Symmetric Measures, Continuous Networks, and Dynamics

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Abstract With view to applications, we here give an explicit correspondence between the following two: (i) the set of symmetric and positive measures ρ on one hand, and (ii) a certain family of generalized Markov transition measures P, with their associated Markov random walk models, on the other. By a generalized Markov transition measure we mean a measurable and measure-valued function P on $(V, \mathcal{B})$, such that for every $x \in V$, $P(x; \cdot)$ is a probability measure on $(V, \mathcal{B})$. Hence, with the use of our correspondence (i)–(ii), we study generalized Markov transitions P and path-space dynamics. Given P, we introduce an associated operator, also denoted by P, and we analyze its spectral theoretic properties with reference to a system of precise L^2 spaces.

Our setting is more general than that of earlier treatments of reversible Markov processes. In a potential theoretic analysis of our processes, we introduce and study an associated energy Hilbert space $\mathcal{H}_E$, not directly linked to the initial L^2-spaces. Its properties are subtle, and our applications include a study of the P-harmonic functions. They may be in $\mathcal{H}_E$, called finite-energy harmonic functions. A second reason for $\mathcal{H}_E$ is that it plays a key role in our introduction of a generalized Green function. (The latter stands in relation to our present measure theoretic Laplace operator in a way that parallels more traditional settings of Green functions from classical potential theory.) A third reason for $\mathcal{H}_E$ is its use in our analysis of path-space dynamics for generalized Markov transition systems.

Keywords Markov operator · Standard measure space · Symmetric measure · Laplace operator · Markov chain · Harmonic function · Finite energy space

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1 Introduction

In this paper, we continue our study of the graph Laplace and Markov operators, initiated in [1], which was based on the key notion of a σ-finite symmetric measure defined on the product space $(V \times V, \mathcal{B} \times \mathcal{B})$ for a standard Borel space $(V, \mathcal{B})$.

Our goal is to extend the basic definitions and results of the theory of weighted networks (known also as electrical or resistance networks) to the case of measure spaces. We briefly recall that, for a countable locally finite connected graph $G = (V, E)$ without loops, one can identify the edge set E with a subset of the Cartesian product $V \times V$ and assign some weight c_{xy} for every point (x, y) in E where c_{xy} is a symmetric positive function. It gives us a symmetric atomic measure ρ on E whose projections on V are the counting measure μ. Then, for a weighted network (V, E, c), one defines the Markov transition probability kernel P and the graph Laplacian $\Delta = c(I - P)$ which are considered as operators acting either in L^2 spaces with respect to the measures μ and $\nu = c\mu$ or in the finite energy space $\mathcal{H}_E$. Their spectral properties are of great interest as well as the study of harmonic functions in the theory of weighted networks.

Our approach to the measurable theory of weighted networks is based on the concept of a *symmetric measure* defined on the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$ where $(V, \mathcal{B})$ is a standard Borel space. (To stress the existing parallels we use the same notation as in discrete case.) In more detail, in the context of measurable dynamics, the state space V is considered very generally; more specifically $(V, \mathcal{B})$ is given, where $\mathcal{B}$ is a specified σ-algebra for V. From $(V, \mathcal{B})$, we then form the corresponding product space, relative to the product σ-algebra on $V \times V$. It is important that our initial measure ρ is not assumed finite, but only σ-finite. Since ρ is assumed symmetric, the respective two marginal measures coincide, here denoted μ, and they will also not be finite; only σ-finite. The σ-finiteness will be a crucial fact in our computations of a number of Radon-Nikodym derivatives and norms of operators and vectors.

We establish an explicit correspondence between (i) symmetric and positive measures ρ on one hand, and (ii) a certain set of generalized Markov transition measures P on the other. More precisely, by a generalized Markov transition measure we mean a measurable and measure-valued function P on $(V, \mathcal{B})$, such that for every x in V, $P(x, \cdot)$ is a probability measure on $(V, \mathcal{B})$. From the generalized Markov transition P, we introduce an associated operator, also denoted by P. Its spectral theoretic properties refer to a certain L^2 space, and they will be made precise in Sect. 3.

In addition to the operator P, we shall also consider a natural transfer operator R (the choice of the letter "R" is for David Ruelle who initiated a variant of our analysis in the context of statistical mechanics); and a measure theoretic Laplacian, or Laplace operator. In the special case when V is countably discrete, our Laplace operator will be analogous to a family of more standard discretized classical Laplace operators. For related results on transfer operators, see e.g. [2–13].

Among the motivations for our present results are the following: A recent study of a variety of graph limits. This research area has both a general flavor, and an application-focus; see below. The latter includes recent papers on graphons; a current and extremely active area. In addition, we are motivated by a number of new operator-theoretic approaches to the study of graph limits, such as the notion of action convergence (see the recent works by Backhausz and Szegedy, [14, 15] and Pensky [16]). While we mention some of these connections inside our paper, our present emphasis is the theoretic foundations for these related developments.

New Results It is important to note that our setting is not restricted to the case of finite measures. In fact, in our discussion of Markov transition dynamics, important examples simply will not allow finite covariant measures. We recall that the theory of weighted networks can serve as a discrete analog of our measurable settings, see [1] where this analogy was discussed in detail. The corresponding symmetric measure on the edge set E is σ-finite as well as the counting measure μ on the set of vertices V. Our definitions of the energy space $\mathcal{H}_E$, Markov operator P, and the graph Laplace operator Δ are direct translations of the corresponding definitions for weighted networks.

To the best of our knowledge, such interpretations of these objects have not been considered earlier. We stress that our approach to Markov processes generated by σ-finite symmetric measures leads with necessity to the study of Markov transition operators defined on infinite σ-finite measure spaces. The existing literature on Markov processes is devoted mostly to the case of probability measure spaces, see, e.g., [17–19].

The notion of Borel equivalence relation defined on a standard Borel space illustrates our setting, and it can be viewed as a rich source of various examples. We refer to the following books and articles: [20–27].

More applications of measurable setting for the study of Markov processes and Laplacians are given in [1]. We mention here the theory of graphons, Dirichlet forms, and the theory of determinantal measures.

With our starting point, a choice of a fixed symmetric and positive measure ρ on a product space, we will then have four natural Hilbert spaces, three are just L^2 spaces, $L^2(\rho)$, and two L^2 spaces referring to the marginal measure μ. The fourth Hilbert space is different. We call it the finite energy Hilbert space $\mathcal{H}_E$. Its use is motivated by potential theory, and it has a more subtle structure among the considered Hilbert spaces. Given ρ, we introduce an associated energy Hilbert space, denoted $\mathcal{H}_E$, but depending on the initially given ρ. This energy Hilbert space $\mathcal{H}_E$ is not directly linked to the initial L^2 spaces, and its properties are quite different. Nonetheless, the energy Hilbert space $\mathcal{H}_E$ will play a key role in our analysis in the main body of our paper. There are many reasons for this. For example, non-constant harmonic functions will not be in L^2; but, in important applications, they may be in $\mathcal{H}_E$; we refer to the latter as finite energy harmonic functions. A second reason for $\mathcal{H}_E$ is that it plays a crucial role in our introduction of a generalized Green's function. The latter stands in relation to our Laplace operator in a way that is parallel to more classical settings of Green's functions from potential theory. A third reason for $\mathcal{H}_E$

is its use in our analysis of path-space dynamics for the Markov transition system, mentioned above.

Organization Our *main results* are proved in Theorems 3.10, 4.7, 4.11, 5.3, 6.2, 6.11, and 7.2.

The paper is organized as follows. Section 2 contains our basic definitions and preliminary results. We discuss here the concepts of standard Borel and standard measure spaces, kernels, irreducible symmetric measures, and disintegration. The transfer operator R, Markov operator P, and graph Laplacian Δ are defined in Sect. 3. We collected a number of results about the spectral properties of these operators that were proved in [1]. Also the reader will find the definition of the finite energy Hilbert space $\mathcal{H}_E$, several results about the structure of the space $\mathcal{H}_E$ and the norm of functions from $\mathcal{H}_E$. We consider also the embedding operator J and prove that J is an isometry. In Sect. 4, we consider the equivalence of Markov operators and the Laplacians generated by equivalent symmetric measures ρ and ρ'. It turns out that, for equivalent symmetric measures ρ and ρ', there exists an isometry for the corresponding energy Hilbert spaces $\mathcal{H}_E(\rho)$ and $\mathcal{H}_E(\rho')$. The notion of reversible Markov processes is discussed in Sect. 5. We relate various properties of the operator P (such as self-ajointness) to this notion and to the notion of a symmetric measure. A number of results about Markov operators acting in the L^2 spaces and energy space $\mathcal{H}_E$ are proved in this section. Section 6 focuses on the case of a transient Markov processes defined by a Markov operator P. We define the path-space measure $\mathbb{P}$ and Green's function $G(x, A)$, and we discuss their properties. Section 7 is devoted to construction of a sequence of discrete weighted networks which can be used to approximate the objects considered for the measurable setting.

In our article we discuss several key notions such as reversible Markov processes, Green's function, transient processes, limit theory (covering boundaries), potential theory, general Dirichlet forms, graph Laplacians, etc. For the benefit of non-experts in these areas, we included a number of general references in the corresponding sections.

2 Basic Definitions and Symmetric Measures

In this section, we briefly describe our main setting and introduce the most important notation. We also recall several results from [1] which will be used here.

2.1 Standard Borel and Measure Spaces

Suppose V is a *Polish space*, i.e., V is a separable completely metrizable topological space. Let $\mathcal{B}$ denote the σ-algebra of Borel sets generated by open sets of V. Then $(V, \mathcal{B})$ is called a *standard Borel space*. The theory of standard Borel spaces is discussed in many recent books, see e.g., [25, 26, 28, 29] and papers [30, 31]. We recall that all uncountable standard Borel spaces are Borel isomorphic, so that one can use any convenient realization of the space V working in the category of measurable spaces. If μ is a continuous (i.e., non-atomic) positive Borel measure on $(V, \mathcal{B})$, then $(V, \mathcal{B}, \mu)$ is called a *standard measure space*. Given $(V, \mathcal{B}, \mu)$, we will call μ a measure for brevity. As a rule, we will deal with non-atomic σ-finite positive measures on $(V, \mathcal{B})$ (unless the opposite is clearly indicated) which take values in the extended real line $\overline{\mathbb{R}}$. We use the name of *standard measure space* for both finite and σ-finite measure spaces. Also the same notation, $\mathcal{B}$, is applied for the σ-algebras of Borel sets and measurable sets of a standard measure space. It should be clear from the context what σ-algebra is considered. Working with a measure space $(V, \mathcal{B}, \mu)$, we always assume that $\mathcal{B}$ is *complete* with respect to μ. By $\mathcal{F}(V, \mathcal{B})$. we denote the space of real-valued bounded Borel functions on $(V, \mathcal{B})$. For $f \in \mathcal{F}(V, \mathcal{B})$ and a Borel measure μ on $(V, \mathcal{B})$, we write

$$\mu(f) = \int_V f \, d\mu.$$

All objects, considered in the context of measure spaces (such as sets, functions, transformations, etc), are determined modulo sets of zero measure. In most cases, we will implicitly use this mod 0 convention not mentioning the sets of zero measure explicitly.

In what follows, we will use (in most cases implicitly) the notion of *measurable fields*. Given a measure space $(V, \mathcal{B}, \mu)$, it is said that $x \mapsto A_x \in \mathcal{B}$ is a *measurable field of sets* if the set

$$\bigcup_{x \in V} \{x\} \times A_x \in \mathcal{B} \times \mathcal{B}.$$

Similarly, one can define a *measurable field of measures* $x \to \mu_x$ on $(V, \mathcal{B})$ requiring $x \mapsto \mu_x(A)$ to be a measurable function for any $A \in \mathcal{B}$.

Consider a σ-finite continuous measure μ on a standard Borel space $(V, \mathcal{B})$. We denote by

$$\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu) = \{A \in \mathcal{B} : \mu(A) < \infty\} \tag{2.1}$$

the algebra of Borel sets of finite measure μ. Clearly, any σ-finite measure μ is uniquely determined by its values on $\mathcal{B}_{\text{fin}}(\mu)$.

The linear space of simple function over sets from $\mathcal{B}_{\text{fin}}(\mu)$ is denoted by

$$\begin{aligned} \mathcal{D}_{\text{fin}}(\mu) &:= \left\{ \sum_{i \in I} a_i \chi_{A_i} : A_i \in \mathcal{B}_{\text{fin}}(\mu),\ a_i \in \mathbb{R},\ |I| < \infty \right\} \\ &= \text{Span}\{\chi_A : A \in \mathcal{B}_{\text{fin}}(\mu)\}, \end{aligned} \tag{2.2}$$

will play an important role in our work since simple functions from $\mathcal{D}_{\text{fin}}(\mu)$ form a norm dense subset in $L^p(\mu)$-space, $p \geq 1$.

2.2 Symmetric Measures, Kernels, and Disintegration

Definition 2.1 Let E be an uncountable Borel subset of the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$ such that:

(i) $(x, y) \in E \iff (y, x) \in E$, i.e. $\theta(E) = E$ where $\theta(x, y) = (y, x)$ is the flip automorphism;
(ii) $E_x := \{y \in V : (x, y) \in E\} \neq \emptyset, \quad \forall x \in X$;
(iii) for every $x \in V$, $(E_x, \mathcal{B}_x)$ is a standard Borel space where $\mathcal{B}_x$ is the σ-algebra of Borel sets induced on E_x from $(V, \mathcal{B})$.

We call E a *symmetric set*.

It follows from (ii) and (iii) that the projection of the symmetric set E on each margin of the product space $(V \times V, \mathcal{B} \times \mathcal{B})$ is V.

We observe that conditions (ii) and (iii) are, strictly speaking, not related to the symmetry property; they are included in Definition 2.1 for convenience, so that we will not have to make additional assumptions. Condition (iii) assumes two cases: the Borel space E_x can be countable or uncountable. We focus mostly on uncountable Borel standard spaces.

There are several natural examples of symmetric sets related to dynamical systems. We mention here the case of a *Borel equivalence relation* E on a standard Borel space $(V, \mathcal{B})$. By definition, E is a Borel subset of $V \times V$ such that $(x, x) \in E$ for all $x \in V$, (x, y) is in E iff (y, x) is in E, and $(x, y) \in E$, $(y, z) \in E$ implies that $(x, z) \in E$. Let $E_x = \{y \in V : (x, y) \in E\}$, then E is partitioned into "vertical fibers" E_x. In particular, it can be the case when every E_x is countable. Then E is called a *countable Borel equivalence relation.*

We say that a symmetric set E is *decomposable* if there exists an uncountable Borel subset $A \subset V$ such that

$$E \subset (A \times A) \cup (A^c \times A^c), \tag{2.3}$$

where $A^c = V \setminus A$.

The meaning of this definition can be clarified for Borel equivalence relations: if E satisfies (2.3), then the set A is *E-invariant*.

We recall several definitions and facts about kernels defined on a measurable space, see e.g. [18, 19]. Given a standard measure space $(V, \mathcal{B})$, we define a *σ-finite kernel k* as a function $k : V \times \mathcal{B} \to \overline{\mathbb{R}}_+$ (where $\overline{\mathbb{R}}_+$ is the extended real line) such that

(i) $x \mapsto k(x, A)$ is measurable for every $A \in \mathcal{B}$;
(ii) for any $x \in V$, $k(x, \cdot)$ is a σ-finite measure on $(V, \mathcal{B})$.

A kernel $k(x, A)$ is called *finite* if $k(x, \cdot)$ is a finite measure on $(V, \mathcal{B})$ for every x. We will also use the notation $k(x, dy)$ for the measure on $(V, \mathcal{B})$.

The definition of a finite kernel can be used to define new measures on the measurable spaces $(V, \mathcal{B})$ and $(V \times V, \mathcal{B} \times \mathcal{B})$.

Given a σ-finite measure space $(V, \mathcal{B}, \mu)$ and a finite kernel $k(x, A)$, we set

$$\kappa(A) = \int_V k(x, A)\, d\mu(x).$$

Then κ is a σ-finite measure on $(V, \mathcal{B})$ (which is also called a *random measure* in the literature).

