeling within ATD the epistemological ideas developed above. Structuralist thinking is characterised by reasoning in terms of classes of objects, relationships between these classes and stability of properties under operations on structures. The application of the structuralist method relies on a *dialectic between the particular and the general*, or in other words between *objects and structures*. The questions and problems are raised to a higher level of generality in order to apply structuralist concepts (e.g., ideal, principal ideal domain, etc.) and tools (e.g., isomorphism theorems, structure theorems, combinatorial of structures, etc.) according to the moto "generalizing is simplifying". In other words, the "structuralist methodology aims at replacing a praxeology [T, ?, ?, $\Theta_{\text{particular}}$] by a structuralist praxeology [T^g, τ, θ, Θ_{general}], where T^g is a generalization of T that allows the use of structuralist techniques" (loc. cit. p. 83). In order to illustrate the theoretical construct of structuralist praxeology, workshop participants were presented the example of the thread on decimal numbers from the online forum mathematiques.net (loc. cit. pp. 83–87). To prove that the ring **D** of decimals is a principal ideal domain, the forum students searched for a proof of the general statement that any subring of **Q** is principal, then investigated whether principality was transferred from a ring to its subrings. The study of these dialogues emphasises the role of the dialectic of objects and structures in the development of structuralist praxeologies: structures are applied as a generalizing-simplifying viewpoint in order to demonstrate properties on mathematical objects and, conversely, a semantic control of the formal general statements on structures is exercised by putting them to the test of known examples (**Q**[x] is principal but **Z**[X] is not).

Inspired by Winsløw's (2008) praxeological formalisation of the concrete to abstract transition in analysis (from calculus to more theoretical tasks involving continuity and differentiability of functions as well as the topology of real numbers), the author proposed a model for the epistemological transition to Abstract Algebra, in two phases: the first phase is concerned with the transition from T to T^{g} described above and leads to the construction of a structuralist praxeology as a fertile strategy to prove properties of concrete objects; the second phase builds on structuralist praxeologies previously developed in order to introduce more abstract and theoretical types of tasks that only consider classes of objects with their structural properties (e.g.: show that a Noetherian integral domain such that every maximal ideal is principal is a principal ideal domain).

A new example related to arithmetic and abelian Group Theory (GT) was then introduced at the workshop. Indeed, Bourbaki cautiously explained that his account of structuralism is a schematic and idealized sketch, and so is the model described above. Its pertinence in the elucidation of teaching-learning phenomena related to GT needs be investigated through concrete case studies. The case of GT was also chosen to relate to other didactic studies. For instance, Bosch et al. (2018) focused "not only on the official raison d'être of GT within university teaching, but also on different possible alternative ones that could motivate or impel the use of GT to solve problematic questions". In particular, they looked for external problems (that is, external to GT) that could lead to the reproduction of a substantial part of GT as a means to ascribe some rational to it. Precisely, they argued that a counting problem (such as that of symmetries of a square) may be a suitable candidate for a reconstruction of elementary GT. This choice is justified by the links between GT and the notions of symmetry and invariant in the historical development of GT and by the role played by Lagrange's theorem as a tool to solve the problem. Nevertheless, it also raises epistemological issues: for instance, "is it substantial enough to motivate the study of the isomorphism theorems"?

The problem discussed at the workshop is classical, external to GT, and may be found in standard textbooks (Perrin, 1996): determine the set of primes p such that -1 is a quadratic residue (congruent to a perfect square) modulo p. Such questions follow from the work of number theorists of the 17th and 18th centuries (Fermat, Euler, Lagrange, Legendre) and have been given a first systematic treatment in Gauss's Disquisitiones Arithmeticae (1801). The answer to this question is called "first supplement to the law of quadratic reciprocity" [$p = 2$ or $p \equiv 1 \pmod 4$]. The ambition is not to cover a substantial part of GT but simply illustrate the structuralist methodology.

The problem may be considered inside arithmetic as a particular theory (in the sense of Bourbaki) in dialectical relationship with GT [through the group $(\mathbf{Z}/p\mathbf{Z})^*$], Field Theory ($\mathbf{F}_p$), or even Ring Theory ($\mathbf{Z}[i]$, $\mathbf{F}_p[X]$) as general structures. Students may stay in arithmetic (as a theory in the sense of ATD), use Fermat's little theorem to prove that $(-1)^{(p-1)/2} \equiv 1 \pmod p$ and deduce the necessity of the condition $p \equiv 1 \pmod 4$. The converse implication is less straightforward (the reader may have a try or look up Gauss's DA art 111) but gains much clarity when translated into structural terms. Indeed, the set Q of quadratic residues modulo p make up a subgroup of index 2 in $\mathbf{F}_p{}^*$ (as the image of the homomorphism $x \mapsto x^2$). It thus contains $(p-1)/2$ elements (and we may recover the preceding result by means of Lagrange's theorem). Moreover, the equation $x^{(p-1)/2} = 1$ in $\mathbf{F}_p$ admits at most $(p-1)/2$ solutions since $\mathbf{F}_p$ is a field. This proves that the set of solutions is exactly Q and the result follows. Another proof uses the argument that a group of even order always possesses an element of order 2 (a well-known result in elementary GT) to conclude that -1 belongs to Q [under the hypothesis $p \equiv 1 \pmod 4$], since it is the unique element of order 2. The problem may thus be related to standard praxeologies from GT (show that a subset is a group, determine its order and use known results on orders of elements) or mixed arguments using a property of polynomials defined over a field.

In this example, the type of task T (find conditions for the existence of a solution of a congruence) is handled with respect to the structural properties of the given congruence equation, and thus gives birth to the structuralist types of tasks (which are not direct generalizations of T) once structures are identified. The first isomorphism theorem, Lagrange's theorem and other general results about groups play the role of technologies. Theories of structures (GT, abelian GT, FT, RT) orient the work and provide the (formalized part of the) theory in the sense of ATD.

3 The Thread on (Z/32Z)* and Abelian Group Theory

Although many mathematical problems from GT may be stated independently on GT as we have just seen, the concepts of GT often offer an adequate framework to state GT problems in a synthetic meaningful way, as well as inspiring new problems as further natural developments. Looking up at lists of problems collected and proposed by mathematicians to students, for instance https://yutsumura.com/welcome-to-problems-in-mathematics/, it is striking that the structuralist scope is put in the fore to give a title to the problems (Prove a Group is Abelian if $(ab)^2 = a^2b^2$; Quotient Group of Abelian Group is Abelian; A Group is Abelian if and only if Squaring is a Group Homomorphism). Therefore, the success of students to solve these problems very much depend on their ability to connect the statements to the core structuralist praxeologies that they previously met, in other words their structuralist praxeological equipment and their understanding of the structuralist methodology.

For the last part of the workshop, participants focused on a problem that was submitted by a student (ianchenmu) to the Math Help Board (MHB) forum. MHB presents itself as an "online community that gives free mathematics help any time of the day about any problem, no matter what the level". As stated by ianchenmu, "the question is to identify isomorphism type for each proper subgroup of $(\mathbf{Z}/32\mathbf{Z})^{\times}$". The student wonders about the meaning of "isomorphism type", guesses that isomorphisms need to be found between each such subgroup and "another group" and questions about the method to proceed. A math scholar and MHB moderator (Klaas_van_Aarsen) helps ianchenmu to advance through the task, and another student (jakncoke) joins in to share his views. The thread may be accessed via this link[1] and a transcript of the collective study process was provided at the workshop.

The study process may be modeled by a study and research path (SRP; Chevallard, 2006) as the author did for the thread on decimal numbers (Hausberger, 2018a; 2019). The chronogenesis is governed by a dialectic between questions and answers while the mesogenesis involves a dialectic of medias and milieus (Chevallard, 2009). Special attention should be paid to the mathematical praxeologies that are developed through the SRP, in particular structuralist praxeologies that, as we have seen, usually appear through the dialectic of objects and structures.

The following questions were proposed for discussion at the workshop: (1) Do you have any idea about the historical origin of such a problem? To which sector and theme of the Abstract Algebra curriculum would you connect this problem to? (2) What are the main praxeologies developed during the study process online? (3) Which structuralist aspects can you identify? What are the objects and structures involved? Can you observe a dialectic between objects and structures? (4) What is your evaluation of the dialectic of medias and milieus, and the global vitality of the study process? (5) What are, according to you, the main conditions that hinder/foster the development of structuralist praxeologies (the ecological question)? If you were moderating the forum, what strategy would you adopt to "link problem solving and learning content" and thus address the "challenges of self-sustained study and research processes" (Bosch & Winsløw, 2016)? What would you retain from Bourbaki's didactic principles? Answers or partial answers to these questions are proposed and discussed in the sequel.

