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New Directions in Function Theory: From Complex to Hypercomplex to Non-Commutative

Chapman University, November 2019

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Here, on the level sand,
Between the sea and the land,
What shall I build or write
Against the fall of the night?

Tell me of runes to grave
That hold the bursting wave,
Or bastions to design
For longer date than mine.

A.E. Housman, Smooth between sea and land, quotation taken from Hardy's book *A mathematical Apology [3, p. 77]*

Preface

In this volume we present 12 refereed papers written by field experts for the conference *New directions in function theory: From complex to hypercomplex to non-commutative*, held at Chapman University from November 21 to November 26, 2019. This conference was part of an on-going series of yearly mathematics conferences and workshops held at Chapman University since 2010 (temporarily interrupted during the fall of 2020 by the corona pandemic). Another example of such work is the volume (see [1]) which the first and fourth editors assembled for a similar occasion, a conference with a different topic held in 2017.

Our 2019 Conference held 42 presentations (see the program below) on a wide range of topics pertaining to the theme of hypercomplex function theory. The papers submitted to this volume can be divided in the following overlapping categories: two papers on hypercomplex analysis, three pertaining to Schur analysis and de Branges spaces, five exploring new aspects of classical function theory, and two related to infinite dimensional analysis. At least three of the works have a very strong signal-processing flavor. More precisely, we have the following classification:

Function theory and harmonic analysis: In the chapter *"Differential Subordinations in Harmonic Mappings"*, authors **M. Aydogan**, **Daoud Bshouty**, **Sanford S. Miller**, and **F.M. Sakar** extend results from the theory of differential subordination (see [6]) to complex harmonic mappings, that is, functions of the form $h(z) + \overline{g(z)}$, where f and g are analytic in a given open subset of the complex plane. In *"Representation Formulae for the Determinant in a Neighborhood of the Identity"*, **Denis Constales** and **Alí Guzman Adan** give a formula for a power series expansion for $\det(I + M)^{-1}$ for suitable matrices M, and provide applications to the Taylor expansion of the Dirac distribution. In *"The Wiener Algebra and Singular Integrals"*, **E. Liflyand** extends to the multivariable case some of his earlier work [5] and gives necessary conditions in terms of singular integrals for a function to be in the Wiener algebra. **Ronen Peretz**, in *"Techniques to Derive Estimates for Integral Means and other Geometric Quantities Related to Conformal Mappings"*, uses the Goluzin inequalities and Riemann sum approximation and obtains estimates on integral means associated to univalent functions. In particular, the author obtains a new and surprising inequality in terms of positive definite kernels for univalent functions in the disk such that $f(0) = 0$ and $f'(0) = 1$. In *"Harmonic Analysis of Some Arithmetical Functions"*, **Ahmed Sebbar** and **Roger Gay** consider analytic number theory and associate Hilbert spaces and a law of composition to arithmetical functions. The notion of functions satisfying a Kubert identity [4], namely functions defined on $\mathbb{Q}/\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ satisfying

$$f(x) = m^{s-1} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)$$

where s is some fixed parameter, which plays an important role.

Schur analysis, de Branges spaces, and function theory: In the paper *"On Parseval Frames of Kernel Functions in de Branges Spaces of Entire Vector Valued Functions"*, **Saud Al-Sadi** and **Eric S. Weber** use Naimark dilation theorem to study Parseval frames and application to multiplexing (transmission of several signals at the same time over a single communications channel). In the chapter **"On the Carathéodory-Fejér Interpolation Problem for Stieltjes Functions"**, by **Vladimir Bolotnikov**, an important point is that the truncated problems with even and odd numbers of conditions lead to quite different type of results in the solution of the stated interpolation problem. Another point of special interest is the Schwartz-Pick type reduction given for Stieltjes functions. In *"Parametrization of the Solution Set of a Matricial Truncated Hamburger Moment Problem by a Schur Type Algorithm"*, authors **Bernd Fritzsche**, **Bernd Kirstein**, **Susanne Kley**, and **Conrad Mädler** develop Schur analysis for the Hamburger moment problem, both on the level of sequences and functions, in the matrix-valued setting. The chapter is a part of a systematic program of creating a Schur analysis approach to matricial versions of truncated classical power moment problems, such as the Hamburger, Stieltjes, and Hausdorff moment problems.

Hypercomplex analysis: In *"The Segal-Bargmann Transform in Clifford Analysis"*, **Swanhild Bernstein** and **Sandra Schufmann** study the Segal-Bargmann transform in its connection to the windowed Fourier transform and time-frequency analysis in the Clifford setting. In *"Complex Ternary Analysis and Applications"*, **Mihaela B. Vajiac** presents a theory of functions on complex ternary algebras. In opposition to the real ternary case (see [2] for the latter), one has a one-dimensional theory in one ternary variable, which has a dual nature: an element that cubes to ± 1 on the one hand, and a theory of one bicomplex variable and one complex variable entangled by algebraic relations on the other hand.

Infinite dimensional analysis and non-commutative theory: In *"Symmetric Measures, Continuous Networks, and Dynamics"*, **Sergey Bezuglyi** and **Palle E.T. Jorgensen** extend the basic definitions and results of the theory of weighted networks (known also as electrical or resistance networks) to the case of measure spaces. In *"Multi Variable Semicircular Processes from $*$-homomorphisms and operators"*, **Ilwoo Cho** and **Palle E.T. Jorgensen** study in non-commutative probability theory new construction of semicircular elements. This is of special importance since the semicircular law plays in the non-commutative setting the role of the Gaussian law in classical (commutative) probability theory; see [7] for an introduction to non-commutative probability.

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Contents

On Parseval Frames of Kernel Functions in de Branges Spaces of Entire Vector Valued Functions

Sa'ud Al-Sa'di and Eric S. Weber

Abstract We consider the existence and structure properties of Parseval frames of kernel functions in vector valued de Branges spaces. We develop some sufficient conditions for Parseval sequences by identifying the main construction with Naimark dilation of frames. The dilation occurs by embedding the de Branges space of vector valued functions into a dilated de Branges space of vector valued functions. The embedding also maps the kernel functions associated with a frame sequence of the original space into a Riesz basis for the embedding space. We also develop some sufficient conditions for a dilated de Branges space to have the Kramer sampling property.

Keywords de Branges Spaces · Entire vector valued functions · Parseval frames · Kramer sampling formula

Mathematics Subject Classification (2000) Primary: 94A20; Secondary 30D10, 47A20

1 Introduction

The theory of de Branges spaces of entire functions can be extended with suitable hypotheses to spaces of entire vector valued functions. Spaces of entire vector valued functions were introduced and extensively studied by Louis de Branges and have been developed in view of the model theory for linear transformations in Hilbert spaces [13]. These spaces have played a central role in applications to direct

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and inverse problems for canonical systems of differential and integral equations and Dirac-Krein systems, see for example [5–7].

The main goal of the present paper is to extend some results on de Branges spaces of scalar valued functions obtained in [3] to de Branges spaces of vector valued functions. We consider the existence and structure properties of Parseval frames of kernel functions in vector valued de Branges spaces. In Sect. 1.3 we shall review some definitions and necessary facts from the theory of reproducing kernel Hilbert spaces of vector valued functions. As a special case of such spaces the de Branges spaces of vector valued functions is reviewed in Sect. 1.4. Sections 2 and 3 are devoted to developing new results on the construction of dilated de Branges spaces of vector valued functions and orthogonality of embeddings within the dilation spaces. We develop some necessary conditions for Parseval sequences in vector valued de Branges spaces by identifying the main construction with Naimark dilation of frames via embedding the de Branges space into a dilated de Branges space. The embedding identifies the kernel functions associated with a frame sequence as a summand for a Riesz basis for the dilated space. We also obtain some sufficient conditions for a dilated de Branges space to have the Kramer sampling property in Sect. 4 as well as results concerning the multiplexing of samples in the dilated space.

1.1 Notation

Some notations are necessary to describe the spaces we will consider here, see [7, 12] for additional information. $\mathbb{C}$ will denote the complex plane, $\mathbb{C}^+$ (resp., $\mathbb{C}^-$) the open upper (resp., lower) half plane, $\mathbb{C}^p$ the complex $p \times 1$ vectors. The notation $\mathbb{C}^{p\times q}$ stands for the set of all $p \times q$ matrices with complex entries, the identity matrix that belongs to $\mathbb{C}^{p\times p}$ will be denoted by I_p. A $\mathbb{C}^p$ vector valued function $f(z)$, defined in a region Ω of the complex plane $\mathbb{C}$, is said to be analytic in Ω if the complex valued function $u^* f(z)$ is analytic in the region for every choice of vector $u \in \mathbb{C}^p$. A continuous $\mathbb{C}^{p\times p}$ matrix valued function $F(z)$, defined in Ω, is said to be analytic in the region if $u^* F(z)v$ is analytic in the region for every choice of vectors u and v in $\mathbb{C}^p$. A matrix valued function with entries that are analytic in the full complex plane is said to be entire matrix valued function. $f^*(z)$ is the Hermitian transpose of the matrix valued function $f(z)$, and $f^\#(z) = f^*(\bar{z})$.
$\mathbb{H}_2^{p\times q}$ is the Hardy space of $p\times q$ matrix valued functions with entries in the classical Hardy space $\mathbb{H}_2$ with respect to $\mathbb{C}^+$, with norm

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} trace\{f^*(x+iy)f(x+iy)\}dx \ < \infty.$$

$(\mathbb{H}_2^{p\times q})^\perp = \{f : f^\# \in \mathbb{H}_2^{q\times p}\}$ (the superscript $\perp$ means that $\mathbb{H}_2^{p\times q}$ and $(\mathbb{H}_2^{p\times q})^\perp$ are orthogonal to each other when regarded as subspaces of $L_2^{p\times q}$). We shall use the symbol $\mathbb{H}_2^p$ for $\mathbb{H}_2^{p\times 1}$, and $(\mathbb{H}_2^q)^\perp$ for $(\mathbb{H}_2^{q\times 1})^\perp$.
$\mathbb{H}_\infty^{p\times q}$ is the Hardy space of holomorphic $p \times q$ matrix valued functions in $\mathbb{C}^+$ with

$$\|f\|_\infty = \sup\{\|f(z)\| : z \in \mathbb{C}^+\} < \infty.$$

The Schur class $\mathcal{S}^{p\times p}$ is the class of $p \times p$ matrix valued functions $s(z)$ that are holomorphic and contractive in $\mathbb{C}^+$, i.e.,

$$I_p - s^*(z)s(z) \succeq 0, \text{ for } z \in \mathbb{C}^+.$$

$\mathcal{S}_{in}^{p\times p}$ is the class of matrix valued functions $f \in \mathcal{S}^{p\times p}$ which are inner, i.e., $I_p - f^*(t)f(t) = 0$ for a.e. point $t \in \mathbb{R}$.

The generalized backward-shift operator R_ω is defined for entire vector valued functions by

$$(R_\omega f)(z) = \begin{cases} \frac{f(z)-f(\omega)}{z-\omega} & \text{if } z \neq \omega \\ f'(\omega) & \text{if } z = \omega \end{cases}$$

for every $z, \omega \in \mathbb{C}$.

1.2 Frame Theory

A sequence $\{f_n\}_{n=1}^\infty$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exists constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}, \tag{1}$$

The constants A and B are called lower and upper frame bounds, respectively. The frames for which $A = B = 1$ are called *Parseval frames*. A frame which is a basis is called a *Riesz basis*. It is easy to see that a Parseval frame $\{f_n\}_{n=1}^\infty$ for a Hilbert space $\mathcal{H}$ is an orthonormal basis if and only if each f_n is a unit vector. If the upper bound in (1) is satisfied, then we say that $\{f_n\}_{n=1}^\infty$ is a *Bessel* sequence with Bessel bound B.

Let $\{f_n\}_{n=1}^\infty$ be a Bessel sequence in $\mathcal{H}$. The analysis operator $\Theta : \mathcal{H} \to \ell^2$, which is bounded because of (1), is defined by

$$\Theta : f \to (\langle f, f_n\rangle);$$

and the synthesis operator $\Theta^* : \ell^2 \to \mathcal{H}$, which is the adjoint operator of Θ, is defined by

$$\Theta^* : (c_n)_{n=1}^{\infty} \to \sum_{n=1}^{\infty} c_n f_n.$$

Additionally, the sum $\sum_{n=1}^{\infty} c_n f_n$ converges in $\mathcal{H}$ for all $(c_n)_{n=1}^{\infty} \in l^2$ (see [14]), and so the synthesis operator is also well defined and bounded.

The operator $S := \Theta^*\Theta : \mathcal{H} \to \mathcal{H}$ is called the frame operator, and we have

$$Sf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n \ , \forall f \in \mathcal{H}.$$

The *canonical dual frame* is denoted by $\{\tilde{f}_n\}_{n=1}^{\infty}$, and is defined by $\tilde{f}_n = S^{-1} f_n$. Furthermore, for each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{n=1}^{\infty} \langle f, f_n \rangle \tilde{f}_n = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n \rangle f_n, \tag{2}$$

with unconditional convergence of these series.

If $\mathbb{F} = \{f_n\}_{n=1}^{\infty}$ and $\mathbb{G} = \{g_n\}_{n=1}^{\infty}$ are two Bessel sequences in $\mathcal{H}$, define the operator

$$\Theta_{\mathbb{G}}^* \Theta_{\mathbb{F}} : \mathcal{H} \to \mathcal{H} : f \to \sum_{n=1}^{\infty} \langle f, f_n \rangle \, g_n.$$

If $\Theta_{\mathbb{G}}^* \Theta_{\mathbb{F}} = 0$ then the two Bessel sequences $\mathbb{F}$ and $\mathbb{G}$ are said to be orthogonal [18]. An extensive study of orthogonal frames can be found in the papers [10, 23]. If $\mathbb{F}$ and $\mathbb{G}$ are both Parseval frames and orthogonal to each other, then for any $f, g \in \mathcal{H}$

$$f = \sum_n (\langle f, f_n \rangle + \langle g, g_n \rangle) f_n, \quad \text{and} \quad g = \sum_n (\langle f, f_n \rangle + \langle g, g_n \rangle) g_n$$

In other words, both functions can be recovered from the summed coefficients $\langle f, f_n \rangle + \langle g, g_n \rangle$. This procedure is called *multiplexing*, and can be used in multiple access communication systems. In the proof of our main results we also need a concept of *similar frames*: two frames $\mathbb{F} = \{f_n\}_{n=1}^{\infty}$ and $\mathbb{G} = \{g_n\}_{n=1}^{\infty}$ are said to be similar if there is an invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $Tf_n = g_n$. Two frames $\mathbb{F}$ and $\mathbb{G}$ are similar if and only if $\Theta_{\mathbb{F}}(\mathcal{H}) = \Theta_{\mathbb{G}}(\mathcal{H})$ [11].

Let P be an orthogonal projection from a Hilbert space $\mathcal{K}$ onto a closed subspace $\mathcal{H}$, and $\{f_n\}$ be a sequence in $\mathcal{K}$. Then $\{Pf_n\}$ is called *orthogonal compression* of

$\{f_n\}$ under P, and $\{f_n\}$ is called an *orthogonal dilation* of $\{Pf_n\}$. A classical fact on dilation of frames, which can be attributed to Han and Larson [17], says that a Parseval frame in a Hilbert space $\mathcal{H}$ is an image of an orthonormal basis under an orthogonal projection of some larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ onto $\mathcal{H}$. This result can be considered as a special case of Naimark's dilation theorem for positive operator valued measures, see [20, 21]. In particular, Han and Larson proved the following result.

Theorem 1 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$. Then*

(i) $\{f_n\}$ is a Parseval frame for $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{e_n\}$ for $\mathcal{K}$ such that if P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ then $f_n = Pe_n$, for all $n \in \mathbb{N}$.
(ii) $\{f_n\}$ is a frame for $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_n\}$ for $\mathcal{K}$ such that if P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ then $f_n = Pu_n$, for all $n \in \mathbb{N}$.

Orthogonality of frames and Naimark dilation of frames are related in the following way (see [8, 17]): If $\{u_n\}$ is a Riesz basis for $\mathcal{K}$ and P is the projection onto $\mathcal{H} \subset \mathcal{K}$, then $\{Pu_n\}$ and $\{(I-P)u_n\}$ are orthogonal frames for $\mathcal{H}$ and $\mathcal{H}^{\perp}$, respectively. Conversely, if $\mathbb{F} = \{f_n\}$ and $\mathbb{G} = \{g_n\}$ are orthogonal frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{f_n + g_n\}$ is a frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Note that the sum of the frames need not be a basis for the direct sum in general–however, it will be provided that

$$\Theta_{\mathbb{F}}(\mathcal{H}_1) \oplus \Theta_{\mathbb{G}}(\mathcal{H}_2) = \ell^2.$$

1.3 Reproducing Kernel Hilbert Spaces of Vector Valued Functions

In this section a number of facts about reproducing kernel Hilbert spaces of vector valued functions that will be used frequently are reviewed briefly; more details and supporting proofs may be found in [4–7]. For related results concerning operator valued reproducing kernel spaces, see e.g. [1, 2].

A Hilbert space $\mathcal{H}$ of $p \times 1$ vector valued functions defined on a subset Ω of $\mathbb{C}$ is said to be a reproducing kernel Hilbert space (RKHS) if there exists a $p \times p$ matrix valued function $K_w(z)$ (for $(z, w) \in \Omega \times \Omega$) such that for every choice of $w \in \Omega$, $u \in \mathbb{C}^p$, and $f \in \mathcal{H}$:

1. $K_w(z)u \in \mathcal{H}$, as a vector valued function of z,
2. The reproducing kernel property

$$\langle f(.), K_w(.)u\rangle_{\mathcal{H}} = \langle f(w), u\rangle_{\mathbb{C}} = u^* f(w) \quad (3)$$

The matrix valued function $K_w(z)$ is called a reproducing kernel (RK) of the RKHS $\mathcal{H}$. The existence and uniqueness of a RK is guaranteed by the Riesz representation theorem [15]. The following properties of RKHS are well known and easily checked, see [16] for more details:

1. $\langle K_w(.)u_1, K_v(.)u_2\rangle_{\mathcal{H}} = u_2^* K_w(v)u_1,$ for all $w, v \in \mathbb{C},\ u_1, u_2 \in \mathbb{C}^p$, and
$$\|K_w u\|^2_{\mathcal{H}} = u^* K_w(w)u. \tag{4}$$
2. $\|f(w)\| \le \|f\|_{\mathcal{H}} \|K_w(w)\|^{1/2}$, for all $w \in \Omega$ and $f \in \mathcal{H}$.
3. The RK is positive in the sense that
$$\sum_{i,j=1}^{n} u_j^* K_{w_i}(w_j)u_i \ge 0 \tag{5}$$
for every choice of points $w_1, \ldots, w_n \in \Omega$ and vectors $u_1, \ldots, u_n \in \mathbb{C}^p$ and every positive integer n. Consequently, the set $\{K_w(.)u\ :\ w \in \Omega,\ u \in \mathbb{C}^p\}$ is total in $\mathcal{H}$, that is
$$\mathcal{H} = \overline{\text{span}}\{K_w(.)u\ :\ w \in \Omega,\ u \in \mathbb{C}^p\}.$$

The following theorem is a matrix version of a theorem of Aronszajn in [4].

Theorem 2 *Let Ω be a subset of $\mathbb{C}$ and let the $p \times p$ matrix valued kernel $K_\omega(z)$ be positive on $\Omega \times \Omega$. Then there is a unique Hilbert space $\mathcal{H}$ of $p \times 1$ vector valued functions $f(z)$ on Ω such that*
$$K_\omega u \in \mathcal{H}, \quad \text{and} \quad \langle f, K_\omega u\rangle_{\mathcal{H}} = u^* f(\omega)$$
for every $\omega \in \Omega$, $u \in \mathbb{C}^p$ and $f \in \mathcal{H}$.

Example 1 ([7]) The Hardy space $\mathbb{H}_2^p$ is a RKHS of $p \times 1$ vector valued functions that are holomorphic in $\mathbb{C}^+$ with RK
$$K_\omega(z) = \frac{I_p}{-2\pi i(z - \bar{\omega})}, \text{ for } z, \omega \in \mathbb{C}^+$$

A RKHS $\mathcal{H}$ of $p \times 1$ vector valued functions is said to have the **Kramer sampling property** if there is a sequence of points $\{w_n\}_{n=1}^\infty \subset \Omega$ and a sequence of vectors $\{\xi_n\}_{n=1}^\infty \in \mathbb{C}^p$, such that the set $\{K_{w_n}(.)\xi_n\}_{n=1}^\infty$ is a complete orthogonal set in $\mathcal{H}$, i.e., every $f \in \mathcal{H}$ can be expressed in the form
$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \langle f, K_{w_n}\xi_n\rangle_{\mathcal{H}} \frac{K_{w_n}(z)\xi_n}{\|K_{w_n}\xi_n\|^2} \\ &= \sum_{n=1}^{\infty} \xi_n^* f(w_n) \frac{K_{w_n}(z)\xi_n}{\|K_{w_n}\xi_n\|^2} \end{aligned}$$

In other words, functions of the space $\mathcal{H}$ are uniquely determined and reconstructible from their samples [19].

The notation

$$\mathcal{H}_\omega = \{f \in \mathcal{H} \,:\, f(\omega) = 0\}$$

for RKHS's $\mathcal{H}$ of entire vector valued functions will be useful.

1.4 de Branges Spaces of Vector Valued Functions

In this section we shall present a number of facts from the theory of de Branges spaces of vector valued functions that will be needed in the sequel. Most of this information can be found in the papers [5–7, 12].

An entire $p \times 2p$ matrix valued function $\mathfrak{E}(z) = [E_-(z) \quad E_+(z)]$ is called an entire **de Branges matrix** with $p \times p$ blocks $E_\pm(z)$ that are matrix valued entire functions, if

$$\det E_+(z) \not\equiv 0, \text{ in } \mathbb{C}, \quad \text{and} \quad \chi_{\mathfrak{E}} := E_+^{-1} E_- \in \mathcal{S}_{in}^{p\times p}. \tag{6}$$

The determinant of an entire matrix valued function is an entire function. Consequently, if the determinant of the entire matrix valued function $E_+(z)$ does not vanish identically, the given entire matrix valued function has invertible values at all but isolated points in the complex plane. Since $E_\pm(z)$ are entire matrix valued functions, the condition in (6) ensures that (see [16])

$$E_+(z)E_+^\#(z) = E_-(z)E_-^\#(z), \text{ for all } z \in \mathbb{C}. \tag{7}$$

Definition 1 Given a de Branges matrix $\mathfrak{E}$, the set of entire $\mathbb{C}^p$ vector valued functions $f(z)$ satisfying

$$E_+^{-1} f \in \mathbb{H}_2^p \quad \text{and} \quad E_-^{-1} f \in (\mathbb{H}_2^p)^\perp \tag{8}$$

is a reproducing kernel Hilbert space with reproducing kernel

$$K_w^{\mathfrak{E}}(z) = \begin{cases} \frac{E_+(z)E_+^*(w) - E_-(z)E_-^*(w)}{2\pi i(\bar{w}-z)} & , \text{ if } z \neq \bar{w} \\ \frac{E_+'(\bar{w})E_+^*(w) - E_-'(\bar{w})E_-^*(w)}{-2\pi i} & , \text{ if } z = \bar{w} \end{cases} \tag{9}$$

with respect to the inner product

$$\langle f, g\rangle_{\mathcal{B}} = \langle E_+^{-1} f, E_+^{-1} g\rangle_{st} = \int_{-\infty}^{\infty} g^*(t)\Delta_{\mathcal{E}}(t) f(t)\, dt, \tag{10}$$

where

$$\Delta_{\mathcal{E}}(t) = \{E_+(t)E_+^*(t)\}^{-1} = \{E_-(t)E_-^*(t)\}^{-1},$$

for all $t \in \mathbb{R}$ at which $\det E_\pm(z) \neq 0$.

The Hilbert space corresponding to the de Branges matrix $\mathfrak{E}$ is called the de Branges space $\mathcal{B}(\mathfrak{E})$; for every $w \in \mathbb{C}$, every $u \in \mathbb{C}^p$, and every $f \in \mathcal{B}(\mathfrak{E})$

1. $K_w^{\mathfrak{E}} u \in \mathcal{B}(\mathfrak{E})$ and
2. $\langle f, K_w^{\mathfrak{E}} u\rangle_{\mathcal{B}(\mathfrak{E})} = u^* f(w)$

Remark 1 If $E(z)$ is a scalar valued entire function which has no real zeros and $|E(z)| > |E(\bar{z})|$ for all $z \in \mathbb{C}^+$, then $\mathcal{B}(\mathfrak{E})$ with $\mathfrak{E} = [E^\#(z) \quad E(z)]$ is just the usual de Branges space corresponding to the de Branges function $E(z)$.

Example 2 ([16]) If $E_+^t(z) = e^{-izt} I_p$ and $E_-^t(z) = e^{izt} I_p$ for $t > 0$, then it is easy to see that $\mathfrak{E}_t(z) = [E_-^t(z) \quad E_+^t(z)]$ is an entire de Branges matrix, and the space $\mathcal{B}(\mathfrak{E}_t)$ is a vector Paley-Wiener space with RK

$$K_w^{\mathfrak{E}_t}(z) = \frac{\sin(z - \bar{w})t}{\pi(z - \bar{w})} I_p.$$

There is a connection between de Branges spaces $\mathcal{B}(\mathfrak{E})$ of entire vector valued functions that are invariant under the action of the generalized backward-shift operator R_ω and the Kramer sampling property, the following is found in [16, Theorem 9.4].

Theorem 3 *Let $\mathcal{H}$ be the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E}$ with RK $K_\omega(z)$. If*

(1) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \mathbb{C}$, and*
(2) *$K_\omega(\omega) \succ 0$ for at least one point $\omega \in \mathbb{C}$,*

then $\mathcal{B}(\mathfrak{E})$ has the Kramer sampling property.

A sufficient condition for the space $\mathcal{H}_\omega$ to be invariant under the operator R_ω is given by the next lemma [16, Lemma 6.4].

Lemma 1 *Let $\mathcal{H}$ be the de Branges space $\mathcal{B}(\mathfrak{E})$ based on the de Branges matrix $\mathfrak{E}$, then:*

(1) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \overline{\mathbb{C}^+}$ at which $E_+(\omega)$ is invertible.*
(2) *$R_\omega \mathcal{H}_\omega \subseteq \mathcal{H}$ for every point $\omega \in \overline{\mathbb{C}^-}$ at which $E_-(\omega)$ is invertible.*

2 The de Branges Space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

In this section a number of results on constructing the dilated de Branges space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ will be obtained. The space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is a simultaneous dilation of two de Branges spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$. We will consider the class of $p \times p$ entire matrix valued functions $F(z)$ such that $\det(F(z)) \not\equiv 0$ in $\mathbb{C}$, and $F^{-1}F^{\#} \in \mathcal{S}_{in}^{p\times p}$. We will denote this class by $\mathbb{N}_{inv}(\mathbb{C}^{p\times p})$. If $F \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, the conditions in (6) and (7) imply that

$$F(z)F^{\#}(z) = F^{\#}(z)F(z), \text{ for all } z \in \mathbb{C}.$$

Hence, the $p \times 2p$ matrix

$$\mathfrak{F} := [F^{\#}(z) \quad F(z)]$$

is a de Branges matrix, with corresponding de Branges space $\mathcal{B}(\mathfrak{F})$.

Example 3 For $n \in \mathbb{N}$, define the family of $2n \times 2n$ entire matrix-valued functions

$$F(z) = \begin{bmatrix} e^{f_1(z)}I_n & 0 \\ 0 & e^{f_2(z)}I_n \end{bmatrix},$$

where $f_1(z) = g_1(z) + \alpha_1 + \beta_1 iz$, $f_2(z) = g_2(z) + \alpha_2 + \beta_2 iz$, for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and entire functions g_1, g_2 which are real on the real line. Then it is readily checked that the matrix valued functions $UF(z)U^*$ belongs to the class $\mathbb{N}_{inv}(\mathbb{C}^{2n\times 2n})$ for any $2n \times 2n$ constant unitary matrix U.

Definition 2 Given de Branges matrices $\mathfrak{F} := [F^{\#}(z) \quad F(z)]$, $\mathfrak{E} = [E_-(z) \quad E_+(z)]$, where $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, we define

$$\mathfrak{F} \diamond \mathfrak{E} := [F^{\#}(z)E_-(z) \quad F(z)E_+(z)].$$

Our main results will utilize the following additional commutation assumption:

$$F^{\#}E_- = E_-F^{\#} \quad \text{and} \quad FE_+ = E_+F. \tag{11}$$

Under this additional assumption on the matrix valued functions F and $E_\pm$ we prove that the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is a RKHS whose kernel can be expressed in terms of the kernels for $\mathcal{B}(\mathfrak{F})$ and $\mathcal{B}(\mathfrak{E})$. Throughout the rest of this paper, unless otherwise specified, we will assume that the de Branges matrices $\mathfrak{F} = [F^{\#}(z) \quad F(z)]$ and $\mathfrak{E} = [E_-(z) \quad E_+(z)]$ with $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$. We begin with a lemma.

Lemma 2 *Assume $\mathfrak{F}$ and $\mathfrak{E}$ satisfy the hypotheses of Definition 2 and Equation (11). Then the following hold:*

(i) $FE_- = E_-F$;
(ii) $F^{\sharp}E_+ = E_+F^{\sharp}$;

(iii) $FE_+^{-1} = E_+^{-1}F$;
(iv) $F^{-1}E_- = E_-F^{-1}$;
(v) $E_-^{-1}(F^\sharp)^{-1}F = (F^\sharp)^{-1}FE_-^{-1}$.

Proof By virtue of $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$, we have that $F^*F = FF^*$ on the real axis. Item (i) holds by Fuglede's Theorem: F^* is normal on the real axis and $F^*E_- = E_-F^*$ holds on the real axis by eq. (11). An analogous argument shows that (ii) holds. Items (iii), respectively (iv), hold because of eq. (11), respectively (i), and a standard Neumann series argument. Item (v) holds by Eq. (11) and (iv). □

Theorem 4 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and eq. (11). Then*

(i) *$\mathfrak{F} \diamond \mathfrak{E}$ is a de Branges matrix, and*
(ii) *the corresponding de Branges space is $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, with RK*

$$K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z) = F(z)K_w^{\mathfrak{E}}(z)F^*(w) + E_-(z)K_w^{\mathfrak{F}}(z)E_-^*(w). \tag{12}$$

Proof Since $\det(E_\pm(z)) \not\equiv 0$, $\det(F(z)) \not\equiv 0$, and $\det(F^\#(z)) \not\equiv 0$ in $\mathbb{C}$, then

$$\det(F^\#(z)E_-(z)) \not\equiv 0 \text{ and } \det(F(z)E_+(z)) \not\equiv 0 \text{ in } \mathbb{C}.$$

To show that the function $\chi_{\mathfrak{F}\diamond\mathfrak{E}} := (FE_+)^{-1}(F^\#E_-) \in \mathcal{S}_{in}^{p\times p}$, we use the fact that both functions $\chi_{\mathfrak{E}} := E_+^{-1}E_-$ and $\chi_{\mathfrak{F}} := F^{-1}F^\#$ belongs to the class $\mathcal{S}_{in}^{p\times p}$. By Lemma 2 (ii), we have $F^\#E_+^{-1} = E_+^{-1}F^\#$. Thus, again using Lemma 2,

$$\begin{aligned}\chi_{\mathfrak{F}\diamond\mathfrak{E}} &= (FE_+)^{-1}(F^\#E_-) = (E_+F)^{-1}(F^\#E_-) \\ &= F^{-1}E_+^{-1}F^\#E_- = F^{-1}F^\#E_+^{-1}E_- = \chi_{\mathfrak{F}}\chi_{\mathfrak{E}}.\end{aligned}$$

This proves that $\mathfrak{F} \diamond \mathfrak{E}$ is a de Branges matrix.

The RK of the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is

$$\begin{aligned}K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z) &= \frac{F(z)E_+(z)(F(w)E_+(w))^* - F^\#(z)E_-(z)(F^\#(w)E_-(w))^*}{2\pi i(\bar{w}-z)} \\ &= \frac{F(z)E_+(z)E_+^*(w)F^*(w) - F^\#(z)E_-(z)E_-^*(w)(F^\#(w))^*}{2\pi i(\bar{w}-z)} \\ &= \frac{F(z)E_+(z)E_+^*(w)F^*(w) - F(z)E_-(z)E_-^*(w)F^*(w)}{2\pi i(\bar{w}-z)} \\ &\quad + \frac{F(z)E_-(z)E_-^*(w)F^*(w) - F^\#(z)E_-(z)E_-^*(w)(F^\#(w))^*}{2\pi i(\bar{w}-z)} \\ &= F(z)K_w^{\mathfrak{E}}(z)F^*(w) + E_-(z)K_w^{\mathfrak{F}}(z)E_-^*(w)\end{aligned}$$

since $FE_- = E_-F$ and $F^\#E_- = E_-F^\#$ by Lemma 2. □

Example 4 Consider the matrix valued function $F(z)$ given in Example 3 and the matrix valued functions $E_+(z)$, $E_-(z)$ given in Example 2, then

$$\mathfrak{F} = [F^\# \quad F], \quad \mathfrak{E} = [E_- \quad E_+]$$

satisfies the conditions of Definition 2.

3 Orthogonality in $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

Now we prove that the spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ can be embedded into the larger space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$.

Proposition 1 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). The operator $\mathcal{I} : \mathcal{B}(\mathfrak{E}) \to \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, defined by $\mathcal{I}(f) = Ff$, is a linear isometry.*

Proof We first prove that $\mathcal{I}$ is well defined, i.e., for every $f \in \mathcal{B}(\mathfrak{E})$, $Ff \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, that is

$$(FE_+)^{-1}Ff \in \mathbb{H}_2^p, \quad \text{and} \quad (F^\# E_-)^{-1}Ff \in (\mathbb{H}_2^p)^\perp,$$

Let $f \in \mathcal{B}(\mathfrak{E})$, then by Definition 1

$$E_+^{-1}f \in \mathbb{H}_2^p, \quad \text{and} \quad E_-^{-1}f \in (\mathbb{H}_2^p)^\perp, \tag{13}$$

hence, $(FE_+)^{-1}Ff = E_+^{-1}f \in \mathbb{H}_2^p$. On the other hand, $(F^\# E_-)^{-1}Ff = E_-^{-1}(F^\#)^{-1}Ff = (F^\#)^{-1}FE_-^{-1}f$ belongs to $(\mathbb{H}_2^p)^\perp$, since $E_-^{-1}f \in (\mathbb{H}_2^p)^\perp$ and $(F^\#)^{-1}F$ is the inverse of a matrix valued inner function.

Let $f_1, f_2 \in \mathcal{B}(\mathfrak{E})$, then

$$\begin{aligned}
\langle \mathcal{I}(f_1), \mathcal{I}(f_2)\rangle_{\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})} &= \int_{-\infty}^{\infty} (F(t)f_2(t))^* \Delta_{\mathfrak{F}\diamond\mathfrak{E}}(t)(F(t)f_1(t))\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t)F^*(t)(FE_+(FE_+)^*)^{-1}(t)F(t)f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t)F^*(t)\,(F^*(t))^{-1}(E_+^*(t))^{-1}E_+^{-1}(t)F^{-1}(t)F(t)f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t)(E_+^*(t))^{-1}E_+^{-1}(t)f_1(t)\, dt \\
&= \int_{-\infty}^{\infty} f_2^*(t)\Delta_{\mathfrak{E}}(t)f_1(t)\, dt = \langle f_1, f_2\rangle_{\mathcal{B}(\mathfrak{E})}.
\end{aligned}$$

□

A similar argument as in the proof of Proposition 1 can be used to proof the next proposition.

Proposition 2 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). The operator $\mathcal{J} : \mathcal{B}(\mathfrak{F}) \to \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, defined by $\mathcal{J}(g(z)) = E_-(z)g(z)$ is a linear isometry.*

Theorem 5 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). The images of the operators $\mathcal{I}$ and $\mathcal{J}$ are orthogonal in $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$.*

Proof Let $f \in \mathcal{B}(\mathfrak{E})$ and $g \in \mathcal{B}(\mathfrak{F})$, then

$$\left\langle (FE_+)^{-1} Ff, (FE_+)^{-1} E_- g \right\rangle = \left\langle (E_+)^{-1} f, (E_+)^{-1} E_- F^{-1} g \right\rangle = 0,$$

because $f \in \mathcal{B}(\mathfrak{E})$ if and only if $E_+^{-1} f \in \mathbb{H}_2^p \ominus (E_+)^{-1} E_- \mathbb{H}_2^p$. □

Remark 2 Given $\omega \in \mathbb{C}$ and $u \in \mathbb{C}^p$ the vector valued function $K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z)u \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ as a function of z. Likewise, $K_w^{\mathfrak{E}}(z)F^*(w)u \in \mathcal{B}(\mathfrak{E})$ and $K_w^{\mathfrak{F}}(z)E_-^*(w)u \in \mathcal{B}(\mathfrak{F})$. It follows from (12) that for any $w \in \mathbb{C}$ and $u \in \mathbb{C}^p$

$$\begin{aligned} K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}(z)u &= F(z)\left(K_\omega^{\mathfrak{E}}(z)F^*(\omega)u\right) + E_-(z)\left(K_w^{\mathfrak{F}}(z)E_-^*(\omega)u\right) \\ &= \mathcal{I}\left(K_w^{\mathfrak{E}}(z)F^*(w)u\right) + \mathcal{J}\left(K_w^{\mathfrak{F}}(z)E_-^*(w)u\right) \end{aligned} \tag{14}$$

Consequently, since the set $\{K_w^{\mathfrak{F}\diamond\mathfrak{E}}(z)u : \ w \in \mathbb{C}, \ u \in \mathbb{C}^p\}$ spans the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, the set

$$\mathcal{I}\left(\{K_w^{\mathfrak{E}}(z)F^*(w)u : w \in \mathbb{C},\ u \in \mathbb{C}^p\}\right) \cup \mathcal{J}\left(\{K_w^{\mathfrak{F}}(z)E_-^*(w)u :\ w \in \mathbb{C},\ u \in \mathbb{C}^p\}\right)$$

spans $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ whenever $\det(F^*(\omega)) \not\equiv 0$ and $\det(E_-^*(\omega)) \not\equiv 0$. Indeed, for any finite set of points $\omega_1, \ldots, \omega_n \in \mathbb{C}$ and vectors $u_1, \ldots, u_n \in \mathbb{C}^p$, then by (12) we have

$$K_{\omega_k}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_k = F(z)K_{\omega_k}^{\mathfrak{E}}(z)F^*(\omega_k)u_k + E_-(z)K_{\omega_k}^{\mathfrak{F}}(z)E_-^*(\omega_k)u_k.$$

Setting $\xi_k = F^*(\omega_k)u_k$ and $\eta_k = E_-^*(\omega_k)u_k$ we get

$$\sum_{k=1}^n K_{\omega_k}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_k = F(z)\left(\sum_{k=1}^n K_{\omega_k}^{\mathfrak{E}}(z)\xi_k\right) + E_-(z)\left(\sum_{k=1}^n K_{\omega_k}^{\mathfrak{F}}(z)\eta_k\right).$$

On the other hand, for any $\omega \in \mathbb{C}$ and $u \in \mathbb{C}^p$, by Eq. (4) we have

$$\begin{aligned} \|K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}u\|^2_{\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})} &= u^* K_\omega^{\mathfrak{F}\diamond\mathfrak{E}}(\omega)u \\ &= u^* F(\omega)K_\omega^{\mathfrak{E}}(\omega)F^*(\omega)u \ + \ u^* E_-(\omega)K_\omega^{\mathfrak{F}}(\omega)E_-^*(\omega)u \\ &= \|K_\omega^{\mathfrak{E}} F^*(\omega)u\|^2_{\mathcal{B}(\mathfrak{E})} \ + \ \|K_\omega^{\mathfrak{F}} E_-^*(\omega)u\|^2_{\mathcal{B}(\mathfrak{F})}. \end{aligned}$$

Let $P_{\mathfrak{E}}$ be the orthogonal projection of $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ onto the image of $\mathcal{I}$, and $P_{\mathfrak{F}}$ be the orthogonal projection of $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ onto the image of $\mathcal{J}$. We have

$$P_{\mathfrak{E}}(h) = Ff_1 \quad \text{and} \quad P_{\mathfrak{F}}(h) = E_- f_2,$$

for some $f_1 \in \mathcal{B}(\mathfrak{E})$ and $f_2 \in \mathcal{B}(\mathfrak{F})$. The next Theorem shows that the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ admits an orthogonal direct sum decomposition using the spaces $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$. For this, we define

$$F\mathcal{B}(\mathfrak{E}) = \{Ff : f \in \mathcal{B}(\mathfrak{E})\}$$
$$E_-\mathcal{B}(\mathfrak{F}) = \{E_- f : f \in \mathcal{B}(\mathfrak{F})\}$$

Theorem 6 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). Then*

$$\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E}) = F\mathcal{B}(\mathfrak{E}) \oplus E_-\mathcal{B}(\mathfrak{F})$$

i.e., for any $h \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$, there exist a unique $f_1 \in \mathcal{B}(\mathfrak{E})$ and $f_2 \in \mathcal{B}(\mathfrak{F})$ such that $h = Ff_1 + E_- f_2$, and

$$\|h\|^2_{\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})} = \|f_1\|^2_{\mathcal{B}(\mathfrak{E})} + \|f_2\|^2_{\mathcal{B}(\mathfrak{F})}.$$

Proof It is easily checked that $K^{(1)}_\omega(z) := F(z)K^{\mathfrak{E}}_\omega(z)F^*(\omega)$ is a reproducing kernel with corresponding RKHS $\mathcal{B}_1 = F\mathcal{B}(\mathfrak{E})$, and $K^{(2)}_\omega(z) := E_-(z)K^{\mathfrak{F}}_\omega(z)E^*_-(\omega)$ is a reproducing kernel with corresponding RKHS $\mathcal{B}_2 = E_-\mathcal{B}(\mathfrak{F})$. Furthermore, Theorem 5 implies that $\mathcal{B}_1 \cap \mathcal{B}_2 = \{0\}$.

Since $K^{(1)}_\omega(z) + K^{(2)}_\omega(z)$ is a RK, and $K^{\mathfrak{F}\diamond\mathfrak{E}}_\omega(z) = K^{(1)}_\omega(z) + K^{(2)}_\omega(z)$, this implies that

$$\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E}) = \mathcal{B}_1 \oplus \mathcal{B}_2 = F\mathcal{B}(\mathfrak{E}) \oplus E_-\mathcal{B}(\mathfrak{F}).$$

It follows that the orthogonal complement of $\mathcal{B}_1$ in $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ is the space $\mathcal{B}_2$. The claim now follows from orthogonality and the isometry properties of $\mathcal{I}$ and $\mathcal{J}$. □

Theorem 7 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). If $\{\omega_n\} \subset \mathbb{C}$ and $\{u_n\} \subset \mathbb{C}^p$ are such that $\left\{\frac{K^{\mathfrak{F}\diamond\mathfrak{E}}_{\omega_n}(.)u_n}{\sqrt{u_n^* K^{\mathfrak{F}\diamond\mathfrak{E}}_{\omega_n}(.)u_n}}\right\}$ is a complete orthonormal set for $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ then*

1. $\left\{\frac{K^{\mathfrak{E}}_{\omega_n}(.)F^*(\omega_n)u_n}{\sqrt{u_n^* K^{\mathfrak{F}\diamond\mathfrak{E}}_{\omega_n}(.)u_n}}\right\}$ *is a Parseval frame for $\mathcal{B}(\mathfrak{E})$, and for every $f \in \mathcal{B}(\mathfrak{E})$*

$$f(z) = \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K^{\mathfrak{E}}_{\omega_n}(z)F^*(\omega_n)u_n}{u_n^* K^{\mathfrak{F}\diamond\mathfrak{E}}_{\omega_n}(\omega_n)u_n}. \tag{15}$$

2. $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}}(.)E_-^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\}$ *is a Parseval frame for* $\mathcal{B}(\mathfrak{F})$*, and for every* $g \in \mathcal{B}(\mathfrak{F})$

$$g(z) = \sum_n u_n^* E_-(\omega_n) g(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}}(z)E_-^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}. \tag{16}$$

Proof By Eq. (14) we have

$$\frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} = \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} + \frac{\mathcal{J}(K_{\omega_n}^{\mathfrak{F}}(.)E_-^*(\omega_n)u_n)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}},$$

hence,

$$P_{\mathfrak{E}}\left(\frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}}\right) = \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}}.$$

Since $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\}$ is an orthonormal set for $\mathcal{B}(\mathfrak{F}\diamond\mathfrak{E})$ and $\mathcal{I}$ is an isometric from $\mathcal{B}(\mathfrak{E})$ onto $\mathcal{I}(\mathcal{B}(\mathfrak{E}))$ then

$$\frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n\right)}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \tag{17}$$

is a Parseval frame for $\mathcal{I}(\mathcal{B}(\mathfrak{E}))$. Applying $\mathcal{I}^*$ to (17) we obtain the first claim. Consequently, given any $f \in \mathcal{B}(\mathfrak{E})$ we have

$$\begin{aligned} f(z) &= \sum_n \left\langle f, \frac{K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \right\rangle_{\mathcal{B}(\mathfrak{E})} \frac{K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \\ &= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}. \end{aligned}$$

Using an analogous argument we obtain the second claim. □

Now we show that the Parseval frames for $\mathcal{B}(\mathfrak{E})$ and $\mathcal{B}(\mathfrak{F})$ given in Theorem 7 are orthogonal.

Theorem 8 *Assume the hypothesis of Theorem 7, then*

1. *For every* $f \in \mathcal{B}(\mathfrak{E})$,

$$\sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}}(.)E_-^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n} = 0. \tag{18}$$

2. *For every* $g \in \mathcal{B}(\mathfrak{F})$,

$$\sum_n u_n^* E_-^*(\omega_n) g(\omega_n) \frac{K_{\omega_n}^{\mathfrak{E}}(.)F^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n} = 0. \tag{19}$$

Proof Let $f \in \mathcal{B}(\mathfrak{E})$. Since $Ff \in \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ and $\left\{ \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}} \right\}$ is a complete orthonormal set for $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ then

$$\begin{aligned}
\mathcal{I}(f)(z) &= F(z)f(z) \\
&= \sum_n \langle Ff, \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(.)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \rangle \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_n}{\sqrt{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(z)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{F(z)K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n + E_-(z)K_{\omega_n}^{\mathfrak{F}}(z)E_-^*(\omega_n)u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n} \\
&= \sum_n u_n^* F(\omega_n) f(\omega_n) \frac{\mathcal{I}\left(K_{\omega_n}^{\mathfrak{E}}(z)F^*(\omega_n)u_n\right) + \mathcal{J}\left(K_{\omega_n}^{\mathfrak{F}}(z)E_-^*(\omega_n)u_n\right)}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n)u_n}
\end{aligned}$$

Applying $\mathcal{J}^*$ to the last line above, and using the fact that $\mathcal{J}^*(Ff) = 0$ we obtain Eq. (18). Similar argument applying $\mathcal{I}^*$ to E_-g yields Eq. (19). □

4 Sampling in the Space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$

The next theorem shows that if a de Branges matrix $\mathfrak{G} = [G_-(z) \quad G_+(z)]$ can be factored as

$$G_-(z) = F^{\#}(z)E_-(z), \quad \text{and} \quad G_+(z) = F(z)E_+(z),$$

with $F(z) \in \mathbb{N}_{inv}(\mathbb{C}^{p\times p})$ and Eq. (11) holds, then the space $\mathcal{B}(\mathfrak{G})$ will have the Kramer sampling property whenever the de Branges space $\mathcal{B}(\mathfrak{E})$ satisfies the conditions of Theorem 3. The sampling problem can be considered dual to the interpolation problem [22]; results concerning interpolation in vector valued reproducing kernel spaces can be found in [9].

Theorem 9 *Let $\mathfrak{F}$ and $\mathfrak{E}$ be two de Branges matrices that satisfy Definition 2 and Eq. (11). Suppose further that* $\det E_+(\cdot)$ *is nonvanishing in $\overline{\mathbb{C}^+}$ and* $\det E_-(\cdot)$ *is nonvanishing in $\overline{\mathbb{C}^-}$. If $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$ for some point $\alpha \in \mathbb{C}$, then the space $\mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ will have the Kramer sampling property.*

Proof Using Theorem 3 it is enough to show that $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \mathbb{C}$, and $K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha) \succ 0$ for the given $\alpha \in \mathbb{C}$.

First, let $\omega \in \overline{\mathbb{C}^+}$ then $F(\omega)E_+(\omega)$ is invertible because $\det E_+(\omega) \neq 0$ by the hypothesis. Hence $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \overline{\mathbb{C}^+}$ by Lemma 1. Similarly, $F(\omega)E_-(\omega)$ is invertible because $\det E_-(\omega) \neq 0$ by the hypothesis, hence $R_\omega \mathcal{B}_\omega(\mathfrak{F} \diamond \mathfrak{E}) \subseteq \mathcal{B}(\mathfrak{F} \diamond \mathfrak{E})$ for every point $\omega \in \overline{\mathbb{C}^-}$.

Let $\alpha \in \mathbb{C}$ be such that $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$. Then $u^* K_\alpha^{\mathfrak{E}}(\alpha)u > 0$ for every nonzero vector $u \in \mathbb{C}^p$. Hence, by Eq. (12) and using the fact that $F^*(\alpha)u \in \mathbb{C}^p$, $E_-^*(\alpha)u \in \mathbb{C}^p$, $K_\alpha^{\mathfrak{E}}(\alpha) \succ 0$, and $K_\alpha^{\mathfrak{F}}(\alpha) \succeq 0$, by (5) we get

$$u^* K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha)u = u^* F(\alpha) K_\alpha^{\mathfrak{E}}(\alpha) F^*(\alpha)u + u^* E_-(\alpha) K_\alpha^{\mathfrak{F}}(\alpha) E_-^*(\alpha)u > 0$$

i.e., $K_\alpha^{\mathfrak{F}\diamond\mathfrak{E}}(\alpha) \succ 0$ for the given $\alpha \in \mathbb{C}$. This completes the proof of the theorem. □

Example 5 Consider the de Branges space $\mathcal{B}(\mathfrak{G})$ with

$$\mathfrak{G} = [G_-(z) \quad G_+(z)]$$

and

$$G_-(z) = F^\#(z)E_-(z), \quad G_+(z) = F(z)E_+(z)$$

where $F(z)$ and $E_\pm(z)$ as in Example 4. Then it is evident that the space $\mathcal{B}(\mathfrak{G})$ have the Kramer sampling property by Theorem 9.

4.1 Multiplexing the Sampled Vector Valued Functions

Multiplexing refers to the transmission of several signals simultaneously over a single communications channel. Generically, multiplexing occurs when two (or more) signals x and y are encoded into X and Y in such a way that x and y can each be recovered from $X + Y$. The signals we consider here are elements of a de Branges space and the encoding involves the sampling of the signal. Specifically, if

$f \in \mathcal{B}(\mathfrak{E})$ and $g \in \mathcal{B}(\mathfrak{F})$, we encode both f and g into the *multiplexed samples*:

$$\{u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\}_n \tag{20}$$

which are transmitted in some fashion. The goal then is to recover f and g from these mixed samples.

Corollary 1 *Assume the hypotheses of Theorem 7, $f \in \mathcal{B}(\mathfrak{E})$ and $g \in \mathcal{B}(\mathfrak{F})$. Given the samples $\{f(\omega_n)\}$ and $\{g(\omega_n)\}$, f and g can be reconstructed from the multiplexed samples in (20) as follows:*

$$f(z) = \sum_n \left(u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\right) \frac{K_{\omega_n}^{\mathfrak{E}}(z) F^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n} \tag{21}$$

$$g(z) = \sum_n \left(u_n^* F(\omega_n) f(\omega_n) + u_n^* E_-^*(\omega_n) g(\omega_n)\right) \frac{K_{\omega_n}^{\mathfrak{F}}(z) E_-^*(\omega_n) u_n}{u_n^* K_{\omega_n}^{\mathfrak{F}\diamond\mathfrak{E}}(\omega_n) u_n}. \tag{22}$$

Proof Equations (21) and (22) follow immediately from Eqs. (15), (16), (18), and (19). □

Conflict of Interest

The authors declare that they have no conflict of interest.

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Differential Subordinations in Harmonic Mappings

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Abstract Differential subordination of analytic functions proved to be useful in many applications. The book of Miller and Mocanu ((2000) Differential Subordinations. In Theory and applications, monographs and textbooks in pure and applied mathematics, vol 225. Marcel Dekker Inc, New York) sums up most of the advancement in the field and the references to the date of its publication. The theory of harmonic mappings can benefit from this theory. We attempt to discuss some aspects of this generalization.

Keywords Harmonic mappings · Differential inequalities · Differential subordination

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1 Introduction

Differential subordination is a global name to differential inequalities and differential subordination of analytic functions in the complex plain. They were first used in the studies of geometric properties of analytic functions as early as the nineteen thirties by Goluzin, Gennadii M. and Robertson, Malcolm S. Their interest grew due to their application to numerous fields including differential equations, partial differential equations, harmonic functions, integral operators, Banach spaces and functions of several complex variables. We shall extend the technique to harmonic mappings in the plane.

Let $w(z)$ be an analytic function in the unit disk $U = \{|z| < 1\}$. Then the tangent vector of $w(re^{it})$; $0 \leq t \leq 2\pi$ at $\zeta = re^{it_0} \in U$ is given by

$$\frac{dw(re^{it})}{dt}|_{t=t_0} = ire^{it_0}w'(re^{it_0}) = i\zeta w'(\zeta)$$

and the direction of the outside normal is

$$-i\frac{dw(re^{it})}{dt}|_{t=t_0} = \zeta w'(\zeta).$$

The earliest problem in differential subordination was introduced by Miller [2]

Problem Let D be a domain in $\mathbb{C}$ and $w(z)$ an analytic function in U. For which continuous functions $h(u, v)$

$$h(w(z), zw'(z)) \in D;\ z \in U \ \Rightarrow\ w(z) \in D;\ z \in U.$$

In this paper we shall concentrate on the case where D is the unit disk U. A typical example is the following

Theorem A (Miller [2, Ex.1], Miller and Mocanu [3]) *Let $w(z)$ be analytic in the unit disk U, and satisfies*

$$|w(z) + zw'(z)| < 1;\ z \in U. \textit{ Then } |w(z)| < 1;\ z \in U. \tag{1}$$

A geometric proof of this result is due.

Proof For each $0 < r < 1$ consider the curve $w(re^{it})$ at the point $\xi = re^{it_0}$ where $|w(\xi)| = w_0 = \max_{|z|=r} |w(z)|$. The circle $|w| = |w_0|$ is tangential to $|w(re^{it})|$ at w_0, and in particular, the exterior normal to $w(re^{it})$ at the point ξ, $\xi w'(\xi)$, is in the direction of $w(\xi)$. Then

$$|w(\xi) + \xi w'(\xi)| = |w(\xi)| + |\xi w'(\xi)| < 1 \tag{2}$$

and thus $|w(\xi)| < 1$. We conclude that $\max_{|z|=r} |w(z)| < 1$ for all $0 < r < 1$ and the result follows. ∎

Definition 1 Let $w = f(z)$ be defined on U. A point denoted ξ in U will always represent the preimage of

$$w_0 = \max_{|z|=r} |f(z)|,$$

for some $0 < r \leq 1$. It is right to call it a Clunie-Jack point.

Theorem 1 *Let* $w(z)$ *be analytic in the unit disk* U, *and satisfies*

$$|w(z) + zw'(z)| < 2;\ z \in U. \textit{ Then } |w(z)| < 1;\ z \in U.$$

Proof Indeed, by Clunie-Jack Lemma ([1] Lemma 1, [4] Lemma 2.2a) we have $|\xi w'(\xi)| \geq 1$. We follow the proof of Theorem A and note that

$$|w(\xi)| = |w(\xi) + \xi w'(\xi)| - |\xi w'(\xi)| < 2 - 1 = 1. \blacksquare \tag{3}$$

In the seminal paper of Miller [*ibid.*], the author investigates the problem above where D is the unit disk. His results rely on two points, namely

(I) A Maximum Principle of the involved function, $w(z)$, and
(II) A lower bound of $|\zeta w'(\zeta)|$ in U (Clunie-Jack Lemma).

In order to generalize the above result for harmonic mappings we need to find the appropriate tools and their estimates.

2 Harmonic Mappings

Let f be a complex harmonic mapping defined in the unit disk U. Then $f(z) = u(z) + iv(z)$ where u, v are real harmonic functions. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)};\ h, g \text{ analytic in } U.$$

This representation is unique if we assume that $g(0) = 0$. The *second dilatation* of $f(z)$ is

$$a(z) = \frac{g'(z)}{h'(z)}$$

which is meromorphic in U. $f(z)$ is sense-preserving if, and only if, $|a(z)| < 1;\ z \in U$, in which case $a(z)$ admit removable singularities and is analytic in U. The mapping $f(z)$ satisfies the Maximum Principle.

For $z = re^{it} \in U$ we set

$$\mathfrak{D} f(z) \equiv -i \frac{\partial f(re^{it})}{\partial t} = -i(h'(re^{it})re^{it}i + \overline{g'(re^{it})re^{it}i}) = zh'(z) - \overline{zg'(z)},$$

which coincide with the normal direction of $F_1(\theta) = f(re^{i\theta})$ at $\theta = t$, and

$$\mathcal{D} f(z) \equiv r \frac{\partial f(re^{it})}{\partial r} = r(h'(re^{it})e^{it} + \overline{g'(re^{it})e^{it}}) = zh'(z) + \overline{zg'(z)},$$

which coincide with the radial direction of $F_2(\rho) = f(\rho^{it})$ at $\rho = r$. For a sense-preserving harmonic mapping f we have

$$\frac{\mathfrak{D} f(z)}{\mathcal{D} f(z)} = \frac{zh'(z) - \overline{zg'(z)}}{zh'(z) + \overline{zg'(z)}} = \frac{1 - \frac{\overline{zg'(z)}}{zh'(z)}}{1 + \frac{\overline{zg'(z)}}{zh'(z)}}$$

and since $\left|\frac{\overline{zg'(z)}}{zh'(z)}\right| = |a(z)| < 1$, therefore

$$\Re\left\{\frac{\mathfrak{D} f(z)}{\mathcal{D} f(z)}\right\} > 0. \tag{4}$$

We shall follow Lemma A in Miller and Mocanu [3] to get an appropriate Clunie-Jack Lemma for Harmonic functions.

Lemma 1 *Let $f(z)$ be a sense-preserving harmonic mapping defined on U of the form $f(z) = h(z) + \overline{g(z)}$; $f(0) = 0$. For fixed r; $0 < r < 1$, let $\xi = re^{it_0}$ satisfy*

$$|f(\xi)| = \max_{|z| \le r} |f(z)|$$

then

$$\frac{\mathfrak{D} f(\xi)}{f(\xi)} = m > 0; \tag{5}$$

and

$$\left|\frac{\mathcal{D} f(\xi)}{f(\xi)}\right| \ge \frac{2}{\pi} \equiv M.$$

Proof For $z = re^{it} \in U$, set $f(z) = R(r,t)e^{i\Phi(r,t)}$. Then

$$\frac{\mathfrak{D} f(z)}{f(z)} = \Phi_t - i\frac{R_t}{R},$$

and since $R(r,t); 0 \le t \le 2\pi$ attains its maximal value at $\xi = re^{it_0}$, we have $R_t(r, t_0) = 0$ and we conclude that

$$\frac{\mathfrak{D} f(\xi)}{f(\xi)} = \Phi_t(r, t_0) = m > 0.$$

Indeed, as explained in the introduction of this section, $\mathfrak{D} f(\xi)$ is in the direction of $f(\xi)$ so that $m \ge 0$. Furthermore, $\mathfrak{D} f(\xi) = zh' - \overline{zg'} \ne 0$ since $J_f = |h'|^2 - |g'|^2 > 0$ in U. Furthermore, from (4) and (5) we conclude that

$$\Re\left\{\frac{\mathcal{D} f(\xi)}{f(\xi)}\right\} > 0. \tag{6}$$

Let $\tilde{f}(z) = \frac{f(\xi z)}{f(\xi)}$ which is harmonic in $\overline{U}$ and satisfies $\tilde{f}(0) = 0$ and for $z \in \partial U$

$$|\tilde{f}(z)| \le \frac{\max_{0 \le t \le 2\pi} |f(e^{it}\xi)|}{|f(\xi)|} \le 1.$$

The Schwarz Lemma for harmonic mappings is $|\tilde{f}(z)| \le \frac{4}{\pi} \arctan |z|$ and in particular we have

$$\lim_{r \uparrow 1} \frac{1 - |\tilde{f}(r)|}{1 - r} \ge \frac{d}{dr}\left(\frac{4}{\pi} \arctan(r)\right)\bigg|_{r=1} = \frac{2}{\pi} = M.$$

Then

$$\frac{\mathcal{D} f(\xi)}{f(\xi)} = \frac{d}{dr}\left(\frac{f(r\xi)}{f(\xi)}\right)\bigg|_{r=1} = \lim_{r \uparrow 1} \frac{f(\xi) - f(r\xi)}{(1-r) f(\xi)}$$
$$= \lim_{r \uparrow 1}\left[1 - \frac{f(r\xi)}{f(\xi)}\right]\frac{1}{1-r}$$

and

$$\left|\frac{\mathcal{D} f(\xi)}{f(\xi)}\right| \ge \lim_{r \uparrow 1} \frac{1 - |\tilde{f}(r)|}{1 - r}$$
$$\ge \frac{2}{\pi} = M. \blacksquare$$

Two generalizations of Theorem A for sense-preserving harmonic mappings are

Corollary 1 *Let $f(z)$ be a sense-preserving harmonic mapping defined on U and satisfies*

$$|f(z) + \mathfrak{D}f(z)| < 1;\ z \in U. \text{ Then } |f(z)| < 1;\ z \in U.$$

Proof Assume to the contrary that the set $S = \{f^{-1}(e^{it}), 0 < t \le 2\pi\}$ is nonempty. Let $r = \min_{z\in S}|z|$. Choose a point $\xi = re^{it_0}$ on $U_r = \{|z| \le r\}$. Then ξ admits the same properties of its alike in Theorem 1. Using (5) we have

$$|f(\xi) + \mathfrak{D}f(\xi)| = |f(\xi) + mf(\xi)| = (1+m)|f(\xi)| = 1 + m > 1$$

a contradiction. ■

Corollary 2 *Let $f(z)$ be a sense-preserving harmonic mapping defined on U and satisfies*

$$|f(z) + \mathcal{D}f(z)| < 1;\ z \in U. \text{ Then } |f(z)| < 1;\ z \in U.$$

Proof We follow the same proof as in Corollary 1. From (6) $\Re\left\{\frac{\mathcal{D}f(\xi)}{f(\xi)}\right\} > 0$. Then

$$|f(\xi) + \mathcal{D}f(\xi)| = \left|1 + \frac{\mathcal{D}f(\xi)}{f(\xi)}\right||f(\xi)| \ge \left(1 + \Re\left\{\frac{\mathcal{D}f(\xi)}{f(\xi)}\right\}\right)|f(\xi)| > 1.$$

a contradiction. ■

In [2, Theorem 1], Miller proves the following subordination principle

Theorem B *Let $h(r, s)$ be a continuous complex function in a domain $D \subset \mathbb{C}^2$ satisfying the following conditions:*

(I) $(0, 0) \in D$ *and* $h(0, 0) < 1$,
(II) $h(e^{it}, ke^{it}) \ge 1$ *when* $(e^{it}, ke^{it}) \in D$ *and* $k \ge 1$ *and* $t \in \mathbb{R}$.

If $w(z)$; $w(0) = 0$, is an analytic function in U and for $z \in U$

(a) $(w(z), zw'(z)) \in D$, *and*
(b) $|h(w(z), zw'(z))| < 1$,

then $|w(z)| < 1;\quad z \in U$.

A generalization of Miller's [2] Theorem 1 is

Theorem 2 *Let $h(r, s)$ be a continuous complex function in a domain $D \subset \mathbb{C}^2$ satisfying the following conditions:*

(I) $(0, 0) \in D$ *and* $h(0, 0) < M = \frac{2}{\pi}$, *and*
(II) $h(\frac{e^{is}}{M}, ke^{it}) \ge 1$ *for all* $t, s \in \mathbb{R}$, $|t - s| \le \frac{\pi}{2}$ *and* $k \ge 1$.

If $f(z);\ \ f(0)=0$, *is a sense-preserving harmonic mapping in* U *and for* $z \in U$

(a) $(f(z), \mathcal{D}f(z)) \in D$, *and*
(b) $|h(f(z), \mathcal{D}f(z))| < 1$,

then $|f(z)| < \frac{1}{M} = \frac{\pi}{2}$; $z \in U$.

Proof Suppose that $\xi = re^{it_0}$ is a point in U such that

$$\max_{|z|\le r} |f(z)| = \max_{|z|=r} |f(z)| = |f(\xi)| = \frac{1}{M} = \frac{\pi}{2}.$$

By Lemma 1 we have $\left|\frac{\mathcal{D}f(\xi)}{f(\xi)}\right| \ge M$ so that $|\mathcal{D}f(\xi)| \ge 1$ or

$$\mathcal{D}f(\xi) = ke^{it};\ \ k \ge 1,\ \ t \in \mathbb{R}.$$

Set $f(\xi) = \frac{e^{is}}{M},\ \ s \in \mathbb{R}$, then by (5) $|t-s| \le \frac{\pi}{2}$. By (b)

$$|h(f(\xi), \mathcal{D}f(\xi))| = |h(\frac{e^{is}}{M}, ke^{it})| < 1,$$

which contradicts (II). Therefore $|f(z)| < \frac{1}{M} = \frac{\pi}{2};\ \ z \in U$. ■

The first theorem by Miller and Mocanu [3] of second order differential subordination of positive real analytic functions is one of many. It turns out that it can be used to get results on second order differential subordination of bounded analytic and harmonic mappings, in the case of linear operator. To simplify the formulas, we shall consider a first order version for harmonic mappings.

Theorem C (Miller and Mocanu [3], Special Case) *Let* $\psi(r, s)$ *be a continuous complex function in a domain* $D \subset \mathbb{C}^2;\ \ r = r_1 + ir_2;\ \ s = s_1 + is_2$, *satisfying the following conditions:*

(I) $(1, 0) \in D$ *and* $\Re\{\psi(1, 0)\} = 1$, *and*
(II) for $(ir_2, s_1) \in D$, *if* $s_1 \le -\frac{1+r_2^2}{2}$ *then* $\Re\{\psi(ir_2, s_1)\} \le 0$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ *be analytic in* U *and if*

$$\Re\{\psi\big(p(z), zp'(z)\big)\} > 0;\ \ z \in U \quad then \quad \Re\{p(z)\} > 0.$$

The next theorem is an improvement of Theorem 2 when ψ is a linear operator.

Theorem 3 *Let* $f(z);\ \ f(0) = 0$, *be a sense-preserving harmonic mapping in* U. *We further assume that*

$$\psi(q(z), \mathcal{D}q(z)) = \alpha q(z) + \beta \mathcal{D}q(z);\ \ \alpha, \beta \in \mathbb{C}$$

and Let $\psi(r, s)$ *be as in Theorem C (In this case it reduces to* $\left|\frac{\Im\{\alpha\}}{\Re\{\beta\}}\right| < 1$*). Then*

$$\textit{if } |\psi(f(z), \mathcal{D}f(z))| < 1;\ z \in U \quad \textit{then} \quad |f(z)| < 1;\ z \in U.$$

For the proof of Theorem 3 we start by observing

Proposition 1 *Let a complex function* $f(z)$ *be defined on some domain* $\Omega \subset \mathbb{C}$*. Then* $|f(z)| < 1$ *in* Ω *if, and only if, for all* $t \in \mathbb{R}$ *we have*

$$\Re\{e^{it} f(z)\} > -1;\ z \in \Omega.$$

Definition 2 Let $f(z) = h + \overline{g}$ be harmonic in U. The shear of $f(z)$ in the direction of the imaginary axis is defined by $[f]_I = h + g$.

In particular $\Re\{f\} = \Re\{h + \overline{g}\} = \Re\{h + g\} = \Re\{[f]_I\}$, therefore

$$|f| < 1 \Rightarrow -1 < \Re\{f\} = \Re\{[f]_I\} < 1.$$

Proposition 2 *Let* f *and* $[f]_I$ *be as in Definition 2. Then* $|f| < 1$ *if, and only if, for all* $t \in \mathbb{R}$

$$-1 < \Re\left\{\left[e^{it} f\right]_I\right\} = \Re\{e^{it} f\} = \Re\{e^{it} h + e^{-it} g\} < 1.$$

Both propositions are trivial and we omit their proofs.

Proof of Theorem 3: Let $f(z) = h(z) + \overline{g(z)}$ where $f(0) = 0$. If $|\psi(f(z), \mathcal{D}f(z))| < 1$, then

$$|e^{it} \psi(f(z), \mathcal{D}f(z))| < 1;\ t \in \mathbb{R}.$$

In particular

$$\Re\{e^{it} \psi(f(z), \mathcal{D}f(z))\} + 1 > 0;\ t \in \mathbb{R}.$$

By assumption we have

$$\Re\{\psi(e^{it} f(z), \mathcal{D}(e^{it} f(z))\} + 1 > 0,$$

and if we set $F(z) = e^{it} f(z) + 1$, then

$$\Re\{\psi(F(z), \mathcal{D}F(z))\} > 0.$$

Using the shear it can be written in the form

$$\Re\{\psi([F(z)]_I, [\mathcal{D}F(z)]_I)\} > 0$$

and since $\psi(r,s)$ satisfy the assumptions in Theorem C and, $[F(z)]_I$ and $[\mathcal{D}F(z)]_I$ are normalized analytic functions as in Theorem C as well, we conclude that for real t

$$\Re\{[F(z)]_I\} = \Re\{[e^{it} f(z) + 1]_I\} > 0 \Leftrightarrow \Re\{e^{it} f(z) + 1\} > 0$$

and therefore, $\Re\{e^{it} f(z)\} > -1;\ \ t \in \mathbb{R}$, which by Proposition 1 implies $|f(z)| < 1$. ■

Remark 1 In Theorem 3

(I) $\mathcal{D}f$ can be replaced by $\mathfrak{D}f$.
(II) α and β could be any two continuous complex functions.

As an example for (II), consider

Theorem D ([4] Theorem 2.4f, p. 39) *Let* $F(z);\ \ F(0) = 1$ *be analytic in* U *and* $Q(z);\ Q(0) = 1$ *be analytic in* U *with* $\Re\{Q\} > 0$. *If in* U

$$\Re\{F(z) + Q(z) \cdot zF'(z)\} > 0,$$

then $\Re\{F(z)\} > 0$.

The same proof of Theorem 3 implies

Theorem 4 *Let* $f(z);\ \ f(0) = 0$, *be a sense-preserving harmonic mapping in* U *and* $Q(z);\ Q(0) = 1$ *be analytic in* U *with* $\Re\{Q\} > 0$. *If in* U

$$|f(z) + Q(z) \cdot \mathcal{D}f(z)| < 1 \quad then \quad |f(z)| < 1.$$

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The Segal–Bargmann Transform in Clifford Analysis

Swanhild Bernstein and Sandra Schufmann

Abstract The Segal–Bargmann transform plays an essential role in signal processing, quantum physics, infinite-dimensional analysis, function theory and further topics. The connection to signal processing is the short-time Fourier transform, which can be used to describe the Segal–Bargmann transform. The classical Segal–Bargmann transform $\mathcal{B}$ maps a square integrable function to a holomorphic function square-integrable with respect to a Gaussian identity. In signal processing terms, a signal from the position space $L_2(\mathbb{R}^m, \mathbb{R})$ is mapped to the phase space of wave functions, or Fock space, $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$. We extend the classical Segal–Bargmann transform to a space of Clifford algebra-valued functions. We show how the Segal–Bargmann transform is related to the short-time Fourier transform and use this connection to demonstrate that $\mathcal{B}$ is unitary up to a constant and maps Sommen's orthonormal Clifford Hermite functions $\left\{\phi_{l,k,j}\right\}$ to an orthonormal basis of the Segal–Bargmann module $\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$. We also lay out that the Segal–Bargmann transform can be expanded to a convergent series with a dictionary of $\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$. In other words, we analyse the signal f in one basis and reconstruct it in a basis of the Segal–Bargmann module.

Keywords Clifford analysis · Bargmann transform · Fock Space · Short-time Fourier transform · Orthonormal basis · Hermite polynomials

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1 Introduction

Due to the importance of the Segal–Bargmann transform, there are various generalizations into quaternion and Clifford analysis. In particular, the Bargmann-Segal transformation has been studied in the theory of slice monogenic functions [2, 11, 12, 22]. Our interest doesn't lie in these theories. We are interested in the importance of the Segal–Bargmann transform in its connection to the windowed Fourier transform and time-frequency analysis.

Time-frequency analysis is an important method in signal processing, because it allows to analyse a given signal simultaneously in the time and frequency domains. A well-known tool is the short-time Fourier transform. Another closely related tool is the Segal–Bargmann transform, which is our main focus in this paper.

The classical Segal–Bargmann transform maps a square integrable function to a holomorphic function square-integrable with respect to a Gaussian identity. In signal processing terms, a signal from the position space $L_2(\mathbb{R}^m, \mathbb{R})$ is mapped to the phase space of wave functions $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$. In the early 1960s, V. Bargmann and I. Segal independently investigated this space [4, 25]. While Bargmann developed a theory about the space and the corresponding transform in the finite-dimensional case, Segal focused primarily on the infinite-dimensional version of the now-called Segal–Bargmann space(s) [20].

The space $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$ has a wide number of applications such as in infinite-dimensional analysis and stochastic distribution theory. As early as 1932, V. Fock introduced a more general, infinite-dimensional version of this space as a quantum states space for an unknown number of particles [16], which is now called *Fock* space. In quantum mechanics, the reproducing kernels of the Fock spaces are the so-called coherent states. Segal and Bargmann showed that an infinite union of the spaces $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$, $m \in \mathbb{N}$, is isomorphic to a certain case of the Fock space, which is why the Segal–Bargmann spaces are sometimes also called Segal–Bargmann-Fock space(s) or only Fock space. For this work, we will stick to the notion of Segal–Bargmann space.

In signal and image processing not only scalar-valued but also quaternion- and Clifford-valued signals are of interest. A monogenic signal [5, 6, 14], for example, consists of a scalar-valued signal and vector components, which are the Riesz transformations of the scalar-valued signal. Other applications deal with colour images of which the colours are separated and considered as components of a Clifford-valued signal, see for example [10, 13, 21, 27].

The main purpose of this paper is to investigate the Segal–Bargmann transform $\mathcal{B}$ of Clifford algebra-valued functions, which has also been the focus of D. Peña Peña, I. Sabadini and F. Sommen [23]. We will define and examine the Segal–Bargmann module $\mathcal{F}^2(\mathbb{C}^m, \mathcal{Cl}_m^{\mathbb{C}})$, a higher-dimensional analogue of the classical Segal–Bargmann space.

It is known that there is a close relationship between the Gabor transform (short-time Fourier transform with a Gaussian window) and the Segal–Bargmann transform. Recently, this connection has been used to filter a signal embedded in white noise [1, 15]. Therefore, we investigate the mapping properties of the Segal–Bargmann transform in the context of Clifford estimators.

We prove that $\mathcal{B}$ is a unitary operator up to a scaling constant, and that it maps an orthonormal basis of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ to an orthonormal basis of the Segal–Bargmann module $\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$. For that, we will use Sommen's Clifford-Hermite functions $\{\phi_{l,k,j}\}$ as an L^2 basis.

We also lay out that the Segal–Bargmann transform can be expanded to a series $\big(\mathcal{B}f\big)(z) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))} \Psi_{l,k,j}(\underline{z})\langle\phi_{l,k,j}, f\rangle$ with a dictionary $\{\Psi_{l,k,j}\}$ of the Segal–Bargmann module and that this series converges absolutely locally uniformly.

The paper is organised as follows. In Sects. 2.1 and 2.2 we give an overview of basic Clifford analysis and of Hilbert Clifford-modules, which replace Hilbert spaces in our context. Section 2.3 deals with a certain class of Clifford-valued functions, the inner spherical monogenics, which are central to the construction of a basis for the function spaces that we deal with. In Sect. 2.4, we present the short-time Fourier transform as in important tool for our work.

After we have established these preliminary notes, we introduce Sommen's generalized Clifford Hermite polynomials and their relevant properties in Sect. 3. In Sect. 4, we formally introduce the Segal–Bargmann transform and the Segal–Bargmann space of the classical, non-Clifford case, before we establish its analogue, the Segal–Bargmann module, in Sect. 5 and show some important properties of the Segal–Bargmann transform of Clifford algebra-valued functions. We conclude our paper with Sect. 6 by constructing a dictionary $\{\Psi_{l,k,j}\}$ for the Segal–Bargmann transform and proving the convergence of the series representation $\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))} \Psi_{l,k,j}(\underline{z})\langle\phi_{l,k,j}, f\rangle$.

2 Preliminaries

2.1 Clifford Algebras

While real Clifford algebras have gained much interest in mathematical research since W. Clifford wrote about them in 1878, cf. [19], complex Clifford algebras are a fairly recent topic of interest. In our work, we deal with both cases. We take notations and properties mainly from [7], in which the real version is displayed, and adopt them to fit the complex case. For that, we work close to J. Ryan's *Complexified clifford analysis* [24], in which a detailed extension of real to complex Clifford algebras is developed.

We will write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Let $n \in \mathbb{N}_0$ and $\mathcal{C}\ell_n^{\mathbb{R}}$ denote the real Clifford algebra over $\mathbb{R}^n$ and $\mathcal{C}\ell_n^{\mathbb{C}}$ the complex Clifford algebra over $\mathbb{C}^n$. Both are based on the multiplication rules

$$\begin{aligned} e_i e_j + e_j e_i &= 0, \ i \neq j, \\ e_j^2 &= -1, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

and have $e_0 \equiv 1$ as their unit element.

An arbitrary element of $\mathcal{C}\ell_n^{\mathbb{R}}$ or $\mathcal{C}\ell_n^{\mathbb{C}}$ is called a Clifford number and is given by

$$a = \sum_A a_A e_A,$$

where $a_A \in \mathbb{R}$ or $a_A \in \mathbb{C}$, resp., and for each $A = (n_1, \ldots, n_l)$ with $1 \leq n_1 < n_2 < \ldots < n_l \leq m$, it is $e_A = e_{n_1} e_{n_2} \ldots e_{n_l}$. The coefficient a_0 is called the scalar part of a and $\underline{a} = \sum_{j=1}^n a_j e_j$ a Clifford vector.

Similar to the complex conjugation $\overline{}^{\mathbb{C}}$, we can define involutions $\overline{}$ for the real and † for the complex Clifford algebra. Let

$$\overline{e}_A = (-1)^{\frac{|A|(|A|+1)}{2}} e_A.$$

Then

$$\overline{a} = \sum_A a_A \overline{e}_A$$

for $a \in \mathcal{C}\ell_n^{\mathbb{R}}$, and

$$a^\dagger = \sum_A \overline{a}_A^{\mathbb{C}} \overline{e}_A$$

for $a \in \mathcal{C}\ell_n^{\mathbb{C}}$.

We refer to [17] and state that $\mathcal{C}\ell_n^{\mathbb{R}}$ becomes a finite dimensional Hilbert space with the inner product

$$(a, b)_0 = [\overline{a} b]_0 = \sum_A a_A b_A$$

for all $a, b \in \mathcal{C}\ell_n^{\mathbb{R}}$, and has Hilbert space norm

$$|a|_0 = \sqrt{(a, a)_0} = \sqrt{\sum_A |a_A|^2}.$$

The inner product on $\mathcal{Cl}_n^{\mathbb{R}}$ extends to a sesqui-linear inner product

$$(a, b)_0 = [a^{\dagger} b]_0 = \sum_A \overline{a_A}^{\mathbb{C}} b_A$$

for $a, b \in \mathcal{Cl}_n^{\mathbb{C}}$.

It can be shown that Clifford algebras are C^*-algebras, see [17].

Proposition 2.1 *Under the involution $a \to a^{\dagger}$ each $\mathcal{Cl}_n^{\mathbb{C}}$ is a complex C^*-algebra which is a complexification of the real C^*-algebra $\mathcal{Cl}_n^{\mathbb{R}}$.*

2.2 Hilbert Clifford-Modules

We want to consider spaces of $\mathcal{Cl}_m^{\mathbb{R}}$- or $C_m^{\mathbb{C}}$-valued functions. For that purpose, we need an anologue to the classical L^2 spaces. Since the elements of a Clifford algebra do not form a field, we work in Clifford-modules. The following two definitions are taken from [7] and adapted for the complex case; the real case is contained implicitly.

Definition 2.2 $X_{(r)}$ is a **unitary right $\mathcal{Cl}_n^{\mathbb{C}}$-module**, when $(X_{(r)}, +)$ is an abelian group and the mapping $(f, a) \to fa$ from $X_{(r)} \times \mathcal{Cl}_n^{\mathbb{C}} \to X_{(r)}$ is defined such that for all $a, b \in \mathcal{Cl}_n^{\mathbb{C}}$ and $f, g \in X_{(r)}$:

1. $f(a + b) = fa + fb$,
2. $f(ab) = (fa)b$,
3. $(f + g)a = fa + ga$,
4. $fe_0 = f$.

We define an inner product on a unitary right $\mathcal{Cl}_n^{\mathbb{C}}$-module as follows.

Definition 2.3 Let $H_{(r)}$ be a unitary right $\mathcal{Cl}_n^{\mathbb{C}}$-module. Then a function $\langle\cdot, \cdot\rangle$: $H_{(r)} \times H_{(r)} \to \mathcal{Cl}_n^{\mathbb{C}}$ is an **inner product** on $H_{(r)}$ if for all $f, g, h \in H_{(r)}$ and $a \in \mathcal{Cl}_n^{\mathbb{C}}$,

1. $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$
2. $\langle f, ga\rangle = \langle f, g\rangle a$
3. $\langle f, g\rangle = \langle g, f\rangle^{\dagger}$
4. $\langle f, f\rangle_0 \in \mathbb{R}_0^+$ and $\langle f, f\rangle_0 = 0$ if and only if $f = 0$
5. $\langle fa, fa\rangle_0 \leq |a|_0^2 \langle f, f\rangle_0$.

The accompanying **norm** on $H_{(r)}$ is $\|f\|^2 = \langle f, f\rangle_0$.

We now give an important property of the inner product.