For a kernel k as above, one can define inductively the sequence of kernels $(k^n : n \geq 1)$ by setting

$$k^n(x, A) = \int_V k^{n-1}(y, A)\, k(x, dy), \qquad n > 1. \tag{2.4}$$

Following [18], we formulate definitions of main properties of a kernel k. We say that a *set* $A \in \mathcal{B}$ *is attainable from* $x \in V$ if there exists $n \geq 1$ such that $k^n(x, A) > 0$, in symbols, we write $x \to A$. A set $A \in \mathcal{B}$ is called *closed* for the kernel k if $k(x, A^c) = 0$ for all $x \in A$. If A is closed, then it follows from (2.4) that $k^n(x, A^c) = 0$ for any $n \in \mathbb{N}$ and $x \in A$. Hence, A is closed if and only if $x \nrightarrow A^c$.

A kernel $k = k(x, A)$ is called *Borel indecomposable* on $(V, \mathcal{B})$ if there do not exist two disjoint non-empty closed subsets A_1 and A_2.

Let $F_x \in \mathcal{B}$ be the support of the measure $k(x, \cdot)$, that is $k(x, V \setminus F_x) = 0$. By $\widetilde{F}_x$, we denote the set $\{x\} \times K_x \subset V \times V$. Then the formula

$$k(A \times B) = \int_A \widetilde{k}(x, B)\, d\mu(x)$$

defines a σ-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ where $\widetilde{k}(x, \cdot) = (\delta_x \times k)(x, \cdot)$. The support of k is the set

$$F := \bigcup_{x \in V} \widetilde{F}_x.$$

We will use below slightly simplified notation identifying the sets F_x and $\widetilde{F}_x$ and the measures $k(x, A)$ and $\widetilde{k}(x, A)$. It will be clear from the context what objects are considered.

As mentioned in Introduction, our approach is based on the study of *symmetric measures* defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, see Definition 2.4. We show that every measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ generates a kernel $x \to \rho_x(A)$, $A \in \mathcal{B}$. This observation is based on the concept of *disintegration* of the measure ρ. We recall here this construction.

Denote by π_1 and π_2 the projections from $V \times V$ onto the first and second factor, respectively. Then $\{\pi_1^{-1}(x) : x \in V\}$ and $\{\pi_2^{-1}(y) : y \in V\}$ are the *measurable partitions* of $V \times V$ into vertical and horizontal fibers, see [1, 22, 32] for more information on properties of measurable partitions. The case of probability measures was studied by Rokhlin in [32], whereas the disintegration of σ-finite measures has been considered somewhat recently. We refer to a result from [33] whose formulation is adapted to our needs.

Theorem 2.2 ([33]) *For a σ-finite measure space $(V, \mathcal{B}, \mu)$, let ρ be a σ-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $\rho \circ \pi_1^{-1} \ll \mu$. Then there exists a unique system of conditional σ-finite measures $(\widetilde{\rho}_x)$ such that*

$$\rho(f) = \int_V \widetilde{\rho}_x(f)\, d\mu(x), \quad f \in \mathcal{F}(V \times V, \mathcal{B} \times \mathcal{B}).$$

In the following remark we collect several facts that clarify the essence of the defined objects.

Remark 2.3

(1) The condition of Theorem 2.2 assumes that a measure μ is prescribed on the Borel space $(V, \mathcal{B})$. If one begins with a measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$, then the measure μ arises as the projection of ρ on $(V, \mathcal{B})$, $\rho \circ \pi_1^{-1} = \mu$.

(2) Let E be a Borel symmetric subset of $(V \times V, \mathcal{B} \times \mathcal{B})$, and let ρ be a measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying the condition of Theorem 2.2. Then E can be partitioned into the fibers $\{x\} \times E_x$. By Theorem 2.2, there exists a unique system of conditional measures $\widetilde{\rho}_x$ such that, for any ρ-integrable function $f(x, y)$, we have

$$\iint_{V \times V} f(x, y)\, d\rho(x, y) = \int_V \widetilde{\rho}_x(f)\, d\mu(x). \tag{2.5}$$

It is obvious that, for μ-a.e. $x \in V$, $\mathrm{supp}(\widetilde{\rho}_x) = \{x\} \times E_x$ (up to a set of zero measure). To simplify the notation, we will write

$$\int_V f\, d\rho_x \ \text{ and } \iint_{V \times V} f\, d\rho$$

though the measures ρ_x and ρ have the supports E_x and E, respectively.

(3) It follows from Theorem 2.2 that the measure ρ determines the measurable field of sets $x \mapsto E_x \subset V$ and measurable field of σ-finite Borel measures $x \mapsto \rho_x$ on $(V, \mathcal{B})$, where the measures ρ_x are defined by the relation

$$\widetilde{\rho}_x = \delta_x \times \rho_x. \tag{2.6}$$

Hence, relation (2.5) can be also written in the following form, used in our subsequent computations,

$$\iint_{V\times V} f(x,y)\,d\rho(x,y) = \int_V \left(\int_V f(x,y)\,d\rho_x(y)\right)\,d\mu(x). \tag{2.7}$$

In other words, we have a measurable family of measures $(x \mapsto \rho_x)$, and it defines a new measure ν on $(V, \mathcal{B})$ by setting

$$\nu(A) := \int_V \rho_x(A)\,d\mu(x), \quad A \in \mathcal{B}. \tag{2.8}$$

Remark that the measure ρ_x is defined on the subset E_x of $(V, \mathcal{B})$, $x \in V$.

Definition 2.4 Let $(V, \mathcal{B})$ be a standard Borel space. We say that a measure ρ on $(V \times V, \mathcal{B} \times B)$ is *symmetric* if

$$\rho(A \times B) = \rho(B \times A), \quad \forall A, B \in \mathcal{B}.$$

In other words, ρ is invariant with respect to the flip automorphism θ.

The following remark contains natural properties of symmetric measures. Some of them were proved in [1], the others are rather obvious.

Remark 2.5

(1) If ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, then the support of ρ, the set $E = E(\rho)$, is symmetric mod 0. Here $E(\rho)$ is defined up to a set of zero measure by the relation $\rho((V \times V) \setminus E) = 0$.

(2) We consider the symmetric measures whose supporting sets E satisfy Definition 2.1. In other words, we require that, for every $x \in V$, the set $E_x \subset E$ is uncountable and therefore is a standard Borel space. The case when E_x is countable arises, in particular, when E is a Borel countable equivalence relation on $(V, \mathcal{B})$. The latter was considered in [1]. For countable sets E_x, $x \in V$, we can take ρ_x as a finite measure which is equivalent to the counting measure, see, e.g. [24, 34, 35] for details.

(3) In general, the notion of a symmetric measure is defined in the context of standard Borel spaces $(V, \mathcal{B})$ and $(V \times V, \mathcal{B} \times \mathcal{B})$. But if a σ-finite measure μ is given on $(V, \mathcal{B})$, then we need to include an additional relation between the projections of ρ on V and the measure μ. Let $\pi_1 : V \times V \to V$ be the projection on the first coordinate. We require that the symmetric measure must satisfy the property $\rho \circ \pi_1^{-1} \ll \mu$, see Theorem 2.2.

(4) The symmetry of the set E allows us to define a "mirror" image of the measure ρ. Let $E^y := \{x \in V : (x, y) \in E\}$, and let $(\widetilde{\rho}^y)$ be the system of conditional

measures with respect to the partition of E into the sets $E^y \times \{y\}$. Then, for the measure

$$\widetilde{\rho} = \int_V \widetilde{\rho}^y d\mu(y),$$

the relation $\rho = \widetilde{\rho}$ holds.

(5) It is worth noting that, in general, when a measure μ is defined on $(V, \mathcal{B})$, the set $E(\rho)$ do not need to be a set of positive measure with respect to the product measure $\mu \times \mu$. In other words, we admit both cases: (a) ρ is equivalent to $\mu \times \mu$, (b) ρ and $\mu \times \mu$ are mutually singular.

Assumption 1 In this paper, we consider the class of symmetric measures ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ which satisfy the following property:

$$0 < c(x) := \rho_x(V) < \infty, \qquad \mu\text{-a.e. } x \in V, \tag{2.9}$$

where $x \mapsto \rho_x$ is the measurable field of measures arising in Theorem 2.2.

Moreover, in most statements, we will assume that $c(x) \in L^1_{\mathrm{loc}}(\mu)$, i.e.,

$$\int_A c(x)\, d\mu(x) < \infty, \qquad \forall A \in \mathcal{B}_{\mathrm{fin}}(\mu).$$

This property of the function $c(x)$ is natural because it corresponds to local finiteness of graphs in the theory of weighted (electric) networks. In several statements, we will require that

$$\left(\forall A \in \mathcal{B}_{\mathrm{fin}}(\mu),\ \int_A c^2\, d\mu < \infty\right) \Longleftrightarrow\ c \in L^2_{\mathrm{loc}}(\mu).$$

We observe also that the case when the function c is bounded leads to bounded Laplace operators and is not interesting for us.

Relation (2.8) defines the measure ν such that the measures μ and ν are equivalent. It is stated in Lemma 2.6 that $c(x)$ is the Radon-Nikodym derivative of ν with respect to μ. If we want to reverse the definition and use ν as a primary measure, then we need to require that the function $c(x)^{-1}$ is locally integrable with respect to ν.

The following (important for us) fact follows from the definition of symmetric measures. We emphasize that formula (2.10) will be used repeatedly in many proofs.

Lemma 2.6

(1) For a symmetric measure ρ and any bounded Borel function f on $(V \times V, \mathcal{B} \times \mathcal{B})$,

$$\iint_{V\times V} f(x, y)\, d\rho(x, y) = \iint_{V\times V} f(y, x)\, d\rho(x, y). \tag{2.10}$$

Equality (2.10) is understood in the sense of the extended real line, i.e., the infinite value of the integral is allowed.

(2) Let ν be defined as in (2.8). Then

$$d\nu(x) = c(x)d\mu(x).$$

2.3 Irreducible Symmetric Measures

We now relate the notions of symmetric measures and kernels. It turns out that one can associate a finite kernel $\mathcal{K}(\rho) = \mathcal{K}$ to any symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$. For this, we use the disintegration of ρ according to Theorem 2.2, $\rho = \int_V \rho_x \, d\mu(x)$, and set $x \to \mathcal{K}(x, A) = \rho_x(A)$.

The definition of sets attainable from $x \in V$ and that of decomposable sets, given above in the context of Borel spaces, can be translated to the case of measure spaces. Below we define the notion of an *irreducible symmetric measure* which will be extensively used in the paper.

Definition 2.7

(1) A kernel $x \to k(x, \cdot)$ is called *irreducible with respect to a σ-finite measure μ on $(V, \mathcal{B})$ (μ-irreducible)* if, for any set A of positive measure μ and μ-a.e. $x \in V$, there exists some n such that $k^n(x, A) > 0$, i.e., any set A of positive measure is attainable from μ-a.e. x, $x \to A$.

(2) A *symmetric measure* ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ is called *irreducible* if the corresponding kernel $\mathcal{K}(\rho) : x \to \rho_x(\cdot)$ is μ-irreducible where μ is the projection of measure ρ.

(3) A symmetric measure ρ (or the kernel $x \to \rho_x(\cdot)$) is called *μ-decomposable* if there exists a Borel subset A of V of positive measure μ such that

$$E \subset (A \times A) \cup (A^c \times A^c) \tag{2.11}$$

where $A^c = V \setminus A$ is also of positive measure. Otherwise, ρ is called *indecomposable*.

Every kernel k, defined on $(V, \mathcal{B})$, generates the *potential kernel*

$$G(k)(x, A) := \sum_{n=0}^{\infty} k^n(x, A)$$

where $k^0(x, A) = \chi_A(x)$. In general, the kernel G may be degenerated admitting only the values 0 and ∞. We will discuss below the role of G in the case of transient Markov processes.

Lemma 2.8 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ with the kernel $\mathcal{K}(x, A) = \rho_x(A)$. Suppose that the support of ρ, the set E, satisfies relation (2.11) where $\mu(A) > 0$ and $\mu(A^c) > 0$, i.e. the kernel $x \mapsto \rho_x(A)$ is μ-decomposable. Then the sets A and A^c are closed and $x \mapsto \rho_x(A)$ is a μ-reducible kernel. The converse statement also holds.*

Proof The first result follows directly from the definitions given above in this subsection. To see that the converse is true, it suffices to note that, for any set B of positive measure, the compliment $\widehat{B}^c$ of the set

$$\widehat{B} := B \cup \{x \in V : x \to B\}$$

is either of zero measure, or closed (recall that $x \to B$ means that there exists n such that $\mathcal{K}^n(x, B) > 0$). If ρ is reducible, then there exists a set A, $\mu(A) > 0$, such that the closed set $\mu(\widehat{A^c})$ has positive measure. The existence of such a set implies that the measure ρ is decomposable. □

It is obvious from this lemma that a decomposable symmetric measure ρ cannot be irreducible. It was proved in [1] that the definitions of an irreducible measure and irreducible kernel agree, see Theorem 6.2 below.

By definition, the projection of the support of an irreducible measure ρ is the set V. Irreducibiliity of symmetric measures means irreducibility of a corresponding Markov process, see details in [1].

In the following statement, we give another approach to the notion of irreducible symmetric measures. Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We use the support of the fiber measure ρ_x, $x \in V$, to characterize an irreducible measure in different terms.

For any fixed $x \in V$, we define a sequence of subsets: $A_0(x) = \{x\}$, $A_1(x) = E_x$,

$$A_n(x) = \bigcup_{y \in A_{n-1}(x)} E_y, \quad n \geq 2.$$

Recall that E_x is the support of the measure ρ_x, and E_x can be identified with the vertical section of the symmetric set E. Note that all the sets $A_n(x)$ are in $\mathcal{B}$ as $x \to E_x$ is a measurable field of sets.

Lemma 2.9 *Given $(V, \mathcal{B}, \mu)$, a symmetric measure ρ is irreducible if and only if for μ-a.e. $x \in V$ and any set $B \in \mathcal{B}$ of positive measure there exists $n \geq 1$ such that*

$$\mu(A_n(x) \cap B) > 0. \tag{2.12}$$

Proof Indeed, the property formulated in (2.12) is another form of $k^n(x, B) > 0$ where the kernel k is defined by $x \to \rho_x$. □

Various aspects of symmetric measures are also discussed in [36, 37]. In particular, one can observe that if symmetric measures ρ and $\overline{\rho}$ are equivalent, then they are simultaneously either irreducible or not.

3 Linear Operators and Hilbert Spaces Associated to Symmetric Measures

While the main structures of our paper (symmetric measures, transfer operators R, Markov transition densities P, and associated Laplace operators Δ) may be naturally formulated in the general context of measurable functions, their spectral theory, and their dynamic-systems properties, only take a precise form after suitable Hilbert spaces are introduced. We will show that the initial structures, reversible Markov processes, and associated Laplace operators, etc., in turn dictate their own natural Hilbert space theoretic context. More precisely, in the section below, we identify the particular L^2 spaces, having the property that respective operators R, P, and Δ become self-adjoint. In addition to these L^2 spaces, we also identify two other Hilbert spaces (details below). They are motivated by parallels to classical potential theory, and to the study of diffusion processes. Moreover, they have discrete counterparts in the study of infinite networks, and of graph Laplacians. But presently, we introduced these two Hilbert spaces in a general measure space context. Continuing conventions from our earlier papers, we shall denote these Hilbert spaces (i) the energy Hilbert space, and (ii) the dissipation Hilbert space. The latter refers to a certain path-space construction, which in turn is built directly from the initial structure, mentioned above, symmetric measure, transfer operator, and Markov transition densities.

3.1 Symmetric Operator R, Markov Operator P, and Laplacian Δ

Suppose $k : V \times \mathcal{B} \to \mathbb{R}_+$ is a finite kernel defined on a standard Borel space $(V, \mathcal{B})$. Then it defines a linear positive (see Remark 3.3) operator $P(k)$ which is determined by the kernel k:

$$P(k)(f)(x) := \int_V f(y)\, k(x, dy). \tag{3.1}$$

It can be easily seen that, for the kernels k^n (see (2.4)), the operator $P(k^n)$, defined as in (3.1), satisfies the property:

$$P(k^n) = P(k)^n, n \in \mathbb{N}.$$

We consider in this section the kernel $\mathcal{K}(\rho)$ generated by a symmetric measure ρ, i.e., $\mathcal{K}(x, A) = \rho_x(A)$.