Considering the formulation of the problem by ianchenmu, two types of tasks emerge: determine the lattice of subgroups of a given finite abelian group G (T_1), determine the isomorphism type (or structure) of a given finite abelian group (T_2). The type of tasks T_2 must be understood in relation to the *fundamental theorem of finite abelian groups* (FTAG), which clarifies the notion of "type": every finite abelian group can be expressed as the direct sum of cyclic subgroups of prime-

[1] https://mathhelpboards.com/linear-abstract-algebra-14/identify-isomorphism-type-each-proper-subgroup-z-32z-3585.html

power order. According to Wussing (2007), the FTAG, was proven by Kronecker in 1870, using a group-theoretic proof, though without stating it in group-theoretic terms. This generalised an earlier result of Gauss from Disquitiones Arithmeticae (1801), which classified quadratic forms. The theorem was stated and proved in the language of groups by Frobenius and Stickelberger in 1878.

The type of tasks T_1 and type of group $G = (\mathbf{Z}/n\mathbf{Z})^{\times}$ both take roots in Galois Theory (GT). Indeed, such groups are the Galois groups of cyclotomic field extensions (generated over $\mathbf{Q}$ by a primitive n-th root of unity), and GT states a correspondence between the lattice of field extensions and the lattice of subgroups (the latter being easier to determine). Finally, the isomorphism types of $(\mathbf{Z}/n\mathbf{Z})^{\times}$ are known: by the Chinese remainder theorem, it is enough to determine the type in the case n is a primer power p^r; moreover, $(\mathbf{Z}/p^r\,\mathbf{Z})^{\times}$ is cyclic except in the case $p = 2$ and $r \geq 2$ in which $(\mathbf{Z}/2^r\,\mathbf{Z})^{\times} \simeq \mathbf{Z}/2^{r-2}\,\mathbf{Z} \times \mathbf{Z}/\mathbf{2}\,\mathbf{Z}$. The parameters p and r (= 5) are thus important didactic variables.

There are introductory as well as advanced textbooks on GT, covering GT or Abstract Algebra in general. The lattice of subgroups is often not a topic covered by elementary textbooks, the classification of groups according to isomorphism types being the culmination of the course. In Dummit and Foote (2003), a more advanced textbook, the lattice of subgroups appears as a topic of Chap. 2 "subgroups" (the theme) within Part I "Group Theory" (sector). The FTAG (generalised to finitely generated abelian groups) and the classification problem are presented in Chap. 5 on "direct and semidirect products and abelian groups".

Several techniques are developed on the forum to solve the main types of tasks T_1 and T_2. Regarding the former: $\tau_{1,\text{Klass}}$: determine cyclic subgroups; if g does not generate G, add other elements; $\tau_{1,\text{ianchenmu}}$: compute $<g>$ for all g in G ["it's so much work"];$\tau_{1a,\text{Jakncoke}}$: if $G = \mathbf{Z}/n\mathbf{Z}$, use the fundamental theorem of cyclic groups; $\tau_{1b,\text{Jakncoke}}$: determine the possible orders of subgroups thanks to Lagrange's theorem and the possible isomorphism types thanks to the FTAG. Then construct subgroups by looking at orders of elements and then *combining elements by means of direct products* [erroneous part of the technique]. The technology includes the definition of a cyclic group, Lagrange's theorem and the fundamental theorems (used implicitly by Jakncoke). Regarding the latter: $\tau_{2a,\text{Klass}}$: if G is cyclic of order n, then the isomorphism type is that of $\mathbf{Z}/n\mathbf{Z}$; $\tau_{2b,\text{Klass}}$: give all possible isomorphism types thanks to the FTAG and conclude by examining orders of elements; $\tau_{2,\text{ianchenmu}}$: if $G = <a, b, c>$ then the isomorphism type is that of the direct product of cyclic groups of orders given by those of the three generators [erroneous technique]. The technology here includes the definitions of the isomorphism type and cartesian product, and the FTAG. In fact, three related auxiliary types of tasks are discussed on the forum: T_3: determine the order of an element g in G; T_4: determine the list of elements of an abelian group $G = <a, b>$; T_5: show that a given group of order 4 is isomorphic to the Klein group V_4.

Several structuralist aspects of the problem may be pointed out: (1) at the level of mathematical concepts, the problem involves: the mathematical structures of group and the partial order $<$ on subgroups; the notion of isomorphism type (abstract structure of a group); the lattice of subgroups that "encodes" part of the group-theoretic information. (2) at the level of methods, the following structuralist principles are encountered: thinking in terms of generators (and relations); thinking up to isomorphism (isomorphisms preserve "structural information"); to decompose into simple components (structure theorems); to combine simple components (direct product).

Let us now focus on the dialectic of objets and structures. The main objects met during the study process are the groups $\mathbf{Z}/32\mathbf{Z}$ and $(\mathbf{Z}/32\mathbf{Z})^{\times}$, and instances of products of cyclic groups (e.g. V_4). The main structures are $<a, b>$ (abstract group on generators) and $\prod \mathbf{Z}/p_i^{r_i}\mathbf{Z}$(FTAG), which serve as abstract models. The dialectical aspects may be described as follows: the structures serve as tools to determine subgroups of $(\mathbf{Z}/32\mathbf{Z})^{\times}$ (by means of generators) and their isomorphism types ($\tau_{2,\text{ianchenmu}}$; the fundamental theorem as a means to predict possible models). Respectively, further elements of the logos on structures emerged (but only implicity) to fulfill technical needs: the implicit erroneous formulas $<a, b> = <a>\cup<b>$ and $<a, b, c> \simeq \mathbf{Z}/o(a)\,\mathbf{Z} \times \mathbf{Z}/o(b)\mathbf{Z} \times \mathbf{Z}/o(c)\,\mathbf{Z}$, which were refuted during the study process. Unfortunately, conditions for such a isomorphism to hold (in the simplest case of two elements) were not discussed, as well as more advanced structuralist questions such as the behavior of the lattice of subgroups under direct product of groups.

Let us now conclude on the ecology of structuralist praxeologies. Our analysis shows that the study process evolved towards more and more basic praxeologies (T_4, T_5) due to a lack in the praxeological equipment of ianchenmu that hindered the possibility to deal with the main tasks. Moreover, the moderator only focused on ianchenmu and didn't help jakncoke develop more advanced praxeologies. The dialectic of medias and milieus is very limited (very few elements of the medias were brought up in the milieu and no media on lattice of subgroups was identified): one hypothesis is that ianchenmu lacked skills to identify pertinent sources and therefore relied on Klauss who played the role of teacher. The dialectic of objects and structures is also limited: structures were applied as tools, but techniques did not lead to further theoretical developments (new general results about structures to test on known examples). The two types of tasks T_1 and T_2 were not related as they could (role of information on the structure of G to construct its lattice of subgroups), and the structuralist methods were not supported by structuralist insights as a metadiscourse (a technology): e.g. the notion of isomorphism type was not clarified by Klauss. Finally, if the FTAG illuminates the role of cyclic groups, it hinders the work on constructing isomorphisms "by hand" to link $<a, b>$ and $\mathbf{Z}/o(a)\,\mathbf{Z} \times \mathbf{Z}/o(b)\mathbf{Z}$ (the issue of the economy of techniques). In Dummit and Foote (2003), the task T_1 on lattice of subgroups is discussed three chapters before the FTAG (generalised) is taught. This suggests, by contrast with structuralist principles, that it would be preferable to separate the two types of tasks, at an early stage of the development of structuralist praxeologies in GT.

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About Two Epistemological Related Aspects in Mathematical Practices of Empirical Sciences

Reinhard Hochmuth and Jana Peters

1 Introduction

Mathematical practices, techniques and algorithms play a significant role in many disciplines (Winsløw et al., 2018). Consequently, mathematical service courses have become part of many study programs. Beyond the service courses, mathematical practices are also developed, adapted, and taught in courses of other disciplines. There, mathematical concepts that are also taught in introductory service courses sometimes have different meanings. Moreover, advanced content, like for example the Gaussian theorem in basic electrical engineering courses, is justified and used long before it is taught in service courses. Another illustrative example is that concepts like the Dirac impulse in signal analysis are often not covered in service courses.

Although the use of mathematics in other disciplines and the issue of mathematical service courses have been discussed for a long time in mathematics education (see for example the third ICMI Study by Howson et al. (1988) and for an actual overview Hochmuth (2020)), it is only more recently that research on mathematical practices in service courses and beyond has been playing an increasing role at international conferences on university mathematics education like CERME (Winsløw et al., 2018), ICME (Biza et al., 2016), INDRUM (Durand-Guerrier et al., 2021), and RUME (Weinberg et al., 2017).