Proposition 2.4 *If* $\langle\cdot,\cdot\rangle$ *is an inner product on a unitary right* $\mathcal{C}\ell_n^{\mathbb{C}}$*-module* $H_{(r)}$ *and* $\|f\|^2 = \langle f, f\rangle_0$ *then*

$$|\langle f, g\rangle|_0 \leq 2^n \, \|f\| \; \|g\|$$

for all $f, g \in H_{(r)}$.

Proof We use the definition of the norm on $\mathcal{C}\ell_n^{\mathbb{C}}$, $|a|_0^2 = \sum_A |a_A|^2$, and the fact that

$$[ae_A]_0 = \left[\sum_B a_B e_B e_A\right]_0 = [a_A e_A e_A]_0 = -a_A \tag{2.1}$$

for all $a \in \mathcal{C}\ell_n^{\mathbb{C}}$. Also, if we consider $H_{(r)}$ to be a vector space over $\mathbb{C}$ with inner product $\langle\cdot,\cdot\rangle_0$, we know that the Cauchy-Schwartz inequality

$$|\langle f, g\rangle_0|^2 \leq \langle f, f\rangle_0 \cdot \langle g, g\rangle_0 = \|f\|^2 \; \|g\|^2 \tag{2.2}$$

has to be true. Now, we get

$$\begin{aligned}
|\langle f, g\rangle|_0^2 &= \sum_A |\langle f, g\rangle_A|^2 \overset{(2.1)}{=} \sum_A \left|[\langle f, g\rangle e_A]_0\right|^2 \\
&\overset{(ii)}{=} \sum_A |\langle f, ge_A\rangle_0|^2 \overset{(2.2)}{\leq} \sum_A \|f\|^2 \; \|ge_A\|^2 \\
&\overset{(v)}{=} \sum_A \|f\|^2 \; \|g\|^2 \cdot |e_A|_0^2 = \sum_A \|f\|^2 \; \|g\|^2 \\
&= 2^n \|f\|^2 \; \|g\|^2.
\end{aligned}$$

□

As an analogue to Hilbert (vector) spaces, we now define Hilbert modules.

Definition 2.5 Let $H_{(r)}$ be a unitary right $\mathcal{C}\ell_n^{\mathbb{C}}$-module provided with an inner product $(\cdot,\cdot)$. Then it is called a **right Hilbert $\mathcal{C}\ell_n^{\mathbb{C}}$-module** if it is complete for the norm topology derived from the inner product.

Let $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$. We now consider the unitary right $\mathcal{C}\ell_m^{\mathbb{R}}$-module of functions from $\mathbb{R}^m$ to $\mathcal{C}\ell_m^{\mathbb{R}}$. A function $f : \Omega \subset \mathbb{R}^m \to \mathcal{C}\ell_m^{\mathbb{R}}$ maps the vector variable $\underline{x} = \sum_{j=1}^m x_j e_j$ to a Clifford number and can be written as

$$f(\underline{x}) = \sum_A e_A f_A(\underline{x}),$$

where $f_A : \mathbb{R}^m \to \mathbb{R}$ [7]. We define an inner product as follows.

Definition 2.6 Let h be a positive function on $\mathbb{R}^m$. Then the inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})}$ is defined as

$$\langle f, g \rangle_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})} = \int_{\mathbb{R}^m} \overline{f(\underline{x})} g(\underline{x}) h(\underline{x}) d\underline{x},$$

where $d\underline{x}$ stands for the Lebesgue measure on $\mathbb{R}^m$, and the associated norm is $||f||^2_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})} = \left[\langle f, f \rangle_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})} \right]_0$.

The unitary right Clifford-module of measurable functions on $\mathbb{R}^m$ for which $||f||_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})} < \infty$ is a right Hilbert Clifford-module, which we denote by $L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})$. In this paper, we will focus on the case where $h(\underline{x}) = 1$. Then the right Hilbert Clifford-module will simply be denoted by $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ and the inner product by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})}$.

We also work on functions with values in a complex Clifford algebra, i.e. $f : \Omega \subset \mathbb{C}^m \to \mathcal{C}\ell_m^{\mathbb{C}}$. For $\underline{z} = \sum_{j=1}^m z_j e_j$, with complex z_j, $j = 1, \ldots, m$, we have

$$f(\underline{z}) = \sum_A e_A f_A(\underline{z})$$

with $f_A : \mathbb{C}^m \to \mathbb{C}$. Analogously to the real case, we can define the right Hilbert Clifford-module $L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})$, where h is a positive function over $\mathbb{C}^m$. Here,

$$\langle f, g \rangle_{L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})} = \int_{\mathbb{C}^m} f^\dagger(\underline{z}) g(\underline{z}) h(\underline{z}) d\underline{x}\, d\underline{y}$$

with $\underline{z} = \underline{x} + i\underline{y}$, where $\dagger$ denotes the involution on $\mathcal{C}\ell_m^{\mathbb{C}}$, cf. page 32. The associated norm is $||f||^2_{L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})} = \left[\langle f, f \rangle_{L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})} \right]_0$. Particularly important to our work will be those spaces $L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})$ for which h is defined as the Gaussian function $h(\underline{z}) = \frac{e^{-|\underline{z}|^2/2}}{\pi^m}$, cf. Sect. 5.

Proposition 2.7

1. *Let* $f \in L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})$. *Then*

$$||f||^2_{L^2(\mathbb{R}^m, h, \mathcal{C}\ell_m^{\mathbb{R}})} = \int_{\mathbb{R}^m} \left| f(\underline{x}) \right|_0^2 h(\underline{x}) d\underline{x}.$$

2. *Let* $f \in L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})$. *Then*

$$||f||^2_{L^2(\mathbb{C}^m, h, \mathcal{C}\ell_m^{\mathbb{C}})} = \int_{\mathbb{C}^m} \left| f(\underline{z}) \right|_0^2 h(\underline{z}) d\underline{x}\, d\underline{y}.$$

Proof We only show (ii) since the real case is equivalent,

$$||f||^2_{L^2(\mathbb{C}^m,h,\mathcal{C}\ell^{\mathbb{C}}_m)} = \left[\langle f, f\rangle_{L^2(\mathbb{C}^m,h,\mathcal{C}\ell^{\mathbb{C}}_m)}\right]_0$$

$$= \int_{\mathbb{C}^m} \left[f^\dagger(\underline{z})f(\underline{z})\right]_0 h(\underline{z})d\underline{x}\,d\underline{y} = \int_{\mathbb{C}^m} \left|f(\underline{z})\right|_0^2 h(\underline{z})d\underline{x}\,d\underline{y}.$$

□

2.3 Inner Spherical Monogenics

Since many of the following results are similar for functions of real and complex Clifford algebras, we will state them for the real case and give the complex case in ().

Of particular importance, when dealing with Clifford algebra-valued functions, is the Dirac operator

$$D_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j} \quad (D_{\underline{z}} = \sum_{j=1}^m e_j \partial_{z_j}).$$

Left nullsolutions of $D_{\underline{x}}$ ($D_{\underline{z}}$) are called (complex) left monogenic functions.

Let $m \in \mathbb{N}$ and $\mathfrak{P}^s$ be the space of scalar-valued polynomials in $\mathbb{R}^m$ ($\mathbb{C}^m$). Then a Clifford polynomial is an element of $\mathfrak{P}^s \otimes \mathcal{C}\ell^{\mathbb{R}}_m$ ($\mathfrak{P}^s \otimes \mathcal{C}\ell^{\mathbb{C}}_m$).

An important class of polynomials are the so called (complex) inner spherical monogenics. A left inner spherical monogenic of order k is a left monogenic homogeneous Clifford polynomial P_k of degree k. The set of all left inner spherical monogenics of order k is denoted by $M^+_l(k)$ and has the dimension [8]

$$\dim(M^+_l(k)) = \binom{m+k-2}{k},$$

(with $\dim(M^+_l(0)) = 1$ for all $m \in \mathbb{N}$).

We will deal with inner spherical monogenics over both $\mathbb{R}^m$ and $\mathbb{C}^m$. To differentiate, we will write $P_k(\underline{x}) : \mathbb{R}^m \to \mathcal{C}\ell^{\mathbb{R}}_m$ and $P_k(\underline{z}) : \mathbb{R}^m \to \mathcal{C}\ell^{\mathbb{C}}_m$.

2.4 Short-time Fourier Transform

An important tool in time-frequency analysis is the short-time Fourier Transform. It allows to analyse a given signal simultaneously in the time and in the frequency domain, because it calculates the Fourier Transform not over the whole signal, but small blocks of it.

Given a signal $f(\underline{t})$ and a window function $\varphi(\underline{t})$, the short-time Fourier Transform $(V_\varphi f)(\underline{t}, \underline{\omega})$ is classically defined as

$$(V_\varphi f)(\underline{t}, \underline{\omega}) = \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} f(\underline{x}) \overline{\varphi(\underline{x} - \underline{t})}^{\mathbb{C}} e^{-i\underline{\omega}\cdot\underline{x}} d\underline{x}.$$

A commonly used window function is the Gaussian window because it provides a very good resolution of the studied signal [18].

It is given by $h(\underline{x}) = e^{-\frac{|\underline{x}|^2}{4}}$.

3 Clifford Hermite Polynomials

We will now consider Clifford Hermite polynomials as a special class of Clifford polynomials.

In the classical case, the Hermite polynomials over $\mathbb{R}$ can be obtained from the Taylor expansion of the function $z \mapsto e^{z^2/2}$,

$$e^{z^2/2} = \sum_{n=0}^{\infty} e^{x^2/2} \frac{t^n}{n!} H_n(ix).$$

They can also be calculated explicitly by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Through a similar expansion for $\mathbb{R}^m$, F. Sommen defined *radial Hermite polynomials* [26], which are explicitly given by

$$H_{k,m}(\underline{x}) = (-1)^k e^{\frac{|\underline{x}|^2}{2}} D_{\underline{x}}^k e^{-\frac{|\underline{x}|^2}{2}}.$$

Since the radial Hermite polynomials only form a basis for a certain kind of L^2 functions, i.e. such functions that are defined on the real line, Sommen developed a more complex set of polynomials, starting from the monogenic extension of $e^{-|\underline{x}|^2/2} P_k(\underline{x})$, where $P_k(\underline{x})$ is a left inner spherical monogenic of degree k, cf. Sect. 2.3. This lead him to what he called the generalized Hermite polynomials, which can be used to construct a basis of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$.

Definition 3.1 The **generalized Clifford Hermite polynomials** $H_{l,m,k}$, $l, k \in \mathbb{N}_0$, are given by

$$H_{l,m,k} P_k(\underline{x}) = e^{-\frac{|\underline{x}|^2}{2}} (-1)^l D_{\underline{x}}^l \left(e^{-\frac{|\underline{x}|^2}{2}} P_k(\underline{x}) \right) \tag{3.1}$$

where $P_k(\underline{x})$ is a left inner spherical monogenic of degree k.

An important property of the generalized Clifford Hermite polynomials is their orthogonality [9].

Theorem 3.2 *Let H_{l,m,k_1} and H_{t,m,k_2} be generalized Clifford Hermite polynomials and $P_{k_1}(\underline{x})$ and $P_{k_2}(\underline{x})$ inner spherical monogenics of order k_1 and k_2, resp. Then*

$$\int_{\mathbb{R}^m} e^{-\frac{|\underline{x}|^2}{2}} \overline{H_{l,m,k_1}(\underline{x}) P_{k_1}(\underline{x})}\, H_{t,m,k_2}(\underline{x}) P_{k_2}(\underline{x})\, d\underline{x} = \gamma_{l,k_1} \delta_{l,t} \delta_{k_1,k_2},$$

with

$$\gamma_{2p,k} = \frac{2^{2p+m/2+k} p! \sqrt{\pi}^m \Gamma\left(\frac{m}{2} + k + p\right)}{\Gamma\left(\frac{m}{2}\right)},$$
$$\gamma_{2p+1,k} = \frac{2^{2p+m/2+k+1} p! \sqrt{\pi}^m \Gamma\left(\frac{m}{2} + k + p + 1\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

Building on the orthogonality, Sommen and his colleagues established an orthonormal basis of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$, cf. [9, 26].

Theorem 3.3 *Let $\gamma_{l,k}$, $k, l \in \mathbb{N}_0$, be as defined in Theorem 3.2. For each $k \in \mathbb{N}_0$, let further $\left\{ P_k^{(j)}(\underline{x}) \right\}_{j=1,2,\ldots,\dim(M_l^+(k))}$ be an orthonormal basis of $M_l^+(k)$.*

$$\left\{ \frac{1}{\sqrt{\gamma_{l,k}}} H_{l,m,k}(\underline{x}) P_k^{(j)}(\underline{x}) e^{-\frac{|\underline{x}|^2}{4}} : l, k \in \mathbb{N}_0, j \le \dim(M_l^+(k)) \right\} \tag{3.2}$$

forms an orthonormal basis of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$.

Each element of (3.2) depends on l, k and the chosen basis of $M_l^+(k)$, which contains $\dim(M_l^+(k)) = \binom{m+k-2}{k}$ elements, cf. Sect. 2.3.

4 Segal–Bargmann Transform

The first very general version of the Segal–Bargmann space goes back to V. Fock's theory of 1932 of the quantum state space of particles [16]. Here, we will consider the more specific finite-dimensional version of the following definitions taken from [23]. We will transfer those definitions to the Clifford case in Sect. 5.

Definition 4.1 The **Segal–Bargmann space** $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$ is defined as the Hilbert space of entire functions f in $\mathbb{C}^m$ which are square-integrable with respect to the $2m$-dimensional Gaussian density, i.e.,

$$\frac{1}{\pi^m} \int_{\mathbb{C}^m} e^{-|\underline{z}|^2} |f(\underline{z})|^2 \, d\underline{x} \, d\underline{y} < \infty, \quad \underline{z} = \underline{x} + i\underline{y}.$$

It is equipped with the inner product

$$\langle f, g \rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} = \frac{1}{\pi^m} \int_{\mathbb{C}^m} e^{-|\underline{z}|^2} \overline{f(\underline{z})} g(\underline{z}) \, d\underline{x} \, d\underline{y}.$$

The Segal–Bargmann transform connects the Bargmann space with the Hilbert space $L^2(\mathbb{R}^m, \mathbb{R})$ by mapping the ladder onto the former.

Definition 4.2 The **Segal–Bargmann transform** $\mathcal{B}$ from $L^2(\mathbb{R}^m, \mathbb{R})$ to $\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$ is defined by

$$\big(\mathcal{B} f\big)(\underline{z}) = \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} e^{-\frac{\underline{z}\cdot\underline{z}}{2} + \underline{x}\cdot\underline{z} - \frac{\underline{x}\cdot\underline{x}}{4}} f(\underline{x}) \, d\underline{x}, \tag{4.1}$$

with $\underline{x} \cdot \underline{z} = \sum_{j=1}^m x_j z_j$, for any $f \in L^2(\mathbb{R}^m, \mathbb{R})$.

The Segal–Bargmann transform is a linear operator. It can also be expressed in terms of a short-time Fourier Transform (cf. Sect. 2.4).

Proposition 4.3 *Let $\mathcal{B}$ be the Segal–Bargmann transform and V_φ the short-time Fourier Transform with window $\varphi(\underline{x}) = e^{-\frac{|\underline{x}|^2}{4}}$. Then for all $f \in L^2(\mathbb{R}^m, \mathbb{R})$,*

$$(V_\varphi f)(2\underline{t}, -\underline{\omega}) = e^{-\frac{|\underline{z}|^2}{2}} e^{i\underline{t}\cdot\underline{\omega}} \big(\mathcal{B} f\big)(\underline{z}), \quad \underline{z} = \underline{t} + i\underline{\omega}.$$

Proof

$$\begin{aligned}
(V_\varphi f)(2\underline{t}, -\underline{\omega}) &= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} f(\underline{x}) e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} f(\underline{x}) e^{-\frac{|\underline{x}|^2}{4} + \underline{x}\cdot\underline{t} - |\underline{t}|^2} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} e^{-\frac{|\underline{t}|^2}{2}} e^{-\frac{|\underline{\omega}|^2}{2}} e^{i\underline{t}\cdot\underline{\omega}} \int_{\mathbb{R}^m} f(\underline{x}) e^{-\frac{|\underline{x}|^2}{4} + \underline{x}\cdot(\underline{t}+i\underline{\omega}) - \frac{(\underline{t}+i\underline{\omega})^2}{2}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} e^{-\frac{|\underline{t}|^2}{2}} e^{-\frac{|\underline{\omega}|^2}{2}} e^{i\underline{t}\cdot\underline{\omega}} \int_{\mathbb{R}^m} f(\underline{x}) e^{-\frac{|\underline{x}|^2}{4} + \underline{x}\cdot\underline{z} - \frac{\underline{z}\cdot\underline{z}}{2}} d\underline{x} \\
&= e^{-\frac{|\underline{z}|^2}{2}} e^{i\underline{t}\cdot\underline{\omega}} \big(\mathcal{B} f\big)(\underline{z})
\end{aligned}$$

□

A well-known property of the Segal–Bargmann transform is that it is a unitary operator up to a scaling constant.

Proposition 4.4 *Let* $\mathcal{B} : L^2(\mathbb{R}^m, \mathbb{R}) \to \mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$ *be the Segal–Bargmann transform (4.1). Then*

$$\langle \mathcal{B}f, \mathcal{B}g\rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} = \frac{1}{\sqrt{2\pi}^m}\langle f, g\rangle_{L^2(\mathbb{R}^m,\mathbb{R})}.$$

Proof We use Proposition 4.3, i.e.

$$\big(\mathcal{B}f\big)(\underline{z}) = e^{\frac{|\underline{z}|^2}{2}} e^{-i\underline{t}\cdot\underline{\omega}}(V_\varphi f)(2\underline{t}, -\underline{\omega}),$$

with $\underline{z} = \underline{t} + i\underline{\omega}$ and $\varphi(\underline{x}) = e^{-\frac{|\underline{x}|^2}{4}}$. Then

$$\begin{aligned}
\langle \mathcal{B}f, \mathcal{B}g\rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} &= \frac{1}{\pi^m}\int\limits_{\mathbb{C}^m} \overline{\big(\mathcal{B}f\big)(\underline{z})}^{\mathbb{C}} \big(\mathcal{B}g\big)(\underline{z}) e^{-|\underline{z}|^2} d\underline{x}\, d\underline{y} \\
&= \frac{1}{\pi^m}\int\limits_{\mathbb{C}^m} e^{\frac{|\underline{z}|^2}{2}} e^{i\underline{t}\cdot\underline{\omega}} \overline{\big(V_\varphi f\big)(2\underline{t}, -\underline{\omega})}^{\mathbb{C}} \\
&\qquad \cdot e^{\frac{|\underline{z}|^2}{2}} e^{-i\underline{t}\cdot\underline{\omega}}\big(V_\varphi g\big)(2\underline{t}, -\underline{\omega}) e^{-|\underline{z}|^2} d\underline{\omega}\, d\underline{t} \\
&= \frac{1}{\pi^m}\int\limits_{\mathbb{C}^m} \overline{\big(V_\varphi f\big)(2\underline{t}, -\underline{\omega})}^{\mathbb{C}} \big(V_\varphi g\big)(2\underline{t}, -\underline{\omega}) d\underline{\omega}\, d\underline{t} \\
&= \frac{1}{\pi^m}\int\limits_{\mathbb{C}^m} \overline{\frac{1}{\sqrt{2\pi}^m}\int\limits_{\mathbb{R}^m} f(\underline{x}) e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x}}^{\mathbb{C}} \\
&\qquad \cdot \frac{1}{\sqrt{2\pi}^m}\int\limits_{\mathbb{R}^m} g(\underline{x}) e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x}\, d\underline{\omega}\, d\underline{t}.
\end{aligned}$$

Let $\varphi(\cdot - 2\underline{t})$ denote the Gaussian window translated by $-2\underline{t}$ and $\mathcal{F}$ the Fourier Transform in $\mathbb{R}^m$. Thus

$$\langle \mathcal{B}f, \mathcal{B}g\rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} = \frac{1}{\pi^m}\int\limits_{\mathbb{C}^m} \overline{\mathcal{F}^{-1}\big(f \cdot \varphi(\cdot - 2\underline{t})\big)(\underline{\omega})}^{\mathbb{C}} \mathcal{F}^{-1}\big(g \cdot \varphi(\cdot - 2\underline{t})\big)(\underline{\omega}) d\underline{\omega}\, d\underline{t}.$$

The Plancherel Theorem now gives us

$$\langle \mathcal{B}f, \mathcal{B}g\rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} = \frac{1}{\pi^m}\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} \overline{f(\underline{x})e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}}}^{\mathbb{C}} g(\underline{x})e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}} d\underline{t}\, d\underline{x}$$
$$= \frac{1}{\pi^m}\int\limits_{\mathbb{R}^m} f(\underline{x})g(\underline{x}) \int\limits_{\mathbb{R}^m} \left(e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}}\right)^2 d\underline{t}\, d\underline{x}.$$

Last, we substitute $\underline{u} = 2\underline{t} - \underline{x}$, and use the fact that $\int\limits_{\mathbb{R}^m} e^{-\frac{|\underline{u}|^2}{2}} d\underline{u} = \sqrt{2\pi}^m$. So,

$$\langle \mathcal{B}f, \mathcal{B}g\rangle_{\mathcal{F}^2(\mathbb{C}^m,\mathbb{C})} = \frac{1}{\pi^m}\int\limits_{\mathbb{R}^m} f(\underline{x})g(\underline{x})d\underline{x} \int\limits_{\mathbb{R}^m} e^{-\frac{|\underline{u}|^2}{2}} \frac{d\underline{u}}{2^m}$$
$$= \frac{1}{\sqrt{2\pi}^m}\langle f, g\rangle_{L^2(\mathbb{R}^m,\mathbb{R})}.$$

□

Another important property of the Segal–Bargmann transform is its invertibility.

Proposition 4.5

1. *$F^2(\mathbb{C}^m, \mathbb{C})$ is the image of $L^2(\mathbb{R}^m, \mathbb{R})$ under the Segal–Bargmann transform.*
2. *The Segal–Bargmann transform is invertible.*

Proof For the proof of (i) we refer to [18]. (ii) then follows directly from the fact that the transform is unitary up to a constant, cf. Proposition 4.4. □

5 Segal–Bargmann Modules

We will now look at how the Segal–Bargmann transform of Definition 4.1 acts on Clifford algebra-valued functions. So, from now on, let f be an element of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$. Then,

$$\left(\mathcal{B}f\right)(\underline{z}) = \frac{1}{\sqrt{2\pi}^m}\int_{\mathbb{R}^m} e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}-\frac{\underline{x}\cdot\underline{x}}{4}} f(\underline{x})\, d\underline{x},$$

is a function with values in the complex Clifford algebra, $\mathcal{B}f : \mathbb{C}^m \to \mathcal{C}\ell_m^{\mathbb{C}}$. Note that Proposition 4.3 holds for functions of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ as well.

Consider the function space $L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})$ as defined in Sect. 2.2. Just as in the real case, the Segal–Bargmann transform of Clifford algebra-valued functions is unitary up to a scaling constant, as the following proposition shows.

Proposition 5.1 *If $f \in L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$, then*

$$\langle \mathcal{B}f, \mathcal{B}g \rangle_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} = \frac{1}{\sqrt{2\pi}^m} \langle f, g \rangle_{L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})}.$$

Proof Since the Segal–Bargmann transform is linear, $f = \sum_A f_A e_A$ implies that $\mathcal{B}f = \sum_A \mathcal{B}f_A e_A$. Hence,

$$\begin{aligned}
\langle \mathcal{B}f, \mathcal{B}g \rangle_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} &= \frac{1}{\pi^m} \int_{\mathbb{C}^m} \big(\mathcal{B}f\big)^{\dagger}(\underline{z}) \big(\mathcal{B}g\big)(\underline{z}) e^{-|\underline{z}|^2} d\underline{x}\, d\underline{y} \\
&= \sum_{A,B} \frac{1}{\pi^m} \int_{\mathbb{C}^m} \overline{\big(\mathcal{B}f_A\big)}^{\mathbb{C}}(\underline{z}) \overline{e_A} \big(\mathcal{B}g_B\big)(\underline{z}) e_B e^{-|\underline{z}|^2} d\underline{x}\, d\underline{y} \\
&= \sum_{A,B} \langle \mathcal{B}f_A, \mathcal{B}g_B \rangle_{\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})} \overline{e_A} e_B
\end{aligned}$$

In Proposition 4.4 we have shown that $\langle \mathcal{B}f, \mathcal{B}g \rangle_{\mathcal{F}^2(\mathbb{C}^m, \mathbb{C})} = \frac{1}{\sqrt{2\pi}^m} \langle f, g \rangle_{L^2(\mathbb{R}^m, \mathbb{R})}$ is true for the classical Segal–Bargmann transform $\mathcal{B} : L^2(\mathbb{R}^m, \mathbb{R}) \to \mathcal{F}^2(\mathbb{C}^m, \mathbb{C})$. Therefore

$$\begin{aligned}
\langle \mathcal{B}f, \mathcal{B}g \rangle_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} &= \sum_{A,B} \frac{1}{\sqrt{2\pi}^m} \langle f_A, g_B \rangle_{L^2(\mathbb{R}^m, \mathbb{R})} \overline{e_A} e_B \\
&= \frac{1}{\sqrt{2\pi}^m} \sum_{A,B} \int_{\mathbb{R}^m} \overline{f_A(\underline{x})} g_B(\underline{x}) \overline{e_A} e_B d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} \overline{f(\underline{x})} g(\underline{x}) d\underline{x} = \frac{1}{\sqrt{2\pi}^m} \langle f, g \rangle_{L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})}
\end{aligned}$$

□

A direct consequence is the following corollary.

Corollary 5.2 *Let* $\|\cdot\|_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} = \sqrt{\left[\langle \cdot, \cdot \rangle_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} \right]_0}$. *Then*

$$\|\mathcal{B}f\|^2_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} = \frac{1}{\sqrt{2\pi}^m} \|f\|^2_{L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})}.$$

Thus $\mathcal{B}$ is an isometry from $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ into $L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})$ up to $\frac{1}{\sqrt{2\pi}^m}$.

In Theorem 5.3, an orthonormal basis of the space $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ was established. The following theorem shows that the Segal–Bargmann transform maps the elements of this basis onto functions $\underline{z}^l P_k(\underline{z})$.

Theorem 5.3 *Let $\mathcal{B}$ be the Segal–Bargmann transform, $H_{l,m,k}$ a generalized Clifford Hermite Polynomial as defined in Definition 3.1 and P_k an inner spherical monogenic of degree k. Then*

$$\left(\mathcal{B}\left(H_{l,m,k}(\underline{x})e^{-\frac{|\underline{x}|^2}{4}} P_k(\underline{x})\right)\right)(\underline{z}) = \underline{z}^l P_k(\underline{z})$$

Proof Our first step follows [23]. Here,

$$\begin{aligned}
&\left(\mathcal{B}\left(H_{l,m,k}(\underline{x})e^{-\frac{|\underline{x}|^2}{4}} P_k(\underline{x})\right)\right)(\underline{z}) \\
&= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}-\frac{\underline{x}\cdot\underline{x}}{2}} H_{l,m,k}(\underline{x}) P_k(\underline{x}) d\underline{x} \\
&\overset{(3.1)}{=} \frac{(-1)^l}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}} D_{\underline{x}}^l \left(e^{-\frac{|\underline{x}|^2}{2}} P_k(\underline{x})\right) d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} D_{\underline{x}}^l \left(e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}}\right) e^{-\frac{|\underline{x}|^2}{2}} P_k(\underline{x}) d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} \underline{z}^l \int_{\mathbb{R}^m} e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}-\frac{\underline{x}\cdot\underline{x}}{4}} P_k(\underline{x}) e^{-\frac{|\underline{x}|^2}{4}} d\underline{x} \\
&= \underline{z}^l \left(\mathcal{B}\left(P_k(\underline{x})e^{-\frac{|\underline{x}|^2}{4}}\right)\right)(\underline{z}).
\end{aligned}$$

Next, we calculate $\mathcal{B}(P_k(\underline{x})e^{-|\underline{x}|^2/4})$ using the windowed Fourier transform. We obtain

$$\begin{aligned}
V_\varphi\left(P_k(\underline{x})e^{-\frac{|\underline{x}|^2}{4}}\right)(2\underline{t}, -\underline{\omega}) &= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} P_k(\underline{x}) e^{-\frac{|\underline{x}|^2}{4}} e^{-\frac{|\underline{x}-2\underline{t}|^2}{4}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} P_k\left(-i\partial_{\underline{\omega}}\right) \int_{\mathbb{R}^m} e^{-\frac{|\underline{x}|^2}{4}} e^{-\frac{|\underline{x}|}{4}+\underline{x}\cdot\underline{t}-|\underline{t}|^2} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} P_k\left(-i\partial_{\underline{\omega}}\right) \int_{\mathbb{R}^m} e^{-\frac{|\underline{x}|}{2}+\underline{x}\cdot\underline{t}-\frac{|\underline{t}|^2}{2}} e^{-\frac{|\underline{t}|}{2}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} e^{-\frac{|\underline{t}|^2}{2}} P_k\left(-i\partial_{\underline{\omega}}\right) \int_{\mathbb{R}^m} e^{-\frac{|\underline{x}-\underline{t}|^2}{2}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x} \\
&= \frac{1}{\sqrt{2\pi}^m} e^{-\frac{|\underline{t}|^2}{2}} P_k\left(-i\partial_{\underline{\omega}}\right) \left(e^{i\underline{\omega}\cdot\underline{t}} \int_{\mathbb{R}^m} e^{-\frac{|\underline{x}|^2}{2}} e^{i\underline{\omega}\cdot\underline{x}} d\underline{x}\right)
\end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}^m}\int_{\mathbb{R}^m} e^{-\frac{|\underline{x}|^2}{2}} e^{i\underline{\omega}\cdot\underline{x}}\, d\underline{x}$ is the inverse Fourier Tranform of $e^{-\frac{|\omega|^2}{2}}$, which is an invariant, we get

$$\begin{aligned} V_\varphi\left(P_k(\underline{x})e^{-\frac{|\underline{x}|^2}{4}}\right)(2\underline{t}, -\underline{\omega}) &= e^{-\frac{|\underline{t}|^2}{2}} P_k\left(-i\,\partial_{\underline{\omega}}\right)\left(e^{i\underline{\omega}\cdot\underline{t}-\frac{|\underline{\omega}|^2}{2}}\right) \\ &= e^{-\frac{|\underline{t}|^2}{2}} P_k\left(-i\,(i\underline{t}-\omega)\right)\left(e^{i\underline{\omega}\cdot\underline{t}-\frac{|\underline{\omega}|^2}{2}}\right) \\ &= e^{-\frac{|\underline{t}|^2}{2}}\left(e^{i\underline{\omega}\cdot\underline{t}-\frac{|\underline{\omega}|^2}{2}}\right) P_k(\underline{z}) \\ &= e^{-\frac{|\underline{z}|^2}{2}} e^{i\underline{\omega}\cdot\underline{t}} P_k(\underline{z}) \end{aligned}$$

with $\underline{z} = \underline{t} + i\underline{\omega}$. Because of Proposition 4.3, this leads to

$$\left(\mathcal{B}\left(P_k(\underline{x})e^{-\frac{|\underline{x}|^2}{4}}\right)\right)(\underline{z}) = P_k(\underline{z}).$$

Together with the first step, the proof is complete. □

We can now define an analogue to the classical Segal–Bargmann space.

Definition 5.4 The closure of

$$\operatorname{span}\left\{\underline{z}^l P_k^{(j)}(\underline{z})\,\middle|\, l, k \in \mathbb{N}_0,\ j = 1, \ldots, \dim(M_l^+(k))\right\}$$

is called **Segal–Bargmann module** $\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$.

Remark 5.5 In this definition and what follows we drop the property that a function of the Segal–Bargmann module (or space) has to be an entire functions. That means we consider the Segal–Bargmann module just as a weighted L^2-module.

A consequence of Theorem 5.3 is the following.

Corollary 5.6 *For all* $l, k \in \mathbb{N}_0$*, let* $\left\{P_k^{(j)}(\underline{x})\right\}_{j=1,2,\ldots,\dim(M_l^+(k))}$ *be an orthonormal basis of* $M_l^+(k)$ *and* $\gamma_{l,k}$ *defined as in Theorem 3.2. Then*

$$\left\{\sqrt{\frac{(2\pi)^m}{\gamma_{l,k}}}\underline{z}^l P_k^{(j)}(\underline{z})\,\middle|\, l, k \in \mathbb{N}_0,\ j = 1, \ldots, \dim(M_l^+(k))\right\}$$

is an orthonormal basis of the Segal–Bargmann module $\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$.

Proof Since the Segal–Bargmann transform is linear, Theorem 5.3 shows that it maps an element

$\phi_{l,k,j}(\underline{x}) = \frac{1}{\sqrt{\gamma_{l,k}}} H_{l,m,k}(\underline{x}) P_k^{(j)}(\underline{x}) e^{-\frac{|\underline{x}|^2}{4}}$

of the orthonormal basis of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ (see Theorem 3.3) onto

$$\left(\mathcal{B}\phi_{l,k,j}\right)(\underline{z}) = \frac{1}{\sqrt{\gamma_{l,k}}} \underline{z}^l P_k^{(j)}(\underline{z}).$$

The statement now follows directly from Proposition 5.1 and Corollary 5.2, which say that

$\left\|\mathcal{B}\phi_{l,k,j}\right\|_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})} = \frac{1}{\sqrt{2\pi}^m} \left\|\phi_{l,k,j}\right\|_{L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})} = \frac{1}{\sqrt{2\pi}^m}$

and $\mathcal{B}$ is unitary up to the scaling constant. □

Theorem 5.7 *The Segal–Bargmann module is the image of $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ under the Segal–Bargmann transform, i.e.*

$$\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}}) = L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right).$$

Proof First, let $F \in \mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$. By construction there has to exist a function $f \in L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ so that $\mathcal{B}f = F$. Since $\mathcal{B}$ is unitary up to a constant, we know that $F \in L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right)$. Hence

$\mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}}) \subseteq L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right)$.

We now show the opposite inclusion. Let $F \in L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right)$. Then F can be written as $F = \sum_A F_A e_A$ with $F_A : \mathcal{C}\ell^m \to \mathbb{C}$ for all A. Since

$$\begin{aligned} \|F\|_{L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right)} &= \langle \sum_A F_A e_A, \sum_B F_B e_B \rangle_0 \\ &= \sum_A \|F_A\|_{L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}}\right)} = \sum_A \|F_A\|_{L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathbb{C}\right)} \end{aligned}$$

is finite if and only if $\|F_A\|_{L^2\left(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathbb{C}\right)}$ is finite for every A, we know that $F_A \in L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathbb{C}) = \mathcal{F}^2(\mathbb{C}_m, \mathbb{C})$, cf. Proposition 4.5(ii).

Proposition 4.5(i) tells us that $\mathcal{B} : L^2(\mathbb{R}^m, \mathbb{R}) \to \mathcal{F}^2(\mathbb{C}_m, \mathbb{C})$ is invertible, so for each A there exists $f_A \in L^2(\mathbb{R}^m, \mathbb{R})$ so that $\mathcal{B} f_A = F_A$. Since $\mathcal{B}$ is linear,

$$F = \sum_A F_A e_A = \sum_A (\mathcal{B} f_A) e_A = \mathcal{B}\left(\sum_A f_A e_A\right),$$

so there exists a function $\sum_A f_A e_A = f \in L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$ such that $\mathcal{B} f = F$. Therefore $F \in \mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$. □

6 A Dictionary for the Segal–Bargmann Transform

In this section, we want to give a series representation for the Segal–Bargmann transform $\mathcal{B}$ on the right Clifford-module $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$. By demonstrating that this representation converges absolutely locally uniformly, we will show that $\mathcal{B} f$ is well-defined and can be represented in kernel form. We work close to R. Bardenet and A. Hardy [3], who have shown similar characteristics of the classical Segal–Bargmann transform on $L^2(\mathbb{R}^m, \mathbb{R})$ and other transforms.

For the rest of this section, we will shorten our notation by writing $L^2 = L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$, $\mathcal{F}^2 = \mathcal{F}^2(\mathbb{C}^m, \mathcal{C}\ell_m^{\mathbb{C}})$, $\langle \cdot, \cdot \rangle_{\mathcal{F}^2} = \langle \cdot, \cdot \rangle_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})}$ and $\| \cdot \|_{\mathcal{F}^2} = \| \cdot \|_{L^2(\mathbb{C}^m, \frac{e^{-|\underline{z}|^2}}{\pi^m}, \mathcal{C}\ell_m^{\mathbb{C}})}$.

Since the set $\left\{\phi_{l,k,j}\right\}_{l,k \in \mathbb{N}_0, j \in \{1,\ldots,\dim(M_l^+(k))\}}$ of Hermite functions

$$\phi_{l,k,j}(\underline{x}) = \frac{1}{\sqrt{\gamma_{l,k}}} H_{l,m,k}(\underline{x}) P_k^{(j)}(\underline{x}) e^{-\frac{|\underline{x}|^2}{4}} \tag{6.1}$$

is a basis of L^2, see Sect. 3, each Clifford algebra-valued square integrable function $f(\underline{x})$ can be expanded as

$$f(\underline{x}) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \phi_{l,k,j}(\underline{x}) \langle \phi_{l,k,j}, f \rangle_{L^2}.$$

Hence,

$$\begin{aligned}
&\big(\mathcal{B} f\big)(\underline{z}) \\
&= \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} \left(\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \phi_{l,k,j}(\underline{x}) \langle \phi_{l,k,j}, f \rangle_{L^2} \right) e^{-\frac{\underline{z}\cdot\underline{z}}{2} + \underline{x}\cdot\underline{z} - \frac{\underline{x}\cdot\underline{x}}{4}} d\underline{x}
\end{aligned}$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\left(\frac{1}{\sqrt{2\pi}^m}\int_{\mathbb{R}^m}\phi_{l,k,j}(\underline{x})e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}-\frac{\underline{x}\cdot\underline{x}}{4}}d\underline{x}\right)\langle\phi_{l,k,j}, f\rangle_{L^2}$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\left(\mathcal{B}\phi_{l,k,j}\right)(\underline{z})\langle\phi_{l,k,j}, f\rangle_{L^2}$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\frac{1}{\sqrt{\gamma_{l,k}}}\underline{z}^l P_k^{(j)}(\underline{z})\langle\phi_{l,k,j}, f\rangle_{L^2}$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\langle\phi_{l,k,j}, f\rangle_{L^2} \tag{6.2}$$

with $\Psi_{l,k,j}(\underline{z}) = \frac{1}{\sqrt{\gamma_{l,k}}}\underline{z}^l P_k^{(j)}(\underline{z})$.

To be able to show convergence of the series expansion (6.2), we need the following two lemmas.

Lemma 6.1 *Let $P_s(\underline{z}) = \sum_{|\alpha|=s} a_\alpha \underline{z}^\alpha$ be a homogeneous $\mathcal{C}\ell_m^{\mathbb{C}}$-polyomial of degree s, with $a_\alpha \in \mathcal{C}\ell_m^{\mathbb{C}}$ for all $|\alpha| = s$. Then*

1. $\left\| P_s(\underline{z})\right\|_{\mathcal{F}^2}^2 = \sum_{|\alpha|=s} |a_\alpha|_0^2\, \alpha!$,
2. $\left|P_s(\underline{z})\right|_0^2 \leq \frac{1}{s!}\left\| P_s(\underline{z})\right\|_{\mathcal{F}^2}^2 \left|\underline{z}\right|_0^{2s}$.

Proof

1. We have

$$\left\| P_s(\underline{z})\right\|_{\mathcal{F}^2}^2 = \left[\langle P_s(\underline{z}), P_s(\underline{z})\rangle_{\mathcal{F}^2}\right]_0 = \frac{1}{\pi^m}\int_{\mathbb{C}^m}\left[P_s^\dagger(\underline{z})P_s(\underline{z})\right]_0 e^{-|\underline{z}|^2}d\underline{x}\,d\underline{y}$$

$$= \frac{1}{\pi^m}\int_{\mathbb{C}^m}\left[\left(\sum_{|\alpha|=s} a_\alpha^\dagger(\underline{z}^{\mathbb{C}})^\alpha\right)\left(\sum_{|\beta|=s} a_\beta \underline{z}^\beta\right)\right]_0 e^{-|\underline{z}|^2}d\underline{x}\,d\underline{y}$$

$$= \frac{1}{\pi^m}\sum_{|\alpha|=s}\sum_{|\beta|=s}\left[a_\alpha^\dagger a_\beta\right]_0\int_{\mathbb{C}^m}(\underline{z}^{\mathbb{C}})^\alpha\underline{z}^\beta e^{-|\underline{z}|^2}d\underline{x}\,d\underline{y}.$$

We solve the integral by transforming the complex coordinates to polar coordinates, i.e. $z_j = r_j e^{i\varphi_j}$, $j = 1, \dots, m$. Then,

$$\| P_s(\underline{z}) \|_{\mathcal{F}^2}^2 = \frac{1}{\pi^m} \sum_{|\alpha|=s} \sum_{|\beta|=s} \left[a_\alpha^\dagger a_\beta \right]_0 \int_{[0,\infty)^m} \int_{[0,2\pi]^m} r_1^{\alpha_1+\beta_1} \dots r_m^{\alpha_m+\beta_m} \cdot e^{i(\beta_1-\alpha_1)} \dots e^{i(\beta_m-\alpha_m)} e^{-r_1^2-\dots-r_m^2} r_1 \dots r_m d\underline{\varphi} d\underline{r}$$

The integral $\int \int \dots d\underline{\varphi} d\underline{r}$ is 0 if $\alpha_j \neq \beta_j$ for any $j = 1, \dots, m$. So, we get with $\int_0^\infty r^{2n+1} e^{-r^2} dr = \frac{n}{2}$,

$$\begin{aligned} \| P_s(\underline{z}) \|_{\mathcal{F}^2}^2 &= \frac{1}{\pi^m} \sum_{|\alpha|=s} \left[a_\alpha^\dagger a_\alpha \right]_0 (2\pi)^m \int_{[0,\infty)^m} r_1^{2\alpha_1+1} \dots r_m^{2\alpha_m+1} e^{-r_1^2-\dots-r_m^2} d\underline{r} \\ &= 2^m \sum_{|\alpha|=s} |a_\alpha|_0^2 \prod_{j=1}^m \frac{\alpha_j!}{2} \\ &= \sum_{|\alpha|=s} |a_\alpha|_0^2 \alpha! \end{aligned}$$

2. We use the generalization of the Binomial theorem,

$$|\underline{z}|_0^{2s} = \left(|z_1|^2 + \dots + |z_m|^2 \right)^s = \sum_{|\alpha|=s} \frac{s!}{\alpha!} |\underline{z}|^{2\alpha}, \tag{6.3}$$

and Cauchy-Schwartz (CS) to get

$$\begin{aligned} \left| P_s(\underline{z}) \right|_0^2 &= \left| \sum_{|\alpha|=s} a_\alpha \underline{z}^\alpha \right|_0^2 \\ &\leq \left(\sum_{|\alpha|=s} \left| a_\alpha \underline{z}^\alpha \right|_0 \right)^2 = \left(\sum_{|\alpha|=s} \sqrt{\frac{\alpha!}{s!}} |a_\alpha|_0 \sqrt{\frac{s!}{\alpha!}} \left| \underline{z}^\alpha \right| \right)^2 \\ &\overset{CS}{\leq} \left(\frac{1}{s!} \sum_{|\alpha|=s} \alpha! \, |a_\alpha|_0^2 \right) \left(\sum_{|\alpha|=s} \frac{s!}{\alpha!} \left| \underline{z}^\alpha \right|^2 \right) \\ &\overset{(i),(6.3)}{=} \frac{1}{s!} \| P_s(\underline{z}) \|_{\mathcal{F}^2}^2 \, |\underline{z}|_0^{2s}. \end{aligned}$$

□

Lemma 6.2 *Let $\Psi_{l,k,j}$ be defined as in (6.2). Then,*

$$\sup_{\underline{z}\in K}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))} |\Psi_{l,k,j}(\underline{z})|_0^2 < \infty$$

for any compact set $K \subset \mathbb{C}^m$.

Proof Let SUP $= \sup_{\underline{z}\in K}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))} |\Psi_{l,k,j}(\underline{z})|_0^2$. We first note that each $\Psi_{l,k,j}(\underline{z}) = \frac{1}{\sqrt{\gamma_{l,k}}}\underline{z}^l P_k^{(j)}(\underline{z})$ is a homogeneous $\mathcal{C}\ell_m^{\mathbb{C}}$-polyomial of degree $l+k$. Hence, with Lemma 6.1(ii), we get

$$\text{SUP} \leq \sup_{\underline{z}\in K}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))} \frac{1}{(l+k)!}\left\|\Psi_{l,k,j}(\underline{z})\right\|_{\mathcal{F}^2}^2 \left|\underline{z}\right|_0^{2l+2k}.$$

We know that $\Psi_{l,k,j}(\underline{z}) = \left(\mathcal{B}\phi_{l,k,j}\right)(\underline{z})$ (cf. Theorem 5.3 and the proof of Corollary 5.6) and that

$$\|\mathcal{B}f\|_{\mathcal{F}^2}^2 = \frac{1}{\sqrt{2\pi}^m}\|f\|_{L^2}^2$$

for all $f \in L^2$ (cf. Corollary 5.2). Hence,

$$\left\|\Psi_{l,k,j}\right\|_{\mathcal{F}^2}^2 = \frac{1}{\sqrt{2\pi}^m}\left\|\phi_{l,k,j}\right\|_{L^2}^2 = \frac{1}{\sqrt{2\pi}^m}.$$

We also know that $\dim(M_l^+(k)) = \binom{m+k-2}{k}$, cf. Sect. 2.3. Together, we get

$$\begin{aligned}\text{SUP} &\leq \frac{1}{\sqrt{2\pi}^m}\sup_{\underline{z}\in K}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+k-2}{k}\frac{1}{(l+k)!}\left|\underline{z}\right|_0^{2l+2k} \\ &\leq \frac{1}{\sqrt{2\pi}^m}\sup_{\underline{z}\in K}\left(\sum_{l=0}^{\infty}\frac{1}{l!}\left|\underline{z}\right|_0^{2l}\right)\left(\sum_{k=0}^{\infty}\binom{m+k-2}{k}\frac{1}{k!}\left|\underline{z}\right|_0^{2k}\right) \\ &= \frac{1}{\sqrt{2\pi}^m}\sup_{\underline{z}\in K}\left(\sum_{l=0}^{\infty}\frac{1}{l!}\left|\underline{z}\right|_0^{2l}\right)\left(\sum_{k=0}^{\infty}\binom{m+k-2}{k}\frac{1}{2^{mk}k!}\left(2^m\left|\underline{z}\right|_0^2\right)^k\right).\end{aligned}$$

It can be shown via induction that $\binom{m+k-2}{k} \le 2^{mk}$ for all $k \in \mathbb{N}_0$. Hence,

$$\begin{aligned}\mathrm{SUP} &\le \frac{1}{\sqrt{2\pi}^m} \sup_{\underline{z}\in K} \left(\sum_{l=0}^{\infty} \frac{1}{l!} \left|\underline{z}\right|_0^{2l} \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(2^m \left|\underline{z}\right|_0^2 \right)^k \right) \\ &= \frac{1}{\sqrt{2\pi}^m} \sup_{\underline{z}\in K} e^{|\underline{z}|_0^2} \cdot e^{2^m |\underline{z}|_0^2} < \infty.\end{aligned}$$

□

We are now fully equipped to show convergence of the series expansion (6.2).

Proposition 6.3 *Let $\phi_{l,k,j}$ be defined as in (6.1) and let $\Psi_{l,k,j}$ be defined as in (6.2). Then, for each compact set $K \subset \mathbb{C}^m$,*

$$\sup_{z\in K} \left| \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \Psi_{l,k,j}(\underline{z}) \langle \phi_{l,k,j}, f \rangle_{L^2} \right|_0 < \infty.$$

Proof Let $\mathrm{SUM} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \Psi_{l,k,j}(\underline{z}) \langle \phi_{l,k,j}, f \rangle_{L^2}$. Since $|\cdot|_0$ is submultiplicative, we have

$$\begin{aligned}|\mathrm{SUM}|_0 &\le \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \left| \Psi_{l,k,j}(\underline{z}) \langle \phi_{l,k,j}, f \rangle_{L^2} \right|_0 \\ &\le \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \left| \Psi_{l,k,j}(\underline{z}) \right|_0 \cdot \left| \langle \phi_{l,k,j}, f \rangle_{L^2} \right|_0\end{aligned}$$

We now use Proposition 2.4 and $\left\| \phi_{l,k,j} \right\|_{L^2} = 1$, so

$$\begin{aligned}|\mathrm{SUM}|_0 &\le \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \left| \Psi_{l,k,j}(\underline{z}) \right|_0 2^m \left\| \phi_{l,k,j} \right\|_{L^2} \|f\|_{L^2} \\ &= 2^m \|f\|_{L^2} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \left| \Psi_{l,k,j}(\underline{z}) \right|_0 \\ &\le 2^m \|f\|_{L^2} \sqrt{ \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_l^+(k))} \left| \Psi_{l,k,j}(\underline{z}) \right|_0^2 }.\end{aligned}$$

Together with Lemma 6.2 the proof is complete. □

Proposition 6.3 shows that $\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\langle\phi_{l,k,j}, f\rangle_{L^2}$ is absolutely convergent locally uniformly in $\underline{z}\in\mathbb{C}^m$. Since $\mathcal{B}f$ is the uniform limit of the triple sum on every compact subset of $\mathbb{C}^m$, it is well-defined and $\mathcal{B}$ can be represented as

$$\begin{aligned}
\big(\mathcal{B}f\big)(\underline{z}) &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\langle\phi_{l,k,j}, f\rangle_{L^2}\\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\int_{\mathbb{R}^m}\overline{\phi_{l,k,j}(\underline{x})}f(\underline{x})d\underline{x}\\
&= \int_{\mathbb{R}^m}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\overline{\phi_{l,k,j}(\underline{x})}f(\underline{x})d\underline{x}\\
&= \left\langle\overline{\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\Psi_{l,k,j}(\underline{z})\overline{\phi_{l,k,j}}}, f\right\rangle_{L^2}\\
&= \left\langle\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\phi_{l,k,j}\overline{\Psi_{l,k,j}(\underline{z})}, f\right\rangle_{L^2}.
\end{aligned}$$

Thus,

$$T(\underline{x},\underline{z}) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_l^+(k))}\phi_{l,k,j}(\underline{x})\overline{\Psi_{l,k,j}}(\underline{z}) = \frac{1}{\sqrt{2\pi}^m}e^{-\frac{\underline{z}\cdot\underline{z}}{2}+\underline{x}\cdot\underline{z}-\frac{\underline{x}\cdot\underline{x}}{4}}$$

is the kernel of the Segal–Bargmann transform on $L^2(\mathbb{R}^m, \mathcal{C}\ell_m^{\mathbb{R}})$.

Conflict of Interest
The authors declare that they have no conflict of interest.

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On the Carathéodory-Fejér Interpolation Problem for Stieltjes Functions

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Abstract The following Carathéodory-Fejér problem is considered: given a point $x_0 < 0$ and numbers $c_0, c_1, \ldots, c_N \geq 0$, to construct a Stieltjes-class function f such that $f^j(x_0)/j! = c_j$ for $j = 0, 1, \ldots, N$. The cases where N is odd or even are quite different. For each case, the solvability criterion is given along with the parametrization of the solution set in the indeterminate case.

1 Introduction

Stieltjes functions appeared in [30] as continued fractions of certain type and as Stieltjes transforms of positive measures on $\mathbb{R}^+ = [0, \infty)$. Being special instances of absolutely monotone, operator monotone, and Pick functions, they have been extensively studied in various contexts [2, 18–21, 28, 32].

We denote by $\mathcal{P}$ the *Pick class* of functions analytic and with nonnegative imaginary part in the upper half-plane $\mathbb{C}^+ = \{z : \Im z > 0\}$ and recall that any such function admits the Nevanlinna–Herglotz representation

$$f(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\mu(t), \qquad \int_{\mathbb{R}} \frac{d\mu(t)}{1 + t^2} < \infty, \tag{1.1}$$

with (uniquely defined by f) $\alpha \in \mathbb{R}$, $\beta \geq 0$ and a positive measure μ subject to the growth condition as above, and that $f \in \mathcal{P}$ if and only if the associated kernel

$$K_f(z, \zeta) := \frac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} \quad \text{is positive definite } (K_f \succeq 0) \text{ on } \mathbb{C}^+.$$

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The *Stieltjes class* $\mathcal{S}$ is defined as the set of all functions $f \in \mathcal{P}$ that have an analytic continuation to $\mathbb{C}\backslash\mathbb{R}^+$ and such that $f(x) \geq 0$ for $x < 0$. Since $\mathcal{S} \subset \mathcal{P}$, any $f \in \mathcal{S}$ is of the form (1.1); the extra condition $f|_{\mathbb{R}^-} \geq 0$ turns out to be equivalent to

$$\operatorname{supp}(\mu) \subset \mathbb{R}_+, \quad \beta = 0 \quad \text{and} \quad \gamma := \alpha - \int_{\mathbb{R}_+} \frac{d\mu(t)}{t+1} \geq 0, \tag{1.2}$$

which eventually (we refer to [17, 21] for details) leads to the following two characterizations of the class $\mathcal{S}$. The first identifies Stieltjes functions vanishing at infinity (i.e., with $\gamma = 0$) with Stieltjes transforms of positive measures on $\mathbb{R}^+$, while the second one characterizes the class $\mathcal{S}$ in terms of two different (although closely related) positive kernels.

Proposition 1.1 *A function f belongs to $\mathcal{S}$ if and only if it is of the form*

$$f(z) = \gamma + \int_{\mathbb{R}^+} \frac{d\mu(t)}{t-z}, \quad \textit{where} \quad \gamma \geq 0, \quad \int_{\mathbb{R}^+} \frac{d\mu(t)}{1+t} < \infty. \tag{1.3}$$

Proposition 1.2 *$f \in \mathcal{S}$ if and only if $f \in \mathcal{P}$ and $zf \in \mathcal{P}$, i.e., the kernels*

$$K_f(z,\zeta) = \frac{f(z) - \overline{f(\zeta)}}{z - \overline{\zeta}} \quad \textit{and} \quad \widetilde{K}_f(z,\zeta) = \frac{zf(z) - \overline{\zeta f(\zeta)}}{z - \overline{\zeta}} \tag{1.4}$$

are positive definite on $\mathbb{C}^+$.

The remarkable "if" part in the last characterization is due to M. Krein. The "only if" part follows from Proposition 1.1: for f of the form (1.3), the kernels (1.4) can be written as

$$K_f(z,\zeta) = \int_{\mathbb{R}^+} \frac{d\mu(t)}{(t-z)(t-\overline{\zeta})}, \quad \widetilde{K}_f(z,\zeta) = \gamma + \int_{\mathbb{R}^+} \frac{t\, d\mu(t)}{(t-z)(t-\overline{\zeta})}, \tag{1.5}$$

so that their positivity not only on $\mathbb{C}^+$, but on the whole $\mathbb{C}\backslash\mathbb{R}^+$ is clear, once we observe that $f(\overline{z}) = \overline{f(z)}$, by the reflection principle and define, by continuity,

$$K_f(z,\overline{z}) = f'(z) \quad \text{and} \quad \widetilde{K}_f(z,\overline{z}) = zf'(z) + f(z) \quad (z \in \mathbb{C}\backslash\mathbb{R}^+).$$

Restrictions of Stieltjes functions to the negative half-axis $\mathbb{R}^- = (-\infty, 0)$ are characterized by the integral representation (1.3) restricted to $\mathbb{R}^-$. Several intrinsic real-valued characterizations of the class $\mathcal{S}|_{\mathbb{R}^-}$ are collected in Proposition 1.3 below. We recall that a function $f : \mathbb{R}^- \to \mathbb{R}$ is called *operator-monotone* on $\mathbb{R}^-$ if $f(A) \preceq f(B)$ for any two real negative definite matrices $A \preceq B \prec 0$, and it is called *absolutely monotone* if $f^{(n)}(x) \geq 0$ for all $n \geq 0$ and $x < 0$.

Proposition 1.3 *Given* $f : (-\infty, 0) \to \mathbb{R}$*, the following are equivalent:*

1. f *is the restriction of a Stieltjes function to* $\mathbb{R}^-$*, i.e., it is of the form*

$$f(x) = \gamma + \int_{\mathbb{R}^+} \frac{d\mu(t)}{t - x} \quad (x < 0), \text{ where } \gamma \text{ and } \mu \text{ are as in (1.3)}. \tag{1.6}$$

2. f *is a nonnegative operator-monotone function on* $\mathbb{R}^-$*.*
3. *the functions* f *and* xf *are operator-monotone on* $\mathbb{R}^-$*.*
4. $f \in C^1$ *and the kernels (1.4) are positive definite on* $\mathbb{R}^-$*.*
5. $f \in C^\infty$ *and for each* $x < 0$*, the Hankel matrices*

$$H_n^f(x) = \left[\frac{f^{(i+j+1)}(x)}{(i+j+1)!}\right]_{i,j=0}^{n-1}, \quad \widetilde{H}_n^f(x) = \left[\frac{xf^{(i+j+1)}(x)}{(i+j+1)!} + \frac{f^{(i+j)}(x)}{(i+j)!}\right]_{i,j=0}^{n-1} \tag{1.7}$$

are positive semidefinite for all $n \geq 1$*.*

6. $f \in C^\infty$ *and* $(x^k f(x))^{(n+k)} \geq 0$ *for all* $x < 0$ *and* $k, n \geq 0$*; equivalently,*

$$\sum_{j=0}^{n} \binom{n}{j} \frac{x^j f^{(k+j)}(x)}{(k+j)!} \geq 0 \quad \text{for all} \quad k, n \geq 0. \tag{1.8}$$

7. $f \in C^\infty$*,* $f(x) \geq 0$ *and* $(x^k f(x))^{(2k-1)} \geq 0$ *for all* $x < 0$ *and* $k \geq 1$*.*

The equivalence (3)⇔(4) is due to Löwner [24], while (3)⇔(5) was established by Dobsch [11]. The implication (2)⇒(1) is essentially due to Löwner: if f is operator monotone on $\mathbb{R}^-$, it admits the Nevanlinna–Herglotz representation (1.1) with $\text{supp}(\mu) \subset \mathbb{R}^+$, and the condition $f(x) \geq 0$ implies the two other relations in (1.2) leading to the representation of f as in part (1). We next observe that the implication (1)⇒(4) follows from representations (1.5) restricted to $z, \zeta \in \mathbb{R}^-$. Furthermore, if f and xf are both operator-monotone on $\mathbb{R}^-$, then their derivatives are nonnegative on $\mathbb{R}^-$ and therefore, $f(x) = (xf(x))' - xf'(x) \geq 0$ for all $x < 0$, which justifies (3)⇒(2). The equivalences (1)⇔(6)⇔(7) are due to Widder [32, Theorem 12.5]; see also [7, 29] for related results.

Note that the equivalence (1)⇔(4) is the real-valued analog of Proposition 1.2. Inequalities in part (6) mean that the functions $(x^k f)^{(k)}$ are absolutely monotone for all $k \geq 0$, thus characterizing the class $\mathcal{S}$ within the class of absolutely monotone functions. Part (7) shows that seemingly weaker conditions still guarantee an absolutely monotone function to be in the class $\mathcal{S}$.

Note that the requirement "for all $x < 0$" in part (7) is essential: for example, the function $f(x) = -\ln(-x)$ does not belong to the Stieltjes class, although it satisfies conditions $f(x) \geq 0$ and $(x^k f(x))^{(2k-1)} \geq 0$ for $k \geq 1$ at each point $x \in [-\frac{1}{e}, 0)$, as is readily seen from the formulas

$$(xf(x))' = -\ln(-x) - 1, \quad (x^k f(x)^{(2k-1)} = (-1)^{k-1}\frac{k!(k-2)!}{x^{k-1}} \quad \text{for } k \geq 2.$$

On the other hand, conditions (1.7) and (1.8) can be (equivalently) restricted to a single point $x_0 < 0$ providing two characterizations of Stieltjes-class functions in terms of their Taylor coefficients.

Proposition 1.4 *Given a point $x_0 < 0$ and a sequence $\{c_k\}_{k\geq 0}$, the power series*

$$f(x) = \sum_{j\geq 0} c_j(x - x_0)^j \tag{1.9}$$

extends to a Stieltjes function if and only if the matrices

$$H_n := \left[c_{i+j+1}\right]_{i,j=0}^{n-1} \succeq 0 \quad \textit{and} \quad \widetilde{H}_n := \left[x_0 c_{i+j+1} + c_{i+j}\right]_{i,j=0}^{n-1} \succeq 0 \tag{1.10}$$

are positive semidefinite for all $n \geq 1$, or equivalently,

$$\sum_{j=0}^{m} \binom{m}{j} c_{k+j} x_0^j \geq 0 \quad \textit{for all} \quad k, m \geq 0. \tag{1.11}$$

The "only if" part follows from Proposition 1.3. For the "if" part, we refer to Bendat-Sherman results [3], according to which inequalities (1.10) guarantee that f of the form (1.7) and the associated function xf are operator-monotone on $(2x_0, 0)$ and hence, extend to Pick functions. Therefore $f \in \mathcal{S}$, by Proposition 1.2. Both (1.10) and (1.11) are equivalent to the existence of a measure $d\sigma \geq 0$ such that

$$c_k = \int_{[0, -\frac{1}{x_0}]} t^k d\sigma(t) \quad \text{for all} \quad k \geq 0, \tag{1.12}$$

and hence, they are equivalent to each other. For $x_0 = -1$, inequalities (1.11) mean that the sequence $\{c_k\}$ is *completely monotone*; the characterization of such sequences in terms of a power moment problem over the interval $[0, 1]$ is due to Hausdorff [15]; we refer to [21] for the general case.

Remark 1.5 Although conditions (1.10) (for all $n \geq 1$) and (1.11) are equivalent, their truncations are not. More precisely,

(1) If (1.10) hold for some n, then inequalities (1.11) hold for all $k + m \leq 2n + 1$.
(2) For any $x_0 < 0$ and $N \geq 1$, there exist $c_0, \ldots, c_N \in \mathbb{R}$ such that inequalities (1.11) hold for all $k + m \leq N$, but $H_2 = \left[\begin{smallmatrix} c_1 & c_2 \\ c_2 & c_3 \end{smallmatrix}\right] \not\succeq 0$.

For part (2), we take (without loss of generality) $x_0 = -1$ and let $c_j = \binom{2N-j}{N}$ for $j = 0, \ldots, N$. Part (1) follows from Theorem 1.6 below and Proposition 1.4. The proof goes through the Stieltjes-class *Carathéodory-Fejér problem* $\mathbf{CFS}_N$ which we now formally introduce:

$\mathbf{CFS}_N$: *Given* $x_0 < 0$ *and* $c_0, \ldots, c_N \in \mathbb{R}$, *find* $f \in \mathcal{S}$ *(if exists) of the form* (1.9), *i.e., such that*

$$\frac{f^{(k)}(x_0)}{k!} = c_k \quad \text{for} \quad k = 0, \ldots, N. \tag{1.13}$$

This problem was originally studied in [8, 9, 27, 31] for functions analytic on the open unit disk $\mathbb{D}$ and bounded by one in modulus (Schur-class functions) or with nonnegative real part (Carathéodory-class functions). The classical Stieltjes moment problem [21, 30] can be viewed as the boundary Carathéodory-Fejér problem at $x_0 = -\infty$. Another boundary case $x_0 = 0$ appears naturally in the context of the *strong Stieltjes moment problem* [16, 19, 26]. The problem (1.13) with $x_0 \notin \mathbb{R}$ has appeared as a particular example in several general interpolation schemes [1, 5, 6, 12]. The problem (1.13) with $N = 2n - 1$ has been specifically addressed in [4, 14].

Theorem 1.6 *The problem* $\mathbf{CFS}_{2n-1}$ *has a solution if and only if conditions* (1.10) *hold. The solution is unique if and only if* H_n *or* $\widetilde{H}_n$ *is singular.*

Along with the matrices H_n and $\widetilde{H}_n$, we may consider the Hankel matrix

$$P_n = \left[c_{i+j}\right]_{i,j=0}^{n-1} = \widetilde{H}_n - x_0 H_n, \tag{1.14}$$

which is positive definite, if at least one of H_n and $\widetilde{H}_n$ is. In the case of the odd problem $\mathbf{CFS}_{2n}$ with given $c_0, \ldots, c_{2n-1}, c_{2n}$, we can define not only H_n and $\widetilde{H}_n$ by formulas (1.10), but also the matrix $P_{n+1} = \left[c_{i+j}\right]_{i,j=0}^{n}$, and all three matrices H_n, $\widetilde{H}_n$, P_{n+1} being positive semidefinite is necessary for the problem $\mathbf{CFS}_{2n}$ to have a solution. If either H_n or $\widetilde{H}_n$ (or both) are singular, the only solution $\mathbf{f}$ of the truncated problem $\mathbf{CFS}_{2n-1}$ is or is not a solution of the problem $\mathbf{CFS}_{2n}$ depending on whether or not $\frac{\mathbf{f}^{(2n)}(x_0)}{(2n)!} = c_{2n}$. The rest is covered below.

Theorem 1.7 *Let us assume that* $H_n \succ 0$, $\widetilde{H}_n \succ 0$, $P_{n+1} \succeq 0$. *The problem* $\mathbf{CFS}_{2n}$ *is indeterminate if and only if* $P_{n+1} \succ 0$ *and* $\left[P_{n+1}^{-1}\right]_{11} > \left[\widetilde{H}_n^{-1}\right]_{11}$, *and it has a unique solution if and only if either* $\operatorname{rank} P_{n+1} = n$ *or* $P_{n+1} \succ 0$ *and* $\left[P_{n+1}^{-1}\right]_{11} = \left[\widetilde{H}_n^{-1}\right]_{11}$.

The main results concerning the problem $\mathbf{CFS}_{2n}$ are recalled in Sect. 2 along with the parametrization of the solution set in the indeterminate case. Parallel results concerning the problem $\mathbf{CFS}_{2n}$ are presented in Sect. 3, where we also discuss linear fractional transformations mapping the Stieltjes class into itself. In Sect. 4, we identify unique solutions of determinate problems $\mathbf{CFS}_N$ as extremal solutions of certain truncated Carathéodory-Fejér problems.

2 The Even Case

In this section we present the results concerning the problem $\mathbf{CFS}_{2n-1}$. We first observe that Hankel matrices H_n and $\widetilde{H}_n$ defined in (1.10) satisfy the identity

$$\widetilde{H}_n - H_n T^* = \mathbf{c}\mathbf{e}^*. \tag{2.1}$$

where $T \in \mathbb{R}^{n\times n}$ and $\mathbf{c}, \mathbf{e} \in \mathbb{R}^n$ are given by

$$T = \begin{bmatrix} x_0 & 0 & \dots & 0 \\ 1 & x_0 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & x_0 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}. \tag{2.2}$$

In the rest of the paper, all functions are real-valued and all matrices are real; by A^* we will mean the transpose of a real matrix A.

Theorem 2.1 *Let us assume that H_n and $\widetilde{H}_n$ are positive definite. Then all solutions f to the problem $\mathbf{CFS}_{2n-1}$ are given by the formula*

$$f = \mathbf{T}_\Psi[h] := \frac{\Psi_{11}h + \Psi_{12}}{\Psi_{21}h + \Psi_{22}}, \qquad \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad h \in \overline{\mathcal{S}} \tag{2.3}$$

where $\overline{\mathcal{S}} := \mathcal{S} \cup \{\infty\}$ and where Ψ_{ij} are rational functions defined by

$$\Psi_{11}(x) = 1 + \mathbf{c}^*(xI - T^*)^{-1} T^* \widetilde{H}_n^{-1}\mathbf{e}, \tag{2.4}$$

$$\Psi_{12}(x) = -\mathbf{c}^*(xI - T^*)^{-1} H_n^{-1}\mathbf{c}, \tag{2.5}$$

$$\Psi_{21}(x) = x\mathbf{e}^*(xI - T^*)^{-1} \widetilde{H}_n^{-1}\mathbf{e}, \tag{2.6}$$

$$\Psi_{22}(x) = 1 - \mathbf{e}^*(xI - T^*)^{-1} H_n^{-1}\mathbf{c}. \tag{2.7}$$

Although the result is known, we present a proof emphasizing details needed later for the odd case $\mathbf{CFS}_{2n}$. The proof consists of four parts.

Step 1: *For any $h \in \mathcal{S}$ and $x < 0$, we have $\varphi(x) := \Psi_{21}(x)h(x) + \Psi_{22}(x) \neq 0$, and hence the formula* (2.3) *makes sense.*

Proof Upon applying the well-known inversion formula

$$(I - XZ^{-1}Y)^{-1} = I + X(Z - YX)^{-1}Y \tag{2.8}$$

(I stand for an identity matrix) to (2.7) and making use of identity (2.1) we get

$$\begin{aligned}\Psi_{22}(x)^{-1} &= (1 - \mathbf{e}^*(xI - T^*)^{-1}H_n^{-1}\mathbf{c})^{-1}\\ &= 1 + \mathbf{e}^*\left[H_n(xI - T^*) - \mathbf{c}\mathbf{e}^*\right]^{-1}\mathbf{c} = 1 - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}.\end{aligned} \tag{2.9}$$

We next combine (2.9) with (2.6) and again use (2.1) to get

$$\begin{aligned}\Psi_{22}(x)^{-1}\Psi_{21}(x) &= x\big(1 - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}\big)\mathbf{e}^*(xI - T^*)^{-1}\widetilde{H}_n^{-1}\mathbf{e}\\ &= x\mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\big(\widetilde{H}_n - xH_n - \mathbf{c}\mathbf{e}^*\big)(xI - T^*)^{-1}\widetilde{H}_n^{-1}\mathbf{e}\\ &= x\mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\big(H_nT^* - xH_n\big)(xI - T^*)^{-1}\widetilde{H}_n^{-1}\mathbf{e}\\ &= -x\mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}H_n\widetilde{H}_n^{-1}\mathbf{e}\\ &= \mathbf{e}^*\widetilde{H}_n^{-1}\mathbf{e} - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{e}.\end{aligned} \tag{2.10}$$

If $x < 0$, then $\widetilde{H}_n - xH_n \succ \widetilde{H}_n \succ 0$, and we see from (2.9) and (2.10) that $\Psi_{22}(x) \neq 0$ and $\frac{\Psi_{21}(x)}{\Psi_{22}(x)} > 0$. Therefore, $\varphi(x) = \Psi_{22}(x)\left(\frac{\Psi_{21}(x)}{\Psi_{22}(x)}h(x) + 1\right) \neq 0$ for any $h \in \mathcal{S}$. □

Let the matrix J and the matrix-function $J_{x,y}$ be defined as

$$J_{x,y} := \begin{bmatrix} 0 & x^{-1} \\ -y^{-1} & 0 \end{bmatrix}, \quad J = J_{1,1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.11}$$

Step 2: *The function $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$ is J-unitary on $\mathbb{R}\backslash\{x_0\}$, while the kernels*

$$K_{\Psi,J}(x,y) = \frac{\Psi(x)J\Psi(y)^* - J}{y - x}, \quad \widetilde{K}_{\Psi,J}(x,y) = \frac{\Psi(x)J_{x,y}\Psi(y)^* - J_{x,y}}{y - x} \tag{2.12}$$

extended by continuity to $x = y$ by $K_{\Psi,J}(x,x) = -\Psi'(x)J\Psi(x)^*$ *and*

$$\widetilde{K}_{\Psi,J}(x,x) = -\frac{1}{x}\Psi'(x)J\Psi(x)^* + \frac{1}{x^2}\begin{bmatrix} 0 & -\Psi_{12}(x) \\ \Psi_{21}(x) & 0 \end{bmatrix}J\Psi(x)^*,$$

are positive definite on $\mathbb{R}\backslash\{x_0\}$.

Proof All statements follow from the identities

$$\Psi(x)J\Psi(y)^* = J + (y-x)\begin{bmatrix}\mathbf{c}^*\\ \mathbf{e}^*\end{bmatrix}(xI-T^*)^{-1}H_n^{-1}(yI-T)^{-1}\begin{bmatrix}\mathbf{c} & \mathbf{e}\end{bmatrix}, \tag{2.13}$$

$$\Psi(x)J_{x,y}\Psi(y)^* = J_{x,y} + (y-x)\begin{bmatrix}\frac{1}{x}\mathbf{c}^*T^*\\ \mathbf{e}^*\end{bmatrix}(xI-T^*)^{-1}\widetilde{H}_n^{-1}(yI-T)^{-1}\begin{bmatrix}\frac{1}{y}T\mathbf{c} & \mathbf{e}\end{bmatrix}.$$

holding for all $x, y \in \mathbb{R}\backslash\{x_0\}$ and verified entry-wise by straightforward computations relying (as in Step 1) solely on the identity (2.1) and explicit formulas (2.4)–(2.7). Letting $y = x$ in (2.13) we conclude that Ψ is J-unitary on $\mathbb{R}\backslash\{x_0\}$. Therefore,

$$\Psi(x)^{-1} = -J\Psi(x)^*J = \begin{bmatrix}\Psi_{22}(x) & -\Psi_{12}(x)\\ -\Psi_{21}(x) & \Psi_{11}(x)\end{bmatrix}, \tag{2.14}$$

and it is clear from (2.14) that $\det\Psi(x) = 1$ for all $x \in \mathbb{R}\backslash\{x_0\}$. Identities (2.13) provide explicit formulas for the kernels (2.12) which show in particular, since $H_n \succ 0$ and $\widetilde{H}_n \succ 0$, that the latter kernels are positive definite on $\mathbb{R}\backslash\{x_0\}$ and the rank of each kernel equals n. □

Remark 2.2 Upon making use of equalities $J^2 = J^* = -I_2$ and (2.14) it is easy to verify that

$$\Psi(y)^{-*}J\Psi(x)^{-1} = -J\Psi(y)J\Psi(x)^*J.$$

Combining the latter equality with (2.13) (with x and y switched) we arrive at the identities

$$\frac{\Psi(y)^{-*}J\Psi(x)^{-1}}{y-x} = \frac{J}{y-x} - \begin{bmatrix}\mathbf{e}^*\\ -\mathbf{c}^*\end{bmatrix}(yI-T^*)^{-1}H_n^{-1}(xI-T)^{-1}\begin{bmatrix}\mathbf{e} & -\mathbf{c}\end{bmatrix}, \tag{2.15}$$

$$\frac{\Psi(y)^{-*}J_{y,x}\Psi(x)^{-1}}{y-x} = \frac{J_{y,x}}{y-x} - \begin{bmatrix}y\mathbf{e}^*\\ -\mathbf{c}^*T^*\end{bmatrix}(yI-T^*)^{-1}\widetilde{H}_n^{-1}(xI-T)^{-1}\begin{bmatrix}x\mathbf{e} & -T\mathbf{c}\end{bmatrix},$$

which will play a crucial role in Step 4 below.

Step 3: *For any $h \in \overline{\mathcal{S}}$, the function $f = \mathbf{T}_\Psi[h]$ belongs to $\mathcal{S}$.*

Proof For a fixed $h \in \mathcal{S}$, let $\varphi = \Psi_{21}h + \Psi_{22}$ and $f = \mathbf{T}_\Psi[h]$. Then the kernels K_f, $\widetilde{K}_f$ given in (1.4) and similar kernels K_h, $\widetilde{K}_h$ associated with h are related as follows:

$$K_f(x,y) = \frac{K_h(x,y)}{\varphi(x)\varphi(y)} + \begin{bmatrix}1 & -f(x)\end{bmatrix}K_{\Psi,J}(x,y)\begin{bmatrix}1\\ -f(y)\end{bmatrix}, \tag{2.16}$$

$$\widetilde{K}_f(x,y) = \frac{\widetilde{K}_h(x,y)}{\varphi(x)\varphi(y)} + \begin{bmatrix}1 & -f(x)\end{bmatrix}\widetilde{K}_{\Psi,J}(x,y)\begin{bmatrix}1\\ -f(y)\end{bmatrix}. \tag{2.17}$$

Indeed, we see from (2.14) and definitions of φ and f, that

$$\begin{bmatrix} 1 & -h \end{bmatrix} \Psi^{-1} = \varphi \begin{bmatrix} 1 & -f \end{bmatrix}. \tag{2.18}$$

Then we compute

$$\begin{aligned} K_f(x,y) &= \begin{bmatrix} 1 & -f(x) \end{bmatrix} \frac{J}{x-y} \begin{bmatrix} 1 \\ -f(y) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -f(x) \end{bmatrix} K_{\Psi,J}(x,y) \begin{bmatrix} 1 \\ -f(y) \end{bmatrix} + \begin{bmatrix} 1 & -f(x) \end{bmatrix} \frac{\Psi(x)J\Psi(y)^*}{x-y} \begin{bmatrix} 1 \\ -f(y) \end{bmatrix}, \end{aligned}$$

and note that the second term on the right equals, on account of (2.18), to

$$\varphi(x)^{-1} \begin{bmatrix} 1 & -h(x) \end{bmatrix} \frac{J}{x-y} \begin{bmatrix} 1 \\ -h(y) \end{bmatrix} \varphi(y)^{-1} = \frac{K_h(x,y)}{\varphi(x)\varphi(y)},$$

which justifies (2.16). The equality (2.17) is verified similarly.

Since $h \in \mathcal{S}$, the kernels K_h and $\widetilde{K}_h$ are positive on $\mathbb{R}^-$. Since the kernels $K_{\Psi,J}$ and $\widetilde{K}_{\Psi,J}$ are positive on $\mathbb{R}\backslash\{x_0\}$ and φ is continuous and non-vanishing on $\mathbb{R}^-\backslash\{x_0\}$, it follows from (2.16) and (2.17) that the kernels K_f and $\widetilde{K}_f$ are positive definite on $\mathbb{R}^-\backslash\{x_0\}$ and hence (by the Chandler's result [10]), they are positive on the whole $\mathbb{R}^-$. Therefore, $f \in \mathcal{S}$, by Proposition 1.3.

Finally, the function $f_K = \mathbf{T}_\Psi[\infty] = \frac{\Psi_{11}}{\Psi_{21}}$ belongs to $\mathcal{S}$, due to relations

$$\begin{aligned} K_{f_K}(x,y) &= \begin{bmatrix} 1 & -f_K(x) \end{bmatrix} K_{\Psi,J}(x,y) \begin{bmatrix} 1 \\ -f_K(y) \end{bmatrix} \succeq 0, \\ \widetilde{K}_{f_K}(x,y) &= \begin{bmatrix} 1 & -f_K(x) \end{bmatrix} \widetilde{K}_{\Psi,J}(x,y) \begin{bmatrix} 1 \\ -f_K(y) \end{bmatrix} \succeq 0, \end{aligned} \tag{2.19}$$

which are easily verified. □

Step 4: *Any $f \in \mathcal{S}$ satisfying conditions* (1.13) *is of the form* (2.3) *for some $h \in \overline{\mathcal{S}}$.*

We will handle this part using the intermediate step characterizing solutions of an interpolation problem in terms of "extended" positive kernels. This approach goes back to Potapov's method of *Fundamental Matrix Inequalities* which in the Stieltjes-class context first appeared in the series of papers [13].

Lemma 2.3 *If $f \in \mathcal{S}$ satisfies conditions* (1.13) *then the kernels*

$$\begin{bmatrix} H_n & (xI-T)^{-1}(\mathbf{e}f(x)-\mathbf{c}) \\ (f(y)\mathbf{e}^*-\mathbf{c}^*)(yI-T^*)^{-1} & K_f(x,y) \end{bmatrix} \succeq 0, \tag{2.20}$$

$$\begin{bmatrix} \widetilde{H}_n & (xI-T)^{-1}(x\mathbf{e}f(x)-T\mathbf{c}) \\ (yf(y)\mathbf{e}^*-\mathbf{c}^*T^*)(yI-T^*)^{-1} & \widetilde{K}_f(x,y) \end{bmatrix} \succeq 0, \tag{2.21}$$

(where K_f, $\widetilde{K}_f$, T, $\mathbf{e}$, $\mathbf{c}$ are defined in (1.4), (2.2)) are positive on $\mathbb{R}_-$. If moreover, H_n and $\widetilde{H}_n$ are positive definite, then f is of the form (2.3) for some $h \in \overline{\mathcal{S}}$.

Proof If $f \in \mathcal{S}$ satisfies conditions (1.13), then it admits the Herglotz representation (1.6) in terms of which conditions (1.13) can be written as

$$\gamma + \int_{\mathbb{R}_+} \frac{d\mu(t)}{t - x_0} = f_0, \quad \int_{\mathbb{R}_+} \frac{d\mu(t)}{(t - x_0)^{k+1}} = f_k \quad (k = 1, \dots, 2n - 1). \tag{2.22}$$

We next combine (2.22) with the equality

$$(xI - T)^{-1}\mathbf{e} = \begin{bmatrix} \frac{1}{x-x_0} & 0 & \dots & 0 \\ \frac{1}{(x-x_0)^2} & \frac{1}{x-x_0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(x-x_0)^n} & \dots & \frac{1}{(x-x_0)^2} & \frac{1}{x-x_0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{x-x_0} \\ \frac{1}{(x-x_0)^2} \\ \vdots \\ \frac{1}{(x-x_0)^n} \end{bmatrix}$$

to represent H_n, $\widetilde{H}_n$ and $\mathbf{c}$ in the compact integral form as follows:

$$H_n = \int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\, d\mu(t)\, \mathbf{e}^*(tI - T^*)^{-1}, \tag{2.23}$$

$$\widetilde{H}_n = \int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\, t d\mu(t)\, \mathbf{e}^*(tI - T^*)^{-1} + \mathbf{e}\gamma\mathbf{e}^*, \tag{2.24}$$

$$\mathbf{c} = \int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\, d\mu(t) + \mathbf{e}\gamma. \tag{2.25}$$

We next consider two positive kernels defined by the integrals

$$\int_{\mathbb{R}_+} \begin{bmatrix} (tI - T)^{-1}\mathbf{e} \\ \frac{1}{t-y} \end{bmatrix} d\mu(t) \begin{bmatrix} \mathbf{e}^*(tI - T^*)^{-1} & \frac{1}{t-x} \end{bmatrix} \succeq 0, \tag{2.26}$$

$$\begin{bmatrix} \mathbf{e} \\ 1 \end{bmatrix} \gamma \begin{bmatrix} \mathbf{e}^* & 1 \end{bmatrix} + \int_{\mathbb{R}_+} \begin{bmatrix} (tI - T)^{-1}\mathbf{e} \\ \frac{1}{t-y} \end{bmatrix} t d\mu(t) \begin{bmatrix} \mathbf{e}^*(tI - T^*)^{-1} & \frac{1}{t-x} \end{bmatrix} \succeq 0 \tag{2.27}$$

and show that these kernels coincide with those in (2.20) and (2.21). The diagonal entries are indeed the same, by (2.23), (2.24) and formulas (1.5) restricted to $x, y \in \mathbb{R}^-$. Multiplying the identity

$$(t - x)^{-1}(tI - T)^{-1} = (xI - T)^{-1}\left((t - x)^{-1}I - (tI - T)^{-1}\right)$$

by $\mathbf{e}$ on the right and integrating the resulting equality against the measure μ, we get, on account of (1.6) and (2.25),

$$\int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\,\frac{d\mu(t)}{t-x} = (xI - T)^{-1}\left(\mathbf{e}f(x) - \mathbf{c}\right). \tag{2.28}$$

Representing $\frac{t}{t-x}$ as $\frac{t}{t-x} = \frac{x}{t-x} + 1$ and making use of (2.28) we get

$$\begin{aligned}\int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\,\frac{t\,d\mu(t)}{t-x} &= x(xI - T)^{-1}\left(\mathbf{e}f(x) - \mathbf{c}\right) + \int_{\mathbb{R}_+} (tI - T)^{-1}\mathbf{e}\,d\mu(t)\\ &= (xI - T)^{-1}\left(x\mathbf{e}f(x) - T\mathbf{c}\right) - \mathbf{e}\gamma.\end{aligned} \tag{2.29}$$

Equalities (2.28) and (2.29) confirm that the off-diagonal entries in kernels (2.26) and (2.27) are the same as those in (2.20) and (2.21). Since the kernels (2.26) are positive definite, the first statement of the lemma follows.