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and ρ a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ supported by a symmetric set E. Let $x \mapsto \rho_x$ be the measurable family of measures on $(V, \mathcal{B})$ that *disintegrates* ρ. Recall that, by Assumption 1, the function

$c(x) = \rho_x(V)$ is finite for μ-a.e. x. As discussed above in Sect. 2.2, the measure ρ produces a finite kernel $\mathcal{K}(\rho)$ which we use to define the following operators.

Definition 3.1 For a symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we define three linear operators R, P and Δ acting on the space of bounded Borel functions $\mathcal{F}(V, \mathcal{B})$.

(i) The *symmetric operator* R:

$$R(f)(x) := \int_V f(y)\, d\rho_x(y) = \rho_x(f). \tag{3.2}$$

(ii) The *Markov operator* P:

$$P(f)(x) = \frac{1}{c(x)} R(f)(x)$$

or

$$P(f)(x) := \frac{1}{c(x)} \int_V f(y)\, d\rho_x(y) = \int_V f(y)\; P(x, dy) \tag{3.3}$$

where $P(x, dy)$ is the probability measure obtained by normalization of $d\rho_x(y)$, i.e.

$$P(x, dy) := \frac{1}{c(x)} d\rho_x(y).$$

In other words, the Markov operator P defines the measurable field $x \mapsto P(x, \cdot)$ of *transition probabilities* on the space $(V, \mathcal{B})$, or a *Markov process*.

(iii) The *graph Laplace operator* Δ:

$$\Delta(f)(x) := \int_V (f(x) - f(y))\, d\rho_x(y) \tag{3.4}$$

or

$$\Delta(f) = c(I - P)(f) = (cI - R)(f). \tag{3.5}$$

Using (2.9), we can write the operator Δ in more symmetric form:

$$\Delta(f) = R(\mathbb{1})f - R(f)$$

where $\mathbb{1}$ is a function identically equal to 1,

Remark 3.2 (*R* as a Transfer Operator) It is worth noting that the operator R can be treated as a transfer operator (see e.g. [38] and the literature cited there).

Let $(V, \mathcal{B}, \mu)$ be a standard measure space, and let σ be a surjective endomorphism of X. Consider the partition ξ of X into the orbits of σ: $y \in Orb_\sigma(x)$

if there are non-negative integers n, m such $\sigma^n(y) = \sigma^m(x)$. Let the partition η be the measurable hull of ξ. Take the system of conditional measures $\{\mu_C\}_{C\in\xi}$ corresponding to the partition η (see Theorem 2.2).

We define a transfer operator R on the standard measure space $(V, \mathcal{B}, \mu)$ by setting

$$R(f)(x) := \int_{C_x} f(y)\, d\mu_{C_x}(y) \tag{3.6}$$

where C_x is the element of η containing x. The domain of R is $L^1(\mu)$ in this example.

As was shown in [38], the operator $R : L^1(\mu) \to L^1(\mu)$ defined by (3.6) is a *transfer operator*, i.e., it satisfies the relation

$$R((f \circ \sigma)g)(x) = f(x)(Rg)(x).$$

To see that our definition of the operator R given in (3.2) agrees with (3.6), it suffices to take the measurable partition η of $V \times V$ into subsets $\{\pi_1^{-1}(x) : x \in V\}$ where π_1 is the projection of $V \times V$ onto V.

Remark 3.3 In this remark we make several comments about the basic properties of the operators R, P, and Δ.

(1) The definition of each of the operators R, P, and Δ depends on a symmetric measure ρ, and, strictly speaking, they must be denoted as $R(\rho)$, $P(\rho)$, and $\Delta(\rho)$. Since most of our results are proved for a fixed measure ρ, we will drop this variable. Below in this section, we discuss the relationships between $P(\rho)$ and $P(\rho')$ when ρ and ρ' are equivalent symmetric measures.
(2) The operators R and P are *positive* in the sense that $R(f) \geq 0$ and $P(f) \geq 0$ whenever $f \geq 0$. Moreover, if $f = \mathbb{1}$, then $P(\mathbb{1}) = \mathbb{1}$ because every measure $P(x, \cdot)$ is probability. Hence, P is a *Markov operator.*
(3) The properties of the graph Laplace operator Δ are formulated in Proposition 3.7, which is given below. All statements from this theorem are proved in [1] (see also [39, 40]). Other aspects of graph Laplace operators in the context of measure spaces are discussed in [41, 42].
(4) Since every measure ρ on $V \times V$ is uniquely determined by its values on a dense subset of functions, it suffices to define ρ on the set of the so-called "cylinder functions" $(f \otimes g)(x, y) := f(x)g(y)$. This observation will be used below when we prove a relation for cylinder functions first.
(5) In general, a positive operator R in $\mathcal{F}(V, \mathcal{B})$ is called *symmetric* if it satisfies the relation:

$$\int_V fR(g)\, d\mu = \int_V R(f)g\, d\mu, \tag{3.7}$$

for any $f, g \in F(V, \mathcal{B})$. It turns out that any symmetric operator R defines a symmetric measure ρ. Indeed, the functional

$$\rho : (f, g) \mapsto \int_V f(x)R(g)(x)\, d\mu(x), \qquad f, g \in F(V, \mathcal{B}), \tag{3.8}$$

determines a measure on $(V, \mathcal{B})$ such that

$$\rho(A \times B) = \int_V \chi_A(x)R(\chi_B)(x)\, d\mu(x).$$

As shown in [1], the operator R is symmetric if and only if the measure ρ, defined in (3.8), is symmetric.

In Definition 3.1, we do not discuss domains of the operators R, P, and Δ. It depends on the space where an operator is considered. In the current paper, we work with L^2-Hilbert spaces defined by the measures μ, ν, and ρ. But the most intriguing is the case of the finite energy space Hilbert space h_E. We discuss the properties of this space as well as those of operators Δ and P acting in $\mathcal{H}_E$ in the forthcoming paper [43]. On the other hand, we have already proved a number of results about these objects in [1]. We find it useful to give here the definitions and some formulas which are used below.

We remark that the finite energy space $\mathcal{H}_E$, see Definition 3.4 can be viewed as a generalization of the energy space considered for discrete weighted networks. They have been extensively studied during last decades.

Definition 3.4 Let $(V, \mathcal{B}, \mu)$ be a standard measure space with σ-finite measure μ. Suppose that ρ is a symmetric measure on the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$. We say that a Borel function $f : V \to \mathbb{R}$ belongs to the *finite energy space* $\mathcal{H}_E = \mathcal{H}_E(\rho)$ if

$$\iint_{V\times V} (f(x) - f(y))^2\, d\rho(x, y) < \infty. \tag{3.9}$$

Remark 3.5

(1) It follows from Definition 3.4 that $\mathcal{H}_E$ is a vector space containing all constant functions. We identify functions f_1 and f_2 such that $f_1 - f_2 = const$ and, with some abuse of notation, the quotient space is also denoted by $\mathcal{H}_E$. So that, we will call elements f of $\mathcal{H}_E$ functions assuming that a representative of the equivalence class f is considered.
(2) Definition 3.4 assumes that a symmetric irreducible measure ρ is fixed on $(V \times V, \mathcal{B} \times \mathcal{B})$. This means that the space of functions f on $(V, \mathcal{B})$ satisfying (3.9) depends on ρ, and, to stress this fact, we will use also the notation $\mathcal{H}_E(\rho)$.

Define the norm in $\mathcal{H}_E$ by setting

$$||f||^2_{\mathcal{H}_E} := \frac{1}{2}\iint_{V\times V}(f(x)-f(y))^2\,d\rho(x,y), \quad f\in\mathcal{H}, \tag{3.10}$$

As proved in [1], $\mathcal{H}_E$ is a *Hilbert space* with respect to the norm $||\cdot||_{\mathcal{H}_E}$.

The description of the structure of the Hilbert space $\mathcal{H}_E$ is a very intriguing problem. We give here a few results proved in [1].

Theorem 3.6 *Let ρ be a symmetric measure on $(V\times V, \mathcal{B}\times\mathcal{B})$ such that $\mu = \rho\circ\pi_1^{-1}$. Suppose $c(x)=\rho_x(V)$ is locally integrable with respect to μ.*

(1) For the measure $d\nu(x) = c(x)d\mu(x)$, we have

$$\mathcal{D}_{\text{fin}}(\mu)\subset\mathcal{D}_{\text{fin}}(\nu)\subset\mathcal{H}_E.$$

Moreover, if $A\in\mathcal{B}_{\text{fin}}(\nu)$, then

$$||\chi_A||^2_{\mathcal{H}_E} = \rho(A\times A^c)\le\int_A c(x)\,d\mu(x) = \nu(A), \tag{3.11}$$

where $A^c := V\setminus A$.

(2) For every $A\in\mathcal{B}_{\text{fin}}(\mu)$, one has $\|\chi_A\|_{\mathcal{H}_E} = \|\chi_{A^c}\|_{\mathcal{H}_E}$. The function χ_A is in $\mathcal{H}_E$ if and only if either $\mu(A)<\infty$ or $\mu(A^c)<\infty$.

(3) The finite energy space $\mathcal{H}_E$ admits the decomposition into the orthogonal sum

$$\mathcal{H} = \overline{\mathcal{D}_{\text{fin}}(\mu)}\oplus\mathcal{H}arm_E \tag{3.12}$$

where the closure of $\mathcal{D}_{\text{fin}}(\mu)$ is taken in the norm of the Hilbert space $\mathcal{H}_E$.

In the following statement we return to the L^2-spaces, and following [1], we formulate a number of properties of the operators, R, P, and Δ that clarify their essence. Here, we focus on the properties of these operators related to L^2-spaces. In the next paper [43], we will mostly consider these operators acting in the finite energy space $\mathcal{H}_E$.

Proposition 3.7 *Let $d\nu(x) = c(x)d\mu(x)$ be the σ-finite measure on $(V,\mathcal{B})$ where μ and $c(x)=\rho_x(V)$ are defined as above. Let the operators R, P, and Δ be defined as in Definition 3.1.*

(1) Suppose that the function $x\mapsto\rho_x(A)\in L^2(\mu)$ for any $A\in\mathcal{B}_{\text{fin}}$. Then R is a symmetric unbounded operator in $L^2(\mu)$, i.e.,

$$\langle g, R(f)\rangle_{L^2(\mu)} = \langle R(g), f\rangle_{L^2(\mu)}.$$

If $c \in L^\infty(\mu)$, then $R : L^2(\mu) \to L^2(\mu)$ is a bounded operator, and

$$||R||_{L^2(\mu)\to L^2(\mu)} \le ||c||_\infty.$$

(2) The operator $R : L^1(\nu) \to L^1(\mu)$ is contractive, i.e.,

$$||R(f)||_{L^1(\mu)} \le ||f||_{L^1(\nu)}, \qquad f \in L^1(\nu).$$

Moreover, for any function $f \in L^1(\nu)$, the formula

$$\int_V R(f)\, d\mu(x) = \int_V f(x)c(x)\, d\mu(x) \tag{3.13}$$

holds. In other words, $\nu = \mu R$, and

$$\frac{d(\mu R)}{d\mu}(x) = c(x).$$

(3) The bounded operator $P : L^2(\nu) \to L^2(\nu)$ is self-adjoint. Moreover, $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$.

(4) The operator P considered in the spaces $L^2(\nu)$ and $L^1(\nu)$ is contractive, i.e.,

$$||P(f)||_{L^2(\nu)} \le ||f||_{L^2(\nu)}, \qquad ||P(f)||_{L^1(\nu)} \le ||f||_{L^1(\nu)}.$$

(5) Spectrum of P in $L^2(\nu)$ is a subset of $[-1, 1]$.

(6) The graph Laplace operator $\Delta : L^2(\mu) \to L^2(\mu)$ is a positive definite essentially self-adjoint operator with domain containing $\mathcal{D}_{\text{fin}}(\mu)$. Moreover,

$$||f||^2_{\mathcal{H}_E} = \int_V f\Delta(f)\, d\mu$$

when the integral in the right hand side exists.

Definition 3.8 A function $f \in \mathcal{F}(V, \mathcal{B})$ is called *harmonic*, if $Pf = f$. Equivalently, f is harmonic if $\Delta f = 0$ or $R(f) = cf$. Similarly, h is *harmonic* for a kernel $x \to k(x, \cdot)$ if

$$\int_V h(y)\, k(x, dy) = h(x).$$

Question As was mentioned above, the definition of operators $R(\rho)$, $P(\rho)$, and $\Delta(\rho)$ is based on a symmetric measure ρ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$. Suppose that another symmetric measure, ρ', which is equivalent to ρ, is defined on $(V \times V, \mathcal{B} \times \mathcal{B})$. It would be interesting to find out what relations between $(R(\rho), P(\rho), \Delta(\rho))$ and $(R(\rho'), P(\rho'), \Delta(\rho'))$ exist. Possibly, this question can be made more precise if we require that both ρ and ρ' are supported by the same symmetric set E and disintegrated with respect to the same measure μ on $(V, \mathcal{B})$.

Remark 3.9 In our further results, the following sets of functions will play an important role. Let $(V, \mathcal{B}, \mu)$ be a σ-measure space, and ρ a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying Assumption 1. Then the measure $d\nu(x) = c(x)d\mu(x)$ is defined on $(V, \mathcal{B})$ and is equivalent to μ where $c(x) = R(\mathbb{1})(x)$. We define $\mathcal{D}_{\text{fin}}(\mu)$ as in (2.2), and, similarly, we set

$$\mathcal{B}_{\text{fin}}(\nu) := \{A \in \mathcal{B} : \nu(A) < \infty\},$$

$$\mathcal{D}_{\text{fin}}(\nu) := \text{Span}\{\chi_A : A \in \mathcal{B}_{\text{fin}}(\nu)\}.$$

It is straightforward to check that Assumption 1 implies

$$\mathcal{D}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{fin}}(\nu).$$

In general, the converse does not hold. But these two sets coincide if and only if Assumption 1 is extended by adding the reverse implication

$$\int_A c(x)\, d\mu(x) \implies \mu(A) < \infty.$$

3.2 Embedding Operator J

We define now a natural embedding J of bounded Borel functions over $(V, \mathcal{B})$ into bounded Borel functions over $(V \times V, \mathcal{B} \times \mathcal{B})$. The operator J will be considered later acting on the corresponding L^2-spaces.

Let

$$(Jf)(x, y) = f(x), \quad f \in \mathcal{F}(V, \mathcal{B}). \tag{3.14}$$

If $(V, \mathcal{B})$ is equipped with a σ-finite measure μ (or $\nu = c\mu$), we can specify J as an operator with domain $L^2(\mu)$ or $L^2(\nu)$).

Theorem 3.10 *For given $(V, \mathcal{B}, \mu)$, let ρ be a symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ and $c(x) = \rho_x(V)$. Then:*

(1) the operator $J : L^2(\nu) \to L^2(\rho)$ is an isometry where $d\nu(x) = c(x)d\mu(x)$;
(2) the co-isometry $J^* : L^2(\rho) \to L^2(\nu)$ acts by the formula

$$(J^*g)(x) = \int_V g(x, y)\, P(x, dy), \qquad g \in L^2(\rho);$$

(3) the operator $J : L^2(\mu) \to L^2(\rho)$ is densely defined (in $L^2(\mu)$) and is, in general, unbounded.

Proof

(1) This fact is proved by the following computation: for any $f \in L^2(\nu)$, one has

$$\begin{aligned}||(Jf)||^2_{L^2(\rho)} &= \iint_{V\times V} (Jf)^2(x,y)\, d\rho(x,y)\\ &= \iint_{V\times V} f^2(x)\, d\rho_x(y) d\mu(x)\\ &= \int_V f^2(x)c(x)\, d\mu(x)\\ &= ||f||^2_{L^2(\nu)}.\end{aligned}$$

(2) To find the co-isometry J^*, we take arbitrary functions $f \in L^2(\nu)$ and $g \in L^2(\rho)$ and compute the inner product using the equality $c(x)P(x,dy) = d\rho_x(y)$:

$$\begin{aligned}\langle Jf, g\rangle_{L^2(\rho)} &= \iint_{V\times V} (Jf)(x,y)g(x,y)\, d\rho(x,y)\\ &= \int_V f(x)\left(\int_V g(x,y)\, d\rho_x(y)\right) d\mu(x)\\ &= \int_V f(x)\left(\int_V g(x,y)\ P(x,dy)\right) d\nu(x)\\ &= \langle f, J^*g\rangle_{L^2(\nu)},\end{aligned}$$

where $J^*g = \int_V g(x,y)\ P(x,dy)$. This proves (2).