Whereas mathematical topics that are relevant in other disciplines, for example differentiation, integration or stochastic distributions, can easily be identified in curricula and textbooks, the respective discipline-related mathematical practices and their respective rationales are often not explicitly known in detail (Winsløw et al., 2018). Due to the differences between mathematical practices and rationales in

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service courses and in major-subject courses, it is often not clear to students which activities and reasoning are allowed, required or forbidden and, in particular, how symbols have to be interpreted with regard to a specific task in major-subject courses (Hochmuth et al., 2014; Alpers, 2017).

In this contribution, we do not approach the study of mathematical practices in other disciplines by looking at students' or lecturers' concrete practices or lecturers' measures supporting students' learning in service courses. Instead, we start out from various historical-philosophical studies on the relationship of mathematics and empirical sciences and from there we explore two epistemological related aspects, that we have partly already investigated in detail in earlier research. One aspect deals with the identification of mathematical objects such as continuous variables and formal quantities with measurable, and therefore finite and discrete, quantities in empirical sciences (Hochmuth, 2019; Hochmuth & Peters, 2020). The second aspect concerns the characterisation of two different ideal-type (Weber, 1904) mathematical discourses and their roles in mathematical practices within empirical sciences (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021). The mathematical discourses connect to what is identified with each other in the sense of the first aspect. In this way they also refer in a certain way to the respective norms and rationales, on the one hand of mathematics and on the other hand of the respective empirical science. In this contribution we want to plausibly demonstrate that both aspects play a role in the use of mathematics in empirical sciences and illustrate this by examples from electrical engineering and psychology.

There is no place here to reflect in detail on the relationship between the two aspects. However, we want to emphasise that both aspects are essentially the result of institutional and societal processes. Each identification must have proved historically adequate and fruitful within the respective empirical science. And the various mathematical discourses are, among other things, the result of the historically specific organisation of the knowledge to be taught, taught and learned in educational institutions.

We have structured this article as follows: in the next section, we embed our research in the context of the ATD. Afterwards, we summarise some epistemological insights and observations from historical-philosophical and epistemological studies. These relate in particular to the two aspects outlined above. The quite abstract assertions are then exemplified for mathematical practices in electrical engineering and for psychology. A short outlook on subsequent research questions concludes our contribution.

2 ATD-Research on the Use of Mathematics in Other Sciences

Artigue et al. (1990) analysed different students' conceptions about differentials linked to mathematics and physics. Requirements regarding the institutional settings are figured out in essentially cognitively understood legitimations and validations of concepts and rituals assigned to mathematics or physics. They have observed, for example, that the idea of approximation works in mathematics as a constitutive moment of some notion and in physics as an excuse of loose reasoning, which reflects an "old conflict between rigorous mathematics and effectiveness in physics" (Artigue et al., 1990, p. 265). In view of teaching goals, the authors have especially suggested to make the various types of situations where differentials are needed more explicit.

Castela and Romo Vázquez (2011) applied and extended notions form ATD (Chevallard, 1992, 1999) in their analysis of mathematical praxeologies in signal and system theory courses. For studying the intrinsic intertwining of mathematics and its use they introduced a distinction between two technological components—a theoretical and a practical component—which reflects among others external didactical transpositions and different modalities of institutional validations. This idea of an internal differentiation of praxeological blocks is further extended and adapted in our analyses of signal and system theory-tasks (Hochmuth & Peters, 2021).

The institutionalised separation of teaching mathematics partially in service courses and in major-subject courses corresponds to a widespread understanding of the use of mathematics in other disciplines essentially as an application of previously constructed mathematical knowledge, an understanding which is coined by Barquero et al. (2013) as "applicationism". This understanding to some extent neglects the intrinsic dialectics between different mathematical practices and underlying needs, something we want to explore in this paper. In contrast, González-Martín and Hernandes-Gomes (2018) address curricular differences between practices, for example, regarding the integral notion and the integral use in calculus and mechanical engineering courses. Such curricular differences were also observed and investigated by Dammann (2016) and, for electrical engineering contexts, by Hennig et al. (2015).

In ATD there are the following two (in fact interrelated) options possible to make use of the idea of higher levels of codetermination: firstly, one could consider the impact of higher levels (for example societal dominant beliefs concerning the relationship between mathematics and other sciences like "applicationism") on the constitution of practices. Secondly, one could inform the analyses of mathematical practices and their institutionalisation by research results from history, sociology and/or philosophy of mathematics and sciences. In the following we mainly focus on the second option and outline a perspective with consequences for further analyses within ATD. Later, in addition to codetermination, we will also discuss the ATD principle of the institutional dependence of knowledge.

3 Epistemological Considerations Regarding the Relationship of Mathematics and Empirical Sciences

Practices in empirical sciences explicitly and/or implicitly claim to show an intrinsic relation to the world. This implies, for example, that their assertions cannot reasonably be justified and understood without recourse to the world. In our everyday life we show a realistic attitude, which means that we act under the premise, that there is conformity between mental images and reality. Philosophical reflections show that this view is highly problematic from an epistemological point of view[1] and, moreover, complicates understanding what the specific truth of empirical knowledge is. In our opinion, didactic studies are also at least occasionally based on the realistic position, such as the modelling cycle. In our contribution, we want to show that elaborate and epistemologically reflected positions are particularly helpful for a better understanding of mathematical practices in empirical sciences. On this basis, especially ATD-related concepts can be complemented and concretised in a suitable way in order to examine institutionalised mathematical practices.

With this in mind, pragmatic (Schlaudt, 2014) and historic-materialistic (Wahsner & von Borzeszkowski, 1992) views seem to be fruitful. According to the pragmatic position, empirical truth relies in "mastering objective means for the achievement of subjective purposes. It shows itself concretely in the agreement of will and ability in the action, in the performatively experienced resistance of the world" (Schlaudt, 2014, p. 11).[2] Accordingly, two readings of physics can be distinguished, a descriptive one "according to which the laws of physics tell us how certain objects behave, and a prescriptive one, according to which the laws are rather rules on how these objects can be manipulated" (Schlaudt, 2014, p. 123), The latter also means, that laws of nature have to be understood as instructions for action. Now an important point in our context is that the "resistance of the world" with regard to assertions from mathematics and empirical sciences like physics, engineering or psychology is quite different. Consequently, also the control of symbolic means for the achievement of purposes is quite different and subject to significantly different validity claims and related discourses. Such a position finally provides a basis for reconstructing the historic-specific societal-institutional constitution of practices and the generation of discourses, as well as for tracing its pedagogical and institutional reproduction.

According to the pragmatic position, and incorporating also historic-materialistic views, measurements in empirical sciences are not seen as the representation of a numerical determination of a property of things, but as something that informs about the behaviour of an object under certain norm conditions. Mathematics abstracts from behaviour and focuses on the pure quantity as well as presupposes the existence of objects in an axiomatic system of relations. But empirical sciences cannot "forget"

[1] See for example Schlaudt, pp. 42.

[2] All text passages originally in German were translated into English by the authors for this contribution.

these constituents. Instead, they are inherently reflected in the practices by constituting, incorporating and framing specific mathematical practices.

In historic specific transformations from dialectic interconnections to dualisms, contradictory conceptual identifications of, for example, infinite and infinitesimal quantities turn out to be particularly important (von Borzeszkowski & Wahsner, 2012). Especially with respect to metrological aspects Wahsner (1981) notes:

> However, natural science (at least this applies to a physical theory) does not make its statements directly about real objects, but about physical quantities and their relationships. These quantities are a means to recognise reality. They are finite determinations and must be, otherwise they cannot be measured. Natural science must therefore operate with these "objects of understanding". This is not metaphysics, but physics based on measurement theory. But these quantities, these objects of understanding are not natural, or are given directly in the imagination. They have to be produced through comparative work, through a comparative work that presupposes a human activity, but above all, it requires the development of a principle of scientific experience, the elaboration of a measurement theory (. . .), a theory that states how the contact between these quantities of the mind and the real objects is established. (p. 200)[3]