To justify the second statement, we assume that $H_n \succ 0$ and $\widetilde{H}_n \succ 0$ and observe that the Schur complements of these blocks in positive kernels (2.20), (2.21) are also positive kernels on $\mathbb{R}^-$:

$$\begin{aligned}&K_f(x,y) - \left(f(y)\mathbf{e}^* - \mathbf{c}^*\right)(yI - T^*)^{-1}H_n^{-1}(xI - T)^{-1}\left(\mathbf{e}f(x) - \mathbf{c}\right) \succeq 0,\\ &\widetilde{K}_f(x,y) - \left(yf(y)\mathbf{e}^* - \mathbf{c}^*T^*\right)(yI - T^*)^{-1}\widetilde{H}_n^{-1}(xI - T)^{-1}\left(x\mathbf{e}f(x) - T\mathbf{c}\right) \succeq 0.\end{aligned} \tag{2.30}$$

Making use of the matrices (2.11), we can rewrite inequalities (2.30) as

$$\begin{aligned}&\begin{bmatrix} f(y) & 1\end{bmatrix}\left\{\frac{J}{y-x} - \begin{bmatrix}\mathbf{e}^*\\ -\mathbf{c}^*\end{bmatrix}(yI - T^*)^{-1}H_n^{-1}(xI - T)^{-1}\begin{bmatrix}\mathbf{e} & -\mathbf{c}\end{bmatrix}\right\}\begin{bmatrix} f(x)\\ 1\end{bmatrix} \succeq 0,\\ &\begin{bmatrix} f(y) & 1\end{bmatrix}\left\{\frac{J_{y,x}}{y-x} - \begin{bmatrix}y\mathbf{e}^*\\ -\mathbf{c}^*T^*\end{bmatrix}(yI - T^*)^{-1}\widetilde{H}_n^{-1}\right.\\ &\qquad\qquad \left.\times (xI - T)^{-1}\begin{bmatrix}x\mathbf{e} & -T\mathbf{c}\end{bmatrix}\right\}\begin{bmatrix} f(x)\\ 1\end{bmatrix} \succeq 0,\end{aligned} \tag{2.31}$$

and furthermore, on account of identities (2.15), as

$$\begin{bmatrix} f(y) & 1\end{bmatrix}\frac{\Psi(y)^{-*}J\Psi(x)^{-1}}{y-x}\begin{bmatrix} f(x)\\ 1\end{bmatrix} \succeq 0 \qquad (x, y \in \mathbb{R}^-), \tag{2.32}$$

$$\begin{bmatrix} f(y) & 1\end{bmatrix}\frac{\Psi(y)^{-*}J_{y,x}\Psi(x)^{-1}}{y-x}\begin{bmatrix} f(x)\\ 1\end{bmatrix} \succeq 0 \qquad (x, y \in \mathbb{R}^-). \tag{2.33}$$

If we define differentiable functions $u, v : \mathbb{R}^- \to \mathbb{R}$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} := \Psi^{-1} \begin{bmatrix} f \\ 1 \end{bmatrix} = \begin{bmatrix} \Psi_{12} f - \Psi_{12} \\ \Psi_{11} - \Psi_{21} f \end{bmatrix}, \tag{2.34}$$

then the kernels (2.32) and (2.33) can be written in terms of u and v as

$$K_{u,v}(x, y) := \frac{u(x)v(y) - v(x)u(y)}{x - y} \succeq 0, \tag{2.35}$$

$$\widetilde{K}_{u,v}(x, y) = \frac{xu(x)v(y) - yv(x)u(y)}{x - y} \succeq 0. \tag{2.36}$$

If $v(x) \neq 0$ for all $x < 0$, then the function $h = \frac{u}{v}$ is well defined on $\mathbb{R}_-$, and it follows from (2.34) that $f = \mathbf{T}_\Psi[h]$. Furthermore, $h \in \mathcal{S}$, since the kernels

$$K_h(x, y) = \frac{K_{u,v}(x, y)}{v(x)v(y)} \quad \text{and} \quad \widetilde{K}_h(x, y) = \frac{\widetilde{K}_{u,v}(x, y)}{v(x)v(y)}$$

are positive definite on $\mathbb{R}^-$. If $v(t) = 0$ for some $t < 0$, then

$$K_{u,v}(t, t) = u'(t)v(t) - u(t)v'(t) = -u(t)v'(t) \geq 0,$$
$$\widetilde{K}_{u,v}(t, t) = (tu'(t) + u(t))v(t) - tu(t)v'(t) = -tu(t)v'(t) \geq 0.$$

Since $t < 0$ and $u(t) \neq 0$ (by (2.34)), the two latter inequalities imply $v'(t) = 0$. Then $K_{u,v}(t, t) = 0$ and, since the kernel $K_{u,v}$ is positive on $\mathbb{R}^-$, we also have $K_{u,v}(x, t) = 0$ for all $x < 0$, which implies that $v(x) = 0$ for all $x < 0$. In this case, we conclude from (2.34) that $f = \frac{\Psi_{11}}{\Psi_{21}} = f_K = \mathbf{T}_\Psi[\infty]$. □

Step 4: *For any $h \in \overline{\mathcal{S}}$, the function $f = \mathbf{T}_\Psi[h]$ satisfies interpolation conditions* (1.13) *for $k = 0, \dots, 2n - 1$.*

We first write the linear fractional formula (2.3) in the Redheffer form.

Remark 2.4 Let $H_n \succ 0$ and $\widetilde{H}_n \succ 0$ be subject to equality (2.1) and let Ψ_{ij} be defined as in (2.4)–(2.7). Then (2.3) can be written as

$$f = \frac{\Psi_{11}h + \Psi_{12}}{\Psi_{21}h + \Psi_{22}} = f_F + \frac{\Upsilon_1^2 h}{1 + \Upsilon_2 h} \tag{2.37}$$

where

$$f_F(x) = \Psi_{12}(x)\Psi_{22}(x)^{-1} = \mathbf{c}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}, \tag{2.38}$$

$$\Upsilon_1(x) = \Psi_{22}(x)^{-1} = 1 - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}, \tag{2.39}$$

$$\Upsilon_2(x) = \Psi_{22}(x)^{-1}\Psi_{21}(x) = \mathbf{e}^*\widetilde{H}_n^{-1}\mathbf{e} - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{e}, \tag{2.40}$$

Indeed, it is readily seen that

$$\frac{\Psi_{11}h + \Psi_{12}}{\Psi_{21}h + \Psi_{22}} = \frac{\Psi_{12}}{\Psi_{22}} + \frac{h \cdot \Psi_{22}^{-2} \cdot \det \Psi}{\Psi_{22}^{-1}\Psi_{21}h + 1},$$

and since $\det \Psi \equiv 1$, the latter equality justifies the representation (2.37) with f_F, Υ_1 and Υ_2 defined by the first equalities in (2.38)–(2.40). The second equalities in (2.39) and (2.40) were confirmed in (2.9) and (2.10), respectively. The second equality in (2.38) follows by combining (2.10) and (2.5) and making use of (2.1):

$$\begin{aligned}
\Psi_{12}(x)\Psi_{22}(x)^{-1} &= -\mathbf{c}^*(xI - T^*)^{-1}H_n^{-1}\mathbf{c}\big(1 - \mathbf{e}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}\big)\\
&= \mathbf{c}^*(xI - T^*)^{-1}H_n^{-1}\big(xH_n - \widetilde{H}_n + \mathbf{c}\mathbf{e}^*\big)\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}\\
&= \mathbf{c}^*(xI - T^*)^{-1}H_n^{-1}(xH_n - H_nT^*)\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c}\\
&= \mathbf{c}^*\big(\widetilde{H}_n - xH_n\big)^{-1}\mathbf{c},
\end{aligned}$$

In what follows, $\mathbf{e}_k$ will denote the k-th column of the identity matrix I_n. We also recall the Hankel matrix $P_n = \widetilde{H}_n - x_0H_n$ (see (1.14)) which is positive definite, if H_n and $\widetilde{H}_n$ are.

Lemma 2.5 *If the matrix P_n is positive definite, then*

(1) *The function f_F given by* (2.38) *admits the Taylor expansion*

$$f_F(x) = c_0 + \ldots + c_{2n-1}(x - x_0)^{2n-1} + \sum_{k=0}^{\infty} \mathbf{e}_n^* H_n(P_n^{-1}H_n)^{k+1}\mathbf{e}_n(x - x_0)^{2n+k}. \tag{2.41}$$

(2) *The function Υ_1 given by* (2.39) *admits the Taylor expansion*

$$\Upsilon_1(x) = -(x - x_0)^n\mathbf{e}_1^*P_n^{-1}H_n\mathbf{e}_n - \sum_{k=n+1}^{\infty}(x - x_0)^k\mathbf{e}_1^*(P_n^{-1}H_n)^{k-n+1}\mathbf{e}_n. \tag{2.42}$$

Furthermore, Υ_1 *has zero of multiplicity* n *at* x_0 *and does not have other zeros if and only if* $\mathbf{e}_1^*P_n^{-1}H_n\mathbf{e}_n \neq 0 \iff H_n \succ 0$. *Otherwise,* $\Upsilon_1 \equiv 0$.

(3) *The function* Υ_2 *given by* (2.40) *admits the Taylor expansion*

$$\Upsilon_2(x) = \mathbf{e}_1^*\widetilde{H}_n^{-1}\mathbf{e}_1 - \mathbf{e}_1^*P_n^{-1}\mathbf{e}_1 - \sum_{k=1}^{\infty}\mathbf{e}_1^*\big(P_n^{-1}H_n\big)^kP_n^{-1}\mathbf{e}_1(x-x_0)^k. \tag{2.43}$$

Proof Making us of the identity $\widetilde{H}_n = P_n + x_0H_n$, we compute the Taylor series expansion of the rational matrix-function $(\widetilde{H}_n - xH_n)^{-1}$ at x_0:

$$\begin{aligned}(\widetilde{H}_n - xH_n)^{-1} &= \big(P_n - (x-x_0)H_n\big)^{-1} \\ &= \big(I - (x-x_0)P_n^{-1}H_n\big)^{-1}P_n^{-1} = \sum_{k=0}^{\infty}(x-x_0)^k\big(P_n^{-1}H_n\big)^kP_n^{-1}.\end{aligned} \tag{2.44}$$

Upon multiplying both parts in (2.44) by $\mathbf{e}_1^*$ on the left and by $\mathbf{e}_1$ on the right, and substituting the outcome into (2.40) we arrive at (2.43). We next multiply both parts in (2.44) $\mathbf{c} = P_n\mathbf{e}_1$ on the right:

$$(\widetilde{H}_n - xH_n)^{-1}\mathbf{c} = \sum_{k=0}^{\infty}(x-x_0)^k\big(P_n^{-1}H_n\big)^k\mathbf{e}_1. \tag{2.45}$$

Since the k-th column of H_n is equal to the $(k+1)$-th column of P_n, we have

$$P_n^{-1}H_n\mathbf{e}_k = \mathbf{e}_{k+1} \quad \text{for} \quad k = 1, \dots, n-1, \tag{2.46}$$

from which we recursively get

$$(P_n^{-1}H_n)^{k-1}\mathbf{e}_1 = \mathbf{e}_k \quad \text{for} \quad k = 1, \dots, n. \tag{2.47}$$

Multiplying the latter equalities and (2.45) by $\mathbf{e}_1^*$ on the left and substituting the resulting equalities into (2.39) we get

$$\begin{aligned}\Upsilon_1(x) &= 1 - \sum_{k=0}^{\infty}(x-x_0)^k\mathbf{e}_1^*\big(P_n^{-1}H_n\big)^k\mathbf{e}_1 \\ &= 1 - \sum_{k=0}^{n-1}\mathbf{e}_1^*\mathbf{e}_k(x-x_0)^k - \sum_{k=n}^{\infty}(x-x_0)^k\mathbf{e}_1^*\big(P_n^{-1}H_n\big)^{k-n+1}\mathbf{e}_n \\ &= -(x-x_0)^n\mathbf{e}_1^*P_n^{-1}H_n\mathbf{e}_n - \sum_{k=n+1}^{\infty}(x-x_0)^k\mathbf{e}_1^*\big(P_n^{-1}H_n\big)^{k-n+1}\mathbf{e}_n,\end{aligned} \tag{2.48}$$

which verifies (2.42). Thus, Υ_1 is a rational function of degree at most n (by (2.39)) and has zero of multiplicity at least n at x_0 (by (2.42)). Therefore, either we have equalities in both cases, and hence, $\mathbf{e}_1^* P_n^{-1} H_n \mathbf{e}_n \neq 0$ and $\Upsilon_1(x) \neq 0$ for all $x \neq x_0$, or $\Upsilon_1 \equiv 0$. On the other hand, $n-1$ leftmost columns in H_n are linearly independent, by (2.46). Thus, H_n is singular if and only if $H_n \mathbf{e}_n$ is a linear combination of $H_n \mathbf{e}_1, \ldots, H_n \mathbf{e}_{n-1}$, which is equivalent (again by (2.46)) to $\mathbf{e}_1^* P_n^{-1} H_n \mathbf{e_n} = 0$. If so, $\Upsilon_1 \equiv 0$, by (2.48). This completes the proof of part (2).

We next multiply the equality(2.45) by $\mathbf{c}^* = \mathbf{e}_1^* P_n$ on the left, arriving, on account of (2.38), at

$$f_F(x) = \sum_{k=0}^{\infty} (x - x_0)^k \mathbf{e}_1^* P_n (P_n^{-1} H_n)^k \mathbf{e}_1.$$

To complete the proof, it remains to verify that

$$\mathbf{e}_1^* P_n (P_n^{-1} H_n)^k \mathbf{e}_1 = c_k \quad \text{for} \quad k = 0, \ldots, 2n-1 \tag{2.49}$$

and

$$\mathbf{e}_1^* P_n \big(P_n^{-1} H_n\big)^{2n+k} \mathbf{e}_1 = \mathbf{e}_n^* (H_n P_n^{-1})^{k+1} H_n \mathbf{e}_n \quad \text{for all} \quad k \geq 0. \tag{2.50}$$

Multiplying equalities (2.47) by $\mathbf{e}_1^* P_n$ on the left we get

$$\mathbf{e}_1^* P_n \big(P_n^{-1} H_n\big)^{k-1} \mathbf{e}_1 = \mathbf{e}_1^* P_n \mathbf{e}_k = c_{k-1} \quad \text{for} \quad k = 1, \ldots, n,$$

verifying the first n equalities in (2.49). Taking adjoints in (2.47) we get

$$\mathbf{e}_1^* (H_n P_n^{-1})^{k-1} = \mathbf{e}_k^*,$$

which being combined with (2.47) (for $k = n$) leads us to

$$\begin{aligned} \mathbf{e}_1^* P_n \big(P_n^{-1} H_n\big)^{n+j} \mathbf{e}_1 &= \mathbf{e}_1^* P_n \big(P_n^{-1} H_n\big)^{j+1} \mathbf{e}_n \\ &= \mathbf{e}_1^* (H_n P_n^{-1})^j H_n \mathbf{e}_n = \mathbf{e}_{j+1}^* H_n \mathbf{e}_n = c_{n+j} \end{aligned}$$

for $j = 0, \ldots, n-1$, thus confirming the remaining equalities in (2.49). We next pursue the last calculation for $j = n + k$ as follows:

$$\mathbf{e}_1^* P_n \big(P_n^{-1} H_n\big)^{2n+k} \mathbf{e}_1 = \mathbf{e}_1^* (H_n P_n^{-1})^{n+k} H_n \mathbf{e}_n = \mathbf{e}_n^* (H_n P_n^{-1})^{k+1} H_n \mathbf{e}_n,$$

thus arriving at (2.42). □

We now complete Step 4: If $H_n \succ 0$, then $\widetilde{H}_n^{-1} \succ (\widetilde{H}_n - x H_n)^{-1}$ for all $x < 0$, and hence we see from the formula (2.40) that $\Upsilon_2(x) > 0$ for all $x < 0$. By part (2) in Lemma 2.5, Υ_1 has zero of multiplicity n at x_0. Therefore, we have

$$\frac{\Upsilon_1^2(x)h(x)}{1+\Upsilon_2(x)h(x)} = O((x-x_0)^{2n})$$

for any $h \in \overline{\mathcal{S}}$, Therefore, for any f of the form (2.37), $f^{(k)}(x_0) = f_F^{(k)}(x_0)$ for $k = 0, \ldots, 2n-1$, which, due to (2.41), completes Step 4.

Combining Remark 2.4 and Lemma 2.5 leads us to the following result.

Lemma 2.6 *Let f be of the form (2.37) for some $h \in \mathcal{S}$. Then*

$$\frac{f^{(2n)}(x_0)}{(2n)!} = \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2 h(x_0)}{1+\boldsymbol{\delta} h(x_0)}, \quad \textit{where} \tag{2.51}$$

$$\boldsymbol{\alpha} = \mathbf{e}_n^* H_n P_n^{-1} H_n \mathbf{e}_n, \quad \boldsymbol{\beta} = \mathbf{e}_1^* P_n^{-1} H_n \mathbf{e}_n, \quad \boldsymbol{\delta} = \mathbf{e}_1^* \widetilde{H}_n^{-1} \mathbf{e}_1 - \mathbf{e}_1^* P_n^{-1} \mathbf{e}_1. \tag{2.52}$$

Proof By formulas (2.41), (2.42) and (2.43),

$$\begin{aligned} f_F(x) &= c_0 + \ldots + c_{2n-1}(x-x_0)^{2n-1} + \boldsymbol{\alpha}(x-x_0)^{2n} + O((x-x_0)^{2n+1}), \\ \Upsilon_1(x) &= -\boldsymbol{\beta}(x-x_0)^n + O((x-x_0)^{n+1}), \quad \Upsilon_2(x) = \boldsymbol{\delta} + O((x-x_0)). \end{aligned} \tag{2.53}$$

Substituting the latter expansions into (2.37) we get the Taylor expansion for f of the form (2.37) at x_0:

$$\begin{aligned} f(x) = c_0 + \ldots + c_{2n-1}(x-x_0)^{2n-1} + \left(\boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2 h(x_0)}{1+\boldsymbol{\delta} h(x_0)}\right)(x-x_0)^{2n} \\ + O((x-x_0)^{2n+1}, \end{aligned} \tag{2.54}$$

which implies (2.51). □

2.1 Extremal Solutions

Let $Sol(\mathbf{CFS}_{2n-1})$ denote the set of all solutions to the problem $\mathbf{CFS}_{2n-1}$. The functions $f_F = \mathbf{T}_\Psi[0] = \frac{\Psi_{12}}{\Psi_{22}}$ and $f_K = \mathbf{T}_\Psi[\infty] = \frac{\Psi_{11}}{\Psi_{21}}$ are extremal elements of this set in the following sense.

Proposition 2.7 *For any $f \in Sol(\mathbf{CFS}_{2n-1})$ different from f_F and f_K,*

$$f_F(x) < f(x) < f_K(x) \quad \textit{for all} \quad x < 0 \ (x \neq x_0), \tag{2.55}$$

$$f_F^{(2n)}(x_0) < f^{(2n)}(x_0) < f_K^{(2n)}(x_0). \tag{2.56}$$

Furthermore, $\{f(x) : f \in Sol(\mathbf{CFS}_{2n-1})\} = [f_F(x), f_K(x)]$ *for each fixed* $x < 0$, *and* $\{f^{(2n)}(x_0) : f \in Sol(\mathbf{CFS}_{2n-1})\} = [f_F^{(2n)}(x_0), f_K^{(2n)}(x_0)]$.

Proof We first observe the equalities

$$f_F(x) + \frac{\Upsilon_1^2(x)}{\Upsilon_2(x)} = f_K(x) \quad \text{and} \quad \frac{f_K^{(2n)}(x_0)}{(2n)!} = \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}},$$

which follow, by (2.52), upon letting $h \equiv \infty$ in (2.37) and (2.51), respectively. Since all solutions of the problem $\mathbf{CFS}_{2n-1}$ are parametrized by the formula (2.37) and since the value $h(x)$ of the parameter $h \in \mathcal{S}$ at x varies in $\mathbb{R}^+$, it suffices to verify that for any fixed $x < 0$, the function

$$y \mapsto f_F(x) + \frac{\Upsilon_1^2(x)y}{1 + \Upsilon_2(x)y}$$

maps $(0, \infty)$ onto $(f_F(x), f_K(x))$. This is indeed the case, since $\Upsilon_2(x) > 0$ and $\Upsilon_1(x) \neq 0$, by Lemma 2.5, The proof of (2.56) is similar: due to formula (2.51), it suffices to show that the function $y \mapsto \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2 y}{1+\boldsymbol{\delta} y}$ maps bijectively $(0, \infty)$ onto the interval $(\boldsymbol{\alpha}, \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}}) = \left(\frac{f_F^{(2n)}(x_0)}{(2n)!}, \frac{f_K^{(2n)}(x_0)}{(2n)!}\right)$. This is again the case, since $\boldsymbol{\delta} = \Upsilon_2(x_0) > 0$ and $\boldsymbol{\beta} \neq 0$, by Lemma 2.5. □

We conclude this section with another feature of extremal solutions.

Remark 2.8 The only $f \in Sol(\mathbf{CFS}_{2n-1})$ for which the extended Hankel matrices $H_{n+1}^f(x_0)$ and $\widetilde{H}_{n+1}^f(x_0)$ (see (1.7)) are both singular, are $f = f_F$ and $f = f_K$.

Proof Relations (2.30) show that the kernels K_{f_F} and $\widetilde{K}_{f_F}$ are of rank n each and hence, the extended Hankel matrices $H_{n+1}^{f_K}(x_0)$ and $\widetilde{H}_{n+1}^{f_K}(x_0)$ are both singular. It follows from integral representations (2.26) and (2.27) that the ranks of the matrix-valued kernels (2.20) and (2.21) are equal to the ranks of the kernels K_f and $\widetilde{K}_f$, respectively. Then by the Schur complement argument as in the proof of Lemma 2.3, we conclude that

$$\text{rank} K_f = n + \text{rank} K_h, \quad \text{rank} \widetilde{K}_f = n + \text{rank} \widetilde{K}_h, \quad \text{if} \quad f = \mathbf{T}_\Psi[h], \ h \in \mathcal{S}. \tag{2.57}$$

If the matrices $H_{n+1}^{f_K}(x_0)$ and $\widetilde{H}_{n+1}^{f_K}(x_0)$ are singular, then $K_h = \widetilde{K}_h \equiv 0$, which hold true only for $h \equiv 0$. □

3 The Odd Case

We now consider the problem $\mathbf{CFS}_{2n}$ with given $c_0, \ldots, c_{2n}$. We still assume that $H_n \succ 0$ and $\widetilde{H}_n \succ 0$ so that the matrix P_n (1.14) is also positive definite. In the present case (that is with the given c_{2n}) we can define extended matrix $P_{n+1} =$

$\left[c_{i+j}\right]_{i,j=0}^{n}$ and the inequality $P_{n+1} \succeq 0$ is another necessary condition for the problem $\mathbf{CFS}_{2n}$ to have a solution. Writing P_{n+1} in the block form as

$$P_{n+1} = \begin{bmatrix} P_n & H_n\mathbf{e}_n \\ \mathbf{e}_n^* H_n & c_{2n} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ \mathbf{e}_n^* H_n P_n^{-1} & 1 \end{bmatrix} \begin{bmatrix} P_n^{-1} & 0 \\ 0 & c_{2n} - \mathbf{e}_1^* H_n P_n^{-1} H_n \mathbf{e}_n \end{bmatrix} \begin{bmatrix} I_n & P_n^{-1} H_n \mathbf{e}_n \\ 0 & 1 \end{bmatrix},$$

and making use of $\boldsymbol{\alpha}$ defined in (2.52), we see that $P_{n+1} \succeq 0$ if and only if

$$c_{2n} - \mathbf{e}_n^* H_n P_n^{-1} H_n \mathbf{e}_n = c_{2n} - \boldsymbol{\alpha} \geq 0 \tag{3.1}$$

and that $P_{n+1} \succ 0$ if and only if the inequality (3.1) is strict. In the latter case, we write P_{n+1}^{-1} in the block form

$$P_{n+1}^{-1} = \begin{bmatrix} I_n & -P_n^{-1} H_n \mathbf{e}_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_n^{-1} & 0 \\ 0 & \frac{1}{c_{2n} - \boldsymbol{\alpha}} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -\mathbf{e}_n^* H_n P_n^{-1} & 1 \end{bmatrix} \succ 0,$$

and compute its leading entry in terms of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}$ defined in (2.52):

$$\begin{aligned} \left[P_{n+1}^{-1}\right]_{11} &= \left[\mathbf{e}_1^* \ -\mathbf{e}_1^* P_n^{-1} H_n \mathbf{e}_n\right] \begin{bmatrix} P_n^{-1} & 0 \\ 0 & \frac{1}{c_{2n} - \boldsymbol{\alpha}} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ -\mathbf{e}_n^* H_n P_n^{-1} \mathbf{e}_1 \end{bmatrix} \\ &= \mathbf{e}_1^* P_n^{-1} \mathbf{e}_1 + \frac{\boldsymbol{\beta}^2}{c_{2n} - \boldsymbol{\alpha}} = \left[\widetilde{H}_n^{-1}\right]_{11} + \frac{\boldsymbol{\beta}^2}{c_{2n} - \boldsymbol{\alpha}} - \boldsymbol{\delta}. \end{aligned}$$

The latter equality shows that if $c_{2n} - \boldsymbol{\alpha} > 0$ (i.e., $P_{n+1} \succ 0$), then

$$c_{2n} \leq \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}} \iff \left[\widetilde{H}_n^{-1}\right]_{11} \leq \left[P_{n+1}^{-1}\right]_{11}.$$

Theorem 3.1 *If $H_n \succ 0$ and $\widetilde{H}_n \succ 0$, then the problem $\mathbf{CFS}_{2n}$ is indeterminate if and only if $P_{n+1} \succ 0$ and $\left[P_{n+1}^{-1}\right]_{11} > \left[\widetilde{H}_n^{-1}\right]_{11}$ or equivalently, if and only if*

$$\boldsymbol{\alpha} < c_{2n} < \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}}, \tag{3.2}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}$ are defined in (2.52). The problem is determinate if and only if either

(1) $c_{2n} = \boldsymbol{\alpha}$ *(i.e.,* $\operatorname{rank} P_{n+1} = n$*), in which case the unique solution of the problem is* $f_F = \frac{\Psi_{12}}{\Psi_{22}}$*. or*
(2) $c_{2n} = \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}}$ *(i.e.,* $P_{n+1} \succ 0$ *and* $\left[P_{n+1}^{-1}\right]_{11} = \left[\widetilde{H}_n^{-1}\right]_{11}$*), in which case the unique solution is* $f_K = \frac{\Psi_{11}}{\Psi_{21}}$.

Proof Solving the Eq. (2.52) for $h(x_0)$ and replacing $\frac{f^{(2n)}(x_0)}{(2n)!}$ by c_{2n}, we then combine Theorem 2.1, Remark 2.4, and Lemma 2.6 to conclude: □

Remark 3.2 A function f is a solution to the problem $\mathbf{CFS}_{2n}$ if and only if it is of the form (2.37) for some $h \in \overline{\mathcal{S}}$ subject to the interpolation condition

$$h(x_0) = h_0 := \frac{c_{2n} - \boldsymbol{\alpha}}{\boldsymbol{\beta}^2 - \boldsymbol{\delta}(c_{2n} - \boldsymbol{\alpha})}. \tag{3.3}$$

If $c_{2n} = \boldsymbol{\alpha}$, then (3.3) amounts to $h(x_0) = 0$ which implies $h \equiv 0$ and hence, the function f_F given by (2.38) is a unique solution to the problem $\mathbf{CFS}_{2n}$. Similarly, the equality $c_{2n} = \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}}$ forces $h \equiv \infty$ and consequently f_K be the unique solution to the problem $\mathbf{CFS}_{2n}$. For any c_{2n} subject to strict inequalities (3.2), the expression on the right side of (3.3) defines a positive number, so that condition (3.3) is satisfied by infinitely many $h \in \mathcal{S}$ and the consequently, the problem $\mathbf{CFS}_{2n}$ has infinitely many solutions. Equivalent formulations of all statements in terms of P_{n+1} and $\widetilde{H}_n$ follow from the discussion preceding the theorem.

3.1 Schwarz-Pick Theorems

To get a parametrization of the set $Sol(\mathbf{CFS}_{2n})$ in the indeterminate case, we need to describe all functions $h \in \mathcal{S}$ subject to the sole interpolation condition (3.3). As was pointed out in [27] in the Schur-class setting, such a description can be derived from the Schwarz-Pick theorem. The Stieltjes-class real-valued Schwarz-Pick theorem is the following.

Theorem 3.3 *Let f be a non-zero Stieltjes function. Then for every $x_0 < 0$, the function*

$$g(x) = \frac{f(x) - f(x_0)}{xf(x) - x_0 f(x_0)} \tag{3.4}$$

belongs to the extended Stieltjes class $\overline{\mathcal{S}}$.

Proof We start with trivial cases. If $f(x) \equiv \gamma \geq 0$ or $f(x) = -\frac{\alpha}{x}$ for some $\alpha > 0$, then the holds true with $g \equiv 0$ and $g \equiv \infty$, respectively. In all other cases,

$$c_0 := f(x_0) > 0, \quad c_1 := f'(x_0) > 0, \quad p := x_0 c_1 + c_0 > 0.$$

Considering f as a solution of the interpolation problem $\mathbf{CFS}_1$ we observe that in this case $H_1 = c_1$, $\widetilde{H}_1 = p$, $T = x_0$, $\mathbf{c} = c_0$, $\mathbf{e} = 1$, so that formulas (2.4)–(2.7) amount to

$$\Psi_{11} = \frac{px - c_1 x_0^2}{(x - x_0)p}, \quad \Psi_{12} = -\frac{c_0^2}{(x - x_0)c_1}, \quad \Psi_{21} = \frac{x}{(x - x_0)p}, \quad \Psi_{22} = \frac{c_1 x - p}{(x - x_0)c_1}.$$

By Theorem 2.1, f admits the representation (2.3), i.e.,

$$f(x)=\frac{\Psi_{11}h+\Psi_{12}}{\Psi_{21}h+\Psi_{22}}=\frac{(x-c_1p^{-1}x_0^2)h(x)-c_0^2c_1^{-1}}{p^{-1}xh(x)+x-pc_1^{-1}}\quad\text{for some}\quad h\in\widetilde{\mathcal{S}}. \tag{3.5}$$

Straightforward calculations (relying on the equality $p=x_0c_1+c_0$ only) show that for f of the form (3.5),

$$\frac{f(x)-f_0}{x-x_0}=\frac{x_0c_1p^{-1}h(x)-c_0}{p^{-1}xh(x)+x-pc_1^{-1}},$$
$$\frac{xf(x)-x_0f_0}{x-x_0}=\frac{xh(x)-c_0pc_1^{-1}}{p_1^{-1}xh(x)+x-pc_1^{-1}}.$$

The ratio of the two latter equalities equals

$$g(x)=\frac{f(x)-f(x_0)}{xf(x)-x_0f(x_0)}=\frac{x_0c_1p^{-1}h(x)-c_0}{xh(x)-c_0pc_1^{-1}},\qquad h\in\overline{\mathcal{S}}. \tag{3.6}$$

If $h=\infty$, then $g(x)=\dfrac{x_0c_1}{px}$ belongs to $\mathcal{S}$ (since $\frac{x_0c_1}{p}<0$). Otherwise, we compute for g given by (3.6) and arbitrary $x,y\in\mathbb{R}^-$,

$$\frac{g(x)-g(y)}{x-y}=\frac{x_0c_1p^{-1}(y-x)h(x)h(y)-x_0c_0(h(x)-h(y))+c_0(xh(x)-yh(y)}{v(x)v(y)(x-y)},$$

where we have set for short, $v(x):=xh(x)-c_0pc_1^{-1}$. The latter equality represents the kernel K_g in terms of the kernels K_h and $\widetilde{K}_h$ (see (1.4)):

$$K_g(x,y)=-x_0c_1p^{-1}\frac{h(x)h(y)}{v(x)v(y)}-\frac{x_0}{v(x)v(y)}K_h(x,y)+\frac{c_0}{v(x)v(y)}\widetilde{K}_h(x,y). \tag{3.7}$$

Since $h\in\mathcal{S}$, the kernels K_h and $\widetilde{K}_h$ are positive definite on $\mathbb{R}^-$. Since c_0,c_1,p are positive and x_0 is negative, the kernel $K_g(x,y)$ in (3.7) is positive definite on $\mathbb{R}^-$ as the sum of three positive definite kernels. Therefore, g is operator-monotone on $\mathbb{R}^-$. Since $h(x)\geq 0$ for all $x<0$ (as $h\in\mathcal{S}$), it follows from (3.6) that $g(x)>0$ for all $x<0$. Therefore, $g\in\mathcal{S}$. □

Corollary 3.4 *Given $x_0 < 0$ and $c_0 > 0$, all functions $f \in \mathcal{S}$ such that $f(x_0) = c_0$ are parametrized by the formula*

$$f(x) = c_0 \cdot \frac{x_0 g(x) - 1}{x g(x) - 1} \tag{3.8}$$

with free parameter $g \in \overline{\mathcal{S}}$.

Proof If $f \in \mathcal{S}$ and $f(x_0) = c_0$, then the function

$$g(x) = \frac{f(x) - c_0}{x f(x) - x_0 c_0}$$

belongs to $\overline{S}$, by Theorem 3.3, Solving the latter equality for f gives (3.8). Conversely, if g is any nonnegative function on $\mathbb{R}^-$, the formula (3.8) defines f which is positive on $\mathbb{R}_-$ and satisfies $f(x_0) = c_0$. For f defined as in (3.8) we have

$$\frac{f(x) - f(y)}{x - y} = \frac{c_0}{v(x)v(y)} \left(-x_0 g(x) g(y) - x_0 \frac{g(x) - g(y)}{x - y} + \frac{x g(x) - y g(y)}{x - y} \right),$$

where we let $v(x) = x g(x) - 1$. In other words,

$$K_f(x, y) = \frac{c_0}{v(x)v(y)} \left(-x_0 g(x) g(y) - x_0 K_g(x, y) + \widetilde{K}_g(x, y) \right). \tag{3.9}$$

Since $g \in \mathcal{S}$, the kernels K_g and $\widetilde{K}_g$ are positive on $\mathbb{R}^-$. Since $x_0 < 0$ and $c_0 > 0$, it now follows from (3.9) that the kernel K_f is positive on $\mathbb{R}^-$. Hence, f is a positive operator-monotone function on $\mathbb{R}^-$ and therefore, $f \in \mathcal{S}$. It remains to note that $g \equiv \infty$ leads via formula (3.8) to the constant Stieltjes function $f \equiv c_0$. □

A slightly different parametrization of the set $Sol(\mathbf{CFS}_0)$ was established in [22] using essentially complex-analytic approach. Note that the complex analog of Theorem 3.3 fails to be true: if f is a Stieltjes function extended to $\mathbb{C}^+ \backslash \mathbb{R}^+$ and x_0 is not real, then the function g defined in (3.4) does not belong to $\mathcal{S}$. It can be shown, however, that for $f \in \mathcal{S}$ (such that both f and zf are not constant functions) there exists $h \in \overline{\mathcal{S}}$ such that

$$\frac{f(z) - f(z_0)}{z f(z) - z_0 f(z_0)} = \frac{1}{d} \cdot \frac{\overline{z}_0 h(z) - d f(z_0)}{z h(z) - d f(z_0)}, \quad d = \frac{z_0 f(z_0) - \overline{z_0 f(z_0)}}{f(z_0) - \overline{f(z_0)}} > 0. \tag{3.10}$$

The latter relation is quite different from (3.4): if we let $z = x \in \mathbb{R}^-$ and $z_0 = x_0 + i\varepsilon$ in (3.10) and then take the limits as $\varepsilon \to 0$, we get

$$\frac{f(x) - f(x_0)}{x f(x) - x_0 f(x_0)} = \frac{1}{d} \cdot \frac{x_0 h(x) - d f(x_0)}{x h(x) - d f(x_0)}, \quad d = \frac{x_0 f'(x_0) + f(x_0)}{f'(x_0)}.$$

3.2 *Parametrization of Sol* $(\mathbf{CFS}_{2n})$

By Corollary 3.4, all functions $h \in \mathcal{S}$ subject to condition (3.3) are parametrized by the linear fractional formula

$$h = \mathbf{T}_{\Phi}[g], \quad g \in \overline{\mathcal{S}}, \quad \Phi(x) = \begin{bmatrix} x_0 & -1 \\ x h_0^{-1} & -h_0^{-1} \end{bmatrix}. \tag{3.11}$$

Combining (3.11) with Remark 3.2 and taking into account that $\mathbf{T}_{\Psi} \circ \mathbf{T}_{\Phi} = \mathbf{T}_{\Psi\Phi}$, we get the following result.

Theorem 3.5 *Assume that* H_n, $\widetilde{H}_n$, P_{n+1} *are positive definite and* $\widetilde{\mathbf{e}}_1^* P_{n+1}^{-1} \widetilde{\mathbf{e}}_1 > \mathbf{e}_1^* \widetilde{H}_n^{-1} \mathbf{e}_1$. *Then all solutions* f *to the problem* $\mathbf{CFS}_{2n}$ *are given by the formula*

$$f = \mathbf{T}_{\mathfrak{A}}[g] := \frac{\mathfrak{A}_{11} g + \mathfrak{A}_{12}}{\mathfrak{A}_{21} g + \mathfrak{A}_{22}}, \quad g \in \overline{\mathcal{S}}, \quad \mathfrak{A} = \begin{bmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} \end{bmatrix} = \Psi\Phi \tag{3.12}$$

where Ψ *and* Φ *are given in* (2.3) *and* (3.11), *respectively.*

3.3 *Extremal Solutions*

We now consider the extremal solutions $\mathfrak{f}_F = \mathbf{T}_{\mathfrak{A}}[0]$ and $\mathfrak{f}_K = \mathbf{T}_{\mathfrak{A}}[\infty]$ of the problem $\mathbf{CFS}_{2n}$. Since $\mathbf{T}_{\Phi}[0] \equiv h_0$ and $\mathbf{T}_{\Phi}[\infty] = \frac{x_0 h_0}{x}$ by (3.11) (where h_0 is given by (3.3)), we have

$$\mathfrak{f}_F = \mathbf{T}_{\Psi}[\mathbf{T}_{\Phi}[0]] = \mathbf{T}_{\Psi}[h_0] = f_F + \frac{\Upsilon_1^2}{h_0^{-1} + \Upsilon_2}, \tag{3.13}$$

$$\mathfrak{f}_K = \mathbf{T}_{\Psi}[\mathbf{T}_{\Phi}[\infty]] = \mathbf{T}_{\Psi}\left[\frac{x_0 h_0}{x}\right] = f_F + \frac{\Upsilon_1^2}{\frac{x}{x_0} h_0^{-1} + \Upsilon_2}, \tag{3.14}$$

where the rightmost equalities in both formulas follow from (2.37). Combining (3.13) and (3.14) gives

$$\mathfrak{f}_K = \mathfrak{f}_F - \frac{(x - x_0) h_0^{-1} \Upsilon_1^2}{(h_0^{-1} + \Upsilon_2)(x h_0^{-1} + x_0 \Upsilon_2)}. \tag{3.15}$$

More generally, letting $h = \mathbf{T}_\Phi[g]$) in (2.37) leads the Redheffer type version of the parametrization formula (3.12):

$$f = \frac{\mathfrak{A}_{11}g + \mathfrak{A}_{12}}{\mathfrak{A}_{21}g + \mathfrak{A}_{22}} = \mathfrak{f}_F + \frac{(x - x_0)\Upsilon_1^2 h_0^{-1} g}{(h_0^{-1} + \Upsilon_2)^2 \left(1 - \frac{x h_0^{-1} + x_0 \Upsilon_2}{h_0^{-1} + \Upsilon_2} g\right)} \tag{3.16}$$

The next proposition is the "odd" counter-part of Proposition 2.7

Proposition 3.6 *For any* $f \in Sol(\mathbf{CFS}_{2n})$ *different from* $\mathfrak{f}_F$ *and* $\mathfrak{f}_K$,

$$\mathfrak{f}_F(x) < f(x) < \mathfrak{f}_K(x) \quad \text{for all} \quad x > x_0, \tag{3.17}$$

$$\mathfrak{f}_K(x) < f(x) < \mathfrak{f}_F(x) \quad \text{for all} \quad x < x_0, \tag{3.18}$$

$$\mathfrak{f}_F^{(2n+1)}(x_0) < f^{(2n+1)}(x_0) < \mathfrak{f}_K^{(2n+1)}(x_0). \tag{3.19}$$

Furthermore, $\{f^{(2n+1)}(x_0) : \ f \in Sol(\mathbf{CFS}_{2n+1})\} = [\mathfrak{f}_F^{(2n)}(x_0), \mathfrak{f}_K^{(2n+1)}(x_0)]$, *and, for each fixed* $x < 0$, $\{f(x) : \ f \in Sol(\mathbf{CFS}_{2n})\}$ *is the open interval with the endpoints* $\mathfrak{f}_F(x)$ *and* $\mathfrak{f}_K(x)$.

Proof Since for any fixed $x < 0$, the function

$$y \mapsto \frac{\Upsilon_1^2 h_0^{-1} y}{(h_0^{-1} + \Upsilon_2)^2 \left(1 - \frac{x h_0^{-1} + x_0 \Upsilon_2}{h_0^{-1} + \Upsilon_2} y\right)}$$

maps $(0, \infty)$ onto $\left(0, -\frac{\Upsilon_1^2 h_0^{-1}}{(h_0^{-1} + \Upsilon_2)(x h_0^{-1} + x_0 \Upsilon_2)}\right)$, the inequalities (3.17) and (3.18) follow from (3.15) and (3.16). Making use of Taylor expansions (2.53), we differentiate equalities (3.15) and (3.16) at x_0 to get

$$\frac{\mathfrak{f}_K^{(2n+1)}(x_0)}{(2n+1)!} = \frac{\mathfrak{f}_F^{(2n+1)}(x_0)}{(2n+1)!} - \frac{\boldsymbol{\beta}^2 h_0^{-1}}{x_0 (h_0^{-1} + \Upsilon_2(x_0))^2}, \tag{3.20}$$

$$\frac{f^{(2n+1)}(x_0)}{(2n+1)!} = \frac{\mathfrak{f}_F^{(2n+1)}(x_0)}{(2n+1)!} + \frac{\boldsymbol{\beta}^2 h_0^{-1} g(x_0)}{(h_0^{-1} + \Upsilon_2(x_0))^2 (1 - x_0 g(x_0))}, \tag{3.21}$$

and inequalities (3.19)follow, since $0 < \frac{g(x_0)}{1 - x_0 g(x_0)} < -\frac{1}{x_0}$ and since the function $y \to \frac{y}{1 - x_0 y}$ maps $(0, \infty)$ onto $(0, -\frac{1}{x_0})$. □

Remark 3.7 Let $\mathfrak{f}_F$ and $\mathfrak{f}_K$ be the extremal solutions of a (indeterminate) problem $Sol(\mathbf{CFS}_{2n})$. Then the matrices (see (1.7)) $\widetilde{H}_{n+1}^{\mathfrak{f}_F}(x_0)$ and $H_{n+1}^{\mathfrak{f}_K}(x_0)$ are positive definite, whereas the matrices $H_{n+1}^{\mathfrak{f}_F}(x_0) \succeq 0$ and $\widetilde{H}_{n+1}^{\mathfrak{f}_K}(x_0) \succeq 0$ are singular.

Proof As has been implicitly mentioned in (3.13), (3.14), the extremal functions $\mathfrak{f}_F$ and $\mathfrak{f}_K$ of various solvable problems $Sol(\mathbf{CFS}_{2n})$ appear as the outcomes of the linear fractional transformation (2.3) and correspond to parameters $h(x) \equiv h_0$ and $h(x) = \frac{x_0 h_0}{x}$, respectively, where $h_0 > 0$ is defined in (3.3). For $h \equiv h_0 > 0$, we have $K_h \equiv 0$ and $\widetilde{K}_h \equiv h_0$. On the other hand, if $h = \frac{x_0 h_0}{x}$, then $K_h(x, y) = -\frac{x_0 h_0}{xy}$ and $\widetilde{K}_h \equiv 0$. By (2.57),

$$\operatorname{rank} K_{\mathfrak{f}_F} = n = \operatorname{rank} \widetilde{K}_{\mathfrak{f}_F} - 1, \quad \operatorname{rank} \widetilde{K}_{\mathfrak{f}_K} = n = \operatorname{rank} K_{\mathfrak{f}_K} - 1,$$

which imply all desired statements. □

3.4 *Linear Fractional Transformations Mapping the Extended Stieltjes Class into Itself*

In this section we discuss three particular examples of linear fractional transformations mapping $\overline{\mathcal{S}}$ into itself and embed them into a more general setting. We start with the following observation.

Proposition 3.8 *Let* $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ *be a rational* 2×2 *matrix-function with* $\det \Theta \not\equiv 0$ *and let us assume that the following two kernels are positive on* $\mathbb{R}^-$*:*

$$\mathfrak{K}_{\Theta,J}(x, y) = \frac{a(x)a(y)\Theta(x)J\Theta(y)^* + b(x)b(y)\Theta(x)J_{x,y}\Theta(y)^* - J}{y - x} \succeq 0, \tag{3.22}$$

$$\widetilde{\mathfrak{K}}_{\Theta,J}(x, y) = \frac{c(x)c(y)\Theta(x)J\Theta(y)^* + d(x)d(y)\Theta(x)J_{x,y}\Theta(y)^* - J_{x,y}}{y - x} \succeq 0, \tag{3.23}$$

where J*,* $J_{x,y}$ *are defined in* (2.11) *and where* a, b, c, d *are rational functions. Then the transformation* $\mathbf{T}_\Theta : h \mapsto \frac{\Theta_{11}h + \Theta_{12}}{\Theta_{21}h + \Theta_{22}}$ *maps* $\overline{\mathcal{S}}$ *into itself.*

Proof We first verify $h = \infty$ and $h = 0$. If $\Theta_{21} \equiv 0$, then $\Theta_{11} \not\equiv 0$, $\Theta_{22} \not\equiv 0$ (since $\det \Theta \not\equiv 0$) and hence, $\mathbf{T}_\Theta[\infty] = \infty$. Similarly, if $\Theta_{22} \equiv 0$, then $\Theta_{12} \not\equiv 0$, $\Theta_{21} \not\equiv 0$, and hence, $\mathbf{T}_\Theta[0] = \infty$. If $\Theta_{21} \not\equiv 0$, then for $f_K := \mathbf{T}_\Theta[\infty] = \frac{\Theta_{11}}{\Theta_{21}}$, we have

$$\begin{bmatrix} 1 & -f_K(x) \end{bmatrix} \Theta(x) = \det \Theta(x) \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Similarly, if $\Theta_{22} \not\equiv 0$, then for the function $f_F := \mathbf{T}_\Theta[0] = \frac{\Theta_{12}}{\Theta_{22}}$, we have

$$\begin{bmatrix} 1 & -f_F(x) \end{bmatrix} \Theta(x) = \frac{\det \Theta(x)}{\Theta_{22}(x)} \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

It follows from the two last relations and (3.22), (3.23) that the functions $f = f_K$ and $f = f_F$ are subject to inequalities

$$K_f(x,y) = \begin{bmatrix} 1 & -f(x) \end{bmatrix} \mathfrak{K}_{\Theta,J}(x,y) \begin{bmatrix} 1 \\ -f(y) \end{bmatrix} \succeq 0,$$
$$\widetilde{K}_f(x,y) = \begin{bmatrix} 1 & -f(x) \end{bmatrix} \widetilde{\mathfrak{K}}_{\Theta,J}(x,y) \begin{bmatrix} 1 \\ -f(y) \end{bmatrix} \succeq 0. \quad \text{for } f = f_K, \ f_F.$$

The positivity of the latter kernels on $\mathbb{R}^-$ implies $f_K, f_F \in \mathcal{S}$, by Proposition 1.3. For any $h \in \mathcal{S}$, we now define two differentiable functions $u, v : \mathbb{R}^- \to \mathbb{R}$ by equivalent formulas

$$\begin{bmatrix} u \\ v \end{bmatrix} = \Theta \begin{bmatrix} h \\ 1 \end{bmatrix} = \begin{bmatrix} \Theta_{11}h + \Theta_{12} \\ \Theta_{21}h + \Theta_{22} \end{bmatrix} \iff \begin{bmatrix} v & -u \end{bmatrix} \Theta = \det \Theta \cdot \begin{bmatrix} 1 & -h \end{bmatrix}. \tag{3.24}$$

The kernels (2.35) and (2.36) associated with this pair are expressed in terms of the kernels (3.22) and (3.23) as follows:

$$K_{u,v}(x,y) = \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \mathfrak{K}_{\Theta,J}(x,y) \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix} \tag{3.25}$$
$$+ \det \Theta(x) \left(a(x) K_h(x,y) a(y) + \frac{b(x)}{x} \widetilde{K}_h(x,y) \frac{b(y)}{y} \right) \det \Theta(y),$$

$$\widetilde{K}_{u,v}(x,y) = x \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \widetilde{\mathfrak{K}}_{\Theta,J}(x,y) \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix} y \tag{3.26}$$
$$+ \det \Theta(x) \left(x c(x) K_h(x,y) c(y) y + d(x) \widetilde{K}_h(x,y) d(y) \right) \det \Theta(y).$$

Indeed, by (2.35) and (3.22),

$$\begin{aligned} K_{u,v}(x,y) &= \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \frac{J}{x-y} \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix} \\ &= \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \mathfrak{K}_{\Theta,J}(x,y) \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix} \\ &\quad + \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \frac{a(x)a(y)\Theta(x) J \Theta(y)^*}{x-y} \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix} \\ &\quad + \begin{bmatrix} v(x) & -u(x) \end{bmatrix} \frac{b(x)b(y)\Theta(x) J_{x,y} \Theta(y)^*}{x-y} \begin{bmatrix} v(y) \\ -u(y) \end{bmatrix}. \end{aligned}$$

By (3.24), the second term on the right side equals

$$a(x)\det\Theta(x)\begin{bmatrix}1 & -h(x)\end{bmatrix}\frac{J}{x-y}\begin{bmatrix}1\\ -h(y)\end{bmatrix}\det\Theta(y)^{\top}a(y)$$
$$= a(x)\det\Theta(x)K_h(x,y)\det\Theta(y)a(y)$$

while the third term equals

$$b(x)\det\Theta(x)\begin{bmatrix}1 & -h(x)\end{bmatrix}\frac{J_{x,y}}{x-y}\begin{bmatrix}1\\ -h(y)\end{bmatrix}\det\Theta(y)^{\top}b(y)$$
$$= \frac{b(x)\det\Theta(x)}{x}\widetilde{K}_h(x,y)\frac{b(y)\det\Theta(y)}{y}.$$

Combining the three latter equalities leads us to (3.25). Relation (3.26) is verified similarly. Since $h\in\mathcal{S}$, the kernels K_h and $\widetilde{K}_h$ are positive on $\mathbb{R}^-$. Then it follows from (3.22)–(3.26) that the kernels $K_{u,v}$ and $\widetilde{K}_{u,v}$ are positive on $\mathbb{R}^-$. As in the proof of Lemma 2.3, we now conclude that either $v(x)\neq 0$ for all $x<0$ or $v\equiv 0$. In the first case, the function $\mathbf{T}_\Theta[h]=\frac{u}{v}$ (by (3.24)) belongs to the Stieltjes class, since the kernels

$$K_{\frac{u}{v}}(x,y)=\frac{K_{u,v}(x,y)}{v(x)v(y)}\quad\text{and}\quad \widetilde{K}_{\frac{u}{v}}(x,y)=\frac{\widetilde{K}_{u,v}(x,y)}{v(x)v(y)}$$

are positive on $\mathbb{R}^-$. To consider the remaining case, we assume that

$$v=\Theta_{21}h+\Theta_{22}\equiv 0,\quad u=\Theta_{21}h+\Theta_{22}\not\equiv 0,\quad h\not\equiv 0;\tag{3.27}$$

the second relation in (3.27) follows from the first, since $\det\Theta\not\equiv 0$, and the last assumption can be made since the case $h\equiv 0$ has been already handled. We will show that assumptions (3.27) are not consistent with the kernels (3.22) and (3.23) be positive.

By (3.27), the kernels (2.35) and (2.36) are equal to zero kernels. The formulas (3.25) and (3.26) represent each of these kernels as the sum of three other positive kernels, from which we conclude that

$$a(x)K_h(x,x)=c(x)K_h(x,x)=b(x)\widetilde{K}_h(x,x)=d(x)\widetilde{K}_h(x,x)\equiv 0.\tag{3.28}$$

and that the bottom diagonal entries in $\mathfrak{K}_{\Theta,J}$ and $\widetilde{\mathfrak{K}}_{\Theta,J}$ are also zero kernels. Therefore, the off-diagonal entries in $\mathfrak{K}_{\Theta,J}$ and $\widetilde{\mathfrak{K}}_{\Theta,J}$ are identical zeros as well:

$$\begin{bmatrix}1 & 0\end{bmatrix}\mathfrak{K}_{\Theta,J}(x,y)\begin{bmatrix}0\\ 1\end{bmatrix}\equiv 0,\quad \begin{bmatrix}1 & 0\end{bmatrix}\widetilde{\mathfrak{K}}_{\Theta,J}(x,y)\begin{bmatrix}0\\ 1\end{bmatrix}\equiv 0.\tag{3.29}$$

Since $h \not\equiv 0$, at least one of the kernels K_h and $\widetilde{K}_h$ is not the zero kernel. If both of them are non-zero, then it follows from (3.28) that $a = b = c = d \equiv 0$, which is not possible, since in this case, the kernels (3.22) and (3.23) are not positive.

If $K_h \equiv 0$ and $\widetilde{K}_h \not\equiv 0$, then it follows from (3.28) that $b = d \equiv 0$, in which case identities (3.29) can be equivalently written as

$$a(x)a(y)\big(\Theta_{11}(x)\Theta_{22}(y) - \Theta_{12}(x)\Theta_{21}(y)\big) - 1 \equiv 0,$$

$$c(x)c(y)\big(\Theta_{11}(x)\Theta_{22}(y) - \Theta_{12}(x)\Theta_{21}(y)\big) - \frac{1}{x} \equiv 0.$$

Combining the latter two identities give $a(x)a(y) \equiv xc(x)c(y)$ which implies $a = c \equiv 0$, which is not possible, as in the previous case. The assumptions $K_h \not\equiv 0$ and $\widetilde{K}_h \equiv 0$ lead to a contradiction in a similar way, which completes the proof. □

Although we assumed that the functions Θ and a, b, c, d in Proposition 3.8 are rational, the same proof goes through for functions differentiable on $\mathbb{R}_-$ except for a discrete set of points at which they have poles.

We next observe that the kernels (2.12) are particular cases of those in (3.22) and (3.23) corresponding to $b = c \equiv 0$ and $a = d \equiv 1$. If Ψ is such that the kernels (2.12) are positive on $\mathbb{R}_-$, then the relation (2.16) guarantees that for any h operator monotone on $\mathbb{R}_-$ (i.e., such that the kernel K_h is positive definite on $\mathbb{R}^-$), the kernel $K_{\mathbf{T}_\Psi[h]}$ is also positive definite on $\mathbb{R}^-$ (in more detail, the function $\frac{\Psi_{22}}{\Psi_{21}}$ belongs to the Stieltjes class, by positivity of the bottom diagonal entries of the kernels (2.12), and hence, any monotone function h may take the same value as the decreasing function $-\frac{\Psi_{22}}{\Psi_{21}}$ at most one point (say $\zeta \in \mathbb{R}^-$); therefore the function ϕ in (2.16) does not vanish on $\mathbb{R}^-\backslash\{\zeta\}$, so that the $K_{\mathbf{T}_\Psi[h]}$ is positive definite on $\mathbb{R}^-\backslash\{\zeta\}$, and therefore, on the whole $\mathbb{R}^-$). We conclude: the linear fractional transformation (2.3) based on a function Ψ such that the kernels (2.12) are positive on $\mathbb{R}^-$, maps the set $\mathcal{M}_{\mathbb{R}^-}$ of operator-monotone functions on $\mathbb{R}^-$ into itself and similarly, it maps the set $\frac{1}{x}\mathcal{M}_{\mathbb{R}^-}$ into itself as well (by (2.17)). For most of classical interpolation problems in the Stieltjes class (in particular, for all problems that can be embedded into general interpolation schemes in [1, 5, 6, 12]) the solution sets are parametrized by linear fractional transformations of the type described above.

Letting $a = d \equiv 0$, $b(x) = x$, $c \equiv 1$ in (3.22) and (3.23) we get the kernels

$$\frac{xy\Theta(x)J_{x,y}\Theta(y)^* - J}{y - x} \succeq 0, \qquad \frac{\Theta(x)J\Theta(y)^* - J_{x,y}}{y - x} \succeq 0 \qquad (x, y \in \mathbb{R}^-). \tag{3.30}$$

It turns out that the linear fractional transformation $\mathbf{T}_\Theta$ based on the function Θ subject to conditions (3.30) maps $\mathcal{M}_{\mathbb{R}^-}$ into $\frac{1}{x}\mathcal{M}_{\mathbb{R}^-}$ and it maps $\frac{1}{x}\mathcal{M}_{\mathbb{R}^-}$ back into $\mathcal{M}_{\mathbb{R}^-}$. A particular example of such function is given by

$$\Theta_0(x) = \begin{bmatrix} 0 & -\frac{1}{x} \\ 1 & 0 \end{bmatrix}: \quad \Theta_0(x)J\Theta_0(y)^* = J_{x,y}, \quad xy\Theta_0(x)J\Theta_0(y)^* = J.$$

Since $\mathbf{T}_{\Theta_0}$ is a bijection on $\overline{\mathcal{S}}$, it follows that any Θ subject to (3.30) is of the form $\Theta = \Theta_0\Psi$ for some Ψ such that the kernels (2.12) are positive on $\mathbb{R}^-$.

We next point out that for the function $\mathfrak{A}$ parametrizing the set $Sol(\mathbf{CFS}_{2n})$ in formula (3.12), neither the kernels $K_{\mathfrak{A},J}$, $\widetilde{K}_{\mathfrak{A},J}$, nor the kernels (3.30) are positive. However, two positive kernels on $\mathbb{R}^-$ associated with $\mathfrak{A}$ exist and are of the form (3.22) and (3.23):

$$\mathfrak{K}_{\mathfrak{A},J}(x,y) = \frac{a(x)a(y)\mathfrak{A}(x)J\mathfrak{A}(y)^* + xyc(x)c(y)\mathfrak{A}(x)J_{x,y}\mathfrak{A}(y)^* - J}{y-x} \succeq 0, \tag{3.31}$$

$$\widetilde{\mathfrak{K}}_{\mathfrak{A},J}(x,y) = \frac{c(x)c(y)\mathfrak{A}(x)J\mathfrak{A}(y)^* + a(x)a(y)\mathfrak{A}(x)J_{x,y}\mathfrak{A}(y)^* - J_{x,y}}{y-x} \succeq 0, \tag{3.32}$$

where

$$a(x) = \frac{\sqrt{-x_0h_0}}{x-x_0}, \quad c(x) = \frac{\sqrt{h_0}}{x-x_0}. \tag{3.33}$$

To justify (3.31) and (3.32), we first compute the inverse of the function Φ in (3.11):

$$\Phi(x)^{-1} = \frac{1}{x-x_0}\begin{bmatrix} -1 & h_0 \\ -x & x_0h_0 \end{bmatrix}$$

and then use it to verify that with a and c defined as in (3.33),

$$\Phi(x)^{-1}J\Phi(y)^{-*} = a(x)a(y)J + xyc(x)c(y)J_{x,y} - (y-x)a(x)a(y)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.34}$$

$$\Phi(x)^{-1}J_{x,y}\Phi(y)^{-*} = c(x)c(y)J + a(x)a(y)J_{x,y} - \frac{(y-x)c(x)c(y)}{xy}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.35}$$

By (3.12), $\mathfrak{A}\Phi^{-1} = \Psi$, where Ψ is defined in (2.3)–(2.7). With this substitution and subsequent use of relations (3.34) and (3.35), the positive kernels (2.12) can be written in terms of the kernels (3.31) and (3.32) as

$$\begin{aligned} K_{\Psi,J}(x,y) &= \frac{\mathfrak{A}(x)\Phi(x)^{-1}J\Phi(y)^{-*}\mathfrak{A}(y)^* - J}{y-x} \\ &= \mathfrak{K}_{\mathfrak{A},J}(x,y) - a(x)a(y)\mathfrak{A}(x)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathfrak{A}(y)^* \preceq \mathfrak{K}_{\mathfrak{A},J}(x,y), \end{aligned}$$

$$\widetilde{K}_{\Psi,J}(x,y) = \frac{\mathfrak{A}(x)\Phi(x)^{-1}J_{x,y}\Phi(y)^{-*}\mathfrak{A}(y)^* J_{x,y} - J_{x,y}}{y-x}$$
$$= \widetilde{\mathfrak{K}}_{\mathfrak{A},J}(x,y) - \frac{c(x)c(y)}{xy}\mathfrak{A}(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathfrak{A}(y)^* \preceq \widetilde{\mathfrak{K}}_{\mathfrak{A},J}(x,y),$$

which imply that the kernels $\mathfrak{K}_{\mathfrak{A},J}$ and $\widetilde{\mathfrak{K}}_{\mathfrak{A},J}$ are also positive.

Proposition 3.8 presents sufficient conditions for a linear transformation $\mathbf{T}_\Theta$ to map the extended Stieltjes class into itself. A follow up question (which we leave open for now) is to which extent conditions (3.22) and (3.23) are necessary and, if they are, is it true that the functions a, b, c, d can always be chosen so that $a(x) = d(x)$ and $b(x) = xc(x)$ for all $x < 0$ (which indeed is the case in all three examples considered above).

4 Determinate Cases

In this section we survey the cases where the problem $\mathbf{CFS}_N$ is determinate and identify the unique solution $\mathbf{f}$ for each case as the extremal solution of certain subproblem. Note that by the identity (2.1),

$$|\text{rank}\widetilde{H}_n - \text{rank}H_n| = |\text{rank}\widetilde{H}_n - \text{rank}(H_nT^*)| \le \text{rank}(\mathbf{e}^*\mathbf{c}) = 1. \tag{4.1}$$

Since $P_n \succeq H_n$ and $P_n - H_n(T^* - x_0 I) = \mathbf{e}^*\mathbf{c}$ we also have

$$\text{rank}H_n \le \text{rank}P_n \le \text{rank}H_n + 1. \tag{4.2}$$

Case 1: The matrix $H_n \succeq 0$ is singular and $\text{rank}H_n = r$. In this case, the measure μ from the integral representation (2.23) is supported by r points and therefore, any solution f to the problem $\mathbf{CFS}_{2n-1}$ is rational with $\deg f = r$, and the leading submatrix H_r of H_n is positive definite. By Kronecker's theorem [23], there is a unique rational function of degree r with the first $2r+1$ Taylor coefficients at x_0 equal $c_0, \dots, c_{2r}$ and therefore, this function is $\mathbf{f}$, the unique solution of the problem $\mathbf{CFS}_{2n-1}$. Since $\text{rank}H_m^{\mathbf{f}}(x_0) = r$ for all $m \ge r$ (again by Kronecker's theorem) all Taylor coefficients $c_m := \frac{\mathbf{f}^{(m)}(x_0)}{m!}$ are uniquely defined by $c_1, \dots, c_{2r}$ via the recursion formula

$$c_m = \begin{bmatrix} c_{m-r} & c_{m-r+1} & \dots & c_{m-1}\end{bmatrix} H_r^{-1}\begin{bmatrix} c_{r+1} \\ \vdots \\ c_{2r}\end{bmatrix}, \quad m > 2r. \tag{4.3}$$

The function $\mathbf{f}$ can be explicitly written (in the spirit of Loewner pencil realizations in [25]) as

$$\mathbf{f}(x) = c_0 + (x - x_0)\mathbf{e}_1^* H_r \left(H_r - (x - x_0)G_r\right)^{-1} H_r\mathbf{e}_1, \tag{4.4}$$

where $\mathbf{e}_1$ denotes the first column of the matrix I_r and $G_r = \left[c_{i+j+2}\right]_{i,j=0}^{r-1}$. Computations similar to those in the proof of Lemma 2.5 verify that the first $2r + 1$ Taylor coefficients of the function on the right side of (4.4) at x_0 are indeed $c_0, \ldots, c_{2r}$.

Case 2: The matrix $\widetilde{H}_n \succeq 0$ is singular and $\operatorname{rank}\widetilde{H}_n = r$. By applying the previous case to $\widetilde{H}_n$ and $x\mathbf{f}(x)$ (rather than H_n and $\mathbf{f}$, we see that the leading submatrix $\widetilde{H}_r$ of $\widetilde{H}_n$ is positive definite and that the only solution $\mathbf{f}$ to the problem $\mathbf{CFS}_{2n-1}$ is recovered from the realization formula

$$x\mathbf{f}(x) = c_0 + x_0c_1 + (x - x_0)\mathbf{e}_1^* \widetilde{H}_r \left(\widetilde{H}_r - (x - x_0)\widetilde{G}_r\right)^{-1} \widetilde{H}_r\mathbf{e}_1,$$

where $\widetilde{G}_r = \left[x_0c_{i+j+3} + c_{i+j+2}\right]_{i,j=0}^{r-1}$. Furthermore, the Taylor coefficients $c_m := \frac{\mathbf{f}^{(m)}(x_0)}{m!}$ are uniquely defined for $m > 2r$ by $c_1, \ldots, c_{2r}$ via the recursion formula

$$c_m = -\frac{c_{m-1}}{x_0} + \left[c_{m-r} + \frac{c_{m-r-1}}{x_0} \;\ldots\; c_{m-1} + \frac{c_{m-2}}{x_0}\right] \widetilde{H}_r^{-1} \begin{bmatrix} x_0c_{r+1}+c_r \\ \vdots \\ x_0c_{2r}+c_{2r-1} \end{bmatrix}. \tag{4.5}$$

The next theorem identifies the unique solution of a determinate problem $\mathbf{CFS}_{2n-1}$ with extremal solutions of its truncations.

Theorem 4.1 *Given $x_0 < 0$ and $c_0, \ldots, c_{2n-1}$, let us assume that $H_n \succeq 0$, $\widetilde{H}_n \succeq 0$. Let $\min\{\operatorname{rank}H_n, \operatorname{rank}\widetilde{H}_n\} = r < n$ and let f_F, f_K and $\mathfrak{f}_F, \mathfrak{f}_K$ be the extremal solutions of the even problem $\mathbf{CFS}_{2r-1}$ and solutions of the odd problem $\mathbf{CFS}_{2r}$, respectively. Then the unique solution $\mathbf{f}$ to the problem $\mathbf{CFS}_{2n-1}$ is equal to*

$$\mathbf{f} = \begin{cases} f_F, & \text{if } \operatorname{rank}H_n = \operatorname{rank}\widetilde{H}_n = \operatorname{rank}P_n = r, \\ f_K, & \text{if } \operatorname{rank}H_n = \operatorname{rank}\widetilde{H}_n = r, \quad \operatorname{rank}P_n = r + 1, \\ \mathfrak{f}_F, & \text{if } \operatorname{rank}H_n = r, \quad \operatorname{rank}\widetilde{H}_n = r + 1, \\ \mathfrak{f}_K, & \text{if } \operatorname{rank}H_n = r + 1, \quad \operatorname{rank}\widetilde{H}_n = r. \end{cases} \tag{4.6}$$

Proof By the definition of r, we have $H_r \succ 0$, $\widetilde{H}_r \succ 0$, and at least one of the matrices H_{r+1} and $\widetilde{H}_{r+1}$ is singular. If both of them are singular, then $\mathbf{f}$ is equal either to f_F or to f_K, by Remark 2.8. If $\operatorname{rank}P_{r+1} = r$, then $\mathbf{f} = f_F$, by Theorem 3.1, and since P_{r+1} is singular, the extended matrix P_n has the same rank r. If $\operatorname{rank}P_{r+1} = r + 1$, then $\mathbf{f} = f_F$ and, in case $r < n - 1$, the matrix P_{r+2} is singular, by (4.2). Therefore, $\operatorname{rank}P_n = \operatorname{rank}P_{r+2} = r + 1$, which completes the verification of the two top cases in (4.6). Two other cases follow by Remark 3.7. By (4.1), the cases listed in (4.6) cover or possible degeneracies of H_n and $\widetilde{H}_n$. □

We next consider the odd problem $\mathbf{CFS}_{2n}$. In each singular case listed in (4.6), the problem $\mathbf{CFS}_{2n}$ has the same unique solution $\mathbf{f}$ if the given c_{2n} happens to be equal to $\frac{\mathbf{f}^{(2n)}(x_0)}{(2n)!}$ and has no solutions otherwise. Below, we classify these cases in terms of the rank of H_n.

Theorem 4.2 *Given $x_0 < 0$ and $c_0, \dots, c_{2n-1}, c_{2n}$, let us assume that $H_n \succeq 0$, $\widetilde{H}_n \succeq 0$ and $P_{n+1} \succeq 0$. The problem $\mathbf{CFS}_{2n}$ has a unique solution if and only if one of the following holds:*

1. $\mathrm{rank} H_n \leq n-2$;
2. $\mathrm{rank} H_n = n-1$ *and* $c_{2n} = \begin{bmatrix} c_{n+1} & c_{n+2} & \dots & c_{2n-1} \end{bmatrix} H_{n-1}^{-1} \begin{bmatrix} c_n \\ \vdots \\ c_{2n-2} \end{bmatrix}$;
3. $H_n \succ 0$, $\mathrm{rank} \widetilde{H}_n = n-1$ *and*

$$c_{2n} = -\frac{c_{2n-1}}{x_0} + \begin{bmatrix} c_{n+1} + \frac{c_n}{x_0} & \dots & c_{2n-1} + \frac{c_{2n-2}}{x_0} \end{bmatrix} \widetilde{H}_{n-1}^{-1} \begin{bmatrix} x_0 c_n + c_{n-1} \\ \vdots \\ x_0 c_{2n-2} + c_{2n-3} \end{bmatrix};$$

4. $H_n \succ 0$, $\widetilde{H}_n \succ 0$ *and* $c_{2n} = \boldsymbol{\alpha}$ *or* $c_{2n} = \boldsymbol{\alpha} + \frac{\boldsymbol{\beta}^2}{\boldsymbol{\delta}}$ *(see (2.52)).*

Proof Let $\mathrm{rank} H_n \leq n-2$ and let $\mathbf{f}$ be the unique solution to the problem $\mathbf{CFS}_{2n-1}$. Then the matrices

$$\left[\frac{\mathbf{f}^{(i+j)}(x_0)}{(i+j)!}\right]_{i,j=0}^{n} \quad \text{and} \quad P_{n+1} = \left[c_{i+j}\right]_{i,j=0}^{n}$$

are positive semidefinite Hankel extension of P_n. Since $\mathrm{rank} H_n \leq n-2$, the matrix P_n is singular (by (4.5)) and therefore, it admits a *unique* positive semidefinite Hankel extension. Therefore, $\frac{\mathbf{f}^{(2n)}(x_0)}{(2n)!} = c_{2n}$ and $\mathbf{f}$ solves the problem $\mathbf{CFS}_{2n}$, which completes the proof of (1). Part (2) follows from **Case 1** considered above and the formula (4.3) for $r = n-1$ and $m = 2n$. Similarly, part (3) follows from **Case 2** and formula (4.5). Part (4) has been covered in Theorem 3.1 □

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Harmonic Analysis of Some Arithmetical Functions

Roger Gay and Ahmed Sebbar

Abstract We study three functions which are power series in the variable z, Dirichlet series in the variable s and with coefficients given by arithmetical functions. A strong point is to relate these functions to some Hilbert spaces. Three main ingredients are used: an estimate of Davenport on sums of Möbius functions, a result of Lucht on convolutions of arithmetical Dirichlet series and the introduction of an operation $\otimes$ on power series, naturally associated with the mentioned Hilbert spaces.

Keywords Arithmetical functions · Franel integral · Riesz basis · Smith determinant

Mathematics Subject Classification (2000) 11M35, 11M41, 43A50

1 Introduction

Several formal trigonometrical expansions of the Analytic Number Theory are of Harmonic Analysis nature. For instance, they are periodic or almost periodic Fourier series of their sums. The main goal of the present paper is to prove a corresponding result for three arithmetical functions called $\mathcal{L}_s$, $\mathcal{M}_s$, $\mathcal{C}_s$. The first is the classical polylogarithm function, the second is built from the Möbius function $\mu(n)$ and the third from the Ramanujan sums. The most salient results of the paper can be summarized as follows. We will study some possible links between $\mathcal{L}_s$, $\mathcal{M}_s$, $\mathcal{C}_s$

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by using a theorem of Lucht [31]. For some arithmetical functions, as for example $\frac{\sigma_s(n)}{n^s}$, $\sigma_s(n) = \sum_{d|n} d^s$, $\Re s > 0$ we study the existence of Ramanujan expansion and give the Ramanujan coefficients. The third objective is to look at the problem from Kubert's identities point of view, first solved by Besicovitch [6], of giving an example of a non-trivial real continuous function f on [0, 1] which is not odd with respect to the point $\frac{1}{2}$ and which has the property that for every positive integer k

$$\sum_{h=0}^{k} f(\frac{h}{k}) = 0.$$

Bateman and Chowla [4, 13] gave the two explicit examples of such functions

$$f_1(t) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos(2\pi nt)$$

$$f_2(t) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \cos(2\pi nt)$$

where μ is the Möbius function and λ is the Liouville's function λ, defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^j$ if n is the product of j (not necessarily distinct) prime numbers. The Liouville's function is a multiplicative function, closely related to the Möbius μ function for they coincide on square-free integers. These two functions share many properties, as we will see in the last section.

We introduce some Hilbert spaces and build some Riesz basis from the function $\mathcal{L}_s$ and determine an bi-orthogonal basis. The characterizations of the Riesz basis highlight some Dirichlet series as well as some extension of the famous Smith determinant. We illustrate the Fourier Analysis aspect through the Ramanujan series and their use in the development of arithmetical functions. The last section briefly presents an opening towards dynamical systems, to emphasize that the path inaugurated by Aurel Wintner, Norbert Wiener and Marc Kac may experience a revival in dynamical systems, as in the conjectures of Chowla and Sarnak.

2 Arithmetical Functions

Lambert series are, by definition, series of the form

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n}, \quad a_n \in \mathbb{C}.$$

They were considered in connection with the convergence of power series. If a series $\sum_{n=1}^{\infty} a_n$ converges, then the Lambert series converges for all $x \neq \pm 1$. Otherwise it converges for those values of x for which the series $\sum_{n=1}^{\infty} a_n x^n$ converges [46], and the references therein. In all that follows, it would be of some interest to highlight three equivalences which will be used more or less explicitly in this paper. Formally we have the following diagram, where f and g are two arithmetical functions

$$f(n) = \sum_{d|n} g(d) \iff \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \iff \sum_{n=1}^{\infty} f(n)x^n = \sum_{n=1}^{\infty} g(n) \frac{x^n}{1-x^n}.$$

This is exactly the essence of our work: we are constantly moving between three aspects: arithmetical convolution, Dirichlet series and power series. This is done through the Riemann zeta function or its inverse. To illustrate this, we give some examples [38] (Part Eight), some of which will be used and all the definitions will be given,

1. If $g(n) = \mu(n)$, the Möbius function, then

$$\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} = x.$$

2. If $g(n) = \lambda(n)$, the Liouville function, the associated Lambert series is the Jacobi theta function

$$\sum_{n=1}^{\infty} \lambda(n) \frac{x^n}{1-x^n} = x + x^4 + x^9 + x^{16} + \cdots.$$

3. If $\Phi(n)$ is Euler's totient function, then for $|x|<1$

$$\sum_{n=1}^{\infty} \Phi(n) \frac{x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

4. If $G_1(x) = \sum_{n=1}^{\infty} g(n) \frac{x^n}{1-x^n}$ and $G_2(x) = \sum_{n=1}^{\infty} g(n)x^n$, then

$$G_1(x) = \sum_{n=1}^{\infty} G_2(x^n).$$

When $g(n)$ is a known arithmetical function, like $\mu(n)$ or $\lambda(n)$ or $\Phi(n)$, the previous relations reflect deep arithmetical identities. On the other hand some elementary functions $g(n)$ can produce non trivial sums. For example if $g(n) = \frac{1}{n}$ and $G_1(x) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1-x^n}$, then

$$e^{-G_1(x)} = \prod_{n=1}^{\infty} (1 - x^n),$$

a well known function in the theory of partitions.

The notion of Kubert's identity is important for us, before defining it we introduce a fundamental function

$$\{t\} = \begin{cases} t - \lfloor t \rfloor - \frac{1}{2} & \text{if} \quad t \neq \lfloor t \rfloor \\ 0 & \text{if} \quad t = \lfloor t \rfloor \end{cases} \tag{2.1}$$

admitting the Fourier expansion

$$\{t\} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi m t)}{m},$$

which extends into a formal summation expansion

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \{nt\} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{A_n}{n} \sin(2\pi n t),$$

where $A_n = \sum_{d|n} a_d$. This reveals a property of the sequence $(\{nt\})_{n \geq 1}$, closely related to the main concern of this paper. We have the well known result

$$\int_0^1 (\{rt\}\{st\})\, dt = \frac{\gcd(r,s)}{12\, \mathrm{lcm}(r,s)} = \frac{\gcd(r,s)^2}{12rs}. \tag{2.2}$$

Another example that we will meet is the expansion, $t \notin 2\pi\mathbb{Z}$

$$\log\left(2\left|\sin\frac{t}{2}\right|\right) = -\sum_{n=1}^{\infty} \frac{\cos nt}{n} \tag{2.3}$$

which leads to the formal identity, for irrational t

$$\sum_{n=1}^{\infty} \frac{c_n \log(2|\sin n\pi t|))}{n} = -\sum_{n=1}^{\infty} \frac{G_n \cos(2n\pi t)}{n},$$

where again $G_n = \sum_{d|n} c_d$. The validity of this equality has been discussed by Davenport in [14] and also by Chowla in [12], who observed that

$$\int_0^1 \log(2|\sin r\pi t|) \log(2|\sin s\pi t|)\, dt = \frac{\pi^2}{12} \frac{\gcd(r,s)^2}{rs}. \tag{2.4}$$

The formulas (2.2) and (2.4) are named Franel integrals in [43]. Beside Number Theory, the functions (2.1) and (2.3) appear in Fourier and Harmonic Analysis where (2.2) and (2.4) find an interpretation. To give the mean idea we cite the following fact: the sequence of functions

$$1, \{t\}, \{2t\}, \cdots \{nt\}, \cdots$$

is a basis for the Hilbert space $(L^2([0, \frac{1}{2}), dt)$, dt being the Lebesgue measure. This kind of results, with very interesting connections with questions in Number Theory, appeared in [22, 48].

Another point of view, which we only briefly evoke here and also in the Sect. 8, is the following: We fix a positive integer p and define on the unit interval the p-Bernoulli map, an extension of (2.1), the function

$$\psi_p(x) = px - \lfloor px \rfloor, \quad \{x\}_p = \psi_p(x) - \frac{1}{2}$$

which admits the Fourier series expansion

$$\{x\}_p = -\sum \frac{2\cos(2n\pi x - \frac{1}{2}p\pi)}{(2n\pi)^p}. \tag{2.5}$$

We look at $\psi_p(x)$ as a one-dynamical system on the space $(0, 1)$, as in [23]. The associated Perron–Frobenius operator P_{ψ_p} is defined by

$$\left(P_{\psi_p} u\right)(x) = \sum_{y \in \psi^{-1}(x)} \frac{u(y)}{|\psi'(y)|} = \frac{1}{p}\left\{u(\frac{x}{p}) + u(\frac{x+1}{p}) + \cdots + u(\frac{x+p-1}{p})\right\}.$$

If u is an eigenvector of P_{ψ_p}, associated to the eigenvalue λ, then

$$\lambda p u(px) = u(x) + u(x + \frac{1}{p}) + \cdots + u(x + \frac{p-1}{p}). \tag{2.6}$$

We see that if, for example, $\lambda = 1$, the eigenfunctions satisfy certain functional equations, similar to those satisfied by the function $\log\Gamma$. We give few fundamental examples:

1. Bernoulli polynomials are given by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)t^n, \quad B_0(x) = 1, \ \ B_1(x) = x-\frac{1}{2}, \ \ B_2(x) = x^2-x+\frac{1}{6}, \cdots$$

   and they satisfy

$$\sum_{n=0}^{\infty}\left(B_n(x)+\cdots+B_n(x+\frac{k-1}{k})\right)t^n = k\sum_{n=0}^{\infty}(\frac{t}{k})^n B_n(kx).$$

   So the eigenvalues are $\lambda = k^{-n}$.
2. Hurwitz zeta function, defined for $\Re s > 1$ by

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^s}$$

   for which we have

$$\zeta(x,s)+\cdots+\zeta(x+\frac{k-1}{k},s) = k^s\zeta(kx,s)$$

   and the eigenvalues are $\lambda = k^{s-1}$. It satisfies for $\Re s > \frac{1}{2}$ the Franel type integral [33]

$$\int_0^1 \zeta(\{ax\},1-s)\zeta(\{bx\},1-s)\,dx = \frac{2\Gamma^2(s)\zeta(2s)}{(2\pi)^{2s}}\left(\frac{\gcd(a,b)}{\operatorname{lcm}(a,b)}\right)^s.$$

   The integral diverges for $\Re s \leq \frac{1}{2}$.
3. The polylogarithm function defined (for $|z| < 1, \Re s \geq 1$ or $|z| \leq 1, \Re s > 1$) by

$$\mathcal{L}_s(z) = \sum_{n\geq 1}\frac{z^k}{k^s}.$$

For $s = k$ an integer the polylogarithm function is related to the Bernoulli polynomial $B_k(X)$ by

$$B_k(\lfloor\theta\rfloor) = -\sum_{n\neq 0}\frac{e^{2i\pi n\theta}}{n^k},$$

which is just (2.5) when $k = p = 1$. In order to study their relation to the Perron–Frobenius operator we introduce a new notion.