(3) To show that (3) holds, we take a Borel function $f \in L^2(\mu)$ and note that

$$||Jf||^2_{L^2(\rho)} = \iint_{V\times V} f^2(x)\, d\rho_x d\mu(x) = \int_V f^2(x)c(x)\, d\mu(x). \tag{3.15}$$

In particular, we have, for $A \in \mathcal{B}_{\mathrm{fin}}$,

$$||J(\chi_A)||^2_{L^2(\rho)} = \int_A c(x)\, d\mu(x),$$

that is, assuming that c is locally integrable, we see that J is well defined on a dense subset of $L^2(\mu)$. Formula (3.15) shows that, for general c, the operator $J : L^2(\mu) \to L^2(\rho)$ is not bounded.

□

4 Equivalence of Symmetric Measures

In this section we focus on the question about relations of Markov operators, and Laplacians, arising from equivalent symmetric measures.

4.1 Equivalence of Markov Operators

Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ which is disintegrated by fiber measures $x \mapsto \rho_x$ over the measure $\mu = \rho \circ \pi^{-1}$. As above, define transition probabilities $x \mapsto P(x, \cdot)$ by setting $c(x)^{-1} d\rho_x(\cdot) = P(x, \cdot)$ where $c(x) = \rho_x(V)$. In other words, $P(x, A) = P(\chi_A)(x)$ where P is the Markov operator, see (3.3).

Having the operator P defined, one can construct a stationary Markov process. Let $\Omega = V \times V \times V \times \cdots = V^{\mathbb{N}_0}$. For $\omega = (\omega_n) \in \Omega$, set

$$X_n : \Omega \to V : X_n(\omega) = \omega_n, \qquad n \in \mathbb{N}_0.$$

These notions are studied in detail in Sect. 5. Here we mention only the notion of *reversibility*, one of the most important properties of Markov operators (processes).

Definition 4.1

(1) A kernel $x \mapsto k(x, \cdot)$ is called *reversible* with respect to a measure μ on $(V, \mathcal{B})$, if for any bounded Borel function $f(x, y)$,

$$\iint_{V \times V} f(x, y) k(x, dy) d\mu(x) = \iint_{V \times V} f(y, x) k(x, dy) d\mu(x).$$

(2) Suppose that $x \mapsto P(x, \cdot)$ is a measurable family of transition probabilities on the space $(V, \mathcal{B}, \mu)$, and let P be the Markov operator determined by $x \mapsto P(x, \cdot)$. It is said that the corresponding Markov process is *reversible* with respect to a measurable functions $c : V \to (0, \infty)$ if, for any sets $A, B \in \mathcal{B}$, the following relation holds:

$$\int_B c(x) P(x, A)\, d\mu(x) = \int_A c(x) P(x, B)\, d\mu(x). \tag{4.1}$$

Denoting $d\nu(x) = c(x) d\mu(x)$, we can rewrite (4.1) in the form that will be used below.

$$\int_V \chi_B(x) P(x, A)\, d\nu(x) = \int_V \chi_A(x) P(x, B)\, d\nu(x).$$

The following result clarifies relationship between symmetric measures ρ and reversible Markov processes. This lemma is a part of more general statement, see Theorem 5.3.

Lemma 4.2 *Let $\rho = \int_V \rho_x \, d\mu$ be a measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $c(x) = \rho_x(V) < \infty$ for all x. Suppose that the Markov operator P is defined according to* (3.3)*. Then the following are equivalent:*

(i) *ρ is symmetric;*
(ii) *(P, c) is reversible.*

In what follows, we will focus on the following *question*: suppose that ρ and ρ' are two *equivalent* symmetric measures such that the corresponding Markov processes (P, c) and (P', c') are reversible. How are they related? More generally, we can ask about relations between all objects whose definition was based on a symmetric measure. They are the Laplacian Δ, symmetric operator R, and finite energy Hilbert space. Some partial answers are given in this and subsequent sections.

Definition 4.3 Let (P, c) be a pair consisting of a positive measurable function $c(x)$ on $(V, \mathcal{B}, \mu)$ and a reversible Markov process $P(x, \cdot)$ satisfying Definition 4.1. We will say that two such pairs (P, c) and (P', c') are *equivalent* if the corresponding symmetric measures ρ and ρ' are equivalent as measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ (see Theorem 5.3). The latter means that there exists a positive measurable function $r(x, y)$ such that

$$d\rho'(x, y) = r(x, y) d\rho(x, y).$$

If the equivalent measures ρ and ρ' satisfy the property $\mu = \rho \circ \pi_1^{-1} = \rho' \circ \pi_1^{-1}$, then we call the pairs (P, c) and (P', c') *strongly equivalent*. In this case, we also call the measures ρ and ρ' *strongly equivalent*.

Remark 4.4

(1) The symmetry of equivalent measures ρ and ρ' implies that the function $r(x, y)$ is symmetric, $r(x, y) = r(y, x)$.
(2) Let the measures ρ and ρ' be strongly equivalent. Then these measures are disintegrated as follows:

$$\rho' = \int_V \rho'_x \, d\mu(x), \qquad \rho = \int_V \rho_x \, d\mu(x).$$

It can be seen that the equivalence of ρ and ρ' implies that the measures ρ_x and ρ'_x are equivalent μ-a.e. Moreover,

$$\frac{d\rho'_x}{d\rho_x}(y) = r_x(y) \tag{4.2}$$

where $r_x(\cdot)$ is obtained from $r(x, \cdot)$ by fixing the variable x.

(3) Conversely, given two (strongly) equivalent measures ρ and ρ', we can construct (strongly) equivalent pairs (P, c) and (P', c') according to the properties formulated in Lemma 4.2 and Theorem 5.3. In other words, if (P, c) defines a reversible Markov process with the symmetric measure ρ, then, for any symmetric measure ρ' equivalent to ρ, we can construct a reversible Markov process (P', c') which is equivalent to (P, c). Note that the functions $c(x) = \rho_x(V)$ and $c'(x) = \rho'_x(V)$ are determined by ρ and ρ' uniquely.

One can prove a more general statement than that given in Remark 4.4 (2).

Lemma 4.5 *Let ρ and ρ' be two symmetric measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $d\rho'(x, y) = r(x, y)d\rho(x, y)$. Suppose that*

$$\rho' = \int_V \rho'_x \, d\mu'(x), \qquad \rho = \int_V \rho_x \, d\mu(x)$$

and the measures μ and μ' on $(V, \mathcal{B})$ are equivalent, i.e., $m(x)d\mu'(x) = d\mu(x)$ for some positive Borel function $m(x)$. Then the measures ρ'_x and ρ_x are equivalent a.e. on V, and

$$\frac{d\rho'_x}{d\rho_x}(y) = m(x)r_x(y). \tag{4.3}$$

Proof (Sketch) The result is deduced as follows:

$$\begin{aligned}
\rho'(A \times B) &= \iint_{A \times B} r(x, y) \, d\rho(x, y) \\
&= \iint_{A \times B} r(x, y) \, d\rho_x(y)d\mu(x) \\
&= \int_A \left(\int_B m(x)r(x, y) \, d\rho_x(y) \right) d\mu'(x).
\end{aligned}$$

On the other hand,

$$\rho'(A \times B) = \int_A \rho'_x(B) \, d\mu'(x).$$

Comparing the above formulas, we obtain that (4.3) holds.

□

Consider a particular case when the Radon-Nikodym derivative $r(x, y)$ of two equivalent measures ρ and ρ' is the product $p(x)q(y)$.

Lemma 4.6 *Let $\rho = \int \rho_x \; d\mu(x)$ and $\rho' = \int \rho'_x \; d\mu'(x)$ be two measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that*

$$\frac{d\rho'}{d\rho}(x, y) = p(x)q(y)$$

for some positive Borel functions p and q. Then, for μ-a.e. $x \in V$, the Radon-Nikodym derivative $\dfrac{d\rho'_x(y)}{d\rho_x(y)}$ satisfies the relation

$$\frac{1}{q(y)}\frac{d\rho'_x(y)}{d\rho_x(y)} = \varphi(x) \tag{4.4}$$

where

$$\varphi(x) = p(x)\frac{d\mu}{d\mu'}(x).$$

Proof The result can be easily deduced from the formula

$$d\rho'_x(y)d\mu'(x) = p(x)q(y)d\rho_x(y)d\mu(x).$$

We leave the details to the reader. □

Relation (4.4) means that the Radon-Nikodym derivative $\dfrac{d\rho'_x}{d\rho_x}(y)$ is proportional to the function $q(y)$ where the coefficient of proportionality is given by $\varphi(x)$. If ρ and ρ' are symmetric measures, then $\dfrac{d\rho'}{d\rho}(x, y) = p(x)p(y)$.

Theorem 4.7 *Let ρ and ρ' be two strongly equivalent measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $d\rho'_x = r_x(y)d\rho_x(y)$ for all $x \in V$. Then the corresponding Markov processes (P, c) and (P', c') are strongly equivalent and*

$$P'(f)(x) = \frac{P(fr_x)(x)}{P(r_x)(x)}. \tag{4.5}$$

Proof We first find $P(r_x)$:

$$\begin{aligned} P(r_x)(x) &= \int_V \frac{d\rho'_x}{d\rho_x}(y)\ P(x, dy) \\ &= \frac{1}{c(x)}\int_V \frac{d\rho'_x}{d\rho_x}(y)\ d\rho_x(y) \\ &= \frac{1}{c(x)}\int_V\ d\rho'_x(y) \qquad (4.6) \\ &= \frac{c'(x)}{c(x)}. \end{aligned}$$

Next, we compute

$$
\begin{aligned}
P'(f)(x) &= \int_V f(y)\; P'(x, dy) \\
&= \frac{1}{c'(x)} \int_V f(y)\; d\rho'_x(y) \\
&= \frac{1}{c'(x)} \int_V f(y) r_x(y)\; d\rho_x(y) \\
&= \frac{c(x)}{c'(x)} \int_V f(y) r_x(y)\; dP(x, dy) \\
&= \frac{c(x)}{c'(x)} P(f r_x)(x)
\end{aligned}
$$

Now, the result follows from (4.6). □

Remark 4.8

(1) Let the symmetric measures ρ and ρ' be strongly equivalent, $d\rho'_x(y) = r_x(y)d\rho_x(y)$. As in (4.6), we can obtain that

$$P'\left(\frac{1}{r_x}\right)(x) = \frac{c(x)}{c'(x)}.$$

Therefore, the following property holds:

$$P(r_x)(x)P'\left(\frac{1}{r_x}\right)(x) = 1$$

(2) Since the notion of equivalence of measures ρ and ρ' is symmetric, we note that the roles of P and P' can be interchanged and the following relation holds:

$$P(f)(x) = \frac{P'\left(f\frac{1}{r_x}\right)(x)}{P'\left(\frac{1}{r_x}\right)(x)}.$$

(3) It follows from the strong equivalence of ρ and ρ' that $r_x(y)$ is integrable with respect to ρ_x and

$$c'(x) = \int_V r_x(y)\; d\rho_x(y).$$

(4) Several useful formulas can be easily deduced from Theorem 4.7. Firstly, formula (4.5) can be rewritten in the form

$$P(fr_x)(x) = c'(x)P'(f)(x)c(x)^{-1}, \tag{4.7}$$

and equivalently, the latter is represented as a relation between Markov kernels:

$$c'(x)P'(x, dy) = c(x)r_x(y)P(x, dy).$$

(5) The same proof as in Theorem 4.7 shows that

$$R'(f)(x) = R(fr_x)(x).$$

(6) In more general setting, assuming that $d\rho'_x(y) = m(x)r_x(y)d\rho_x(y)$ where $m(x)$ is as in (4.3), we deduce that

$$P(fr_x)(x)m(x) = c'(x)P'(f)(x)c(x)^{-1}.$$

Similarly, one can show that

$$R'(f)(x) = m(x)R(fr_x)(x)$$

where the operator R' is defined by $x \mapsto \rho'_x$.

(7) Suppose that, for given pair (P, c), the operator P' is defined by (4.7), and let $d\nu'(x) = c'(x)d\mu(x)$. Then we claim that $\nu'P' = \nu'$:

$$\begin{aligned}
\int_V P'(f)(x)\, d\nu'(x) &= \int_V c(x)P(fr_x)(x)c'(x)^{-1}c'x)\, d\mu(x)\\
&= \int_V P(fr_x)(x)\, d\nu(x)\\
&= \int_V \left(\int_V (fr_x)(y)P(x, dy)\right)\, d\nu(x)\\
&= \iint_{V\times V} f(y)\frac{d\rho'_x}{d\rho_x}(y)c(x)^{-1}d\rho_x(y)c(x)d\mu(x)\\
&= \iint_{V\times V} f(y)\, d\rho'_x(y)d\mu(x)\\
&= \iint_{V\times V} f(x)\, d\rho'(x, y)\\
&= \int_V f(x)c'(x)\, d\mu(x)\\
&= \int f(x)\, d\nu'(x).
\end{aligned}$$

4.2 On the Laplacians Δ and Δ'

In the remaining part of this section, we will discuss relations between the Laplace operators Δ and Δ' acting in the finite energy Hilbert spaces $\mathcal{H}_E(\rho)$ and $\mathcal{H}_E(\rho')$ respectively.

Let $\Delta'(f)$ be the Laplace operator defined by a symmetric measure ρ' on $(V \times V, \mathcal{B} \times \mathcal{B})$. We can find out how Δ' and Δ are related.

Proposition 4.9 *Let ρ and ρ' be two equivalent symmetric measures on $(V\times V, \mathcal{B}\times \mathcal{B})$ such that $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. Then*

$$\Delta'(f) = cqf(P(q) - q) + q\Delta(qf).$$

In particular, when q is harmonic for P, then

$$\Delta'(f) = q\Delta(qf). \tag{4.8}$$

Moreover,

$$\Delta'(f) = 0 \iff P(qf) = fP(q),$$

and assuming that $P(q) = q$, we have

$$f \in \mathcal{H}arm(\Delta') \iff qf \in \mathcal{H}arm(\Delta).$$

Proof

(1) By definition of the operator Δ, we have

$$\begin{aligned}
\Delta'(f)(x) &= \int_V (f(x) - f(y))\, d\rho'_x(y) \\
&= \int_V (f(x) - f(y))q(x)q(y)\, d\rho_x(y) \\
&= \int_V (f(x) - f(y))c(x)q(x)q(y)\, dP(x, dy) \\
&= c(x)q(x)f(x)\int_V q(y)\ P(x, dy) - c(x)q(x)\int_V q(y)f(y)\ P(x, dy) \\
&= c(x)q(x)\left[f(x)P(q)(x) - P(qf)(x)\right].
\end{aligned} \tag{4.9}$$

Add and subtract cq^2f to the right hand side of (4.9). Then, regrouping the terms, we obtain

$$\Delta'(f) = cq[qf - P(qf)] + cqf(P(q) - q) = q\Delta(qf) + cqf(P(q) - q).$$

This means that, in case when $P(q) = q$, the Laplace operators Δ and Δ' are related as in (4.8).