One aspect of mathematical knowledge is that a statement is true if it can be derived logically from true statements within the mathematical system. Truth (valid knowledge) is thus essentially determined inner-theoretical.[4] This is different in empirical sciences. Here, truth (valid knowledge) must always establish a reference beyond theory. Empirical sciences cannot be divided into an empirical (non-mathematical) part that regulates the relationship to reality and a mathematical part that is free of this relationship to reality. This phenomenon is made explicit in the investigations of Wahsner and von Borszeskowski on the relationship of mathematics and physics, but also by our investigations, especially with regard to the electrotechnical mathematics discourse. Such a decomposition, which in our opinion is ultimately not possible, would justify considering mathematical practices in empirical sciences as exclusively inner-mathematically justified actions. Accordingly, studies of mathematical practices in empirical sciences that ignore the empirical reference, including mathematics, to reality would imply this separation in an

[3]"Doch die Naturwissenschaft (wenigstens gilt dies für eine physikalische Theorie) trifft ihre Aussagen nicht unmittelbar über die wirklichen Gegenstände, sondern über Physikalische Größen und deren Beziehungen. Diese Größen sind ein Mittel, um die Wirklichkeit zu erkennen. Sie sind endliche Bestimmungen und müssen es sein, sonst kann man sie nicht messen. Die Naturwissenschaft muss daher mit diesen „Verstandesgegenständen" operieren. Es ist dies keine Metaphysik, sondern meßtheoretisch begründete Physik. Doch diese Größen, diese Verstandesgegenstände sind nicht naturgegeben, bzw. unmittelbar in der Vorstellung gegeben. Sie müssen durch Vergleichsarbeit erzeugt werden, durch eine Vergleichsarbeit, die die handelnde Tätigkeit des Menschen voraussetzt, vor allem aber die Entwicklung eines Prinzips wissenschaftlicher Erfahrung bedingt, die Ausarbeitung einer Meßtheorie (. . .), einer Theorie, die aussagt, wie der Kontakt zwischen diesen Verstandesgrößen und den realen Gegenständen hergestellt wird."

[4]Of course our description of mathematical knowledge is not comprehensive. But for our considerations, the point of inner-theoretical validation of knowledge is decisive. The same holds for validation of knowledge in empirical sciences.

empirical un-mathematical part and a mathematical part justified within mathematics. Examples are modelling cycles (Blum & Leiss, 2005) and "applicationsm" (Barquero et al., 2013). The fact that in empirical sciences mathematical practices also have a constitutive relation to reality have to be taken into account in didactic analyses. ATD enables this consideration, among other things, through the principle of institutional dependence on knowledge. In the institutions that can be assigned to the empirical sciences (e.g. lectures in the engineering sciences, research institutions, etc.), mathematical practices are justified, substantiated, validated and constituted differently than in academic mathematics. The core of this otherness is the empirical reference, which is constituted differently for each individual science.

In summary, we would like to point out that the epistemological questions outlined above have to be resolved in every mathematically based empirical science. Their relevance could be examined especially with regard to mathematical practices and, in the sense of ATD, to characterisations of technological-theoretical blocks of mathematical praxeologies in empirical sciences. Their consideration in electrical engineering is the focus of the next section. Afterwards, we briefly turn to mathematical practices in psychology.

4 About Mathematical Practices in Electrical Engineering

In various papers we have analysed mathematical practices in electrical engineering with a focus on the technological-theoretical block (Hochmuth et al., 2014; Hochmuth & Peters, 2020, 2021; Hochmuth & Schreiber, 2015a, 2015b, 2016; Peters et al., 2017; Peters & Hochmuth, 2021), and also in quantum mechanics (Hochmuth, 2019). A sociological and philosophical informed view has been crucial for the analyses of justifications, validations and their discursive constitution. In the following, we will elaborate on the two above mentioned epistemology related aspects. Firstly, the two different mathematical discourses and their roles in the use of mathematics within empirical sciences. And secondly, the identification of mathematical formal quantities with measurable quantities in empirical sciences.

The technological-theoretical blocks of mathematical practices in engineering and in mathematics differ in terms of general characteristics (empirical truth vs. deductive-logical truth, the ontology of objects etc.) and concrete contents. In previous studies we have been able to reconstruct two ideal-type mathematical discourses in relation to mathematical practices: an Electrotechnical Mathematics Discourse (ET) and a Higher Mathematics Discourse (HM) (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021). In ATD the logos is considered as a discourse on praxis, but as praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is also related to the institutional discourse. In the following, we will describe both mathematical discourses within the context of complex numbers. In the concrete studies just cited on mathematical practices in signal theory we have presented this in more detail, and in a broader context.

The mathematical knowledge associated with the HM-discourse is characterised by an inner-mathematical conception of terms and statements without concrete references to reality, a generalisation-oriented rational of academic mathematics and a concentration on calculation rules. To describe the mathematical knowledge concerning complex numbers we refer to the textbook by Strampp (2012). This book represents a standard approach to complex numbers. It is used as a course literature for a consolidated two-semester standard course on higher mathematics for engineers which is held every year at the University of Kassel and thus represents an important reference point for our previous analyses with regard to mathematical practices. Complex numbers are covered in the first semester in the context of Linear Algebra and are introduced as a field extension of real numbers, motivated by the solvability of the equation $x^2 + 1 = 0$. Field extension is not introduced as a formal algebraic concept. Strampp (2012) just states that the real numbers are extended by a number i with the property $i^2 = -1$ and that after the extension, all field axioms which are relevant for calculating with real numbers shall continue to exist (p. 59). This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations, resulting in considerations about the general solution of algebraic equations, the fundamental theorem of algebra and Vieta's formula. In doing so, however, no formal concepts are introduced and proven, but rather calculation rules for complex numbers are derived and presented. Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, an orientation towards the inner-mathematical, generalisation-oriented rational of academic mathematics can also be observed. In addition to the previously mentioned more algebraic view on complex numbers, the chapter contains another, geometric, orientation based on an analogy to vectors. However, the vector concept is also distinguished from complex numbers: "We speak of phasors[5] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This multiplication extends the multiplication of real numbers." (p. 60) This HM-phasor concept differs from the phasor concept in electrical engineering, described below, but refers to it.[6] The geometrical representation of complex

[5]We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrotechnical aspects play no role in the course and Stramp (2012) does not refer to them either. Another possible translation of Zeiger, without the connection to engineering concepts would be pointer. But we decided to use phasor for the following reason: In German, the term Zeiger is used both in electrical engineering and in mathematics courses for engineers, but with different meanings (reference to electrotechnical concepts vs. geometrical object with no further references to reality). By using the term Zeiger instead of vector Strampp (2012) can thus establish a connection to the electrotechnical courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different institutional contexts is in jeopardy of being lost through translation.

[6]The textbooks by Fettweis (1996) and Frey and Bossert (2009) cover signal and system theory, the context for our second example, the introduction of the Dirac impulse. Complex numbers are also very important in signal and system theory, especially in the context of amplitude modulation, see for example (Peters & Hochmuth, 2021).

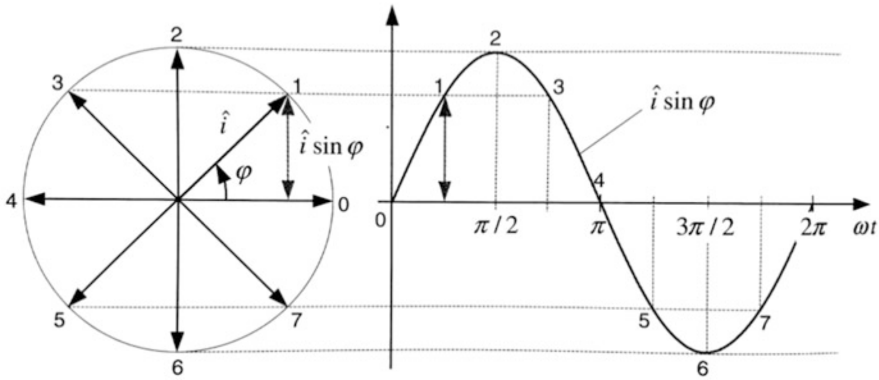

Fig. 1 Relationship between phasor and time-dependent function (Albach, 2011, p. 32)

numbers as arrows in the Argand diagram is used as a visualisation of calculation rules.

To characterise the ET-discourse we first note that electrical engineering as an empirical science inherently includes a reference to reality. However, a look at various textbooks, e.g. Fettweis (1996) and Frey and Bossert (2009) (see footnote 5), shows a large variance in the explication of this reference to reality, which is accompanied by a variance in the degree of mathematical formalisation. For a short overview of how complex numbers are treated in electrical engineering courses we refer to the standard textbook by Albach (2011). In Albach (2011) phasors [Zeiger] are introduced in the context of alternating currents and voltages. The first introduction is without reference to complex numbers: Here a phasor is an arrow with a specific length and a specific angle with respect to a reference angle. This arrow can be related to a time-dependent sinusoidal[7] function, see Fig. 1.