Definition 2.1 According to Kubert [29, 34] we say that a function $f(x)$, where x varies over $\mathbb{Q}/\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$, satisfies a Kubert identity if it verifies the functional equations

$$f(x) = m^{s-1} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \qquad (\star_s)$$

for every positive integer m. Here s is some fixed parameter.

It is apparent that this definition is more restrictive than the one given for eigenfunctions of the Perron–Frobenius operator. The derivative of a differentiable function satisfying $(\star_s)$ satisfies $(\star_{s-1})$. A very instructive example is given by the following example: From the polynomial relation

$$X^n - 1 = \prod_{\eta^n=1} (\eta X - 1)$$

we deduce that

$$e^{2i\pi nx} - 1 = \prod_{k=0}^{n-1} (e^{2i\pi(x+\frac{k}{n})} - 1), \quad x \in \mathbb{Q}/\mathbb{Z},\ x \neq 0$$

so that if $f(x) := \log|e^{2i\pi x} - 1|$, then

$$f(nx) = \sum_{k=0}^{n-1} \log|2\sin\pi(x + \frac{k}{n})| = \sum_{k=0}^{n-1} f(x + \frac{k}{n}). \tag{2.7}$$

This property of the function $f(x) := \log|e^{2i\pi x} - 1|$ is connected with the Franel type equality (2.4).

Let $\mathcal{K}_s$, $s \in \mathbb{C}$, be the complex vector space of all continuous maps $f : (0,1) \to \mathbb{C}$ which satisfy the Kubert identity $(\star_s)$ for every positive integer m and every x in $(0,1)$. It is easy to see directly that the function $\mathcal{L}_s(z)$ satisfies the relation $(\star_s)$. More precisely [34, p.287]

Theorem 2.2 *The complex vector space $\mathcal{K}_s$ has dimension 2, spanned by one even element ($f(x) = f(1-x)$) and one odd element ($f(x) = -f(1-x)$). Each $f(x) \in \mathcal{K}_s$ is real analytic.*

This is an interesting interpretation of an important result. In fact if

$$\mathfrak{l}(x) = \mathcal{L}_s(e^{2i\pi x})$$

we should have, according to this theorem, a linear combination

$$\mathfrak{l}(x) = A_s\zeta(1-s, x) + B_s\zeta(1-s, 1-x).$$

The values of the coefficients are given in [34, p.308]

$$A_s = \frac{i(2\pi)^s e^{-\frac{i\pi s}{2}}}{2\Gamma(s)\sin(\pi s)}, \quad B_s = \frac{-i(2\pi)^s e^{\frac{i\pi s}{2}}}{2\Gamma(s)\sin(\pi s)}.$$

This is precisely an another formulation of Lerch's transformation formula for the function

$$\Phi(z, s, \nu) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\nu)^s}, \quad |z| < 1, \ \nu \neq 0, -1, -2, \cdots$$

which is [18, p.29]

$$\Phi(z, s, \nu) =$$
$$iz^{-\nu}(2\pi)^{s-1}\Gamma(1-s)\left\{e^{\frac{-i\pi s}{2}}\Phi[e^{-2i\pi\nu}, 1-s, \frac{(\log z)}{2i\pi}] - e^{i\pi(\frac{s}{2}+2\nu)}\Phi[e^{2i\pi\nu}, 1-s, 1-\frac{(\log z)}{2i\pi}]\right\}$$

and which reduces to the functional equation of the Riemann zeta function when $z = 1, \nu = 0, \Re s > 1$.

Remark 2.3 The theorem of Milnor stated above asserts that every function in the space $\mathcal{K}_s$ is real analytic. These functions are eigenfunctions corresponding to the eigenvalue $\lambda = 1$ of the Perron–Frobenius operator. However the later has eigenfunctions corresponding to the eigenvalue $\lambda = \frac{1}{2}$ which are continuous but nowhere differentiable. As mentioned in [23, p.361] the Tagaki function (or the blancmange function) $T(x)$ is an example of a such function. This function is defined by

$$T(x) = \sum_{n=1}^{\infty} \Psi(2^n x) - \frac{1}{2},$$

where

$$\Psi(x) = \inf\{|x-n|, n \in \mathbb{Z}\} = \left|x - 2\left\lfloor \frac{x+1}{2} \right\rfloor\right|$$

is the sawtooth function, periodic of period 1.

3 Three Power Series

The essential of the analytic properties of the polylogarithm function $\mathcal{L}_s(z) = \sum_{n\geq 1} \frac{z^k}{k^s}$ come from the integral representation

$$\mathcal{L}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z}\, dt, \quad \Re s > 0,\ z \notin (1, \infty).$$

Let $\vartheta = z\frac{d}{dz}$ be Boole's differential operator. We define an inverse of ϑ by

$$\vartheta^{-1} f(z) = \int_0^z f(u) \frac{du}{u}$$

defined on the class of analytic functions near the origin, and vanishing at the origin. For $s = n$ a positive integer we have the symbolic representation as an iterated integral

$$\mathcal{L}_n(z) = \vartheta^{-n} \frac{z}{1-z}. \qquad (3.1)$$

To define the next function we recall first the definition of the Möbius arithmetical function

$$\mu(n) = \mu_n = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n \text{ has one or more repeated prime factors} \\ (-1)^k \text{ if } n \text{ is the product of } k \text{ prime factors.} \end{cases}$$

The importance of the Möbius function lies in the following inversion

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} f(d)\mu(\frac{n}{d}) = \sum_{d|n} f(\frac{n}{d})\mu(d). \qquad (3.2)$$

The generalized Möbius inversion may be formulated as follows: If t varies on the half-line t > 0, and if either $g(t) = O(t^{-1-\eta})$ holds for some $\eta > 0$ or $h(t) = O(t^{-1-\delta})$ holds for some $\delta > 0$ then

$$h(t) = \sum_{n=1}^{\infty} g(nt)$$

is equivalent to

$$g(t) = \sum_{n=1}^{\infty} \mu(n) h(nt).$$

The main objective of this section is the study the relationships between the three functions $\mathcal{L}_s(z)$, $\mathcal{M}_s(z)$ and $\mathcal{C}_s(z)$ defined by the following power series:

1. $\mathcal{L}_s(z) = \sum_{k\geq 1} \frac{z^k}{k^s}$, $|z| \leq 1$, $\Re s > 1$ or $|z| < 1$, $\Re s \geq 0$,
2. $\mathcal{M}_s(z) = \sum_{k\geq 1} \mu_k \frac{z^k}{k^s}$, $|z| \leq 1$ $\Re s > 1$ or $|z| < 1$, $\Re s \geq 0$. This series is most known when $s = 0$ and $|z| < 1$. It amounts to the series $\sum_{k\geq 1} \mu_k z^k$ studied by Bateman and Chowla [4, 13]. They use the crucial estimates for sums of Möbius functions values, due to Davenport [14]: For every $A > 0$, there exists a constant $D(A)$ such that, uniformly for $|z| \leq 1$

$$\left| \sum_{0<j\leq x} \mu(j) z^j \right| \leq D(A) \log(x+1)^{-A}. \tag{3.3}$$

3. $\mathcal{C}_{s,l}(z) = \sum_{k\geq 1} c_k(l) \frac{z^k}{k^s}$ $|z| \leq 1$ $\Re s > 1$ or $|z| < 1$, $\Re s \geq 0$,

where $c_k(l)$ is the Ramanujan sum

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{n} e^{2i\pi \frac{an}{q}} = \sum_{\substack{a=1 \\ (a,q)=1}}^{n} \cos(2\pi \frac{an}{q}).$$

As we will see the series $\mathcal{C}_{s,l}(z) = \sum_{k\geq 1} c_k(l) \frac{z^k}{k^s}$ is most known only when $z = 1$ and $\Re s > 1$. Its sum was given by Ramanujan, and simplified methods were found by Estermann and others.

For fixed n, $c_q(n)$ is a multiplicative function: if q_1, q_2 are relatively prime, then

$$c_{q_1}(n) c_{q_2}(n) = c_{q_1 q_2}(n).$$

Moreover $c_q(n)$ is a periodic function of n with period q. When $(m, k) = 1$ we have $c_k(m) = \mu_k$, and when $(m, k) = k$ we have $c_k(m) = \Phi(m)$, Φ being the Euler's totient function, with for every positive integer N, $\Phi(N)$ is the number of positive integers less than or equal to N and relatively prime to N. More generally Hölder

[25] showed that Ramanujan's sum can also be expressed in closed form as follows:

$$c_k(m) = \frac{\Phi(m)}{\Phi\left(\frac{m}{(k,m)}\right)}\mu\left(\frac{m}{(k,m)}\right).$$

Three well known properties of the Φ-function are important for further extension. For every positive integer N

$$N = \sum_{d|N} \Phi(d)$$

$$\Phi(N) = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right). \tag{3.4}$$

An important property of the Ramanujan coefficients is their orthogonality, that can be used to compute the Ramanujan coefficients

Lemma 3.1 (Orthogonality Relations) *Let Φ the Euler's totient function, then*

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n\le x} c_r(n)c_s(n) = \Phi(r)$$

if $r = s$ and zero otherwise. More generally

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n\le x} c_r(n)c_s(n+h) = c_r(h)$$

if $r = s$ and zero otherwise.

The functions $\mathcal{M}_s, \mathcal{L}_s$, tough different in nature, share the same difference-differential equation

$$z\frac{\partial}{\partial z} f(z,s) = f(z,s-1), \tag{3.5}$$

but the series $\mathcal{M}_s(z)$ does not seem to have attracted much attention. For the particular case $z = 1$ and $\Re s > 1$ we have with $\sigma_{s-1}(n) = \sum_{d|n} d^{s-1}$

$$\mathcal{L}_s(1) = \zeta(s), \quad \mathcal{M}_s(1) = \frac{1}{\zeta(s)}, \quad \mathcal{C}_{s,l}(1) = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}.$$

The last equality can be understood in the framework of Ramanujan-Fourier series: Given an arithmetical function $a : \mathbb{N} \to \mathbb{C}$, the Ramanujan-Fourier series for a is the series

$$a(n) = \sum_{q=1}^{\infty} a_q c_q(n), \quad n \in \mathbb{N}.$$

The coefficients can be computed by using the previous orthogonality relations, but we will be concerned by a different kind of approach.

Let $f,\ g : \mathbb{N} \to \mathbb{C}$ two arithmetical functions. The Dirichlet convolution product of f, g is defined by

$$f \star g(n) = \sum_{d|n} f(d) g(\frac{n}{d}),\ n \in \mathbb{N}.$$

For example if $1 : n \to n$ is the identity arithmetical function, the inversion formula (3.2) is just

$$f = g \star 1 \iff g = f \star \mu.$$

To study the expansion in Ramanujan series, or to find relations between the three series $\mathcal{L}_s(z)$, $\mathcal{M}_s(z)$ and $\mathcal{C}_{s,l}(z)$ we will make use of a result of H. Delange [16] and a result of L. Lucht [31].

Theorem 3.2 (Delange) *Suppose that*

$$f(n) = \sum_{d|n} g(d) = (g \star 1)(n)$$

and that

$$\sum_{n=1}^{\infty} 2^{\omega(n)} \frac{|g(n)|}{n} < \infty, \tag{3.6}$$

where $\omega(n)$ is the number of distinct prime divisors of n. Then f admits a Ramanujan expansion with

$$\hat{f}(q) = \sum_{m=1}^{\infty} \frac{g(qm)}{qm}.$$

More precisely for each n the sum

$$\sum_{q=1}^{\infty} \hat{f}(q) c_q(n)$$

is absolutely convergent and is equal to $f(n)$.

We can weaken this statement by observing that from the prime decomposition of a positive integer $n = \prod_{i=1}^{k} p_i^{e_i}$, $e_i \geq 1$, the number of divisors $d(n)$ and the number of prime divisors $\omega(n)$ are related by

$$d(n) = (e_1 + 1) + \cdots + (e_k + 1) \geq 2^{\omega(n)}.$$

So the condition (3.6) can be weakened by asking only

$$\sum_{n=1}^{\infty} d(n) \frac{|g(n)|}{n} < \infty. \tag{3.7}$$

To give an application of this theorem we begin by a review of some additional properties of Ramanujan sums. First of all for $z = 1$ the sum is the celebrated formula of Ramanujan [40] (p.199): For $k > 1$ and $s \in \mathbb{C} \setminus \{1\}$ with $\Re s > 0$

$$\sum_{m=1}^{\infty} \frac{c_k(m)}{m^s} = \zeta(s) \sum_{d|k} d^{1-s} \mu(\frac{k}{d}), \tag{3.8}$$

and even for $s = 1$ we have [40] (p.199)

$$\sum_{m=1}^{\infty} \frac{c_k(m)}{m} = -\sum_{d|k} \mu(\frac{k}{d}) \log d = -\Lambda(k),$$

where $k > 1$ and $\Lambda(k)$ is Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m \text{ for some } m \geq 1 \\ 0 \qquad\qquad \text{else} \end{cases}$$

Another formula of Ramanujan [40] (p.185) is

$$\sum_{k=1}^{\infty} \frac{c_k(m)}{k^s} = \frac{\sigma_{1-s}(m)}{\zeta(s)}, \tag{3.9}$$

valid for $\Re s > 1$ and also for $s = 1$. In this case the sum vanishes. According to Ramanujan this assertion is equivalent to the Prime Number Theorem

This formula results directly from Delange's theorem. The next role will be played by the following lemma.

Lemma 3.3 (Romanoff [42]) *If $k < n$ and $f(u)$ is any function defined on the positive integers then*

$$\sum_{d|n} \mu(\frac{n}{d}) f((d,k)) = 0. \tag{3.10}$$

This Lemma can be proved, for example, by establishing that

$$\sum_{d|n} \mu(\frac{n}{d}) f((d,k)) = \sum_{\delta} f(\delta) \sum_{\substack{d|n \\ (d,k)=\delta}} \mu(\frac{n}{d}).$$

This lemma is very important of the following reason. Let (x_n) be a sequence in a Hilbert space $\mathcal{H}$ with an inner product $\langle , \rangle$ and let $g : \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\} \to \mathbb{C}$ be an arithmetical function such the $\langle x_n, x_m \rangle = g(n,m)$. Then the sequence

$$y_n = \sum_{d|n} \mu(\frac{n}{d}) x_d$$

is an orthogonal sequence. Another important result is the following

Theorem 3.4 (Lucht) *Let $g : \mathbb{N} \to \mathbb{C}$ be an arbitrary arithmetical function. Then the following assertions are equivalent:*

1. *The series $\hat{g}(k) = \sum_{n \geq 1} g(n) c_n(k)$ converges (absolutely) for every $k \in \mathbb{N}^*$ and determine an arithmetical function $\hat{g}$.*
2. *The series $\gamma(k) = k \sum_{n \geq 1} \mu(n) g(kn)$ converges (absolutely) for every $k \in \mathbb{N}^*$ and determine an arithmetical function γ.*

In the case of convergence we have the convolution products $\gamma = \mu \star \hat{g}$ or $1 \star \gamma = \hat{g}$.

As a first application we take $g(n) = g_{z,s}(n) = \dfrac{z^n}{n^s}$, with $|z| \leq 1$, $\Re s > 1$ or $|z| < 1$, $\Re s \geq 0$ to obtain by the theorem of Lucht:

$$\hat{g}(k) = \hat{g}_z(k) = \sum_{n \geq 1} c_n(k) \frac{z^n}{n^s} = \mathcal{C}_{s,k}(z).$$

Hence

$$\gamma(k) = \gamma_{z,s}(k) = \frac{1}{k^{s-1}} \sum_{n \geq 1} \mu_n \frac{z^{nk}}{n^s} = \frac{1}{k^{s-1}} \mathcal{M}_s(z^k).$$

For $s = 1$ the uniform convergence on the unit closed disk of the series $\sum_{n\geq 1} \frac{\mu(n)}{n} z^n$ results from the uniform convergence on $\mathbb{R}$ of the series $\sum_{n\geq 1} \frac{\mu(n)}{n} e^{2i\pi\theta}$, resulting from (3.3) and the maximum principle. For the sake of completeness we give all the details of this important result. The following lemma, due to Cahen and Jensen, is classical [21]:

Lemma 3.5 *If the Dirichlet series* $f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}$ *is convergent for* s_0*, then it is also convergent for any* s *in the cone* $|\arg(s - s_0)| \leq \alpha < \frac{\pi}{2}$ *(Stolz angle). Furthermore the series is uniformly convergent on any compact set of the cone, as well as any of its derivatives and*

$$f^{(n)}(s) = (-1)^n \sum a_n \lambda_k^n e^{-\lambda_n s}.$$

Accordingly

$$\lim_{s\to 1} \mathcal{M}_s(e^{2i\pi\theta}) = \mathcal{M}_1(e^{2i\pi\theta}).$$

The Ramanujan-Fourier transform of the arithmetical function $g(n) = \frac{z^n}{n}$ is

$$\hat{g}(k) = \sum_{n\geq 1} c_n(k) \frac{z^n}{n}$$

which converges for $|z| < 1$. It also converges for $|z| = 1$. In fact

$$\gamma(k) = k \sum_{n\geq 1} \mu(n) \frac{z^{nk}}{nk} = \sum_{n\geq 1} \mu(n) \frac{z^{nk}}{n}$$

converge uniformly on the closed unit disc by Davenport's estimate (3.3). By using 3.4 we obtain the convergence of the series $\hat{g}(k)$ with $\gamma(k) = \mathcal{M}_1(z^k)$, and finally

$$\sum_{n\geq 1} c_n(l) \frac{z^n}{n} = \sum_{d|l} \mathcal{M}_1(z^d)$$

for $|z| = 1$, by using a Möbius inversion of $\gamma = \mu \star \hat{g}$. The maximum principle asserts that this equality continues to be valid even for $|z| \leq 1$.

In the same vein we have:

$$\mathcal{C}_{s,l}(z) = \sum_{d|l} d^{1-s} \mathcal{M}_s(z^d),$$

which is an interesting link between the two series $\mathcal{C}_{s,l}(z)$ and $\mathcal{M}_s(z)$. The following lemma is elementary and it is just a variation of (2.7).

Lemma 3.6 *We have*

$$\sum_{1 \le h \le m} e^{2i\pi \frac{nhk}{m}} = \begin{cases} m \text{ if } m|nk \\ 0 \text{ otherwise.} \end{cases},$$

or, equivalently, for all $n, m \in \mathbb{N}^*$ and $w, z \in \mathbb{C}$, we have

$$\sum_{w : w^m = z} w^n = m z^{\frac{n}{m}}$$

if $m|n$ and zero otherwise.

Remark 3.7 The meaning of $nk \in m\mathbb{N}^*$ is the following: We first observe that $k\mathbb{Z} \cap m\mathbb{Z} = \text{lcm}(k, m)\mathbb{Z}$, hence

$$nk \in \text{lcm}(k, m)\mathbb{N}^* \iff n \in \delta(k, m)\mathbb{N}^*, \quad \delta(k, m) := \frac{\text{lcm}(k, m)}{k}$$

and

$$\sum_{1 \le h \le m} e^{2i\pi \frac{nhk}{m}} = \begin{cases} m \text{ if } n \in \delta(k, m)\mathbb{N}^* \\ 0 \text{ otherwise} \end{cases} \tag{3.11}$$

If we choose $z = e^{2i\pi \frac{h}{m}}$ with some fixed $m \in \mathbb{N}^*$ and $1 \le h \le m$, and denote simply

$$g_{z,1}(n) = g_{\frac{h}{m}}(n) = \frac{e^{2i\pi \frac{nh}{m}}}{n}, \quad \gamma_{z,1} = \gamma_{\frac{h}{m}}$$

we get

$$\gamma_{\frac{h}{m}}(k) = k \sum_{n \ge 1} \mu(n) g_{\frac{h}{m}}(kn) = k \sum_{n \ge 1} \mu(n) \frac{e^{2i\pi \frac{nkh}{m}}}{nk} = \mathcal{M}_1(e^{2i\pi \frac{hk}{m}})$$

and

$$\sum_{1 \le h \le m} \gamma_{\frac{h}{m}}(k) = m \sum_{n \in \delta(k,m)\mathbb{N}^*} \frac{\mu(n)}{n}$$

or

$$\sum_{1 \le h \le m} \mathcal{M}_1(e^{2i\pi\frac{hk}{m}}) = m \sum_{n \in \delta(k,m)\mathbb{N}^*} \frac{\mu(n)}{n} = 0.$$

As the argument usually used in the proof of the last equality is very present in the present study we give a slightly more general result. We are going to show the following lemma

Lemma 3.8 *We have for* $\Re s > 1$

$$\sum_{n \ge 1} \frac{\mu(qn)}{n^s} = \frac{\mu(q)q^s}{\Phi_s(q)\zeta(s)}, \tag{3.12}$$

with

$$\Phi_s(q) = q^s \prod_{p|q}(1 - p^{-s}) = \sum_{d|q} d^s \mu(\frac{q}{d}).$$

In particular

1. $\displaystyle\sum_{n \ge 1} \frac{\mu(qn)}{n} = 0$ *for every* $q \in \mathbb{N}^*$.
2. *For* $d \in \mathbb{N}^*$ *we have* $\displaystyle\sum_{n \in d\mathbb{N}^*} \frac{\mu(n)}{n} = 0$

Indeed

$$\frac{1}{\zeta(s)} = \sum_{n \ge 1} \frac{\mu(n)}{n^s} = \prod_p (1-p^{-s}) = \prod_{p|q}(1-p^{-s}).\prod_{p\nmid q}(1-p^{-s}) = \frac{\Phi_s(q)}{q^s} \sum_{n \ge 1,\,(n,q)=1} \frac{\mu(n)}{n^s}$$

$$\frac{\Phi_s(q)}{\mu(q)q^s} \sum_{n \ge 1,\,(n,q)=1} \frac{\mu(qn)}{n^s}.$$

The lemma is obtained by using the Lemma 3.5, or the classical Ingham's Tauberian theorem [28] (p.133). One can show that

$$\sum_{n \in \delta(k,m)\mathbb{N}^*} \frac{\mu(n)}{n} = \mu_{\delta(k,m)} \lim_{s \to 1} \left(\frac{1}{\zeta(s) \prod_{p|\delta(k,m)}} (1 - \frac{1}{p^s}) \right)^{-1} = 0.$$

It is easily seen that $\sum_{1\le h\le m} \gamma_{\frac{h}{m}}(k) = 0$. Hence

$$\sum_{1\le h\le m} \mathcal{M}_1(e^{2i\pi \frac{hk}{m}}) = 0.$$

From the relation $\mu \star \sum_{1\le h\le m} \hat{g}_{\frac{h}{m}} = 0$ we conclude again, by Cahen-Jensen lemma and Möbius inversion, that

$$\lim_{s\to 1} \sum_{j\ge 1} \frac{c_{jm}(k)}{j^s} = \sum_{j\ge 1} \frac{c_{jm}(k)}{j} = 0.$$

Indeed we have $\sum_{1\le h\le m} \hat{g}_{\frac{h}{m}} = 0$, so for every $k \in \mathbb{N}^*$

$$\lim_{s\to 1} m^{1-s} \sum_{j\ge 1} \frac{c_{jm}(k)}{j^s} = \lim_{s\to 1} \sum_{j\ge 1} \frac{c_{jm}(k)}{j^s} = 0$$

since

$$\sum_{1\le h\le m} \sum_{n\ge 1} \frac{c_n(k)}{n^s} e^{2i\pi \frac{nk}{m}} = m^{1-s} \sum_{j\ge 1} \frac{c_{jm}(k)}{j^s}.$$

We also have

$$\lim_{s\to 1} \sum_{1\le h\le m} \mathcal{C}_{s,l}(e^{2i\pi \frac{h}{m}}) = \sum_{d|l} \left(\sum_{1\le h\le m} \mathcal{M}_1(E^{2i\pi \frac{hd}{m}}) \right) = 0$$

by using

$$\mathcal{C}_{s,l}(z) = \sum_{d|l} d^{1-s} \mathcal{M}_s(z^d).$$

This lemma, applied to $\gamma_{\frac{h}{m}(k)} = \mathcal{M}_1(e^{2i\pi \frac{hk}{m}})$ and with $\delta(k,m) = \dfrac{\mathrm{lcm}(k,m)}{k}$, gives

$$\sum_{1\le h\le m} \mathcal{M}_1(e^{2i\pi \frac{hk}{m}}) = m \sum_{j\ge 1} \frac{\mu_{j\delta(k,m)}}{j\delta(k,m)} = \frac{m}{\delta(k,m)} \sum_{j\ge 1} \frac{\mu_{j\delta(k,m)}}{j} = 0.$$

Thus we have a nontrivial example of a function solving the initial Besicovich question. A natural question suggested by the Lemma 3.8 is: Find all the sequences $a = (a_n)_{n\geq 1}$ satisfying the relations $\sum_{j\geq 1} a_{jm} = 0$ for every $m \in \mathbb{N}^*$. According to [39] (p. 294) if

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} a_{jn} = 0,\ j = 1, 2, \cdots$$

then $a_n = 0$ for every $n \geq 1$. As was pointed out in [39] (p.294) the absolute convergence is necessary. We give here an example [32] (p.219) completing (3.12) and showing the necessity of the absolute convergence. It is

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{|\mu(m)|}{m^s} = \frac{n^s \zeta(s)}{\psi_n(s)\zeta(2s)}, \tag{3.13}$$

with

$$\psi_n(s) = \sum_{d|n} d^s |\mu(\frac{n}{d})|.$$

It is interesting to consider the following question later a more general question: For which sequences $a = (a_n)_{n\geq 1} \in l^2$ we have

$$\lim_{\substack{s\to 1 \\ \Re s>1}} \frac{1}{m^{s-1}} \sum_{n\geq 1} a_n \left(\sum_{j\geq 1} \frac{\mu_{j\delta(m,k)}}{j^s} \right) = 0.$$

As we saw in 3.8 we have $\lim_{\substack{s\to 1 \\ \Re s>1}} \sum_{j\geq 1} \frac{\mu_{j\delta(m,k)}}{j^s} = 0$ for every $k \geq 1$.

4 Three Examples of Hilbert Spaces

4.1 Preliminaries

We propose to introduce a binary operation $\otimes$ to combine two power series. Let D be the open unit disk and

$$H_0^2(D) = \left\{ \sum_{n\geq 1} a_n z^n,\ a_n \in \mathbb{C},\ \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

Let **H** the space of functions defined almost everywhere on $\mathbb{R}$, odd and 2-periodic, such that $f_{|(0,1)} \in L^2(0,1)$ (that is we will consider only Fourier sine series expansion of any $f \in L^2(0,1)$) and finally let

$$H^2 = \left\{ \sum_{n\geq 1} \frac{a_n}{n^s},\ a_n \in \mathbb{C},\ \sum_{n\geq 1} |a_n|^2 < \infty \right\}.$$

It is remarkable that from the Hilbert point of view these three spaces are isomorphic but the analytical properties are different but enrich each other. The power series $\mathcal{L}_s$ and $\mathcal{M}_s$ belong to $H_0^2(D)$ provided that $\Re s > \frac{1}{2}$ and will play a preponderant role. We need the following definition [46]

Definition 4.1 Let $A(z) = \sum_{p=1}^{\infty} a_p z^p$ and $B(z) = \sum_{p=1}^{\infty} b_p z^p$ be two power series. We define their Dirichlet product as

$$\sum_{p\geq 1} a_p z^p \otimes \sum_{p\geq 1} b_p z^p = \sum_{p\geq 1} a_p (\sum_{q\geq 1} b_q z^{qp}) = \sum_{q\geq 1} b_q (\sum_{p\geq 1} a_p z^{pq}) = \sum_{n\geq 1} (a \star b)_n z^n$$

where $a \star b$ stands for the Dirichlet convolution of the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. It is clear that the identity element for the binary operation $\otimes$ is $e(z) = z$.

It should be noted that this product comes from the natural formal product of the two Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{z^n}$ and $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$. In other words the map

$$S : \left(H_0^2(D), +, \otimes\right) \longrightarrow (\mathbf{H}, +, .)\,, \quad S(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} \frac{a_n}{z^n} \tag{4.1}$$

is a ring homomorphism.

It is possible to define, transferring $\otimes$ by the map S, a product on the set of Lambert series. But, instead, we give few examples, in particular to show how to compute $\mathcal{M}_0(z) \otimes \mathcal{M}_0(z)$ and evoke the problem that $\mathcal{C}_{s,l}(z) \otimes \mathcal{C}_{s,l}(z)$ poses.

First, we have two useful properties

1. $\mathcal{L}_s(z^m) \otimes \mathcal{M}_s(z^n) = z^{mn}, \quad m, n \in \mathbb{N}^*$.
2. The functions $\mathcal{L}_s(z)$ and $\mathcal{M}_s(z)$ are mutual inverses for the operation $\otimes$.

Second, According to [32] (p.40) we introduce $d(n,k)$ the number of ways of expressing n as a product of k positive factors (of which any number may be unity), expressions in which the order of the factors is different being regarded as distinct. It is a multiplicative function satisfying the functional equation

$$d(n, k+1) = \sum_{d|n} d(n, k).$$

In particular, $d(n,2) = d(n) = \sum_{d|n} 1$, the number of divisors of n. By a simple induction we have

$$\underbrace{\mathcal{L}_s(z) \otimes \cdots \otimes \mathcal{L}_s(z)}_{k \text{ times}} = \sum_{n=1}^{\infty} \frac{d(n,k)}{n^s} z^n.$$

Third, we compute the square of the Möbius function μ for the convolution. For any real number α, we denote by μ_α the multiplicative function defined for all primes p and positive integer k by Dickson [17]

$$\mu_\alpha(1) = 1, \quad \mu_\alpha(p^k) = (-1)^k \binom{\alpha}{k},$$

with

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Then $\mu_1 = \mu$, the Möbius function, $\mu_{-1} = 1$, the constant arithmetical function 1, and $\mu_0 = e$ with $e(1) = 1, e(n) = 0$ if $n > 1$, the neutral element for the Dirichlet convolution. The function μ_α may be defined even for complex α since it is a polynomial in α [9]. It satisfies

$$\mu_{\alpha+\beta} = \mu_\alpha \star \mu_\beta$$

for all real numbers α and β. Let n_a be the number of simple prime divisors of n, that is those primes whose square do not divide n, then

$$\mu \star \mu(n) = (-2)^{n_a}.$$

For sake of completeness we give a very quick proof of this result. Since $\mu \star \mu$ is multiplicative, it is enough to know $\mu \star \mu(p^e)$ for a prime p. But

$$\begin{aligned}\mu \star \mu(p^e) &= \sum_{k=0}^{e} \mu(p^k)\mu(p^{e-k}) \\ &= \mu(p^e) + \mu(p)\mu(p^{e-1}) + \cdots + \mu(p^{e-1})\mu(p) + \mu(p^e).\end{aligned}$$

If $e \geq 3$ and $0 \leq k \leq e$, one of the integers k, $e-k$ is greater than 2, so $\mu(p^k)\mu(p^{e-k}) = 0$ and $\mu(p^e) = 0$.

If $e = 2$

$$\mu \star \mu(p^2) = \mu(p^2) + \mu(p)\mu(p) + \mu(p^2) = \mu(p)\mu(p) = 1.$$

If $e = 1$, $\mu(p) = -1$, $\mu \star \mu(p) = \mu(p) + \mu(p) = -2$. So only the simple prime divisors of n contribute, each by -2. This proves the formula above. It follows that

Proposition 4.2

$$\mathcal{M}_0(z) \otimes \mathcal{M}_0(z) = \sum_{n=1}^{\infty} (-2)^{n_a} z^n.$$

This shows the surprising and not so known formula

$$\frac{1}{\zeta^2(s)} = \sum_{n=1}^{\infty} \frac{(-2)^{n_a}}{n^s}.$$

The same method gives

$$\mathcal{M}_s(z) \otimes \mathcal{M}_s(z) = \sum_{n=1}^{\infty} \frac{(-2)^{n_a}}{n^s} z^n.$$

If we set

$$\underbrace{\mathcal{M}_s(z) \otimes \cdots \otimes \mathcal{M}_s(z)}_{k \text{ times}} = \sum_{n=1}^{\infty} \frac{d'(n,k)}{n^s} z^n$$

we get from the equality $\dfrac{1}{\zeta^{k+1}(s)} = \dfrac{1}{\zeta^k(s)} \dfrac{1}{\zeta(s)}$ the relation

$$d'(n, k+1) = \sum_{d'|n} d'(n,k)\mu(k), \quad d'(n,2) = (-2)^{n_a} = \mu \star \mu(n).$$

As far as we know the map $n \to d'(n,k)$ does not seem to have been studied to the point like what we have, for example, in the estimate (3.3).

Remark 4.3 The situation for the series $\mathcal{C}_{s,l}(z)$ is not as manageable as it is for $\mathcal{L}_s(z)$ and $\mathcal{M}_s(z)$. The product $\mathcal{C}_{s,l}(z) \otimes \mathcal{C}_{s,l}(z)$ is not apparently easy to compute, as we have

$$c_{q_1}(n) c_{q_2}(n) = c_{q_1 q_2}(n)$$

only when q_1, q_2 are relatively prime. We modify the binary operation $\otimes$ by defining for two arithmetical function $f = f(n), g = g(n)$ the following operation [32] (p.154)

$$(f \sqcup g)_n = \sum_{\substack{pq=n \\ (p,q)=1}} f(p)f(q),$$

known as the unitary product, and extend it to powers series by

$$\sum_{n\geq 1} f(n)z^n \boxtimes \sum_{n\geq 1} g(n)z^n = \sum_{n\geq 1}(f \sqcup g)_n z^n.$$

With $f(n) = \dfrac{c_n(l)}{n^s}$ we get

$$(f \sqcup f)_n = \sum_{\substack{pq=n \\ (p,q)=1}} f(p)f(q) = \tilde{d}(n)\frac{c_n(s)}{n^s},$$

where $\tilde{d}(n) = \displaystyle\sum_{(p,q)=1,\ pq=n} 1$, so that

$$\mathcal{C}_{s,l}(z) \boxtimes \mathcal{C}_{s,l}(z) = \sum_{n\geq 1} c_n(l)\frac{z^n}{n^s} \boxtimes \sum_{n\geq 1} c_n(l)\frac{z^n}{n^s} = \sum_{n\geq 1} \tilde{d}(n)c_n(l)\frac{z^n}{n^s}.$$

The arithmetical function $\tilde{d}(n)$ is known as the unitary divisor function. It coincide with $d(n)$ if n is square free.

To understand the Hilbert space structure of $H_0^2(D)$ we recall that the Bergman space $B(D)$ is the space of holomorphic functions f in D for which the integral

$$(f, f) = \int\int_D |f(z)|^2 dx\, dy < \infty.$$

The system of functions $\{1, z, z^2, \cdots\}$ is an orthogonal set. Indeed we have

$$(z^n, z^m) = \int\int_{|z|<1} z^n \bar{z}^m dxdy = \frac{1}{2i(m+1)}\int_{|z|=1} z^n \bar{z}^{m+1} dz = \frac{1}{2(m+1)}\int_0^{2\pi} e^{i(n-m)\theta}\, d\theta.$$

The orthonormalized set is

$$v_n(z) = \sqrt{\frac{n+1}{\pi}} z^n.$$

The Fourier coefficients of $f \in B(D)$,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

are

$$b_n = \sqrt{\frac{n+1}{\pi}} \int\int_{|z|<1} f(z)\bar{z}^n dxdy = \lim_{r\to 1} \sqrt{\frac{n+1}{\pi}} \int\int_{|z|<r} f(z)\bar{z}^n dxdy = \sqrt{\frac{\pi}{n+1}} a_n$$

so that the norm given in the space $H_0^2(D) \subset B(D)$ can be written in terms of the Fourier coefficients

$$\sum_{n=0}^{\infty} |a_n|^2 = \pi \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}.$$

The following lemma from [46] is the analogue of the classical Cauchy's theorem for the new binary operation $\otimes$

Lemma 4.4 (Spira) *If $f(z) = \sum_{p\geq 1} a_p z^p$ and $g(z) = \sum_{p\geq 1} b_p z^p$ are two holomorphic functions on the open disk D then so is $f \otimes g$. Furthermore if R_f, R_g, $R_{f\otimes g}$ are the radius of convergence of f, g, $f \otimes g$ repectively, then*

$$\min(1, R_f, R_g) \leq R_{f\otimes g}.$$

Proof For every fixed $n \geq 3$ and $2 \leq p \leq n-1$ we have $p + \frac{n}{p} \leq n$ for each divisor p of n. In fact It suffices to show it when $2 \leq p \leq \frac{n}{2}$, and in this case $p + \frac{n}{p} \leq \frac{n}{2} + \frac{n}{2} = n$. For $|z| \leq 1$ we have $|z|^n \leq |z|^p |z|^{\frac{n}{p}}$ and for large N:

$$\begin{aligned} \sum_{2\leq n\leq N} \Big(\sum_{p|n,\ 2\leq p\leq n-1} |a_p||b_{\frac{n}{p}}| \Big) |z|^n &\leq \sum_{2\leq n\leq N} \Big(\sum_{\substack{p|n \\ 2\leq p\leq n-1}} |a_p||b_{\frac{n}{p}}| \Big) |z|^p |z|^{\frac{n}{p}} \\ &\leq \Big(\sum_{2\leq p\leq N-1} |a_p||z|^p \Big) \Big(\sum_{2\leq q\leq N-1} |b_q||z|^q \Big) \\ &\leq \Big(\sum_{p\geq 1} |a_p||z|^p \Big) \Big(\sum_{q\geq 1} |b_q||z|^q \Big). \end{aligned} \tag{4.2}$$

This means that for large N:

$$\begin{aligned}
&\Big(\sum_{1\le p\le N}|a_p||z|^p\Big)\otimes\Big(\sum_{1\le q\le N}|b_q||z|^q\Big)\\
&\le |a_1|\sum_{q\ge 1}|b_q||z|^q+|b_1|\sum_{p\ge 1}|a_p||z|^p+\Big(\sum_{1\le p\le N}|a_p||z|^p\Big)\Big(\sum_{1\le q\le N}|b_q||z|^q\Big)\\
&\le |a_1|\sum_{q\ge 1}|b_q||z|^q+|b_1|\sum_{p\ge 1}|a_p||z|^p+\Big(\sum_{p\ge 1}|a_p||z|^p\Big)\Big(\sum_{q\ge 1}|b_q||z|^q\Big).
\end{aligned} \tag{4.3}$$

Hence the conclusion. □

4.2 Historic Facts

1. Around 1944 A. Wintner [48] shows that for $u(t) = \{t\}$, the sequence $u(nt)$ is total in $L^2(0, \frac{1}{2})$ and observes, for the eventual totality of a sequence $(\varphi(nt))_{n\ge 1}$, the possibility to express the conditions in terms of the Möbius inversion. He uses the associated Dirichlet series and shows that the sequence of dilates $\varphi_\tau(nt)$ for $\varphi_\tau(t) = \sqrt{2}\sum_{n\ge 1}\frac{\sin nt}{n^\tau}$ with $\Re\tau > \frac{1}{2}$ is total in $\in L^2(0, 1)$.
2. In 1945 A. Beurling [8] considers the problem of deciding if the system of the dilates $(\psi(nt))_{n\ge 1}$ of a function $\psi \in L^2(0, 1)$ is a total system in $L^2(0, 1)$. To the development $\sum_{n\ge 1} a_n\varphi(nt)$ of the function ψ in the basis $\varphi_n(t) = \sqrt{2}\sin\pi nt = \varphi(nt)$ with $\varphi(t) = \sqrt{2}\sin\pi t$, he associates the Dirichlet series $f(s) = \sum_{n\ge 1}\frac{a_n}{n^s}$, which converges for $\Re s > \frac{1}{2}$, and studies the initial problem using the properties of the Dirichlet series f.
3. From 1990, very intensive research was carried out to link and exploit the correspondence between the two aspects (Fourier series $F \overset{S}{\longleftrightarrow} G$ Dirichlet series) [5, 15, 24, 26, 35, 39] $\cdots$ and the references therein.

   In particular we quote from [24] the following

Theorem 4.5 *Let $\varphi \in L^2(0, 1)$ having the following Fourier expansion*

$$\varphi(t) = \sqrt{2}\sum_{n\ge 1} a_n \sin(\pi nt),$$

then the following are equivalent:

1. *The sequence of dilates* $(\varphi_n)_{n\geq 1}$ *of* φ *form a Riesz basis of* $L^2\,(0, 1)$.
2. *The generating function* $S\varphi(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ *belongs to*

$$\mathcal{H}^\infty = \mathbf{H}^\infty(\{s \in \mathbb{C}, \Re s > 0\}) \cap \mathcal{D}$$

as well as its reciprocal $\frac{1}{S\varphi(s)}$, $\mathcal{D}$ *is the ring of convergent Dirichlet series on the half plane* $\{s \in \mathbb{C}, \Re s > 0\}$.

In particular the dilates of φ_τ *form a Riesz basis of* $L^2(0, 1)$ *if and only if* $\Re\tau > 1$. *In this case* $S\varphi(s) = \zeta(s + \overline{\tau}) = \sum_{n\geq 1} \frac{1}{n^{s+\overline{\tau}}}$ *and* $S^{-1}(s) = \sum_{n\geq 1} \frac{\mu(n)}{n^{s+\overline{\tau}}}$.

Furthermore

Lemma 4.6 *The three rings* $\mathbb{C}([[z]])$, $\mathcal{O}(D)$ *and* $\mathcal{O}_0$ *equipped with the binary composition* $\otimes$ *are commutative ring, with neutral element* z. *The ring of arithmetical functions, equipped with Dirichlet convolution, is an integral domain, factorial, local and isomorphic to the ring of Dirichlet series. The same is true of the ring* $\mathcal{D}$.

We recall that a convergent Dirichlet series is a series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ having a finite abscissa of convergence. This is equivalent to $a_n = O(n^k)$ for some real positive k.

Definition 4.7 Let H be a separable Hilbert space. A basis (x_n) is a Riesz basis for H if it is equivalent to some (and therefore every) orthonormal basis (y_n) for H, that is if there exists a topological isomorphism $L : H \to H$ such that $Lx_n = y_n$ for all n.

The system $\{e^{int},\ n \in \mathbb{Z}\}$ is a Riesz basis for $L^2[-\pi, \pi]$ and a conditional basis for $L^p[-\pi, \pi]$ with $1 < p < \infty,\ p \neq 2$. In evocation of the polylogarithm function, we cite the following example of Babenko given in [2, 44] (p.428, Example 14.4): The systems $\{|t|^{-|\beta|}e^{int},\ n \in \mathbb{Z}\}$ and $\{|t|^{|\beta|}e^{int},\ n \in \mathbb{Z}\}$ with $0 < \beta < \frac{1}{2}$ are bounded conditional bases for $L^2[-\pi, \pi]$ that are not a Riesz basis. Naturally all Riesz bases are equivalent, as do all orthonormal bases of Hilbert spaces. The adjoint mapping $L^* : H \to H$ is a Hilbert space epimorphism. The sequence $x_n^* = L^*x_n$ is the biorthogonal sequence of $(x_n)_{n\geq 1}$. We are going to see that the sequence $(\mathcal{L}_\tau(z^n)$ is a Riesz basis of $H^2(D)$ for $\Re\tau > 1$, and the corresponding biorthogonal sequence is $(\psi_n(z))_{n\geq 1} = (\psi_{n,\tau}(z))_{n\geq 1}$ where $\psi_n(z) = \frac{1}{n^\tau} \sum_{d|n} \mu_{\frac{n}{d}} d^\tau z^d$.

We will use the following characterization of Riesz sequences, due to N.K. Bari [3, 8, 24]:

Lemma 4.8 *Let H be a Hilbert separable space and $B = (x_n)_{n\geq 1}$ be a sequence in H. B is a Riesz basis in H if and only if*

1. *every $x \in H$ can be expanded as $x = \sum_n a_n x_n$*
2. *There exist two constants $0 < c < C < \infty$ such that for every sequence $(a_n)_{n\geq 1}$ with finite support we have:*

$$c \sum_{n\geq 1} |a_n|^2 \leq \| \sum_{n\geq 1} a_n x_n \|^2 \leq C \sum_{n\geq 1} |a_n|^2.$$

The following lemma uses ideas from [48], see also [8, 24].

Lemma 4.9 *We have the following equalities for $\Re s > 1$*

1. $(\mathcal{L}_s(z^m)|\mathcal{L}_s(z^n)) = \sum_{\substack{k,l\geq 1\\ km=ln}} \frac{1}{k^s l^s} = \frac{(\gcd(m,n))^{2s}}{(mn)^s} \zeta(2s).$
2. $(\mathcal{M}_s(z^m)|\mathcal{L}_s(z^n)) = \sum_{\substack{k,l\geq 1\\ km=ln}} \frac{\mu_k}{k^s l^s} = \frac{(\gcd(m,n))^{2s}}{(mn)^s} \mu_{\delta(m,n)} \sum_{\substack{j\geq 1\\ (j,\delta(m,n))=1}} \frac{\mu_j}{j^s}.$

with, if $f(z) = \sum_{n=0}^{\infty} a_n z^n, \; g(z) = \sum_{n=0}^{\infty} b_n z^n,$

$$(f(z)|g(z)) = \sum_{n=0}^{\infty} a_n \bar{b}_n.$$

Remark 4.10 The basic example is provided by the Hilbert space $L^2(0,\pi)$ and the dilates (u_n) of the function $u(x) = \sum_{k\geq 1} \frac{\sin kx}{k^s}$, with

$$(u_m|u_n) = \frac{\pi}{2} \sum_{\substack{k,l\geq 1\\ km=ln}} \frac{1}{k^s l^s} = \frac{\pi}{2} \zeta(2s) \frac{(\gcd(m,n))^s}{(mn)^s}.$$

Lemma 4.11 *Let $(c_n)_{n\geq 1}$ a sequence with finite support of complex numbers. Then*

1. $\| \sum_{n\geq 1} c_n \mathcal{L}_s(z^n) \|^2 = \sum_{m,n\geq 1} c_m \overline{c}_n (\mathcal{L}_s(z^m)|\mathcal{L}_s(z^n)) = \zeta(2s) \sum_{m,n\geq 1} \frac{(\gcd(m,n))^{2s}}{(mn)^s} c_m \overline{c}_n.$

2.

$$\|\sum_{n\geq 1} c_n \mathcal{M}_s(z^n)\|^2 = \sum_{m,n\geq 1} c_m \overline{c}_n (\mathcal{L}_s(z^m)|\mathcal{L}_s(z^n))$$

$$= \sum_{m,n\geq 1} \frac{(\gcd(m,n))^{2s}}{(mn)^s} c_m \overline{c}_n \mu_{\frac{n}{\operatorname{lcm}(m,n)}} \sum_{\substack{j\geq 1 \\ (j,\,\delta(n,m))=1}} \frac{\mu_j}{j^{2s}}.$$

Remark 4.12 We thus see appearing the $N \times N$ symmetric square matrices

$$M_{s,N} = \left(\frac{(\gcd(m,n))^{2s})}{(mn)^s}\right)_{1\leq m,n\leq N}.$$

It is possible to compute the determinant of the matrix $M_{s,N}$. We recall first that the Smith determinant is defined to be

$$\Delta_N = \det\left(\gcd(m,n)\right)_{1\leq m,\, n\leq N}$$

and its value, in terms of the Euler's totient function Φ, is [45]

$$\Delta_N = \Phi(1)\Phi(2)\cdots\Phi(N).$$

The determinant $\Delta_N^{(r)} = \det\left(\gcd(m,n)^r\right)_{1\leq m,\, n\leq N}$ where r is a real number, was also evaluated by Smith in [45]. To explain the value of $\Delta_N^{(r)}$ we introduce the Jordan's totient function J_k given by [1, 45, 47]

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right),$$

where p ranges through the prime divisors of n. We have $J_1(n) = \Phi(n)$. Furthermore

$$\sum_{d|n} J_k(d) = n^k.$$

which may be written as convolution product as

$$J_k(n) \star 1 = n^k$$

and by a Möbius inversion

$$J_k(n) = \mu(n) \star n^k.$$

The Dirichlet generating function of series for J_k is

$$\sum_{n\geq 1}\frac{J_k(n)}{n^s}=\frac{\zeta(s-k)}{\zeta(s)}.$$

Similarly to the case of $\Delta=\Delta^{(1)}$, we have the formula

$$\Delta_N^{(r)}=J_r(1)J_r(2)\cdots J_r(N).$$

Since the determinant is a multilinear form we obtain

$$\det\ M_{s,N}=\frac{1}{(N!)^{2s}}J_{2s}(1)J_{2s}(2)\cdots J_{2s}(N),$$

so the matrices $M_{s,N}$ are invertible. These statements remain valid for every $s\in\mathbb{C}$ by analytic continuation. The matrices $M_{s,N}$ are also positive for $s\in(1,+\infty)$ and according to [30, 48], the smallest eigenvalue of $\lambda_N(s)$ and the largest eigenvalue $\Lambda_N(s)$ of $M_{s,M}$ satisfy

$$\frac{\zeta(2s)}{\zeta(s)^2}\leq\lambda_N(s)\leq\Lambda_N(s)\leq\frac{\zeta(s)^2}{\zeta(2s)}. \tag{4.4}$$

We deduce that for a sequence $a=(a_n)_{n\geq 1}\in l^2$ we have [30]

$$\frac{\zeta(2s)}{(\zeta(s))^2}\sum_{1\leq n\leq N}|a_n|^2\leq\sum_{1\leq m,n\leq N}\frac{(\gcd(m,n))^{2s}}{(mn)^s}a_m\overline{a}_n\leq\frac{(\zeta(s))^2}{\zeta(2s)}\sum_{1\leq n\leq N}|a_n|^2.$$

This proves the following

Proposition 4.13 *If $s>1$, the sequence $(\mathcal{L}_s(z^n))_{n\geq 1}$ is a Riesz basis of $H^2(D)$.*

By direct computation we see that the associated biorthogonal basis is $(\psi_n(z))_{n\geq 1}$ where

$$\psi_n(z)=\psi_{n,s}(z)=\frac{1}{n^s}\sum_{d|n}\mu_{\frac{n}{d}}d^s z^d. \tag{4.5}$$

The two extreme factors in (4.4) have interesting Dirichlet series expansion [32] (p.227)

$$\frac{\zeta^2(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{\theta(n)}{n^s},\qquad\frac{\zeta(2s)}{\zeta^2(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)\theta(n)}{n^s}.$$

The function $\theta(n)$ is defined by

$$\theta(n) = 2^{\omega(n)},$$

where $\omega(n)$ is the number of different prime factors of n. It is a multiplicative function, also related to the Möbius function by $\theta(n) = \sum_{d|n} |\mu(d)|$.

Remark 4.14 It is worth noting that the positivity of the matrix $M_{s,N}$ can be deduced from the Franel integral (2.2). If for a suitable real function f we have two real s, s' such that for every $1 \leq m, n \leq N$

$$\int_0^1 f(mx)f(nx)dx = \frac{\gcd(m,n)^s}{m^{s'}n^{s'}}$$

then for $c_1, \cdots, c_N \in \mathbb{C}$

$$\sum_{1\leq m,n\leq N} \frac{\gcd(m,n)^s}{m^{s'}n^{s'}} c_m \bar{c}_n = \int_0^1 \left| \sum_{p=1}^{p=N} f(px)c_p \right|^2 dx \geq 0$$

and

$$\sum_{1\leq m,n\leq N} \gcd(m,n)^s c_m \bar{c}_n = \int_0^1 \left| \sum_{p=1}^{p=N} p^{s'} f(px)c_p \right|^2 dx \geq 0.$$

We have the definite positivity if the functions $x \to f(px)$, $1 \leq p \leq N$ are linearly independent.

4.3 Multipliers

We consider new spaces of Dirichlet series.

1. The space $\mathcal{H}^2 = \left\{ \sum_{n\geq 1} \frac{a_n}{n^s},\ a = (a_n)_{n\in\mathbb{N}^*} \in l^2 \right\}$ corresponding to the spaces $L^2(0,1)$ et $H^2(D)$.
2. The space $\mathcal{H}^\infty = \mathbf{H}^\infty(\{s \in \mathbb{C}, \Re s > 0\}) \cap \mathcal{D}$ equipped with the usual norm $\| \ \|_\infty$, defined on the space of measurable and bounded functions defined on $\{s \in \mathbb{C} : \Re s > 0\}$.

It is easily shown that $\mathcal{H}^\infty \subset \mathcal{H}^2$ and that $\|f\|^2 \leq \|f\|_\infty$ for $f \in \mathcal{H}^\infty$. The set of multipliers of $\mathcal{H}^2$ can be identified with $\mathcal{H}^\infty$. The norm of the multiplier, as operator, $M_\varphi : \mathcal{H}^2 \ni f \longrightarrow \varphi f \in \mathcal{H}^2$ is $\|M_\varphi\| = \|\varphi\|_\infty$. We also have the following interesting property: Let $\varphi \in \mathcal{H}^\infty$, the multiplier operator M_φ is an isomorphism of $\mathcal{H}^2$ if and only if $\varphi^{-1} \in \mathcal{H}^\infty$. In this case $\|M_\varphi^{-1}\| = \|\varphi^{-1}\|_\infty$. In the correspondence between power series and Dirichlet series the multiplier set of $H_0^2(D)$ for $\otimes$ is identified with the set of power series $\varphi(s) = \sum_{n\geq 1} \alpha_n z^n$ such that the function $\varphi(s) = \sum_{n\geq 1} \frac{\alpha_n}{n^s}$ belongs to $\mathcal{H}^\infty$. An illustration of this fact is given by the polylogarithm function: For $\Re\tau > 1$ the series $\mathcal{L}_\tau(z)$ and $\mathcal{M}_\tau(z)$ are reciprocal multipliers of $H_0^2(D)$ for the operation $\otimes$. Moreover the image of the multiplier $\mathcal{L}_s(z)$ by the map S is the translate of the zeta function $\zeta(s+\tau) = \sum_{n\geq 1} \frac{1}{n^{s+\tau}}$.

Example 4.15 We now give an example of the expansion of a given $g \in H_0^2(D)$ in the Riesz basis $(\mathcal{L}_s(z^n))_{n\geq 1}$. We need to find a sequence $(\alpha_n)_{n\geq 1}$ such that $g(z) = \sum_{k\geq 1} \alpha_k \mathcal{L}_s(z^k)$. If $f(z)$ is the formal power series $f(z) = \sum_{k\geq 1} \alpha_k z^k$, then $g(z)$ should be $g(z) = f(z) \otimes \mathcal{L}_s(z)$ or $f(z) = g(z) \otimes \mathcal{M}_s(z)$. According to the Lemma 4.1 the convergence radius R_f of f satisfies $1 = \min(1, R_{\mathcal{M}_s}, R_g) \leq R_f$ and

$$f(z) = \sum_{n\geq 1} \Big(\sum_{d|n} \frac{\mu_{\frac{n}{d}}}{(\frac{n}{d})^s} a_d \Big) z^n = \sum_{n\geq 1} \frac{1}{n^s} \Big(\sum_{d|n} \mu_{\frac{n}{d}} d^s a_d \Big) z^n.$$

We thus see that in terms of the biorthogonal basis (4.5), $\alpha_n = (g|\psi_n)$, naturally enough.

4.4 *On the Estermann's Function*

The Estermann zeta function $E(s, a, z)$ is defined by the Dirichlet series

$$E(s, a, z) = \sum_{n\geq 1} \frac{\sigma_a(n)}{n^s} z^n \quad \Re s > 1 + \Re a, \ |z| \leq 1, \tag{4.6}$$

where, as already denoted, $\sigma_a(n) = \sum_{d|n} d^a$, $a \in \mathbb{C}$. This Dirichlet series is closely related to Ramanujan sums. This series (4.6) can be given in terms of the polylogarithm function, or more precisely can be expanded with respect to the Riesz basis $\{\mathcal{L}_s(z^n),\ n \geq 1\}$. In fact if $a > 0$ then

$$E(s, a, z) = \sum_{p\geq 1} \frac{1}{p^{s-a}} \mathcal{L}_s(z^p), \quad |z| \leq 1,\ \Re s > \max(1, 1+a).$$

This can be shown either by observing that

$$(E|\psi_p) = \frac{1}{p^s}(\mu \star \sigma_a)(p) = \frac{1}{p^{s-a}}$$

by using $\sigma_a = I \star u^a$ with $I(n) = 1$ (the constant arithmetical function) and $u^a(n) = n^a$ for every $n \in \mathbb{N}^*$, or by showing that

$$E(s, a, \bullet) \otimes \mathcal{M}_s = \mathcal{L}_{s-a}.$$

Assume that $a < 0$ and recall first that for $\Re(s + \tau) > 1$ then (3.8)

$$\zeta(s+\tau) \sum_{d|k} \mu_{\frac{k}{d}} d^{1-s-\tau} = \sum_{n\geq 1} \frac{c_k(n)}{n^{s+\tau}}.$$

We deduce that for $t > 0$ we have

$$\begin{aligned}
\sum_{n\geq 1} \frac{c_k(n)}{n^\tau} e^{-nt} &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \sum_{n\geq 1} \frac{c_k(n)}{n^{s+\tau}} \frac{ds}{t^s} \\
&= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \sum_{n\geq 1} \zeta(s+\tau) \sum_{d|k} \mu_{\frac{k}{d}} d^{1-\tau} \frac{ds}{(dt)^s} \\
&= \sum_{d|k} \mu_{\frac{k}{d}} d^{1-\tau} \sum_{n\geq 1} \frac{e^{-ndt}}{n^\tau}.
\end{aligned}$$

We obtain, by analytic continuation, that

$$\sum_{n\geq 1} \frac{c_k(n)}{n^\tau} z^n = \sum_{d|k} \mu_{\frac{k}{d}} d^{1-\tau} \mathcal{L}_\tau(z^d).$$

Since $a < 0$, we get from (3.9)

$$\sigma_a(n) = \zeta(1-a) \sum_{k\geq 1} \frac{c_k(n)}{k^{1-a}}.$$

Thus

$$
\begin{aligned}
E(s,a,z) &= \sum_{n\geq 1}\frac{\sigma_a(n)}{n^s}z^n \\
&= \zeta(1-a)\sum_{k\geq 1}\frac{1}{k^{1-a}}\left(\sum_{n\geq 1}\frac{c_k(n)}{n^s}z^n\right) \\
&= \zeta(1-a)\sum_{k\geq 1}\frac{1}{k^{1-a}}\left(\sum_{d|k}\mu_{\frac{k}{d}}d^{1-s}\mathcal{L}_s(z^d)\right).
\end{aligned}
$$

5 A Link with Kubertt Identities

Let $f(z) = \sum_{n\geq 1} a_n z^n$ be a power series, convergent for $|z| \leq 1$. The condition

$$\sum_{1\leq h\leq m} f(e^{2i\pi\frac{h}{m}}) = 0$$

is equivalent to

$$\sum_{n\geq 1} a_n\left(\sum_{1\leq h\leq m} e^{2i\pi\frac{nh}{m}}\right) = m\sum_{j\geq 1} a_{jm} = 0.$$

Furthermore if we assume that

$$\sum_{1\leq h\leq m} f(e^{2i\pi\frac{h}{m}}) = 0$$

for every $m \in \mathbb{N}^*$, then for every $m \in \mathbb{N}^*$

$$\sum_{j\geq 1} a_{jm} = 0.$$

The function $\mathcal{M}_1$ satisfies this property. Furthermore according to [39] (p.294) if $a = (a_n)_{n\in\mathbb{N}^*}$ is a non zero sequence satisfying this property, then $a \notin \ell^1$, the space of sequences whose series is absolutely convergent. This last condition is also satisfied by $\mathcal{M}_1$. In fact if $|\mu(n)| = \mu^2(n)$ is the characteristic function of squarefree integers, then [32] (p.227)

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$

and $\lim_{s\to 1} \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \infty$. Obviously this also results from (3.13).

One wonders for what $f \in \mathcal{O}_0$ the product $f \otimes \mathcal{M}_1$ will verify Kubert identity. Naturally, quite strong convergence hypothesis on the sequence $a = (a_k)_{k\geq 1}$ will be required. We recall that

$$f \in H^2(D) \to f \otimes \mathcal{M}_\tau \in H^2(D)$$

is an isomorphism. If $f(z) = \sum_{k\geq 1} a_k z^k$ with suitable convergence on the circle $\mathbf{U} = \{|z| = 1\}$ we can write

$$\sum_{1\leq h\leq m} (f \otimes \mathcal{M}_\tau)(e^{2i\pi\frac{h}{m}}) = \sum_{k\geq 1} a_k \left(\sum_{1\leq h\leq m} \mathcal{M}_\tau(e^{2i\pi\frac{h}{m}}) \right) = \frac{1}{m^{\tau-1}} \sum_{k\geq 1} a_k \sum_{j\geq 1} \frac{\mu_{jD(m,k)}}{j^\tau}.$$

We need two lemmas, the first is strongly inspired by Pollack [37].

Lemma 5.1 *Let $D \in \mathbb{N}^*$. For every $1 < \tau < \frac{3}{2}$ we have*

$$\left| \sum_{j\geq 1} \frac{\mu_{jD}}{j^\tau} \right| = \left| \frac{\mu(D)}{\zeta(\tau)} \prod_{p|D} (1 - p^{-\tau}) \right| \leq e \frac{\tau - 1}{\zeta(\tau)}.$$

Proof We introduce

$$P_D(\tau) = \prod_{p|D} (1 - p^{-\tau}), \qquad \tau > 1$$

so we have

$$\log P_D(\tau) = -\sum_{p|D} \log(1 - p^{-\tau}) = \sum_{p|D} \frac{1}{p^\tau} + \sum_{p|D} \sum_{k\geq 2} \frac{1}{kp^{k\tau}}.$$

Now

$$\sum_{p|D} \sum_{k\geq 2} \frac{1}{kp^{k\tau}} \leq \sum_{p|D} \sum_{k\geq 2} \frac{1}{kp^{k\tau}} \leq \frac{1}{2} \sum_{p|D} \sum_{k\geq 2} \frac{1}{p^{k\tau}}$$

$$\leq \frac{1}{2} \sum_{p|D} \sum_{k\geq 2} \frac{1}{p^k} = \frac{1}{2} \sum_{p|D} \frac{1}{p(p-1)} \leq \frac{1}{2}.$$

And

$$\log P_D(\tau) = -\sum_{p|D} \log(1 - p^{-\tau}) \leq \sum_{p|D} \frac{1}{p^\tau} + \frac{1}{2}.$$

The same ideas give

$$0 < \log \zeta(\tau) - \sum_p \frac{1}{p^\tau} < \frac{1}{2} \quad \tau > 1.$$

On the other hand it is easily seen that

$$1 < (\tau - 1)\zeta(\tau) < \tau, \quad \tau > 1$$

and for $\tau < \frac{3}{2}$,

$$0 < \log \zeta(\tau) + \log(\tau - 1) < \log \tau < \log \frac{3}{2}.$$

By putting together these results we arrive to

$$-\frac{1}{2} < \sum_p \frac{1}{p^\tau} + \log(\tau - 1) < \log \frac{3}{2} < \frac{1}{2}.$$

Hence, for $\tau \in]1, \frac{3}{2}[$, we have

$$|\sum_p \frac{1}{p^\tau} + \log(\tau - 1)| \leq \frac{1}{2}$$

or

$$\sum_p \frac{1}{p^\tau} \leq \frac{1}{2} + \log(\tau - 1), \quad 1 < \tau < \frac{3}{2}.$$

Finally, using $\log P_D(\tau) \leq \sum_p \frac{1}{p^s} + \frac{1}{2}$, we find that

$$\log P_D(\tau) \leq 1 + \log(\tau - 1)$$

or

$$P_D(\tau) \leq e(\tau - 1), \quad 1 < \tau < \frac{3}{2}.$$

Hence the lemma. □

Before to state the following lemma we would like to make a comparative remark. Let $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ be the circle, and let $f : \mathbf{T} \to \mathbb{C}$ be an integrable function. If the Fourier coefficients $\hat{f}(n)$ satisfy

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 < \infty,$$

then Carleson theorem [10] asserts that the sequence $(S_n(x)_{n\geq 0},\quad S_n(x) = \sum_{|k|\leq n} \hat{f}(k)e^{2i\pi kx}$ converges to $f(x)$ for almost all $x \in \mathbf{T}$. In particular for every τ with $\Re\tau > \frac{1}{2}$, $\sum_{n\geq 1} \frac{\mu(n)z^n}{n^\tau}$ converges for almost every $z \in \mathbf{T}$. The next lemma is a form of Jensen lemma adapted to the Dirichlet series $\sum_{n\geq 1} \frac{\mu(n)z^n}{n^\tau}$, $|z| \leq 1$ and $\Re\tau > 1$. It differs considerably from what we got from Carleson's theorem.

Lemma 5.2 *For every τ, $\Re\tau > 1$, the series $\sum_{n\geq 1} \frac{\mu(n)}{n^\tau} z^n$ converges uniformly on the angle*

$$|z| \leq 1,\ |\arg(z-1)| < \frac{\pi}{2} - \delta,$$

where $\delta \in (0, \frac{\pi}{2})$ is fixed

Proof Let $a_n(z) = \mu(n)\, z^n$ be the coefficients of this Dirichlet series. By using the fundamental estimate of Davenport (3.3) we can show that $b_p(z) = \sum_{n\geq p} a_n(z)$ converges uniformly to zero on the unit circle, hence on the closed unit disc, by the maximum principle. For every $\varepsilon > 0$ there exists $p_\varepsilon \in \mathbb{N}^*$ such that $p \geq p_\varepsilon$ we have $|b_p(z)| < \varepsilon$. Let

$$S_N = S_N(z,s) = \sum_{1\leq n\leq N} \frac{a_n(z)}{n^s},\ |z| \leq 1,\ \Re s > 0,\ |\arg s| \leq \frac{\pi}{2} - \delta.$$

For $Q > P \geq p_\varepsilon$, we have by partial summation

$$S_Q - S_{P-1} = \frac{b_Q}{Q^s} + b_{Q-1}\Big(\frac{1}{(Q-1)^s} - \frac{1}{Q^s}\Big) + \cdots + b_P\Big(\frac{1}{P^s} - \frac{1}{(P+1)^s}\Big) - \frac{b_{P-1}}{P^s}.$$

and

$$|S_Q - S_{P-1}| \leq \varepsilon\left(\frac{1}{|Q^s|} + \frac{1}{|P^s|} + \left|\frac{1}{(Q-1)^s} - \frac{1}{|Q^s|}\right| = \cdots + \left|\frac{1}{P^s} - \frac{1}{|(P+1)^s|}\right|\right).$$

Now

$$\frac{1}{(k+1)^s} - \frac{1}{k^s} = s \int_{\log k}^{\log(k+1)} e^{-\lambda s} d\lambda,$$

so, with $\sigma = \Re s$

$$\left| \frac{1}{(k+1)^s} - \frac{1}{k^s} \right| \le \frac{|s|}{\sigma} \Big(\frac{1}{k^\sigma} - \frac{1}{(k+1)^\sigma} \Big)$$

and

$$|S_Q - S_{P-1}| \le 2\varepsilon(1 + \frac{1}{\sin\delta}),$$

hence the lemma. □

As a consequence we find that $\lim_{\tau \to 1, \tau > 1} \mathcal{M}_\tau(z) = \mathcal{M}_1(z)$ uniformly with respect to z in the closed unit disk. So, if the sequence $a = (a_k)_{k \ge 1}$ is reasonable we will obtain that the function $f \otimes \mathcal{M}_1$, with $f(z) = \sum_{k\ge1} a_k z^k$, satisfies the property of Besicovich. For example, if $a \in l^1$ (which also ensures that $f \in H^2(D)$).

Theorem 5.3 *If* $f(z) = \sum_{k\ge1} a_k z^k$ *with* $a = (a_k)_{k\ge1} \in l^1$, *then*

$$\sum_{1 \le h \le m} (f \otimes \mathcal{M}_1)(e^{2i\pi \frac{h}{m}}) = 0.$$

Proof The two lemmas above ensure the possibility of passing to the limit $\tau \to 1,\ 1 < \tau < \frac{3}{2}$ in the following relation:

$$\sum_{1\le h\le m} (f \otimes \mathcal{M}_\tau)(e^{2i\pi\frac{h}{m}}) = \sum_{k\ge1} a_k \left(\sum_{1\le h\le m} \mathcal{M}_\tau(e^{2i\pi\frac{h}{m}}) \right) = \frac{1}{m^{\tau-1}} \sum_{k\ge1} a_k \sum_{j\ge1} \frac{\mu_{jD(m,k)}}{j^\tau}.$$

□

Remark 5.4 Let $f(z) = \sum_{k\ge2} a_k z^k$ be a power series of radius at least equal to $R > 0$. For $|z| < R$ and $\Re s > \frac{1}{2}$ we have [27, 36]

$$\sum_{n\ge1} f(\frac{z}{n^s}) = \sum_{k\ge2} a_k \zeta(ks) z^k.$$

Proof For $z \in D(0, R)$ we choose $\varepsilon > 0$ and $\alpha > 0$ such that $|z| < R - \alpha < R - \varepsilon < R$. There is $k_0 \in \mathbb{N}^*$ such that, for $k > k_0$ we have $|a_k||z|^k < \dfrac{(R-\alpha)^k}{(R-\varepsilon)^k}$, which ensures the convergence of the series $\sum_{k\ge2} |a_k||z|^k$. Since $\Re(ks) > 1$ for

$\Re s > \frac{1}{2}$ and $k \geq 2$, the series $\sum_{k\geq 2} |a_k||z|^k \sum_{n\geq 1} \frac{1}{n^{ks}}$ is summable. We can therefore apply Fubini's theorem to exchange the summations. □

As an example we take $a_{2n} = (-1)^n \frac{1}{2n!}$ and $a_{2n+1} = 0$ for every $n \geq 1$. This gives

$$f(z) = \sum_{k\geq 2} a_k z^k = \sum_{k\geq 1} \frac{(-1)^k}{2k!} z^{2k} = \cos z - 1$$

which converges for every $z \in \mathbb{C}$ and insure, with $s = 1$,

$$\sum_{j\geq 1} \Big(\cos(\frac{z}{j}) - 1\Big) = \sum_{k\geq 1} (-1)^k \frac{\zeta(2k)}{2k!} z^{2k}. \tag{5.1}$$

We investigated some functions related to the right side of (5.1) in [43].

6 Asymptotic Expansion

In this section we give the asymptotic expansion of the coefficients of the power series in $H_0^2(D)$ corresponding to $\sum_{1\leq \nu\leq N} c_\nu \{\frac{\theta_\nu}{x}\}$, $\sum_{1\leq \nu\leq N} c_\nu \theta_\nu = 0$ as element of $L^2(0, 1)$. This is given by the following result

Theorem 6.1 *The n-th coefficient a_n of the power series belonging to $H^2(D)$ and corresponding to $\sum_{1\leq \nu\leq N} c_\nu \{\frac{\theta_\nu}{x}\}$ with $\sum_{1\leq \nu\leq N} c_\nu \theta_\nu = 0$ is given by*

$$a_n = \frac{\sqrt{2}}{\pi n} \Big(\sum_{1\leq \nu\leq N} c_\nu (n\pi\theta_\nu)^{\sigma_0} \Big) o_{\sigma_0}(n)$$

with σ_0 such that $\frac{2}{3} < \sigma_0 < 1$ and

$$o_{\sigma_0}(n) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \Gamma(-(\sigma_0 + i\tau)) \zeta(\sigma_0 + i\tau) \cos(\frac{\pi}{2}(\sigma_0 + i\tau)) n^{i\tau} d\tau$$

tending to zero as n tends to $+\infty$.

Proof We look for an asymptotic expansion of the function $f(x) = \sum_{l\geq 1}(-1)^l \frac{\zeta(2l)}{2l!} x^{2l}$ when x tends to $+\infty$. For this we consider the line integral

$$I_{M,T}(x) = \frac{1}{2i\pi}\int_{\gamma_{M,T}} \Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s ds,$$

where $\gamma_{M,T}$ is the rectangular circuit which sides are parallel to the axis, and whose vertices are the points $\sigma_0 \pm iT$ and $M + \frac{1}{2} \pm iT$ for $M \in \mathbb{N}^*$, $T > 0$. We recall that

1. For every fixed $\sigma \in \mathbb{R}$ there exists $C_\sigma > 0$ such that $|\Gamma(\sigma + i\tau)| \leq C_\sigma |\tau|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|\tau|}$.
2. For $0 \leq \sigma \leq 1$ and $\varepsilon > 0$, there exists $D_\varepsilon > 0$ such that $|\zeta(\sigma + i\tau)| \leq D_\varepsilon |\tau|^{\frac{1-\sigma}{2}+\varepsilon}$.
3. We have $|\cos\frac{\pi}{2}(\sigma + i\tau)| \leq e^{\frac{\pi}{2}|\tau|}$.

Hence for $s = \sigma_0 + i\tau$ with $\sigma_0 \in (\frac{2}{3}, 1)$ we have the estimate

$$\left|\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s\right| \leq C_{-\sigma_0} C_\varepsilon \frac{x^{\sigma_0}}{|\tau|^{\sigma_0+\frac{\sigma_0}{2}-\varepsilon}},$$

with $\sigma_0 + \frac{\sigma_0}{2} - \varepsilon > 1$ for $\varepsilon > 0$ small enough, since $\sigma_0 + \frac{\sigma_0}{2} > \frac{2}{3} + \frac{1}{3} = 1$. □

6.1 The Integral on the Line $\Re s = \sigma_0$

We recall that the function

$$\mathbb{R} \ni \tau \to \Gamma(-(\sigma_0 + i\tau))\zeta(\sigma_0 + i\tau)\cos(\frac{\pi}{2}(\sigma_0 + i\tau))x^{\sigma_0} \in L^1(\mathbb{R}).$$

6.2 Integrals on $\Re s = M + \frac{1}{2}$ and on Horizontal Lines

For every $s \in \mathbb{C}$ and $M \in \mathbb{N}^*$ such that $\Re s = M + \frac{1}{2}$ we have $s - (-\overline{s}) = 2\Re s = 2M + 1$ and

$$\Gamma(-s) = \frac{\Gamma(-s + 2M + 1)}{(-s)(1-s)\ldots(2M-s)} = \frac{\overline{\Gamma(s)}}{(-s)(1-s)\ldots(2M-s)}.$$

According to the Stirling formula, for $0 \leq |\arg s| \leq \frac{\pi}{2}$

$$|\Gamma(s)| \leq C|s|^{\sigma-\frac{1}{2}} e^{-\sigma} e^{-\frac{\pi}{2}|\tau|}.$$

Then, with $\Re s = \sigma = M + \frac{1}{2}$, we have $|\Gamma(s)| \leq C|s|^M e^{-(M+\frac{1}{2})} e^{-\frac{\pi}{2}|\tau|}$. Hence

$$|\Gamma(-s)| \leq \frac{C|s|^M e^{-(M+\frac{1}{2})} e^{-\frac{\pi}{2}|\tau|}}{|s(s-1)\dots(s-2M)|}.$$

This guarantees the integrability of the modulus of the function $\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s$ on the line $\Re s = M + \frac{1}{2}$ as $|\zeta(s)| \leq 2$, $|\cos(\frac{\pi}{2}s)| \leq e^{\frac{\pi}{2}|\tau|}$ and so the modulus of the integrand is less than

$$\frac{C|s|^M e^{-(M+\frac{1}{2})} x^{M+\frac{1}{2}}}{|s(s-1)\dots(s-2M)|} \leq C_M \frac{1}{|\tau|^{M+1}},$$

C_M being a positive constant, independent of M. In the same vein we can obtain an upper bound of $\dfrac{|s|^M}{|s(s-1)\dots(s-2M)|}$ on $\Re s = M + \frac{1}{2}$ by grouping $(s-1)$ and $(s-2M)$, $(s-2)$ and $(s-(2M-1))$ ..., $(s-M)$ and $(s-(M+1))$ to obtain

$$\frac{|s|^M}{|s(s-1)\dots(s-2M)|} = \frac{|s|^{M-1}}{(\tau^2+(M-1)^2)\dots(\tau^2+\frac{1}{4})} \leq \frac{|\tau|^{M-1}}{|\tau|^{2M}} = \frac{1}{|\tau|^{M+1}}.$$

We thus obtain the absolute convergence of the integral on $\Re s = M + \frac{1}{2}$ and, in the same way, the limit to zero of the integrals on the horizontal segments by using a majorization, uniform in $\sigma \in [\sigma_0, M + \frac{1}{2}]$, of $\dfrac{|s|^M}{|s(s-1)\dots(s-2M)|}$.

6.3 *Evaluation of the Residues*

1. For $n \geq 2$. The poles of $\Gamma(-s)$ greater than σ_0 are $s = n, n \geq 1$ and are of respective residues $\dfrac{(-1)^n}{n!}$. We find that if n is odd the residue of $\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s$ at n vanishes, and if $n = 2l$ is even, the residue is $-(-1)^l \dfrac{\zeta(2l)}{2l!} x^{2l}$.
2. For $n = 1$ we have a double pole and by computing

$$\lim_{s\to 1} \frac{d}{ds}\big((s-1)^2\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s\big)$$

we find that the residue at 1 is $-\pi x$.

The intermediate result we obtained is the equality

$$-\pi x - \sum_{1\le l\le M} (-1)^l \frac{\zeta(2l)}{2l!} x^{2l} = \frac{1}{2i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s\,ds$$

$$-\frac{1}{2i\pi}\int_{M+\frac{1}{2}-i\infty}^{M+\frac{1}{2}+i\infty} \Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s\,ds.$$

We need to analyze $I_M(x) = \frac{1}{2i\pi}\int_{M+\frac{1}{2}-i\infty}^{M+\frac{1}{2}+i\infty} \Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s\,ds$. We write $\cos(\frac{\pi}{2}s) = \frac{(e^{i\frac{\pi}{2}})^s + (e^{-i\frac{\pi}{2}})^s}{2}$ and making the legitimate interchange of summation and integration we get

$$I_M(x) = \frac{1}{2}\sum_{n\ge 1}\left(\frac{1}{2i\pi}\int_{M+\frac{1}{2}-i\infty}^{M+\frac{1}{2}+\infty} \Gamma(-s)\Big(\frac{e^{i\frac{\pi}{2}}x}{n}\Big)^s ds + \frac{1}{2i\pi}\int_{M+\frac{1}{2}-i\infty}^{M+\frac{1}{2}+\infty} \Gamma(-s)\Big(\frac{e^{-i\frac{\pi}{2}}x}{n}\Big)^s ds\right),$$

or

$$I_M(x) = \frac{1}{2}\sum_{n\ge 1}\left(J_M(\frac{ix}{n}) + J_M(\frac{-ix}{n})\right)$$

with

$$J_M(z) = \frac{1}{2i\pi}\int_{M+\frac{1}{2}-\infty}^{M+\frac{1}{2}+\infty} \Gamma(-s)z^s\,ds = \frac{1}{2i\pi}\int_{-(M+\frac{1}{2})-\infty}^{-(M+\frac{1}{2})+\infty} \Gamma(s)z^{-s}\,ds.$$

According to [19] (7.3, p.348), the inverse Mellin transform of $e^{-ias}\Gamma(s)$, with the conditions $|\Re a| \le \frac{\pi}{2},\ -m < \Re s < 1-m,\ m = 1, 2, \ldots$, is the function $e^{-te^{ia}} - \sum_{0\le r\le m-1} \frac{(-te^{ia})^r}{r!}$. For $a = -\frac{\pi}{2} + i\log\frac{n}{x}$ we have $|\Re a| = \frac{\pi}{2}$, $e^{ia} = -i\frac{x}{n}$ and

$$J_M(i\frac{x}{n}) = \sum_{r\ge M+1} (-i)^r \frac{x^r}{n^r r!}, \quad J_M(-i\frac{x}{n}) = \sum_{r\ge M+1} (i)^r \frac{x^r}{n^r r!}.$$

Thus

$$I_M(x) = \sum_{n\ge 1}\left(\sum_{r\ge M+1} (i^r + (-i)^r)\frac{x^r}{n^r r!}\right).$$

Since

$$\sum_{n\geq 1}\sum_{r\geq M+1}\frac{|x|^r}{n^r r!}\leq e^{|x|}\sum_{n\geq 1}\frac{1}{n^{M+1}}=\zeta(M+1)e^{|x|}<+\infty$$

and $\lim_{M\to+\infty}\sum_{r\geq M+1}(i^r+(-i)^r)\frac{x^r}{n^r r!}=0$ we get

$$\lim_{M\to+\infty} I_M(x)=0,$$

and finally

$$\sum_{l\geq 1}(-1)^l\frac{\zeta(2l)}{2l!}x^{2l}=\frac{1}{2i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s ds+\pi x.$$

We also see from the properties of the Fourier transform of an integrable function

$$\begin{aligned}\frac{1}{2i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}&\Gamma(-s)\zeta(s)\cos(\frac{\pi}{2}s)x^s ds\\ &=\frac{x^{\sigma_0}}{2\pi}\int_{\mathbb{R}}\Gamma(-\sigma_0-i\tau)\zeta(\sigma_0+i\tau)\cos(\frac{\pi}{2}(\sigma_0+i\tau))e^{i(\log x)\tau}d\tau\\ &=x^{\sigma_0}o_{\sigma_0}(x)\end{aligned}$$

If we replace x by $\pi\nu\theta_\nu$, $1\leq\nu\leq N$ and summing over ν, the terms coming from πx disappear, since $\sum_{1\leq\nu\leq N}c_\nu\theta_\nu=0$. This therefore gives the statement of the theorem.