(2) Now we can apply (1) to prove the formulas given in (2). From the last expression in (4.9), we see that f is harmonic with respect to Δ' if and only if $P(qf) = fP(q)$. □

Corollary 4.10 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let q be a harmonic function for the Markov operator P generated by ρ. Define the symmetric measure ρ' such that $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. Let P' be the corresponding Markov operator produced by ρ'. Then we have the map*

$$\mathcal{H}arm(P') \times \mathcal{H}arm(P) \ni (f, q) \mapsto fq \in \mathcal{H}arm(P).$$

Proof It follows from the definition of the measure ρ' that

$$c'(x) = \int_V d\rho'_x(y) = \int_V q(x)q(y)\, d\rho_x(y) = q(x)R(q)(x).$$

Since q is harmonic, i.e., $R(q) = cq$, we obtain that

$$c'(x) = c(x)q^2(x). \tag{4.10}$$

Let f be any function harmonic with respect to the operator P'. Then

$$\begin{aligned} f(x) &= \int_V f(y)\ P'(x, dy) \\ &= \frac{1}{c'(x)} \int_V f(y)\ d\rho'_x(y) \\ &= \frac{q(x)}{c'(x)} \int_V f(y)q(y)d\rho_x(y) \\ &= \frac{q(x)}{c'(x)} \int_V f(y)q(y)c(x)P(x, dy) \\ &= \frac{q(x)c(x)}{c'(x)} P(qf)(x) \end{aligned}$$

It follows from (4.10) that $f = q^{-1}P(qf)$, and we are done. □

We remark that in the proved statement we temporarily extended the notion of symmetric measures to the case of *signed symmetric measures* assuming that the P-harmonic function q can be negative.

Theorem 4.11 *Suppose that* ρ' *and* ρ *are two symmetric measures such that* $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. *If* q *is harmonic for the Laplace operator* Δ, *then the operator*

$$Q : \mathcal{H}_E(\rho') \to \mathcal{H}_E(\rho) : Q(f) = qf$$

is an isometry.

Proof We need to show that, for any $f \in \mathcal{H}_E(\rho')$,

$$||f||_{\mathcal{H}_E(\rho')} = ||qf||_{\mathcal{H}_E(\rho)}.$$

In the computation given below, we use the following: the definition of the norm in the finite energy space, the symmetry of the measures ρ and ρ', and the relation $R(q) = cq$ that holds for harmonic functions because

$$\Delta(q)(x) = c(x)q(x) - R(q)(x).$$

Then we compute

$$\begin{aligned}
||f||^2_{\mathcal{H}_E(\rho')} - ||qf||^2_{\mathcal{H}_E(\rho)} &= \frac{1}{2}\iint_{V\times V} (f(x) - f(y))^2\, d\rho'(x, y) \\
&\quad - \iint_{V\times V} (q(x)f(x) - q(y)f(y))^2\, d\rho(x, y) \\
&= \iint_{V\times V} [(f(x) - f(y))^2 q(x)q(y) \\
&\quad - (q(x)f(x) - q(y)f(y))^2]\, d\rho(x, y) \\
&= \iint_{V\times V} [f^2(x)q(x)q(y) - q^2(x)f^2(x)] \\
&\quad + [f^2(y)q(x)q(y) - q^2(y)f^2(y)]\, d\rho(x, y) \\
&= 2\iint_{V\times V} [f^2(x)q(x)q(y) - q^2(x)f^2(x)]\, d\rho_x(y)d\mu(x) \\
&= 2\int_V f^2(x)q(x)[R(q)(x) - c(x)q(x)]\, d\mu(x) \\
&= 0.
\end{aligned}$$

This computation shows that $Q(f) = qf \in \mathcal{H}_E(\rho)$ and Q preserves the norm. □

Continuing the above theme, consider the Laplace operator Δ acting in $L^2(\mu)$. We recall that $\Delta : L^2(\mu) \to L^2(\mu)$ is a positive definite self-adjoint operator according to Proposition 3.7.

Proposition 4.12 *Suppose ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ and the Laplacian $\Delta = \Delta(\rho)$ is defined by (3.4). Let q and f be functions on $(V, \mathcal{B}, \mu)$ from the domain of Δ such that qf is also in the domain of Δ. Then*

$$\int_V \Delta(qf)\, d\mu = \int_V q\Delta(f)\, d\mu - \int_V f\Delta(q)\, d\mu. \tag{4.11}$$

If q and f are in $L^2(\mu)$, then $\int_V \Delta(qf)\, d\mu = 0$.

Proof By definition of Δ, we have

$$\begin{aligned}\Delta(qf) &= \int_V [(qf(x) - qf(y)]\, d\rho_x(y)\\ &= \int_V (q(x)f(x) - q(x)f(y) + q(x)f(y) - q(y)f(y))\, d\rho_x(y)\\ &= q(x)\Delta(f) - \int_V f(y)(q(x) - q(y))\, d\rho_x(y)\end{aligned}$$

Then

$$\begin{aligned}\int_V \Delta(qf)(x)\, d\mu(x) &= \int_V q\Delta(f)\, d\mu(x) + \iint_{V\times V} f(y)(q(x) - q(y))\, d\rho_x(y)d\mu(x)\\ &= \int_V q\Delta(f)\, d\mu(x) + \iint_{V\times V} f(x)(q(y) - q(x))\, d\rho_x(y)d\mu(x)\\ &= \int_V q\Delta(f)\, d\mu(x) - \int_V f\Delta(q)\, d\mu(x)\end{aligned}$$

and (4.11) is proved.

If the functions q and f are in $L^2(\mu)$ (in particular, q and f can be taken from the dense subset $\mathcal{D}_{\text{fin}}(\mu)$), then we can use the fact that Δ is essentially self-adjoint and conclude that

$$\int_V \Delta(qf)(x)\, d\mu(x) = \langle q, \Delta(f)\rangle_{L^2(\mu)} - \langle \Delta(q), f\rangle_{L^2(\mu)} = 0.$$

□

We immediately deduce the following fact from Proposition 4.12.

Corollary 4.13

(1) If functions f and f^2 are in the domain of Δ, then

$$\int_V \Delta(f^2)\, d\mu = 0.$$

(2) If f is a harmonic function for Δ, then $\Delta(f^2) = 0$, and therefore f^2 is also harmonic.

Proof

(1) is an obvious consequence of Proposition 4.12. To show that (2) holds, we use that $\Delta(f) = c(f - P(f))$ and P is a positive operator. This means that $P(f) \geq 0$ whenever $f \geq 0$. By Schwarz' inequality for positive operators, we have $P(f^2)(x) \geq P(f)^2(x)$, and therefore

$$\begin{aligned}\Delta(f^2) &= c(f^2 - P(f^2))\\ &\leq c(f^2 - P(f)^2)\\ &= c(f - P(f))(f + P(f))\\ &= 0.\end{aligned}$$

The fact that f^2 is harmonic follows from (1) and the proved inequality in (2).

□

5 Reversible Markov Process Generated by Symmetric Measures

In this section, we consider Markov processes generated by a Markov operator which is determined by a symmetric irreducible measure ρ on the standard Borel space $(V \times V, \mathcal{B} \times \mathcal{B})$ such that the margin measure μ on $(V, \mathcal{B})$ is *σ-finite*. Our first theme is *reversible Markov processes*. For the benefit of non-specialist readers, we cite the following sources: [44–46]. We refer also to [47–49]. In the second part of this section, we will assume that this Markov process is *transient* (see the definition below). The reader can find vast literature on the theory of transient Markov processes, we refer to [17–19, 50–57].

5.1 Reversible Markov Processes

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ which is disintegrated with respect to $(\rho_x, x \in V)$ and μ according to (2.5). By assumption, $c(x) = \rho_x(V)$ is locally integrable. We recall (see Definition 3.1) that, in this setting, a Markov operator P is defined on $\mathcal{F}(V, \mathcal{B})$ by the probability kernel $x \mapsto P(x, \cdot)$. This operator P acts by the formula

$$P(f)(x) = \int_V f(y)\ P(x, dy) \tag{5.1}$$

where $P(x, dy) = c(x)^{-1} d\rho_x(y)$. Then the operator P is positive and normalized, i.e., $P(\mathbb{1}) = \mathbb{1}$. As mentioned above in Proposition 3.7, the fact that ρ is symmetric is equivalent to self-adjointness of P as an operator in $L^2(\nu)$. It follows also that P preserves the measure $\nu = c\mu$. Furthermore, we can use the kernel $x \to P(x, \cdot) = P_1(x, \cdot)$ to define the sequence of probability kernels (transition probabilities) $(P_n(x, \cdot) : n \in \mathbb{N})$ in accordance with (2.4). These kernels satisfy the equality

$$P_{n+m}(x, A) = \int_V P_n(y, A) P_m(x, dy), \qquad n, m \in \mathbb{N}.$$

Therefore one has

$$P^n(f)(x) = \int_V f(y)\ P_n(x, dy), \qquad n \in \mathbb{N},$$

and this relation defines the sequence of probability measures (P_n) by setting $P_0(x, A) = \delta_A(x) = \chi_A(x)$ and

$$P_n(x, A) = P^n(\chi_A) = \int_V \chi_A(y)\ P_n(x, dy), \qquad A \in \mathcal{B}, n \in \mathbb{N}.$$

We use the notation $P(x, A)$ for $P_1(x, A)$.

For the Markov operator P, one can define one more sequence of measures. We use the formula

$$\rho_n(A \times B) = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)}, \tag{5.2}$$

to define the measures ρ_n, $n \in \mathbb{N}$, on the Borel space $(V \times V, \mathcal{B} \times \mathcal{B})$ (here $\rho_1 = \rho$).

Lemma 5.1

(1) Every measure ρ_n, $n \in \mathbb{N}$, is symmetric on $(V \times V, \mathcal{B} \times \mathcal{B})$, and ρ_n is equivalent to ρ.

(2) $\rho_x^{(n)}(V) = c(x), \forall n \in \mathbb{N}$.

(3)

$$d\rho_n(x, y) = c(x)P_n(x, dy)d\mu(x) = P_n(x, dy)d\nu(x). \tag{5.3}$$

(4)

$$\rho_n(A \times B) = \langle \chi_A, RP^{n-1}(\chi_B)\rangle_{L^2(\mu)}.$$

Proof The assertions of the lemma are rather obvious. We only mention two simple facts: $\rho_n(A \times V) = \rho(A \times V)$ for every n, and, since the operator P^n is self-adjoint in $L^2(\nu)$, the measure ρ_n is symmetric. □

Definition 5.2 Suppose that $x \mapsto P(x, \cdot)$ is a measurable family of transition probabilities on the space $(V, \mathcal{B}, \mu)$, and let P be the Markov operator determined by $x \mapsto P(x, \cdot)$. It is said that the corresponding Markov process is *reversible* with respect to a measurable function $c : x \to (0, \infty)$ on $(V, \mathcal{B})$ if, for any sets $A, B \in \mathcal{B}$, the following relation holds:

$$\int_B c(x)P(x, A)\, d\mu(x) = \int_A c(x)P(x, B)\, d\mu(x). \tag{5.4}$$

As shown in [1], the reversibility for the Markov process (P_n) is equivalent to the following properties (here we give an extended and more comprehensive formulation):

Theorem 5.3 *Let $(V, \mathcal{B}, \mu)$ be a standard σ-finite measure space, $x \mapsto c(x) \in (0, \infty)$ a measurable function, $c \in L^1_{\mathrm{loc}}(\mu)$. Suppose that $x \mapsto P(x, \cdot)$ is a probability kernel. The following are equivalent:*

- *(i) $x \mapsto P(x, \cdot)$ is reversible (i.e., it satisfies (5.4));*
- *(ii) $x \to P_n(x, \cdot)$ is reversible for any $n \geq 1$;*
- *(iii) the Markov operator P defined by $x \to P(x, \cdot)$ is self-adjoint on $L^2(\nu)$ and $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$;*
- *(iv)*

$$c(x)P(x, dy)d\mu(x) = c(y)P(y, dx)d\mu(y);$$

- *(v) the operator R defined by the relation $R(f)(x) = c(x)P(f)(x)$ is symmetric (see Remark 3.3);*
- *(vi) the measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ defined by*

$$\rho(A \times B) = \int_V \chi_A R(\chi_B)\, d\mu = \int_V c(x)\chi_A P(\chi_B)\, d\mu$$

is symmetric;

- *(vii) for every $n \in \mathbb{N}$, the measure ρ_n defined by (5.2) is symmetric;*

(viii) for any Borel sets $A_1, \dots, A_n \in \mathcal{B}_{\text{fin}}(\mu)$,

$$\int_V \mathbb{P}_x(X_0 \in A_0, \dots, X_n \in A_n)\, d\nu(x) = \int_V \mathbb{P}_x(X_0 \in A_n, \dots, X_n \in A_0)\, d\nu(x),$$

where the random variables $X_1, \dots, X_n$ *are defined below in Remark 5.4 (5) and the sets* $A_0, A_1, \dots, A_n$ *are written in the reverse order in the right hand side.*

Proof We refer to [1] where most of these properties are discussed. We prove (*viii*) here. Indeed, it can be seen that

$$\int_V \mathbb{P}_x(X_0 \in A_0, \dots, X_n \in A_n)\, d\nu(x) = \int_V \chi_{A_0} P(\chi_{A_1} P(\chi_{A_2} \cdots P(\chi_{A_n}) \cdots))(x)\, d\nu(x). \tag{5.5}$$

Since P is self-adjoint on $L^2(\nu)$, we can repeatedly use the relation $\int_V f P(g) d\nu = \int_V P(f) g d\nu$ and rewrite (5.5) as follows:

$$\begin{aligned} &\int_V \chi_{A_0} P(\chi_{A_1} P(\chi_{A_2} \cdots P(\chi_{A_n}) \cdots))(x)\, d\nu(x) \\ &= \int_V \chi_{A_n} P(\chi_{A_{n-1}} P(\chi_{A_{n-2}} \cdots P(\chi_{A_0}) \cdots))(x)\, d\nu(x) \\ &= \int_V \mathbb{P}_x(X_0 \in A_n, \dots, X_n \in A_0)\, d\nu(x). \end{aligned}$$

The fact that property (*viii*) implies that P is reversible is proved by using the density of simple functions in $L^2(\nu)$. □

We discuss the notion of reversibility in the following Remark where we included several direct consequences of Definition 4.1 and Theorem 5.3.

Remark 5.4

(1) Let $x \mapsto P(x, \cdot)$ be a Borel field of probability measures over a standard Borel space $(V, \mathcal{B})$. This field of transition probabilities generates the Markov operator P such that $P(\mathbb{1}) = 1$. It follows from Theorem 5.3 that one can define the notion of reversible Markov process $x \mapsto P(x, \cdot)$ with respect to a σ-finite measure ν: It is said that $((x \mapsto P(x, \cdot)), \nu)$ is *reversible* if P is a self-adjoint operator in $L^2(\nu)$. This definition is equivalent to the property

$$\int_A P(x, B)\, d\nu = \int_B P(x, A)\, d\nu.$$

Equally, one can consider the notion of reversibility for $P(x, \cdot)$ with respect to a symmetric measure ρ. Theorem 5.3 states the equivalence of these approaches.

(2) Based on (1), the following *question* is raised naturally: *Given* $x \mapsto P(x, \cdot)$ *as above, under what condition the set*

$$\mathcal{S}(P) := \{\nu : P \text{ is self-adjoint in } L^2(\nu)\}$$

is non-empty?

(3) The following observation is a direct consequence of Theorem 5.3. Let $P(x, A) = P(\chi_A)(x)$ be the probability kernel defined by a normalized Markov operator P acting on Borel functions over $(V, \mathcal{B}, \mu)$. To answer the question about the existence of a P-invariant measure $\nu \sim \mu$ such that (P, ν) is reversible, it suffices to construct a locally integrable function c satisfying (5.4). It can be done by pointing out a symmetric measure ρ such that $\rho_x(V) = c(x)$ and the projection of ρ onto V is the measure μ.

(4) There exists a stronger version of reversible Markov processes. Let P be a Markov operator acting on $\mathcal{F}(V, \mathcal{B})$ such that, for any $A, B \in \mathcal{B}_{\text{fin}}(\mu)$,

$$\chi_A P(\chi_B) = \chi_B P(\chi_A).$$

Then, for any positive Borel function $c \in L^1_{\text{loc}}(\mu)$, the measure $d\nu(x) = c(x)d\mu(x)$ belongs to $\mathcal{S}(P)$. Indeed, it suffices to define the symmetric measure ρ according to Theorem 5.3 (vi) and then apply statement (ii).

(5) We give here one more interpretation of the definition of reversible Markov processes. For this, we use the notation to be introduced in Sect. 6. Let

$$\Omega = V \times V \times V \cdots$$

be the path space of the Markov process (P_n), and let $X_n : \Omega \to V$ be the random variable defined by $X_n(\omega) = \omega_n$. Given a measure ν on V, we can reformulate the definition of reversible Markov operator as follows:

$$dist(X_0 \mid X_1 \in A) = dist(X_1 \mid X_0 \in A).$$

The meaning of the above formula is clarified in Proposition 6.4.