Current and voltage ratios in electrical networks can be displayed and analysed graphically in phasor diagrams without using differential equations. On the basis of Kirchhoff's rules for the analysis of electric circuits, geometric calculation rules for phasors are derived, which are analogous to the calculation rules for vectors. For the purpose of a mathematical description of phasors, the plane in which phasors are drawn, can be considered as the complex plane. The phasor is now understood as a complex quantity that symbolises the time-dependent voltage (see Albach, 2011, p. 42). Whereas in the HM-discourse phasors are used to graphically illustrate the properties of complex numbers, in electrical engineering phasors are arrows that represent measurable, time-dependent quantities such as alternating voltages or currents. Complex numbers are then used for the convenient mathematical description of phasors, justifying the compatibility of the rules for manipulating phasors and the calculation rules of complex numbers via physical relations.

[7]Circuits are operated with sinusoidal current- and voltage forms in the power supply network as well as in many other important areas.

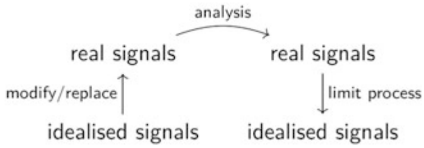

Fig. 2 Illustration of the interplay between idealised and real signals

With the above explanations we have shown how phasors and complex numbers are constituted as different epistemological objects in the mathematical discourses. In other publications we have shown how both discourses can also be reconstructed empirically on the basis of tasks, lecturer sample solutions (Peters & Hochmuth, 2021) and student work (Hochmuth & Peters, 2021).

In the following, we will use the introduction of the Dirac impulse in signal theory as an example for the principle of identifying infinitesimal formal mathematical quantities with finite measurable quantities. Furthermore, we show how this principle also interacts with the two mathematical discourses described above.

In previous work (Hochmuth & Peters, 2020) we analysed the introduction of the Dirac impulse in the signal theory textbook by Fettweis (1996): Thereby we have taken up Fettweis' distinction between idealised and real signals and highlighted a general principle (see diagram in Fig. 2), which plays an important role in justifications by Fettweis and is used by Dirac (1958) in a similar way.

Here we want to draw attention to the connection between the principle, illustrated in Fig. 2, and the two mathematical discourses outlined above. On this basis we will then show how this connection is also helpful for the reconstruction of a passage from the textbook by Frey and Bossert (2009).

In Fettweis's approach, the real signals represent irregular transmissions on the one hand, i.e. they refer to empirical objects, and on the other hand to functions with pleasant mathematical properties such as sufficient differentiability. Thus, they form a central link that enables further identifications, allow specific justifications in this context and connect discourses. The ideas associated with idealised signals, on the other hand, refer to irregular (according to Fettweis) mathematical objects, such as the Heaviside function or the Dirac impulse, as well as to signals which as such do not exist in reality, but only approximately. Here, too, references to mathematics and empirical sciences are brought together, and mathematical discourses can start from these. We now illustrate these general remarks with the example of a passage from the signal theory textbook by Frey and Bossert (2009).

Frey and Bossert (pp. 208) formulate the goal to differentiate the (in the usual sense) non-differentiable Heaviside function, which is defined by:

$$\varepsilon(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

The Heaviside function represents (according to Fettweis) an idealised signal. In order to apply HM-practices, the Heaviside function is represented approximately by the sequence of differentiable functions (real signals):

$$f_a(t) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{t}{a}\right) + \frac{\pi}{2}\right), a > 0.$$

The approximation can be interpreted in the sense of the HM-discourse as pointwise convergence. The derivation of the Heaviside function is then derived with the following steps:

$$\frac{d}{dt}\varepsilon(t) = \frac{d}{dt}\lim_{a\to 0} f_a(t) = \lim_{a\to 0}\frac{d}{dt} f_a(t) = \lim_{a\to 0}\frac{1}{\pi}\frac{a}{a^2+t^2} = \{ \begin{matrix} 0, & t \neq 0 \\ \infty, & t = 0 \end{matrix}.$$

The HM concept of pointwise convergence also allows to understand the equal sign in the last step. The central step in the argumentation is the transition from the second to the third term, which can be clarified with reference to the above principle and through this mediated interplay between the two mathematical discourses. From a mathematical point of view, the step from the second to the third term, i.e. the permutation of the limit values on the basis of the pointwise convergence of function sequences, is not permissible. This permutation can therefore not be justified in the HM discourse. This is where the principle comes into play: The derivative of the idealised signal cannot be calculated. Therefore, it is approximated and replaced by real signals. These are differentiated. Then finally the limit of the calculated derivatives is calculated. The above principle thus allows the step from the second to the third term to be broken up in such a way that HM discourse elements can be made effective. Here the principle proves to be an expression of the ET-discourse. Thus, in the step from the second to the third term an interesting interplay between the two mathematical discourses, mediated by the principle, results.

Of course, the discussed step could be rewritten in terms of distribution theory (also partially addressed by Frey and Bossert, pp. 110), so that it could be justified in this interpretation purely mathematically. But the central point in our argumentation is not that distribution theory is not used here as the basis for justification, which would correspond to a rather deficit-oriented view. Rather, our point is that the electrotechnical mathematics discourse, in its reference to empirical objects, not only allows for a justification of the step, but also establishes a reference of symbols and argumentation to empirical objects and contexts. A purely distribution-theoretical argumentation could not make this possible. To do so, it would have to be supplemented in a suitable way by electrotechnical means, which would of course be possible in principle.

5 About Mathematical Practices in Psychology

We consider psychology as another example of mathematical practices in an empirical science. Here we focus on psychometric tests, as used in diagnostics in the form of performance, intelligence, ability or development tests.[8] Such tests are also used or developed in empirical research projects in university mathematics education (see e.g. Kuklinski et al., 2018; Hochmuth et al., 2019). The aim of such tests is, in particular:

> to capture inter-individual differences in behaviour and experience as well as intra-individual characteristics and changes, including their relevant conditions, in such a way that sufficiently precise predictions of future behaviour and experience as well as possible changes in defined situations become possible.[9] (Amelang & Zielinski, 2002, Sect. 1.1)

In general, model assumptions underlying the tests can be distinguished in terms of traits and behaviour diagnostics. In the first case, the description of experience and behaviour in the form of traits is crucial, whereby traits are represented by hypothetical constructs that are derived from and refer to observable behaviour. In the second case, personality traits act as intervening variables, which are usually determined as the probability that a person with certain traits will exhibit certain behavioural tendencies. Theory and empirical knowledge are interdependent in test development and application: On the one hand, theories are available in the form of descriptions and conceptualisations of psychological constructs (e.g. motivation, self-efficacy, intelligence) and are usually embedded in broader theoretical contexts: they form the basis of quantitative models. On the other hand, modelling and testing also create opportunities for observation. These allow, for example, theoretically suggested hypotheses to be empirically confirmed or refuted. In theoretical preliminary considerations, the aim is to determine the situational test conditions as precisely and objectively as possible (e.g. also selection of suitable cohorts, suitable item formulations). The tests to be developed should ideally be sensitive to interesting factors and robust against interfering factors. Which factors and constructs come into view is determined by the underlying theories and their basic concepts. Each concrete test development, both in the traits and in behaviour diagnostics, now includes the transformation of theoretical constructs or factors into variables. This transformation is often referred to as operationalisation. The aim of

[8] Psychometric tests, of course, represent only a small section of psychology as an empirical science. It should at least be noted that throughout the history of psychology there have been repeated controversies about how to define the specific subject of psychology and what this means in terms of feasible and appropriate scientific methods. For example, the controversy of explanation-understanding is to be mentioned (see e.g. Riedel, 1978). There are, for example, many relationships between this controversy and our discussion in this section. For space reasons alone we cannot go into this here.

[9] "…interindividuelle Unterschiede im Verhalten und Erleben sowie intraindividuelle Merkmale und Veränderungen einschließlich ihrer jeweils relevanten Bedingungen so zu erfassen, hinlänglich präzise Vorhersagen künftigen Verhaltens und Erlebens sowie deren evtl. Veränderungen in definierten Situationen möglich werden".

operationalisation is then to formulate assumptions of interrelationships between variables, e.g. between independent and dependent ones. Based on variable-related data, the assumptions are tested by means of stochastic procedures. The goal of operationalisation is thus to enable statistically processable and assessable findings.