7 Analytic Continuation of $\mathcal{M}_s$ Outside the Unit Disk

One of the purposes of this paragraph is to show that the unit circle is a natural boundary for $\mathcal{M}_s$ for $\Re s>0$. This can be easily done when $s=k>0$ is an integer. Since $(\mu(n))_{n\geq 1}=(\mu_n)_{n\geq 1}$ is a non-periodic sequence taking values in the finite set $\{-1,0,1\}$ and the power series $\mathcal{M}_0$ is a non rational function then, by Polya-Carlson theorem, the unit circle is a natural boundary. When $k\geq 1$ we can write

$$\left(z\frac{d}{dz}\right)^k\mathcal{M}_k=\mathcal{M}_0,$$

showing that $\mathcal{M}_0$ is analytic where $\mathcal{M}_k$ is analytic (compare with (3.1)). Hence the unit circle is a natural boundary for $\mathcal{M}_k$ for every integer k. Alternatively we have

$$\mathcal{M}_k(z) = \frac{1}{\Gamma k)} \int_0^\infty \mathcal{M}_0(e^{-t}z)t^{k-1}dt.$$

We introduce the operators $T_s, \Re s > 0$ defined on $\mathcal{M}_0$ by a Mellin's integral

$$T_s(\mathcal{M}_0)(z) = \frac{1}{\Gamma(s)} \mathcal{M}\left(\mathcal{M}_0(e^{-\bullet}z)\right)(s), \quad |z| < 1.$$

One can verifies that

$$\mathcal{M}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n\geq 1} \mu_n (e^{-t}z)^n t^{s-1} dt, \quad |z| < 1.$$

Let $U \subset \mathbb{C}$ be a star-shiped open set containing the origin. Let

$$\mathcal{O}_0(U) = \{f \text{ holomorphic on } U,\ f(0) = 0\}$$

on which we define the family of operators

$$T_s(f)(z) = \frac{1}{\Gamma(s)} \mathcal{M}\left(f(e^{-\bullet}z)\right)(s), \quad z \in U,$$

and show that it has a semi-group property (moreover holomorphic).

Proposition 7.1 *The family of operators* $(T_s)_{\Re s>0}$ *defined on* $\mathcal{O}_0(U)$ *by*

$$T_s(f)(z) = \frac{1}{\Gamma(s)} \int_0^\infty f(e^{-t}z)t^{s-1}dt, \quad z \in U,$$

verifies

$$T_{s_1+s_2} = T_{s_1} \circ T_{s_2}.$$

Note first that $\lim_{t\to+\infty} f(e^{-t}z) = f(0) = 0$ which ensures, for $f \not\equiv 0$, the existence of a integer $N \geq 1$ and a constant $a_N \neq 0$ such that $f(e^{-t}z) = e^{-Nt}(a_N z^N + \cdots)$ for t large and guarantees the convergence of the integral. One checks the holomorphicity of $T_s(f)$ in U as usual. To show the semi-group property we consider $f \in \mathcal{O}_0(U)$, $z \in U$, $s_1, s_2 \in \mathbb{C}$, $\Re s_i > 0$, $i = 1, 2$, then

$$\begin{aligned} T_{s_2}(T_{s_1}(f))(z) &= \frac{1}{\Gamma(s_2)} \int_0^\infty T_{s_1}(f)(e^{-t}z)t^{s_2-1}dt \\ &= \frac{1}{\Gamma(s_2)} \int_0^\infty \left(\frac{1}{\Gamma(s_1} \int_0^\infty f(e^{-u}e^{-t}z)u^{s_1-1}du \right) t^{s_2-1}dt \\ &= \frac{1}{\Gamma(s_2)\Gamma(s_1)} \int\int_{[0,\infty)^2} f(e^{-(u+t)}z)u^{s_1-1}t^{s_2-1}du\,dt. \end{aligned}$$

We set $t + u = v$, $t - u = w$ and $\Delta = \{(u, v) : v \geq 0,\ |w| \leq v\}$ and obtain

$$T_{s_2}(T_{s_1}(f))(z) = \frac{1}{2\Gamma(s_2)\Gamma(s_1)} \int_0^\infty \left[f(e^{-v}z) \int_{-v}^{v} \left(\frac{v-w}{2}\right)^{s_1-1} \left(\frac{v+w}{2}\right)^{s_2-1} dw \right] dv.$$

The integral

$$I = \int_{-v}^{v} \left(\frac{v-w}{2}\right)^{s_1-1} \left(\frac{v+w}{2}\right)^{s_2-1} dw$$

is of Euler's type. We set $w = \rho v$ with $|\rho| \leq 1$ and obtain

$$I = \frac{v^{s_1+s_2-1}}{2^{s_1+s_2-2}} \int_{-1}^{1} (1 - \rho^{s_1-1})(1+\rho)^{s_2-1} d\rho.$$

Now with $x = 1 + \rho = 2y$ we have

$$\begin{aligned} \int_{-1}^{1} (1 - \rho^{s_1-1})(1+\rho)^{s_2-1} d\rho &= \int_0^2 x^{s_2-1}(2-x)^{s_1-1}dx \\ &= 2^{s_1+s_2-1} \int_0^1 y^{s_2-1}(1-y)^{s_1-1}dy \\ &= 2^{s_1+s_2-1} \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)}. \end{aligned}$$

That is

$$I = 2\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} v^{s_1+s_2-1}$$

and

$$T_{s_2}(T_{s_1}(f))(z) = \frac{1}{\Gamma(s_1+s_2)} \int_0^\infty f(e^{-v}z)v^{s_1+s_2-1}dv = T_{s_1+s_2}(f)(z).$$

The proposition is proved.

It is worth noting that this proof is contained in essence in (3.5), a representation of the Boole's differential operator $\vartheta = z\dfrac{d}{dz}$ acting on the z-variable, by a translation on the s-variable. As a simple but illustrative example we take $U = D$ (or $\overline{D} \subset U$), D being the unit disk. If $f(z) = \sum_{n\geq 1} a_n z^n$, $z \in D$, then

$$T_s(f)(z) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

and

$$T_{s_1+s_2}(f)(z) = \sum_{n\geq 1} \frac{a_n}{n^{s_1+s_2}} z^n$$

is also equal to

$$T_{s_2}(T_{s_1}(f))(z) = \sum_{n\geq 1} \frac{1}{n_{s_2}}\Big(\frac{a_n}{n^{s_1}}\Big)z^n = T_{s_1}(T_{s_2}(f))(z).$$

We can now finish the proof that the unit circle is a natural boundary for $\mathcal{M}_s$. If $k > \Re s$ is an integer, we write $k = s + \sigma$ with $\Re\sigma > 0$. Assume that $\mathcal{M}_s$ extends holomorphically to an open set U, containing D, strictly larger than D. Without any loss of generality we can assume that U is star-shiped with respect to the origin, then

$$\mathcal{M}_k = \mathcal{M}_{s+\sigma} = T_{s+\sigma}(\mathcal{M}_0) = T_\sigma(\mathcal{M}_s)$$

extends to, which contradicts what have been said on the non holomorphic extendability of $\mathcal{M}_k(z)$.

8 Conclusion

By way of conclusion we would like to come back to what was the motivation of this work, namely the Besicovitch question, and to mention that in fact it results from the identities of Kubert by means, in general, of a deep result of Number Theory as, for example, the Prime Number Theorem. We have also mentioned, very briefly, the occurrence of the Perron–Frobenius operator and the interpretation that can be drawn from it on the identities of Kubert. Our aim of this section is to point out the interest in combining Number Theory, Harmonic Analysis and Dynamical Systems in the study of arithmetic functions. The two twin functions of Möbius $\mu(n)$ and of Liouville $\lambda(n)$ share so many of these properties. For example, with the function (2.1) we have, as showed by Davenport [14] and used in [20, 42]

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\{nt\} = -\frac{1}{\pi}\sin 2\pi x,$$

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}\{nt\} = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2\pi n^2 x}{n^2}.$$

The second equality makes a link with what Riemann gave as an example of a continuous non-differentiable function. It is natural to define, similarly to $\mathcal{M}_s(z)$, the function

$$\mathcal{N}_s(z) = \sum_{n\geq 1} \frac{\lambda(n)}{n^s} z^n.$$

One of the main ideas of this work can be formulated in the following theorem and its corollary

Theorem 8.1 *Let*

$$\mathfrak{mi}_s(\theta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{2i\pi n\theta}, \qquad \mathfrak{ni}_s(\theta) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} e^{2i\pi n\theta}, \quad \theta \in \mathbb{R}.$$

Then for every positive k we have

$$\sum_{h=1}^{k} \mathfrak{mi}_s(h/k) = \frac{\mu(k)}{k^{s-1}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \qquad \sum_{h=1}^{k} \mathfrak{ni}_s(h/k) = \frac{\lambda(k)}{k^{s-1}} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

As we have seen this results from the following facts

$$\mathfrak{mi}_s(h/k) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{h=1}^{k} e^{2iphn/k} = \frac{\mu(k)}{k^{s-1}} \sum_{n=1,(n,k)=1}^{\infty} \frac{\mu(n)}{n^s}.$$

and

$$\sum_{n=1,\,(n,k)=1}^{\infty} \frac{\mu(n)}{n} = \lim_{s\to 1^+} \sum_{n=1,\,(n,k)}^{\infty} \frac{\mu(n)}{n^s}$$

$$= \lim_{s\to 1^+} \prod_{p\nmid k} (1-p^{-s}) = \lim_{s\to 1^+} \left\{ \zeta(s) \prod_{p|k} (1-p^{-s}) \right\}^{-1} = 0.$$

and, similarly for $\Re s > 1$

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = \lim_{s \to 1^+} \frac{\zeta(2s)}{\zeta(s)} = 0. \tag{8.1}$$

Corollary 8.2 *The functions $\Re \mathfrak{mi}_s(\theta)$, $\Re \mathfrak{ni}_s(\theta)$ are non-trivial real-valued continuous functions f on the real line which have period unity, are even, and for every positive integer k have the property*

$$\sum_{h=1}^{n} f(h/k) = 0.$$

Furthermore, one can prove directly that $\sum (-1)^n \frac{\lambda(n)}{n} = 0$. Indeed

$$\sum_{n=1}^{\infty} (-1)^n \frac{\lambda(n)}{n^s} + \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = 2 \sum_{n=1}^{\infty} \frac{\lambda(2n)}{(2n)^s}.$$

Since $\lambda(2) = -1$ and λ is multiplicative we obtain

$$\frac{\lambda(2n)}{(2n)^s} = -\frac{1}{2^s} \frac{\lambda(n)}{n^s}$$

so that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\lambda(n)}{n^s} = (-1 - \frac{2}{2^s}) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},$$

or

$$\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{\lambda(n)}{n^s} = (1 + \frac{2}{2^s}) \frac{\zeta(2s)}{\zeta(s)}.$$

We conclude by taking the limit $s \to 1$ as in (8.1).

It is generally believed that the values of the Möbius and Liouville functions enjoy various randomness properties. One manifestation of this principle is an old conjecture of Chowla [13] asserting that for all $l \in \mathbb{N}$ and all distinct $n_1, n_2, \cdots, n_l \in \mathbb{N}$ and for every $\epsilon_1, \epsilon_2, \cdots, \epsilon_l \in \{1, 2\}$ we have

$$\sum_{m=1}^{M} \mu^{\epsilon_1}(m+n_1)\cdots\mu^{\epsilon_l}(m+n_l) = o(M)$$

$$\sum_{m=1}^{M} \lambda^{\epsilon_1}(m+n_1)\cdots\lambda^{\epsilon_l}(m+n_l) = o(M).$$

According to P. Sarnak we say that a sequence $a(n)_{n\geq}1$ is deterministic if there exists a topological dynamical system (X, T) with zero topological entropy, a point $x \in X$, and a continuous function $f : X \to \mathbb{C}$ such that for all $n \geq 1$, $a(n) = f(T^n(x))$. Sarnak's conjecture states that for every deterministic sequence $a(n)_{n\geq}1$ we have

$$\sum_{m=1}^{M} \mu(n)a(n) = o(M).$$

The case of X is a point corresponds to the estimate $\sum_{m=1}^{M} \mu(n) = o(M)$, an equivalent form of the Prime Number Theorem. When $X = \mathbb{R}/\mathbb{Z}$ and $T(x) = x + \alpha$ (modulo 1), Sarnak's conjecture results from Davenports's estimate. We refer to [11] for an extended report on these innovative ideas.

We end this section by giving the graphs of the two remarkable functions

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\cos(2\pi nt), \quad \sum_{n=1}^{\infty} \frac{\lambda(n)}{n}\cos(2\pi nt).$$

These graphs evoke a hidden fractal structure, which deserves to be studied in depth (Figs. 1 and 2).

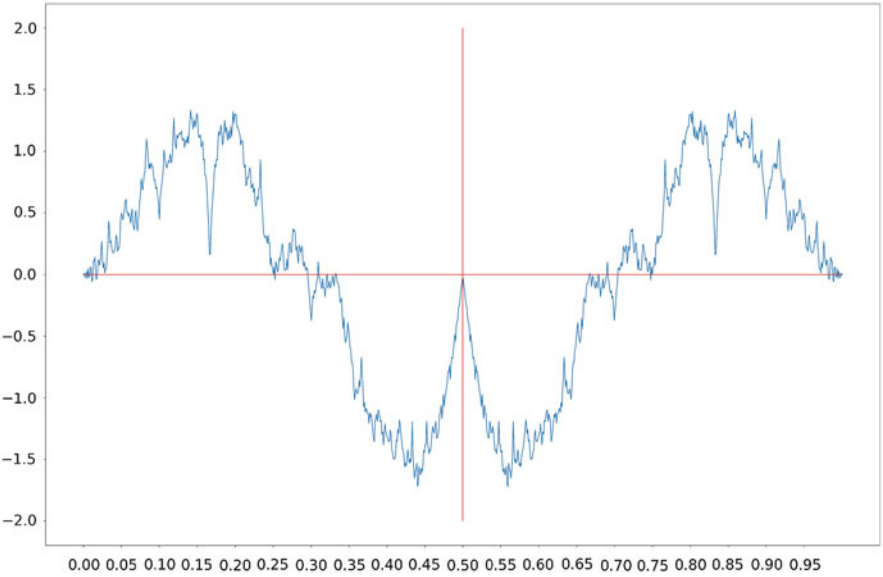

Fig. 1 Graph of $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cos(2\pi nt)$

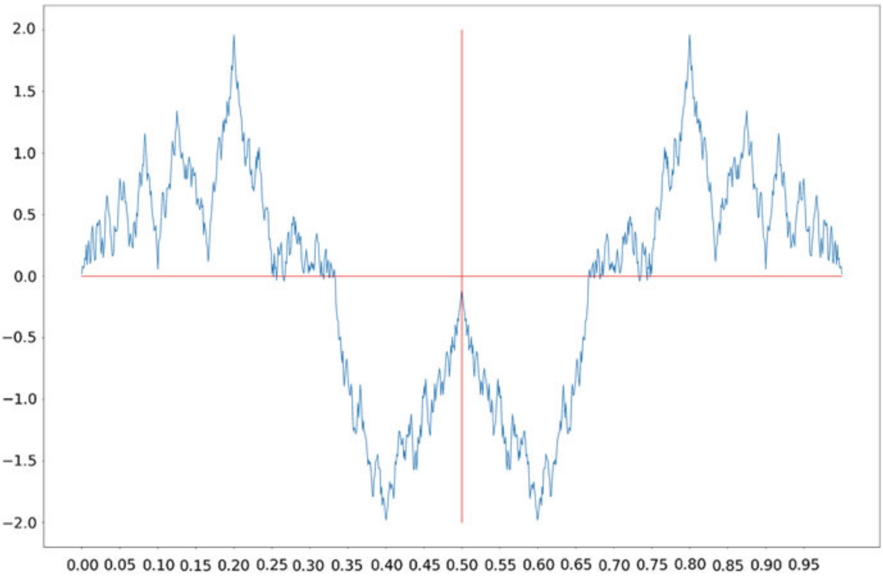

Fig. 2 Graph of $\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} \cos(2\pi nt)$

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Symmetric Measures, Continuous Networks, and Dynamics

Sergey Bezuglyi and Palle E. T. Jorgensen

Abstract With view to applications, we here give an explicit correspondence between the following two: (i) the set of symmetric and positive measures ρ on one hand, and (ii) a certain family of generalized Markov transition measures P, with their associated Markov random walk models, on the other. By a generalized Markov transition measure we mean a measurable and measure-valued function P on $(V, \mathcal{B})$, such that for every $x \in V$, $P(x; \cdot)$ is a probability measure on $(V, \mathcal{B})$. Hence, with the use of our correspondence (i)–(ii), we study generalized Markov transitions P and path-space dynamics. Given P, we introduce an associated operator, also denoted by P, and we analyze its spectral theoretic properties with reference to a system of precise L^2 spaces.

Our setting is more general than that of earlier treatments of reversible Markov processes. In a potential theoretic analysis of our processes, we introduce and study an associated energy Hilbert space $\mathcal{H}_E$, not directly linked to the initial L^2-spaces. Its properties are subtle, and our applications include a study of the P-harmonic functions. They may be in $\mathcal{H}_E$, called finite-energy harmonic functions. A second reason for $\mathcal{H}_E$ is that it plays a key role in our introduction of a generalized Green function. (The latter stands in relation to our present measure theoretic Laplace operator in a way that parallels more traditional settings of Green functions from classical potential theory.) A third reason for $\mathcal{H}_E$ is its use in our analysis of path-space dynamics for generalized Markov transition systems.

Keywords Markov operator · Standard measure space · Symmetric measure · Laplace operator · Markov chain · Harmonic function · Finite energy space

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1 Introduction

In this paper, we continue our study of the graph Laplace and Markov operators, initiated in [1], which was based on the key notion of a σ-finite symmetric measure defined on the product space $(V \times V, \mathcal{B} \times \mathcal{B})$ for a standard Borel space $(V, \mathcal{B})$.

Our goal is to extend the basic definitions and results of the theory of weighted networks (known also as electrical or resistance networks) to the case of measure spaces. We briefly recall that, for a countable locally finite connected graph $G = (V, E)$ without loops, one can identify the edge set E with a subset of the Cartesian product $V \times V$ and assign some weight c_{xy} for every point (x, y) in E where c_{xy} is a symmetric positive function. It gives us a symmetric atomic measure ρ on E whose projections on V are the counting measure μ. Then, for a weighted network (V, E, c), one defines the Markov transition probability kernel P and the graph Laplacian $\Delta = c(I - P)$ which are considered as operators acting either in L^2 spaces with respect to the measures μ and $\nu = c\mu$ or in the finite energy space $\mathcal{H}_E$. Their spectral properties are of great interest as well as the study of harmonic functions in the theory of weighted networks.

Our approach to the measurable theory of weighted networks is based on the concept of a *symmetric measure* defined on the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$ where $(V, \mathcal{B})$ is a standard Borel space. (To stress the existing parallels we use the same notation as in discrete case.) In more detail, in the context of measurable dynamics, the state space V is considered very generally; more specifically $(V, \mathcal{B})$ is given, where $\mathcal{B}$ is a specified σ-algebra for V. From $(V, \mathcal{B})$, we then form the corresponding product space, relative to the product σ-algebra on $V \times V$. It is important that our initial measure ρ is not assumed finite, but only σ-finite. Since ρ is assumed symmetric, the respective two marginal measures coincide, here denoted μ, and they will also not be finite; only σ-finite. The σ-finiteness will be a crucial fact in our computations of a number of Radon-Nikodym derivatives and norms of operators and vectors.

We establish an explicit correspondence between (i) symmetric and positive measures ρ on one hand, and (ii) a certain set of generalized Markov transition measures P on the other. More precisely, by a generalized Markov transition measure we mean a measurable and measure-valued function P on $(V, \mathcal{B})$, such that for every x in V, $P(x, \cdot)$ is a probability measure on $(V, \mathcal{B})$. From the generalized Markov transition P, we introduce an associated operator, also denoted by P. Its spectral theoretic properties refer to a certain L^2 space, and they will be made precise in Sect. 3.

In addition to the operator P, we shall also consider a natural transfer operator R (the choice of the letter "R" is for David Ruelle who initiated a variant of our analysis in the context of statistical mechanics); and a measure theoretic Laplacian, or Laplace operator. In the special case when V is countably discrete, our Laplace operator will be analogous to a family of more standard discretized classical Laplace operators. For related results on transfer operators, see e.g. [2–13].

Among the motivations for our present results are the following: A recent study of a variety of graph limits. This research area has both a general flavor, and an application-focus; see below. The latter includes recent papers on graphons; a current and extremely active area. In addition, we are motivated by a number of new operator-theoretic approaches to the study of graph limits, such as the notion of action convergence (see the recent works by Backhausz and Szegedy, [14, 15] and Pensky [16]). While we mention some of these connections inside our paper, our present emphasis is the theoretic foundations for these related developments.

New Results It is important to note that our setting is not restricted to the case of finite measures. In fact, in our discussion of Markov transition dynamics, important examples simply will not allow finite covariant measures. We recall that the theory of weighted networks can serve as a discrete analog of our measurable settings, see [1] where this analogy was discussed in detail. The corresponding symmetric measure on the edge set E is σ-finite as well as the counting measure μ on the set of vertices V. Our definitions of the energy space $\mathcal{H}_E$, Markov operator P, and the graph Laplace operator Δ are direct translations of the corresponding definitions for weighted networks.

To the best of our knowledge, such interpretations of these objects have not been considered earlier. We stress that our approach to Markov processes generated by σ-finite symmetric measures leads with necessity to the study of Markov transition operators defined on infinite σ-finite measure spaces. The existing literature on Markov processes is devoted mostly to the case of probability measure spaces, see, e.g., [17–19].

The notion of Borel equivalence relation defined on a standard Borel space illustrates our setting, and it can be viewed as a rich source of various examples. We refer to the following books and articles: [20–27].

More applications of measurable setting for the study of Markov processes and Laplacians are given in [1]. We mention here the theory of graphons, Dirichlet forms, and the theory of determinantal measures.

With our starting point, a choice of a fixed symmetric and positive measure ρ on a product space, we will then have four natural Hilbert spaces, three are just L^2 spaces, $L^2(\rho)$, and two L^2 spaces referring to the marginal measure μ. The fourth Hilbert space is different. We call it the finite energy Hilbert space $\mathcal{H}_E$. Its use is motivated by potential theory, and it has a more subtle structure among the considered Hilbert spaces. Given ρ, we introduce an associated energy Hilbert space, denoted $\mathcal{H}_E$, but depending on the initially given ρ. This energy Hilbert space $\mathcal{H}_E$ is not directly linked to the initial L^2 spaces, and its properties are quite different. Nonetheless, the energy Hilbert space $\mathcal{H}_E$ will play a key role in our analysis in the main body of our paper. There are many reasons for this. For example, non-constant harmonic functions will not be in L^2; but, in important applications, they may be in $\mathcal{H}_E$; we refer to the latter as finite energy harmonic functions. A second reason for $\mathcal{H}_E$ is that it plays a crucial role in our introduction of a generalized Green's function. The latter stands in relation to our Laplace operator in a way that is parallel to more classical settings of Green's functions from potential theory. A third reason for $\mathcal{H}_E$

is its use in our analysis of path-space dynamics for the Markov transition system, mentioned above.

Organization Our *main results* are proved in Theorems 3.10, 4.7, 4.11, 5.3, 6.2, 6.11, and 7.2.

The paper is organized as follows. Section 2 contains our basic definitions and preliminary results. We discuss here the concepts of standard Borel and standard measure spaces, kernels, irreducible symmetric measures, and disintegration. The transfer operator R, Markov operator P, and graph Laplacian Δ are defined in Sect. 3. We collected a number of results about the spectral properties of these operators that were proved in [1]. Also the reader will find the definition of the finite energy Hilbert space $\mathcal{H}_E$, several results about the structure of the space $\mathcal{H}_E$ and the norm of functions from $\mathcal{H}_E$. We consider also the embedding operator J and prove that J is an isometry. In Sect. 4, we consider the equivalence of Markov operators and the Laplacians generated by equivalent symmetric measures ρ and ρ'. It turns out that, for equivalent symmetric measures ρ and ρ', there exists an isometry for the corresponding energy Hilbert spaces $\mathcal{H}_E(\rho)$ and $\mathcal{H}_E(\rho')$. The notion of reversible Markov processes is discussed in Sect. 5. We relate various properties of the operator P (such as self-ajointness) to this notion and to the notion of a symmetric measure. A number of results about Markov operators acting in the L^2 spaces and energy space $\mathcal{H}_E$ are proved in this section. Section 6 focuses on the case of a transient Markov processes defined by a Markov operator P. We define the path-space measure $\mathbb{P}$ and Green's function $G(x, A)$, and we discuss their properties. Section 7 is devoted to construction of a sequence of discrete weighted networks which can be used to approximate the objects considered for the measurable setting.

In our article we discuss several key notions such as reversible Markov processes, Green's function, transient processes, limit theory (covering boundaries), potential theory, general Dirichlet forms, graph Laplacians, etc. For the benefit of non-experts in these areas, we included a number of general references in the corresponding sections.

2 Basic Definitions and Symmetric Measures

In this section, we briefly describe our main setting and introduce the most important notation. We also recall several results from [1] which will be used here.

2.1 Standard Borel and Measure Spaces

Suppose V is a *Polish space*, i.e., V is a separable completely metrizable topological space. Let $\mathcal{B}$ denote the σ-algebra of Borel sets generated by open sets of V. Then $(V, \mathcal{B})$ is called a *standard Borel space*. The theory of standard Borel spaces is discussed in many recent books, see e.g., [25, 26, 28, 29] and papers [30, 31]. We recall that all uncountable standard Borel spaces are Borel isomorphic, so that one can use any convenient realization of the space V working in the category of measurable spaces. If μ is a continuous (i.e., non-atomic) positive Borel measure on $(V, \mathcal{B})$, then $(V, \mathcal{B}, \mu)$ is called a *standard measure space*. Given $(V, \mathcal{B}, \mu)$, we will call μ a measure for brevity. As a rule, we will deal with non-atomic σ-finite positive measures on $(V, \mathcal{B})$ (unless the opposite is clearly indicated) which take values in the extended real line $\overline{\mathbb{R}}$. We use the name of *standard measure space* for both finite and σ-finite measure spaces. Also the same notation, $\mathcal{B}$, is applied for the σ-algebras of Borel sets and measurable sets of a standard measure space. It should be clear from the context what σ-algebra is considered. Working with a measure space $(V, \mathcal{B}, \mu)$, we always assume that $\mathcal{B}$ is *complete* with respect to μ. By $\mathcal{F}(V, \mathcal{B})$. we denote the space of real-valued bounded Borel functions on $(V, \mathcal{B})$. For $f \in \mathcal{F}(V, \mathcal{B})$ and a Borel measure μ on $(V, \mathcal{B})$, we write

$$\mu(f) = \int_V f \, d\mu.$$

All objects, considered in the context of measure spaces (such as sets, functions, transformations, etc), are determined modulo sets of zero measure. In most cases, we will implicitly use this mod 0 convention not mentioning the sets of zero measure explicitly.

In what follows, we will use (in most cases implicitly) the notion of *measurable fields*. Given a measure space $(V, \mathcal{B}, \mu)$, it is said that $x \mapsto A_x \in \mathcal{B}$ is a *measurable field of sets* if the set

$$\bigcup_{x \in V} \{x\} \times A_x \in \mathcal{B} \times \mathcal{B}.$$

Similarly, one can define a *measurable field of measures* $x \to \mu_x$ on $(V, \mathcal{B})$ requiring $x \mapsto \mu_x(A)$ to be a measurable function for any $A \in \mathcal{B}$.

Consider a σ-finite continuous measure μ on a standard Borel space $(V, \mathcal{B})$. We denote by

$$\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu) = \{A \in \mathcal{B} : \mu(A) < \infty\} \tag{2.1}$$

the algebra of Borel sets of finite measure μ. Clearly, any σ-finite measure μ is uniquely determined by its values on $\mathcal{B}_{\text{fin}}(\mu)$.

The linear space of simple function over sets from $\mathcal{B}_{\text{fin}}(\mu)$ is denoted by

$$\begin{aligned}\mathcal{D}_{\text{fin}}(\mu) &:= \left\{ \sum_{i \in I} a_i \chi_{A_i} : A_i \in \mathcal{B}_{\text{fin}}(\mu),\ a_i \in \mathbb{R},\ |I| < \infty \right\} \\ &= \text{Span}\{\chi_A : A \in \mathcal{B}_{\text{fin}}(\mu)\},\end{aligned} \tag{2.2}$$

will play an important role in our work since simple functions from $\mathcal{D}_{\text{fin}}(\mu)$ form a norm dense subset in $L^p(\mu)$-space, $p \geq 1$.

2.2 Symmetric Measures, Kernels, and Disintegration

Definition 2.1 Let E be an uncountable Borel subset of the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$ such that:

(i) $(x, y) \in E \iff (y, x) \in E$, i.e. $\theta(E) = E$ where $\theta(x, y) = (y, x)$ is the flip automorphism;
(ii) $E_x := \{y \in V : (x, y) \in E\} \neq \emptyset, \quad \forall x \in X$;
(iii) for every $x \in V$, $(E_x, \mathcal{B}_x)$ is a standard Borel space where $\mathcal{B}_x$ is the σ-algebra of Borel sets induced on E_x from $(V, \mathcal{B})$.

We call E a *symmetric set*.

It follows from (ii) and (iii) that the projection of the symmetric set E on each margin of the product space $(V \times V, \mathcal{B} \times \mathcal{B})$ is V.

We observe that conditions (ii) and (iii) are, strictly speaking, not related to the symmetry property; they are included in Definition 2.1 for convenience, so that we will not have to make additional assumptions. Condition (iii) assumes two cases: the Borel space E_x can be countable or uncountable. We focus mostly on uncountable Borel standard spaces.

There are several natural examples of symmetric sets related to dynamical systems. We mention here the case of a *Borel equivalence relation* E on a standard Borel space $(V, \mathcal{B})$. By definition, E is a Borel subset of $V \times V$ such that $(x, x) \in E$ for all $x \in V$, (x, y) is in E iff (y, x) is in E, and $(x, y) \in E$, $(y, z) \in E$ implies that $(x, z) \in E$. Let $E_x = \{y \in V : (x, y) \in E\}$, then E is partitioned into "vertical fibers" E_x. In particular, it can be the case when every E_x is countable. Then E is called a *countable Borel equivalence relation.*

We say that a symmetric set E is *decomposable* if there exists an uncountable Borel subset $A \subset V$ such that

$$E \subset (A \times A) \cup (A^c \times A^c), \tag{2.3}$$

where $A^c = V \setminus A$.

The meaning of this definition can be clarified for Borel equivalence relations: if E satisfies (2.3), then the set A is *E-invariant*.

We recall several definitions and facts about kernels defined on a measurable space, see e.g. [18, 19]. Given a standard measure space $(V, \mathcal{B})$, we define a *σ-finite kernel* k as a function $k : V \times \mathcal{B} \to \overline{\mathbb{R}}_+$ (where $\overline{\mathbb{R}}_+$ is the extended real line) such that

(i) $x \mapsto k(x, A)$ is measurable for every $A \in \mathcal{B}$;
(ii) for any $x \in V$, $k(x, \cdot)$ is a σ-finite measure on $(V, \mathcal{B})$.

A kernel $k(x, A)$ is called *finite* if $k(x, \cdot)$ is a finite measure on $(V, \mathcal{B})$ for every x. We will also use the notation $k(x, dy)$ for the measure on $(V, \mathcal{B})$.

The definition of a finite kernel can be used to define new measures on the measurable spaces $(V, \mathcal{B})$ and $(V \times V, \mathcal{B} \times \mathcal{B})$.

Given a σ-finite measure space $(V, \mathcal{B}, \mu)$ and a finite kernel $k(x, A)$, we set

$$\kappa(A) = \int_V k(x, A)\, d\mu(x).$$

Then κ is a σ-finite measure on $(V, \mathcal{B})$ (which is also called a *random measure* in the literature).

For a kernel k as above, one can define inductively the sequence of kernels $(k^n : n \geq 1)$ by setting

$$k^n(x, A) = \int_V k^{n-1}(y, A)\, k(x, dy), \qquad n > 1. \tag{2.4}$$

Following [18], we formulate definitions of main properties of a kernel k. We say that a *set* $A \in \mathcal{B}$ *is attainable from* $x \in V$ if there exists $n \geq 1$ such that $k^n(x, A) > 0$, in symbols, we write $x \to A$. A set $A \in \mathcal{B}$ is called *closed* for the kernel k if $k(x, A^c) = 0$ for all $x \in A$. If A is closed, then it follows from (2.4) that $k^n(x, A^c) = 0$ for any $n \in \mathbb{N}$ and $x \in A$. Hence, A is closed if and only if $x \nrightarrow A^c$.

A kernel $k = k(x, A)$ is called *Borel indecomposable* on $(V, \mathcal{B})$ if there do not exist two disjoint non-empty closed subsets A_1 and A_2.

Let $F_x \in \mathcal{B}$ be the support of the measure $k(x, \cdot)$, that is $k(x, V \setminus F_x) = 0$. By $\widetilde{F}_x$, we denote the set $\{x\} \times K_x \subset V \times V$. Then the formula

$$k(A \times B) = \int_A \widetilde{k}(x, B)\, d\mu(x)$$

defines a σ-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ where $\widetilde{k}(x, \cdot) = (\delta_x \times k)(x, \cdot)$. The support of k is the set

$$F := \bigcup_{x \in V} \widetilde{F}_x.$$

We will use below slightly simplified notation identifying the sets F_x and $\widetilde{F}_x$ and the measures $k(x, A)$ and $\widetilde{k}(x, A)$. It will be clear from the context what objects are considered.

As mentioned in Introduction, our approach is based on the study of *symmetric measures* defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, see Definition 2.4. We show that every measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ generates a kernel $x \to \rho_x(A), A \in \mathcal{B}$. This observation is based on the concept of *disintegration* of the measure ρ. We recall here this construction.

Denote by π_1 and π_2 the projections from $V \times V$ onto the first and second factor, respectively. Then $\{\pi_1^{-1}(x) : x \in V\}$ and $\{\pi_2^{-1}(y) : y \in V\}$ are the *measurable partitions* of $V \times V$ into vertical and horizontal fibers, see [1, 22, 32] for more information on properties of measurable partitions. The case of probability measures was studied by Rokhlin in [32], whereas the disintegration of σ-finite measures has been considered somewhat recently. We refer to a result from [33] whose formulation is adapted to our needs.

Theorem 2.2 ([33]) *For a σ-finite measure space $(V, \mathcal{B}, \mu)$, let ρ be a σ-finite measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $\rho \circ \pi_1^{-1} \ll \mu$. Then there exists a unique system of conditional σ-finite measures $(\widetilde{\rho}_x)$ such that*

$$\rho(f) = \int_V \widetilde{\rho}_x(f)\, d\mu(x), \quad f \in \mathcal{F}(V \times V, \mathcal{B} \times \mathcal{B}).$$

In the following remark we collect several facts that clarify the essence of the defined objects.

Remark 2.3

(1) The condition of Theorem 2.2 assumes that a measure μ is prescribed on the Borel space $(V, \mathcal{B})$. If one begins with a measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$, then the measure μ arises as the projection of ρ on $(V, \mathcal{B})$, $\rho \circ \pi_1^{-1} = \mu$.

(2) Let E be a Borel symmetric subset of $(V \times V, \mathcal{B} \times \mathcal{B})$, and let ρ be a measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying the condition of Theorem 2.2. Then E can be partitioned into the fibers $\{x\} \times E_x$. By Theorem 2.2, there exists a unique system of conditional measures $\widetilde{\rho}_x$ such that, for any ρ-integrable function $f(x, y)$, we have

$$\iint_{V \times V} f(x, y)\, d\rho(x, y) = \int_V \widetilde{\rho}_x(f)\, d\mu(x). \tag{2.5}$$

It is obvious that, for μ-a.e. $x \in V$, $\mathrm{supp}(\widetilde{\rho}_x) = \{x\} \times E_x$ (up to a set of zero measure). To simplify the notation, we will write

$$\int_V f\, d\rho_x \text{ and } \iint_{V \times V} f\, d\rho$$

though the measures ρ_x and ρ have the supports E_x and E, respectively.

(3) It follows from Theorem 2.2 that the measure ρ determines the measurable field of sets $x \mapsto E_x \subset V$ and measurable field of σ-finite Borel measures $x \mapsto \rho_x$ on $(V, \mathcal{B})$, where the measures ρ_x are defined by the relation

$$\widetilde{\rho}_x = \delta_x \times \rho_x. \tag{2.6}$$

Hence, relation (2.5) can be also written in the following form, used in our subsequent computations,

$$\iint_{V\times V} f(x,y)\, d\rho(x,y) = \int_V \left(\int_V f(x,y)\, d\rho_x(y)\right) d\mu(x). \tag{2.7}$$

In other words, we have a measurable family of measures ($x \mapsto \rho_x$), and it defines a new measure ν on $(V, \mathcal{B})$ by setting

$$\nu(A) := \int_V \rho_x(A)\, d\mu(x), \quad A \in \mathcal{B}. \tag{2.8}$$

Remark that the measure ρ_x is defined on the subset E_x of $(V, \mathcal{B})$, $x \in V$.

Definition 2.4 Let $(V, \mathcal{B})$ be a standard Borel space. We say that a measure ρ on $(V \times V, \mathcal{B} \times B)$ is *symmetric* if

$$\rho(A \times B) = \rho(B \times A), \quad \forall A, B \in \mathcal{B}.$$

In other words, ρ is invariant with respect to the flip automorphism θ.

The following remark contains natural properties of symmetric measures. Some of them were proved in [1], the others are rather obvious.

Remark 2.5

(1) If ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, then the support of ρ, the set $E = E(\rho)$, is symmetric mod 0. Here $E(\rho)$ is defined up to a set of zero measure by the relation $\rho((V \times V) \setminus E) = 0$.
(2) We consider the symmetric measures whose supporting sets E satisfy Definition 2.1. In other words, we require that, for every $x \in V$, the set $E_x \subset E$ is uncountable and therefore is a standard Borel space. The case when E_x is countable arises, in particular, when E is a Borel countable equivalence relation on $(V, \mathcal{B})$. The latter was considered in [1]. For countable sets E_x, $x \in V$, we can take ρ_x as a finite measure which is equivalent to the counting measure, see, e.g. [24, 34, 35] for details.
(3) In general, the notion of a symmetric measure is defined in the context of standard Borel spaces $(V, \mathcal{B})$ and $(V \times V, \mathcal{B} \times \mathcal{B})$. But if a σ-finite measure μ is given on $(V, \mathcal{B})$, then we need to include an additional relation between the projections of ρ on V and the measure μ. Let $\pi_1 : V \times V \to V$ be the projection on the first coordinate. We require that the symmetric measure must satisfy the property $\rho \circ \pi_1^{-1} \ll \mu$, see Theorem 2.2.
(4) The symmetry of the set E allows us to define a "mirror" image of the measure ρ. Let $E^y := \{x \in V : (x, y) \in E\}$, and let $(\widetilde{\rho}^y)$ be the system of conditional

measures with respect to the partition of E into the sets $E^y \times \{y\}$. Then, for the measure

$$\widetilde{\rho} = \int_V \widetilde{\rho}^y \, d\mu(y),$$

the relation $\rho = \widetilde{\rho}$ holds.

(5) It is worth noting that, in general, when a measure μ is defined on $(V, \mathcal{B})$, the set $E(\rho)$ do not need to be a set of positive measure with respect to the product measure $\mu \times \mu$. In other words, we admit both cases: (a) ρ is equivalent to $\mu \times \mu$, (b) ρ and $\mu \times \mu$ are mutually singular.

Assumption 1 In this paper, we consider the class of symmetric measures ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ which satisfy the following property:

$$0 < c(x) := \rho_x(V) < \infty, \qquad \mu\text{-a.e. } x \in V, \tag{2.9}$$

where $x \mapsto \rho_x$ is the measurable field of measures arising in Theorem 2.2.

Moreover, in most statements, we will assume that $c(x) \in L^1_{\text{loc}}(\mu)$, i.e.,

$$\int_A c(x) \, d\mu(x) < \infty, \qquad \forall A \in \mathcal{B}_{\text{fin}}(\mu).$$

This property of the function $c(x)$ is natural because it corresponds to local finiteness of graphs in the theory of weighted (electric) networks. In several statements, we will require that

$$\left(\forall A \in \mathcal{B}_{\text{fin}}(\mu), \ \int_A c^2 \, d\mu < \infty \right) \Longleftrightarrow c \in L^2_{\text{loc}}(\mu).$$

We observe also that the case when the function c is bounded leads to bounded Laplace operators and is not interesting for us.

Relation (2.8) defines the measure ν such that the measures μ and ν are equivalent. It is stated in Lemma 2.6 that $c(x)$ is the Radon-Nikodym derivative of ν with respect to μ. If we want to reverse the definition and use ν as a primary measure, then we need to require that the function $c(x)^{-1}$ is locally integrable with respect to ν.

The following (important for us) fact follows from the definition of symmetric measures. We emphasize that formula (2.10) will be used repeatedly in many proofs.

Lemma 2.6

(1) For a symmetric measure ρ and any bounded Borel function f on $(V \times V, \mathcal{B} \times \mathcal{B})$,

$$\iint_{V \times V} f(x, y) \, d\rho(x, y) = \iint_{V \times V} f(y, x) \, d\rho(x, y). \tag{2.10}$$

Equality (2.10) is understood in the sense of the extended real line, i.e., the infinite value of the integral is allowed.

(2) Let ν be defined as in (2.8). Then

$$d\nu(x) = c(x)d\mu(x).$$

2.3 Irreducible Symmetric Measures

We now relate the notions of symmetric measures and kernels. It turns out that one can associate a finite kernel $\mathcal{K}(\rho) = \mathcal{K}$ to any symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$. For this, we use the disintegration of ρ according to Theorem 2.2, $\rho = \int_V \rho_x \, d\mu(x)$, and set $x \to \mathcal{K}(x, A) = \rho_x(A)$.

The definition of sets attainable from $x \in V$ and that of decomposable sets, given above in the context of Borel spaces, can be translated to the case of measure spaces. Below we define the notion of an *irreducible symmetric measure* which will be extensively used in the paper.

Definition 2.7

(1) A kernel $x \to k(x, \cdot)$ is called *irreducible with respect to a σ-finite measure μ on $(V, \mathcal{B})$ (μ-irreducible)* if, for any set A of positive measure μ and μ-a.e. $x \in V$, there exists some n such that $k^n(x, A) > 0$, i.e., any set A of positive measure is attainable from μ-a.e. x, $x \to A$.

(2) A *symmetric measure* ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ is called *irreducible* if the corresponding kernel $\mathcal{K}(\rho) : x \to \rho_x(\cdot)$ is μ-irreducible where μ is the projection of measure ρ.

(3) A symmetric measure ρ (or the kernel $x \to \rho_x(\cdot)$) is called *μ-decomposable* if there exists a Borel subset A of V of positive measure μ such that

$$E \subset (A \times A) \cup (A^c \times A^c) \tag{2.11}$$

where $A^c = V \setminus A$ is also of positive measure. Otherwise, ρ is called *indecomposable*.

Every kernel k, defined on $(V, \mathcal{B})$, generates the *potential kernel*

$$G(k)(x, A) := \sum_{n=0}^{\infty} k^n(x, A)$$

where $k^0(x, A) = \chi_A(x)$. In general, the kernel G may be degenerated admitting only the values 0 and ∞. We will discuss below the role of G in the case of transient Markov processes.

Lemma 2.8 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ with the kernel $\mathcal{K}(x, A) = \rho_x(A)$. Suppose that the support of ρ, the set E, satisfies relation (2.11) where $\mu(A) > 0$ and $\mu(A^c) > 0$, i.e. the kernel $x \mapsto \rho_x(A)$ is μ-decomposable. Then the sets A and A^c are closed and $x \mapsto \rho_x(A)$ is a μ-reducible kernel. The converse statement also holds.*

Proof The first result follows directly from the definitions given above in this subsection. To see that the converse is true, it suffices to note that, for any set B of positive measure, the compliment $\widehat{B}^c$ of the set

$$\widehat{B} := B \cup \{x \in V : x \to B\}$$

is either of zero measure, or closed (recall that $x \to B$ means that there exists n such that $\mathcal{K}^n(x, B) > 0$). If ρ is reducible, then there exists a set A, $\mu(A) > 0$, such that the closed set $\mu(\widehat{A}^c)$ has positive measure. The existence of such a set implies that the measure ρ is decomposable. □

It is obvious from this lemma that a decomposable symmetric measure ρ cannot be irreducible. It was proved in [1] that the definitions of an irreducible measure and irreducible kernel agree, see Theorem 6.2 below.

By definition, the projection of the support of an irreducible measure ρ is the set V. Irreducibiliity of symmetric measures means irreducibility of a corresponding Markov process, see details in [1].

In the following statement, we give another approach to the notion of irreducible symmetric measures. Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We use the support of the fiber measure ρ_x, $x \in V$, to characterize an irreducible measure in different terms.

For any fixed $x \in V$, we define a sequence of subsets: $A_0(x) = \{x\}$, $A_1(x) = E_x$,

$$A_n(x) = \bigcup_{y \in A_{n-1}(x)} E_y, \quad n \geq 2.$$

Recall that E_x is the support of the measure ρ_x, and E_x can be identified with the vertical section of the symmetric set E. Note that all the sets $A_n(x)$ are in $\mathcal{B}$ as $x \to E_x$ is a measurable field of sets.

Lemma 2.9 *Given $(V, \mathcal{B}, \mu)$, a symmetric measure ρ is irreducible if and only if for μ-a.e. $x \in V$ and any set $B \in \mathcal{B}$ of positive measure there exists $n \geq 1$ such that*

$$\mu(A_n(x) \cap B) > 0. \tag{2.12}$$

Proof Indeed, the property formulated in (2.12) is another form of $k^n(x, B) > 0$ where the kernel k is defined by $x \to \rho_x$. □

Various aspects of symmetric measures are also discussed in [36, 37]. In particular, one can observe that if symmetric measures ρ and $\overline{\rho}$ are equivalent, then they are simultaneously either irreducible or not.

3 Linear Operators and Hilbert Spaces Associated to Symmetric Measures

While the main structures of our paper (symmetric measures, transfer operators R, Markov transition densities P, and associated Laplace operators Δ) may be naturally formulated in the general context of measurable functions, their spectral theory, and their dynamic-systems properties, only take a precise form after suitable Hilbert spaces are introduced. We will show that the initial structures, reversible Markov processes, and associated Laplace operators, etc., in turn dictate their own natural Hilbert space theoretic context. More precisely, in the section below, we identify the particular L^2 spaces, having the property that respective operators R, P, and Δ become self-adjoint. In addition to these L^2 spaces, we also identify two other Hilbert spaces (details below). They are motivated by parallels to classical potential theory, and to the study of diffusion processes. Moreover, they have discrete counterparts in the study of infinite networks, and of graph Laplacians. But presently, we introduced these two Hilbert spaces in a general measure space context. Continuing conventions from our earlier papers, we shall denote these Hilbert spaces (i) the energy Hilbert space, and (ii) the dissipation Hilbert space. The latter refers to a certain path-space construction, which in turn is built directly from the initial structure, mentioned above, symmetric measure, transfer operator, and Markov transition densities.

3.1 Symmetric Operator R, Markov Operator P, and Laplacian Δ

Suppose $k : V \times \mathcal{B} \to \mathbb{R}_+$ is a finite kernel defined on a standard Borel space $(V, \mathcal{B})$. Then it defines a linear positive (see Remark 3.3) operator $P(k)$ which is determined by the kernel k:

$$P(k)(f)(x) := \int_V f(y)\, k(x, dy). \tag{3.1}$$

It can be easily seen that, for the kernels k^n (see (2.4)), the operator $P(k^n)$, defined as in (3.1), satisfies the property:

$$P(k^n) = P(k)^n, n \in \mathbb{N}.$$

We consider in this section the kernel $\mathcal{K}(\rho)$ generated by a symmetric measure ρ, i.e., $\mathcal{K}(x, A) = \rho_x(A)$.

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and ρ a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ supported by a symmetric set E. Let $x \mapsto \rho_x$ be the measurable family of measures on $(V, \mathcal{B})$ that *disintegrates* ρ. Recall that, by Assumption 1, the function

$c(x) = \rho_x(V)$ is finite for μ-a.e. x. As discussed above in Sect. 2.2, the measure ρ produces a finite kernel $\mathcal{K}(\rho)$ which we use to define the following operators.

Definition 3.1 For a symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$, we define three linear operators R, P and Δ acting on the space of bounded Borel functions $\mathcal{F}(V, \mathcal{B})$.

(i) The *symmetric operator* R:

$$R(f)(x) := \int_V f(y)\, d\rho_x(y) = \rho_x(f). \tag{3.2}$$

(ii) The *Markov operator* P:

$$P(f)(x) = \frac{1}{c(x)} R(f)(x)$$

or

$$P(f)(x) := \frac{1}{c(x)} \int_V f(y)\, d\rho_x(y) = \int_V f(y)\ P(x, dy) \tag{3.3}$$

where $P(x, dy)$ is the probability measure obtained by normalization of $d\rho_x(y)$, i.e.

$$P(x, dy) := \frac{1}{c(x)} d\rho_x(y).$$

In other words, the Markov operator P defines the measurable field $x \mapsto P(x, \cdot)$ of *transition probabilities* on the space $(V, \mathcal{B})$, or a *Markov process*.

(iii) The *graph Laplace operator* Δ:

$$\Delta(f)(x) := \int_V (f(x) - f(y))\, d\rho_x(y) \tag{3.4}$$

or

$$\Delta(f) = c(I - P)(f) = (cI - R)(f). \tag{3.5}$$

Using (2.9), we can write the operator Δ in more symmetric form:

$$\Delta(f) = R(\mathbb{1})f - R(f)$$

where $\mathbb{1}$ is a function identically equal to 1,

Remark 3.2 (*R* as a Transfer Operator) It is worth noting that the operator R can be treated as a transfer operator (see e.g. [38] and the literature cited there).

Let $(V, \mathcal{B}, \mu)$ be a standard measure space, and let σ be a surjective endomorphism of X. Consider the partition ξ of X into the orbits of σ: $y \in Orb_\sigma(x)$

if there are non-negative integers n, m such $\sigma^n(y) = \sigma^m(x)$. Let the partition η be the measurable hull of ξ. Take the system of conditional measures $\{\mu_C\}_{C\in\xi}$ corresponding to the partition η (see Theorem 2.2).

We define a transfer operator R on the standard measure space $(V, \mathcal{B}, \mu)$ by setting

$$R(f)(x) := \int_{C_x} f(y)\, d\mu_{C_x}(y) \tag{3.6}$$

where C_x is the element of η containing x. The domain of R is $L^1(\mu)$ in this example.

As was shown in [38], the operator $R : L^1(\mu) \to L^1(\mu)$ defined by (3.6) is a *transfer operator*, i.e., it satisfies the relation

$$R((f \circ \sigma)g)(x) = f(x)(Rg)(x).$$

To see that our definition of the operator R given in (3.2) agrees with (3.6), it suffices to take the measurable partition η of $V \times V$ into subsets $\{\pi_1^{-1}(x) : x \in V\}$ where π_1 is the projection of $V \times V$ onto V.

Remark 3.3 In this remark we make several comments about the basic properties of the operators R, P, and Δ.

(1) The definition of each of the operators R, P, and Δ depends on a symmetric measure ρ, and, strictly speaking, they must be denoted as $R(\rho)$, $P(\rho)$, and $\Delta(\rho)$. Since most of our results are proved for a fixed measure ρ, we will drop this variable. Below in this section, we discuss the relationships between $P(\rho)$ and $P(\rho')$ when ρ and ρ' are equivalent symmetric measures.
(2) The operators R and P are *positive* in the sense that $R(f) \geq 0$ and $P(f) \geq 0$ whenever $f \geq 0$. Moreover, if $f = \mathbb{1}$, then $P(\mathbb{1}) = \mathbb{1}$ because every measure $P(x, \cdot)$ is probability. Hence, P is a *Markov operator.*
(3) The properties of the graph Laplace operator Δ are formulated in Proposition 3.7, which is given below. All statements from this theorem are proved in [1] (see also [39, 40]). Other aspects of graph Laplace operators in the context of measure spaces are discussed in [41, 42].
(4) Since every measure ρ on $V \times V$ is uniquely determined by its values on a dense subset of functions, it suffices to define ρ on the set of the so-called "cylinder functions" $(f \otimes g)(x, y) := f(x)g(y)$. This observation will be used below when we prove a relation for cylinder functions first.
(5) In general, a positive operator R in $\mathcal{F}(V, \mathcal{B})$ is called *symmetric* if it satisfies the relation:

$$\int_V fR(g)\, d\mu = \int_V R(f)g\, d\mu, \tag{3.7}$$

for any $f, g \in F(V, \mathcal{B})$. It turns out that any symmetric operator R defines a symmetric measure ρ. Indeed, the functional

$$\rho : (f, g) \mapsto \int_V f(x)R(g)(x)\, d\mu(x), \qquad f, g \in F(V, \mathcal{B}), \tag{3.8}$$

determines a measure on $(V, \mathcal{B})$ such that

$$\rho(A \times B) = \int_V \chi_A(x)R(\chi_B)(x)\, d\mu(x).$$

As shown in [1], the operator R is symmetric if and only if the measure ρ, defined in (3.8), is symmetric.

In Definition 3.1, we do not discuss domains of the operators R, P, and Δ. It depends on the space where an operator is considered. In the current paper, we work with L^2-Hilbert spaces defined by the measures μ, ν, and ρ. But the most intriguing is the case of the finite energy space Hilbert space h_E. We discuss the properties of this space as well as those of operators Δ and P acting in $\mathcal{H}_E$ in the forthcoming paper [43]. On the other hand, we have already proved a number of results about these objects in [1]. We find it useful to give here the definitions and some formulas which are used below.

We remark that the finite energy space $\mathcal{H}_E$, see Definition 3.4 can be viewed as a generalization of the energy space considered for discrete weighted networks. They have been extensively studied during last decades.

Definition 3.4 Let $(V, \mathcal{B}, \mu)$ be a standard measure space with σ-finite measure μ. Suppose that ρ is a symmetric measure on the Cartesian product $(V \times V, \mathcal{B} \times \mathcal{B})$. We say that a Borel function $f : V \to \mathbb{R}$ belongs to the *finite energy space* $\mathcal{H}_E = \mathcal{H}_E(\rho)$ if

$$\iint_{V\times V} (f(x) - f(y))^2\, d\rho(x, y) < \infty. \tag{3.9}$$

Remark 3.5

(1) It follows from Definition 3.4 that $\mathcal{H}_E$ is a vector space containing all constant functions. We identify functions f_1 and f_2 such that $f_1 - f_2 = const$ and, with some abuse of notation, the quotient space is also denoted by $\mathcal{H}_E$. So that, we will call elements f of $\mathcal{H}_E$ functions assuming that a representative of the equivalence class f is considered.

(2) Definition 3.4 assumes that a symmetric irreducible measure ρ is fixed on $(V \times V, \mathcal{B} \times \mathcal{B})$. This means that the space of functions f on $(V, \mathcal{B})$ satisfying (3.9) depends on ρ, and, to stress this fact, we will use also the notation $\mathcal{H}_E(\rho)$.

Define the norm in $\mathcal{H}_E$ by setting

$$||f||^2_{\mathcal{H}_E} := \frac{1}{2}\iint_{V\times V} (f(x) - f(y))^2 \, d\rho(x, y), \quad f \in \mathcal{H}, \tag{3.10}$$

As proved in [1], $\mathcal{H}_E$ is a *Hilbert space* with respect to the norm $||\cdot||_{\mathcal{H}_E}$.

The description of the structure of the Hilbert space $\mathcal{H}_E$ is a very intriguing problem. We give here a few results proved in [1].

Theorem 3.6 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $\mu = \rho \circ \pi_1^{-1}$. Suppose $c(x) = \rho_x(V)$ is locally integrable with respect to μ.*

(1) For the measure $d\nu(x) = c(x)d\mu(x)$, we have

$$\mathcal{D}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{fin}}(\nu) \subset \mathcal{H}_E.$$

Moreover, if $A \in \mathcal{B}_{\text{fin}}(\nu)$, then

$$||\chi_A||^2_{\mathcal{H}_E} = \rho(A \times A^c) \leq \int_A c(x)\, d\mu(x) = \nu(A), \tag{3.11}$$

where $A^c := V \setminus A$.

(2) For every $A \in \mathcal{B}_{\text{fin}}(\mu)$, one has $\|\chi_A\|_{\mathcal{H}_E} = \|\chi_{A^c}\|_{\mathcal{H}_E}$. The function χ_A is in $\mathcal{H}_E$ if and only if either $\mu(A) < \infty$ or $\mu(A^c) < \infty$.

(3) The finite energy space $\mathcal{H}_E$ admits the decomposition into the orthogonal sum

$$\mathcal{H} = \overline{\mathcal{D}_{\text{fin}}(\mu)} \oplus \mathcal{H}arm_E \tag{3.12}$$

where the closure of $\mathcal{D}_{\text{fin}}(\mu)$ is taken in the norm of the Hilbert space $\mathcal{H}_E$.

In the following statement we return to the L^2-spaces, and following [1], we formulate a number of properties of the operators, R, P, and Δ that clarify their essence. Here, we focus on the properties of these operators related to L^2-spaces. In the next paper [43], we will mostly consider these operators acting in the finite energy space $\mathcal{H}_E$.

Proposition 3.7 *Let $d\nu(x) = c(x)d\mu(x)$ be the σ-finite measure on $(V, \mathcal{B})$ where μ and $c(x) = \rho_x(V)$ are defined as above. Let the operators R, P, and Δ be defined as in Definition 3.1.*

(1) Suppose that the function $x \mapsto \rho_x(A) \in L^2(\mu)$ for any $A \in \mathcal{B}_{\text{fin}}$. Then R is a symmetric unbounded operator in $L^2(\mu)$, i.e.,

$$\langle g, R(f)\rangle_{L^2(\mu)} = \langle R(g), f\rangle_{L^2(\mu)}.$$

If $c \in L^\infty(\mu)$, then $R : L^2(\mu) \to L^2(\mu)$ is a bounded operator, and

$$||R||_{L^2(\mu)\to L^2(\mu)} \le ||c||_\infty.$$

(2) The operator $R : L^1(\nu) \to L^1(\mu)$ is contractive, i.e.,

$$||R(f)||_{L^1(\mu)} \le ||f||_{L^1(\nu)}, \qquad f \in L^1(\nu).$$

Moreover, for any function $f \in L^1(\nu)$, the formula

$$\int_V R(f)\; d\mu(x) = \int_V f(x)c(x)\; d\mu(x) \tag{3.13}$$

holds. In other words, $\nu = \mu R$, and

$$\frac{d(\mu R)}{d\mu}(x) = c(x).$$

(3) The bounded operator $P : L^2(\nu) \to L^2(\nu)$ is self-adjoint. Moreover, $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$.

(4) The operator P considered in the spaces $L^2(\nu)$ and $L^1(\nu)$ is contractive, i.e.,

$$||P(f)||_{L^2(\nu)} \le ||f||_{L^2(\nu)}, \qquad ||P(f)||_{L^1(\nu)} \le ||f||_{L^1(\nu)}.$$

(5) Spectrum of P in $L^2(\nu)$ is a subset of $[-1, 1]$.

(6) The graph Laplace operator $\Delta : L^2(\mu) \to L^2(\mu)$ is a positive definite essentially self-adjoint operator with domain containing $\mathcal{D}_{\text{fin}}(\mu)$. Moreover,

$$||f||^2_{\mathcal{H}_E} = \int_V f\Delta(f)\; d\mu$$

when the integral in the right hand side exists.

Definition 3.8 A function $f \in \mathcal{F}(V, \mathcal{B})$ is called *harmonic*, if $Pf = f$. Equivalently, f is harmonic if $\Delta f = 0$ or $R(f) = cf$. Similarly, h is *harmonic* for a kernel $x \to k(x, \cdot)$ if

$$\int_V h(y)\; k(x, dy) = h(x).$$

Question As was mentioned above, the definition of operators $R(\rho)$, $P(\rho)$, and $\Delta(\rho)$ is based on a symmetric measure ρ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$. Suppose that another symmetric measure, ρ', which is equivalent to ρ, is defined on $(V \times V, \mathcal{B} \times \mathcal{B})$. It would be interesting to find out what relations between $(R(\rho), P(\rho), \Delta(\rho))$ and $(R(\rho'), P(\rho'), \Delta(\rho'))$ exist. Possibly, this question can be made more precise if we require that both ρ and ρ' are supported by the same symmetric set E and disintegrated with respect to the same measure μ on $(V, \mathcal{B})$.

Remark 3.9 In our further results, the following sets of functions will play an important role. Let $(V, \mathcal{B}, \mu)$ be a σ-measure space, and ρ a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ satisfying Assumption 1. Then the measure $d\nu(x) = c(x)d\mu(x)$ is defined on $(V, \mathcal{B})$ and is equivalent to μ where $c(x) = R(\mathbb{1})(x)$. We define $\mathcal{D}_{\text{fin}}(\mu)$ as in (2.2), and, similarly, we set

$$\mathcal{B}_{\text{fin}}(\nu) := \{A \in \mathcal{B} : \nu(A) < \infty\},$$

$$\mathcal{D}_{\text{fin}}(\nu) := \text{Span}\{\chi_A : A \in \mathcal{B}_{\text{fin}}(\nu)\}.$$

It is straightforward to check that Assumption 1 implies

$$\mathcal{D}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{fin}}(\nu).$$

In general, the converse does not hold. But these two sets coincide if and only if Assumption 1 is extended by adding the reverse implication

$$\int_A c(x)\, d\mu(x) \implies \mu(A) < \infty.$$

3.2 Embedding Operator J

We define now a natural embedding J of bounded Borel functions over $(V, \mathcal{B})$ into bounded Borel functions over $(V \times V, \mathcal{B} \times \mathcal{B})$. The operator J will be considered later acting on the corresponding L^2-spaces.

Let

$$(Jf)(x, y) = f(x), \quad f \in \mathcal{F}(V, \mathcal{B}). \tag{3.14}$$

If $(V, \mathcal{B})$ is equipped with a σ-finite measure μ (or $\nu = c\mu$), we can specify J as an operator with domain $L^2(\mu)$ or $L^2(\nu)$).

Theorem 3.10 *For given $(V, \mathcal{B}, \mu)$, let ρ be a symmetric measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ and $c(x) = \rho_x(V)$. Then:*

(1) the operator $J : L^2(\nu) \to L^2(\rho)$ is an isometry where $d\nu(x) = c(x)d\mu(x)$;
(2) the co-isometry $J^* : L^2(\rho) \to L^2(\nu)$ acts by the formula

$$(J^*g)(x) = \int_V g(x, y)\, P(x, dy), \qquad g \in L^2(\rho);$$

(3) *the operator* $J : L^2(\mu) \to L^2(\rho)$ *is densely defined (in* $L^2(\mu)$*) and is, in general, unbounded.*

Proof

(1) This fact is proved by the following computation: for any $f \in L^2(\nu)$, one has

$$\begin{aligned}
||(Jf)||^2_{L^2(\rho)} &= \iint_{V\times V} (Jf)^2(x,y)\, d\rho(x,y) \\
&= \iint_{V\times V} f^2(x)\, d\rho_x(y) d\mu(x) \\
&= \int_V f^2(x)c(x)\, d\mu(x) \\
&= ||f||^2_{L^2(\nu)}.
\end{aligned}$$

(2) To find the co-isometry J^*, we take arbitrary functions $f \in L^2(\nu)$ and $g \in L^2(\rho)$ and compute the inner product using the equality $c(x)P(x,dy) = d\rho_x(y)$:

$$\begin{aligned}
\langle Jf, g\rangle_{L^2(\rho)} &= \iint_{V\times V} (Jf)(x,y)g(x,y)\, d\rho(x,y) \\
&= \int_V f(x) \left(\int_V g(x,y)\, d\rho_x(y)\right) d\mu(x) \\
&= \int_V f(x) \left(\int_V g(x,y)\ P(x,dy)\right) d\nu(x) \\
&= \langle f, J^* g\rangle_{L^2(\nu)},
\end{aligned}$$

where $J^* g = \int_V g(x,y)\ P(x,dy)$. This proves (2).

(3) To show that (3) holds, we take a Borel function $f \in L^2(\mu)$ and note that

$$||Jf||^2_{L^2(\rho)} = \iint_{V\times V} f^2(x)\, d\rho_x d\mu(x) = \int_V f^2(x)c(x)\, d\mu(x). \tag{3.15}$$

In particular, we have, for $A \in \mathcal{B}_{\text{fin}}$,

$$||J(\chi_A)||^2_{L^2(\rho)} = \int_A c(x)\, d\mu(x),$$

that is, assuming that c is locally integrable, we see that J is well defined on a dense subset of $L^2(\mu)$. Formula (3.15) shows that, for general c, the operator $J : L^2(\mu) \to L^2(\rho)$ is not bounded.

□

4 Equivalence of Symmetric Measures

In this section we focus on the question about relations of Markov operators, and Laplacians, arising from equivalent symmetric measures.

4.1 Equivalence of Markov Operators

Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ which is disintegrated by fiber measures $x \mapsto \rho_x$ over the measure $\mu = \rho \circ \pi^{-1}$. As above, define transition probabilities $x \mapsto P(x, \cdot)$ by setting $c(x)^{-1} d\rho_x(\cdot) = P(x, \cdot)$ where $c(x) = \rho_x(V)$. In other words, $P(x, A) = P(\chi_A)(x)$ where P is the Markov operator, see (3.3).

Having the operator P defined, one can construct a stationary Markov process. Let $\Omega = V \times V \times V \times \cdots = V^{\mathbb{N}_0}$. For $\omega = (\omega_n) \in \Omega$, set

$$X_n : \Omega \to V : X_n(\omega) = \omega_n, \qquad n \in \mathbb{N}_0.$$

These notions are studied in detail in Sect. 5. Here we mention only the notion of *reversibility*, one of the most important properties of Markov operators (processes).

Definition 4.1

(1) A kernel $x \mapsto k(x, \cdot)$ is called *reversible* with respect to a measure μ on $(V, \mathcal{B})$, if for any bounded Borel function $f(x, y)$,

$$\iint_{V \times V} f(x, y) k(x, dy) d\mu(x) = \iint_{V \times V} f(y, x) k(x, dy) d\mu(x).$$

(2) Suppose that $x \mapsto P(x, \cdot)$ is a measurable family of transition probabilities on the space $(V, \mathcal{B}, \mu)$, and let P be the Markov operator determined by $x \mapsto P(x, \cdot)$. It is said that the corresponding Markov process is *reversible* with respect to a measurable functions $c : V \to (0, \infty)$ if, for any sets $A, B \in \mathcal{B}$, the following relation holds:

$$\int_B c(x) P(x, A)\, d\mu(x) = \int_A c(x) P(x, B)\, d\mu(x). \tag{4.1}$$

Denoting $d\nu(x) = c(x) d\mu(x)$, we can rewrite (4.1) in the form that will be used below.

$$\int_V \chi_B(x) P(x, A)\, d\nu(x) = \int_V \chi_A(x) P(x, B)\, d\nu(x).$$

The following result clarifies relationship between symmetric measures ρ and reversible Markov processes. This lemma is a part of more general statement, see Theorem 5.3.

Lemma 4.2 *Let $\rho = \int_V \rho_x \, d\mu$ be a measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $c(x) = \rho_x(V) < \infty$ for all x. Suppose that the Markov operator P is defined according to* (3.3)*. Then the following are equivalent:*

(i) *ρ is symmetric;*
(ii) *(P, c) is reversible.*

In what follows, we will focus on the following *question*: suppose that ρ and ρ' are two *equivalent* symmetric measures such that the corresponding Markov processes (P, c) and (P', c') are reversible. How are they related? More generally, we can ask about relations between all objects whose definition was based on a symmetric measure. They are the Laplacian Δ, symmetric operator R, and finite energy Hilbert space. Some partial answers are given in this and subsequent sections.

Definition 4.3 Let (P, c) be a pair consisting of a positive measurable function $c(x)$ on $(V, \mathcal{B}, \mu)$ and a reversible Markov process $P(x, \cdot)$ satisfying Definition 4.1. We will say that two such pairs (P, c) and (P', c') are *equivalent* if the corresponding symmetric measures ρ and ρ' are equivalent as measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ (see Theorem 5.3). The latter means that there exists a positive measurable function $r(x, y)$ such that

$$d\rho'(x, y) = r(x, y) d\rho(x, y).$$

If the equivalent measures ρ and ρ' satisfy the property $\mu = \rho \circ \pi_1^{-1} = \rho' \circ \pi_1^{-1}$, then we call the pairs (P, c) and (P', c') *strongly equivalent*. In this case, we also call the measures ρ and ρ' *strongly equivalent*.

Remark 4.4

(1) The symmetry of equivalent measures ρ and ρ' implies that the function $r(x, y)$ is symmetric, $r(x, y) = r(y, x)$.
(2) Let the measures ρ and ρ' be strongly equivalent. Then these measures are disintegrated as follows:

$$\rho' = \int_V \rho'_x \, d\mu(x), \qquad \rho = \int_V \rho_x \, d\mu(x).$$

It can be seen that the equivalence of ρ and ρ' implies that the measures ρ_x and ρ'_x are equivalent μ-a.e. Moreover,

$$\frac{d\rho'_x}{d\rho_x}(y) = r_x(y) \tag{4.2}$$

where $r_x(\cdot)$ is obtained from $r(x, \cdot)$ by fixing the variable x.

(3) Conversely, given two (strongly) equivalent measures ρ and ρ', we can construct (strongly) equivalent pairs (P, c) and (P', c') according to the properties formulated in Lemma 4.2 and Theorem 5.3. In other words, if (P, c) defines a reversible Markov process with the symmetric measure ρ, then, for any symmetric measure ρ' equivalent to ρ, we can construct a reversible Markov process (P', c') which is equivalent to (P, c). Note that the functions $c(x) = \rho_x(V)$ and $c'(x) = \rho'_x(V)$ are determined by ρ and ρ' uniquely.

One can prove a more general statement than that given in Remark 4.4 (2).

Lemma 4.5 *Let ρ and ρ' be two symmetric measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $d\rho'(x, y) = r(x, y)d\rho(x, y)$. Suppose that*

$$\rho' = \int_V \rho'_x \, d\mu'(x), \qquad \rho = \int_V \rho_x \, d\mu(x)$$

and the measures μ and μ' on $(V, \mathcal{B})$ are equivalent, i.e., $m(x)d\mu'(x) = d\mu(x)$ for some positive Borel function $m(x)$. Then the measures ρ'_x and ρ_x are equivalent a.e. on V, and

$$\frac{d\rho'_x}{d\rho_x}(y) = m(x)r_x(y). \tag{4.3}$$

Proof (Sketch) The result is deduced as follows:

$$\begin{aligned}\rho'(A \times B) &= \iint_{A\times B} r(x, y) \, d\rho(x, y)\\ &= \iint_{A\times B} r(x, y) \, d\rho_x(y)d\mu(x)\\ &= \int_A \left(\int_B m(x)r(x, y) \, d\rho_x(y)\right) d\mu'(x).\end{aligned}$$

On the other hand,

$$\rho'(A \times B) = \int_A \rho'_x(B) \, d\mu'(x).$$

Comparing the above formulas, we obtain that (4.3) holds.

□

Consider a particular case when the Radon-Nikodym derivative $r(x, y)$ of two equivalent measures ρ and ρ' is the product $p(x)q(y)$.

Lemma 4.6 *Let $\rho = \int \rho_x \; d\mu(x)$ and $\rho' = \int \rho'_x \; d\mu'(x)$ be two measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that*

$$\frac{d\rho'}{d\rho}(x, y) = p(x)q(y)$$

for some positive Borel functions p and q. Then, for μ-a.e. $x \in V$, the Radon-Nikodym derivative $\dfrac{d\rho'_x(y)}{d\rho_x(y)}$ satisfies the relation

$$\frac{1}{q(y)}\frac{d\rho'_x(y)}{d\rho_x(y)} = \varphi(x) \tag{4.4}$$

where

$$\varphi(x) = p(x)\frac{d\mu}{d\mu'}(x).$$

Proof The result can be easily deduced from the formula

$$d\rho'_x(y)d\mu'(x) = p(x)q(y)d\rho_x(y)d\mu(x).$$

We leave the details to the reader. □

Relation (4.4) means that the Radon-Nikodym derivative $\dfrac{d\rho'_x}{d\rho_x}(y)$ is proportional to the function $q(y)$ where the coefficient of proportionality is given by $\varphi(x)$. If ρ and ρ' are symmetric measures, then $\dfrac{d\rho'}{d\rho}(x, y) = p(x)p(y)$.

Theorem 4.7 *Let ρ and ρ' be two strongly equivalent measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $d\rho'_x = r_x(y)d\rho_x(y)$ for all $x \in V$. Then the corresponding Markov processes (P, c) and (P', c') are strongly equivalent and*

$$P'(f)(x) = \frac{P(fr_x)(x)}{P(r_x)(x)}. \tag{4.5}$$

Proof We first find $P(r_x)$:

$$\begin{aligned} P(r_x)(x) &= \int_V \frac{d\rho'_x}{d\rho_x}(y)\ P(x, dy) \\ &= \frac{1}{c(x)}\int_V \frac{d\rho'_x}{d\rho_x}(y)\ d\rho_x(y) \\ &= \frac{1}{c(x)}\int_V\ d\rho'_x(y) \qquad (4.6) \\ &= \frac{c'(x)}{c(x)}. \end{aligned}$$

Next, we compute

$$\begin{aligned} P'(f)(x) &= \int_V f(y)\ P'(x,dy) \\ &= \frac{1}{c'(x)} \int_V f(y)\ d\rho'_x(y) \\ &= \frac{1}{c'(x)} \int_V f(y) r_x(y)\ d\rho_x(y) \\ &= \frac{c(x)}{c'(x)} \int_V f(y) r_x(y)\ dP(x,dy) \\ &= \frac{c(x)}{c'(x)} P(f r_x)(x) \end{aligned}$$

Now, the result follows from (4.6). □

Remark 4.8

(1) Let the symmetric measures ρ and ρ' be strongly equivalent, $d\rho'_x(y) = r_x(y)d\rho_x(y)$. As in (4.6), we can obtain that

$$P'\left(\frac{1}{r_x}\right)(x) = \frac{c(x)}{c'(x)}.$$

Therefore, the following property holds:

$$P(r_x)(x)P'\left(\frac{1}{r_x}\right)(x) = 1$$

(2) Since the notion of equivalence of measures ρ and ρ' is symmetric, we note that the roles of P and P' can be interchanged and the following relation holds:

$$P(f)(x) = \frac{P'\left(f\frac{1}{r_x}\right)(x)}{P'\left(\frac{1}{r_x}\right)(x)}.$$

(3) It follows from the strong equivalence of ρ and ρ' that $r_x(y)$ is integrable with respect to ρ_x and

$$c'(x) = \int_V r_x(y)\ d\rho_x(y).$$

(4) Several useful formulas can be easily deduced from Theorem 4.7. Firstly, formula (4.5) can be rewritten in the form

$$P(fr_x)(x) = c'(x)P'(f)(x)c(x)^{-1}, \tag{4.7}$$

and equivalently, the latter is represented as a relation between Markov kernels:

$$c'(x)P'(x, dy) = c(x)r_x(y)P(x, dy).$$

(5) The same proof as in Theorem 4.7 shows that

$$R'(f)(x) = R(fr_x)(x).$$

(6) In more general setting, assuming that $d\rho'_x(y) = m(x)r_x(y)d\rho_x(y)$ where $m(x)$ is as in (4.3), we deduce that

$$P(fr_x)(x)m(x) = c'(x)P'(f)(x)c(x)^{-1}.$$

Similarly, one can show that

$$R'(f)(x) = m(x)R(fr_x)(x)$$

where the operator R' is defined by $x \mapsto \rho'_x$.

(7) Suppose that, for given pair (P, c), the operator P' is defined by (4.7), and let $d\nu'(x) = c'(x)d\mu(x)$. Then we claim that $\nu' P' = \nu'$:

$$\begin{aligned}
\int_V P'(f)(x)\, d\nu'(x) &= \int_V c(x)P(fr_x)(x)c'(x)^{-1}c'x)\, d\mu(x)\\
&= \int_V P(fr_x)(x)\, d\nu(x)\\
&= \int_V \left(\int_V (fr_x)(y)P(x, dy)\right)\, d\nu(x)\\
&= \iint_{V\times V} f(y)\frac{d\rho'_x}{d\rho_x}(y)c(x)^{-1}d\rho_x(y)c(x)d\mu(x)\\
&= \iint_{V\times V} f(y)\, d\rho'_x(y)d\mu(x)\\
&= \iint_{V\times V} f(x)\, d\rho'(x, y)\\
&= \int_V f(x)c'(x)\, d\mu(x)\\
&= \int f(x)\, d\nu'(x).
\end{aligned}$$

4.2 On the Laplacians Δ and Δ′

In the remaining part of this section, we will discuss relations between the Laplace operators Δ and Δ' acting in the finite energy Hilbert spaces $\mathcal{H}_E(\rho)$ and $\mathcal{H}_E(\rho')$ respectively.

Let $\Delta'(f)$ be the Laplace operator defined by a symmetric measure ρ' on $(V \times V, \mathcal{B} \times \mathcal{B})$. We can find out how Δ' and Δ are related.

Proposition 4.9 *Let ρ and ρ' be two equivalent symmetric measures on $(V \times V, \mathcal{B} \times \mathcal{B})$ such that $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. Then*

$$\Delta'(f) = cqf(P(q) - q) + q\Delta(qf).$$

In particular, when q is harmonic for P, then

$$\Delta'(f) = q\Delta(qf). \tag{4.8}$$

Moreover,

$$\Delta'(f) = 0 \iff P(qf) = fP(q),$$

and assuming that $P(q) = q$, we have

$$f \in \mathcal{H}arm(\Delta') \iff qf \in \mathcal{H}arm(\Delta).$$

Proof

(1) By definition of the operator Δ, we have

$$\begin{aligned}
\Delta'(f)(x) &= \int_V (f(x) - f(y))\, d\rho'_x(y) \\
&= \int_V (f(x) - f(y))q(x)q(y)\, d\rho_x(y) \\
&= \int_V (f(x) - f(y))c(x)q(x)q(y)\, dP(x, dy) \\
&= c(x)q(x)f(x) \int_V q(y)\ P(x, dy) - c(x)q(x) \int_V q(y)f(y)\ P(x, dy) \\
&= c(x)q(x)\left[f(x)P(q)(x) - P(qf)(x)\right].
\end{aligned} \tag{4.9}$$

Add and subtract $cq^2 f$ to the right hand side of (4.9). Then, regrouping the terms, we obtain

$$\Delta'(f) = cq[qf - P(qf)] + cqf(P(q) - q) = q\Delta(qf) + cqf(P(q) - q).$$

This means that, in case when $P(q) = q$, the Laplace operators Δ and Δ' are related as in (4.8).

(2) Now we can apply (1) to prove the formulas given in (2). From the last expression in (4.9), we see that f is harmonic with respect to Δ' if and only if $P(qf) = f P(q)$. □

Corollary 4.10 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let q be a harmonic function for the Markov operator P generated by ρ. Define the symmetric measure ρ' such that $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. Let P' be the corresponding Markov operator produced by ρ'. Then we have the map*

$$\mathcal{H}arm(P') \times \mathcal{H}arm(P) \ni (f, q) \mapsto fq \in \mathcal{H}arm(P).$$

Proof It follows from the definition of the measure ρ' that

$$c'(x) = \int_V d\rho'_x(y) = \int_V q(x)q(y)\, d\rho_x(y) = q(x)R(q)(x).$$

Since q is harmonic, i.e., $R(q) = cq$, we obtain that

$$c'(x) = c(x)q^2(x). \tag{4.10}$$

Let f be any function harmonic with respect to the operator P'. Then

$$\begin{aligned} f(x) &= \int_V f(y)\ P'(x, dy) \\ &= \frac{1}{c'(x)} \int_V f(y)\ d\rho'_x(y) \\ &= \frac{q(x)}{c'(x)} \int_V f(y)q(y)d\rho_x(y) \\ &= \frac{q(x)}{c'(x)} \int_V f(y)q(y)c(x)P(x, dy) \\ &= \frac{q(x)c(x)}{c'(x)} P(qf)(x) \end{aligned}$$

It follows from (4.10) that $f = q^{-1}P(qf)$, and we are done. □

We remark that in the proved statement we temporarily extended the notion of symmetric measures to the case of *signed symmetric measures* assuming that the P-harmonic function q can be negative.

Theorem 4.11 *Suppose that ρ' and ρ are two symmetric measures such that $d\rho'(x, y) = q(x)q(y)d\rho(x, y)$. If q is harmonic for the Laplace operator Δ, then the operator*

$$Q : \mathcal{H}_E(\rho') \to \mathcal{H}_E(\rho) : Q(f) = qf$$

is an isometry.

Proof We need to show that, for any $f \in \mathcal{H}_E(\rho')$,

$$||f||_{\mathcal{H}_E(\rho')} = ||qf||_{\mathcal{H}_E(\rho)}.$$

In the computation given below, we use the following: the definition of the norm in the finite energy space, the symmetry of the measures ρ and ρ', and the relation $R(q) = cq$ that holds for harmonic functions because

$$\Delta(q)(x) = c(x)q(x) - R(q)(x).$$

Then we compute

$$\begin{aligned}
||f||^2_{\mathcal{H}_E(\rho')} - ||qf||^2_{\mathcal{H}_E(\rho)} &= \frac{1}{2}\iint_{V\times V} (f(x) - f(y))^2 \, d\rho'(x, y) \\
&\quad - \iint_{V\times V} (q(x)f(x) - q(y)f(y))^2 \, d\rho(x, y) \\
&= \iint_{V\times V} [(f(x) - f(y))^2 q(x)q(y) \\
&\quad - (q(x)f(x) - q(y)f(y))^2] \, d\rho(x, y) \\
&= \iint_{V\times V} [f^2(x)q(x)q(y) - q^2(x)f^2(x)] \\
&\quad + [f^2(y)q(x)q(y) - q^2(y)f^2(y)] \, d\rho(x, y) \\
&= 2\iint_{V\times V} [f^2(x)q(x)q(y) - q^2(x)f^2(x)] \, d\rho_x(y)d\mu(x) \\
&= 2\int_V f^2(x)q(x)[R(q)(x) - c(x)q(x)] \, d\mu(x) \\
&= 0.
\end{aligned}$$

This computation shows that $Q(f) = qf \in \mathcal{H}_E(\rho)$ and Q preserves the norm. □

Continuing the above theme, consider the Laplace operator Δ acting in $L^2(\mu)$. We recall that $\Delta : L^2(\mu) \to L^2(\mu)$ is a positive definite self-adjoint operator according to Proposition 3.7.