(6) Suppose now that a non-symmetric measure ρ is given on the space $(V \times V, \mathcal{B} \times \mathcal{B})$, i.e, $\rho(A \times B) \neq \rho(B \times A)$, in general. However, we will assume that ρ is equivalent to $\rho \circ \theta$ where $\theta(x, y) = (y, x)$. Then, using the same approach as above, we can define the following objects: margin measures $\mu_i := \rho \circ \pi_i^{-1}$, $i = 1, 2,$, fiber measures $d\rho_x(\cdot)$ and $d\rho^x(\cdot)$ (see Remark 2.5), and functions $c_1(x) = \rho_x(V)$, $c_2(x) = \rho^x(V)$.

Define now the *symmetric measure* $\rho^\#$ generated by ρ as follows

$$\rho^\# := \frac{1}{2}(\rho + \rho \circ \theta).$$

Then

$$\rho^{\#}(A \times B) = \frac{1}{2}(\rho(A \times B) + \rho(B \times A)).$$

Clearly, $\rho^{\#}$ is equivalent to ρ.

Let $E \subset V \times V$ be the support of ρ. Then $E^{\#} = E \cup \theta(E)$ is the support of the symmetric measure $\rho^{\#}$. The disintegration of $\rho = \int_V \rho_x \, d\mu_1(x)$ with respect to the partition $\{x\} \times E_x$ defines the disintegration of $\rho^{\#}$. For $\mu^{\#} := \dfrac{1}{2}(\mu_1 + \mu_2)$, we obtain that

$$\rho^{\#} = \int_V (\rho_x + \rho^x) \, d\mu^{\#}.$$

Having the symmetric measure $\rho^{\#}$ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, we can introduce the operators $R^{\#}$ and $P^{\#}$ as in (3.2) and (3.3). It turns out that, for $f \in \mathcal{F}(V, \mathcal{B})$,

$$R^{\#}(f)(x) = R_1(f)(x) + R_2(f)(x)$$

where

$$R_1(f) = \int_V f(y) \, d\rho_x(y), \qquad R_2(f) = \int_V f(y) \, d\rho^x(y).$$

Similarly,

$$P^{\#}(f)(x) = \frac{1}{c^{\#}(x)} R^{\#}(f)(x)$$

where

$$c^{\#}(x) = \rho_x(V) + \rho^x(V).$$

Then we can define the measure $d\nu^{\#}(x) = c^{\#}(x) d\mu(x)$ such that the operator

$$P^{\#}(f)(x) = \int_V f(y) \frac{1}{c^{\#}(x)} \, d\rho_x^{\#}(y)$$

is self-adjoint in $L^2(\nu^{\#})$. By Theorem 5.3, we obtain that the Markov process generated by $x \mapsto P^{\#}(x, \cdot)$ is *reversible* where $P^{\#}(x, A) = P^{\#}(\chi_A)(x)$.

5.2 Properties of Markov Operators

In this subsection, we discuss some properties of the Markov operator P, which is defined by relation (3.3). The operator P is considered acting in Hilbert spaces $L^2(\mu)$, $L^1(\nu)$, and $\mathcal{H}_E$ where $d\nu(x) = c(x)d\mu(x)$ and $\mathcal{H}_E$ is the energy space.

We begin with the following simple observations whose proofs are obvious and can be omitted. Remind that $\mathcal{B}_{\text{fin}}(\mu)$ is the family of Borel subsets of finite measure μ, and $\mathcal{D}_{\text{fin}} = \mathcal{D}_{\text{fin}}(\mu)$ is the linear subspace generated by the characteristic functions χ_A, $A \in \mathcal{B}_{\text{fin}}$.

Remark 5.5

(1) If $c \in L^1_{\text{loc}}(\mu)$, then

$$\mathcal{B}_{\text{fin}}(\mu) \subset \mathcal{B}_{\text{fin}}(\nu).$$

The converse is not true.

(2) We observe that if both functions, $c(x)$ and $c(x)^{-1}$ are in $L^1_{\text{loc}}(\mu)$, then

$$\mathcal{B}_{\text{fin}}(\mu) = \mathcal{B}_{\text{fin}}(\nu).$$

(3) The following property holds for $c \in L^1_{\text{loc}}(\mu)$:

$$\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu) \cap L^2(\nu) \cap \mathcal{H}_E \tag{5.6}$$

(this should be understood that functions from $\mathcal{D}_{\text{fin}}$ are representatives of elements from $\mathcal{H}_E$).

(4) We recall that

$$\|\chi_A\|^2_{\mathcal{H}_E} = \rho(A \times A^c) \tag{5.7}$$

where ρ is a symmetric measure used in the definition $\mathcal{H}_E$. This fact is proved in [1].

Lemma 5.6 *If $c \in L^1_{\text{loc}}(\mu)$, then $\mathcal{D}_{\text{fin}}(\mu)$ is dense in $L^1(\nu)$ and $L^2(\nu)$.*

Proof (Sketch) We show the density of $\mathcal{D}_{\text{fin}}(\mu)$ in $L^1(\nu)$ only. It suffices to check that, for every $B \in \mathcal{B}_{\text{fin}}(\nu)$, the characteristic function χ_B can be approximated in $L^1(\nu)$ by simple functions from $\mathcal{D}_{\text{fin}}(\mu)$, i.e., for every $\varepsilon > 0$, there exists some $s(x) \in \mathcal{D}_{\text{fin}}(\mu)$ such that $||\chi_B - s||_{L^1(\nu)} < \varepsilon$. Without loss of generality, we can assume that $s(x) \le \chi_B(x)$. Then

$$||\chi_B - s||_{L^1(\nu)} = \int_V (\chi_B - s(x))\, d\nu(x) = \int_B c(x)(1 - s(x))\, d\mu(x).$$

Since c is μ-integrable on B, one can take a subset $B_0 \subset B$ such that

$$\int_B c\, d\mu - \int_{B_0} c\, d\mu < \varepsilon.$$

The result follows. □

Next, let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let P be the operator acting on bounded Borel functions by the formula

$$P(f)(x) = \int_V f(y) P(x, dy)$$

where $c(x)P(x, dy) = d\rho_x(y)$.

In the next statement we collect several properties of the Markov operator P considered in various spaces.

Proposition 5.7 *Let $(V, \mathcal{B}, \mu)$, ν, and ρ be as above. Then, for any $A \in \mathcal{B}_{\text{fin}}$,*

(a) $P(\chi_A) \in L^1(\mu) \iff \dfrac{\rho_x(A)}{c(x)} \in L^1(\mu) \implies P(\chi_A) \in L^2(\mu)$*;*
(b) if the function $x \mapsto \int_V c(y)^{-1}\, d\rho_x(y)$ is locally integrable, then P is a densely defined operator in $L^2(\mu)$;
(c) if $c \in L^1_{\text{loc}}(\mu)$, then

$$P(\chi_A) \in L^1(\nu) \cap L^2(\nu);$$

(d) the measures μ and μP are equivalent if and only if the function c^{-1} is integrable on (E_x, ρ_x) for μ-a.e. $x \in V$. The Radon-Nikodym derivative can be found by the formula:

$$\frac{d(\mu P)}{d\mu}(x) = \int_V \frac{1}{c(y)}\, d\rho_x(y).$$

***Proof** (Sketch)*

(a) The fact that $P(\chi_A)$ is in $L^2(\mu)$ follows from the Schwarz inequality for positive operators,

$$P(\chi_A)^2 \le P(\chi_A^2) = P(\chi_A).$$

The criterion for integrability of the function $P(\chi_A)$ is proved as follows:

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\mu(x) &= \iint_{V\times V} \chi_A(y)P(x,dy)\, d\mu(x)\\ &= \iint_{V\times V} \frac{\chi_A(y)}{c(x)}\, d\rho_x(y)d\mu(x)\\ &= \int_V \frac{\rho_x(A)}{c(x)}\, d\mu(x).\end{aligned}$$

It follows from (a) that the same computation can be used to show that $P(\chi_A)$ is in $L^2(\mu)$ whenever

$$\frac{\rho_x(A)}{c(x)} \in L^1(\mu).$$

(b) To prove this result, we refer to the proof of (b) and use the symmetry of the measure ρ:

$$P(\chi_A) \in L^2(\mu) \iff P(\chi_A) \in L^1(\mu)$$

and

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\mu(x) &= \iint_{V\times V} \frac{\chi_A(y)}{c(x)}\, d\rho_x(y)d\mu(x)\\ &= \iint_{V\times V} \frac{\chi_A(x)}{c(y)}\, d\rho_x(y)d\mu(x)\\ &= \int_A \left(\int_V \frac{\chi_A(x)}{c(y)}\, d\rho_x(y)\right) d\mu(x).\end{aligned}$$

It gives the desired statement.

(c) Suppose $c(x) \in L^1_{\text{loc}}(\mu)$. Then, using the symmetry of the measure ρ and relation (2.7), we obtain

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\nu(x) &= \int_V \left(\int_V \chi_A(y)\frac{1}{c(x)}\, d\rho_x(y)\right)\, c(x)d\mu(x)\\ &= \iint_{V\times V} \chi_A(x)\, d\rho_x(y)d\mu(x)\\ &= \int_V \chi_A(x)c(x)\, d\mu(x)\\ &= \int_A c(x)\, d\mu(x) < \infty,\end{aligned}$$

i.e., $P(\chi_A) \in L^1(\nu)$. The fact that $P(\chi_A) \in L^2(\nu)$ is proved as in (a).

(d) The statement will follow from the following chain of equalities:

$$\begin{aligned}(\mu P)(A) &= \int_V \chi_A \, d(\mu P) \\ &= \int_V P(\chi_A)\, d\mu \\ &= \int_V \left(\int_V \chi_A(y) P(x, dy) \right) d\mu(x) \\ &= \iint_{VtV} \chi_A(y) \frac{1}{c(x)}\, d\rho_x(y) d\mu(x) \\ &= \int_V \chi_A(x) \left(\int_V \frac{1}{c(y)}\, d\rho_x(y) \right) d\mu(x) \\ &= \int_A \left(\int_V \frac{1}{c(y)}\, d\rho_x(y) \right) d\mu(x) \\ &= \int_A \frac{d(\mu P)}{d\mu}(x)\, d\mu(x)\end{aligned}$$

where

$$\frac{d(\mu P)}{d\mu}(x) = \int_V \frac{1}{c(y)}\, d\rho_x(y).$$

□

Clearly, Proposition 5.7 can be extended to functions from $\mathcal{D}_{\text{fin}}$.

Lemma 5.8

(1) Let P be a self-adjoint Markov operator in $L^2(\nu)$. Suppose that $c \in L^1_{\text{loc}}(\mu)$. Then, for $A \in \mathcal{B}_{\text{fin}}(\mu)$,

$$||P^n(\chi_A)||^2_{L^2(\nu)} = \rho_{2n}(A \times A), \quad n \in \mathbb{N}, \tag{5.8}$$

where measures ρ_n are defined in (5.2).

(2) Moreover, for all $n \in \mathbb{N}$,

$$\int_A c\, d\mu = ||\chi_A||^2_{\mathcal{H}_E(\rho_{2n})} + ||P^n(\chi_A)||^2_{L^2(\nu)}.$$

Proof

(1) We recall that if P is a self-adjoint operator in the space $L^2(\nu)$, then $\nu P = \nu$. Hence,

$$
\begin{aligned}
||P^n(\chi_A)||^2_{L^2(\nu)} &= \langle P^n(\chi_A), P^n(\chi_A)\rangle_{L^2(\nu)} \\
&= \langle \chi_A, P^{2n}(\chi_A)\rangle_{L^2(\nu)} \\
&= \rho_{2n}(A \times A).
\end{aligned}
$$

(2) Since $\rho_x^{(n)}(V) = c(x)$ for all $n \in \mathbb{N}$, we can easily deduce from (1) the following equality (we use here formula (5.7)):

$$
\begin{aligned}
||\chi_A||^2_{\mathcal{H}_E(\rho_n)} &= \rho_n(A \times A^c) \\
&= \rho_n(A \times V) - \rho_n(A \times A) \\
&= \int_A c\, d\mu - \rho_n(A \times A).
\end{aligned}
$$

□

Remark 5.9 It is interesting to compare formula (5.8) with a similar result for $||P^n(\chi_A)||^2_{\mathcal{H}_E}$ proved in [1], see also (3.11) in Theorem 3.6.

$$
\|P^n(\chi_A)\|^2_{\mathcal{H}_E} = \rho_{2n}(A \times A) - \rho_{2n+1}(A \times A), \qquad n \in \mathbb{N}.
$$

Hence, it follows that

$$
\|P^n(\chi_A)\|^2_{\mathcal{H}_E} = ||P^n(\chi_A)||^2_{L^2(\nu)} - \rho_{2n+1}(A \times A).
$$

5.3 *More on the Embedding Operator J*

In this subsection, we return to the study of the operator J defined in (3.14), see Sect. 3.2. We recall that the operator J is an isometry if considered acting from $L^2(\nu)$ to $L^2(\rho)$, and it is an unbounded operator from $L^2(\mu)$ to $L^2(\rho)$. Here we focus on relations between J and other operators we study in the paper.

Lemma 5.10 *For any $A \in \mathcal{B}_{\mathrm{fin}}(\mu)$, we have*

$$
||J(P(\chi_A))||^2_{L^2(\rho)} \le ||\chi_A||^2_{L^2(\nu)}.
$$

Proof Indeed, we use Schwarz' inequality for P to show that

$$\begin{aligned}
\iint_{V\times V} J(P(\chi_A))^2(x,y)\, d\rho(x,y) &= \int_V P(\chi_A)^2(x)\, d\rho(x,y)\\
&\le \int_V P(\chi_A)(x)\, d\rho(x,y)\\
&= \int_V c(x)P(\chi_A)(x)\, d\mu(x)\\
&= \iint_{V\times V} \chi_A(y)\, d\rho_x(y)d\mu(x)\\
&= \iint_{V\times V} \chi_A(x)\, d\rho_x(y)d\mu(x)\\
&= \int_A c(x)\, d\mu(x)\\
&= ||\chi_A||^2_{L^2(\nu)}.
\end{aligned}$$

□

As an illustration of properties of this embedding J, we note that the function $J(c^{-1})(x, y)$ is not integrable with respect to ρ but is locally integrable.

Another useful relation that compares norms of functions is contained in the following inequality.

Lemma 5.11 *Let f be a function from the finite energy space such that f and $\Delta(f)$ belong to $L^2(\mu)$. Then*

$$||Jf||^2_{L^2(\rho)} \ge \frac{1}{2}||f||^2_{\mathcal{H}_E}.$$

Proof The proof follows from [1, Corollary 7.4] and Proposition 3.7 (6):

$$\begin{aligned}
\iint_{V\times V} (Jf)^2(x,y)\, d\rho(x,y) &= \iint_{V\times V} f^2(x)\, d\rho_x(y)d\mu(x)\\
&= \int_V f^2(x)c(x)\, d\mu(x)\\
&\ge \frac{1}{2}\langle f, \Delta f\rangle_{L^2(\mu)}\\
&= \frac{1}{2}||f||^2_{\mathcal{H}_E}.
\end{aligned}$$

□

In the remaining part of this section, we consider the Markov operator P as an operator acting on functions from the energy space $\mathcal{H}_E$.

Proposition 5.12 *Assume that $c \in L^1_{loc}(\mu)$. Then, for every $A \in \mathcal{B}_{fin}(\mu)$, we have*

$$(JP)(\chi_A)(x, y) \in \mathcal{H}_E.$$

Proof We need to show that the energy norm of $J(P(\chi_A))$ is finite. By Theorem 3.6, we find that

$$\begin{aligned}||(JP)(\chi_A)||^2_{L^2(\rho)} &= \frac{1}{2}\iint_{V\times V}(P(\chi_A)(x) - P(\chi_A)(y))^2\, d\rho(x, y)\\ &= \iint_{V\times V}(P(\chi_A)^2(x) - P(\chi_A)(x)P(\chi_A)(y))\, d\rho(x, y).\end{aligned}$$

To see that the last integral is finite, we first show that $(JP)(\chi_A)$ is in $L^2(\rho)$:

$$\begin{aligned}\iint_{V\times V} P(\chi_A)^2(x)\, d\rho(x, y) &\le \iint_{V\times V} P(\chi_A)(x)\, d\rho_x(y)d\mu(x)\\ &= \int_V P(\chi_A)(x)c(x)\, d\mu(x)\\ &= \nu(A)\\ &= \int_A c(x)\, d\mu(x).\end{aligned}$$

The latter is finite.