To make this possible, operationalisation must ensure that the events and reference variables that are considered quantifiable and measurable by variables meet conditions such as random variability:

> Only if a result could in principle also have occurred by chance does the statement that in the present case (according to agreed criteria) it is more frequently than simply random has empirical substance.[10] (Holzkamp, 1994, p. 85)

Random variability must therefore be ensured in the psychological design of experiments. Which constructs or factors in which situations are suitable for transformation into variables is a central specific contribution of the science of psychology and cannot be answered mathematically alone. In contrast to electrical engineering, for example, such transformations or "identifications" of psychological constructs and variables are controversial in psychology (cf. e.g. Echterhoff et al., 2013, pp. 39–42). From a historical point of view, operationalisation represents, among other things, a starting point for the formulation of a fundamental critique of the type of psychological research outlined here: For example, the assumptions of connections formulated in this way would often be "secondary constructions of abstract generality . . . that have very little to do with the real connections/contradictions that should actually be up for debate [in psychology; the authors], with which the research findings, because they `bypass the problem´, always seem somehow trivial, meaningless, indifferent" (Holzkamp, 1994, p. 82).[11] Currently, a large part of psychological research is oriented towards the methodological approach sketched up. With regard to this, it should have become plausible that it includes, in its operationalisation, an identification of mathematical objects with quantities that are considered measurable and quantitative.[12]

[10]"Nur wenn ein Resultat prinzipiell auch zufällig zustande gekommen sein könnte, hat die Aussage, dass es im vorliegenden Fall (nach vereinbarten Kriterien) 'überzufällig' ist, einen empirischen Gehalt."

[11]"sekundäre Konstruktionen von abstrakter Allgemeinheit . . . die mit den wirklichen Zusammenhängen/Widersprüchen, um die es eigentlich [in der Psychology; the authors] gehen sollte, nicht viel zu tun haben, womit die Forschungsbefunde, weil sie `am Problem vorbei´ gehen, stets irgendwie als trivial, nichtssagend, gleichgültig anmuten". One can also compare Blumer's introduction of a theory of symbolic actionism, which was explicitly founded as an alternative to "variable psychology". ATD, with its focus on mathematics, also distinguishes itself to a certain extent from research in didactics that is essentially psychologically based. It goes without saying that this does not mean that psychometric tests in psychological or didactic research to investigate specific questions, such as the effects of interventions in the face of large cohorts, are rejected.

[12]In psychology, operationalisations often do not take place in a single step. Therefore, a sequence of such steps could be distinguished in detail. The last step, with regard to mathematical objects, would then be particularly close to considerations in electrical engineering. In a certain sense, the preceding steps would then be comprised in that step. Of course, the basic principle can only be roughly described here.

The second epistemology related aspect is the emergence of different mathematical discourses in mathematical practices in psychology, especially statistics, that can also be observed in many ways. This concerns the specific selection of stochastic models oriented to particular psychological research questions, but also their implementation in detail. This can be shown particularly well with path models and structural equation models (cf. e.g. Renner et al., 2012). These do not result only from quantitative calculations, but they are also usually based on theoretical psychological considerations. Only this subsequently enables the psychological interpretation of the calculated results.

The considerations about mathematical practices in psychology of this section cannot, of course, replace a praxeological study based on concrete empirical material. However, they point out that this could also be fruitful with regard to the two epistemology related aspects highlighted in this paper. Finally, we would like to briefly add that our reflections on psychology are also compatible with the pragmatic position referred to at the beginning: according to this position, psychological measurements (e.g. of an intelligence test) would not be understood as quantitative definitions of personal characteristics, but rather as something that makes statements about the behaviour of a person under certain conditions.

6 Outlook

In this contribution we examined a few ideas of how ATD-analyses could be informed by epistemological-philosophical insights. Aspects from historic-materialistic studies by Wahsner and von Borzeszkowski and philosophical-pragmatic considerations by Schlaudt were used to discuss relationship of mathematics and empirical sciences. Links to concepts of the ATD were drawn via the scale of levels of codetermination and the institutional dependence of knowledge. We have illustrated our considerations by examples of mathematical practices in electrical engineering and psychology.

The philosophical-epistemological reflections on mathematical practices indicate that they (partially) ground in major issues related to the interrelationships between empirical sciences and pure mathematics and their historic-specific manifestation in societal institutionalised teaching learning contexts. On this basis, especially ATD-related concepts can be complemented and concretised in a suitable way in order to examine institutionalised mathematical practices of teaching and learning in educational institutions and to draw conclusions for teaching innovations. In particular, implementations of well-meant measures might produce unsatisfactory or unintended effects as long as institutional, pedagogical and epistemological conditions are not sufficiently well understood. Beyond an analysis of technological-theoretical blocks of mathematical practices in empirical sciences like electrical engineering, physics but also psychology, sociology, etc., the philosophical-epistemological considerations further allow to question notions often used in mathematical education research that claim to make essential aspects of such mathematical practices didactically accessible. Examples of these questions are: Which aspects of those mathematical practices are covered and which are not covered by approaches applying modelling cycles (Blum & Leiss, 2005) or "Grundvorstellungen" (Greefrath et al., 2016)? How could these approaches be reinterpreted by ATD terms, if relevant, in order to complement them appropriately with regard to ignored aspects? And even more critically: How are those approaches and their deficits regarding the issue of mathematical practices in empirical sciences related to societal dominating reflections on teaching-learning issues? The latter question is (partially) further connected to questions regarding the level of external didactical transformations (see e.g. Bosch et al., 2021): How are the investigated issues reflected in the construction of study programs and module structures? And finally, regarding consequences for teaching: in addition to an emphasis on the explication of identifications and the rationales of different mathematical discourses in lectures and texts, the construction of suitable rich tasks with interesting opening questions for the establishment of SRPs could be an interesting and useful way. In connection with suitable initial problems for SRPs, the mathematical practices of the empirical sciences, which have historically been widely recognised as adequate, could and should indeed prove useful for their elaboration and solution by students.

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Describing Mathematical Activity: Dynamic and Static Aspects

Ignasi Florensa and Catarina Lucas

1 Describing Inquiry: A Challenge for Research

During the past decades, there has been an important movement claiming for a change in school institutions and more precisely in the organization of teaching and learning processes. The main characteristics of this new pedagogical paradigm are usually described by general principles such as active learning, student-centred processes, open and contextualized activities and independent learning among others (Felder & Brent, 2005; Prince, 2004; Prince & Felder, 2006).

The anthropological theory of the didactic (ATD) assume as one of its educational ends to promote a shift in the pedagogical dominant paradigm, moving from the old paradigm of "visiting works" towards the new paradigm of "questioning the world". This new paradigm is based on these four concepts: *inquiry* and *being herbartian, procognitive* and *exoteric* (Chevallard, 2015).

One of the basic assumptions of the epistemological approach in didactics and, in consequence, of the ATD is the strong dependence existing between the way knowledge is conceived in school institutions and the didactic phenomena arising in their implementation. The design, implementation and analysis of study processes and their associated phenomena cannot be detached from a deep epistemological analysis.

The design, implementation and analysis of inquiry study processes is not different and presents diverse specificities at the epistemological level: knowledge shifts from a static, individual and structured form to a more dynamic, collective and

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provisional structure. The systematic observation and description of these study processes is a challenging issue for any theoretical approach in didactics. The ATD framework has developed diverse tools enabling researchers to describe knowledge involved in study processes as well as its dynamic evolution.

In this workshop we will present and mobilise three of these tools in order to illustrate their use and potentialities. First, we will present the praxeological analysis, the Herbartian schema and the question-answer maps. Secondly, we will use these tools to analyse an SRP experienced by the participants regarding the forecasting of a flu epidemy.

2 Fostering This Transition: Study and Research Paths

Study and Research Paths (SRPs) are a teaching format proposed by the ATD to foster the transition from the paradigm of visiting works to the paradigm of questioning the world. The design, implementation and analysis of these teaching formats is based on the Didactic Engineering methodology (Artigue, 2014; Barquero & Bosch, 2015). This methodology includes the consideration of the prevailing epistemology in the school institution, the didactic phenomena that will be studied, the selection of a generating question that will initiate the process of study as well as the analysis of the whole process.

Specifically, SRPs in are initiated by an open question posed to a community of study (a set of students X and a set of guides of the study, Y) that will generate moments of study of available information in the media, and moments of research and development of new solutions to generate an answer to the initial question. The implementation of an SRP under the ATD perspective is often twofold. On the one hand, SRPs can be considered as a tool to reach the education ends of the ATD: fostering the paradigm shift. On the other hand, and from a research perspective, SRPs are implemented following the didactic engineering methodology to empirically validate how an alternative conception of knowledge overcomes (and to what extend) a specific undesired didactic phenomenon. This double character of SRPs is crucial: they are research tools enabling researchers to generate answers to their research questions, and also teaching tools to implement new study processes.