Proposition 4.12 *Suppose ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ and the Laplacian $\Delta = \Delta(\rho)$ is defined by (3.4). Let q and f be functions on $(V, \mathcal{B}, \mu)$ from the domain of Δ such that qf is also in the domain of Δ. Then*

$$\int_V \Delta(qf)\, d\mu = \int_V q\Delta(f)\, d\mu - \int_V f\Delta(q)\, d\mu. \tag{4.11}$$

If q and f are in $L^2(\mu)$, then $\int_V \Delta(qf)\, d\mu = 0$.

Proof By definition of Δ, we have

$$\begin{aligned}\Delta(qf) &= \int_V [(qf(x) - qf(y)]\, d\rho_x(y)\\ &= \int_V (q(x)f(x) - q(x)f(y) + q(x)f(y) - q(y)f(y))\, d\rho_x(y)\\ &= q(x)\Delta(f) - \int_V f(y)(q(x) - q(y))\, d\rho_x(y)\end{aligned}$$

Then

$$\begin{aligned}\int_V \Delta(qf)(x)\, d\mu(x) &= \int_V q\Delta(f)\, d\mu(x) + \iint_{V\times V} f(y)(q(x) - q(y))\, d\rho_x(y)d\mu(x)\\ &= \int_V q\Delta(f)\, d\mu(x) + \iint_{V\times V} f(x)(q(y) - q(x))\, d\rho_x(y)d\mu(x)\\ &= \int_V q\Delta(f)\, d\mu(x) - \int_V f\Delta(q)\, d\mu(x)\end{aligned}$$

and (4.11) is proved.

If the functions q and f are in $L^2(\mu)$ (in particular, q and f can be taken from the dense subset $\mathcal{D}_{\text{fin}}(\mu)$), then we can use the fact that Δ is essentially self-adjoint and conclude that

$$\int_V \Delta(qf)(x)\, d\mu(x) = \langle q, \Delta(f)\rangle_{L^2(\mu)} - \langle \Delta(q), f\rangle_{L^2(\mu)} = 0.$$

□

We immediately deduce the following fact from Proposition 4.12.

Corollary 4.13

(1) *If functions f and f^2 are in the domain of Δ, then*

$$\int_V \Delta(f^2)\, d\mu = 0.$$

(2) *If f is a harmonic function for Δ, then $\Delta(f^2) = 0$, and therefore f^2 is also harmonic.*

Proof

(1) is an obvious consequence of Proposition 4.12. To show that (2) holds, we use that $\Delta(f) = c(f - P(f))$ and P is a positive operator. This means that $P(f) \geq 0$ whenever $f \geq 0$. By Schwarz' inequality for positive operators, we have $P(f^2)(x) \geq P(f)^2(x)$, and therefore

$$\begin{aligned} \Delta(f^2) &= c(f^2 - P(f^2)) \\ &\leq c(f^2 - P(f)^2) \\ &= c(f - P(f))(f + P(f)) \\ &= 0. \end{aligned}$$

The fact that f^2 is harmonic follows from (1) and the proved inequality in (2).

□

5 Reversible Markov Process Generated by Symmetric Measures

In this section, we consider Markov processes generated by a Markov operator which is determined by a symmetric irreducible measure ρ on the standard Borel space $(V \times V, \mathcal{B} \times \mathcal{B})$ such that the margin measure μ on $(V, \mathcal{B})$ is *σ-finite*. Our first theme is *reversible Markov processes*. For the benefit of non-specialist readers, we cite the following sources: [44–46]. We refer also to [47–49]. In the second part of this section, we will assume that this Markov process is *transient* (see the definition below). The reader can find vast literature on the theory of transient Markov processes, we refer to [17–19, 50–57].

5.1 Reversible Markov Processes

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$ which is disintegrated with respect to $(\rho_x, x \in V)$ and μ according to (2.5). By assumption, $c(x) = \rho_x(V)$ is locally integrable. We recall (see Definition 3.1) that, in this setting, a Markov operator P is defined on $\mathcal{F}(V, \mathcal{B})$ by the probability kernel $x \mapsto P(x, \cdot)$. This operator P acts by the formula

$$P(f)(x) = \int_V f(y)\ P(x, dy) \tag{5.1}$$

where $P(x, dy) = c(x)^{-1} d\rho_x(y)$. Then the operator P is positive and normalized, i.e., $P(\mathbb{1}) = \mathbb{1}$. As mentioned above in Proposition 3.7, the fact that ρ is symmetric is equivalent to self-adjointness of P as an operator in $L^2(\nu)$. It follows also that P preserves the measure $\nu = c\mu$. Furthermore, we can use the kernel $x \to P(x, \cdot) = P_1(x, \cdot)$ to define the sequence of probability kernels (transition probabilities) $(P_n(x, \cdot) : n \in \mathbb{N})$ in accordance with (2.4). These kernels satisfy the equality

$$P_{n+m}(x, A) = \int_V P_n(y, A) P_m(x, dy), \qquad n, m \in \mathbb{N}.$$

Therefore one has

$$P^n(f)(x) = \int_V f(y)\ P_n(x, dy), \qquad n \in \mathbb{N},$$

and this relation defines the sequence of probability measures (P_n) by setting $P_0(x, A) = \delta_A(x) = \chi_A(x)$ and

$$P_n(x, A) = P^n(\chi_A) = \int_V \chi_A(y)\ P_n(x, dy), \qquad A \in \mathcal{B}, n \in \mathbb{N}.$$

We use the notation $P(x, A)$ for $P_1(x, A)$.

For the Markov operator P, one can define one more sequence of measures. We use the formula

$$\rho_n(A \times B) = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)}, \tag{5.2}$$

to define the measures ρ_n, $n \in \mathbb{N}$, on the Borel space $(V \times V, \mathcal{B} \times \mathcal{B})$ (here $\rho_1 = \rho$).

Lemma 5.1

(1) Every measure ρ_n, $n \in \mathbb{N}$, is symmetric on $(V \times V, \mathcal{B} \times \mathcal{B})$, and ρ_n is equivalent to ρ.

(2) $\rho_x^{(n)}(V) = c(x), \forall n \in \mathbb{N}$.

(3)

$$d\rho_n(x, y) = c(x)P_n(x, dy)d\mu(x) = P_n(x, dy)d\nu(x). \tag{5.3}$$

(4)

$$\rho_n(A \times B) = \langle \chi_A, RP^{n-1}(\chi_B)\rangle_{L^2(\mu)}.$$

Proof The assertions of the lemma are rather obvious. We only mention two simple facts: $\rho_n(A \times V) = \rho(A \times V)$ for every n, and, since the operator P^n is self-adjoint in $L^2(\nu)$, the measure ρ_n is symmetric. □

Definition 5.2 Suppose that $x \mapsto P(x, \cdot)$ is a measurable family of transition probabilities on the space $(V, \mathcal{B}, \mu)$, and let P be the Markov operator determined by $x \mapsto P(x, \cdot)$. It is said that the corresponding Markov process is *reversible* with respect to a measurable function $c : x \to (0, \infty)$ on $(V, \mathcal{B})$ if, for any sets $A, B \in \mathcal{B}$, the following relation holds:

$$\int_B c(x)P(x, A)\, d\mu(x) = \int_A c(x)P(x, B)\, d\mu(x). \tag{5.4}$$

As shown in [1], the reversibility for the Markov process (P_n) is equivalent to the following properties (here we give an extended and more comprehensive formulation):

Theorem 5.3 *Let $(V, \mathcal{B}, \mu)$ be a standard σ-finite measure space, $x \mapsto c(x) \in (0, \infty)$ a measurable function, $c \in L^1_{\text{loc}}(\mu)$. Suppose that $x \mapsto P(x, \cdot)$ is a probability kernel. The following are equivalent:*

(i) *$x \mapsto P(x, \cdot)$ is reversible (i.e., it satisfies (5.4));*
(ii) *$x \to P_n(x, \cdot)$ is reversible for any $n \geq 1$;*
(iii) *the Markov operator P defined by $x \to P(x, \cdot)$ is self-adjoint on $L^2(\nu)$ and $\nu P = \nu$ where $d\nu(x) = c(x)d\mu(x)$;*
(iv)

$$c(x)P(x, dy)d\mu(x) = c(y)P(y, dx)d\mu(y);$$

(v) *the operator R defined by the relation $R(f)(x) = c(x)P(f)(x)$ is symmetric (see Remark 3.3);*
(vi) *the measure ρ on $(V \times V, \mathcal{B} \times \mathcal{B})$ defined by*

$$\rho(A \times B) = \int_V \chi_A R(\chi_B)\, d\mu = \int_V c(x)\chi_A P(\chi_B)\, d\mu$$

is symmetric;
(vii) *for every $n \in \mathbb{N}$, the measure ρ_n defined by (5.2) is symmetric;*

(viii) for any Borel sets $A_1, \ldots, A_n \in \mathcal{B}_{\text{fin}}(\mu)$,

$$\int_V \mathbb{P}_x(X_0 \in A_0, \ldots, X_n \in A_n)\, d\nu(x) = \int_V \mathbb{P}_x(X_0 \in A_n, \ldots, X_n \in A_0)\, d\nu(x),$$

where the random variables $X_1, \ldots, X_n$ *are defined below in Remark 5.4 (5) and the sets* $A_0, A_1, \ldots, A_n$ *are written in the reverse order in the right hand side.*

Proof We refer to [1] where most of these properties are discussed. We prove (*viii*) here. Indeed, it can be seen that

$$\int_V \mathbb{P}_x(X_0 \in A_0, \ldots, X_n \in A_n)\, d\nu(x) = \int_V \chi_{A_0} P(\chi_{A_1} P(\chi_{A_2} \cdots P(\chi_{A_n}) \cdots))(x)\, d\nu(x). \tag{5.5}$$

Since P is self-adjoint on $L^2(\nu)$, we can repeatedly use the relation $\int_V f P(g) d\nu = \int_V P(f) g d\nu$ and rewrite (5.5) as follows:

$$\begin{aligned}
&\int_V \chi_{A_0} P(\chi_{A_1} P(\chi_{A_2} \cdots P(\chi_{A_n}) \cdots))(x)\, d\nu(x) \\
&= \int_V \chi_{A_n} P(\chi_{A_{n-1}} P(\chi_{A_{n-2}} \cdots P(\chi_{A_0}) \cdots))(x)\, d\nu(x) \\
&= \int_V \mathbb{P}_x(X_0 \in A_n, \ldots, X_n \in A_0)\, d\nu(x).
\end{aligned}$$

The fact that property (*viii*) implies that P is reversible is proved by using the density of simple functions in $L^2(\nu)$. □

We discuss the notion of reversibility in the following Remark where we included several direct consequences of Definition 4.1 and Theorem 5.3.

Remark 5.4

(1) Let $x \mapsto P(x, \cdot)$ be a Borel field of probability measures over a standard Borel space $(V, \mathcal{B})$. This field of transition probabilities generates the Markov operator P such that $P(\mathbb{1}) = 1$. It follows from Theorem 5.3 that one can define the notion of reversible Markov process $x \mapsto P(x, \cdot)$ with respect to a σ-finite measure ν: It is said that $((x \mapsto P(x, \cdot)), \nu)$ is *reversible* if P is a self-adjoint operator in $L^2(\nu)$. This definition is equivalent to the property

$$\int_A P(x, B)\, d\nu = \int_B P(x, A)\, d\nu.$$

Equally, one can consider the notion of reversibility for $P(x, \cdot)$ with respect to a symmetric measure ρ. Theorem 5.3 states the equivalence of these approaches.

(2) Based on (1), the following *question* is raised naturally: *Given* $x \mapsto P(x, \cdot)$ *as above, under what condition the set*

$$\mathcal{S}(P) := \{\nu : P \text{ is self-adjoint in } L^2(\nu)\}$$

is non-empty?

(3) The following observation is a direct consequence of Theorem 5.3. Let $P(x, A) = P(\chi_A)(x)$ be the probability kernel defined by a normalized Markov operator P acting on Borel functions over $(V, \mathcal{B}, \mu)$. To answer the question about the existence of a P-invariant measure $\nu \sim \mu$ such that (P, ν) is reversible, it suffices to construct a locally integrable function c satisfying (5.4). It can be done by pointing out a symmetric measure ρ such that $\rho_x(V) = c(x)$ and the projection of ρ onto V is the measure μ.

(4) There exists a stronger version of reversible Markov processes. Let P be a Markov operator acting on $\mathcal{F}(V, \mathcal{B})$ such that, for any $A, B \in \mathcal{B}_{\text{fin}}(\mu)$,

$$\chi_A P(\chi_B) = \chi_B P(\chi_A).$$

Then, for any positive Borel function $c \in L^1_{\text{loc}}(\mu)$, the measure $d\nu(x) = c(x) d\mu(x)$ belongs to $\mathcal{S}(P)$. Indeed, it suffices to define the symmetric measure ρ according to Theorem 5.3 (vi) and then apply statement (ii).

(5) We give here one more interpretation of the definition of reversible Markov processes. For this, we use the notation to be introduced in Sect. 6. Let

$$\Omega = V \times V \times V \cdots$$

be the path space of the Markov process (P_n), and let $X_n : \Omega \to V$ be the random variable defined by $X_n(\omega) = \omega_n$. Given a measure ν on V, we can reformulate the definition of reversible Markov operator as follows:

$$dist(X_0 \mid X_1 \in A) = dist(X_1 \mid X_0 \in A).$$

The meaning of the above formula is clarified in Proposition 6.4.

(6) Suppose now that a non-symmetric measure ρ is given on the space $(V \times V, \mathcal{B} \times \mathcal{B})$, i.e, $\rho(A \times B) \neq \rho(B \times A)$, in general. However, we will assume that ρ is equivalent to $\rho \circ \theta$ where $\theta(x, y) = (y, x)$. Then, using the same approach as above, we can define the following objects: margin measures $\mu_i := \rho \circ \pi_i^{-1}$, $i = 1, 2,$, fiber measures $d\rho_x(\cdot)$ and $d\rho^x(\cdot)$ (see Remark 2.5), and functions $c_1(x) = \rho_x(V)$, $c_2(x) = \rho^x(V)$.

Define now the *symmetric measure* $\rho^\#$ generated by ρ as follows

$$\rho^\# := \frac{1}{2}(\rho + \rho \circ \theta).$$

Then

$$\rho^{\#}(A \times B) = \frac{1}{2}(\rho(A \times B) + \rho(B \times A)).$$

Clearly, $\rho^{\#}$ is equivalent to ρ.

Let $E \subset V \times V$ be the support of ρ. Then $E^{\#} = E \cup \theta(E)$ is the support of the symmetric measure $\rho^{\#}$. The disintegration of $\rho = \int_V \rho_x \, d\mu_1(x)$ with respect to the partition $\{x\} \times E_x$ defines the disintegration of $\rho^{\#}$. For $\mu^{\#} := \frac{1}{2}(\mu_1 + \mu_2)$, we obtain that

$$\rho^{\#} = \int_V (\rho_x + \rho^x) \, d\mu^{\#}.$$

Having the symmetric measure $\rho^{\#}$ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, we can introduce the operators $R^{\#}$ and $P^{\#}$ as in (3.2) and (3.3). It turns out that, for $f \in \mathcal{F}(V, \mathcal{B})$,

$$R^{\#}(f)(x) = R_1(f)(x) + R_2(f)(x)$$

where

$$R_1(f) = \int_V f(y) \, d\rho_x(y), \qquad R_2(f) = \int_V f(y) \, d\rho^x(y).$$

Similarly,

$$P^{\#}(f)(x) = \frac{1}{c^{\#}(x)} R^{\#}(f)(x)$$

where

$$c^{\#}(x) = \rho_x(V) + \rho^x(V).$$

Then we can define the measure $d\nu^{\#}(x) = c^{\#}(x) d\mu(x)$ such that the operator

$$P^{\#}(f)(x) = \int_V f(y) \frac{1}{c^{\#}(x)} \, d\rho^{\#}_x(y)$$

is self-adjoint in $L^2(\nu^{\#})$. By Theorem 5.3, we obtain that the Markov process generated by $x \mapsto P^{\#}(x, \cdot)$ is *reversible* where $P^{\#}(x, A) = P^{\#}(\chi_A)(x)$.

5.2 Properties of Markov Operators

In this subsection, we discuss some properties of the Markov operator P, which is defined by relation (3.3). The operator P is considered acting in Hilbert spaces $L^2(\mu)$, $L^1(\nu)$, and $\mathcal{H}_E$ where $d\nu(x) = c(x)d\mu(x)$ and $\mathcal{H}_E$ is the energy space.

We begin with the following simple observations whose proofs are obvious and can be omitted. Remind that $\mathcal{B}_{\text{fin}}(\mu)$ is the family of Borel subsets of finite measure μ, and $\mathcal{D}_{\text{fin}} = \mathcal{D}_{\text{fin}}(\mu)$ is the linear subspace generated by the characteristic functions χ_A, $A \in \mathcal{B}_{\text{fin}}$.

Remark 5.5

(1) If $c \in L^1_{\text{loc}}(\mu)$, then

$$\mathcal{B}_{\text{fin}}(\mu) \subset \mathcal{B}_{\text{fin}}(\nu).$$

The converse is not true.

(2) We observe that if both functions, $c(x)$ and $c(x)^{-1}$ are in $L^1_{\text{loc}}(\mu)$, then

$$\mathcal{B}_{\text{fin}}(\mu) = \mathcal{B}_{\text{fin}}(\nu).$$

(3) The following property holds for $c \in L^1_{\text{loc}}(\mu)$:

$$\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu) \cap L^2(\nu) \cap \mathcal{H}_E \tag{5.6}$$

(this should be understood that functions from $\mathcal{D}_{\text{fin}}$ are representatives of elements from $\mathcal{H}_E$).

(4) We recall that

$$\|\chi_A\|^2_{\mathcal{H}_E} = \rho(A \times A^c) \tag{5.7}$$

where ρ is a symmetric measure used in the definition $\mathcal{H}_E$. This fact is proved in [1].

Lemma 5.6 *If $c \in L^1_{\text{loc}}(\mu)$, then $\mathcal{D}_{\text{fin}}(\mu)$ is dense in $L^1(\nu)$ and $L^2(\nu)$.*

Proof (Sketch) We show the density of $\mathcal{D}_{\text{fin}}(\mu)$ in $L^1(\nu)$ only. It suffices to check that, for every $B \in \mathcal{B}_{\text{fin}}(\nu)$, the characteristic function χ_B can be approximated in $L^1(\nu)$ by simple functions from $\mathcal{D}_{\text{fin}}(\mu)$, i.e., for every $\varepsilon > 0$, there exists some $s(x) \in \mathcal{D}_{\text{fin}}(\mu)$ such that $||\chi_B - s||_{L^1(\nu)} < \varepsilon$. Without loss of generality, we can assume that $s(x) \leq \chi_B(x)$. Then

$$||\chi_B - s||_{L^1(\nu)} = \int_V (\chi_B - s(x))\, d\nu(x) = \int_B c(x)(1 - s(x))\, d\mu(x).$$

Since c is μ-integrable on B, one can take a subset $B_0 \subset B$ such that

$$\int_B c\, d\mu - \int_{B_0} c\, d\mu < \varepsilon.$$

The result follows. □

Next, let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let P be the operator acting on bounded Borel functions by the formula

$$P(f)(x) = \int_V f(y) P(x, dy)$$

where $c(x)P(x, dy) = d\rho_x(y)$.

In the next statement we collect several properties of the Markov operator P considered in various spaces.

Proposition 5.7 *Let $(V, \mathcal{B}, \mu)$, ν, and ρ be as above. Then, for any $A \in \mathcal{B}_{\mathrm{fin}}$,*

(a) $P(\chi_A) \in L^1(\mu) \iff \dfrac{\rho_x(A)}{c(x)} \in L^1(\mu) \implies P(\chi_A) \in L^2(\mu)$*;*

(b) if the function $x \mapsto \int_V c(y)^{-1}\, d\rho_x(y)$ is locally integrable, then P is a densely defined operator in $L^2(\mu)$;

(c) if $c \in L^1_{\mathrm{loc}}(\mu)$, then

$$P(\chi_A) \in L^1(\nu) \cap L^2(\nu);$$

(d) the measures μ and μP are equivalent if and only if the function c^{-1} is integrable on (E_x, ρ_x) for μ-a.e. $x \in V$. The Radon-Nikodym derivative can be found by the formula:

$$\frac{d(\mu P)}{d\mu}(x) = \int_V \frac{1}{c(y)}\, d\rho_x(y).$$

***Proof** (Sketch)*

(a) The fact that $P(\chi_A)$ is in $L^2(\mu)$ follows from the Schwarz inequality for positive operators,

$$P(\chi_A)^2 \leq P(\chi_A^2) = P(\chi_A).$$

The criterion for integrability of the function $P(\chi_A)$ is proved as follows:

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\mu(x) &= \iint_{V\times V} \chi_A(y) P(x, dy)\, d\mu(x)\\ &= \iint_{V\times V} \frac{\chi_A(y)}{c(x)}\, d\rho_x(y) d\mu(x)\\ &= \int_V \frac{\rho_x(A)}{c(x)}\, d\mu(x).\end{aligned}$$

It follows from (a) that the same computation can be used to show that $P(\chi_A)$ is in $L^2(\mu)$ whenever

$$\frac{\rho_x(A)}{c(x)} \in L^1(\mu).$$

(b) To prove this result, we refer to the proof of (b) and use the symmetry of the measure ρ:

$$P(\chi_A) \in L^2(\mu) \iff P(\chi_A) \in L^1(\mu)$$

and

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\mu(x) &= \iint_{V\times V} \frac{\chi_A(y)}{c(x)}\, d\rho_x(y) d\mu(x)\\ &= \iint_{V\times V} \frac{\chi_A(x)}{c(y)}\, d\rho_x(y) d\mu(x)\\ &= \int_A \left(\int_V \frac{\chi_A(x)}{c(y)}\, d\rho_x(y)\right) d\mu(x).\end{aligned}$$

It gives the desired statement.

(c) Suppose $c(x) \in L^1_{\mathrm{loc}}(\mu)$. Then, using the symmetry of the measure ρ and relation (2.7), we obtain

$$\begin{aligned}\int_V P(\chi_A)(x)\, d\nu(x) &= \int_V \left(\int_V \chi_A(y) \frac{1}{c(x)}\, d\rho_x(y)\right)\, c(x) d\mu(x)\\ &= \iint_{V\times V} \chi_A(x)\, d\rho_x(y) d\mu(x)\\ &= \int_V \chi_A(x) c(x)\, d\mu(x)\\ &= \int_A c(x)\, d\mu(x) < \infty,\end{aligned}$$

i.e., $P(\chi_A) \in L^1(\nu)$. The fact that $P(\chi_A) \in L^2(\nu)$ is proved as in (a).

(d) The statement will follow from the following chain of equalities:

$$\begin{aligned}(\mu P)(A) &= \int_V \chi_A \, d(\mu P)\\ &= \int_V P(\chi_A)\, d\mu\\ &= \int_V \left(\int_V \chi_A(y) P(x, dy)\right) d\mu(x)\\ &= \iint_{V t V} \chi_A(y) \frac{1}{c(x)} \, d\rho_x(y) d\mu(x)\\ &= \int_V \chi_A(x) \left(\int_V \frac{1}{c(y)} \, d\rho_x(y)\right) d\mu(x)\\ &= \int_A \left(\int_V \frac{1}{c(y)} \, d\rho_x(y)\right) d\mu(x)\\ &= \int_A \frac{d(\mu P)}{d\mu}(x) \, d\mu(x)\end{aligned}$$

where

$$\frac{d(\mu P)}{d\mu}(x) = \int_V \frac{1}{c(y)} \, d\rho_x(y).$$

□

Clearly, Proposition 5.7 can be extended to functions from $\mathcal{D}_{\text{fin}}$.

Lemma 5.8

(1) Let P be a self-adjoint Markov operator in $L^2(\nu)$. Suppose that $c \in L^1_{\text{loc}}(\mu)$. Then, for $A \in \mathcal{B}_{\text{fin}}(\mu)$,

$$||P^n(\chi_A)||^2_{L^2(\nu)} = \rho_{2n}(A \times A), \quad n \in \mathbb{N}, \tag{5.8}$$

where measures ρ_n are defined in (5.2).

(2) Moreover, for all $n \in \mathbb{N}$,

$$\int_A c \, d\mu = ||\chi_A||^2_{\mathcal{H}_E(\rho_{2n})} + ||P^n(\chi_A)||^2_{L^2(\nu)}.$$

Proof

(1) We recall that if P is a self-adjoint operator in the space $L^2(\nu)$, then $\nu P = \nu$. Hence,

$$\begin{aligned}||P^n(\chi_A)||^2_{L^2(\nu)} =&\langle P^n(\chi_A), P^n(\chi_A)\rangle_{L^2(\nu)}\\ =&\langle \chi_A, P^{2n}(\chi_A)\rangle_{L^2(\nu)}\\ =&\rho_{2n}(A\times A).\end{aligned}$$

(2) Since $\rho_x^{(n)}(V) = c(x)$ for all $n \in \mathbb{N}$, we can easily deduce from (1) the following equality (we use here formula (5.7)):

$$\begin{aligned}||\chi_A||^2_{\mathcal{H}_E(\rho_n)} =&\rho_n(A\times A^c)\\ =&\rho_n(A\times V) - \rho_n(A\times A)\\ =&\int_A c\, d\mu - \rho_n(A\times A).\end{aligned}$$

□

Remark 5.9 It is interesting to compare formula (5.8) with a similar result for $||P^n(\chi_A)||^2_{\mathcal{H}_E}$ proved in [1], see also (3.11) in Theorem 3.6.

$$\|P^n(\chi_A)\|^2_{\mathcal{H}_E} = \rho_{2n}(A\times A) - \rho_{2n+1}(A\times A), \qquad n\in\mathbb{N}.$$

Hence, it follows that

$$\|P^n(\chi_A)\|^2_{\mathcal{H}_E} = ||P^n(\chi_A)||^2_{L^2(\nu)} - \rho_{2n+1}(A\times A).$$

5.3 *More on the Embedding Operator J*

In this subsection, we return to the study of the operator J defined in (3.14), see Sect. 3.2. We recall that the operator J is an isometry if considered acting from $L^2(\nu)$ to $L^2(\rho)$, and it is an unbounded operator from $L^2(\mu)$ to $L^2(\rho)$. Here we focus on relations between J and other operators we study in the paper.

Lemma 5.10 *For any $A \in \mathcal{B}_{\mathrm{fin}}(\mu)$, we have*

$$||J(P(\chi_A))||^2_{L^2(\rho)} \le ||\chi_A||^2_{L^2(\nu)}.$$

Proof Indeed, we use Schwarz' inequality for P to show that

$$\begin{aligned}\iint_{V\times V} J(P(\chi_A))^2(x,y)\, d\rho(x,y) &= \int_V P(\chi_A)^2(x)\, d\rho(x,y)\\ &\le \int_V P(\chi_A)(x)\, d\rho(x,y)\\ &= \int_V c(x)P(\chi_A)(x)\, d\mu(x)\\ &= \iint_{V\times V} \chi_A(y)\, d\rho_x(y)d\mu(x)\\ &= \iint_{V\times V} \chi_A(x)\, d\rho_x(y)d\mu(x)\\ &= \int_A c(x)\, d\mu(x)\\ &= ||\chi_A||^2_{L^2(\nu)}.\end{aligned}$$

□

As an illustration of properties of this embedding J, we note that the function $J(c^{-1})(x,y)$ is not integrable with respect to ρ but is locally integrable.

Another useful relation that compares norms of functions is contained in the following inequality.

Lemma 5.11 *Let f be a function from the finite energy space such that f and $\Delta(f)$ belong to $L^2(\mu)$. Then*

$$||Jf||^2_{L^2(\rho)} \ge \frac{1}{2}||f||^2_{\mathcal{H}_E}.$$

Proof The proof follows from [1, Corollary 7.4] and Proposition 3.7 (6):

$$\begin{aligned}\iint_{V\times V} (Jf)^2(x,y)\, d\rho(x,y) &= \iint_{V\times V} f^2(x)\, d\rho_x(y)d\mu(x)\\ &= \int_V f^2(x)c(x)\, d\mu(x)\\ &\ge \frac{1}{2}\langle f, \Delta f\rangle_{L^2(\mu)}\\ &= \frac{1}{2}||f||^2_{\mathcal{H}_E}.\end{aligned}$$

□

In the remaining part of this section, we consider the Markov operator P as an operator acting on functions from the energy space $\mathcal{H}_E$.

Proposition 5.12 *Assume that $c \in L^1_{\text{loc}}(\mu)$. Then, for every $A \in \mathcal{B}_{\text{fin}}(\mu)$, we have*

$$(JP)(\chi_A)(x, y) \in \mathcal{H}_E.$$

Proof We need to show that the energy norm of $J(P(\chi_A))$ is finite. By Theorem 3.6, we find that

$$\begin{aligned} ||(JP)(\chi_A)||^2_{L^2(\rho)} &= \frac{1}{2} \iint_{V\times V} (P(\chi_A)(x) - P(\chi_A)(y))^2 \, d\rho(x, y) \\ &= \iint_{V\times V} (P(\chi_A)^2(x) - P(\chi_A)(x)P(\chi_A)(y)) \, d\rho(x, y). \end{aligned}$$

To see that the last integral is finite, we first show that $(JP)(\chi_A)$ is in $L^2(\rho)$:

$$\begin{aligned} \iint_{V\times V} P(\chi_A)^2(x) \, d\rho(x, y) &\le \iint_{V\times V} P(\chi_A)(x) \, d\rho_x(y) d\mu(x) \\ &= \int_V P(\chi_A)(x)c(x) \, d\mu(x) \\ &= \nu(A) \\ &= \int_A c(x) \, d\mu(x). \end{aligned}$$

The latter is finite.

Similarly, one can check that $\iint_{V\times V} P(\chi_A)(x)P(\chi_A)(y) \, d\rho(x, y)$ is also finite. We leave the proof for the reader. □

Consider a new operator, denoted by ∂, which acts from the energy space $\mathcal{H}_R$ to $L^2(\rho)$:

$$(\partial f)(x, y) = \frac{1}{\sqrt{2}}(f(x) - f(y)), \qquad f \in \mathcal{H}_E \tag{5.9}$$

Remark that in the theory of electrical networks the analogous transformation is called a voltage drop operator.

Lemma 5.13 *The operator $\partial : \mathcal{H}_E \to L^2(\rho)$ defined by (5.9) is an isometry.*

Proof The proof is obvious because

$$||f||^2_{\mathcal{H}_E} = \frac{1}{2} \iint_{V\times V} (f(x) - f(y))^2 \, d\rho(x, y) = ||(\partial f)||^2_{L^2(\rho)}.$$

□

Since $J : L^2(\nu) \to L^2(\rho)$ is an isometry, then the co-isometry J^* sends $L^2(\rho)$ to $L^2(\nu)$ according to the formula

$$(J^* g)(x) = \int_V g(x, \cdot)\ P(x, \cdot)$$

where $g \in L^2(\rho)$.

In the following proposition, we formulate a relation between operators P, J^*, and ∂.

Proposition 5.14 *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}_E & \xrightarrow{\widetilde{\Delta}} & L^2(\nu) \\ \searrow_{\partial} & & \nearrow_{J^*} \\ & L^2(\rho) & \end{array}$$

where $\widetilde{\Delta} = (\sqrt{2}c)^{-1}\Delta = (\sqrt{2})^{-1}(I - P)$.

Proof The proof is mainly based on Theorem 3.10 and the definition of ∂. We have

$$\begin{aligned}
(J^*\partial f)(x) =& \frac{1}{\sqrt{2}} J^*(f(x) - f(y)) \\
=& \frac{1}{\sqrt{2}} \int_V (f(x) - f(y))\ P(x, dy) \\
=& \frac{1}{\sqrt{2}} (f(x) - P(f)(x)) \\
=& \frac{1}{\sqrt{2}} c(x)\Delta(f)(x).
\end{aligned}$$

□

In the next statement, we present several properties of the operator $I - P$.

Corollary 5.15

(1)

$$(I - P)\mathcal{H}_E \subset L^2(\nu),$$

(2) The operator $I - P$ acting from $\mathcal{H}_E$ to $L^2(\nu)$ is contractive.
(3) For the operator $\Delta = c(I - P)$, the following holds

$$\Delta(\mathcal{H}_E) \subset cL^2(\nu).$$

Proof Assertion (1) is a direct consequence of Proposition 5.14 (this result was already mentioned in [1]).

To see that (2) holds, we recall the formula for the norm of a function in the finite energy space $\mathcal{H}_E$:

$$\|f\|^2_{\mathcal{H}_E} = \frac{1}{2}\left(\|f - P(f)\|^2_{L^2(\nu)} + \int_V \mathrm{Var}_x(f \circ X_1)\, d\nu\right),$$

where the meaning of random variables X_n is explained in Sect. 6 below.

(3) is obvious. □

6 Transient Markov Processes and Symmetric Measures

Transient Markov processes and Green's functions are central themes in the theory of Markov chains that have been studied in a numerous books and papers. For the benefit of non-specialist readers, we cite the following sources [17, 58–60]. More interesting results can be found in [61–63].

In this section we consider Green's functions $G_A(x)$ of transient Markov processes and relate the symmetric measures ρ_n to the norm of G_A in the finite energy space.

6.1 Path-space Measure

We denote by Ω the infinite Cartesian product $V \times V \times \cdots = V^{\mathbb{N}_0}$. Let $(X_n(\omega) : n = 0, 1, \dots)$ be the sequence of random variables $X_n : \Omega \to V$ such that $X_n(\omega) = \omega_n$. We call Ω as the path space of the Markov process (P_n). Let Ω_x, $x \in V$, be the set of infinite paths beginning at x:

$$\Omega_x := \{\omega \in \Omega : X_0(\omega) = x\}.$$

Clearly, $\Omega = \coprod_{x \in V} \Omega_x$.

A subset $\{\omega \in \Omega : X_0(\omega) \in A_0, \dots X_k(\omega) \in A_k\}$ is called a *cylinder set* defined by Borel sets $A_0, A_1, \dots, A_k$ taken from $\mathcal{B}$, $k \in \mathbb{N}_0$. The collection of cylinder sets generates the σ-algebra $\mathcal{C}$ of Borel subsets of Ω, and $(\Omega, \mathcal{C})$ is a standard Borel space. Then the functions $X_n : \Omega \to V$ are Borel.

Define a probability measure $\mathbb{P}_x$ on Ω_x. For this, denote by $\mathcal{F}_{\leq n}$ the increasing sequence of σ-subalgebras such that $\mathcal{F}_{\leq n}$ is the smallest subalgebra for which the functions $X_0, X_1, \dots, X_n$ are Borel. For a cylinder set $(A_1, \dots, A_n)$ from $\mathcal{F}_{\le n}$ we set

$$\mathbb{P}_x(X_1 \in A_1, \dots, X_n \in A_n) = \int_{A_1} \cdots \int_{A_{n-1}} P(y_{n-1}, A_n) P(y_{n-2}, dy_{n-1}) \cdots P(x, dy_1). \tag{6.1}$$

Then $\mathbb{P}_x$ extends to the Borel sets on Ω_x by the Kolmogorov extension theorem [64].

The values of $\mathbb{P}_x$ can be written as

$$\mathbb{P}_x(X_1 \in A_1, \dots, X_n \in A_n) = P(\chi_{A_1} P(\chi_{A_2} P(\cdots \ P(\chi_{A_{n-1}} P(\chi_{A_n})) \cdots)))(x). \tag{6.2}$$

The joint distribution of the random variables X_i is given by

$$d\mathbb{P}_x(X_1, \dots, X_n)^{-1} = P(x, dy_1) P(y_1, dy_2) \cdots P(y_{n-1}, dy_n). \tag{6.3}$$

Lemma 6.1 *The measure space $(\Omega_x, \mathbb{P}_x)$ is a standard probability measure space for μ-a.e. $x \in V$.*

On the measurable space $(\Omega, \mathcal{C})$, define a σ-finite measure λ by

$$\lambda := \int_V \mathbb{P}_x \, d\nu(x) \tag{6.4}$$

(λ is infinite if and only if the measure ν is infinite).

By $\mathcal{F}_n$, we denote the σ-subalgebra $X_n^{-1}(\mathcal{B})$. Since $X_n^{-1}(\mathcal{B})$ is a σ-subalgebra of $\mathcal{C}$, there exists a projection

$$E_n : L^2(V, \mathcal{C}, \lambda) \to L^2(\Omega, X_n^{-1}(\mathcal{B}), \lambda).$$

The projection E_n is called the *conditional expectation* with respect to $X_n^{-1}(\mathcal{B})$ and satisfies the property:

$$E_n(f \circ X_n) = f \circ X_n. \tag{6.5}$$

We proved in [1] that the Markov process P_n is irreducible if the initial symmetric measure is irreducible. More precisely, the statement is as follows.

Theorem 6.2 *Let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$, and let A and B be any two sets from $\mathcal{B}_{\text{fin}}(\mu)$. Then*

$$\rho_n(A \times B) = \langle \chi_A, P^n(\chi_B) \rangle_{L^2(\nu)} = \lambda(X_0 \in A, X_n \in B), \quad n \in \mathbb{N}. \tag{6.6}$$

The Markov process (P_n) is irreducible if and only if the measure ρ is irreducible.

In other words, relation (6.6) can be interpreted in the following way: for the Markov process (P_n), the "probability" to get in B for n steps starting somewhere in A is exactly $\rho_n(A \times B) > 0$.

To see that (6.6) holds, one uses the definition of the measure λ and formulas (6.1) and (6.2).

Corollary 6.3 *Let $A_0, A_1, \dots, A_n$ be a finite sequence of subsets from $\mathcal{B}_{\text{fin}}$. Then*

$$\mathbb{P}_x(X_1 \in A_1, \dots, X_n \in A_n) \mid x \in A_0) > 0 \iff \rho(A_{i-1} \times A_i) > 0$$

for $i = 1, \dots, n$.

It is worth noting that the concept of reversible Markov processes can be formulated in terms of the measure λ, roughly speaking λ must be a symmetric distribution.

Proposition 6.4 *Let the measure λ on Ω be defined by (6.4). The Markov operator P is reversible if and only if, for any sets $A_0, \dots, A_n$ from $\mathcal{B}_{\text{fin}}(\mu)$ and any $n \in \mathbb{N}$,*

$$\lambda(X_0 \in A_0, \dots, X_n \in A_n) = \lambda(X_0 \in A_n, \dots, X_n \in A_0).$$

Proof The proof uses Theorem 5.3 (viii). In the proof we assume for simplicity that $n = 2$; the general case is proved similarly. We recall that P is reversible if and only is the Markov operator P is self-adjoint in $L^2(\nu)$. We compute applying (6.1):

$$\begin{aligned}\lambda(X_0 \in A_0 \mid X_1 \in A_1) &= \int_{A_0} \mathbb{P}_x(X_1 \in A_1)\, d\nu(x)\\ &= \int_V \chi_{A_0}(x) P(\chi_{A_1})(x)\, d\nu(x)\\ &= \int_V \chi_{A_1}(x) P(\chi_{A_0})(x)\, d\nu(x)\\ &= \lambda(X_0 \in A_1 \mid X_1 \in A_0).\end{aligned}$$

It proves the statement. □

In the next statement we relate harmonic functions to martingales. Recall first the definition of a martingale.

Let $(X_n : n \in \mathbb{N})$ be the Markov chain on Ω with values in $(V, \mathcal{B})$ defined by $X_n(\omega) = \omega_n$. We recall that the space Ω is represented as the disjoint union of subsets $\Omega_x := \{\omega \in \Omega : \omega_0 = x\}, x \in V$. Let $(\Phi_n : n \in \mathbb{N}_0)$ be a sequence of real-valued random variables defined on Ω. Then it generates a sequence of measurable fields of random variables $x \to \Phi_n(x), x \in V$, defined on the corresponding subset Ω_x. Let $\mathcal{C}_n$ be the σ-algebra of subsets of Ω generated by $\Phi_n^{-1}(B)$, $B \in \mathcal{B}$. Denote by $\mathcal{C}_{\leq n}$ the smallest σ-subalgebra such that the functions $\Phi_i, i = 1, \dots n$, are Borel measurable. These σ-algebras induce σ-algebras $\mathcal{C}_{\leq n}(x)$ on every Ω_x.

It is said that the sequence (Φ_n) is a *martingale* if

$$\mathbb{E}_x(\Phi_{n+k}(x) \mid \mathcal{C}_{\leq n}(x)) = \Phi_n(x), \quad \forall k.$$

Here $\mathbb{E}_x$ is the conditional expectation with respect to the probability path measure $\mathbb{P}_x$, see (6.1).

Proposition 6.5 *Let P be the Markov operator defined by a symmetric measure ρ. For the objects defined above, the following are equivalent:*

(i) a Borel function h on $(V, \mathcal{B})$ is harmonic with respect to the Markov operator P;
(ii) the sequence $(h \circ X_n : n \in \mathbb{N}_0)$ is a martingale.

Proof It follows from the definition of the Markov chain (X_n), path space measure $\mathbb{P}_x$, and [2, Proposition 2.24] that, for any Borel function f,

$$\mathbb{E}_x(f \circ X_{n+m} \mid \mathcal{C}_{\leq n}(x)) = \mathbb{E}_x(f \circ X_{n+m} \mid \mathcal{C}_n(x)) = P^m(f) \circ X_n.$$

Hence, we see that a function h is harmonic if and only if

$$\mathbb{E}_x(h \circ X_{n+m} \mid \mathcal{C}_{\leq n}(x)) = h \circ X_n,$$

i.e., $(h \circ X_n)$ is a martingale. □

6.2 Green's Functions

In this section, we will work with transient Markov processes. We first define a Green's function $G(x, A)$. Our main goal is to study Green's functions as elements of the energy space.

Definition 6.6 Let

$$G(x, A) = \sum_{n=0}^{\infty} P_n(x, A), \qquad A \in \mathcal{B}_{\text{fin}}(\mu), x \in V.$$

The Markov process is called *transient* if, for every $A \in \mathcal{B}_{\text{fin}}$, the function $G(x, A)$ is finite μ-a.e. on V.

In this subsection, we will always assume that the Markov process (P_n) is transient.

Lemma 6.7 *Let ρ be an irreducible symmetric measure. Suppose $A \in \mathcal{B}_{\text{fin}}$ be a set such that $G(x, A)$ is finite a.e. Then, for any $B \in \mathcal{B}_{\text{fin}}$, the function $G(x, B)$ is finite for μ-a.e. $x \in V$.*

Proof The proof of this result is straightforward and mainly based on the definition of irreducible measure, see also Lemma 2.9. □

Lemma 6.8 *Let $A \in \mathcal{B}_{\text{fin}}$ and let P be a Markov operator defined by a symmetric measure ρ. Then the function $x \mapsto P_n(x, A) = P^n(\chi_A)(x)$ belongs to $\mathcal{H}_E$ and*

$$\|P_n(\cdot, A)\|^2_{\mathcal{H}_E} = \rho_{2n}(A \times A) - \rho_{2n+1}(A \times A), \qquad n \in \mathbb{N}.$$

Proof The proof is based on the facts that ν is P-invariant, ρ is symmetric, and on the definition of the norm in the energy space which are used in the following computation:

$$\begin{aligned}
||P_n(x, A)||^2_{\mathcal{H}_E} &= \iint_{V \times V} P_n(x, A)(P_n(x, A) - P_n(y, A))\, d\rho(x, y) \\
&= \iint_{V \times V} P_n(x, A)(P_n(x, A) - P_n(y, A))c(x)P(x, dy)\, d\mu(x) \\
&= \int_V \left[P_n(x, A)^2 - P_n(x, A)\int_V P_n(y, A)P(x, dy)\right] d\nu(x) \\
&= \int_V \left[P_n(x, A)^2 - P_n(x, A)P_{n+1}(x, A)\right] d\nu(x) \\
&= \int_V P_n(x, A)(P_n(x, A) - P_{n+1}(x, A))\, d\nu(x) \\
&= \int_V \chi_A(x)P^n(P^n(\chi_A) - P^{n+1}(\chi_A))(x)\, d\nu(x) \\
&= \langle \chi_A(x), P^{2n}(\chi_A)(x)\rangle_{L^2(\nu)} - \langle \chi_A(x), P^{2n+1}(\chi_A)(x)\rangle_{L^2(\nu)} \\
&= \rho_{2n}(A \times A) - \rho_{2n+1}(A \times A).
\end{aligned}$$

□

Remark 6.9 As a curious observation, we mention that, for any $A \in \mathcal{B}_{\text{fin}}$,

$$\rho_{2n}(A \times A) > \rho_{2n+1}(A \times A).$$

It is worth noting that the above formula cannot be extended to direct products of sets A and B from $\mathcal{B}_{\text{fin}}(\mu)$. In particular, one can prove that the relation

$$\rho_2(A \times B) < \rho(A \times B)$$

implies that $P(\chi_B - P(\chi_B)) > 0$ a.e. Therefore there would exist a harmonic function in $L^2(\nu)$ which is a contradiction.

Fix a set $A \in \mathcal{B}_{\text{fin}}$, then we have the family of measurable functions $G_A(x) := G(x, A)$ indexed by sets of finite measure.

Lemma 6.10 *For a set $A \in \mathcal{B}_{\text{fin}}$, the equality*

$$c(x)(I - P)(G_A)(x) = c(x)\chi_A(x)$$

holds. Equivalently,

$$\Delta G_A(x) = c(x)\chi_A(x).$$

Proof We compute using the definition of Green's function and the fact that the series $\sum_n P_n(x, A)$ is convergent for all x and all $A \in \mathcal{B}_{\text{fin}}(\mu)$:

$$\begin{aligned} c(x)(I - P)G_A(x) &= c(x)(I - P)\sum_{n=0}^{\infty} P_n(x, A) \\ &= c(x)\sum_{n=0}^{\infty} P_n(x, A) - c(x)\sum_{n=1}^{\infty} P_n(x, A) \\ &= c(x)\chi_A(x). \end{aligned}$$

□

Theorem 6.11 *For the objects defined above, we have the following properties.*

(1) For any sets $A, B \in \mathcal{B}_{\text{fin}}$, we have

$$\langle G_A, G_B\rangle_{\mathcal{H}_E} = \sum_{n=0}^{\infty} \rho_n(A \times B); \tag{6.7}$$

and, in particular,

$$\|G_A(x)\|^2_{\mathcal{H}_E} = \sum_{n=1}^{\infty} \rho_n(A \times A). \tag{6.8}$$

(2) For any $f \in \mathcal{H}_E$ and $A \in \mathcal{B}_{\text{fin}}(\mu)$,

$$\langle f, G_A\rangle_{\mathcal{H}_E} = \int_A f\, d\nu.$$

Furthermore, if

$$\mathcal{G} := \text{span}\{G_A(\cdot) : A \in \mathcal{B}_{\text{fin}}\}, \tag{6.9}$$

then $\mathcal{G}$ is dense in the energy space $\mathcal{H}_E$.

Proof

(1) We prove (6.8) here. Relation (6.7) is proved similarly. One has

$$\begin{aligned}
\|G_A(x)\|^2_{\mathcal{H}_E} &= \iint_{V\times V} (G_A(x) - P_A(y))^2 \, d\rho(x,y) \\
&= \iint_{V\times V} G_A(x)(G_A(x) - G_A(y)) \, d\rho(x,y) \\
&= \iint_{V\times V} G_A(x)(G_A(x) - P_A(y))c(x)P(x,dy)d\mu(x)) \\
&= \int_V G_A(x)[G_A(x) - P(G_A)(x)]c(x) \, d\mu(x)) \\
&= \int_V G_A(x)[\sum_{n=0}^{\infty} P^n(\chi_A)(x) - \sum_{n=0}^{\infty} P^{n+1}(\chi_A)(x)c(x) \, d\mu(x)) \\
&= \int_V \sum_{n=0}^{\infty} P^n(\chi_A)(x)\chi_A(x) \, d\nu(x) \\
&= \sum_{n=0}^{\infty} \langle \chi_A, P^n(\chi_A\rangle_{L^2(\nu)} \\
&= \sum_{n=0}^{\infty} \rho_n(A\times A).
\end{aligned}$$

For (2),

$$\begin{aligned}
\langle f, G_A\rangle_{\mathcal{H}_E} &= \frac{1}{2}\iint_{V\times V} (f(x) - f(y))(G_A(x) - G_A(y)) \, d\rho(x,y) \\
&= \iint_{V\times V} (f(x)G_A(x) - f(x)G_A(y)) \, d\rho(x,y) \\
&= \int_V \left[f(x)G_A(x)c(x) - f(x)\left(\int_V G_A(y)P(x,dy)\right)c(x)\right] d\mu(x) \\
&= \int_V f(x)c(x)\left[\sum_{n=o}^{\infty} P^n(\chi_A)(x) - \sum_{n=o}^{\infty} P^{n+1}(\chi_A)(x)\right] d\mu(x) \\
&= \int_V f(x)\chi_A(x)c(x) \, d\mu(x) \\
&= \int_A f \, d\nu.
\end{aligned}$$

It follows from the proved relation that if $\langle f, G_A\rangle_{\mathcal{H}_E} = 0$ for all $A \in \mathcal{B}_{\text{fin}}(\mu)$, then $f = 0$, and $\mathcal{G}$ is dense in $\mathcal{H}_E$. □

Let $\mathcal{D}_{\text{fin}}(\mu) \subset L^2(\mu)$ denote, as usual, the space spanned by characteristic functions, and let $\mathcal{G}$ be as in (6.9). Then the following two operators, J and K, are densely defined

$$J : \chi_A \mapsto \chi_A : \mathcal{D}_{\text{fin}} \to \mathcal{H}_E, \qquad K : G_A \mapsto c(I - P)(G_A) : \mathcal{G} \to L^2(\mu) \tag{6.10}$$

where $A \in \mathcal{B}_{\text{fin}}(\mu)$.

Proposition 6.12 *The operators J and K form a symmetric pair, i.e.,*

$$\langle J\varphi, f\rangle_{\mathcal{H}_E} = \langle \varphi, K(f)\rangle_{L^2(\mu)} \tag{6.11}$$

where $\varphi \in \mathcal{D}_{\text{fin}}$ and $f \in \mathcal{G}$.

Proof To prove (6.11) it suffices to check that it holds for $\varphi = \chi_A$ and $f = G_B$ where $A, B \in \mathcal{B}_{\text{fin}}(\mu)$. For these functions, we show that the both inner products are equal to $\nu(A \cap B)$.

By Lemma 6.10, we have

$$\begin{aligned}\langle \chi_A, K(G_B)\rangle_{L^2(\mu)} &= \langle \chi_A, c\chi_B\rangle_{L^2(\mu)}\\ &= \int_V \chi_A c \chi_B \, d\mu\\ &= \nu(A \cap B).\end{aligned}$$

On the other hand, for the same functions φ and f, we compute the inner product in the finite energy Hilbert space using the symmetry of ρ:

$$\begin{aligned}\langle J(\chi_A), G_B\rangle_{\mathcal{H}_E} &= \frac{1}{2}\iint_{V\times V} (\chi_A(x) - \chi_A(y))(G_B(x) - G_B(y)) \, d\rho(x, y)\\ &= \iint_{V\times V} (\chi_A(x)G_B(x) - \chi_A(x)G_B(y)) \, d\rho(x, y)\\ &= \iint_{V\times V} [\chi_A(x)\sum_{n=0}^{\infty} P^n(\chi_B)(x)\\ &\qquad - \chi_A(x)\sum_{n=0}^{\infty} P^n(\chi_B)(y)]c(x)P(x, dy)d\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_V [\chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(x) \\
&\qquad - \chi_A(x) \sum_{n=0}^{\infty} \int_V P^n(\chi_B)(y) P(x, dy)] c(x) d\mu(x) \\
&= \int_V [\chi_A(x) \sum_{n=0}^{\infty} P^n(\chi_B)(x) - \chi_A(x) \sum_{n=1}^{\infty} P^n(\chi_B)] \, d\nu(x) \\
&= \int_V \chi_A(x) \chi_B(x) \, d\nu(x) \\
&= \nu(A \cap B).
\end{aligned}
$$

□

Corollary 6.13 *The finite energy Hilbert space admits the orthogonal decomposition*

$$\mathcal{H}_E = \overline{J(\mathcal{D}_{\text{fin}}(\mu))} \oplus \mathcal{H}arm.$$

In particular, for every $B \in \mathcal{B}_{\text{fin}}(\mu)$, we have $G_B = G_1 \oplus G_2$, where $G_1 \in \overline{J(\mathcal{D}_{\text{fin}}(\mu))}$ is always non-zero.

Proof Indeed, if one assumed that $G_1 = 0$, then we would have that G_B is orthogonal to $\overline{J(\mathcal{D}_{\text{fin}}(\mu))}$. This contradicts Theorem 6.11. □

We conclude this section with the following result that was proved in [1]:

Theorem 6.14 *Let (P_n) be a transient Markov process, and let $G(x, A)$ be the corresponding Green's function. Then, for any $f \in \mathcal{H}_E$, we have the decomposition*

$$f = G(\varphi) + h$$

where h is a harmonic function and $\varphi \in L^2(\nu)$.

7 Discretization of the Graph $\mathcal{B}_{\text{fin}}(\mu)$

In this section, we show that our basic setting (a symmetric measure on the Cartesian product $(V, \mathcal{B})$) can be realized as a limit of discrete graphs. This approach naturally leads to the notion of *graphons*. The reader can find necessary information in the following books [65–67] and articles [68–70].

Let $(V, \mathcal{B}, \mu)$ be a σ-finite measure space, and let ρ be a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We will associate with $(V, \mathcal{B}, \mu)$ and ρ a sequence of countably infinite graphs $\mathcal{G}_n$ equipped with conductance functions c_n such that the weighted graphs $(\mathcal{G}_n, c_n)$ can be viewed as a discretization of of the uncountable graph $\mathcal{B}_{\text{fin}}$ considered in [1].

We first recall a few facts from [1].

Lemma 7.1 *Suppose that $c(x) \in L^1_{\text{loc}}(\mu)$. Then, for any set $A \in \mathcal{B}_{\text{fin}}$,*

$$\rho(A \times A^c) < \infty \tag{7.1}$$

where $A^c = V \setminus A$. The converse is not true, in general.

We can view at the set $\mathcal{B}_{\text{fin}} = \mathcal{B}_{\text{fin}}(\mu)$ as an uncountable graph $\mathcal{G}$ whose vertices are sets A from $\mathcal{B}_{\text{fin}}$ and edges are defined as follows. For a symmetric measure ρ defined on $(V \times V, \mathcal{B} \times \mathcal{B})$, we say that two sets A and B from $\mathcal{B}_{\text{fin}}$ are connected by an edge e if $\rho(A \times B) > 0$.

This definition is extended to get finite paths in the graph $\mathcal{G}$. It is said that there exists a finite path in the graph $\mathcal{G}$ from A to B if there exists a sequence $\{A_i : i = 0, \dots, n\}$ of sets from $\mathcal{B}_{\text{fin}}$ (vertices of $\mathcal{G}$) such that $A_0 = A$, $A_n = B$ and $\rho(A_i \times A_{i+1}) > 0, i = 0, \dots n-1$.

Theorem 7.2 *Let $(V, \mathcal{B}, \mu)$ be as above, and let ρ be a symmetric irreducible measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. Then any two sets A and B from the graph $\mathcal{G}$ are connected by a finite path, i.e., the graph $\mathcal{G}$ is connected.*

Proof We will show that there exists a finite sequence $(A_i : 0 \le i \le n)$ of disjoint subsets from $\mathcal{B}_{\text{fin}}$ such that $A_0 = A$, $\rho(A_i \times A_{i+1}) > 0$, and $\rho(A_n \times B) > 0$, $i = 0, \dots, n-1$.

If $\rho(A \times B) > 0$, then nothing to prove, so that we can assume that $\rho(A \times B) = 0$.

Let $\xi = (C_i : i \in \mathbb{N})$ be a partition of V into disjoint subsets of positive finite measure such that $C_i \in \mathcal{B}_{\text{fin}}$ for all i. Without loss of generality, we can assume that the sets A and B are included in ξ. Let for definiteness, $A = C_0$.

Since $\rho(A \times A^c) > 0$ (by Lemma 7.1), there exists a set $C_{i_1} \in \xi$ such that $\rho(A \times C_{i_1}) > 0$ and $\rho(A \times C_j) = 0$ for all $0 < j < i_1$. Set

$$A_1 := \bigcup_{0 < j \le i_1} C_j.$$

It is clear that $A_1 \in \mathcal{B}_{\text{fin}}$ and $\rho(A_0 \times A_1) > 0$. If $\rho(A_1 \times B) > 0$, then we are done. If not, we proceed as follows. Because of the property $\rho(A_1 \times A_1^c) > 0$, there exists some $i_2 > i_1$ such that $\rho(A_1 \times C_{i_2}) > 0$ and $\rho(A_1 \times C_j) = 0$ for all $i_1 < j < i_2$. Set

$$A_2 := \bigcup_{i_1 < j \le i_2} C_j.$$

Then $\rho(A_1 \times A_2) > 0$, and we check whether $\rho(A_2 \times B) > 0$. If not, we continue in the same manner by constructing consequently disjoint sets A_i satisfying the property $\rho(A_i \times A_{i+1}) > 0$. Since B is an element of ξ, this process will terminate. This means that there exists some n such that $A_n \supset B$. This argument proves the proposition. □

Given a σ-finite measure space $(V, \mathcal{B}, \mu)$, consider a sequence of measurable partition $\{\xi_n\}_{n\in\mathbb{N}}$ such that

(i) $\xi_n = (A_n(i) : i \in \mathbb{N})$, $\bigsqcup_i A_n(i) = V$, $A_n(i) \in \mathcal{B}_{\text{fin}}(\mu)$;
(ii) ξ_{n+1} refines ξ_n, i.e., every element $A_n(i)$ of the partition ξ_n is the union of some elements of ξ_{n+1}: $A_n(i) = \bigcup_{j\in\Lambda_n(i)} A_{n+1}(j)$ where $\Lambda_n(i)$ is a finite subset of $\mathbb{N}$;
(iii) the set $\{A_n(i) : i \in \mathbb{N}, n \in \mathbb{N}\}$ generates the Borel σ-algebra $\mathcal{B}$.

If for every i, the cardinality of the set Λ_i is bigger than one, we say that ξ_{n+1} refines ξ_n strictly.

It is well known, see e.g. [26], that, for any point $x \in V$, there exists a sequence $i_n(x)$ such that $A_{n+1}(i_{n+1}(x)) \subset A_n((i_n)(x))$ and

$$\{x\} = \bigcap_{n\in\mathbb{N}} A_n(i_n(x)) \tag{7.2}$$

Suppose ρ is a symmetric measure on $(V \times V, \mathcal{B} \times \mathcal{B})$. We define a sequence of non-negative Borel functions $c^{(n)}$ on $(V \times V, \mathcal{B} \times \mathcal{B})$ by setting

$$c_{xy}^{(n)} := \rho(A_n(i_n(x)) \times A_n(i_n(y)))$$

for any x, y from V. Clearly, $c_{xy}^{(n)}$ is a piecewise constant function.

Lemma 7.3 *For a given sequence of strictly refining partitions* (ξ_n)*, the sequence of functions* $(c_{xy}^{(n)})$ *is monotone decreasing.*

Proof The proof is straightforward. For $x, y \in V$, let the sequences $(A_n(i_n(x)))$ and $(A_n(j_n(y)))$ shrink to the points x and y, respectively, according to (7.2). By assumption of the lemma, $A_{n+1}(i_{n+1}(x))$ is a proper subset of $A_n(i_n(x))$. Hence,

$$\begin{aligned} c_{xy}^{(n+1)} &= \rho(A_{n+1}(i_{n+1}(x)) \times A_{n+1}(j_{n+1}(y)) \\ &< \rho(A_n(i_n(x)) \times A_n(j_n(y))) \\ &= c_{xy}^{(n)}. \end{aligned}$$

□

We now can define a sequence of discrete graphs (weighted networks) $G_n = (V_n, E_n, w_n)$. The vertex set V_n is formed by the atoms of the partition ξ_n, i.e., by the sets $\{A_n(i) : i \in \mathbb{N}_0\}$; therefore V_n can be identified with $\mathbb{N}_0$. The set of edges

E_n consists of pairs (i, j) such that

$$(i, j) \in E_n \iff \rho(A_n(i) \times A_n(j)) > 0.$$

The weight function is $w_n(i, j) = \rho(A_n(i) \times A_n(j))$.

Lemma 7.4 *Let ρ be a symmetric irreducible measures on $(V \times V, \mathcal{B} \times \mathcal{B})$. Then the weighted graph G_n is connected for every n.*

It follows from Lemma 7.3 that

$$c_{xy} = \lim_{n\to\infty} c_{xy}^{(n)}$$

exists and is a Borel non-negative function. Since the measure ρ is symmetric, we conclude that $c_{xy} = c_{yx}$.

Next, we define

$$c^{(n)}(x) = \sum_j \rho(A_n(i_n(x)) \times A_n(j)) = \sum_{y\sim_n x} c_{xy}^{(n)}$$

where $x \sim_n y$ if and only if $c_{xy}^{(n)} > 0$. It can be seen that

$$c^{(n)}(x) = \rho(A_n(i_n(x)) \times V). \tag{7.3}$$

Using the proved results, we can deduce the following statement.

Theorem 7.5 *The sequence $(c^{(n)}(x))$ is monotone decreasing for every $x \in V$ and*

$$c(x) := \lim_{n\to\infty} c^{(n)}(x) = \rho_x(V).$$

Proof Indeed, we see from (7.3) that

$$c^{(n+1)}(x) = \rho(A_{n+1}(i_{n+1}(x)) \times V) < \rho(A_n(i_n(x)) \times V) = c^{(n)}(x).$$

Hence, the Borel function $c(x)$ is well defined for every x. Because $\bigcap_n A_n(i_n(x)) = \{x\}$, we obtain that $c(x) = \rho_x(V)$. □

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Multi Variable Semicircular Processes From $*$-Homomorphisms and Operators

Ilwoo Cho and Palle E. T. Jorgensen

Abstract In this paper, we (i) consider a Banach $*$-probability space $\mathbb{L}_Q^{(N)}$ generated by mutually free finitely, or countable-infinitely many semicircular elements, induced by mutually orthogonal projections in a C^*-probability space, (ii) construct certain $*$-homomorphisms acting on $\mathbb{L}_Q^{(N)}$, determined by the shift processes acting on the index set $\{1, \ldots, N\}$ of semicircular elements, and the corresponding Banach-space operators induced by them, and (iii) study how the $*$-homomorphisms and Banach-space operators of (ii) deform the original free-distributional data on $\mathbb{L}_Q^{(N)}$.

Keywords Free probability · Projections · (Weighted-)Semicircular elements · Banach $*$-Probability spaces · Integer shifts · Restricted integer shifts · Shift operators

Mathematics Subject Classification (2000) 46L10, 46L40, 46L53, 46L54, 47L15, 47L30, 47L55

1 Introduction

We shall use the term *free probability* in the customary sense, as an extension of classical *measure theory* (including *probability theory*). *Random variables* in measure theory are functions in commutative function systems; by contract, in free theory, *free random variables* are *operators* in noncommutative $*$-algebras (e.g., C^*-algebras, or von Neumann algebras, or Banach $*$-algebras). Noncommutative

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probability was initiated by D. Voiculescu in response to questions in *quantum physics*, and in operator algebra theory, notably, in the free group factor isomorphism problem (e.g., [20, 26] and [30]). In classical theories, *independence* is a key feature, however, in free probability theory, there is a counterpart, the *freeness*, or *free independence* (e.g., [3, 4, 23, 25] and [30]). Freeness parallels the more familiar and classical independence, but it is much more subtle. Its use entails new tools from multiple areas of mathematics (e.g., [8–11, 16, 17] and [18]). While in classical probability, we know that, on account of the central limit theorem, the *Gaussian law* (or the *Gaussian distribution*) is universal. In free probability, the counterpart is *the semicircular law*.

A *Banach $*$-algebra Y* is a complete topological $*$-algebra equipped with the norm topology. If there is a linear functional ψ on Y, then the pair (Y, ψ) is called a *Banach $*$-probability space*. Then the free-distributional data on (Y, ψ) are determined by joint free moments of operators of Y up to ψ. Here, we are interested in $*$-homomorphisms on Y, preserving free-probabilistic information on (Y, ψ). Since they are (multiplicative) linear transformations on Y, one can regard them as operators in the *operator space $B(Y)$* of all bounded linear transformations on Y, by understanding the Banach $*$-algebra Y as a *Banach space*. Our main purpose is to investigate how such Banach-space operators affect the free probability on (Y, ψ), where Y is generated by multi semicircular elements.

1.1 Motivations

There are many ways to construct semicircular elements (e.g., [1, 5, 7, 9, 17, 20, 22, 27–30]) in topological $*$-probability spaces (e.g., *C^*-probability spaces*, or *W^*-probability spaces*, or *Banach $*$-probability spaces*, etc.). Our construction of semicircular elements is motivated by that of *weighted-semicircular elements* of [11], from the analysis on the *p-adic number fields $\mathbb{Q}_p$*, for *primes p* (e.g., [13, 24]), which is different from earlier works.

Like the (weighted-)semicircularity of [11], semicircular elements from mutually orthogonal $|\mathbb{Z}|$-many *projections* in a C^*-probability space, and the corresponding *semicircular law* are considered here (See Sects. 3, 4, and 5 below; also, see [8, 9]). Meanwhile, free distributions of *free reduced words* in mutually free, multi semicircular elements are re-characterized, estimated, and asymptotically estimated in terms of their *joint free moments* in [10] (See Sect. 6 below). By using the techniques of [10], we study free probability on our structures, and consider how certain Banach-space operators deform original free-distributional data.

1.2 Overview

In Sect. 2, we briefly introduce free probability, free distributions of operators characterized by free moments, or free cumulants. In Sects. 3, 4, and 5, we give a new construction of (weighted-)semicircular elements. This is done with the use of a new induction procedure from countable systems of projections. The focus of Sect. 6 is a new tool, which we call *semicircular free filterizations*. It is used in turn to give a new characterization of specific free (weighted-)semicircular families. In Sects. 7 and 8, we study *integer-shifts* on semicircular free filterizations as ∗-homomorphisms on them; and more generally, *integer-shift operators*, generated by them. Section 9 deals with deformed free-distributional data under the actions of integer-shifts, and associated shift operators.

2 Preliminaries

For free probability theory, see [23, 27] (and the cited papers therein). *Free probability* is the noncommutative analogue of classical *measure theory* (including *probability theory*) and *statistical analysis*. The classical *independence* is replaced by so-called the *freeness*, by replacing *measures* on sets to *linear functionals* on algebras. It is an important branch not only of operator theory (e.g., [4, 6, 7, 10, 17, 19, 20]), but also of applied mathematics (e.g., [8, 9, 11, 15, 18, 21, 29, 30]).

We use the combinatorial approaches [23] of free probability. Without detailed introduction, *Free moments* and *free cumulants* of operators will be computed to verify free distributions of them. Also, (free-probabilistic) *free product* (in the sense of [23] and [27]) is used without definition.

Let B be a (noncommutative) topological ∗-algebra (a C^*-algebra, or a von Neumann algebra, or a Banach ∗-algebra, etc.), and suppose φ is a linear functional on B. Then the pair (B, φ) is said to be a topological (free) ∗-probability space (a C^*-probability space, respectively, a W^*-probability space, respectively, a Banach ∗-probability space, etc.). If one regards an operator $x \in B$ as an element of (B, φ), it is called a *free random variable*.

For any arbitrarily chosen free random variables $x_1, \ldots, x_N \in (B, \varphi)$, the *free distribution* of them is characterized by the *joint free moments*

$$\bigcup_{n=1}^{\infty}\left(\bigcup_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}^n}\left(\bigcup_{(r_1,\ldots,r_n)\in\{1,*\}^n}\varphi\left(x_{i_1}^{r_1}x_{i_2}^{r_2}\ldots x_{i_n}^{r_n}\right)\right)\right).$$

Equivalently, if $k_\bullet(\ldots)$ is the *free cumulant on B in terms of φ* (in the sense of [23]), then the free distribution of $x_1, \ldots, x_N$ is characterized by the *joint free cumulants*

$$\bigcup_{n=1}^{\infty}\left(\bigcup_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}^n}\left(\bigcup_{(r_1,\ldots,r_n)\in\{1,*\}^n}k_n\left(x_{i_1}^{r_1},\ldots,x_{i_n}^{r_n}\right)\right)\right),$$

via the *Möbius inversion* of [23]. For instance, if $x \in (B, \varphi)$ is a self-adjoint free random variable, equivalently, if $x \in B$ is self-adjoint in the sense that: $x^* = x$, where x^* is the adjoint of x in B (e.g., [14]), then the free distribution of x is fully characterized by

$$\text{the free moment sequence } \left(\varphi(x^n)\right)_{n=1}^{\infty},$$

or

$$\text{the free cumulant sequence } (k_n(x, \ \dots, x))_{n=1}^{\infty}.$$

Recall that two free random variables x_1 and x_2 are *free in* (B, φ), if and only if all "mixed" free cumulants of them vanish (See e.g., [23]), i.e., for any mixed n-tuple $(i_1, \dots, i_n) \in \{1, 2\}^n$, for $n > 1$ in $\mathbb{N}$,

$$k_n\left(x_{i_1}^{r_1}, \ \dots, x_{i_n}^{r_n}\right) = 0,$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$.

3 The Banach $*$-Algebra $\mathfrak{L}_Q$

Let (B, φ) be an arbitrary topological $*$-probability space.

A self-adjoint free random variable a is weighted-semicircular in (B, φ) with its weight $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (in short, t_0-semicircular), if a satisfies the free cumulant computations,

$$k_n(a, \dots, a) = \begin{cases} k_2(a, \ a) = t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_\bullet(\dots)$ is the free cumulant on B in terms of φ under the Möbius inversion of [23].

If $t_0 = 1$ in (3.1), the 1-semicircular element a is said to be semicircular in (B, φ), i.e., a is semicircular in (B, φ), if a satisfies

$$k_n(a, \dots, a) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

By the *Möbius inversion* of [23], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \overset{def}{=} \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, and c_k are the k-th *Catalan numbers*,

$$c_k = \frac{1}{k+1} \binom{2k}{k} = \frac{1}{k+1} \frac{(2k)!}{k!(2k-k)!} = \frac{(2k)!}{k!(k+1)!},$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Similarly, a free random variable a is semicircular in (B, φ), if and only if a is 1-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

by (3.3), for all $n \in \mathbb{N}$, where ω_n and $c_{\frac{n}{2}}$ are in the sense of (3.3).

So, we use the t_0-semicircularity (3.1) (or the semicircularity (3.2)) and its characterization (3.3) (resp., (3.4)) alternatively from below.

Throughout this paper, we fix a C^*-probability space (A, ψ), containing $|\mathbb{Z}|$-many projections $\{q_j\}_{j \in \mathbb{Z}}$ in the C^*-algebra A, i.e., the operators q_j satisfy

$$q_j^* = q_j = q_j^2 \text{ in } A,$$

for all $j \in \mathbb{Z}$. Note that there do exist such C^* -probabilistic structures arising naturally (e.g., [11, 14]), or artificially (e.g., [8, 9]).

Assume further that these projections $\{q_j\}_{j \in \mathbb{Z}}$ are *mutually orthogonal* in A, in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \text{ for all } i, j \in \mathbb{Z}, \tag{3.5}$$

where δ is the *Kronecker delta*.

Now, we fix a family $\{q_j\}_{j \in \mathbb{Z}}$ of the projections (3.5),

$$\mathbf{Q} = \{q_j \text{ of } (3.5) : j \in \mathbb{Z}\} \text{ in } A. \tag{3.6}$$

And let Q be the C^*-*subalgebra of* A generated by the family $\mathbf{Q}$ of (3.6),

$$Q \overset{def}{=} C^* (\mathbf{Q}) \subseteq A. \tag{3.7}$$

Let Q be the C^*-subalgebra (3.7) of A. Then

$$Q \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot q_j\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{3.8}$$

in A, where $\oplus$ is the direct product of C^*-algebras.

Proof The proof of (3.8) is immediate from the mutual-orthogonality (3.5). □

Define now linear functionals ψ_j on the C^*-algebra Q by

$$\psi_j\ (q_i) = \delta_{ij} \psi(q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.9}$$

for all $j \in \mathbb{Z}$, where ψ is the linear functional of our fixed C^*-probability space (A, ψ). The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.9) are well-defined on Q by (3.8).

Assumption In the rest of this paper, we assume that

$$\psi(q_j) \neq 0 \text{ in } \mathbb{C}, \forall q_j \in \mathbf{Q}.$$ □

Then, as an independent C^*-algebra, the C^*-subalgebra Q of A forms C^*-probability spaces (Q, ψ_j), where ψ_j are the linear functionals (3.9).

Define now bounded linear transformations $\mathbf{c}$ and $\mathbf{a}$ acting on the C^*-algebra Q, by linear morphisms satisfying

$$\mathbf{c}\left(q_j\right) = q_{j+1}, \text{ and } \mathbf{a}\left(q_j\right) = q_{j-1}, \tag{3.10}$$

for all $j \in \mathbb{Z}$. Then $\mathbf{c}$ and $\mathbf{a}$ are well-defined bounded operators "on Q," by (3.8).

They are *Banach-space operators*, contained in the *operator space* $B(Q)$, consisting of all bounded linear transformations on Q, if we regard Q as a Banach space equipped with its C^*-norm (e.g., [12]). We call these Banach-space operators $\mathbf{c}$ and $\mathbf{a}$ of (3.10), the *creation*, respectively, the *annihilation* on Q.

Define the *radial operator* $\mathbf{l} \in B(Q)$, by

$$\mathbf{l} = \mathbf{c} + \mathbf{a} \text{ on } Q. \tag{3.11}$$

And then, construct a subspace $\mathfrak{L}$ of $B(Q)$ by

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{\mathbf{l}\}]}^{\|.\|}, \tag{3.12}$$

generated by the radial operator $\mathbf{l}$ of (3.11), under the operator norm $\|.\|$ of $B(Q)$, where $\overline{X}^{\|.\|}$ are the *operator-norm closures* of subsets X of $B(Q)$ (e.g., [12]). By (3.12), this subspace $\mathfrak{L}$ forms a Banach algebra in $B(Q)$.

On this Banach algebra $\mathfrak{L}$ of (3.12), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n \mathbf{l}^n\right)^* = \sum_{n=0}^{\infty} \overline{t_n} \mathbf{l}^n \text{ in } \mathfrak{L}, \tag{3.13}$$

where $\overline{z}$ are the *conjugates of* $z \in \mathbb{C}$.

Then this operation (3.13) is a well-defined *adjoint on* $\mathfrak{L}$ (See [8]), and hence, every element of $\mathfrak{L}$ is *adjointable* in $B(Q)$ (e.g., [12]). So, the Banach algebra $\mathfrak{L}$ of (3.12) forms a *Banach ∗-algebra* in $B(Q)$.

Construct now the *tensor product Banach ∗-algebra* $\mathfrak{L}_Q$,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q, \tag{3.14}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach ∗-algebras.

We call the tensor product Banach ∗-algebra $\mathfrak{L}_Q$ of (3.14), the radial projection (Banach ∗-)algebra on Q.

4 Weighted-Semicircular Elements Induced by Q

Let $\mathfrak{L}_Q$ be the radial projection algebra (3.14). Remark that, if

$$u_j \overset{def}{=} \mathbf{l} \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = \left(\mathbf{l} \otimes q_j\right)^n = \mathbf{l}^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^n = q_j$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$. So, by (3.8), (3.12), and (3.14), such operators $\{u_j\}_{j \in \mathbb{Z}}$ of (4.1) generate $\mathfrak{L}_Q$.

Now, define linear morphisms

$$E_{j,Q} : \mathfrak{L}_Q \to \mathfrak{L}_Q$$

by linear transformations satisfying

$$E_{j,Q}\left(u_i^n\right) \overset{def}{=} \begin{cases} \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)} u_j^n & \text{if } i = j \\ \\ 0_{\mathfrak{L}_Q}, \text{ the zero element of } \mathfrak{L}_Q & \text{otherwise,} \end{cases} \tag{4.2}$$

for all $n \in \mathbb{N}$, $i,\ j \in \mathbb{Z}$, where $[\frac{n}{2}]$ means the *minimal integer* greater than or equal to $\frac{n}{2}$. The linear transformations $E_{j,Q}$ of (4.13) are well-defined linear transformations on $\mathfrak{L}_Q$, by the cyclicity (3.12) of a tensor factor $\mathfrak{L}$ of $\mathfrak{L}_Q$, and the structure theorem (3.8) of the other tensor factor Q of $\mathfrak{L}_Q$.

Define now linear functionals τ_j on $\mathfrak{L}_Q$ by a linear morphism satisfying

$$\begin{aligned} \tau_j\left(u_k^n\right) &= \tau_j\left(\mathbf{l}^n \otimes q_k\right) \\ &= \psi_j\left(E_{j,Q}\left(\mathbf{l}^n(q_k)\right)\right), \end{aligned} \tag{4.3}$$

for all $j,\, k \in \mathbb{Z}$, for all $n \in \mathbb{N}$.

Fix $j \in \mathbb{Z}$, and let $u_k = \mathbf{l} \otimes q_k$ be the k-th generating operators of the Banach $*$-probability space $(\mathfrak{L}_Q, \varphi_j)$ of (4.3), for all $k \in \mathbb{Z}$. Then

$$\tau_j\left(u_j^n\right) = \delta_{k,j}\omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}}, \tag{4.4}$$

where ω_n and $c_{\frac{n}{2}}$ are in the sense of (3.3) for all $n \in \mathbb{N}$.

Proof By the definition (3.11) of the radial operator $\mathbf{l}$ on $\mathfrak{L}_Q$, one has

$$\mathbf{l}^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{c}^k \mathbf{a}^{n-k} on \mathfrak{L}_Q,$$

with axiomatization:

$$\mathbf{l}^0 = \mathbf{c}^0 = \mathbf{a}^0 = 1_{\mathfrak{L}_Q}, \text{ the identity operator on } \mathfrak{L}_Q,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for all } n, k \in \mathbb{N}_0,$$

because

$$\mathbf{c}^{n_1}\mathbf{a}^{n_2} = \mathbf{a}^{n_2}\mathbf{c}^{n_1}, \forall n_1, n_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

So, one can verify that $\mathbf{l}^{2n-1}$ does not contain $1_{\mathfrak{L}_Q}$-terms, but $\mathbf{l}^{2n}$ contains the $1_{\mathfrak{L}_Q}$-term,

$$\binom{2n}{n} \cdot 1_{\mathfrak{L}_Q}.$$

So, by the straightforward computation on (4.3), we obtain the formula (4.4), since

$$\psi(q_j) = \psi_j(q_j) \neq 0, forall j \in \mathbb{Z},$$

by (3.9) and (4.2). See [9] for more details. □

The well-defined Banach ∗-probability spaces

$$\mathfrak{L}_Q(j) \overset{denote}{=} \left(\mathfrak{L}_Q,\ \tau_j\right) \tag{4.5}$$

are called the j-th filter of $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$, where τ_j are as in (4.3).

The following theorem is proven by (4.4).

Let $\mathfrak{L}_Q(j)$ be the "j-th" filter of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$, and let u_j be the "j-th" generating operator (4.1) of $\mathfrak{L}_Q(j)$. Then u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$. Meanwhile, if $k \neq j$ in $\mathbb{Z}$, then the k-th generating operators u_k of $\mathfrak{L}_Q$ have the zero free distribution in $\mathfrak{L}_Q(j)$.

Proof It is not hard to check the generating operator u_k are self-adjoint in $\mathfrak{L}_Q$. So, by (4.4), a generating operator u_k is $\psi(q_j)^2$-semicircular, if and only if $k = j$; otherwise, $\{u_k\}_{k\neq j}$ follow the zero free distribution. □

5 Semicircular Elements Induced by Q

As in Sect. 4, let $\mathfrak{L}_Q(j)$ be the j-th filter for $j \in \mathbb{Z}$.

Let $U_j = \frac{1}{\psi(q_j)} u_j$ be a free random variable $\mathfrak{L}_Q(j)$ for $j \in \mathbb{Z}$, where u_j is the j-th generating operator of $\mathfrak{L}_Q$. If

$$\psi(q_j) \in \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\} in \mathbb{C}^{\times}, \tag{5.1}$$

then U_j is semicircular in $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$.

Proof Since $\psi(q_j) \in \mathbb{R}^{\times}$, the operator U_j is self-adjoint in $\mathfrak{L}_Q(j)$, by the self-adjointness of $u_j \in \mathfrak{L}_Q$. So,

$$\tau_j\left(U_j^n\right) = \frac{1}{\psi(q_j)^n}\tau_j\left(u_j^n\right) = \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by the $\psi(q_j)^2$-semicircularity (4.4) of u_j in $\mathfrak{L}_Q(j)$. Therefore, by (3.4), the free random variable U_j is semicircular in $\mathfrak{L}_Q(j)$, under the condition (5.1). □

Assumption 2 From below, we automatically assume that

$$\psi(q_j) \in \mathbb{R}^{\times} in \mathbb{C}, \text{ for } q_j \in \mathbf{Q},$$

for all $j \in \mathbb{Z}$. □

6 On the Free Filterization $\mathfrak{L}_Q(\mathbb{Z})$

In this section, we construct the free product Banach $*$-probability space $\mathfrak{L}_Q(\mathbb{Z})$ of the free filters $\{\mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}}$, and the corresponding sub-structure $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$ generated by a free semicircular family $\{U_j \in \mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}}$ of $\mathfrak{L}_Q(\mathbb{Z})$, and study free-distributional data on $\mathbb{L}_Q$.