Similarly, one can check that $\iint_{V\times V} P(\chi_A)(x)P(\chi_A)(y)\, d\rho(x, y)$ is also finite. We leave the proof for the reader. □

Consider a new operator, denoted by ∂, which acts from the energy space $\mathcal{H}_R$ to $L^2(\rho)$:

$$(\partial f)(x, y) = \frac{1}{\sqrt{2}}(f(x) - f(y)), \qquad f \in \mathcal{H}_E \tag{5.9}$$

Remark that in the theory of electrical networks the analogous transformation is called a voltage drop operator.

Lemma 5.13 *The operator $\partial : \mathcal{H}_E \to L^2(\rho)$ defined by (5.9) is an isometry.*

Proof The proof is obvious because

$$||f||^2_{\mathcal{H}_E} = \frac{1}{2}\iint_{V\times V}(f(x) - f(y))^2\, d\rho(x, y) = ||(\partial f)||^2_{L^2(\rho)}.$$

□

Since $J : L^2(\nu) \to L^2(\rho)$ is an isometry, then the co-isometry J^* sends $L^2(\rho)$ to $L^2(\nu)$ according to the formula

$$(J^* g)(x) = \int_V g(x, \cdot)\ P(x, \cdot)$$

where $g \in L^2(\rho)$.

In the following proposition, we formulate a relation between operators P, J^*, and ∂.

Proposition 5.14 *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}_E & \xrightarrow{\widetilde{\Delta}} & L^2(\nu) \\ \searrow_{\partial} & & \nearrow_{J^*} \\ & L^2(\rho) & \end{array}$$

where $\widetilde{\Delta} = (\sqrt{2}c)^{-1}\Delta = (\sqrt{2})^{-1}(I - P)$.

Proof The proof is mainly based on Theorem 3.10 and the definition of ∂. We have

$$\begin{aligned}(J^*\partial f)(x) =& \frac{1}{\sqrt{2}} J^*(f(x) - f(y)) \\ =& \frac{1}{\sqrt{2}} \int_V (f(x) - f(y))\ P(x, dy) \\ =& \frac{1}{\sqrt{2}} (f(x) - P(f)(x)) \\ =& \frac{1}{\sqrt{2}} c(x)\Delta(f)(x).\end{aligned}$$

□

In the next statement, we present several properties of the operator $I - P$.

Corollary 5.15

(1)

$$(I - P)\mathcal{H}_E \subset L^2(\nu),$$

(2) The operator $I - P$ acting from $\mathcal{H}_E$ to $L^2(\nu)$ is contractive.
(3) For the operator $\Delta = c(I - P)$, the following holds

$$\Delta(\mathcal{H}_E) \subset cL^2(\nu).$$

Proof Assertion (1) is a direct consequence of Proposition 5.14 (this result was already mentioned in [1]).

To see that (2) holds, we recall the formula for the norm of a function in the finite energy space $\mathcal{H}_E$:

$$\|f\|^2_{\mathcal{H}_E} = \frac{1}{2}\left(\|f - P(f)\|^2_{L^2(\nu)} + \int_V \mathrm{Var}_x(f \circ X_1)\, d\nu\right),$$

where the meaning of random variables X_n is explained in Sect. 6 below.

(3) is obvious. □

6 Transient Markov Processes and Symmetric Measures

Transient Markov processes and Green's functions are central themes in the theory of Markov chains that have been studied in a numerous books and papers. For the benefit of non-specialist readers, we cite the following sources [17, 58–60]. More interesting results can be found in [61–63].

In this section we consider Green's functions $G_A(x)$ of transient Markov processes and relate the symmetric measures ρ_n to the norm of G_A in the finite energy space.

6.1 Path-space Measure

We denote by Ω the infinite Cartesian product $V \times V \times \cdots = V^{\mathbb{N}_0}$. Let $(X_n(\omega) : n = 0, 1, \dots)$ be the sequence of random variables $X_n : \Omega \to V$ such that $X_n(\omega) = \omega_n$. We call Ω as the path space of the Markov process (P_n). Let Ω_x, $x \in V$, be the set of infinite paths beginning at x:

$$\Omega_x := \{\omega \in \Omega : X_0(\omega) = x\}.$$

Clearly, $\Omega = \coprod_{x \in V} \Omega_x$.

A subset $\{\omega \in \Omega : X_0(\omega) \in A_0, \dots X_k(\omega) \in A_k\}$ is called a *cylinder set* defined by Borel sets $A_0, A_1, \dots, A_k$ taken from $\mathcal{B}$, $k \in \mathbb{N}_0$. The collection of cylinder sets generates the σ-algebra $\mathcal{C}$ of Borel subsets of Ω, and $(\Omega, \mathcal{C})$ is a standard Borel space. Then the functions $X_n : \Omega \to V$ are Borel.

Define a probability measure $\mathbb{P}_x$ on Ω_x. For this, denote by $\mathcal{F}_{\le n}$ the increasing sequence of σ-subalgebras such that $\mathcal{F}_{\le n}$ is the smallest subalgebra for which the functions $X_0, X_1, \dots, X_n$ are Borel. For a cylinder set $(A_1, \dots, A_n)$ from $\mathcal{F}_{\le n}$ we set

$$\mathbb{P}_x(X_1 \in A_1, \dots, X_n \in A_n) = \int_{A_1} \cdots \int_{A_{n-1}} P(y_{n-1}, A_n) P(y_{n-2}, dy_{n-1}) \cdots P(x, dy_1). \tag{6.1}$$

Then $\mathbb{P}_x$ extends to the Borel sets on Ω_x by the Kolmogorov extension theorem [64].

The values of $\mathbb{P}_x$ can be written as

$$\mathbb{P}_x(X_1 \in A_1, \dots, X_n \in A_n) = P(\chi_{A_1} P(\chi_{A_2} P(\cdots\ P(\chi_{A_{n-1}} P(\chi_{A_n}))\cdots)))(x). \tag{6.2}$$

The joint distribution of the random variables X_i is given by

$$d\mathbb{P}_x(X_1, \dots, X_n)^{-1} = P(x, dy_1) P(y_1, dy_2) \cdots P(y_{n-1}, dy_n). \tag{6.3}$$

Lemma 6.1 *The measure space $(\Omega_x, \mathbb{P}_x)$ is a standard probability measure space for μ-a.e. $x \in V$.*

On the measurable space $(\Omega, \mathcal{C})$, define a σ-finite measure λ by

$$\lambda := \int_V \mathbb{P}_x \, d\nu(x) \tag{6.4}$$

(λ is infinite if and only if the measure ν is infinite).

By $\mathcal{F}_n$, we denote the σ-subalgebra $X_n^{-1}(\mathcal{B})$. Since $X_n^{-1}(\mathcal{B})$ is a σ-subalgebra of $\mathcal{C}$, there exists a projection

$$E_n : L^2(V, \mathcal{C}, \lambda) \to L^2(\Omega, X_n^{-1}(\mathcal{B}), \lambda).$$

The projection E_n is called the *conditional expectation* with respect to $X_n^{-1}(\mathcal{B})$ and satisfies the property:

$$E_n(f \circ X_n) = f \circ X_n. \tag{6.5}$$

We proved in [1] that the Markov process P_n is irreducible if the initial symmetric measure is irreducible. More precisely, the statement is as follows.

Theorem 6.2 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let A and B be any two sets from $\mathcal{B}_{\rm fin}(\mu)$. Then*

$$\rho_n(A \times B) = \langle \chi_A, P^n(\chi_B)\rangle_{L^2(\nu)} = \lambda(X_0 \in A, X_n \in B), \quad n \in \mathbb{N}. \tag{6.6}$$

The Markov process (P_n) is irreducible if and only if the measure ρ is irreducible.

In other words, relation (6.6) can be interpreted in the following way: for the Markov process (P_n), the "probability" to get in B for n steps starting somewhere in A is exactly $\rho_n(A \times B) > 0$.

To see that (6.6) holds, one uses the definition of the measure λ and formulas (6.1) and (6.2).

Corollary 6.3 *Let $A_0, A_1, \ldots, A_n$ be a finite sequence of subsets from $\mathcal{B}_{\text{fin}}$. Then*

$$\mathbb{P}_x(X_1 \in A_1, \ldots, X_n \in A_n) \mid x \in A_0) > 0 \iff \rho(A_{i-1} \times A_i) > 0$$

for $i = 1, \ldots, n$.

It is worth noting that the concept of reversible Markov processes can be formulated in terms of the measure λ, roughly speaking λ must be a symmetric distribution.

Proposition 6.4 *Let the measure λ on Ω be defined by (6.4). The Markov operator P is reversible if and only if, for any sets $A_0, \ldots, A_n$ from $\mathcal{B}_{\text{fin}}(\mu)$ and any $n \in \mathbb{N}$,*

$$\lambda(X_0 \in A_0, \ldots, X_n \in A_n) = \lambda(X_0 \in A_n, \ldots, X_n \in A_0).$$

Proof The proof uses Theorem 5.3 (viii). In the proof we assume for simplicity that $n = 2$; the general case is proved similarly. We recall that P is reversible if and only is the Markov operator P is self-adjoint in $L^2(\nu)$. We compute applying (6.1):

$$\begin{aligned}\lambda(X_0 \in A_0 \mid X_1 \in A_1) &= \int_{A_0} \mathbb{P}_x(X_1 \in A_1)\, d\nu(x)\\ &= \int_V \chi_{A_0}(x) P(\chi_{A_1})(x)\, d\nu(x)\\ &= \int_V \chi_{A_1}(x) P(\chi_{A_0})(x)\, d\nu(x)\\ &= \lambda(X_0 \in A_1 \mid X_1 \in A_0).\end{aligned}$$

It proves the statement. □

In the next statement we relate harmonic functions to martingales. Recall first the definition of a martingale.

Let $(X_n : n \in \mathbb{N})$ be the Markov chain on Ω with values in $(V, \mathcal{B})$ defined by $X_n(\omega) = \omega_n$. We recall that the space Ω is represented as the disjoint union of subsets $\Omega_x := \{\omega \in \Omega : \omega_0 = x\}, x \in V$. Let $(\Phi_n : n \in \mathbb{N}_0)$ be a sequence of real-valued random variables defined on Ω. Then it generates a sequence of measurable fields of random variables $x \to \Phi_n(x), x \in V$, defined on the corresponding subset Ω_x. Let $\mathcal{C}_n$ be the σ-algebra of subsets of Ω generated by $\Phi_n^{-1}(B)$, $B \in \mathcal{B}$. Denote by $\mathcal{C}_{\leq n}$ the smallest σ-subalgebra such that the functions $\Phi_i, i = 1, \ldots n$, are Borel measurable. These σ-algebras induce σ-algebras $\mathcal{C}_{\leq n}(x)$ on every Ω_x.

It is said that the sequence (Φ_n) is a *martingale* if

$$\mathbb{E}_x(\Phi_{n+k}(x) \mid \mathcal{C}_{\leq n}(x)) = \Phi_n(x), \quad \forall k.$$

Here $\mathbb{E}_x$ is the conditional expectation with respect to the probability path measure $\mathbb{P}_x$, see (6.1).

Proposition 6.5 *Let P be the Markov operator defined by a symmetric measure ρ. For the objects defined above, the following are equivalent:*

(i) a Borel function h on $(V, \mathcal{B})$ is harmonic with respect to the Markov operator P;
(ii) the sequence $(h \circ X_n : n \in \mathbb{N}_0)$ is a martingale.

Proof It follows from the definition of the Markov chain (X_n), path space measure $\mathbb{P}_x$, and [2, Proposition 2.24] that, for any Borel function f,

$$\mathbb{E}_x(f \circ X_{n+m} \mid \mathcal{C}_{\leq n}(x)) = \mathbb{E}_x(f \circ X_{n+m} \mid \mathcal{C}_n(x)) = P^m(f) \circ X_n.$$

Hence, we see that a function h is harmonic if and only if

$$\mathbb{E}_x(h \circ X_{n+m} \mid \mathcal{C}_{\leq n}(x)) = h \circ X_n,$$

i.e., $(h \circ X_n)$ is a martingale. □

6.2 Green's Functions

In this section, we will work with transient Markov processes. We first define a Green's function $G(x, A)$. Our main goal is to study Green's functions as elements of the energy space.

Definition 6.6 Let

$$G(x, A) = \sum_{n=0}^{\infty} P_n(x, A), \qquad A \in \mathcal{B}_{\text{fin}}(\mu), x \in V.$$

The Markov process is called *transient* if, for every $A \in \mathcal{B}_{\text{fin}}$, the function $G(x, A)$ is finite μ-a.e. on V.

In this subsection, we will always assume that the Markov process (P_n) is transient.

Lemma 6.7 *Let ρ be an irreducible symmetric measure. Suppose $A \in \mathcal{B}_{\text{fin}}$ be a set such that $G(x, A)$ is finite a.e. Then, for any $B \in \mathcal{B}_{\text{fin}}$, the function $G(x, B)$ is finite for μ-a.e. $x \in V$.*

Proof The proof of this result is straightforward and mainly based on the definition of irreducible measure, see also Lemma 2.9. □

Lemma 6.8 *Let $A \in \mathcal{B}_{\text{fin}}$ and let P be a Markov operator defined by a symmetric measure ρ. Then the function $x \mapsto P_n(x, A) = P^n(\chi_A)(x)$ belongs to $\mathcal{H}_E$ and*

$$\|P_n(\cdot, A)\|^2_{\mathcal{H}_E} = \rho_{2n}(A \times A) - \rho_{2n+1}(A \times A), \qquad n \in \mathbb{N}.$$

Proof The proof is based on the facts that ν is P-invariant, ρ is symmetric, and on the definition of the norm in the energy space which are used in the following computation:

$$\begin{aligned}
||P_n(x, A)||^2_{\mathcal{H}_E} &= \iint_{V \times V} P_n(x, A)(P_n(x, A) - P_n(y, A))\, d\rho(x, y)\\
&= \iint_{V \times V} P_n(x, A)(P_n(x, A) - P_n(y, A))c(x)P(x, dy)\, d\mu(x)\\
&= \int_V \left[P_n(x, A)^2 - P_n(x, A)\int_V P_n(y, A)P(x, dy)\right] d\nu(x)\\
&= \int_V \left[P_n(x, A)^2 - P_n(x, A)P_{n+1}(x, A)\right] d\nu(x)\\
&= \int_V P_n(x, A)(P_n(x, A) - P_{n+1}(x, A))\, d\nu(x)\\
&= \int_V \chi_A(x)P^n(P^n(\chi_A) - P^{n+1}(\chi_A))(x)\, d\nu(x)\\
&=\langle \chi_A(x), P^{2n}(\chi_A)(x)\rangle_{L^2(\nu)} - \langle \chi_A(x), P^{2n+1}(\chi_A)(x)\rangle_{L^2(\nu)}\\
&=\rho_{2n}(A \times A) - \rho_{2n+1}(A \times A).
\end{aligned}$$

□

Remark 6.9 As a curious observation, we mention that, for any $A \in \mathcal{B}_{\text{fin}}$,

$$\rho_{2n}(A \times A) > \rho_{2n+1}(A \times A).$$

It is worth noting that the above formula cannot be extended to direct products of sets A and B from $\mathcal{B}_{\text{fin}}(\mu)$. In particular, one can prove that the relation

$$\rho_2(A \times B) < \rho(A \times B)$$

implies that $P(\chi_B - P(\chi_B)) > 0$ a.e. Therefore there would exist a harmonic function in $L^2(\nu)$ which is a contradiction.

Fix a set $A \in \mathcal{B}_{\text{fin}}$, then we have the family of measurable functions $G_A(x) := G(x, A)$ indexed by sets of finite measure.