3 Modelling and Describing Knowledge Within the ATD: Praxeological Analysis

As said before, the crucial role of knowledge in ATD forces researchers to avoid blindly accepting the way knowledge is conceived in a specific institution. ATD proposes to model knowledge in terms of praxeologies. Praxeologies are living entities evolving and changing according to the institutions where they exist. They are defined by a set of four elements [$T/\tau/\theta/\Theta$], according to the ATD principle that any activity combines a “practice part” (or “know-how”) known as the praxis block, and a “knowledge part” (or “know-that”) known as the logos block. The praxis block involves a specific kind of tasks (T) and a set of associated techniques (τ) enabling to develop the tasks. The logos block includes the technology (θ) and a theory (Θ) justifying and interpreting (in a more or less formal and explicit form) the praxis, its raison d’être and its results (Chevallard, 1999).

Praxeologies enable researchers to systematically describe knowledge existing in a school setting and to describe the different elements of the praxeology and its degree of explicitness. In addition, this way to characterise knowledge allow researchers to detach from the traditional way of conceiving knowledge in school institution, a way usually based on finished and closed elements (such as notions, concepts, properties, procedures and definitions in mathematics). Praxeologies can be used to describe the type of tasks that are actually associated to this piece of knowledge, the way these tasks are carried out, how they are described and justified, how they are related to other pieces of knowledge and also its raison d’être, that is, the main questions this piece of knowledge allegedly helps to addressing.

4 Describing Study Processes: Herbartian Schema

Chevallard (2008) developed the notion of Herbatian schema (see Fig. 1) as a representation facilitating researchers to describe and analyse different aspects of study processes (Bosch & Winsløw, 2016).

The first part of the schema represents the didactic system $S(X; Y; Q_0)$ formed by a set of students (X) a set of guides of study (Y) that together face the task to generate an answer to an open question Q_0. The second part of the schema describe the process of elaboration of an answer ($A^{\heartsuit}$) of the community of study to the generating question Q_0 This part is composed of two elements interacting through a dialectic: the questions ($Q_1, Q_2, \ldots, Q_m$) and answers and works ($A^{\diamond}_{m+1}, A^{\diamond}_{m+2}, \ldots, A^{\diamond}_{n}$ and $W_{n+1}, W_{n+2}, \ldots, W_p$). The hallmarked answers and works are preexisting developed knowledge in different institutions that the community of study will access in different media (Bosch & Winsløw, 2016). This information obtained is then studied, deconstructed and adapted to the (sub)question addressed and incorporated to the milieu. The potential of the Herbartian schema is not only its capacity to systematically model inquiry study process by easily incorporating the question-answer and media-milieu dialectics but also its adequacy to compare any study process ranging from traditional lectures to more innovative formats. For example, in a more transmissive—traditional setting in where lectures are central, the Herbartian schema reveals that only one answer will be available to the community of study and that this answer will coincide with the one presented by the teacher. In

$$[S(X;Y;Q_0) \Rsh \{Q_1, Q_2, \ldots, Q_m; A^{\diamond}_{m+1}, A^{\diamond}_{m+2}, \ldots, A^{\diamond}_{n}; W_{n+1}, W_{n+2}, \ldots, W_p\}] \hookrightarrow A^{\heartsuit}$$

Fig. 1 Herbartian scheme

addition, the question leading to the knowledge will remain in the shadow. In contrast, in an open study process initiated by a question the community of study will search for available answers, will modify and incorporate them. The difference between teaching formats is made explicit in terms of diversity of elements of the Herbartian schema mobilised during the process.

5 Describing Study Processes: Question-Answer Maps

Winsløw et al. (2013) present the question-answer maps (Q-A maps) as a research tool to model “mathematical knowledge from a didactical perspective”. Q-A maps are rooted tree representations of the inquiry followed in an SRP. They start from the generating question and include all the partial answers and the derived questions appearing during the whole process.

The use of Q-A maps by researchers in the ATD has spread in the past decade with two main uses. On the one hand, they have been used as the materialisation of a REM (Barquero, 2009; Lucas, 2015). On the other hand, Q-A maps have been used as tools to describe and model study processes by, for instance, Barquero et al. (2008) when implementing an SRP about population growth, and by Jessen and Winsløw (2011) during the implementation of a bidisciplinary SRP in mathematics and history. In addition, Q-A maps have also been incorporated in teacher professional development as a tool to structure and describe knowledge involved in SRPs as well as a communication tool with students involved in these kind of study processes (Florensa et al., 2018).

6 A Possible Raison d’Être for Elemental Differential Calculus

Lucas and colleagues (Fonseca et al., 2014; Lucas, 2015) taking as starting point the work of Ruiz-Munzon (Ruiz-Munzón, 2010), propose in their works a reference epistemological model (REM) for elemental differential calculus at the transition between high school and university. This REM considers that the activity around functional modelling should be the raison d’être of elemental differential calculus. The ATD considers that any modelling process can be described in four steps (not necessarily in this chronological order): delimitation of the system to be modelled, construction of the mathematical model, technical work in the model, and interpreting this work within the system.

In her work Lucas (2015) also propose diverse SRPs based on this REM that were experienced with first year students of Nuclear Medicine in the Biomathematics course. One of the SRPs was initiated by the following generating question: “How to prepare and administer a radiopharmaceutical to diagnose a thyroid cancer?”. The derived questions that appeared during the implementation of this SRP involved data analysis of medicament concentration in diverse patients and the study of its average rate of change, among other aspects. The students also worked on the selection of continuous functions modelling the phenomena.

We consider this work as an important contribution to the ATD, in particular, on the development of mathematical modelling at higher education. In consequence, we have chosen to illustrate the use of the ATD tools (praxeological analysis, Herbartian schema and question-answer maps) using a mathematical activity based on the REM developed by Lucas (2015).

7 The Workshop: Describing Mathematical Activity with ATD Tools

The structure of the workshop was the following (see Table 1): it started with a presentation of the three ATD tools that the workshop intends to illustrate. Secondly, we organized the participants in groups of 6 and proposed them to address a generating question during 60 min. Then the groups used the ATD tools to analyse their own modelling activity and we finished by the presentation of the work developed by the groups and a general discussion.

The generating question proposed to the participants was selected based on the REM on elemental differential calculus developed by Lucas (2015). The question was: “We provide you the incidence index of the flu outbreak of the last winters in Spain. We also give you the data of the first weeks of 2018 winter. Can you forecast when the outbreak peak will take place? How intense will be the outbreak of 2018?”. Data was provided using a spreadsheet and a graphic (see Fig. 2).

During this activity each member of the team will also use one of the ATD tools to describe the activity in where they are involved.

Table 1 Activities of the workshop

	Time (min)
ATD tools to describe study processes: Praxeological analysis, Q-A maps and Herbartian schema	20
Presenting the SRP Q_0	10
Group working on the Q_0	60
Analysing the inquiry process, deploying the ATD tools	50
Presentation of the results of each group	20
General discussion	20

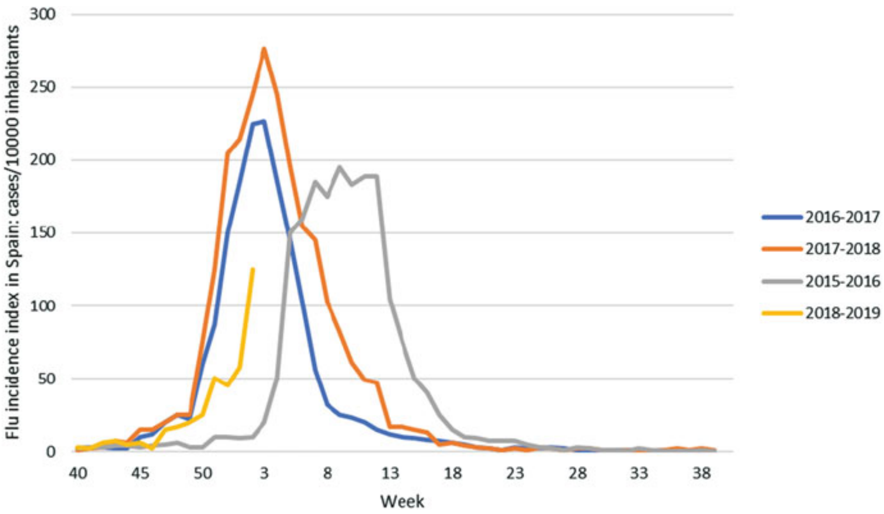

Fig. 2 Flu incidence index in Spain in 2015–2016, 2016–2017, 2017–2018, 2018–2019 winters

8 Results

Participants mobilised the three ATD tools to analyse their activity. Regarding the use of Q-A maps, two of the groups worked with the Q-A maps. While group 1 (see Fig. 3) included questions and answers involving diverse aspects, group 2 (see Fig. 4) only worked on questions and answers involving the SIR model. As is emphasized in previous research (Florensa et al., 2018; Winsløw et al., 2013) works the Q-A maps model this dynamic nature of knowledge involved in inquiry and its evolution during the study process.