6.1 The Semicircular Filterization $\mathbb{L}_Q$

Let (A, ψ) be the fixed C^*-probability space containing a family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections with

$$\psi(q_j) \in \mathbb{R}^{\times}, \text{ for all } j \in \mathbb{Z},$$

and let $\mathfrak{L}_Q(j)$ be the j-th free filters of Q, for all $j \in \mathbb{Z}$. For the system

$$\{\mathfrak{L}_Q(j) : j \in \mathbb{Z}\}$$

of Banach $*$-probability spaces, define the *free product Banach $*$-probability space* $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\begin{aligned} \mathfrak{L}_Q(\mathbb{Z}) &\overset{denote}{=} \left(\mathfrak{L}_Q(\mathbb{Z}), \ \tau\right) \\ &\overset{def}{=} \underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_Q(j) = \left(\underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_{Q,j}, \ \underset{j \in \mathbb{Z}}{\star} \tau_j \right), \end{aligned} \tag{6.1.1}$$

with

$$\mathfrak{L}_Q(\mathbb{Z}) = \underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_{Q,j}, \text{ with } \mathfrak{L}_{Q,j} = \mathfrak{L}_Q, \forall j \in \mathbb{Z},$$

and

$$\tau = \underset{j \in \mathbb{Z}}{\star} \tau_j \text{ on } \mathfrak{L}_Q(\mathbb{Z}).$$

For more about free-probabilistic free products, see [23, 27].

Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free product Banach $*$-probability space (6.1.1) of the system $\{\mathfrak{L}_Q(j)\}_{j\in\mathbb{Z}}$ of all free filters of $\mathfrak{L}_Q$. Then it is said to be the free filterization of $Q \subset (A, \psi)$.

Now, construct two subsets $\mathcal{X}$ and $\mathcal{S}$ of $\mathfrak{L}_Q(\mathbb{Z})$,

$$\mathcal{X} = \{u_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}, \tag{6.1.2}$$

and

$$\mathcal{S} = \{U_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}.$$

Recall that a subset $\mathcal{Y}$ of an arbitrary topological $*$-probability space (B, φ) is said to be a *free (weighted-)semicircular family* in (B, φ), if all elements of $\mathcal{Y}$ are not only mutually free from each other, but also (weighted-)semicircular in (B, φ). (e.g., [10, 27]).

Let $\mathcal{X}$ and $\mathcal{S}$ be in the sense of (6.1.2) in $\mathfrak{L}_Q(\mathbb{Z})$.

(6.1.3) The family $\mathcal{X}$ is a free weighted-semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$.
(6.1.4) The family $\mathcal{S}$ is a free semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$.

Proof The proof of the statements (6.1.3) is done by (4.4) and (6.1.1). The statement (6.1.4) is shown by Theorem 5.1 and (6.1.1). □

By (4.4), the only "j-th" generating operators u_j of the free blocks $\mathfrak{L}_Q(j)$ provide non-vanishing free-distributional data on the free filterization $\mathfrak{L}_Q(\mathbb{Z})$. Thus, we restrict our interests to the Banach $*$-subalgebra $\mathbb{L}_Q$ of the free filterization $\mathfrak{L}_Q(\mathbb{Z})$.

Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free filterization of Q. Define a Banach $*$-subalgebra $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\mathbb{L}_Q \overset{def}{=} \overline{\mathbb{C}[\mathcal{X}]}, \tag{6.1.5}$$

where $\mathcal{X}$ is the free weighted-semicircular family (6.1.3) in $\mathfrak{L}_Q(\mathbb{Z})$, and $\overline{Y}$ are the Banach-topology closures of the subsets Y of $\mathfrak{L}_Q(\mathbb{Z})$. Canonically, construct the Banach $*$-probability space,

$$\mathbb{L}_Q \overset{denote}{=} \left(\mathbb{L}_Q,\ \tau = \tau\mid_{\mathbb{L}_Q}\right), \tag{6.1.6}$$

in $\mathfrak{L}_Q(\mathbb{Z}) = \left(\mathfrak{L}_Q(\mathbb{Z}),\ \tau\right)$.

We call the Banach $*$-algebra $\mathbb{L}_Q$ of (6.1.5), or the Banach $*$-probability space $\mathbb{L}_Q$ of (6.1.6), the semicircular (free-sub-)filterization of $\mathfrak{L}_Q(\mathbb{Z})$.

By (6.1.5) and (6.1.6), the operators of $\mathbb{L}_Q$ are the free random variables of $\mathfrak{L}_Q(\mathbb{Z})$, with possible non-zero free distributions.

Let $\mathbb{L}_Q$ be the semicircular filterization (6.1.5). Then

$$\mathbb{L}_Q = \overline{\mathbb{C}[\mathcal{S}]} \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{u_j\}]} \overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{j\in\mathbb{Z}}{\star}\{u_j\}\right]}, \tag{6.1.7}$$

in $\mathfrak{L}_Q(\mathbb{Z})$, where "$\overset{*\text{-iso}}{=}$" means "being Banach-$*$-isomorphic," and where ($\star$) in the first $*$-isomorphic relation of (6.1.7) means the free-probabilistic free product of [23, 27], and ($\star$) in the second $*$-isomorphic relation of (6.1.7) is the pure-algebraic free product inducing noncommutative free words in $\mathcal{X}$.

Proof The relation (6.1.7) is proven by (6.1.3) and (6.1.4). See [9] for details. □

6.2 Free-Distributional Data Induced by Semicircular Elements

Throughout this section, let (B, φ) be an arbitrary topological $*$-probability space, and suppose there are mutually free, N-many semicircular elements $x_1, \ldots, x_N$ in (B, φ), for $N \in \mathbb{N} \setminus \{1\}$.

By the self-adjointness of these semicircular elements $x_1, \ldots, x_N \in (B, \varphi)$, the free distribution, say

$$\rho \overset{denote}{=} \rho_{x_1,\ldots,x_N}, \tag{6.2.1}$$

of them are characterized by the joint free-moments

$$\bigcup_{n=1}^{\infty} \left(\bigcup_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}^n} \left\{\varphi\left(x_{i_1}x_{i_2}\ldots x_{i_n}\right)\right\} \right) \tag{6.2.1$'$}$$

(e.g., [23, 27]). More precisely, the free distribution ρ of (6.2.1), is characterized by the free-moments

$$\bigcup_{l=1}^{N} \left\{\varphi(x_l^n)\right\}_{n=1}^{\infty}, \tag{6.2.2}$$

and the "mixed" free-moments,

$$\bigcup_{s=2}^{\infty} \left\{ \varphi\left(x_{i_1}^{n_1}x_{i_2}^{n_2}\ldots x_{i_s}^{n_s}\right) \middle| \begin{array}{c} (i_1,\ldots,i_s)\in\{1,\ldots,N\}^s \\ \text{are mixed in } \{1,\ldots,N\}, \\ \text{for all } n_1,\ldots,n_s \in \mathbb{N} \end{array} \right\}, \tag{6.2.3}$$

by $(6.2.1)'$. In this section, to characterize the free distribution ρ of (6.2.1), we investigate the free-distributional data (6.2.2) and (6.2.3).

The free-distributional data (6.2.2) of the free distribution ρ of (6.2.1) are characterized by the semicircularity. i.e.,

$$\varphi\left(x_l^n\right) = \omega_n c_{\frac{n}{2}}, \text{ for all } n \in \mathbb{N}, \tag{6.2.4}$$

for all $l = 1, \ldots, N$.

Proof The formula (6.2.4) is proven by (3.4). □

Now, we concentrate on studying the free-distributional data (6.2.3) of the free distribution ρ of (6.2.1). For any $s \in \mathbb{N} \setminus \{1\}$, we fix a mixed s-tuple I_s,

$$I_s \overset{denote}{=} (i_1, \ldots, i_s) \in \{1, \ldots, N\}^s, \tag{6.2.5}$$

in $\{1, \ldots, N\}$ in the sense that there exists at least one entry i_{k_0} in I_s such that $i_{k_0} \neq i_l$, for some $l \neq k_0$ in $\{1, \ldots, s\}$. For example,

$$I_8 = (1, 1, 3, 2, 3, 2, 2, 1),$$

in $\{1, 2, 3, 4, 5\}^8$.

From the sequence I_s of (6.2.5), define a set

$$I_s] = \{i_1, i_2, \ldots, i_s\}, \tag{6.2.6}$$

without considering repetitions on identical entries. For instance, if I_8 is as above,

$$[I_8] = \{i_1, i_2, \ldots, i_8\},$$

with $i_1 = i_2 = i_8 = 1$, $i_4 = i_6 = i_7 = 2$, and $i_3 = i_5 = 3$.

Then from the set $[I_s]$ of (6.2.6), one can define a unique "noncrossing" partition $\pi_{(I_s)}$ of the lattice $NC\left([I_s]\right)$ of [23], such that (i) starting from the very first entry i_1, construct the largest block V_1 of $\pi_{(I_s)}$, satisfying

$$V_1 = \left(i_{j_1} = i_1,\ i_{j_2},\ \ldots, i_{j_{|V_1|}}\right) \in \pi_{(I_s)}, \tag{6.2.7}$$

$\Longleftrightarrow$

$$i_{j_1} = i_{j_2} = \ldots = i_{j_{|V_1|}} = i_1,$$

and do this process for the very next entry other than $i_{j_1}, \ldots, i_{j_{|V_1|}}$, step-by-step, until they end; (ii) such a partition $\pi_{(I_s)}$ of (i) has to be "maximal" in $NC\left([I_s]\right)$, satisfying the processes of (i) (e.g., [23]). For example, if I_8 and $[I_8]$ are as above, then there exists a noncrossing partition

$$\begin{aligned}\pi_{(I_8)} &= \{(i_1,\ i_2,\ i_8),\ (i_3,\ i_5),\ (i_4),\ (i_6,\ i_7),\ (i_5)\} \\ &= \{(1,\ 1,\ 1),\ (3,\ 3),\ (2),\ (2,\ 2),\ (3)\},\end{aligned}$$

in $NC([I_8])$, satisfying the above conditions (i) and (ii). Note that, even though $i_4 = i_6 = i_7 = 2$, we have to take the separated blocks (i_4) and (i_6, i_7) (after taking (i_3, i_5)), to avoid "crossing."

Now, suppose $\pi_{(I_s)} \in NC([I_s])$ is the noncrossing partition (6.2.7) over the set $[I_s]$ of (6.2.6), and let

$$\pi_{(I_s)} = \{V_1, \ldots, V_t\},$$

where $t \leq s$ and $V_k \in \pi_{(I_s)}$ are the blocks of (ii), satisfying (i), for $k = 1, \ldots, t$.

Then the partition $\pi_{(I_s)}$ is regarded as the joint partition,

$$\pi_{(I_s)} = 1_{|V_1|} \vee 1_{|V_2|} \vee \ldots \vee 1_{|V_t|}, \tag{6.2.8}$$

where $1_{|V_k|}$ are the maximal elements of $NC(V_k)$, for all $k = 1, \ldots, t$, by regarding blocks V_k as independent sets.

Let I_s be in the sense of (6.2.5), and let $x_{i_1}, \ldots, x_{i_s}$ be the corresponding semicircular elements of (B, φ) induced by I_s. Define a free random variable $X[I_s]$ by

$$X[I_s] \overset{def}{=} \prod_{l=1}^{s} x_{i_l} \in (B, \varphi). \tag{6.2.9}$$

If $X[I_s]$ is in the sense of (6.2.9), then

$$\varphi(X[I_s]) = \sum_{\pi \in NC([I_s])} k_\pi$$

by the Möbius inversion, where

$$k_\pi = \prod_{V \in \pi} k_V,$$

with $k_V = k_{|V|}\left(x_{j_{l_1}}, \ldots, x_{j_{l_{|V|}}}\right)$, whenever $V = \left(j_{l_1}, \ldots, j_{l_{|V|}}\right)$, where $k_\bullet(\ldots)$ is the free cumulant on B in terms of φ, and hence, it goes to

$$= \sum_{\pi \in NC([I_s]),\, \pi \leq \pi_{(I_s)}} k_\pi$$

by the mutual-freeness of $x_1, \ldots, x_N$ in (B, φ)

$$= \sum_{(\theta_1, \ldots, \theta_t) \in NC(V_1) \times \ldots \times NC(V_t)} k_{\theta_1 \vee \ldots \vee \theta_t}$$

by (6.2.8)

$$= \sum_{(\theta_1,\ldots,\theta_t)\in NC_2(V_1)\times\ldots\times NC_2(V_t)} k_{\theta_1\vee\ldots\vee\theta_t} \tag{6.2.10}$$

by the semicircularity (3.2) of $x_{i_1}, \ldots, x_{i_s}$ in (A, φ), where $NC_2(X)$ is the subset of the noncrossing-partition lattice $NC(X)$,

$$NC_2(X) = \{\pi \in NC(X) : \forall V \in \pi, |V| = 2\}, \tag{6.2.11}$$

over countable finite sets X.

By (6.2.10) and (6.2.11), if there is at least one $k_0 \in \{1, \ldots, t\}$, such that $\left|V_{k_0}\right|$ is odd in $\mathbb{N}$, then

$$\varphi\left(X[I_s]\right) = 0,$$

by (3.2), where $X[I_s]$ is the free random variable (6.2.9) of (B, φ).

So, the formula (6.2.10) is non-zero, only if

$$|V_k| \in 2\mathbb{N}, \text{ for all } k = 1, \ldots, t, \tag{6.2.12}$$

where $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$.

Moreover, if the condition (6.2.12) is satisfied, then the summands $k_{\theta_1\vee\ldots\vee\theta_t}$ of (6.2.10) satisfy that

$$k_{\theta_1\vee\ldots\vee\theta_t} = \prod_{V\in\theta_1\vee\ldots\vee\theta_t} k_V = \prod_{V\in\theta_1\vee\ldots\vee\theta_t}\left(\prod_{i=1}^{t} 1^{\#(\theta_i)}\right) = 1, \tag{6.2.13}$$

by the semicircularity (3.2), where $\#(\theta_i)$ are the number of blocks of θ_i, for all $i = 1, \ldots, t$. Therefore, if the condition (6.2.12) holds, then

$$\begin{aligned}\varphi\left(X[I_s]\right) &= \sum_{(\theta_1,\ldots,\theta_t)\in NC_2([V_1])\times\ldots\times NC_2([V_t])} 1 \\ &= \left|NC_2\left(V_1\right) \times \ldots \times NC_2\left(V_t\right)\right|,\end{aligned} \tag{6.2.14}$$

by (6.2.10) and (6.2.13), where $|Y|$ are the cardinalities of sets Y.

Let I_s be an s-tuple (6.2.5), and let $X[I_s] = \prod_{l=1}^{s} x_{i_l}$ be the corresponding free random variable (6.2.9) of (B, φ). If

$$\pi_{(I_s)} = 1_{|V_1|} \vee \ldots \vee 1_{|V_t|},$$

in the sense of (6.2.7) and (6.2.8), then

$$\varphi\left(X[I_s]\right) = \begin{cases} \prod_{i=1}^{t} c_{\frac{|V_i|}{2}} & \text{if } |V_k| \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.15}$$

Proof Under hypothesis, by (6.2.14)

$$\varphi\left(X[I_s]\right) = \begin{cases} \left|NC_2\left(V_1\right) \times \ldots \times NC_2\left(V_t\right)\right| & \text{if } |V_k| \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise.} \end{cases}$$

Recall that, for every countable set X, with $|X| \in 2\mathbb{N}$, the subset

$$NC_2\left(X\right) = \{\theta \in NC(X) : \forall V \in \theta, |V| = 2\}$$

is equipotent (or bijective) to the noncrossing-partition lattice $NC\left(\frac{|X|}{2}\right)$ over $\{1, \ldots, \frac{|X|}{2}\}$ (e.g., [8] and [11]). i.e., if $|V_k| \in 2\mathbb{N}$, then

$$\left|NC_2\left(V_k\right)\right| = \left|NC\left(\frac{|V_k|}{2}\right)\right|, \tag{6.2.16}$$

for all $k = 1, \ldots, t$. So, we have

$$\begin{aligned} \varphi\left(X[I_s]\right) &= \begin{cases} \left|NC\left(\frac{|V_1|}{2}\right) \times \ldots \times NC\left(\frac{|V_t|}{2}\right)\right| & \text{if } |V_k| \in 2\mathbb{N}, \\ & \text{for all } k = 1, \ldots, t \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \prod_{l=1}^{t} c_{\frac{|V_l|}{2}} & \text{if } |V_l| \in 2\mathbb{N}, \\ & \text{for all } l = 1, \ldots, t \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{6.2.17}$$

by (6.2.16), because $|NC(X)| = c_{|X|}$, for all finite sets X (e.g., [10, 11, 19, 23]). Therefore, the formula (6.2.15) holds by (6.2.17). □

6.3 Free-Distributional Data on $\mathbb{L}_Q$

Let $\mathbb{L}_Q$ be our semicircular filterization (6.1.5) generated by the free semicircular family $\mathcal{S}$ of (6.1.3). By the structure theorem (6.1.7), all free random variables of $\mathbb{L}_Q$ are the limits of linear combinations of free reduced words

$$W = \prod_{l=1}^{N} U_{j_l}^{n_l}, \text{ for } U_{j_l} \in \mathcal{S}, \forall l = 1, \ldots, N, \tag{6.3.1}$$

in $\mathcal{S}$, for all $N \in \mathbb{N}$, where $n_1, \ldots, n_N \in \mathbb{N}$, and the N-tuple $(j_1, \ldots, j_N)$ is an alternating in $\mathbb{Z}$ in the sense that:

$$j_1 \neq j_2, j_2 \neq j_3, \ldots, j_{N-1} \neq j_N.$$

Let W be a free reduced word (6.3.1) of $\mathbb{L}_Q$ in $\mathcal{S}$.

(6.3.2) If $N = 1$ in (6.3.1), then $\tau(W)$ is characterized by (6.2.4).
(6.3.3) If $N > 1$ in (6.3.1), then $\tau(W)$ is determined by (6.2.15).

Proof The statement (6.3.2) (or (6.3.3)) is proven by (6.2.4) (resp., (6.2.15)), by the universality (3.4) (or (3.2)) of the semicircular law. □

The above theorem fully characterizes the free-distributional data on the semicircular filterization $\mathbb{L}_Q$, by (6.3.2) and (6.3.3).

7 Shifts on $\mathbb{Z}$ and Integer-Shifts on $\mathbb{L}_Q$

In this section, let (A, ψ) be the fixed C^*-probability space containing a family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually-orthogonal projections q_j's having

$$\psi(q_j) \in \mathbb{R}^{\times}, \text{ for all } j \in \mathbb{Z},$$

and let $\mathbb{L}_Q$ be the semicircular filterization (6.1.6).

7.1 (±)-Shifts on $\mathbb{Z}$

Define functions h_+ and h_- on $\mathbb{Z}$ by

$$h_+(j) = j + 1, \tag{7.1.1}$$

and

$$h_{-}(j) = j - 1,$$

for all $j \in \mathbb{Z}$. By the definition (7.1.1), these two functions $h_{\pm}$ are well-defined bijections, which are functional inverses from each other on $\mathbb{Z}$.

For these bijections $h_{\pm}$ of (7.1.1), define the bijections $h_{\pm}^{(n)}$ by

$$h_{\pm}^{(n)} = \underbrace{h_{\pm} \circ h_{\pm} \circ \cdots \circ h_{\pm}}_{n\text{-times}}, \text{ on } \mathbb{Z} \tag{7.1.2}$$

for all $n \in \mathbb{N}$, with identities, $h_{\pm}^{(1)} = h_{\pm}$. It is not difficult to check that

$$h_{\pm}^{(n)}(j) = j \pm n, \text{ for all } j \in \mathbb{Z},$$

for all $n \in \mathbb{N}$. We call the bijections $h_{\pm}^{(n)}$, the n-($\pm$)-shifts on $\mathbb{Z}$.

7.2 Integer-Shifts on $\mathbb{L}_Q$

Let $\mathbb{L}_Q$ be the semicircular filterization, and let $h_{\pm}^{(n)}$ be n-($\pm$)-shifts of (7.1.2) on $\mathbb{Z}$, for all $n \in \mathbb{N}$. Define a "multiplicative" bounded linear transformations $\beta_{\pm}$ on $\mathbb{L}_Q$ by morphisms satisfying that:

$$\beta_{\pm}\left(U_j\right) = U_{h_{\pm}(j)}, \tag{7.2.1}$$

for all $U_j \in \mathcal{S}$, where $\mathcal{S}$ is the free semicircular family (6.1.4).

Let $Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \in \mathbb{L}_Q$, for $U_{j_1}, \ldots, U_{j_N} \in \mathcal{S}$, and $n_1, \ldots, n_N \in \mathbb{N}$, for $N \in \mathbb{N}$. Then

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} U_{j_l \pm 1}^{n_l}. \tag{7.2.2}$$

Proof Let Y be given as above in $\mathbb{L}_Q$. Then, by the multiplicativity of the linear transformations $\beta_{\pm}$ of (7.2.1), one has that

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} \beta_{\pm}\left(U_{j_l}^{n_l}\right) = \prod_{l=1}^{N} \left(\beta_{\pm}\left(U_{j_l}\right)\right)^{n_l} = \prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}.$$

Therefore, the formula (7.2.2) holds. □

By (7.2.2), the free-reduced-word-ness on $\mathbb{L}_Q$ in the generator set $\mathcal{S}$ is preserved by the actions of $\beta_\pm$. Indeed, if an arbitrary N-tuple $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, then the N-tuples $(h_\pm(j_1), \ldots, h_\pm(j_N))$ are alternating in $\mathbb{Z}$, too, for all $N \in \mathbb{N}$.

The linear morphisms $\beta_\pm$ of (7.2.1) are $*$-isomorphisms on $\mathbb{L}_Q$.

Proof By (6.1.7), all elements of the semicircular filterization $\mathbb{L}_Q$ are the limits of linear combinations of free reduced words in the free semicircular family $\mathcal{S}$. So, let's focus on free reduced words of $\mathbb{L}_Q$ in $\mathcal{S}$.

Let $(j_1, \ldots, j_N)$ be an alternating N-tuple in $\mathbb{Z}$ for $N \in \mathbb{N}$, and

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l}, \text{ for } n_1, \ldots, n_N \in \mathbb{N}.$$

By the alternating-ness of $(j_1, \ldots, j_N)$, the above operator Y is a free reduced word with its length-N in $\mathbb{L}_Q$ by (6.1.7). So, by (7.2.2),

$$\beta_\pm(Y) = \prod_{l=1}^{N} U_{h_\pm(j_l)}^{n_l}, \tag{7.2.3}$$

are free reduced words with their lengths-N in $\mathbb{L}_Q$ too. i.e., these multiplicative linear transformations $\beta_\pm$ of (7.2.1) are generator-preserving by (6.1.7), and hence, they are bounded and bijective on $\mathbb{L}_Q$.

Consider now that if Y is as above, then

$$\beta_\pm(Y^*) = \beta_\pm\left(\prod_{l=1}^{N} U_{j_{N-l+1}}^{n_{N-l+1}}\right)$$

by the self-adjointness of $U_{j_1}, \ldots, U_{j_N}$

$$= \prod_{l=1}^{N} U_{h_\pm(j_{N-l+1})}^{n_{N-l+1}}$$

by (7.2.2)

$$= \left(\prod_{l=1}^{N} U_{h_\pm(j_l)}^{n_l}\right)^* = (\beta_\pm(Y))^*. \tag{7.2.4}$$

By (6.1.7) and (7.2.4),

$$\beta_\pm(S^*) = (\beta_\pm(S))^*, \text{ for all } S \in \mathbb{L}_Q.$$

Thus, the bijective bounded multiplicative linear transformations $\beta_\pm$ of (7.2.1) are adjoint-preserving. i.e., they are well-defined $*$-isomorphisms on $\mathbb{L}_Q$. □

The above lemma shows that the $(\pm)$-shifts $h_\pm$ of (7.1.1) on $\mathbb{Z}$ induce the corresponding $*$-isomorphisms $\beta_\pm$ of (7.2.2) on $\mathbb{L}_Q$.

The $*$-isomorphisms $\beta_{\pm}$ of (7.2.1) are called the ($\pm$)-integer-shifts on $\mathbb{L}_Q$.

Let $\beta_{\pm}$ be ($\pm$)-integer-shifts on $\mathbb{L}_Q$. Then one can define the iterated product (or composition) $\beta_{\pm}^n$ of them by

$$\beta_{\pm}^n = \underbrace{\beta_{\pm}\beta_{\pm}\beta_{\pm}\cdots\beta_{\pm}}_{n\text{-times}} on \mathbb{L}_Q, \tag{7.2.5}$$

for all $n \in \mathbb{N}_0$, with axiomatization:

$$\beta_{\pm}^0 = 1_{\mathbb{L}_Q}, \text{ the identity } *\text{-isomorphism on } \mathbb{L}_Q.$$

We call $\beta_{\pm}^n$, the *n-($\pm$)-integer-shifts on* $\mathbb{L}_Q$, for all $n \in \mathbb{N}_0$. Since $\beta_{\pm}$ are $*$-isomorphisms on $\mathbb{L}_Q$, the n-($\pm$)-integer-shifts $\beta_{\pm}^n$ of (7.2.5) are $*$-isomorphisms on $\mathbb{L}_Q$ too, for all $n \in \mathbb{N}_0$.

It is not difficult to check that

$$\beta_{+}^n\beta_{-}^n = 1_{\mathbb{L}_Q} = \beta_{-}^n\beta_{+}^n on \mathbb{L}_Q, \tag{7.2.6}$$

i.e.,

$$\left(\beta_e^n\right)^{-1} = \beta_{-e}^n,$$

for all $e \in \{\pm\}$, and $n \in \mathbb{N}_0$, where f^{-1} mean the *inverses of* f, where

$$-e = \begin{cases} - & \text{if } e = + \\ + & \text{if } e = -. \end{cases}$$

If $\beta_{e_1}^{n_1}$, $\beta_{e_2}^{n_2}$ are in the sense of (7.2.5), for $e_1, e_2 \in \{\pm\}$, and $n_1, n_2 \in \mathbb{N}_0$, then

$$\beta_{e_1}^{n_1}\beta_{e_2}^{n_2} = \beta_{sgn(e_1n_1e_2n_2)}^{|e_1n_1e_2n_2|} \text{ on } \mathbb{L}_Q, \tag{7.2.7}$$

where

$$en = \begin{cases} +n & \text{if } e = + \\ -n & \text{if } e = -, \end{cases}$$

for all $n \in \mathbb{N}_0$, and where $|.|$ is the *absolute value on* $\mathbb{Z}$, and sgn is the *sign map on* $\mathbb{Z}$, defined by

$$sgn(j) = \begin{cases} + & \text{if } j \geq 0 \\ - & \text{if } j < 0, \end{cases}$$

for all $j \in \mathbb{Z}$.

By (7.2.7), one can check that

$$\left(\beta_{e_1}^{n_1}\beta_{e_2}^{n_2}\right)\beta_{e_3}^{n_3} = \beta_{sgn(e_1n_1e_2n_2)}^{|e_1n_1e_2n_2|}\beta_{e_3}^{n_3}$$
$$= \beta_{sgn(e_1n_1e_2n_2e_3n_3)}^{|e_1n_1e_2n_2e_3n_3|} = \beta_{e_1}^{n_1}\beta_{sgn(e_2n_2e_3n_3)}^{|e_2n_2e_3n_3|}$$
$$= \beta_{e_1}^{n_1}\left(\beta_{e_2}^{n_2}\beta_{e_3}^{n_3}\right), \tag{7.2.8}$$

on $\mathbb{L}_Q$, for $e_l \in \{\pm\}$, and $n_l \in \mathbb{N}_0$, for all $l = 1, 2, 3$.

Now, consider the set $\mathfrak{B}$ of all n-($\pm$)-shifts $\beta_{\pm}^{n}$ on $\mathbb{L}_Q$, i.e.,

$$\mathfrak{B} = \{\beta_{\pm}^{n}\}_{n\in\mathbb{N}_0}. \tag{7.2.9}$$

Let $Aut(\mathbb{L}_Q)$ be the *automorphism group of* $\mathbb{L}_Q$,

$$Aut\left(\mathbb{L}_Q\right) = \left(\left\{\alpha : \mathbb{L}_Q \to \mathbb{L}_Q \;\middle|\; \begin{array}{c}\alpha \text{ are} \\ *\text{-isomorphisms} \\ \text{on } \mathbb{L}_Q\end{array}\right\}, \cdot\right), \tag{7.2.10}$$

consisting of all ∗-isomorphisms on $\mathbb{L}_Q$, where the operation $(\cdot)$ is the product (or composition) of ∗-isomorphisms.

Let $\mathfrak{B}$ be the subset (7.2.9) of the automorphism group $Aut(\mathbb{L}_Q)$ of (7.2.10). Then

$$\mathfrak{B} \text{ is a subgroup of } Aut(\mathbb{L}_Q). \tag{7.2.11}$$

Proof Let $\mathfrak{B}$ be in the sense of (7.2.9). Then, by (7.2.7), the operation $(\cdot)$ is closed on $\mathfrak{B}$. So, the algebraic pair $\mathfrak{B} = (\mathfrak{B}, \cdot)$ is well-constructed as a sub-structure of $Aut(\mathbb{L}_Q)$. By (7.2.8), this operation is associative on $\mathfrak{B}$.

Since $\beta_{+}^{0} = 1_{\mathbb{L}_Q} = \beta_{-}^{0}$ in $\mathfrak{B}$, and since

$$\beta_{e}^{n} \cdot 1_{\mathbb{L}_Q} = \beta_{e}^{n} = 1_{\mathbb{L}_Q} \cdot \beta_{e}^{n} on \mathbb{L}_Q,$$

by (7.2.8), for all $e \in \{\pm\}$, and $n \in \mathbb{N}_0$, the set $\mathfrak{B}$ contains its $(\cdot)$-identity $1_{\mathbb{L}_Q}$.

Finally, by (7.2.6), all elements $\beta_{\pm}^{n} \in \mathfrak{B}$ have their unique $(\cdot)$-inverses $\beta_{\mp}^{n} \in \mathfrak{B}$. So, the subset $\mathfrak{B}$ forms a group in $Aut(\mathbb{L}_Q)$. □

By (7.2.11), the system $\mathfrak{B}$ of (7.2.9) is a group.

Let $\mathfrak{B}$ be the subgroup (7.2.9) of the automorphism group $Aut(\mathbb{L}_Q)$. Then

$$\mathfrak{B} \overset{\text{Group}}{=} (\mathbb{Z}, +), \tag{7.2.12}$$

where "$\overset{\text{Group}}{=}$" means "being group-isomorphic," where $(\mathbb{Z}, +)$ is the infinite cyclic abelian group.

Proof Define now a function $\Phi : \mathbb{Z} \to \mathfrak{B}$ by

$$\Phi : j \in \mathbb{Z} \longmapsto \beta_{sgn(j)}^{|j|} \in \mathfrak{B}. - \tag{7.2.13}$$

Then it is not difficult to check this function Φ of (7.2.13) is a group-isomorphism from $\mathbb{Z}$ onto $\mathfrak{B}$. Therefore, the relation (7.2.12) holds. □

The above theorem characterizes the group-structure of $\mathfrak{B}$ by (7.2.12).

We call the group $\mathfrak{B}$, the integer-shift group on $\mathbb{L}_Q$.

7.3 *Free Distributions on $\mathbb{L}_Q$ Affected by $\mathfrak{B}$*

Let $\mathfrak{B}$ be the integer-shift group (7.2.9) acting on the semicircular filterization $\mathbb{L}_Q$. In this section, we consider how our $*$-isomorphisms $\beta_{\pm}^{n} \in \mathfrak{B}$ affect the free probability on the semicircular filterization $\mathbb{L}_Q$.

Take an arbitrary free reduced word Y

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l} of \mathbb{L}_Q \tag{7.3.1}$$

in the free semicircular family $\mathcal{S}$, for $N \in \mathbb{N}$, where the N-tuple $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, and $n_1, \ldots, n_N \in \mathbb{N}$.

Let Y be a free reduced word (7.3.1) of $\mathbb{L}_Q$ in $\mathcal{S}$. Then

$$\tau\left(\beta_e^k(Y)\right) = \tau(Y), \text{ for all } \beta_e^k \in \mathfrak{B}. \tag{7.3.2}$$

Proof First assume that $N = 1$, and hence, $Y = U_{j_1}^{n_1}$ in $\mathbb{L}_Q$. Then, by the semicircularity of $U_{j_1}, U_{j_1 ek} \in \mathcal{S}$ in $\mathbb{L}_Q$, one has that

$$\tau\left(\beta_e^k(Y)\right) = \tau\left(U_{jek}^{n_1}\right) = \omega_{n_1} c_{\frac{n_1}{2}} = \tau\left(U_{j_1}^{n_1}\right), \tag{7.3.3}$$

for all $\beta_e^k \in \mathfrak{B}$. Therefore, the statement (7.3.2) holds.

Assume now that $N > 1$ in $\mathbb{N}$. Then $\beta_e^k(Y) = \prod_{l=1}^{N} U_{j_l ek}^{n_k}$ is a free reduced word with the same length-N in $\mathbb{L}_Q$, for all $\beta_e^k \in \mathfrak{B}$. Now, let

$$I_s = \left(\underbrace{j_1, \ldots, j_1}_{n_1\text{-times}}, \underbrace{j_2, \ldots, j_2}_{n_2\text{-times}}, \ldots, \underbrace{j_N, \ldots, j_N}_{n_N\text{-times}}\right)$$

be the s-tuple in the sense of (6.2.5), satisfying

$$Y = X[I_s] in \mathbb{L}_Q, \text{ for some } s \geq N,$$

where $X[I_s]$ is in the sense of (6.2.9).

Similarly, let

$$I_{s'} = \left(\underbrace{j_1 ek, \ldots, j_1 ek}_{n_1\text{-times}}, \underbrace{j_2 ek, \ldots, j_2 ek}_{n_2\text{-times}}, \ldots, \underbrace{j_N ek, \ldots, j_N ek}_{n_N\text{-times}} \right)$$

be the s'-tuple of (6.2.5), satisfying

$$\beta_e^k(Y) = X[I_{s'}] \text{ in } \mathbb{L}_Q,$$

for $\beta_e^k \in \mathfrak{B}$, where $X[I_{s'}]$ is in the sense of (6.2.9).

Since Y and $\beta_e^k(Y)$ are the free reduced words with same lengths-N, one has

$$s = s' in \mathbb{N}, and \pi(I_s) = \pi(I_{s'}) in NC([I_s]),$$

where $\pi(I_s)$ and $\pi(I_{s'})$ are the noncrossing partitions of (6.2.7).

By the semicircularity of $U_{j_1}, \ldots, U_{j_N}, U_{j_1 ek}, \ldots, U_{j_N ek} \in \mathcal{S}$ in $\mathbb{L}_Q$, we have

$$\tau\left(\beta_e^k(Y)\right) = \tau\left(X[I_{s'}]\right) = \tau\left(X[I_s]\right) = \tau(Y),$$

by (6.2.15), for all $\beta_e^k \in \mathfrak{B}$. So, the relation (7.3.2) holds. □

The above theorem shows that $\mathfrak{B}$ preserves the free probability on $\mathbb{L}_Q$. i.e.,

$$\tau\left(\beta_e^k(T)\right) = \tau(T), \text{ for all } T \in \mathbb{L}_Q,$$

by (6.1.7) and (6.3.2), for all $\beta_e^k \in \mathfrak{B}$.

8 Semicircular Elements Induced by Multi Projections

In this section, we show that if there are N-many, mutually orthogonal projections in an arbitrary C^*-probability space, then there exists a corresponding free semicircular family $\mathcal{S}^{(N)}$ of mutually free, N-many semicircular elements, induced by the projections in a certain Banach ∗-probability space $\mathbb{L}_Q^{(N)}$, for any

$$N \in \mathbb{N}_{>1}^{\infty} = (\mathbb{N} \setminus \{1\}) \cup \{\infty\}.$$

We consider how the integer-shift group $\mathfrak{B}$ acts on $\left\{\mathbb{L}_Q^{(N)}\right\}_{N \in \mathbb{N}_{>1}^{\infty}}$.

8.1 A Free Semicircular Family $\mathcal{S}^{(N)}$ Induced by N-many Projections

Let (A_o, ψ_o) be a C^*-probability space containing its N-many, mutually orthogonal projections

$$\mathbf{Q}_o = \{q_1^o, \ldots, q_N^o\} \tag{8.1.1}$$

for $N \in \mathbb{N}_{>1}^{\infty}$, and let

$$Q_o = C^* (\mathbf{Q}_o) \subseteq A_o \tag{8.1.2}$$

be the C^*-subalgebra of A generated by the family $\mathbf{Q}_o$ of (8.1.1).

Suppose

$$\psi_o \left(q_k^o\right) \in \mathbb{R}^{\times} \text{ in } \mathbb{C}, \forall k = 1, \ldots, N. \tag{8.1.3}$$

Let Q_o be the C^*-subalgebra (8.1.2) of A_o. Then

$$Q_o \overset{*\text{-iso}}{=} \underset{l=1}{\overset{N}{\oplus}} \left(\mathbb{C} \cdot q_l^o\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus N}. \tag{8.1.4}$$

Proof By the mutual orthogonality on the generator set $\mathbf{Q}_o$ of Q_o, the structure theorem (8.1.4) is shown. □

Suppose there is a C^*-probability space (A, ψ) containing the family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal $|\mathbb{Z}|$-many projections q_j's, satisfying

$$\psi \left(q_j\right) \in \mathbb{R}^{\times} in \mathbb{C}, \text{ for all } j \in \mathbb{Z}. \tag{8.1.5}$$

Assume further that there exist projections $q_{j_1}, \ldots, q_{j_N} \in \mathbf{Q}$, such that

$$\psi \left(q_{j_l}\right) = \psi_o \left(q_l^o\right) in \mathbb{R}^{\times}, \tag{8.1.6}$$

for all $l = 1, \ldots, N$, where ψ_o is the linear functional on the C^*-algebra A_o, satisfying (8.1.3). For convenience, without loss of generality, we re-index the subfamily

$$\{q_{j_1}, \ldots, q_{j_N}\} of \mathbf{Q} \tag{8.1.7}$$

by

$$\{q_1, \ldots, q_N\} \text{ in } \mathbf{Q}.$$

Let Q_o be the C^*-subalgebra (8.1.2) of a fixed C^*-probability space (A_o, ψ_o). If there exists a C^*-probability space (A, ψ), satisfying (8.1.6) under the re-indexing process (8.1.7), then there exists a Banach ∗-subalgebra

$$\mathbb{L}_Q^{(N)} \overset{*\text{-iso}}{=} \underset{l=1}{\overset{N}{\star}} \overline{\mathbb{C}[\{U_l\}]}$$

of the semicircular filterization $\mathbb{L}_Q$ of (6.1.5).

Proof Let $Q = C^*(\mathbf{Q})$ be the C^*-subalgebra (3.7) of the C^*-probability space (A, ψ) satisfying (8.1.6), under (8.1.7). First, define a linear morphism

$$\Psi : Q_o \to Q$$

by

$$\Psi\left(\sum_{l=1}^{N} t_l q_l^o\right) \overset{def}{=} \sum_{l=1}^{N} t_l q_l + \sum_{j \in \mathbb{Z} \setminus \{1,\ldots,N\}} 0 \cdot q_j.$$

Then it is a well-defined injective (embedding) ∗-homomorphism from Q_o into Q, by (3.8) and (8.1.4). So, one can have the semicircular elements

$$U_l = \mathbf{1} \otimes q_l = \mathbf{1} \otimes \Psi\left(q_l^o\right) \in \mathbb{L}_Q, \tag{8.1.8}$$

in the free semicircular family $\mathcal{S}$ generating $\mathbb{L}_Q$, for all $l = 1, \ldots, N$.

By (8.1.8), one can define Banach ∗-subalgebra of $\mathbb{L}_Q$ by

$$\begin{aligned} \mathbb{L}_Q^{(N)} &\overset{def}{=} \overline{\mathbb{C}[\{U_1, \ldots, U_N\}]} \\ &\overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\{\mathbf{1} \otimes \Psi(q_l^o) : l = 1, \ldots, N\}\right]} \\ &\overset{*\text{-iso}}{=} \underset{l=1}{\overset{N}{\star}} \overline{\mathbb{C}[\{\mathbf{1} \otimes \Psi(q_j^o)\}]} \overset{*\text{-iso}}{=} \underset{l=1}{\overset{N}{\star}} \overline{\mathbb{C}[\{U_l\}]} \end{aligned} \tag{8.1.9}$$

So, the family $\mathbf{Q}_o$ of (8.1.1) induces a Banach ∗-probability space,

$$\mathbb{L}_Q^{(N)} \overset{denote}{=} \left(\mathbb{L}_Q^{(N)}, \; \tau = \tau \mid_{\mathbb{L}_Q^{(N)}}\right),$$

generated by the free semicircular family

$$\mathcal{S}^{(N)} = \{U_l = \mathbf{1} \otimes \Psi(q_l^o)\}_{l=1}^{N}, \tag{8.1.10}$$

as a free-probabilistic sub-structure of $\mathbb{L}_Q$. □

As we briefly discussed in [8], whenever such a family $\mathbf{Q}_o$ of (8.1.1) in a C^*-probability space (A_o, ψ_o) is fixed, in fact, there does exists a C^*-probability space

(A, ψ), having a family **Q** of mutually orthogonal $|\mathbb{Z}|$-many projections, such that the condition (8.1.6) holds artificially-or-naturally. i.e., whenever a family $\mathbf{Q}_o$ of (8.1.1) is fixed in a C^*-probability space (A_o, ψ_o), there does exist a family **Q** of mutually orthogonal $|\mathbb{Z}|$-many projections in a C^*-probability space (Q, ψ) (or (A, ψ) with $Q \subseteq A$), such that Q automatically satisfies (8.1.6) (and (8.1.7)).

By the above lemma and remark, one obtains the following theorem.

Let (A_o, ψ_o) be an arbitrary C^*-probability space containing mutually orthogonal N-many projections $q_1, \ldots, q_N$, satisfying (8.1.3), for $N \in \mathbb{N}_{>1}^{\infty}$. Then there exists a free semicircular family $\mathcal{S}^{(N)}$ induced by $\{q_k\}_{k=1}^{N}$, generating a Banach $*$-probability space $\mathbb{L}_Q^{(N)}$, as a free-probabilistic sub-structure of our semicircular filterization $\mathbb{L}_Q$.

Proof The proof is done by (8.1.8), (8.1.9), (8.1.10), and Remark 8.1. □

8.2 Restricted Integer-Shifts On $\mathbb{L}_Q^{(N)}$

For an arbitrarily fixed $N \in \mathbb{N}_{>1}^{\infty}$, let

$$\mathbb{L}_Q^{(N)} = (\mathbb{L}_Q^{(N)}, \tau), \text{ with } \tau = \tau \mid_{\mathbb{L}_Q^{(N)}},$$

be the Banach $*$-probability space (8.1.9) generated by the free semicircular family $\mathcal{S}^{(N)}$ of (8.1.10).

Since the integer-shift group $\mathfrak{B}$ of (7.2.9) acts on $\mathbb{L}_Q$, one can restrict the action on $\mathbb{L}_Q$ to that on $\mathbb{L}_Q^{(N)}$.

Let $\beta_e^k \in \mathfrak{B}$ be an integer-shift on $\mathbb{L}_Q$, and let $U_l \in \mathcal{S}^{(N)}$ be a semicircular element, generating $\mathbb{L}_Q^{(N)}$, for $l = 1, \ldots, N$. Denote the restriction $\beta_e^k \mid_{\mathbb{L}_Q^{(N)}}$ simply by β_e^k. If $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, then

$$\beta_e^k (U_l) = \begin{cases} U_{lek} & \text{if } 1 \leq lek \leq N \\ O & \text{otherwise,} \end{cases} \tag{8.2.1}$$

in $\mathbb{L}_Q^{(N)}$, where O is the zero element of $\mathbb{L}_Q^{(N)}$. Meanwhile, if $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$, then

$$\beta_e^k (U_l) = \begin{cases} U_{l+k} & \text{if } e = + \\ U_{l-k} & \text{if } e = -, \text{ and } l > k \\ O & \text{if } e = -, \text{ and } l \leq k, \end{cases} \tag{8.2.2}$$

in $\mathbb{L}_Q^{(N)}$.

Proof First, assume that $N < \infty$ in $\mathbb{N}^{\infty}_{>1}$, and fix $l \in \{1, \ldots, N\}$ arbitrarily, and let β_e^k be the restriction $\beta_e^k \mid_{\mathbb{L}_Q^{(N)}}$ on $\mathbb{L}_Q^{(N)}$, for $\beta_e^k \in \mathfrak{B}$. Then, for a semicircular elements $U_l \in \mathcal{S}^{(N)}$, generating $\mathbb{L}_Q^{(N)}$, one has that: if $e = +$, then

$$\beta_e^k\,(U_l) = \begin{cases} U_{l+k} & \text{if } l + k \leq N \\ O & \text{if } l + k > N; \end{cases} \tag{8.2.3}$$

and if $e = -$, then

$$\beta_e^k\,(U_l) = \begin{cases} U_{l-k} & \text{if } l - k \geq 1 \\ O & \text{if } l - k < 1, \end{cases} \tag{8.2.4}$$

in $\mathbb{L}_Q^{(N)}$. By (8.2.3) and (8.2.4),

$$\beta_e^k(U_l) = \begin{cases} U_{lek} & \text{if } 1 \leq lek \leq N \\ O & \text{otherwise,} \end{cases} \tag{8.2.5}$$

in $\mathbb{L}_Q^{(N)}$. Therefore, the formula (8.2.1) holds by (8.2.5).

Now, assume that $N = \infty$ in $\mathbb{N}^{\infty}_{>1}$. Then, the restriction β_e^k on $\mathbb{L}_Q^{(N)}$ satisfies that: if $e = +$, then

$$\beta_e^k\,(U_l) = U_{l+k}; \tag{8.2.6}$$

if $e = -$, then

$$\beta_e^k(U_l) = \begin{cases} U_{l-k} & \text{if } l - k \geq 1 \\ O & \text{if } l - k < 1, \end{cases} \tag{8.2.7}$$

in $\mathbb{L}_Q^{(N)}$. Therefore, the formula (8.2.2) is shown by (8.2.6) and (8.2.7). □

The above lemma not only shows how the restricted action of the integer-shift group $\mathfrak{B}$ on $\mathbb{L}_Q^{(N)}$ acts on the free generator set $\mathcal{S}^{(N)}$ of $\mathbb{L}_Q^{(N)}$, but also demonstrates that the restrictions are no longer ∗-isomorphisms on $\mathbb{L}_Q^{(N)}$.

Let B be an arbitrary topological ∗-algebra. Then the (∗-)*homomorphism semigroup* $Hom(B)$ is defined to be the semigroup (under product)

$$Hom(B) = \{f : f is a * \text{-homomorphism on } B\}.$$

Since the zero map on B is contained in $Hom(B)$, it cannot be a group (under product), however, it forms a well-defined semigroup (or a *monoid* containing its identity, the identity map on B). Of course, $Aut(B)$ is a subset of $Hom(B)$.

Notation From below, we denote the family of restricted integer-shifts on $\mathbb{L}_Q^{(N)}$ by $\mathfrak{B}^{(N)}$, i.e.,

$$\mathfrak{B}^{(N)} = \left\{ \beta_e^k \mid_{\mathbb{L}_Q^{(N)}} \left| \begin{array}{c} \beta_e^k \in \mathfrak{B}, \text{ with} \\ e \in \{\pm\}, \ k \in \mathbb{N}_0 \end{array} \right. \right\}. \tag{8.2.8}$$

Also, for convenience, we denote the restrictions $\beta_e^k \mid_{\mathbb{L}_Q^{(N)}} \in \mathfrak{B}^{(N)}$ by β_e^k. □

Let $\mathfrak{B}^{(N)}$ be the set (8.2.8) of restricted integer-shifts on $\mathbb{L}_Q^{(N)}$. Then

$$\mathfrak{B}^{(N)} \subseteq Hom\left(\mathbb{L}_Q^{(N)}\right). \tag{8.2.9}$$

Proof First, assume that $N < 1$ in $\mathbb{N}_{>1}^{\infty}$. If $\beta_e^k \in \mathfrak{B}^{(N)}$ satisfies

$$1ek < 1, or Nek > N, \tag{8.2.10}$$

then such a restricted integer-shift β_e^k satisfies that

$$\beta_e^k\left(U_l\right) = O \text{ in } \mathbb{L}_Q^{(N)}, \text{ for all } l = 1, \ldots, N.$$

So, if (8.2.10) holds, then β_e^k is identified with the zero $*$-homomorphism $0_Q^{(N)}$ on $\mathbb{L}_Q^{(N)}$, i.e.,

$$\beta_e^k(T) = 0_Q^{(N)}(T) = O \text{ in } \mathbb{L}_Q^{(N)},$$

for all $T \in \mathbb{L}_Q^{(N)}$, by (8.1.9), (8.2.1), (8.2.8), and (8.2.10).

And hence, all elements β_e^k of $\mathfrak{B}^{(N)}$ satisfying (8.2.10) satisfy

$$\beta_e^k = 0_Q^{(N)} \in Hom\left(\mathbb{L}_Q^{(N)}\right). \tag{8.2.11}$$

Suppose that $\beta_e^k \in \mathfrak{B}^{(N)}$ satisfies

$$1 \leq lek \leq N, \text{ for some } l \in \{1, \ldots, N\}. \tag{8.2.12}$$

Then, by (8.2.1), the morphism β_e^k is a well-defined non-zero $*$-homomorphism on $\mathbb{L}_Q^{(N)}$, i.e., if (8.2.12) holds, then

$$\beta_e^k \in Hom(\mathbb{L}_Q^{(N)}). \tag{8.2.13}$$

So, if $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, then

$$\mathfrak{B}^{(N)} \subseteq Hom\left(\mathbb{L}_Q^{(N)}\right), \tag{8.2.14}$$

by (8.2.11) and (8.2.13).

Assume now that $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$. If $\beta_+^k \in \mathfrak{B}^{(N)}$, then

$$\beta_+^k \in Hom\left(\mathbb{L}_Q^{(N)}\right), \tag{8.2.15}$$

by (8.2.2); if $\beta_-^k \in \mathfrak{B}^{(N)}$, then

$$\beta_-^k \in Hom\left(\mathbb{L}_Q^{(N)}\right), \tag{8.2.16}$$

again by (8.2.2). Therefore, if $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$, then

$$\mathfrak{B}^{(N)} \subseteq Hom\left(\mathbb{L}_Q^{(N)}\right), \tag{8.2.17}$$

by (8.2.15) and (8.2.16).

In conclusion, if $\mathfrak{B}^{(N)}$ is the family (8.2.8) of restricted integer-shifts on $\mathbb{L}_Q^{(N)}$, then

$$\mathfrak{B}^{(N)} \subseteq Hom\left(\mathbb{L}_Q^{(N)}\right), \forall N \in \mathbb{N}_{>1}^{\infty}, \tag{8.2.18}$$

by (8.2.14) and (8.2.17). i.e., the relation (8.2.9) holds, by (8.2.18). □

The set-inclusion (8.2.9) shows that all restricted integer-shifts of $\mathfrak{B}^{(N)}$ are well-defined ∗-homomorphisms on $\mathbb{L}_Q^{(N)}$. However, by (8.2.11), they cannot be ∗-isomorphisms on $\mathbb{L}_Q^{(N)}$, in general.

By (8.2.11), if $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, then many restricted integer-shifts $\beta_e^k \in \mathfrak{B}^{(N)}$ become the zero ∗-homomorphism $0_Q^{(N)}$ of $Hom(\mathbb{L}_Q^{(N)})$. Consider the case where $N = \infty$. If $N = \infty$, then all restricted integer-shifts $\beta_e^k \in \mathfrak{B}^{(\infty)}$ are non-zero in $Hom(\mathbb{L}_Q^{(\infty)})$. Indeed, since there are infinitely many generators $U_1, U_2, U_3, \ldots$ of $\mathbb{L}_Q^{(N)}$, there always exists $n \in \mathbb{N}$, such that $\beta_e^k(U_n) \neq O$ in $\mathbb{L}_Q^{(N)}$, by the Zorn's lemma. In conclusion,

$$0_Q^{(N)} \in \mathfrak{B}^{(N)} \Longleftrightarrow N < \infty in \mathbb{N}_{>1}^{\infty}.$$

We call $\mathfrak{B}^{(N)}$, the restricted(-integer)-shift family on $\mathbb{L}_Q^{(N)}$.

As we discussed above, the restricted-shift family $\mathfrak{B}^{(N)}$ is not a group in general. Let $\mathfrak{B}^{(N)}$ be the restricted-shift family (8.2.8) on $\mathbb{L}_Q^{(N)}$, for $N \in \mathbb{N}_{>1}^{\infty}$. Then

$$\mathfrak{B}^{(N)} \text{ is a sub-semigroup of } Hom\left(\mathbb{L}_Q^{(N)}\right). \tag{8.2.19}$$

Proof Let $\beta_{e_1}^{k_1}, \beta_{e_2}^{k_2} \in \mathfrak{B}^{(N)}$. Then

$$\beta_{e_1}^{k_1}\beta_{e_2}^{k_2} = \begin{cases} \beta_{sgn(e_1k_1e_2k_2)}^{|e_1k_1e_2k_2|} & \text{or} \\ 0_Q^{(N)}, & (\text{if } N < \infty) \end{cases} \tag{8.2.20}$$

in $\mathfrak{B}^{(N)}$. So, the pair $\left(\mathfrak{B}^{(N)}, \ \cdot\right)$ forms a well-defined algebraic structure.

Observe that

$$\left(\beta_{e_1}^{k_1}\beta_{e_2}^{k_2}\right)\beta_{e_3}^{k_3} = \beta_{sgn(e_1k_1e_2k_2e_3k_3)}^{|e_1k_1e_2k_2e_3k_3|} = \beta_{e_1}^{k_1}\left(\beta_{e_2}^{k_2}\beta_{e_3}^{k_3}\right),$$

by (8.2.20), for all $\beta_{e_i}^{k_i} \in \mathfrak{B}^{(N)}$, for all $i = 1, 2, 3$. Therefore, $\mathfrak{B}^{(N)}$ is a sub-semigroup of $Hom\left(\mathbb{L}_Q^{(N)}\right)$. □

Since $\mathfrak{B}^{(N)}$ contains the identity map $1_Q^{(N)} \in Hom\left(\mathbb{L}_Q^{(N)}\right)$, it actually forms a monoid by (8.2.19).

9 Restricted-Shift Families $\mathfrak{B}^{(N)}$ Acting on $\mathbb{L}_Q^{(N)}$

In this section, we fix $N \in \mathbb{N}_{>1}^{\infty}$, and consider how the restricted-shift family $\mathfrak{B}^{(N)}$ deform the original free-distributional data on the Banach $*$-probability space $\mathbb{L}_Q^{(N)}$ of (8.1.9), for $N \in \mathbb{N}_{>1}^{\infty}$.

Let $\beta_e^k \in \mathfrak{B}^{(N)}$ be a restricted shift, and let $U_l \in \mathcal{S}^{(N)}$ be a generating semicircular element of $\mathbb{L}_Q^{(N)}$, for $l \in \{1, \ldots, N\}$. Then the free random variable $W_l = \beta_e^k(U_l) \in \mathbb{L}_Q^{(N)}$ is either a semicircular element in $\mathcal{S}^{(N)}$, or the zero free random variable O of $\mathbb{L}_Q^{(N)}$.

Proof Under hypothesis,

$$W_l = \begin{cases} U_{lek} \in \mathcal{S}^{(N)} & \text{if } 1 \leq lek \leq N \\ O, & \text{otherwise,} \end{cases}$$

in $\mathbb{L}_Q^{(N)}$, by (8.2.1) and (8.2.2). So, if $W_l = U_{lek} \in \mathcal{S}^{(N)}$, then it is semicircular, while, if $W_l = O$, then it follows the zero free distribution in $\mathbb{L}_Q^{(N)}$. □

The above lemma characterizes how the semigroup-action of the restricted-shift family $\mathfrak{B}^{(N)}$ affects the free probability on $\mathbb{L}_Q^{(N)}$. Indeed, one has the following theorem.

Let $U_{l_1}, \ldots, U_{l_s} \in \mathcal{S}^{(N)}$ in $\mathbb{L}_Q^{(N)}$, for

$$I_s = (l_1, \ldots, l_s) \in \{1, \ldots, N\}^s,$$

for $s \in \mathbb{N}$, without considering repetition, and let $\beta_e^k \in \mathfrak{B}^{(N)}$. Define a free random variable $X[I_s]$ by

$$X[I_s] = \prod_{l=1}^{s} U_{l_s} \in \mathbb{L}_Q^{(N)}. \tag{9.0.1}$$

Then one has either

$$\tau\left(\beta_e^k\left(X[I_s]\right)\right) = \tau\left(X[I_s]\right), \text{ satisfying (6.2.15)} \tag{9.0.2}$$

or

$$\tau\left(\beta_e^k\left(X[I_s]\right)\right) = 0.$$

Proof Let $X[I_s] \in \mathbb{L}_Q^{(N)}$ be in the sense of (9.0.1). Then it is a free (reduced, or non-reduced) word in $\mathcal{S}^{(N)}$, by (8.1.9).

Assume first that there exists at least one entry l_p in the s-tuple I_s such that

$$\beta_e^k\left(U_{l_p}\right) = U_{l_pek} \notin \mathcal{S}^{(N)}, \Longleftrightarrow \beta_e^k\left(U_{l_p}\right) = O in \mathbb{L}_Q^{(N)}.$$

Then $\beta_e^k\left(X[I_s]\right) = O$, because it contains a factor $\beta_e^k\left(U_{l_p}^{n_p}\right)$, for some $n_p \in \mathbb{N}$, by (8.2.1), and hence,

$$\tau\left(\beta_e^k\left(X[I_s]\right)\right) = 0.$$

Meanwhile, if $1 \leq l_i + k \leq N$, for all $i = 1, \ldots, s$, i.e., if

$$\beta_e^k\left(U_{l_i}\right) = U_{l_i+k} \neq O in \mathcal{S}^{(N)} \subset \mathbb{L}_Q^{(N)}, \forall i = 1, \ldots, s,$$

then

$$\tau\left(\beta_e^k\left(X[I_s]\right)\right) = \tau\left(X[I_s]\right),$$

by Lemma 9.1. So, the formula (9.0.2) holds. □

9.1 Banach-Space Operators on $\mathbb{L}_Q^{(N)}$ Generated by $\mathfrak{B}^{(N)}$

Let $\mathfrak{B}^{(N)}$ be the restricted-shift family on $\mathbb{L}_Q^{(N)}$, for a fixed $N \in \mathbb{N}_{>1}^{\infty}$. Since all restricted-shifts β_e^k are $*$-homomorphisms on $\mathbb{L}_Q^{(N)}$ by (8.2.19), they are bounded (multiplicative) linear transformations on $\mathbb{L}_Q^{(N)}$, and hence, they are understood as Banach-space operators on the Banach space $\mathbb{L}_Q^{(N)}$. i.e.,

$$\mathfrak{B}^{(N)} \subset Hom(\mathbb{L}_Q^{(N)}) \subseteq B(\mathbb{L}_Q^{(N)}), \tag{9.1.1}$$

where $B\left(\mathbb{L}_Q^{(N)}\right)$ is the operator space of [12].

By (9.1.1), we now regard the restricted-shift family $\mathfrak{B}^{(N)}$ as a subset of the operator space $B\left(\mathbb{L}_Q^{(N)}\right)$. Define a (closed) subspace $\mathcal{A}_N$ of $B\left(\mathbb{L}_Q^{(N)}\right)$ by the topological vector space spanned by the semigroup $\mathfrak{B}^{(N)}$,

$$\mathcal{A}_N \overset{def}{=} \overline{span_{\mathbb{C}}\left(\mathfrak{B}^{(N)}\right)}^{\|.\|}, \tag{9.1.2}$$

where $\overline{Z}^{\|.\|}$ are the operator-norm closures of subsets Z of $B\left(\mathbb{L}_Q^{(N)}\right)$.

Note that, since $\mathcal{B}^{(N)}$ is a semigroup embedded in $B\left(\mathbb{L}_Q^{(N)}\right)$, in fact, the subspace $\mathcal{A}_N$ of (9.1.2) is identified with

$$\mathcal{A}_N = \overline{\mathbb{C}\left[\mathfrak{B}^{(N)}\right]}^{\|.\|},$$

in $B\left(\mathbb{L}_Q^{(N)}\right)$. i.e., the subspace $\mathcal{A}_N$ of (9.1.2) is a Banach algebra in $B\left(\mathbb{L}_Q^{(N)}\right)$.

Define now a unary operation $(*)$ on $\mathcal{A}_N$ by

$$\left(\sum_{\beta_e^k \in \mathfrak{B}^{(N)}} t_e^k \beta_e^k\right)^* = \sum_{\beta_e^k \in \mathfrak{B}^{(N)}} \overline{t_e^k} \beta_{-e}^k, \tag{9.1.3}$$

where

$$t_e^k = t_{\beta_e^k} \in \mathbb{C}, \text{ with their conjugates } \overline{t_e^k},$$

for all $e \in \{\pm\}$, $k \in \mathbb{N}_0$. Then it is not hard to check the operation (9.1.3) is an adjoint on $\mathcal{A}_N$.

Every operator of $\mathcal{A}_N$ is an adjointable operator in $B\left(\mathbb{L}_Q^{(N)}\right)$.

Proof Since the operation $(*)$ of (9.1.3) is an adjoint, the Banach algebra $\mathcal{A}_N$ is a Banach $*$-algebra in $B\left(\mathbb{L}_Q^{(N)}\right)$. So, every Banach-space operator $T \in \mathcal{A}_N$ is adjointable (in the sense of [12]) with its adjoint $T^* \in \mathcal{A}_N$. □

The Banach $*$-algebra $\mathcal{A}_N$ of (9.1.2) is called the (restricted-)shift-operator algebra on $\mathbb{L}_Q^{(N)}$. All elements of $\mathcal{A}_N$ are said to be (restricted-)shift operators on $\mathbb{L}_Q^{(N)}$.

9.2 *The Shift Operators β_e^k of $\mathcal{A}_N$*

In this section, we concentrate on studying integer-shifts $\beta_e^k \in \mathfrak{B}^{(N)}$, as shift operators of $\mathcal{A}_N$ acting on $\mathbb{L}_Q^{(N)}$, for $N \in \mathbb{L}_Q^{(N)}$.

9.2.1 Case where $N < \infty$

In this sub-section, we assume $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, and let $\mathcal{A}_N$ be the shift-operator algebra in $B\left(\mathbb{L}_Q^{(N)}\right)$. Note that the shift operator $\beta_e^k \in \mathcal{A}_N$ satisfies

$$\beta_+^k\,(U_l) \in \{O, U_l, U_{l+1}, \ldots, U_N\}, \tag{9.2.1}$$

and

$$\beta_-^k\,(U_l) \in \{O, U_1, \ldots, U_{l-1}, U_l\},$$

in $\mathbb{L}_Q^{(N)}$, by (8.2.1), for all $k \in \mathbb{N}_0$, and $l = 1, \ldots, N < \infty$.

By (9.2.1), if $k > N$ in $\mathbb{N}_0$, then

$$\beta_+^k\,(U_l) = O, \text{ for all } l = 1, \ldots, N,$$

and

$$\beta_-^k\,(U_l) = O, \text{ for all } l = 1, \ldots, N,$$

i.e., if $k > N$ in $\mathbb{N}_0$, then

$$\beta_e^k = 0_Q^{(N)} \text{ in } \mathfrak{B}^{(N)} \subset \mathcal{A}_N. \tag{9.2.2}$$

as in (8.2.12).

Let $t\beta_e \in \mathcal{A}_N$ be a shift operator for $t \in \mathbb{C}^\times$, and $e \in \{\pm\}$. Then

$$(t\beta_e)^k = 0_Q^{(N)} \text{ in } \mathcal{A}_N \Longleftrightarrow k > N in \mathbb{N}_0. \tag{9.2.3}$$

Proof ($\Leftarrow$) Let $t \in \mathbb{C}^\times$, and $t\beta_e \in \mathcal{A}_N$, a shift operator, where $\beta_e \in \mathfrak{B}^{(N)}$. Suppose $k > N$ in $\mathbb{N}_0$. Then, by (9.2.2)

$$(t\beta_e)^k = t^k \beta_e^k = t^k 0_Q^{(N)} = 0_Q^{(N)}.$$

($\Rightarrow$) Assume now that $(t\beta_e)^k = 0_Q^{(N)}$ in $\mathcal{A}_N$, for any arbitrary $t \in \mathbb{C}^\times$. Then

$$(t\beta_e)^k = t^k \beta_e^k = 0_Q^{(N)} \Longleftrightarrow \beta_e^k = 0_Q^{(N)} \text{ in } \mathfrak{B}^{(N)} \subset \mathcal{A}_N.$$

So, we focus on β_e^k. If $k \leq N$ in $\mathbb{N}_0$, then

$$\beta_e^k (U_l) \neq O, forsomel \in \{1, \ldots, N\},$$

by (9.2.1), implying that

$$\beta_e^k \neq 0_Q^{(N)} \in \mathfrak{B}^{(N)}, \text{ in } \mathcal{A}_N.$$

Therefore, if $\beta_e^k = 0_Q^{(N)}$ in $\mathcal{A}_N$, then $k > N$ in $\mathbb{N}_0$. □

The above theorem shows that, if $N < \infty$ in $\mathbb{N}_{>1}^\infty$, then the sequence $(t\beta_e^n)_{n=1}^\infty$ converges to $0_Q^{(N)} \in \mathcal{A}_N$, under the operator-norm topology for $\mathcal{A}_N$ (inherited from that for $B\left(\mathbb{L}_Q^{(N)}\right)$), as $n \to \infty$. i.e., the shift operators $t\beta_e \in \mathcal{A}_N$ (for all $t \in \mathbb{C}^\times$) act like nilpotent operators (e.g., [14]), whenever $N < \infty$.

9.2.2 Case where $N = \infty$

In this sub-section, let $N = \infty$ in $\mathbb{N}_{>1}^\infty$, and let $\mathcal{A}_\infty$ be the shift-operator algebra acting on $\mathbb{L}_Q^{(\infty)}$. One can get that

$$\beta_+^k (U_l) \in \{U_l, U_{l+1}, U_{l+2}, U_{l+3}, \ldots\}, \tag{9.2.4}$$

and

$$\beta_-^k (U_l) \in \{O, U_1, \ldots, U_l\},$$

by (8.2.2), for all $k \in \mathbb{N}_0$, for all $l \in \mathbb{N}$.

Let $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$, and $\beta_e \in \mathcal{A}_\infty$, a shift operator for $e \in \{\pm\}$. Then

$$(\beta_e)^n \neq 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \tag{9.2.5}$$

for all $n \in \mathbb{N}_0$. Meanwhile, if $e = -$, then

$$\lim_{n\to\infty} (\beta_-)^n (U_l) = O \text{ in } \mathbb{L}_Q^{(\infty)},$$

for $U_l \in \mathcal{S}^{(\infty)}$ in $\mathbb{L}_Q^{(\infty)}$, for all $l \in \mathbb{N}$, where the limit "$\lim\limits_{n\to\infty}$" is taken under the operator-norm topology for $\mathcal{A}_\infty$. i.e.,

$$(\beta_-)^n \overset{\text{strong}}{\longrightarrow} 0_Q^{(\infty)} in \mathcal{A}_\infty, \text{ as } n \to \infty, \tag{9.2.6}$$

where "$\overset{\text{strong}}{\longrightarrow}$" means "being strongly convergent to."

Proof For all $n \in \mathbb{N}_0$, we have that

$$\beta_e^n \left(\mathcal{S}^{(\infty)}\right) \supsetneqq \{O\},$$

by (9.2.4), since there are infinitely many generating elements $U_l \in \mathcal{S}^{(\infty)}$ of $\mathcal{A}_\infty$. It implies that

$$\beta_e^n \neq 0_Q^{(\infty)} in \mathcal{A}_\infty, \text{ for all } n \in \mathbb{N}_0,$$

because $\mathfrak{B}^{(\infty)}$ generates $\mathcal{A}_\infty$. Therefore, the relation (9.2.5) holds.

Now, fix $l \in \mathbb{N}$, and take $U_l \in \mathcal{S}^{(\infty)}$ in $\mathbb{L}_Q^{(\infty)}$, and let $\beta_- \in \mathcal{A}_\infty$ be the $(-)$-restricted-shift of $\mathfrak{B}^{(\infty)}$. Then

$$\begin{aligned}\lim_{n\to\infty} (\beta_-)^n (U_l) &= \lim_{n\to\infty} \beta_-^n (U_l) \\ &= \lim_{n\to\infty} U_{l-n} = O,\end{aligned} \tag{9.2.7}$$

in $\mathbb{L}_Q^{(\infty)}$, by (9.2.4), where "$\lim\limits_{n\to\infty}$" is taken under the operator-norm topology for $\mathcal{A}_\infty$. It implies that

$$\lim_{n\to\infty} \beta_-^n(W) = O \text{ in } \mathbb{L}_Q^{(\infty)}, \tag{9.2.8}$$

for all free reduced words W of $\mathbb{L}_Q^{(\infty)}$ in the generating free semicircular family $\mathcal{S}^{(\infty)}$, by (9.2.7). So,

$$\lim_{n\to\infty} \beta_-^n (T) = O \text{ in } \mathbb{L}_Q^{(\infty)}, \forall T \in \mathbb{L}_Q^{(\infty)}, \tag{9.2.9}$$

by (9.2.8). So, the strong-convergence (9.2.6) holds by (9.2.9). □

The above theorem shows that the shift operators $\beta_e^n \in \mathfrak{B}^{(\infty)}$ of $\mathcal{A}_\infty$ satisfy

$$\beta_e^n \neq 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \forall e \in \{\pm\} \text{ and } n \in \mathbb{N}_0,$$

but

$$\beta_-^n \overset{\text{strong}}{\longrightarrow} 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \text{ as } n \to \infty,$$

by (9.2.5) and (9.2.6).

Let $t \in \mathbb{D}^\times$ in $\mathbb{C}$, and take a shift operator $t\beta_e \in \mathcal{A}_\infty$, for $e \in \{\pm\}$, where

$$\mathbb{D}^\times = \{z \in \mathbb{C} : 0 < |z| \leq 1\}.$$

Then

$$(t\beta_e)^n \neq 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \text{ for all } n \in \mathbb{N}_0, \tag{9.2.10}$$

while

$$(t\beta_-)^n \overset{\text{strong}}{\longrightarrow} 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \text{ as } n \to \infty.$$

Proof The proof of (9.2.10) is done by (9.2.6). □

Actually, the strong-convergence (9.2.10) is refined as follows; under the same hypothesis with the above corollary,

$$(t\beta_-)^n \longrightarrow 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \text{ as } n \to \infty,$$

if $t \in \mathbb{D}^\times \setminus \mathbb{T}$ in $\mathbb{C}$, where $\mathbb{T}$ is the unit circle of $\mathbb{C}$, and

$$(t\beta_-)^n \overset{\text{strong}}{\longrightarrow} 0_Q^{(\infty)} \text{ in } \mathcal{A}_\infty, \text{ as } n \to \infty,$$

if $t \in \mathbb{T} \subset \mathbb{D}^\times$ in $\mathbb{C}$, where "$\to$" means "being convergent to, under operator-norm-topology for $\mathcal{A}_\infty$," which imply (9.2.10), anyway.

9.2.3 Free Probability on $\mathbb{L}_Q^{(N)}$ Under the Action of β_e

Here, we consider how the shift operators $t\beta_e \in \mathcal{A}_N$, for $t \in \mathbb{C}$, affect the free probability on $\mathbb{L}_Q^{(N)}$, for an arbitrarily fixed $N \in \mathbb{N}_{>1}^{\infty}$.

Let $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, and let $t\beta_e \in \mathcal{A}_N$ be a shift operator for $t \in \mathbb{C}^{\times}$, and $\beta_e \in \mathfrak{B}^{(N)}$. Then

$$\tau\left(\left((t\beta_e)^k(U_l)\right)^n\right) = \begin{cases} \omega_n t^{kn} c_{\frac{n}{2}} & \text{if } 1 \leq lek \leq N \\ 0 & \text{otherwise,} \end{cases} \tag{9.2.11}$$

for all $k \in \mathbb{N}_0$, $n \in \mathbb{N}$.

Proof By (9.2.3), $(t\beta_e)^k = 0_Q^{(N)}$ in $\mathcal{A}_N$, whenever $k > N$ in $\mathbb{N}_0$. So, if $k > N$, then

$$\begin{aligned} \tau\left(\left(t^k\beta_e^k(U_l)\right)^n\right) &= \tau\left(\left(0_Q^{(N)}(U_l)\right)^n\right) \\ &= \tau\left(O^n\right) = 0, \end{aligned} \tag{9.2.12}$$

for all $n \in \mathbb{N}$.

Suppose now that: if $k \leq N$ in $\mathbb{N}_0$, then

$$\begin{aligned} \tau\left(\left(t^k\beta_e^k(U_l)\right)^n\right) &= t^{kn}\tau\left(\left(\beta_e^k(U_l)\right)^n\right) \\ &= \begin{cases} t^{kn}\tau\left(U_{lek}^n\right) & \text{if } 1 \leq lek \leq N \\ t^{kn}\tau\left(O^n\right) & \text{otherwise} \end{cases} \\ &= \begin{cases} \omega_n t^{kn} c_{\frac{n}{2}} & \text{if } 1 \leq lek \leq N \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{9.2.13}$$

for all $n \in \mathbb{N}$. So, the formula (9.2.11) holds by (9.2.12) and (9.2.13). □

By (9.2.11), the following corollary is obtained.

Under the same hypothesis with the above theorem, if

$$t \in \mathbb{R}^{\times} \text{ and } k \in \mathbb{N}, \text{ and } 1 \leq lek \leq N,$$

then the element $(t\beta_e)^k(U_l)$ is t^{2k}-semicircular in $\mathbb{L}_Q^{(N)}$, for all $l = 1, \ldots, N$.

Proof By assumption, if $t \in \mathbb{R}^{\times}$, then

$$(t\beta_e)^k(U_l) = t^k U_{lek},$$

and

$$\left((t\beta_e)^k (U_l)\right)^* = \overline{t_k} U_{lek}^* = t^k U_{lek},$$

in $\mathbb{L}_Q^{(N)}$. i.e., it is self-adjoint in $\mathbb{L}_Q^{(N)}$.

So, $(t\beta_e)^k (U_l)$ is t^{2k}-semicircular in $\mathbb{L}_Q^{(N)}$, by (3.3) and (9.2.11). □

The above theorem and corollary illustrate that how the action of shift operators $t\beta_e \in \mathcal{A}_N$ deform the semicircular law on $\mathbb{L}_Q^{(N)}$, whenever $N < \infty$ in $\mathbb{N}_>^\infty$.

Let $N = \infty$, and $t\beta_e \in \mathcal{A}_\infty$, for $t \in \mathbb{D}^\times$ in $\mathbb{C}$, and $\beta_e \in \mathfrak{B}^{(\infty)}$. If $U_l \in \mathcal{S}^{(\infty)}$ is a semicircular element of $\mathbb{L}_Q^{(\infty)}$, then

$$\tau\left(\left((t\beta_e)^k(U_l)\right)^n\right) = \begin{cases} 0 & \text{if } e = - \text{ and } l \le k \\ \omega_n t^{kn} c_{\frac{n}{2}} & \text{otherwise,} \end{cases} \tag{9.2.14}$$

for all $n \in \mathbb{N}$. In particular, if

$$\mathcal{B}_{t,l}^- \overset{def}{=} \left\{(t\beta_-)^k (U_l) \in \mathbb{L}_Q^{(\infty)} : k \in \mathbb{N}\right\}, \tag{9.2.15}$$

in $\mathbb{L}_Q^{(\infty)}$, for $t \in \mathbb{D}^\times$ and $l \in \mathbb{N}$, then the asymptotic free distribution of this family $\mathcal{B}_{t,l}^-$ is the zero free distribution in $\mathbb{L}_Q^{(\infty)}$, as $k \to \infty$

Proof Under hypothesis, one has that

$$\tau\left(\left((t\beta_e)^k(U_l)\right)^n\right) = \tau\left((t^k U_{lek})^n\right) = \tau\left(t^{kn} U_{lek}^n\right)$$
$$= \begin{cases} \omega_n t^{kn} c_{\frac{n}{2}} & \text{if } e = + \\ \omega_n t^{kn} c_{\frac{n}{2}} & \text{if } e = - \text{ and } l > k \\ 0 & \text{if } e = - \text{ and } l \le k, \end{cases} \tag{9.2.16}$$

for all $n \in \mathbb{N}$. So, the formula (9.2.14) holds by (9.2.16).

Recall that, under the same hypothesis,

$$(t\beta_e)^k \overset{\text{strong}}{\longrightarrow} 0_Q^{(\infty)} in \mathcal{A}_\infty, \text{ as } k \to \infty, \tag{9.2.17}$$

by (9.2.10). Thus,

$$(t\beta_-)^k (U_l) \to O \text{ in } \mathbb{L}_Q^{(\infty)}, \text{ as } k \to \infty,$$

by (9.2.17), implying that, for all $n \in \mathbb{N}$,

$$\lim_{k\to\infty} \tau\left(\left((t\beta_-)^k (U_l)\right)^n\right) = \tau\left(\lim_{k\to\infty}\left((t\beta_-)^k (U_l)\right)^n\right)$$
$$= \tau\left(\left(\left(\lim_{k\to\infty}(t\beta_-)^k\right)(U_l)\right)^n\right) = \tau\left(O^n\right) = 0. \tag{9.2.18}$$

It means that the family $\mathcal{B}_{t,l}^-$ of (9.2.15) has the asymptotic free distribution, the zero free distribution in $\mathbb{L}_Q^{(\infty)}$, by (9.2.18). □

The above theorem illustrates how our shift operators $t\beta_e \in \mathcal{A}_\infty$ deform the semicircular law on $\mathbb{L}_Q^{(\infty)}$, for $t \in \mathbb{D}^\times$.

9.3 *The Shift Operators of $\mathcal{A}_N$*

More general to Sect. 9.2, we now are interested in general forms of shift operators T of $\mathcal{A}_N$,

$$T = \sum_{\beta_e^k \in \mathfrak{B}^{(N)}} t_e^k \beta_e^k, \text{ with } t_e^k = t_{\beta_e^k} \in \mathbb{C}. \tag{9.3.1}$$

9.3.1 Case where $N < \infty$

Assume first that $N < \infty$ in $\mathbb{N}_{>1}^\infty$. Then, by (9.2.3), every shift operator T of (9.3.1) can be re-expressed by

$$T = \sum_{s=1}^{N} t_{-s}\beta_-^s + t_0 1_Q^{(N)} + \sum_{k=1}^{N} t_k \beta_+^k, \tag{9.3.2}$$

for

$$t_{-N}, \ldots, t_{-1}, t_0, t_1, \ldots, t_N \in \mathbb{C},$$

in the shift-operator algebra $\mathcal{A}_N$ in $B\left(\mathbb{L}_Q^{(N)}\right)$, where $1_Q^{(N)} = 1_{\mathbb{L}_Q^{(N)}} \in \mathcal{A}_N$, the identity operator on $\mathbb{L}_Q^{(N)}$.

Then one can get that, for any $U_l \in \mathcal{S}^{(N)}$ in $\mathbb{L}_Q^{(N)}$, for $l = 1, \ldots, N$,

$$T\left(U_l\right) = \sum_{s=1}^{N} t_{-s}\beta_{-}^{s}\left(U_l\right) + t_0 U_l + \sum_{k=1}^{N} t_k \beta_{+}^{k}\left(U_l\right)$$
$$= \sum_{s=1}^{N} t_{-s} W_{l-s} + t_0 U_l + \sum_{k=1}^{N} t_k W_{l+k}$$

satisfying

$$W_{l+k} = \begin{cases} U_{l+k} & \text{if } 1 \leq l+k \leq N \\ O & \text{otherwise,} \end{cases}$$

similarly,

$$W_{l-s} = \begin{cases} U_{s-l} & \text{if } 1 \leq l-s \leq N \\ O & \text{otherwise,} \end{cases}$$

for all $s, k = 1, \ldots, N$, for $e \in \{\pm\}$, and hence, it goes to

$$= \sum_{s=1}^{l-1} t_{-s} U_{l-s} + t_0 U_l + \sum_{k=1}^{N-l} t_k U_{l+k}. \tag{9.3.3}$$

Let $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, and let $T \in \mathcal{A}_N$ be a shift operator (9.3.2). If $U_l \in \mathcal{S}^{(N)}$ in $\mathbb{L}_Q^{(N)}$, for $l \in \{1, \ldots, N\}$, then there exists a $\mathbb{C}$-quantity $z_{T,l} \in \mathbb{C}$, such that

$$T\left(U_l^n\right) = z_{T,l}\left(\omega_n c_{\frac{n}{2}}\right), \forall n \in \mathbb{N}. \tag{9.3.4}$$

In particular,

$$z_{T,l} = \sum_{s=1}^{l-1} t_{-}^{s} + t_{+}^{0} + \sum_{k=1}^{N-l} t_k in \mathbb{C}. \tag{9.3.5}$$

Proof Since $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, for any semicircular element $U_l \in \mathcal{S}^{(N)}$,

$$T\left(U_l^n\right) = \sum_{s=l}^{l-1} t_{-s} U_{l-s}^n + t_{+}^{0} U_l^n + \sum_{k=1}^{N-l} t_k U_{l+k}^n, \tag{9.3.6}$$

by (9.3.3), for all $n \in \mathbb{N}$. Thus, one has that

$$\tau\left(T\left(U_l^n\right)\right) = \sum_{s=l}^{l-1} t_{-s}\tau\left(U_{l-s}^n\right) + t_+^0\tau\left(U_l^n\right) + \sum_{k=1}^{N-l} t_k\tau\left(U_{l+k}^n\right)$$

$$= \sum_{s=1}^{l-1} t_{-s}\left(\omega_n c_{\frac{n}{2}}\right) + t_+^0\left(\omega_n c_{\frac{n}{2}}\right) + \sum_{k=1}^{N-l} t_k\left(\omega_n c_{\frac{n}{2}}\right)$$

by the semicircularity of U_l, $U_{l\pm k} \in \mathcal{S}^{(N)}$ in $\mathbb{L}_Q^{(N)}$

$$= \left(\omega_n c_{\frac{n}{2}}\right)\left(\sum_{s=1}^{l-1} t_{-s} + t_0 + \sum_{k=1}^{N-l} t_k\right), \tag{9.3.7}$$

for all $n \in \mathbb{N}$. So, the formula (9.3.4) holds by (9.3.7). □

The above theorem illustrates how shift operators $T \in \mathcal{A}_N$ of (9.3.1) deform the original free-distributional data on $\mathbb{L}_Q^{(N)}$, whenever $N < \infty$ in $\mathbb{N}_{>1}^{\infty}$, by (6.2.15).