Lemma 6.10 *For a set* $A \in \mathcal{B}_{\text{fin}}$*, the equality*

$$c(x)(I - P)(G_A)(x) = c(x)\chi_A(x)$$

holds. Equivalently,

$$\Delta G_A(x) = c(x)\chi_A(x).$$

Proof We compute using the definition of Green's function and the fact that the series $\sum_n P_n(x, A)$ is convergent for all x and all $A \in \mathcal{B}_{\text{fin}}(\mu)$:

$$\begin{aligned} c(x)(I - P)G_A(x) &= c(x)(I - P)\sum_{n=0}^{\infty} P_n(x, A) \\ &= c(x)\sum_{n=0}^{\infty} P_n(x, A) - c(x)\sum_{n=1}^{\infty} P_n(x, A) \\ &= c(x)\chi_A(x). \end{aligned}$$

□

Theorem 6.11 *For the objects defined above, we have the following properties.*

(1) For any sets $A, B \in \mathcal{B}_{\text{fin}}$*, we have*

$$\langle G_A, G_B\rangle_{\mathcal{H}_E} = \sum_{n=0}^{\infty} \rho_n(A \times B); \tag{6.7}$$

and, in particular,

$$\|G_A(x)\|^2_{\mathcal{H}_E} = \sum_{n=1}^{\infty} \rho_n(A \times A). \tag{6.8}$$

(2) For any $f \in \mathcal{H}_E$ *and* $A \in \mathcal{B}_{\text{fin}}(\mu)$*,*

$$\langle f, G_A\rangle_{\mathcal{H}_E} = \int_A f \, d\nu.$$

Furthermore, if

$$\mathcal{G} := \text{span}\{G_A(\cdot) : A \in \mathcal{B}_{\text{fin}}\}, \tag{6.9}$$

then $\mathcal{G}$ *is dense in the energy space* $\mathcal{H}_E$*.*

Proof

(1) We prove (6.8) here. Relation (6.7) is proved similarly. One has

$$
\begin{aligned}
\|G_A(x)\|^2_{\mathcal{H}_E} &= \iint_{V\times V} (G_A(x) - P_A(y))^2 \, d\rho(x,y) \\
&= \iint_{V\times V} G_A(x)(G_A(x) - G_A(y)) \, d\rho(x,y) \\
&= \iint_{V\times V} G_A(x)(G_A(x) - P_A(y))c(x)P(x,dy)d\mu(x)) \\
&= \int_V G_A(x)[G_A(x) - P(G_A)(x)]c(x) \, d\mu(x)) \\
&= \int_V G_A(x)[\sum_{n=0}^{\infty} P^n(\chi_A)(x) - \sum_{n=0}^{\infty} P^{n+1}(\chi_A)(x)c(x) \, d\mu(x)) \\
&= \int_V \sum_{n=0}^{\infty} P^n(\chi_A)(x)\chi_A(x) \, d\nu(x) \\
&= \sum_{n=0}^{\infty} \langle \chi_A, P^n(\chi_A\rangle_{L^2(\nu)} \\
&= \sum_{n=0}^{\infty} \rho_n(A\times A).
\end{aligned}
$$

For (2),

$$
\begin{aligned}
\langle f, G_A\rangle_{\mathcal{H}_E} &= \frac{1}{2}\iint_{V\times V} (f(x) - f(y))(G_A(x) - G_A(y)) \, d\rho(x,y) \\
&= \iint_{V\times V} (f(x)G_A(x) - f(x)G_A(y)) \, d\rho(x,y) \\
&= \int_V \left[f(x)G_A(x)c(x) - f(x)\left(\int_V G_A(y)P(x,dy)\right)c(x)\right] d\mu(x) \\
&= \int_V f(x)c(x)\left[\sum_{n=o}^{\infty} P^n(\chi_A)(x) - \sum_{n=o}^{\infty} P^{n+1}(\chi_A)(x)\right] d\mu(x) \\
&= \int_V f(x)\chi_A(x)c(x) \, d\mu(x) \\
&= \int_A f \, d\nu.
\end{aligned}
$$

It follows from the proved relation that if $\langle f, G_A\rangle_{\mathcal{H}_E} = 0$ for all $A \in \mathcal{B}_{\text{fin}}(\mu)$, then $f = 0$, and $\mathcal{G}$ is dense in $\mathcal{H}_E$. □

Let $\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu)$ denote, as usual, the space spanned by characteristic functions, and let $\mathcal{G}$ be as in (6.9). Then the following two operators, J and K, are densely defined

$$J : \chi_A \mapsto \chi_A : \mathcal{D}_{\text{fin}} \to \mathcal{H}_E, \qquad K : G_A \mapsto c(I - P)(G_A) : \mathcal{G} \to L^2(\mu) \tag{6.10}$$

where $A \in \mathcal{B}_{\text{fin}}(\mu)$.

Proposition 6.12 *The operators J and K form a symmetric pair, i.e.,*

$$\langle J\varphi, f\rangle_{\mathcal{H}_E} = \langle \varphi, K(f)\rangle_{L^2(\mu)} \tag{6.11}$$

where $\varphi \in \mathcal{D}_{\text{fin}}$ and $f \in \mathcal{G}$.

Proof To prove (6.11) it suffices to check that it holds for $\varphi = \chi_A$ and $f = G_B$ where $A, B \in \mathcal{B}_{\text{fin}}(\mu)$. For these functions, we show that the both inner products are equal to $\nu(A \cap B)$.

By Lemma 6.10, we have

$$\begin{aligned}
\langle \chi_A, K(G_B)\rangle_{L^2(\mu)} &= \langle \chi_A, c\chi_B\rangle_{L^2(\mu)} \\
&= \int_V \chi_A c \chi_B \, d\mu \\
&= \nu(A \cap B).
\end{aligned}$$

On the other hand, for the same functions φ and f, we compute the inner product in the finite energy Hilbert space using the symmetry of ρ:

$$\begin{aligned}
\langle J(\chi_A), G_B\rangle_{\mathcal{H}_E} &= \frac{1}{2}\iint_{V\times V} (\chi_A(x) - \chi_A(y))(G_B(x) - G_B(y))\, d\rho(x, y) \\
&= \iint_{V\times V} (\chi_A(x)G_B(x) - \chi_A(x)G_B(y))\, d\rho(x, y) \\
&= \iint_{V\times V} [\chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(x) \\
&\qquad - \chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(y)] c(x) P(x, dy) d\mu(x)
\end{aligned}$$

$$= \int_V [\chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(x)$$

$$- \chi_A(x) \sum_{n=0}^{\infty} \int_V P^n(\chi_B)(y) P(x, dy)] c(x) d\mu(x)$$

$$= \int_V [\chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(x) - \chi_A(x) \sum_{n=1}^{\infty} P^n(\chi_B)]\, d\nu(x)$$

$$= \int_V \chi_A(x)\chi_B(x)\, d\nu(x)$$

$$= \nu(A \cap B).$$

□

Corollary 6.13 *The finite energy Hilbert space admits the orthogonal decomposition*

$$\mathcal{H}_E = \overline{J(\mathcal{D}_{\text{fin}}(\mu))} \oplus \mathcal{H}arm.$$

In particular, for every $B \in \mathcal{B}_{\text{fin}}(\mu)$*, we have* $G_B = G_1 \oplus G_2$*, where* $G_1 \in \overline{J(\mathcal{D}_{\text{fin}}(\mu))}$ *is always non-zero.*

Proof Indeed, if one assumed that $G_1 = 0$, then we would have that G_B is orthogonal to $\overline{J(\mathcal{D}_{\text{fin}}(\mu))}$. This contradicts Theorem 6.11. □

We conclude this section with the following result that was proved in [1]:

Theorem 6.14 *Let* (P_n) *be a transient Markov process, and let* $G(x, A)$ *be the corresponding Green's function. Then, for any* $f \in \mathcal{H}_E$*, we have the decomposition*

$$f = G(\varphi) + h$$

where h *is a harmonic function and* $\varphi \in L^2(\nu)$.

7 Discretization of the Graph $\mathcal{B}_{\text{fin}}(\mu)$

In this section, we show that our basic setting (a symmetric measure on the Cartesian product $(V, \mathcal{B})$) can be realized as a limit of discrete graphs. This approach naturally leads to the notion of *graphons*. The reader can find necessary information in the following books [65–67] and articles [68–70].

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We will associate with $(V, \mathcal{B}, \mu)$ and ρ a sequence of countably infinite graphs $\mathcal{G}_n$ equipped with conductance functions c_n such that the weighted graphs $(\mathcal{G}_n, c_n)$ can be viewed as a discretization of of the uncountable graph $\mathcal{B}_{\rm fin}$ considered in [1].

We first recall a few facts from [1].

Lemma 7.1 *Suppose that $c(x) \in L^1_{\rm loc}(\mu)$. Then, for any set $A \in \mathcal{B}_{\rm fin}$,*

$$\rho(A \times A^c) < \infty \tag{7.1}$$

where $A^c = V \setminus A$. The converse is not true, in general.

We can view at the set $\mathcal{B}_{\rm fin} = \mathcal{B}_{\rm fin}(\mu)$ as an uncountable graph $\mathcal{G}$ whose vertices are sets A from $\mathcal{B}_{\rm fin}$ and edges are defined as follows. For a symmetric measure ρ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, we say that two sets A and B from $\mathcal{B}_{\rm fin}$ are connected by an edge e if $\rho(A \times B) > 0$.

This definition is extended to get finite paths in the graph $\mathcal{G}$. It is said that there exists a finite path in the graph $\mathcal{G}$ from A to B if there exists a sequence $\{A_i : i = 0, \dots, n\}$ of sets from $\mathcal{B}_{\rm fin}$ (vertices of $\mathcal{G}$) such that $A_0 = A$, $A_n = B$ and $\rho(A_i \times A_{i+1}) > 0, i = 0, \dots n-1$.

Theorem 7.2 *Let $(V, \mathcal{B}, \mu)$ be as above, and let ρ be a symmetric irreducible measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. Then any two sets A and B from the graph $\mathcal{G}$ are connected by a finite path, i.e., the graph $\mathcal{G}$ is connected.*

Proof We will show that there exists a finite sequence $(A_i : 0 \le i \le n)$ of disjoint subsets from $\mathcal{B}_{\rm fin}$ such that $A_0 = A$, $\rho(A_i \times A_{i+1}) > 0$, and $\rho(A_n \times B) > 0$, $i = 0, \dots, n-1$.

If $\rho(A \times B) > 0$, then nothing to prove, so that we can assume that $\rho(A \times B) = 0$.

Let $\xi = (C_i : i \in \mathbb{N})$ be a partition of V into disjoint subsets of positive finite measure such that $C_i \in \mathcal{B}_{\rm fin}$ for all i. Without loss of generality, we can assume that the sets A and B are included in ξ. Let for definiteness, $A = C_0$.

Since $\rho(A \times A^c) > 0$ (by Lemma 7.1), there exists a set $C_{i_1} \in \xi$ such that $\rho(A \times C_{i_1}) > 0$ and $\rho(A \times C_j) = 0$ for all $0 < j < i_1$. Set

$$A_1 := \bigcup_{0<j\le i_1} C_j.$$

It is clear that $A_1 \in \mathcal{B}_{\rm fin}$ and $\rho(A_0 \times A_1) > 0$. If $\rho(A_1 \times B) > 0$, then we are done. If not, we proceed as follows. Because of the property $\rho(A_1 \times A_1^c) > 0$, there exists some $i_2 > i_1$ such that $\rho(A_1 \times C_{i_2}) > 0$ and $\rho(A_1 \times C_j) = 0$ for all $i_1 < j < i_2$. Set

$$A_2 := \bigcup_{i_1<j\le i_2} C_j.$$

Then $\rho(A_1 \times A_2) > 0$, and we check whether $\rho(A_2 \times B) > 0$. If not, we continue in the same manner by constructing consequently disjoint sets A_i satisfying the property $\rho(A_i \times A_{i+1}) > 0$. Since B is an element of ξ, this process will terminate. This means that there exists some n such that $A_n \supset B$. This argument proves the proposition. □

Given a σ-finite measure space $(V, \mathcal{B}, \mu)$, consider a sequence of measurable partition $\{\xi_n\}_{n\in\mathbb{N}}$ such that

(i) $\xi_n = (A_n(i) : i \in \mathbb{N}),\ \bigsqcup_i A_n(i) = V,\ A_n(i) \in \mathcal{B}_{\rm fin}(\mu)$;
(ii) ξ_{n+1} refines ξ_n, i.e., every element $A_n(i)$ of the partition ξ_n is the union of some elements of ξ_{n+1}: $A_n(i) = \bigcup_{j\in\Lambda_n(i)} A_{n+1}(j)$ where $\Lambda_n(i)$ is a finite subset of $\mathbb{N}$;
(iii) the set $\{A_n(i) : i \in \mathbb{N}, n \in \mathbb{N}\}$ generates the Borel σ-algebra $\mathcal{B}$.

If for every i, the cardinality of the set Λ_i is bigger than one, we say that ξ_{n+1} refines ξ_n strictly.

It is well known, see e.g. [26], that, for any point $x \in V$, there exists a sequence $i_n(x)$ such that $A_{n+1}(i_{n+1}(x)) \subset A_n((i_n)(x))$ and

$$\{x\} = \bigcap_{n\in\mathbb{N}} A_n(i_n(x)) \tag{7.2}$$

Suppose ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We define a sequence of non-negative Borel functions $c^{(n)}$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ by setting

$$c^{(n)}_{xy} := \rho(A_n(i_n(x)) \times A_n(i_n(y)))$$

for any x, y from V. Clearly, $c^{(n)}_{xy}$ is a piecewise constant function.

Lemma 7.3 *For a given sequence of strictly refining partitions* (ξ_n)*, the sequence of functions* $(c^{(n)}_{xy})$ *is monotone decreasing.*

Proof The proof is straightforward. For $x, y \in V$, let the sequences $(A_n(i_n(x)))$ and $(A_n(j_n(y)))$ shrink to the points x and y, respectively, according to (7.2). By assumption of the lemma, $A_{n+1}(i_{n+1}(x))$ is a proper subset of $A_n(i_n(x))$. Hence,

$$\begin{aligned} c^{(n+1)}_{xy} &= \rho(A_{n+1}(i_{n+1}(x)) \times A_{n+1}(j_{n+1}(y)) \\ &< \rho(A_n(i_n(x)) \times A_n(j_n(y))) \\ &= c^{(n)}_{xy}. \end{aligned}$$

□

We now can define a sequence of discrete graphs (weighted networks) $G_n = (V_n, E_n, w_n)$. The vertex set V_n is formed by the atoms of the partition ξ_n, i.e., by the sets $\{A_n(i) : i \in \mathbb{N}_0\}$; therefore V_n can be identified with $\mathbb{N}_0$. The set of edges

E_n consists of pairs (i, j) such that

$$(i, j) \in E_n \iff \rho(A_n(i) \times A_n(j)) > 0.$$

The weight function is $w_n(i, j) = \rho(A_n(i) \times A_n(j))$.

Lemma 7.4 *Let ρ be a symmetric irreducible measures on $(V \times V, \mathcal{B} \times \mathcal{B})$. Then the weighted graph G_n is connected for every n.*

It follows from Lemma 7.3 that

$$c_{xy} = \lim_{n\to\infty} c_{xy}^{(n)}$$

exists and is a Borel non-negative function. Since the measure ρ is symmetric, we conclude that $c_{xy} = c_{yx}$.

Next, we define

$$c^{(n)}(x) = \sum_j \rho(A_n(i_n(x)) \times A_n(j)) = \sum_{y\sim_n x} c_{xy}^{(n)}$$

where $x \sim_n y$ if and only if $c_{xy}^{(n)} > 0$. It can be seen that

$$c^{(n)}(x) = \rho(A_n(i_n(x)) \times V). \tag{7.3}$$

Using the proved results, we can deduce the following statement.

Theorem 7.5 *The sequence $(c^{(n)}(x))$ is monotone decreasing for every $x \in V$ and*

$$c(x) := \lim_{n\to\infty} c^{(n)}(x) = \rho_x(V).$$

Proof Indeed, we see from (7.3) that

$$c^{(n+1)}(x) = \rho(A_{n+1}(i_{n+1}(x)) \times V) < \rho(A_n(i_n(x)) \times V) = c^{(n)}(x).$$

Hence, the Borel function $c(x)$ is well defined for every x. Because $\bigcap_n A_n(i_n(x)) = \{x\}$, we obtain that $c(x) = \rho_x(V)$. □

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