Regarding the use of the Herbartian schema, group number 3 used it and described in a very detailed way the elements of the media and how they evolved. In particular, they identified the following hallmarked works: standard distribution, average, best fitting techniques, variance, SIR model. In addition, the use of the Herbartian schema made also explicit data they used (provided by the workshop animators) but also complementary date they found (such as previous years data). The elements that the group used to validate their answers are also made explicit thanks to the use of the schema: data of the end of the 2018 epidemy, found in the Health Ministry website is an illustrating example.

Finally, two groups (group 3 and 4) developed a praxeological analysis of their activity. Both groups have only analysed a short extract of their activity: the fine grain praxeological analysis allow them to describe exhaustively their activity but, in contrast, they highlight the difficulty to connect the different types of tasks and techniques mobilised. As an example, we present here the analysis of one of the praxeologies involved according to group 4:

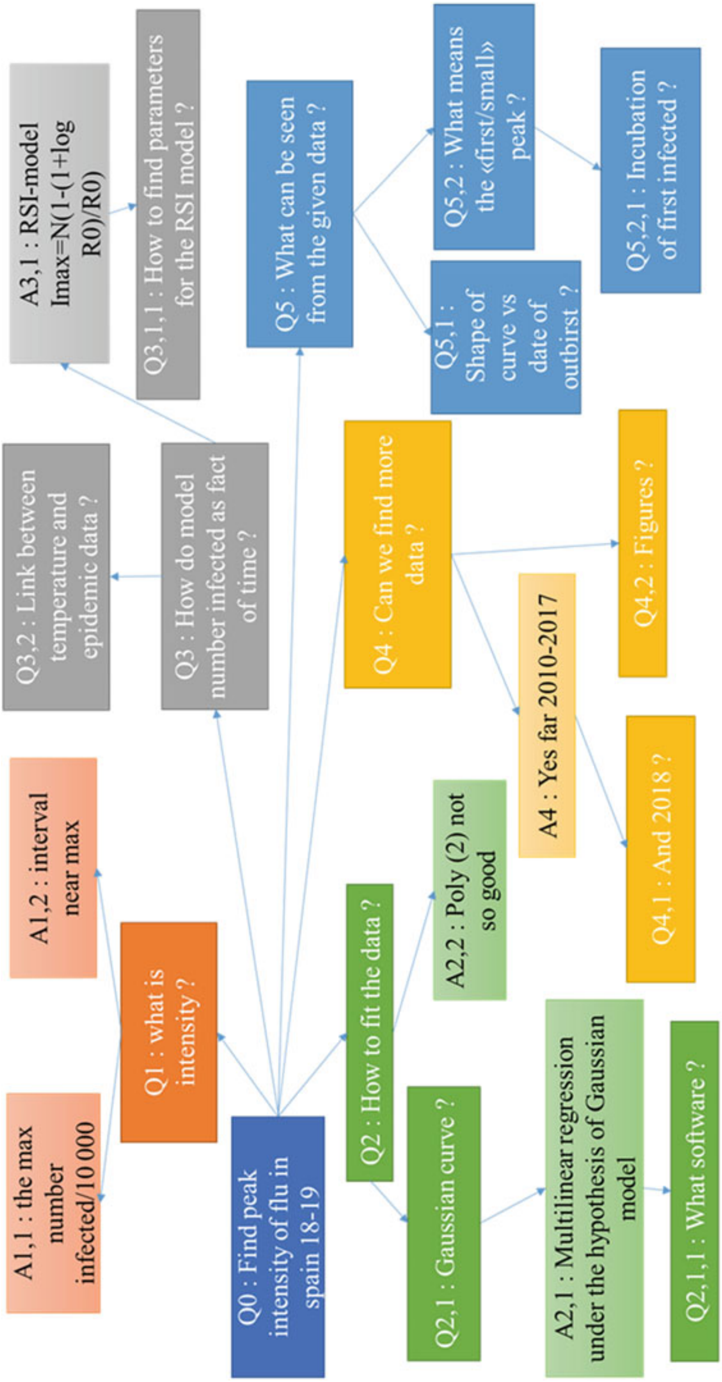

Fig. 3 Q-A map of group 1

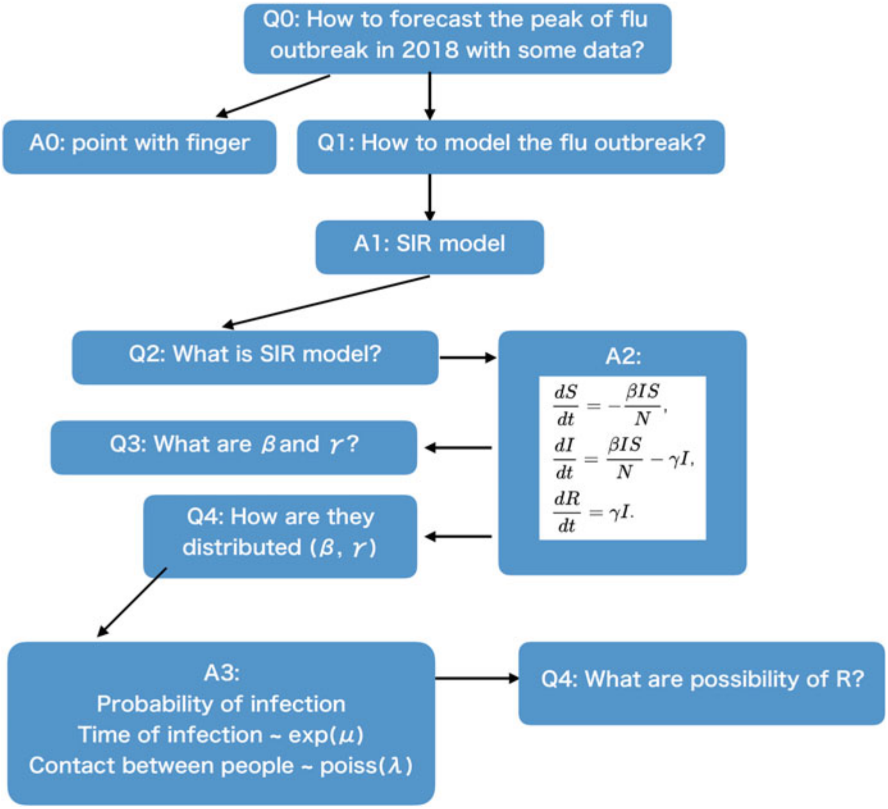

Fig. 4 Q-A map of group 2

Type of task T_1: Provided the incidence index of the flu outbreak of the last n winters in a country, and provided the data of the first weeks of the current winter: (a) Forecast when the outbreak peak will take place and, (b) forecast how intense will be the outbreak of the current year.

One possible strategy (**involving several techniques):** To "play" with the data provided: to consider the table describing the variation of infected along time, and to consider the table describing the relative rate of infected along time.

Another possible strategy/**technique** $\boldsymbol{\tau}_1$: to look in internet at disposable and suitable works. We finded the SIR (Susceptible-Infected-Recovered) model, which involves the functions S(t), I(t), R(t), and the parameters N, \gamma y \beta.

Technology $\boldsymbol{\theta}_1$ **corresponding to** $\boldsymbol{\tau}_1$: Description of the parameters N, \gamma and \beta in terms of their properties in relation with the function S(t), I(t) and R(t).

Theory $\boldsymbol{\Theta}_1$ **underlying** $\boldsymbol{\theta}_1$ **corresponding to** $\boldsymbol{\tau}_1$: Mathematical elements but also notions and properties concerning the notion of epidemy, etc.

The diversity of tools developed within the ATD framework enable researchers to describe inquiry study processes under different approaches. Each tool mobilised during the workshop seem to emphasize a certain didactic-epistemological aspect of the SRP. While Q-A maps highlight the dynamic and the evolution of knowledge in a study process, the Herbartian schema allow researchers to analyse the elements involved in the study as well as their providers. Finally, the praxeological analysis allow a fine grain analysis where each praxeology is detailed in terms of the 4t model. The selection of the mobilised tool in research works should be done according to the studied didactic phenomena.

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