9.3.2 Case where $N = \infty$

In this sub-section, let $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$, and let $T \in \mathcal{A}_\infty$ be a shift operator (9.3.1) on $\mathbb{L}_Q^{(\infty)}$. Similar to (9.3.2),

$$T = \sum_{s=1}^{\infty} t_{-s}\beta_-^s + t_0 1_Q^{(\infty)} + \sum_{k=1}^{\infty} t_k\beta_+^k, \tag{9.3.8}$$

in $\mathcal{A}_\infty$, for t_{-s}, $t_k \in \mathbb{C}$, for all $s, k \in \mathbb{N}$, by (9.2.4).

Fix a semicircular element $U_l \in \mathcal{S}^{(\infty)}$ of $\mathbb{L}_Q^{(\infty)}$, for $l \in \mathbb{N}$. Then

$$T\left(U_l\right) = \sum_{s=1}^{\infty} t_{-s}W_{l-s} + t_0U_l + \sum_{k=1}^{\infty} t_kU_{l+k}$$

satisfying

$$W_{l-s} = \begin{cases} U_{l-s} & \text{if } l-s \geq 1 \\ O & \text{otherwise,} \end{cases}$$

and hence, it goes to

$$= \sum_{s=1}^{l-1} t_{-s} U_{l-s} + t_0 U_l + \sum_{k=1}^{\infty} t_k U_{l+k}, \tag{9.3.9}$$

in $\mathbb{L}_Q^{(\infty)}$.

Let $N = \infty$ in $\mathbb{N}_{>1}^{\infty}$, and $T \in \mathcal{A}_\infty$, a shift operator (9.3.8) on $\mathbb{L}_Q^{(\infty)}$, and let $U_l \in \mathcal{S}^{(\infty)}$ in $\mathbb{L}_Q^{(\infty)}$, for $l \in \mathbb{N}$. Then there exists $z_{T,l} \in \mathbb{C}$, such that

$$\tau\left(T\left(U_l^n\right)\right) = z_{T,l}\left(\omega_n c_{\frac{n}{2}}\right), \forall n \in \mathbb{N}. \tag{9.3.10}$$

In particular,

$$z_{T,l} = \sum_{s=1}^{l-1} t_-^s + t_+^0 + \sum_{k=1}^{\infty} t_+^k, \text{ in } \mathbb{C}. \tag{9.3.11}$$

Proof Under hypothesis, if $U_l \in \mathcal{S}^{(\infty)}$ in $\mathbb{L}_Q^{(\infty)}$, for $l \in \mathbb{N}$, then

$$\tau\left(T\left(U_l^n\right)\right) = \tau\left(\sum_{s=1}^{l-1} t_{-s} U_{l-s}^n + t_0 U_l^n + \sum_{k=1}^{\infty} t_k U_{l+k}^n\right)$$

by (9.3.9)

$$= \sum_{s=1}^{l-1} t_{-s}\left(\omega_n c_{\frac{n}{2}}\right) + t_0\left(\omega_n c_{\frac{n}{2}}\right) + \sum_{k=1}^{\infty} t_k\left(\omega_n c_{\frac{n}{2}}\right)$$

by the semicircularity of $U_{l-s}, U_l, U_{l+k} \in \mathcal{S}^{(\infty)}$

$$= \left(\omega_n c_{\frac{n}{2}}\right)\left(\sum_{s=1}^{l-1} t_{-s} + t_0 + \sum_{k=1}^{\infty} t_k\right), \tag{9.3.12}$$

for all $n \in \mathbb{N}$. So, the formula (9.3.10) holds by (9.3.12). □

The above theorem illustrates how a shift operator $T \in \mathcal{A}_\infty$ affects the free probability on $\mathbb{L}_Q^{(\infty)}$, with help of (6.2.15).

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Representation Formulae for the Determinant in a Neighborhood of the Identity

Denis Constales and Alí Guzmán Adán

Abstract We prove an integral representation and a power series expansion for the function $\det(A)^{-1}$ in a small neighborhood of the identity matrix. Both results are closely linked to the formula for the change of coordinates of the Dirac delta distribution in $\mathbb{R}^m$.

Mathematics Subject Classification (2000) 15A15, 41A58, 30E20

Keywords Determinant · Taylor series · Complex analysis · Dirac distribution

1 Introduction

In this manuscript we prove two representation formulas for the function $\det(A)^{-1}$ in a small neighborhood of the identity matrix. Let us start by describing our results.

Let $\mathbb{C}^{k\times k}$ the algebra of complex matrices $M = \{m_{r,\ell}\}_{r,\ell=1\dots,k}$ of order $(k \times k)$ with identity $\mathbb{1}_k$. The Frobenius norm of $M \in \mathbb{C}^{k\times k}$ is defined as

$$\|M\| = (\sum_{r,\ell=1}^{k} |m_{r,\ell}|^2)^{1/2},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}$. Associated to each row vector $M_r = (m_{r,1}, \dots, m_{r,k})$, we consider a multi-index $\alpha_r = (\alpha_{r,1}, \dots, \alpha_{r,k}) \in \mathbb{N}_0^k$ where $\mathbb{N}_0$

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denotes the set of non-negative integers. As usual, we denote $|\alpha_r| = \alpha_{r,1} + \ldots + \alpha_{r,k}$, $\alpha_r! = \alpha_{r,1}! \cdots \alpha_{r,k}!$ and $M_r^{\alpha_r} = m_{r,1}^{\alpha_{r,1}} \cdots m_{r,k}^{\alpha_{r,k}}$. We shall also consider the multi-index sum $\alpha_1 + \ldots + \alpha_k = (\sum_r \alpha_{r,1}, \ldots, \sum_r \alpha_{r,k})$. In general, for multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ we have $I \pm J = (i_1 \pm j_1, \ldots, i_k \pm j_k)$. We say that $I \leq J$ if $i_r \leq j_r$ for all $r = 1, \ldots, k$.

The multi-indices α_r give rise to the multi-index matrix $\boldsymbol{\alpha} = \{\alpha_{r,\ell}\}_{r,\ell=1\ldots,k} \in \mathbb{N}_0^{k\times k}$. We thus denote $|\alpha| = |\alpha_1| + \ldots + |\alpha_k|$, $\alpha! = \alpha_1! \cdots \alpha_k!$ and $M^{\boldsymbol{\alpha}} = M_1^{\alpha_1} \cdots M_k^{\alpha_k}$. We also introduce the differential operator

$$\partial_M^\alpha = \prod_{r,\ell=1}^{k} \partial_{m_{r,\ell}}^{\alpha_{r,\ell}},$$

which is the so-called Fischer dual of the monomial M^α.

With the above notation, the Taylor series of the function $\det(\mathbb{1}_k + M)^{-1}$ around the point $M = 0$ can be written as

$$\frac{1}{\det(\mathbb{1}_k + M)} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^{k\times k}} \frac{M^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \, \partial_M^\alpha \left[\frac{1}{\det(\mathbb{1}_k + M)}\right]\Bigg|_{M=0}. \tag{1.1}$$

Our first goal is to explicitly write down the above formula, i.e. to compute the derivatives $\partial_M^\alpha \left[\frac{1}{\det(\mathbb{1}_k+M)}\right]\Big|_{M=0}$. In particular, we will prove the following result.

Theorem 1.1 *Let $M \in \mathbb{C}^{k\times k}$ be such that $\|M\| \leq 1/k$. Then the Taylor series (1.1) converges and has the form*

$$\frac{1}{\det(\mathbb{1}_k + M)} = \sum_{J\in\mathbb{N}_0^k} (-1)^{|J|} J! \sum_{\substack{\alpha_1+\ldots+\alpha_k=J\\ |\alpha_r|=j_r}} \frac{M_1^{\alpha_1} \cdots M_k^{\alpha_k}}{\alpha_1! \cdots \alpha_k!}, \tag{1.2}$$

where the above sum runs over all multi-indices $J = (j_1, \ldots, j_k) \in \mathbb{N}_0^k$ and all multi-index matrices $\boldsymbol{\alpha}$ such that $\alpha_{1,r} + \cdots + \alpha_{k,r} = |\alpha_r| = j_r$ for all $r = 1, \ldots, k$. This is, $\boldsymbol{\alpha} \in \mathbb{N}_0^{k\times k}$ is such that the sum of its r-th row equals the sum of its r-th column for all $r = 1, \ldots, k$.

This constitutes a generalization to higher dimensions of the convergence of the geometric series

$$\sum_{j=0}^{\infty} (-1)^j z^j = \frac{1}{1+z}, \quad \text{when} \quad z \in \mathbb{C},\ |z| < 1.$$

It also provides a detailed expression for the full expansion of the well-known formula (see e.g. [2])

$$\frac{1}{\det(\mathbb{1}_k + M)} = \exp\left(-\ln\left(\det(\mathbb{1}_k + M)\right)\right) = \exp\left(\sum_{j=0}^{\infty}(-1)^j \frac{\mathrm{tr}(M^j)}{j}\right),$$

where $\mathrm{tr}(A)$ is the usual matrix trace of A. For a detailed account on this and other matrix analysis results we refer the reader (without claiming completeness) to the works [1, 4, 6].

Theorem 1.1 also yields that $\partial_M^\alpha \left[\frac{1}{\det(\mathbb{1}_k+M)}\right]\Big|_{M=0}$ is different from zero if and only if the sum per row equals the sum per column in the multi-index matrix α, namely:

$$\partial_M^\alpha \left[\frac{1}{\det(\mathbb{1}_k + M)}\right]\Bigg|_{M=0} = \begin{cases} (-1)^{|\alpha|}\, |\alpha_1|! \cdots |\alpha_k|! & \text{if} \quad \alpha_1 + \ldots + \alpha_k = (|\alpha_1|, \ldots, |\alpha_k|), \\ 0 & \text{otherwise.} \end{cases}$$

In addition, it allows to compute the inverse of the characteristic polynomial $\det(M - \lambda\mathbb{1}_k)$ as a power series of λ. Indeed, if $|\lambda| \leq (k\|M\|)^{-1}$, one obtains from (1.2) that

$$\frac{1}{\det(M - \lambda\mathbb{1}_k)} = \sum_{j=0}^{\infty} c_j\, \lambda^{k-j},$$

where

$$c_j = (-1)^{j+k} \sum_{|J|=j} J! \sum_{\substack{\alpha_1+\ldots+\alpha_k=J \\ |\alpha_r|=j_r}} \frac{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}{\alpha_1!\cdots\alpha_k!}.$$

The main motivation for formula (1.2) comes for the Taylor series expansion of the Dirac distribution, see Sect. 2. However, in order to rigorously prove this result, we will need the following integral representation for $\det(\mathbb{1}_k + M)^{-1}$.

Theorem 1.2 *Let $A \in \mathbb{C}^{k\times k}$ be a matrix such that $\|A - \mathbb{1}_k\| < 1/k$ and consider the linear transformation $A\underline{z} = \underline{w}$, where $\underline{z} = (z_1, \ldots, z_k)^T$ and $\underline{w} = (w_1, \ldots, w_k)^T$ are vector variables in $\mathbb{C}^k$. Then*

$$\frac{1}{\det(A)} = \frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{1}{w_1\cdots w_k}\, dz_1\cdots dz_k, \tag{1.3}$$

where $\oint_{(\partial D)^k} = \oint_{\partial D} \cdots \oint_{\partial D}$ *and* $\oint_{\partial D}$ *denotes the contour integral along the boundary* $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ *of the unit disk.*

Remark 1.1 As the statements in the Theorems 1.1 and 1.2 announce, the condition $\|A - \mathbb{1}_k\| < 1/k$ implies that $\det(A) \neq 0$. Indeed, the power series $A^{-1} = (\mathbb{1}_k + (A - \mathbb{1}_k))^{-1} = \sum_{\ell=0}^{\infty}(-1)^{\ell}(A - \mathbb{1}_k)^{\ell}$ converges absolutely if $k > 1$. Therefore, the inverse matrix A^{-1} exists. For $k = 1$, it is clear that $\|A - 1\| < 1$ implies $A \neq 0$.

The proximity of $A = \mathbb{1}_k + M$ to $\mathbb{1}_k$ is important for these results to hold. For example, in the class of nilpotent matrices (that are somehow close to the origin) formula (1.2) does not hold in general. Indeed, if M is nilpotent then $\mathbb{1}_k + M$ is invertible, however the series (1.2) may not converge. See, for example, the case where $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The statement of Theorem 1.2 also announces that none of the w_j's becomes zero as $\underline{z}$ varies in the boundary $(\partial D)^k$ of the unit polydisc. This is another reason why the proximity of A to $\mathbb{1}_k$ is crucial. As a matter of fact, $(\partial D)^k$ can be continuously deformed into $A(\partial D)^k$ without ever touching one of the axis $z_j = 0$, see our Lemma 5.2.

This paper is organized as follows. In Sect. 2, we informally discuss the main motivations for Theorem 1.1. This result is closely connected to the formula for the change of coordinates of the Dirac delta distribution. Most of the heuristic and motivational reasoning in that section is not completely rigorous. In the later sections, all of our results will be rigorously proved. Our strategy to prove Theorem 1.1 is to show first that it is equivalent to Theorem 1.2 and then proceed to prove the latter. In Sect. 3, we prove that the left-hand side expressions in (1.2) and (1.3) coincide when $A = \mathbb{1}_k + M$, completing in this way the first step of our strategy. The second step is completed in Sect. 4, where we prove Theorem 1.2 by means of the method of Gaussian elimination. Finally, in Sect. 5, we provide an alternative proof by showing that the manifolds $(\partial D)^k$ and $A(\partial D)^k$ are in the same homology class in the space $(\mathbb{C} \setminus \{0\})^k$ and therefore, integrals of closed differential k-forms over these manifolds remain invariant. At the end, we briefly discuss the connection between Theorem 1.2 and the formula for the change of coordinates of the delta distribution.

2 Motivations and Informal Discussion

In this section we sketch the main motivation behind formula (1.2). This does not lead to a rigorous proof but it shows how this result is linked with the change of coordinates of the Dirac delta distribution. In the next sections, we will rigorously prove Theorems 1.1 and 1.2.

Given the real vector variable $\underline{x} = (x_1, \ldots, x_k)^T$, we consider the Dirac distribution $\delta(\underline{x}) = \delta(x_1)\ldots\delta(x_k)$ in the k-dimensional Euclidean space $\mathbb{R}^k$. For any non-singular real matrix $\mathbb{1}_k + M \in \mathrm{GL}(k, \mathbb{R})$ it is known that (see e.g. [3])

$$\delta(\underline{x} + M\underline{x}) = \delta\left((\mathbb{1}_k + M)\,\underline{x}\right) = \frac{\delta(\underline{x})}{|\det(\mathbb{1}_k + M)|}. \tag{2.1}$$

On the other hand, if we formally write down the Taylor series expansion of the left-hand side, we get

$$\delta(\underline{x} + M\underline{x}) = \sum_{J \in \mathbb{N}_0^k} \frac{\left(M\underline{x}\right)^J}{J!}\, \delta^{(J)}(\underline{x}),$$

where $\delta^{(J)}(\underline{x}) = \delta^{(j_1)}(x_1)\ldots\delta^{(j_k)}(x_k)$ for the multi-index $J = (j_1, \ldots, j_k)$. We recall that $\left(M\underline{x}\right)^J = \left(M\underline{x}\right)_1^{j_1}\ldots\left(M\underline{x}\right)_k^{j_k}$ where $\left(M\underline{x}\right)_r = \sum_{\ell=1}^{k} m_{r,\ell}x_\ell$ is the r-th component of the vector $M\underline{x}$. By the multinomial theorem we have

$$\frac{\left(M\underline{x}\right)_r^{j_r}}{j_r!} = \sum_{|\alpha_r|=j_r} \frac{M_r^{\alpha_r}\, \underline{x}^{\alpha_r}}{\alpha_r!}, \quad \text{with} \quad \underline{x}^{\alpha_r} = x_1^{\alpha_{r,1}}\cdots x_k^{\alpha_{r,k}},$$

and therefore

$$\delta(\underline{x} + M\underline{x}) = \sum_{J \in \mathbb{N}_0^k} \left(\sum_{|\alpha_r|=j_r} \frac{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}{\alpha_1!\cdots\alpha_k!} \underline{x}^{\alpha_1+\ldots\alpha_k} \right) \delta^{(J)}(\underline{x}). \tag{2.2}$$

Let us consider $I = \alpha_1 + \cdots \alpha_k \in \mathbb{N}_0^k$, it is a known result that (see e.g. [3])

$$\underline{x}^I\,\delta^J = \begin{cases} (-1)^I \frac{J!}{(J-I)!}\delta^{(J-I)}(\underline{x}), & \text{if } I \le J, \\ 0, & \text{otherwise.} \end{cases}$$

But our multi-index I satisfies $|I| = |J|$. Thus, in this case, the condition $I \le J$ implies $I = J$. Therefore, formula (2.2) can be rewritten as

$$\delta(\underline{x} + M\underline{x}) = \left(\sum_{J \in \mathbb{N}_0^k} (-1)^{|J|} J! \sum_{\substack{\alpha_1+\ldots+\alpha_k=J \\ |\alpha_r|=j_r}} \frac{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}{\alpha_1!\cdots\alpha_k!} \right) \delta(\underline{x}).$$

Comparing this with (2.1), it follows that formula (1.2) should hold for some suitable class of matrices M. The statement of Theorem 1.1 is stronger than this guess. In fact, it explicitly describes a class of matrices for which this formula holds and it states the result for complex matrices. In Sect. 5, we shall make the link between the other representation formula (1.3) and the change of coordinates in the Dirac distribution.

3 An Intermediate Step

Before rigorously proving Theorem 1.1, we will show that the statements in the Theorems 1.1 and 1.2 are equivalent. Let $M \in \mathbb{C}^{k\times k}$ be as in Theorem 1.1 and let us denote the sum in the right-hand side of (1.2) by

$$R(M) = \sum_{J\in\mathbb{N}_0^k} (-1)^{|J|} J! \sum_{\substack{\alpha_1+\ldots+\alpha_k=J\\ |\alpha_r|=j_r}} \frac{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}{\alpha_1!\cdots\alpha_k!}.$$

Now consider the following *"more relaxed"* version of $R(M)$

$$S(M) = \sum_{J\in\mathbb{N}_0^k} (-1)^{|J|} J! \sum_{|\alpha_r|=j_r} \frac{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}{\alpha_1!\cdots\alpha_k!}. \tag{3.1}$$

While the sum $S(M)$ runs over all multi-index matrices $\boldsymbol{\alpha} \in \mathbb{N}_0^{k\times k}$, the sum $R(M)$ considers only those $\boldsymbol{\alpha} \in \mathbb{N}_0^{k\times k}$ for which the sum of its r-th row equals the sum of its r-th column for all $r = 1,\ldots,k$. Using the multinomial theorem, we can write $S(M)$ in terms of the following geometric series,

$$\begin{aligned} S(M) &= \sum_{J\in\mathbb{N}_0^k} \prod_{r=1}^k \left(-\sum_{\ell=1}^k m_{r,\ell}\right)^{j_r} \\ &= \prod_{r=1}^k \left(\sum_{j_r=0}^\infty \left(-\sum_{\ell=1}^k m_{r,\ell}\right)^{j_r}\right) \\ &= \prod_{r=1}^k \frac{1}{1+\sum_{\ell=1}^k m_{r,\ell}}. \end{aligned} \tag{3.2}$$

We recall that the power series $\sum_{j_r=0}^{\infty}\left(-\sum_{\ell=1}^{k} m_{r,\ell}\right)^{j_r}$ converges uniformly to $\left(1+\sum_{\ell=1}^{k} m_{r,\ell}\right)^{-1}$ since $\|M\| \leq \frac{1}{k}$. Indeed, $\left|\sum_{\ell=1}^{k} m_{r,\ell}\right| \leq \sum_{\ell=1}^{k} |m_{r,\ell}| < k\frac{1}{k} = 1$. This reasoning also shows that the series $R(M)$ converges absolutely for $\|M\| \leq \frac{1}{k}$.

Our strategy is to apply some transformations to the sum $S(M)$ in order to recover $R(M)$. To that end we first consider the transformation $M \mapsto D^{-1}MD$ where $D = \mathrm{diag}(z_1, \dots, z_k)$ is a diagonal matrix whose diagonal entries are in the unit circle, i.e. $z_1, \dots, z_k \in \partial D$. The matrix $D^{-1}MD$ is the result of multiplying the r-th row of M by z_r^{-1} and the r-th column by z_r, $r = 1, \dots, k$. Then for every entry this transformation can be written as

$$m_{r,\ell} \mapsto z_r^{-1} z_\ell\, m_{r,\ell}, \qquad r, \ell = 1, \dots, k. \tag{3.3}$$

Let us examine how the sum written in (3.1) reads after this transformation. Observe that

$$M_r^{\alpha_r} = m_{r,1}^{\alpha_{r,1}} \cdots m_{r,k}^{\alpha_{r,k}} \quad \mapsto \quad z_r^{-j_r} z_1^{\alpha_{r,1}} \cdots z_k^{\alpha_{r,k}}\, M_r^{\alpha_r},$$

which yields

$$M_1^{\alpha_1} \cdots M_k^{\alpha_k} \quad \mapsto \quad \left(\prod_{r=1}^{k} z_r^{\sum_\ell \alpha_{\ell,r} - j_r}\right) M_1^{\alpha_1} \cdots M_k^{\alpha_k}.$$

The only summands in (3.1) that remain independent of the z_r's are those satisfying $\sum_{\ell=1}^{k} \alpha_{\ell,r} = j_r$ for all $r = 1, \dots, k$, or equivalently, $\alpha_1 + \ldots + \alpha_k = J$. These are exactly the terms that appear in $R(M)$. Hence we can write

$$S(D^{-1}MD) = R(M) + T(D^{-1}MD),$$

where $T(D^{-1}MD)$ is a sum of elements of the form $z_1^{\lambda 1} \cdots z_k^{\lambda_k} c$ such that c is independent of the z_r's and at least one of the powers $\lambda_r \in \mathbb{Z}$ is different from zero.

Using Cauchy's integral theorem we easily find that

$$\frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{T(D^{-1}MD)}{z_1 \cdots z_k}\, dz_1 \cdots dz_k = 0$$

and

$$\frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{R(M)}{z_1 \cdots z_k}\, dz_1 \cdots dz_k = R(M) \frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{dz_1 \cdots dz_k}{z_1 \cdots z_k} = R(M).$$

Thus

$$\frac{1}{(2\pi i)^k}\oint_{(\partial D)^k}\frac{S(D^{-1}MD)}{z_1\cdots z_k}\,dz_1\cdots dz_k = R(M).$$

On the other hand, using formulas (3.2) and (3.3), we get

$$S(D^{-1}MD)=\prod_{r=1}^{k}\frac{1}{1+\sum_{\ell=1}^{k} z_r^{-1}z_\ell\, m_{r,\ell}}=\prod_{r=1}^{k}\frac{z_r}{z_r+\sum_{\ell=1}^{k} z_\ell\, m_{r,\ell}}=\frac{z_1\cdots z_k}{w_1\cdots w_k}.$$

Here the vectors $\underline{z}=(z_1,\ldots,z_k)^T$ and $\underline{w}=(w_1,\ldots,w_k)^T$ are as in Theorem 1.2, i.e. $A\underline{z}=\underline{w}$ with $A=\mathbb{1}_k+M$. Finally, combining the last two formulas, we obtain

$$R(M)=\frac{1}{(2\pi i)^k}\oint_{(\partial D)^k}\frac{1}{w_1\cdots w_k}\,dz_1\cdots dz_k,$$

which proves that the left-hand side expressions in (1.2) and (1.3) coincide when $A=\mathbb{1}_k+M$.

4 Proofs of the Main Theorems

We now proceed to prove Theorem 1.2. To that end, we first need the following lemma.

Lemma 4.1 *Let $A=\{a_{r,\ell}\}_{r,\ell=1,\ldots,k}$ be a matrix in $\mathbb{C}^{k\times k}$ such that $\|A-\mathbb{1}_k\|<1/k$ and consider the linear transformation $A\underline{z}=\underline{w}$, where $\underline{z}=(z_1,\ldots,z_k)^T\in(\partial D)^k$ and $\underline{w}=(w_1,\ldots,w_k)^T$. Then*

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{dz_1}{w_r}=\begin{cases}\dfrac{1}{a_{1,1}} & \text{if } r=1,\\ 0 & \text{if } r=2,\ldots,k.\end{cases}$$

Proof Let us start by considering the case $r=2,\ldots,k$. If $a_{r,1}=0$, then w_r does not depend on z_1 and the integral is automatically zero. If $a_{r,1}\neq 0$, we have

$$\frac{1}{w_r}=\frac{1}{a_{r,1}\left(z_1+\sum_{\ell=2}^{k}\frac{a_{r,\ell}}{a_{r,1}}z_\ell\right)}.$$

This is a function of z_1 with only one singularity, namely $z_1 = -\sum_{\ell=2}^{k} \frac{a_{r,\ell}}{a_{r,1}} z_\ell$, which lies outside of the unit disk $D = \{z_1 \in \mathbb{C} : |z_1| \leq 1\}$. Indeed, by the triangular inequality we obtain

$$\begin{aligned}\left|\sum_{\ell=2}^{k} \frac{a_{r,\ell}}{a_{r,1}} z_\ell\right| &= \frac{1}{|a_{r,1}|}\left|\sum_{\ell=2}^{k} a_{r,\ell} z_\ell\right| \\ &\geq \frac{1}{|a_{r,1}|}\left(|a_{r,r}| - \left|\sum_{\ell\neq 1,r} a_{r,\ell} z_\ell\right|\right) \\ &\geq \frac{1}{|a_{r,1}|}\left(|a_{r,r}| - \sum_{\ell\neq 1,r} \left|a_{r,\ell}\right|\right).\end{aligned}$$

Now we recall that $|a_{r,\ell} - \delta_{r,\ell}| < \frac{1}{k}$. In particular, this implies that $\frac{1}{|a_{r,1}|} > k$, $|a_{r,r}| > \frac{k-1}{k}$ and $-|a_{r,\ell}| > -\frac{1}{k}$ $(\ell \neq r)$. We thus obtain

$$\left|\sum_{\ell=2}^{k} \frac{a_{r,\ell}}{a_{r,1}} z_\ell\right| > k\left(\frac{k-1}{k} - \frac{k-2}{k}\right) = 1.$$

This means that $\frac{1}{w_r}$ is a holomorphic function inside the unit disk and therefore $\frac{1}{2\pi i}\oint_{\partial D} \frac{dz_1}{w_r} = 0$.

In the case where $r = 1$, it suffices to show that the isolated singularity $z_1 = -\sum_{\ell=2}^{k} \frac{a_{1,\ell}}{a_{1,1}} z_\ell$ of $\frac{1}{w_1}$ is inside of the unit disk. One easily observes that

$$\left|\sum_{\ell=2}^{k} \frac{a_{1,\ell}}{a_{1,1}} z_\ell\right| \leq \sum_{\ell=2}^{k} \frac{|a_{1,\ell}|}{|a_{1,1}|} < \sum_{\ell=2}^{k} \frac{\frac{1}{k}}{1-\frac{1}{k}} = 1.$$

Then, by the residue theorem we obtain $\frac{1}{2\pi i}\oint_{\partial D} \frac{dz_1}{w_1} = \frac{1}{a_{1,1}}$. □

Proof of Theorem 1.2 We proceed by induction on $k \in \mathbb{N}$. For $k = 1$ we have $w_1 = a_{1,1} z_1$ with $|a_{1,1} - 1| < 1$. It is then clear that $\frac{1}{2\pi i}\oint_{\partial D} \frac{dz_1}{a_{1,1} z_1} = \frac{1}{a_{1,1}} = \frac{1}{\det(A)}$.

Let us assume that formula (1.3) is true for $k - 1 \in \mathbb{N}$, and let us prove that it also holds for k. To that end we first decompose the function $\frac{1}{w_1 \cdots w_k}$ as a sum of partial fractions with respect to z_1, i.e.

$$\frac{1}{w_1 \cdots w_k} = \frac{\lambda_1}{w_1} + \cdots + \frac{\lambda_k}{w_k}$$

where $\lambda_1, \ldots, \lambda_k$ do not depend on z_1. From Lemma 4.1 we obtain

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{dz_1}{w_1\cdots w_k}=\frac{\lambda_1}{a_{1,1}}.$$

We recall that λ_1 is the residue of the function $\frac{a_{1,1}}{w_1\cdots w_k}$ at the singularity

$$z_1=-\sum_{\ell=2}^{k}\frac{a_{1,\ell}}{a_{1,1}}z_\ell.$$

Thus λ_1 can be easily computed to be

$$\lambda_1=\left.\frac{1}{w_2\cdots w_k}\right|_{z_1=-\sum_{\ell=2}^{k}\frac{a_{1,\ell}}{a_{1,1}}z_\ell}=\frac{1}{\widetilde{w}_2\cdots\widetilde{w}_k}$$

where $\widetilde{w}_r$ is the value of w_r when substituting $z_1=-\sum_{\ell=2}^{k}\frac{a_{1,\ell}}{a_{1,1}}z_\ell$, $r=2,\ldots,k$. Further computations yield

$$\widetilde{w}_r=-\frac{a_{r,1}}{a_{1,1}}\sum_{\ell=2}^{k}a_{1,\ell}z_\ell+\sum_{\ell=2}^{k}a_{r,\ell}z_\ell=\sum_{\ell=2}^{k}b_{r,\ell}z_\ell,\quad \text{where}\quad b_{r,\ell}=a_{r,\ell}-\frac{a_{r,1}}{a_{1,1}}a_{1,\ell}.$$

In this way we have obtained

$$\frac{1}{(2\pi i)^k}\oint_{(\partial D)^k}\frac{1}{w_1\cdots w_k}dz_1\cdots dz_k=\frac{1}{a_{1,1}}\frac{1}{(2\pi i)^{k-1}}\oint_{(\partial D)^{k-1}}\frac{1}{\widetilde{w}_2\cdots\widetilde{w}_k}dz_2\cdots dz_k, \tag{4.1}$$

where $\begin{pmatrix} b_{2,2} & \ldots & b_{2,k} \\ \vdots & \ddots & \vdots \\ b_{k,2} & \ldots & b_{k,k}\end{pmatrix}\begin{pmatrix} z_2 \\ \vdots \\ z_k\end{pmatrix}=\begin{pmatrix}\widetilde{w}_2 \\ \vdots \\ \widetilde{w}_k\end{pmatrix}$ and the matrix $B=\{b_{r,\ell}\}_{r,\ell=2,\ldots,k}$ satisfies $\|B-\mathbb{1}_{k-1}\|\leq\frac{1}{k-1}$. Indeed, the matrix $B-\mathbb{1}_{k-1}$ has entries $b_{r,\ell}-\delta_{r,\ell}=(a_{r,\ell}-\delta_{r,\ell})-\frac{1}{a_{1,1}}a_{r,1}a_{1,\ell}$ with $r,\ell=2,\ldots,k$. If we donote by C the matrix $\{a_{r,1}a_{1,\ell}\}_{r,\ell=2,\ldots,k}$, we get

$$\|B-\mathbb{1}_{k-1}\|\leq\|A-\mathbb{1}_k\|+\frac{1}{|a_{1,1}|}\|C\|.$$

But $\|C\| = \left(\sum_{r,\ell=2}^{k} |a_{r,1}|^2 |a_{1,\ell}|^2\right)^{\frac{1}{2}} = \left(\sum_{r=2}^{k} |a_{r,1}|^2\right)^{\frac{1}{2}} \left(\sum_{\ell=2}^{k} |a_{1,\ell}|^2\right)^{\frac{1}{2}} < \frac{1}{k^2}$ which implies

$$\|B - \mathbb{1}_{k-1}\| \leq \frac{1}{k} + \frac{k}{k-1}\frac{1}{k^2} = \frac{1}{k-1}.$$

Now, applying our induction hypothesis on (4.1), we obtain

$$\frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{1}{w_1 \cdots w_k} \, dz_1 \cdots dz_k = \frac{1}{a_{1,1} \det(B)}.$$

Thus it suffices to prove that $a_{1,1} \det(B) = \det(A)$. This easily follows from the Gaussian elimination process. Indeed, if we add to the r-th row in the matrix A the first row multiplied by $-\frac{a_{r,1}}{a_{1,1}}$ $(r = 2, \ldots, k)$, we obtain the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,k} \\ 0 & b_{2,2} & \ldots & b_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{k,2} & \ldots & b_{k,k} \end{pmatrix}.$$

This matrix has the same determinant as A. We thus obtain that $\det(A) = a_{1,1} \det(B)$, which proves the result. □

5 An Alternative Proof

We now provide an alternative proof for Theorem 1.2. In particular, we will prove the following slightly generalized result.

Theorem 5.1 *Consider* $A\underline{z} = \underline{w}$ *as in Theorem 1.2 and let* $f(\underline{z})$ *be a* $\mathbb{C}$*-valued holomorphic function in* $\mathbb{C}^k$*. Then*

$$\frac{f(0)}{\det(A)} = \frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{f(\underline{z})}{w_1 \cdots w_k} \, dz_1 \cdots dz_k. \tag{5.1}$$

We first observe that $A\underline{z} = \underline{w}$ yields $dw_1 \cdots dw_k = \det(A)\, dz_1 \cdots dz_k$. Effectuating this change of variables in the right-hand side of (5.1) we obtain

$$\frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{f(\underline{z})}{w_1 \cdots w_k} \, dz_1 \cdots dz_k = \frac{1}{\det(A)} \frac{1}{(2\pi i)^k} \oint_{A(\partial D)^k} \frac{f(A^{-1}\underline{w})}{w_1 \cdots w_k} \, dw_1 \cdots dw_k. \tag{5.2}$$

Thus it suffices to prove that $\frac{1}{(2\pi i)^k}\oint_{A(\partial D)^k}\frac{f(A^{-1}\underline{w})}{w_1\cdots w_k}\,dw_1\cdots dw_k = f(0)$. The idea of this proof is to note that $\frac{f(A^{-1}\underline{w})}{w_1\cdots w_k}\,dw_1\cdots dw_k$ is a closed differential form on $(\mathbb{C}\setminus\{0\})^k$. Therefore, by Stoke's theorem, the above integral remains invariant when taken over any other manifold in the same homology class of $(\partial D)^k$ in $(\mathbb{C}\setminus\{0\})^k$, see e.g. [7]. This is summarized in the following two lemmas.

Lemma 5.2 *Let $A\in\mathbb{C}^{k\times k}$ be a matrix such that $\|A-\mathbb{1}_k\|<1/k$. Then $(\partial D)^k$ and $A(\partial D)^k$ belong to the same homology class in $(\mathbb{C}\setminus\{0\})^k$, i.e. there exists a continuous map $A(t):[0,1]\to\mathbb{C}^{k\times k}$ such that $A(0)=\mathbb{1}_k$, $A(1)=A$ and $A(t)(\partial D)^k\subset(\mathbb{C}\setminus\{0\})^k$ for every $t\in[0,1]$.*

Proof Let us consider the map $A(t)=\mathbb{1}_k+t(A-\mathbb{1}_k)$ and $\underline{w}(t)=A(t)\underline{z}$ with $\underline{z}\in(\partial D)^k$ arbitrary. We need to prove that $\underline{w}(t)\in(\mathbb{C}\setminus\{0\})^k$ for every $t\in[0,1]$, or equivalently, that every entry $w_r(t)$ of $\underline{w}(t)$ $(r=1,\dots,k)$ is different from zero.

We recall that $w_r(t)=z_r+t\sum_{j=1}^k(a_{r,j}-\delta_{r,j})z_j$. Then by the triangular inequality we have

$$1=|z_r|=\left|w_r(t)-t\sum_{j=1}^k(a_{r,j}-\delta_{r,j})z_j\right|$$
$$\le|w_r(t)|+t\sum_{j=1}^k|a_{r,j}-\delta_{r,j}|<|w_r(t)|+t\sum_{j=1}^k\frac{1}{k}.$$

Thus $|w_r(t)|>1-t\ge 0$ and therefore $w_r(t)\neq 0$. □

Lemma 5.3 *Let $\gamma(\underline{w})$ be a $\mathbb{C}$-valued holomorphic function in an open region $\Omega\subset\mathbb{C}^k$. Then the differential form $\gamma dw_1\cdots dw_k$ is closed in Ω, i.e. $d(\gamma dw_1\cdots dw_k)=0$ where d is the exterior derivative.*

Proof It is easily seen that the exterior derivative can be written as $d=\partial+\overline{\partial}$ where $\partial=\sum_{j=1}^n\partial_{w_j}dw_j$ and $\overline{\partial}=\sum_{j=1}^n\partial_{\overline{w}_j}d\overline{w}_j$ are given in terms of the classical Cauchy–Riemann operators $\partial_{w_j},\partial_{\overline{w}_j}$ and the complex differentials $dw_j, d\overline{w}_j$. If we write $w_j=x_j+iy_j$ (with x_j, y_j being real variables) then

$$\partial_{w_j}=\frac{1}{2}(\partial_{x_j}-i\partial_{y_j}),\quad dw_j=dx_j+idy_j,\quad \partial_{\overline{w}_j}=\frac{1}{2}(\partial_{x_j}+i\partial_{y_j}),\quad d\overline{w}_j=dx_j-idy_j.$$

It is clear that $\partial(\gamma dw_1\cdots dw_k)=0$ and, since γ is holomorphic, we also have that $\overline{\partial}(\gamma dw_1\cdots dw_k)=\sum_{j=1}^n\partial_{\overline{w}_j}[\gamma]d\overline{w}_j dw_1\cdots dw_k=0$. Therefore $d(\gamma dw_1\cdots dw_k)=0$. □

Using the previous lemma we easily observe that $\frac{f(A^{-1}\underline{w})}{w_1\cdots w_k}\,dw_1\cdots dw_k$ is a closed differential form on $(\mathbb{C}\setminus\{0\})^k$. This means that its integrals over the homologous manifolds $(\partial D)^k$ and $A(\partial D)^k$ are equal. Going back to formula (5.2), we finally obtain from Cauchy's theorem that

$$\frac{1}{(2\pi i)^k}\oint_{(\partial D)^k}\frac{f(\underline{z})}{w_1\cdots w_k}\,dz_1\cdots dz_k = \frac{1}{\det(A)}\frac{1}{(2\pi i)^k}\oint_{A(\partial D)^k}\frac{f(A^{-1}\underline{w})}{w_1\cdots w_k}\,dw_1\cdots dw_k$$
$$= \frac{1}{\det(A)}\frac{1}{(2\pi i)^k}\oint_{(\partial D)^k}\frac{f(A^{-1}\underline{w})}{w_1\cdots w_k}\,dw_1\cdots dw_k$$
$$= \frac{f(0)}{\det(A)}.$$

5.1 Connection with the Dirac Distribution

Let us consider the $2k$-dimensional real vector variables $\underline{x} = (x_1, \dots, x_{2k})^T$ and $\underline{y} = (y_1, \dots, y_{2k})^T$ associated to the complex vector variables $\underline{z}$ and $\underline{w}$ in $\mathbb{C}^k$ by means of

$$\underline{z} = (x_1 + i x_{k+1}, \dots, x_k + i x_{2k})^T, \quad \text{and} \quad \underline{w} = (y_1 + i y_{k+1}, \dots, y_k + i y_{2k})^T,$$

respectively. Equivalently we may write $\underline{z} = P\underline{x}$ and $\underline{w} = P\underline{y}$ where $P = (\mathbb{1}_k | i\,\mathbb{1}_k) \in \mathbb{C}^{k\times 2k}$. Associated to any complex-linear transformation $\underline{w} = A\underline{z}$, one finds a real-linear transformation $\underline{y} = \Psi(A)\underline{x}$, where $\Psi : \mathbb{C}^{k\times k} \to \mathbb{R}^{2k\times 2k}$ is an algebra morphism given by

$$\Psi(A_1 + i A_2) = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \qquad A_1, A_2 \in \mathbb{R}^{k\times k}.$$

The determinants of the matrices A and $\Psi(A)$ are linked by the relation $\det(\Psi(A)) = |\det(A)|^2$. Indeed, if one considers the matrices $D = \begin{pmatrix} \mathbb{1}_k & i\,\mathbb{1}_k \\ 0 & \mathbb{1}_k \end{pmatrix}$ and its inverse $D^{-1} = \begin{pmatrix} \mathbb{1}_k & -i\,\mathbb{1}_k \\ 0 & \mathbb{1}_k \end{pmatrix}$, one obtains

$$D\,\Psi(A)\,D^{-1} = \begin{pmatrix} A_1 + i A_2 & 0 \\ A_2 & A_1 - i A_2 \end{pmatrix},$$

and therefore $\det(\Psi(A)) = \det(D\,\Psi(A)\,D^{-1}) = \det(A_1 + i A_2)\det(A_1 - i A_2) = |\det(A)|^2$.

Let us define $\delta(\underline{z}) = \delta(\underline{x}) = \delta(x_1)\dots\delta(x_{2k})$ and $\delta(\underline{w}) = \delta(\underline{y}) = \delta(y_1)\dots\delta(y_{2k})$. Then (see e.g. [3])

$$\delta(\underline{w}) = \frac{\delta(\underline{z})}{\det(\Psi(A))} = \frac{\delta(\underline{z})}{|\det(A)|^2}. \tag{5.3}$$

In Sect. 2, we showed how this formula is linked with Theorem 1.1. In this section, we shall make the relation between formula (5.3) and Theorem 5.1 explicit.

From Theorem 5.1, we have for every holomorphic function $f(\underline{z})$ that

$$\left\langle \frac{\delta(\underline{z})}{\det(A)}, f \right\rangle = \frac{f(0)}{\det(A)} = \frac{1}{\det(A)} \frac{1}{(2\pi i)^k} \oint_{(\partial D)^k} \frac{f(\underline{z})}{w_1 \cdots w_k} dw_1 \cdots dw_k. \tag{5.4}$$

We now recall that Green's theorem can be written, in terms of the Cauchy–Riemann $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ operator of the complex variable $z = x + iy$, as

$$\oint_{\partial D} g\, dz = 2i \iint_D \partial_{\overline{z}}[g]\, dx dy,$$

where g is a differentiable function in a neighborhood of the unit disc D. Applying Green's theorem in each variable w_j in (5.4), we obtain

$$\left\langle \frac{\delta(\underline{z})}{\det(A)}, f \right\rangle = \frac{1}{\pi^k \det(A)} \int_{D^k} \partial_{\overline{w_1}} \cdots \partial_{\overline{w_{2k}}} \left[\frac{1}{w_1 \cdots w_k} \right] f(\underline{z})\, dy_1 \cdots dy_{2k}.$$

Now, we recall that $(\pi z)^{-1}$ is the fundamental solution of $\partial_{\overline{z}}$, see e.g. [5]. Then we can substitute in the above formula $\partial_{\overline{w_j}} \left[\frac{1}{w_j} \right] = \pi\, \delta(w_j) = \pi \delta(y_j)\delta(y_{k+j})$. Finally, we obtain

$$\begin{aligned} \left\langle \frac{\delta(\underline{z})}{\det(A)}, f \right\rangle &= \frac{1}{\det(A)} \int_{D^k} \delta(\underline{w}) f(\underline{z})\, dy_1 \cdots dy_{2k} \\ &= \overline{\det(A)} \int_{A^{-1}D^k} \delta(\underline{w}) f(\underline{z})\, dx_1 \cdots dx_{2k} \\ &= \left\langle \overline{\det(A)}\, \delta(\underline{w}), f \right\rangle, \end{aligned}$$

which yields (5.3). In the second equality we have used the fact that $dy_1 \cdots dy_{2k} = \det(\Psi(A))\, dx_1 \cdots dx_{2k} = \det(A)\overline{\det(A)}\, dx_1 \cdots dx_{2k}$.

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Parametrization of the Solution Set of a Matricial Truncated Hamburger Moment Problem by a Schur Type Algorithm

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Abstract This paper contains a Schur analytic approach to a truncated matricial moment problem of Hamburger type, which is studied in the most general case. It is shown that a Schur type algorithm constructed by the authors for a related moment problem can be suitably modified to obtain a full description of the solution set with the aid of a linear fractional transformation with polynomial generating matrix-valued function. The main feature of our Schur type algorithm consists of an appropriate synthesis of two different versions of types of algorithms, namely on the one side an algebraic one working for sequences of complex matrices and on the other side a function theoretic one applied to special classes of holomorphic matrix functions.

Keywords Matricial Hamburger moment problem · Stieltjes transform · Schur type algorithm

Mathematics Subject Classification (2000) Primary 44A60; Secondary 30E05

1 Introduction

This paper is part of a systematic program of creating a Schur analytic approach to matricial versions of truncated classical power moment problems, which was developed by the authors in the last decade. The essential feature of our concept can be described as a detailed study of the structure of sequences of moment matrices by Schur type algorithms on the one side combined with the construction of concordant Schur type algorithms for special classes of holomorphic matrix-valued functions in several domains which are determined by the choice of the moment problem

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under consideration. For each of the generalized versions of the classical moment problems (named after Hamburger, Stieltjes and Hausdorff) we were able to give a parametrization of the set of solutions in the most general case. What concerns the matricial versions of truncated Stieltjes type moment problems we refer to [16, 17, 20, 22] whereas the matricial version of the truncated Hausdorff moment problem was investigated in [18, 21, 23, 24].

Our Schur analytic approach to the truncated matricial Hamburger moment problem was begun in [15], where we treated the particular version of the truncated problem which is connected with equality of all prescribed matricial moments. There we handled simultaneously both the case of an even as well as the case of an odd number of prescribed moments. In this paper we concentrate on the last still remaining case, namely on the truncated matricial moment problem for a given sequence $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, where the matrix s_{2n} is required to satisfy an inequality in the sense of Löwner semi-ordering for Hermitian matrices. In the so-called non-degenerate case this problem was already studied by I. V. Kovalishina [27]. She used the FMI method due to V. P. Potapov who interpreted Schur type algorithm from the view of multiplicative decomposition of rational J-elementary factors as a finite product of rational J-elementary factors with poles of order one. In this way, I. V. Kovalishina [27] treated Schur analytic aspects of the moment problem under study. It should be mentioned that even in the non-degenerate case our Schur type algorithm does not coincide with that multiplicative decomposition of the resolvent matrix derived in [27]. The considerations in [27] formed the starting point for the investigations of Chen/Hu [4] where a function theoretic version of a Schur type algorithm was presented. This algorithm uses the Drazin generalized inverse of matrices. The parameters in the linear-fractional transformation description of the set of solutions are pairs of meromorphic matrix functions. Against the background of a later computation of the Weyl matrix balls associated with a truncated matricial Hamburger moment problem in the most general case we strive for a parametrization of the solution set which is based on pairs of holomorphic matrix-valued functions. The use of the Moore–Penrose inverse is a key instrument in our construction of Schur type algorithm. The elementary step of our algorithm is taken from [15].

2 Notation and Preliminaries

First we state some notation. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. Further, for every choice of $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\alpha,\beta}$ be the set of all integers k such that $\alpha \leq k \leq \beta$. Throughout this paper, if not explicitly mentioned otherwise, then let $p, q, r \in \mathbb{N}$. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p \times q}$ represents the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p \times 1}$. The notation $\mathbb{C}_{\mathrm{H}}^{q \times q}$ is used to denote the set of all

Hermitian complex $q \times q$ matrices. We write $\mathbb{C}_{\geq}^{q\times q}$ and $\mathbb{C}_{>}^{q\times q}$ to designate the set of all non-negative Hermitian complex $q \times q$ matrices and the set of all positive Hermitian complex $q \times q$ matrices, respectively.

If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping defined on $\mathfrak{A}$ with values in $\mathbb{C}_{\geq}^{q\times q}$ is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$ and the notation $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ stands for the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$. (Appendix B is aimed to state some basic results on non-negative Hermitian measures.) If $\mu = (\mu_{jk})_{j,k=1}^{q}$ is a non-negative Hermitian $q \times q$ measure on a measurable space $(\Omega, \mathfrak{A})$, then we use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ to denote the set of all Borel-measurable functions $f: \Omega \to \mathbb{C}$ for which $\int_\Omega |f| \mathrm{d}\nu_{jk} < \infty$ holds true for every choice of j and k in $\mathbb{Z}_{1,q}$, where ν_{jk} is the variation of the complex measure μ_{jk} (see also Lemma B.1). If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, then let $\int_\Omega f \mathrm{d}\mu := (\int_\Omega f \mathrm{d}\mu_{jk})_{j,k=1}^{q}$ and we also write $\int_\Omega f(\omega)\mu(\mathrm{d}\omega)$ for this integral.

Let $\mathfrak{B}_\mathbb{R}$ (resp. $\mathfrak{B}_\mathbb{C}$) be the σ-algebra of all Borel subsets of $\mathbb{R}$ (or $\mathbb{C}$, respectively). For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borel subsets of Ω and let $\mathcal{M}_{\geq}^{q}(\Omega)$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{B})$, i.e., $\mathcal{M}_{\geq}^{q}(\Omega)$ is short for $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{B}_\Omega)$. Furthermore, for all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$ and for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}_{\geq,\kappa}^{q}(\Omega)$ be the set of all $\sigma \in \mathcal{M}_{\geq}^{q}(\Omega)$ such that, for all $j \in \mathbb{Z}_{0,\kappa}$, the function $f_j : \Omega \to \mathbb{C}$ defined by $f_j(\omega) := \omega^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$.

Remark 2.1 Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $\sigma \in \mathcal{M}_{\geq,\kappa}^{q}(\Omega)$. Then, for each $j \in \mathbb{Z}_{0,\kappa}$, the integral $s_j^{(\sigma)} := \int_\Omega \omega^j \sigma(\mathrm{d}\omega)$ is well defined. In view of [20, Rem. B.2], one easily can check that $(s_j^{(\sigma)})^* = s_j^{(\sigma)}$ holds true for all $j \in \mathbb{Z}_{0,\kappa}$.

Obviously, once more considering an arbitrary $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, we have

$$\mathcal{M}_{\geq,\infty}^{q}(\Omega) \subseteq \mathcal{M}_{\geq,l}^{q}(\Omega) \subseteq \mathcal{M}_{\geq,k}^{q}(\Omega) \subseteq \mathcal{M}_{\geq,0}^{q}(\Omega) = \mathcal{M}_{\geq}^{q}(\Omega)$$

for every choice of non-negative integers k and l with $k \leq l$. If Ω is a bounded set belonging to $\mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, then it is readily checked that $\mathcal{M}_{\geq,\infty}^{q}(\Omega) = \mathcal{M}_{\geq}^{q}(\Omega)$ is valid. We will consider the following types of a so-called matricial Hamburger power moment problems:

Problem ($\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{m}, \leq]$) Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{m}, \leq]$ of all $\sigma \in \mathcal{M}_{\geq,m}^{q}(\mathbb{R})$ for which the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m \geq 1$, for which additionally $s_j = s_j^{(\sigma)}$ is fulfilled for all $j \in \mathbb{Z}_{0,m-1}$.

Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{m}, \leq]$ is connected to a further type of truncated moment problem considered (for particular cases of the set Ω), for example, in [4, 15]:

Problem ($\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$ of all $\sigma \in \mathcal{M}_{\geq,\kappa}^{q}(\mathbb{R})$ for which $s_j = s_j^{(\sigma)}$ is fulfilled for all $j \in \mathbb{Z}_{0,\kappa}$.

To recall the criteria of solvability of the matricial Hamburger problem in detail as well as for our further consideration, we introduce certain sets of sequences of complex $q \times q$ matrices which are determined by properties of particular block Hankel matrices built of them. If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n}$ is a sequence of complex $q \times q$ matrices, then $(s_j)_{j=0}^{2n}$ is called $\mathbb{R}$-*non-negative definite* ($\mathbb{R}$-*positive definite*, respectively) if the block Hankel matrix

$$H_n := (s_{j+k})_{j,k=0}^{n} = \begin{pmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ s_n & s_{n+1} & \dots & s_{2n} \end{pmatrix} \tag{2.1}$$

is non-negative Hermitian (positive Hermitian, respectively). For all $n \in \mathbb{N}_0$, we will write $\mathcal{H}_{q,2n}^{\geq}$ (or $\mathcal{H}_{q,2n}^{>}$, respectively) for the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices which are $\mathbb{R}$-non-negative definite ($\mathbb{R}$-positive definite, respectively). If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ (or $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$, respectively), then, for each $m \in \mathbb{Z}_{0,n}$, the sequence $(s_j)_{j=0}^{2m}$ obviously belongs to $\mathcal{H}_{q,2m}^{\geq}$ (or $\mathcal{H}_{q,2m}^{>}$, respectively). Thus, let $\mathcal{H}_{q,\infty}^{\geq}$ (or $\mathcal{H}_{q,\infty}^{>}$, respectively) be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$ (or $\mathcal{H}_{q,2n}^{>}$, respectively). A solvability criterion for Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ is the following:

Theorem 2.2 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$.*

There are different proofs of Theorem 2.2, namely in [4, Thm. 3.2] and [10, Thm. 4.16]. A parametrization of the solution set $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ was given in [27, Thm. $\mathscr{H}$] for the non-degenerate case, i. e., if the sequence $(s_j)_{j=0}^{2n}$ of prescribed matricial moments is $\mathbb{R}$-positive definite. In the general case of a given $\mathbb{R}$-non-negative definite sequence $(s_j)_{j=0}^{2n}$, parametrizations of $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ can be found in [2, Thm. 4.6], [4, Thm. 4.5], and [33, Ch. 1].

For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices s_{2n+1} and s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ to denote the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. For each $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\geq,\mathrm{e}}$ are called $\mathbb{R}$-*non-negative definite extendable* sequences. For technical reasons, we set $\mathcal{H}_{q,\infty}^{\geq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\geq}$.

Remark 2.3 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathcal{H}_{q,2\kappa}^{>} \subseteq \mathcal{H}_{q,2\kappa}^{\geq,\mathrm{e}} \subseteq \mathcal{H}_{q,2\kappa}^{\geq}$.

If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, then $\mathcal{H}^{>}_{q,2\kappa} \neq \mathcal{H}^{\geq,\mathrm{e}}_{q,2\kappa}$. Furthermore, $\mathcal{H}^{\geq,\mathrm{e}}_{q,0} = \mathcal{H}^{\geq}_{q,0}$ whereas $\mathcal{H}^{\geq,\mathrm{e}}_{q,2\kappa} \neq \mathcal{H}^{\geq}_{q,2\kappa}$ for all $\kappa \in \mathbb{N} \cup \{\infty\}$. The following result is essential for a parametrization of the set $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$:

Theorem 2.4 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n}$. Then there exists a unique sequence $(\tilde{s}_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n}$ such that $\mathcal{M}^q_{\geq}[\mathbb{R}; (\tilde{s}_j)_{j=0}^{2n}, \leq] = \mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$.*

The existence of such a sequence $(\tilde{s}_j)_{j=0}^{2n}$ was formulated first in [2, Lem. 2.12]. In [33, Satz 1.22], one can find a complete proof for the existence of such a sequence $(\tilde{s}_j)_{j=0}^{2n}$. A complete proof of Theorem 2.4 can be found in [10, Thm. 7.3]. A general principle which stands behind the construction of the sequence $(\tilde{s}_j)_{j=0}^{2n}$ was uncovered in [19]. This concept is connected with a special kind of Schur complement. Furthermore, observe that necessary and sufficient conditions for the case that Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ has a unique solution are given in [10, Theorems 8.4 and 8.5]. The solvability of Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$ can be characterized as follows:

Theorem 2.5 *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$ is not empty if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,\kappa}$.*

A proof of Theorem 2.5 can be found in [10, Thm. 4.17]. This proof modifies an idea presented in [2, Lem. 2.10], where κ is an even non-negative integer. In the case of an even non-negative integer κ, a proof is also given in [4, Thm. 3.1]. Moreover, if $\kappa = \infty$, a proof is given in [11, Thm. 6.6]. Furthermore, observe that necessary and sufficient conditions for the case that Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{m}, \leq]$, where m is an arbitrarily given non-negative integer, has a unique solution are given in [10, Theorems 8.7 and 8.9].

In the so-called non-degenerate situation, a parametrization of the solution set of $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$, where n is an arbitrarily given positive integer, was worked out by H. Dym in [9]. This was done for arbitrarily given sequences $(s_j)_{j=0}^{2n} \in \mathcal{H}^{>}_{q,2n}$ by using the theory of Hilbert spaces with a reproducing kernel. Applying a Schur type algorithm, in [4, Thm. 4] a parametrization of $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$ was shown for given sequences $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n}$. Alternatively, a description of this solution set was presented in [1, Thm. 4], using operator-theoretic methods. In the case $\kappa = 2n+1$ with some non-negative integer n, a parametrization of the solution set of $\mathcal{M}^1_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n+1}, =]$, i. e., in the scalar case, was found in [7, Sec. 3]. The matricial case $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n+1}, =]$ with an arbitrarily given positive integer q could be handled by using a two-step Schur type algorithm in [15]. In the case $\kappa = \infty$, only assuming additional conditions, a parametrization was found previously (cf. [5, 27]).

At the end of this introductory section, we give some further notation. We will write I_q to denote the identity matrix in $\mathbb{C}^{q \times q}$, whereas $0_{p \times q}$ is the null matrix belonging to $\mathbb{C}^{p \times q}$. If the size of an identity matrix or a null matrix is obvious, then we also will omit the indices. For each $A \in \mathbb{C}^{p\times q}$, let $\mathcal{R}(A)$ be the column space of A, let $\mathcal{N}(A)$ be the null space of A, and let $\operatorname{rank} A$ be the rank of A. For each $A \in \mathbb{C}^{q\times q}$, we will use $\operatorname{Im} A$ to denote the imaginary part of A: $\operatorname{Im} A := \frac{1}{2\mathrm{i}}(A - A^*)$. Furthermore, for each $A \in \mathbb{C}^{p\times q}$, let $\|A\|_{\mathrm{S}}$ be the operator norm of A. A complex $p \times q$ matrix A is said to be contractive if $\|A\|_{\mathrm{S}} \leq 1$. If $A \in \mathbb{C}^{q\times q}$, then $\det A$ denotes the determinant of A. For each $A \in \mathbb{C}^{q\times p}$, let A^+ be the Moore–Penrose inverse of A. If $p_1, p_2, q_1, q_2 \in \mathbb{N}$ and if $A_j \in \mathbb{C}^{p_j\times q_j}$ for every choice of $j \in \{1, 2\}$, then let $\operatorname{diag}(A_1, A_2) := \begin{pmatrix} A_1 & 0_{p_1\times q_2} \\ 0_{p_2\times q_1} & A_2 \end{pmatrix}$. Furthermore, within the set $\mathbb{C}_{\mathrm{H}}^{q\times q}$, we use the Löwner semi-ordering: If A and B are complex Hermitian $q \times q$ matrices, then we will write $A \leq B$ (or $B \geq A$) to indicate that $B - A$ is a non-negative Hermitian matrix.

For all $x, y \in \mathbb{C}^q$, by $\langle x, y\rangle_{\mathrm{E}}$ we denote the (left-hand side) Euclidean inner product of x and y, i. e., we have $\langle x, y\rangle_{\mathrm{E}} := y^*x$. If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^q$, then let $\mathcal{M}^\perp$ be the set of all vectors in $\mathbb{C}^q$ which are orthogonal to $\mathcal{M}$ (with respect to the Euclidean inner product $\langle ., .\rangle_{\mathrm{E}}$). If $\mathcal{U}$ and $\mathcal{W}$ are orthogonal subspaces of $\mathbb{C}^q$, then we will say that $\mathcal{U} \oplus \mathcal{W}$ is the orthogonal sum of $\mathcal{U}$ and $\mathcal{W}$. If $\mathcal{U}$ is a subspace of $\mathbb{C}^q$, then let $\mathbb{P}_{\mathcal{U}}$ be the orthoprojection matrix onto $\mathcal{U}$.

We consider the set

$$\Pi_+ := \{z \in \mathbb{C} : \operatorname{Im} z \in (0, \infty)\}.$$

We will call a subset $\mathcal{D}$ of Π_+ a discrete subset of Π_+ if $\mathcal{D}$ does not have an accumulation point in Π_+. If f is a meromorphic function defined on a non-empty open subset of the complex plane, then we use $\mathbb{H}_f$ to denote the set of all points w at which f is holomorphic.

3 Particular Classes of Holomorphic Matrix Functions

We will reformulate the matricial moment problems under consideration as equivalent interpolation problems for particular classes of holomorphic matrix-valued functions. For this reason, we introduce in this section the corresponding function classes and summarize some of their essential properties needed in the sequel. Most of this material is taken from [12, 15]. In this section, we consider some special classes of holomorphic matrix-valued functions. First we turn our attention to a (well-studied) class of matrix-valued functions.

A matrix-valued function $S : \Pi_+ \to \mathbb{C}^{q\times q}$ is called $q \times q$ *Schur function in* Π_+ if S is both holomorphic and contractive in Π_+, i. e., if S is holomorphic in Π_+ and if $\|S(z)\|_{\mathrm{S}} \leq 1$ is fulfilled for each $z \in \Pi_+$. The set of all $q \times q$ Schur functions in Π_+ will be denoted by $\mathcal{S}_{q\times q}(\Pi_+)$.

The class $\mathcal{R}_q(\Pi_+)$ of all $q \times q$ *Herglotz–Nevanlinna functions in the upper half-plane* Π_+ consists of all matrix-valued functions $F : \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in Π_+ and which satisfy $\operatorname{Im} F(z) \in \mathbb{C}_{\geq}^{q\times q}$ for all $z \in \Pi_+$.

Detailed observations about matrix-valued Herglotz–Nevanlinna functions can be found in [12, 25]. Especially, the functions belonging to $\mathcal{R}_q(\Pi_+)$ admit a well-known integral representation. Before we formulate this version of a famous result due to R. Nevanlinna, we observe that, for every choice of $\nu \in \mathcal{M}_{\geq}^q(\mathbb{R})$ and $z \in \mathbb{C} \setminus \mathbb{R}$, from Lemma B.1 one easily can see that $f_z\colon \mathbb{R} \to \mathbb{C}$ given by $f_z(x) := (1 + xz)/(x - z)$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \nu; \mathbb{C})$.

Theorem 3.1 (Nevanlinna)

(a) *For each* $F \in \mathcal{R}_q(\Pi_+)$*, there exists a unique triple* $(\alpha, \beta, \nu) \in \mathbb{C}_{\mathrm{H}}^{q\times q} \times \mathbb{C}_{\geq}^{q\times q} \times \mathcal{M}_{\geq}^q(\mathbb{R})$ *such that*

$$F(z) = \alpha + \beta z + \int_{\mathbb{R}} \frac{1 + xz}{x - z}\nu(\mathrm{d}x) \qquad \text{for each } z \in \Pi_+. \tag{3.1}$$

(b) *If* $\alpha \in \mathbb{C}_{\mathrm{H}}^{q\times q}$*, if* $\beta \in \mathbb{C}_{\geq}^{q\times q}$*, and if* $\nu \in \mathcal{M}_{\geq}^q(\mathbb{R})$*, then* $F\colon \Pi_+ \to \mathbb{C}^{q\times q}$ *defined by* (3.1) *belongs to* $\mathcal{R}_q(\Pi_+)$.

For each $F \in \mathcal{R}_q(\Pi_+)$, the unique triple $(\alpha, \beta, \nu) \in \mathbb{C}_{\mathrm{H}}^{q\times q} \times \mathbb{C}_{\geq}^{q\times q} \times \mathcal{M}_{\geq}^q(\mathbb{R})$ for which the representation (3.1) holds true is called the *Nevanlinna parametrization of* F and we also write $(\alpha_F, \beta_F, \nu_F)$ instead of (α, β, ν). In particular, ν_F is said to be the *Nevanlinna measure of* F. For our following consideration, it seems to be useful to state some further known results concerning Herglotz–Nevanlinna functions. We start with the following result:

Lemma 3.2 *Let* $\mathcal{D}$ *be a discrete subset of* Π_+ *and let* $F\colon \Pi_+ \setminus \mathcal{D} \to \mathbb{C}^{q\times q}$ *be a matrix-valued function holomorphic in* $\Pi_+ \setminus \mathcal{D}$ *such that* $\operatorname{Im} F(z) \in \mathbb{C}_{\geq}^{q\times q}$ *is valid for all* $z \in \Pi_+ \setminus \mathcal{D}$*. For each* $z \in \mathcal{D}$*, then* F *has a removable singularity at* z *and the extended matrix-valued function belongs to the class* $\mathcal{R}_q(\Pi_+)$.

A proof of Lemma 3.2 easily can be obtained, e. g., using [8, Lem. 2.1.9]. If F belongs to $\mathcal{R}_1(\Pi_+)$, then $\sigma_F\colon \mathfrak{B}_{\mathbb{R}} \to [0, +\infty]$ defined by

$$\sigma_F(B) := \int_B (1 + x^2)\nu_F(\mathrm{d}x) \qquad \text{for all } B \in \mathfrak{B}_{\mathbb{R}} \tag{3.2}$$

describes a measure which is called the *spectral measure* of F. By $\mathcal{R}'_q(\Pi_+)$ we denote the set of all $F \in \mathcal{R}_q(\Pi_+)$ for which $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) := 1 + x^2$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \nu_F; \mathbb{R})$. Since the trace measure τ of ν_F fulfils $\tau(\mathbb{R}) < \infty$, Lemma B.1 yields $\mathcal{R}'_q(\Pi_+) = \{F \in \mathcal{R}_q(\Pi_+)\colon \nu_F \in \mathcal{M}_{\geq,2}^q(\mathbb{R})\}$. If F belongs to $\mathcal{R}'_q(\Pi_+)$, then $\sigma_F\colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}_{\geq}^{q\times q}$ given by (3.2) is a well-defined non-negative Hermitian $q \times q$ measure belonging to $\mathcal{M}_{\geq}^q(\mathbb{R})$ which is said to be the *matricial spectral measure* of F. Clearly, considering functions belonging to the class $\mathcal{R}'_1(\Pi_+)$, the notations ‘spectral measure’ and ‘matricial spectral measure’ coincide. Observe that [20, Rem. B.1] shows that, for each $F \in \mathcal{R}'_q(\Pi_+)$, the matricial spectral measure σ_F of F fulfils $\sigma_F(B) = \int_B(\sqrt{1 + x^2}I_q)\nu_F(\mathrm{d}x)(\sqrt{1 + x^2}I_q)^*$

for all $B \in \mathfrak{B}_{\mathbb{R}}$. Observe that a particular integral representation for functions belonging to the class $\mathcal{R}'_q(\Pi_+)$ is given in [12, Thm. 4.3]. For our consideration, the class $\mathcal{R}_{0,q}(\Pi_+)$ given by

$$\mathcal{R}_{0,q}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) \colon \sup_{y \in [1,\infty)} y \| F(\mathrm{i}y) \|_S < \infty \right\},$$

plays a key role. The class $\mathcal{R}_{0,q}(\Pi_+)$ is a subclass of $\mathcal{R}'_q(\Pi_+)$ (see, e. g., [32, Lem. 8.4]). Furthermore, the functions belonging to $\mathcal{R}_{0,q}(\Pi_+)$ admit a special integral representation. Before we formulate this result, let us observe that, for every choice of $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R})$ and $z \in \mathbb{C} \setminus \mathbb{R}$, one can easily see that the function $h_z \colon \mathbb{R} \to \mathbb{C}$ given by $h_z(x) := (x - z)^{-1}$ describes a bounded and continuous function which, in particular, belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \mu; \mathbb{C})$ (see also Lemma B.1). Now we are able to formulate a well-known matricial generalization of a classical result due to R. Nevanlinna [29].

Theorem 3.3

(a) *For each $F \in \mathcal{R}_{0,q}(\Pi_+)$, the matrix-valued function F belongs to $\mathcal{R}'_q(\Pi_+)$ and there is a unique $\sigma \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that*

$$F(z) = \int_{\mathbb{R}} \frac{1}{x - z} \sigma(\mathrm{d}x) \qquad \textit{for each } z \in \Pi_+, \tag{3.3}$$

namely, the matricial spectral measure σ_F of F.

(b) *If $F \colon \Pi_+ \to \mathbb{C}^{q \times q}$ is a matrix-valued function for which there exists a non-negative Hermitian measure $\sigma \in \mathcal{M}^q_{\geq}(\mathbb{R})$ such that (3.3) holds true, then F belongs to $\mathcal{R}_{0,q}(\Pi_+)$.*

Theorem 3.3 can be proved by using its well-known scalar version in the case $q = 1$ as well as the fact that, for each $F \in \mathcal{R}_{0,q}(\Pi_+)$ and each $u \in \mathbb{C}^q$, the function $u^* F u$ belongs to $\mathcal{R}_{0,1}(\Pi_+)$, Lemma B.1, and [12, Lem. B.3]. If $F \in \mathcal{R}_{0,q}(\Pi_+)$, then the unique $\sigma \in \mathcal{M}^q_{\geq}(\mathbb{R})$ for which (3.3) holds true is also called the $\mathbb{R}$-*Stieltjes measure of* F. Consequently, for each $F \in \mathcal{R}_{0,q}(\Pi_+)$, the notions $\mathbb{R}$-Stieltjes measure and matricial spectral measure coincide. If $\sigma \in \mathcal{M}^q_{\geq}(\mathbb{R})$ is given, then $F \colon \Pi_+ \to \mathbb{C}^{q \times q}$ defined by (3.3) is said to be the $\mathbb{R}$-*Stieltjes transform of* σ.

Remark 3.4 In view of Theorem 3.3, now one can reformulate Problems $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ and $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$ in the language of $\mathbb{R}$-Stieltjes transforms:

Problem ($\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$) Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ of all matrix-valued functions $F \in \mathcal{R}_{0,q}(\Pi_+)$ the $\mathbb{R}$-Stieltjes measure of which belongs to $\mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$.

Problem $(\mathsf{R}[\Pi_+; (s_j)_{j=0}^{\kappa}, =])$ Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{\kappa}, =]$ of all matrix-valued functions $F \in \mathcal{R}_{0,q}(\Pi_+)$ the $\mathbb{R}$-Stieltjes measure of which belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$.

The following result contains an important property of the class $\mathcal{R}_{0,q}(\Pi_+)$.

Lemma 3.5 *Let $F \in \mathcal{R}_{0,q}(\Pi_+)$. Then, for each $z \in \Pi_+$, the equations $\mathcal{R}(F(z)) = \mathcal{R}(\sigma_F(\mathbb{R}))$ and $\mathcal{N}(F(z)) = \mathcal{N}(\sigma_F(\mathbb{R}))$ hold true.*

There is a proof of Lemma 3.5, e. g., in [12, Lem. 8.2 and Prop. 8.9]. In the following, we introduce a variety of further subclasses of $\mathcal{R}_q(\Pi_+)$. We summarize basic facts about these subclasses which mostly are characterized by growth properties on the positive imaginary axis. Within our consideration, essentially we refer to the representations in [12, Sections 4–8] and [15, Sections 3 and 4]. Let

$$\mathcal{R}_q^{[-2]}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) \colon \lim_{y \to \infty} \left(\frac{1}{y} \|F(\mathrm{i}y)\|_{\mathrm{S}} \right) = 0 \right\}$$

and

$$\mathcal{R}_q^{[-1]}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) \colon \int_{[1,\infty)} \frac{1}{y} \|\mathrm{Im}\, F(\mathrm{i}y)\|_{\mathrm{S}} \tilde{\lambda}(\mathrm{d}y) < \infty \right\},$$

where $\tilde{\lambda}$ is used for the Lebesgue measure defined on $\mathfrak{B}_{[1,\infty)}$.

Remark 3.6 ([12, Lem. 5.1 and Remarks 5.2 and 5.3]) Let $F \in \mathcal{R}_q^{[-1]}(\Pi_+)$ with Nevanlinna parametrization $(\alpha_F, \beta_F, \nu_F)$. Then $\beta_F = 0_{q \times q}$, $\nu_F \in \mathcal{M}_{\geq,1}^{q}(\mathbb{R})$, and $h \colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) := (x^2+1)/(|x|+1)$ belongs to $\mathcal{L}^1(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, \nu_F; \mathbb{R})$. Let $\gamma_F := \alpha_F - s_1^{(\nu_F)}$ and let $\mu_F \colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q \times q}$ be given by

$$\mu_F(B) := \int_B \frac{x^2+1}{|x|+1} \nu_F(\mathrm{d}x). \tag{3.4}$$

Then $\gamma_F \in \mathbb{C}_{\mathrm{H}}^{q \times q}$ and $\mu_F \in \mathcal{M}_{\geq}^{q}(\mathbb{R})$. Furthermore, $\mathcal{R}(\mu_F(\mathbb{R})) = \mathcal{R}(\nu_F(\mathbb{R}))$ and $\mathcal{N}(\mu_F(\mathbb{R})) = \mathcal{N}(\nu_F(\mathbb{R}))$.

We continue to use the notation γ_F and μ_F, explained in Remark 3.6. Let

$$\mathcal{R}_{-1,q}(\Pi_+) := \left\{ F \in \mathcal{R}_q^{[-1]}(\Pi_+) \colon \gamma_F = 0_{q \times q} \right\}. \tag{3.5}$$

Proposition 3.7 ([12, Propositions 8.8 and 8.9]) *The class $\mathcal{R}_{0,q}(\Pi_+)$ admits the representation $\mathcal{R}_{0,q}(\Pi_+) = \mathcal{R}_{-1,q}(\Pi_+) \cap \mathcal{R}_q'(\Pi_+)$.*

Lemma 3.8 *Let* $F \in \mathcal{R}_{0,q}(\Pi_+)$*. Then* $F \in \mathcal{R}_q^{[-1]}(\Pi_+)$ *and* $\mathcal{N}(\mu_F(\mathbb{R})) = \mathcal{N}(\sigma_F(\mathbb{R}))$.

Proof From Proposition 3.7 we can infer $F \in \mathcal{R}_{-1,q}(\Pi_+)$. According to (3.5), then $F \in \mathcal{R}_q^{[-1]}(\Pi_+)$. Let $z_0 \in \Pi_+$. We can apply [12, Lem. 8.1] to obtain $\mathcal{N}(F(z_0)) = \mathcal{N}(\mu_F(\mathbb{R}))$. Since Lemma 3.5 yields $\mathcal{N}(F(z_0)) = \mathcal{N}(\sigma_F(\mathbb{R}))$, consequently $\mathcal{N}(\mu_F(\mathbb{R})) = \mathcal{N}(\sigma_F(\mathbb{R}))$ follows. □

From [12, Prop. 3.7] we know that, for an arbitrary function $F \in \mathcal{R}_q(\Pi_+)$, the null space of $F(z)$ is independent of the choice of $z \in \Pi_+$. Keeping this in mind, we consider a special subclass of $\mathcal{R}_{-1,q}(\Pi_+)$ which is characterized by the interrelation between this constant null space and the null space of a prescribed matrix $A \in \mathbb{C}^{p\times q}$. Indeed, for all $A \in \mathbb{C}^{p\times q}$, let

$$\mathcal{P}_q^{\text{odd}}[A] := \left\{F \in \mathcal{R}_{-1,q}(\Pi_+) \colon \mathcal{N}(A) \subseteq \mathcal{N}(\mu_F(\mathbb{R}))\right\}, \tag{3.6}$$

where $\mu_F \colon \mathfrak{B}_{\mathbb{R}} \to \mathbb{C}^{q\times q}$ is given by (3.4). Observe that, for each $A \in \mathbb{C}^{p\times q}$, the matrix-valued function $F \colon \Pi_+ \to \mathbb{C}^{q\times q}$ defined by $F(z) := 0_{q\times q}$ belongs to $\mathcal{P}_q^{\text{odd}}[A]$ (cf.[15, Example 4.2]). Moreover, if $A \in \mathbb{C}^{p\times q}$ satisfies $\mathcal{N}(A) = \{0_{q\times 1}\}$, then $\mathcal{P}_q^{\text{odd}}[A] = \mathcal{R}_{-1,q}(\Pi_+)$ (cf.[15, Rem. 4.1]).

4 Some Facts on Nevanlinna Pairs and their Interrelation to Matricial Schur Functions

In this section, we state some results on certain pairs of matrix-valued functions meromorphic in Π_+. These special pairs take over the role of the free parameters within the parametrization of the set of solutions to the matricial power moment problems. Before we recall the definition of this well-known class of so-called Nevanlinna pairs, we observe the following well-known fact:

Remark 4.1 The matrix $\widetilde{J}_q$ given by

$$\widetilde{J}_q := \begin{pmatrix} 0_{q\times q} & -\mathrm{i}I_q \\ \mathrm{i}I_q & 0_{q\times q} \end{pmatrix} \tag{4.1}$$

obviously is a $2q \times 2q$ signature matrix, i. e., $\widetilde{J}_q^* = \widetilde{J}_q$ and $\widetilde{J}_q^2 = I_{2q}$ hold true. Moreover,

$$\begin{pmatrix} A \\ B \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}^* \begin{pmatrix} 0 & \mathrm{i}I_q \\ -\mathrm{i}I_q & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mathrm{i}(A^*B - B^*A) = 2\,\mathrm{Im}(B^*A)$$

for all $A, B \in \mathbb{C}^{q\times q}$. In particular, $\binom{A}{I_q}^*(-\widetilde{J}_q)\binom{A}{I_q} = 2\,\mathrm{Im}(A)$ for each $A \in \mathbb{C}^{q\times q}$.

Definition 4.2 Let ϕ and ψ be $q \times q$ matrix-valued functions meromorphic in Π_+. The pair $[\phi; \psi]$ is called $q \times q$ *Nevanlinna pair in* Π_+ if there is a discrete subset $\mathcal{D}$ of Π_+ such that the following three conditions are fulfilled:

(i) ϕ and ψ are holomorphic in $\Pi_+ \setminus \mathcal{D}$.
(ii) $\operatorname{rank}\binom{\phi(w)}{\psi(w)} = q$ for each $w \in \Pi_+ \setminus \mathcal{D}$.
(iii) $\binom{\phi(w)}{\psi(w)}^*(-\widetilde{J}_q)\binom{\phi(w)}{\psi(w)} \in \mathbb{C}_{\geq}^{q\times q}$ for each $w \in \Pi_+ \setminus \mathcal{D}$.

We denote the set of all $q \times q$ Nevanlinna pairs in Π_+ by $\mathcal{PR}_q(\Pi_+)$.

Remark 4.3 Remark A.5 shows that condition (ii) of Definition 4.2 equivalently can be replaced by the following condition (ii'):

(ii') $\det[\psi(w) - \mathrm{i}\phi(w)] \neq 0$ for all $w \in \Pi_+ \setminus \mathcal{D}$.

Remark 4.4 Let $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$. For each $q \times q$ matrix-valued function g meromorphic in Π_+ such that the function $\det g$ does not vanish identically, one can easily see that the pair $[\phi g; \psi g]$ belongs to $\mathcal{PR}_q(\Pi_+)$ as well. Two $q \times q$ Nevanlinna pairs $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ in Π_+ are said to be *equivalent* if there exist a $q \times q$ matrix-valued function g meromorphic in Π_+ and a discrete subset $\mathcal{D}$ of Π_+ such that ϕ_1, ψ_1, ϕ_2, ψ_2, and g are holomorphic in $\Pi_+ \setminus \mathcal{D}$ and that $\det g(w) \neq 0$ as well as $\phi_2(w) = \phi_1(w)g(w)$ and $\psi_2(w) = \psi_1(w)g(w)$ hold true for each $w \in \Pi_+ \setminus \mathcal{D}$. Indeed, it is readily checked that this relation defines an equivalence relation on $\mathcal{PR}_q(\Pi_+)$. For each $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$, let $\langle[\phi; \psi]\rangle$ denote the equivalence class generated by $[\phi; \psi]$. Furthermore, if $\mathcal{M}$ is a non-empty subset of $\mathcal{PR}_q(\Pi_+)$, then let $\langle\mathcal{M}\rangle := \{\langle[\phi; \psi]\rangle : [\phi; \psi] \in \mathcal{M}\}$.

Remark 4.5 Let $\psi_0 : \Pi_+ \to \mathbb{C}^{q\times q}$ be given by $\psi_0(w) := I_q$. Then, for each $F \in \mathcal{R}_q(\Pi_+)$, the pair $[F; \psi_0]$ belongs to $\mathcal{PR}_q(\Pi_+)$.

Now we state a well-known interrelation between the classes $\mathcal{PR}_q(\Pi_+)$ and $\mathcal{S}_{q\times q}(\Pi_+)$.

Lemma 4.6

(a) *For each* $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$, *the function* $\det(\psi - \mathrm{i}\phi)$ *does not vanish identically and*

$$S := (\psi + \mathrm{i}\phi)(\psi - \mathrm{i}\phi)^{-1}$$

belongs to $\mathcal{S}_{q\times q}(\Pi_+)$.

(b) *For each* $S \in \mathcal{S}_{q\times q}(\Pi_+)$, *the pair* $[\phi; \psi]$ *given by* $\phi := \mathrm{i}(I_q - S)$ *and* $\psi := I_q + S$ *belongs to* $\mathcal{PR}_q(\Pi_+)$, *where the matrix-valued functions* ϕ *and* ψ *are holomorphic in* Π_+ *and fulfil, for all* $w \in \Pi_+$ *the inequality* $\det[\psi(w) - \mathrm{i}\phi(w)] \neq 0$ *and the equation*

$$S(w) = [\psi(w) + \mathrm{i}\phi(w)]\,[\psi(w) - \mathrm{i}\phi(w)]^{-1}.$$

(c) *Let* $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{PR}_q(\Pi_+)$. *Then* $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ *if and only if* $(\psi_1 + \mathrm{i}\phi_1)(\psi_1 - \mathrm{i}\phi_1)^{-1} = (\psi_2 + \mathrm{i}\phi_2)(\psi_2 - \mathrm{i}\phi_2)^{-1}$.

In view of Remarks A.6 and A.7 as well as Riemann's theorem on removable singularities, a detailed proof of Lemma 4.6 is given, e. g., in [33, Lem. 1.7]. Now we want to study special subclasses of the class $\mathcal{PR}_q(\Pi_+)$.

Notation 4.7 Let $M \in \mathbb{C}^{q\times p}$. We denote by $\mathcal{P}[M]$ the set of all pairs $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$ such that $\mathbb{P}_{\mathcal{R}(M)}\phi = \phi$ is fulfilled.

It should be mentioned that in our generic applications of Notation 4.7 the role of the matrix M is taken by the matrix s_0 in a sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$.

Example 4.8 Let $M \in \mathbb{C}^{q\times p}$. Remark 4.5 shows then that the pair $[\phi_0; \psi_0]$ given by $\phi_0(w) := 0_{q\times q}$ and $\psi_0(w) := I_q$ for all $w \in \Pi_+$ belongs to $\mathcal{P}[M]$.

Remark 4.9 Let $M \in \mathbb{C}^{q\times p}$ be such that $\operatorname{rank} M = q$. Then $\mathcal{R}(M) = \mathbb{C}^q$, $\mathbb{P}_{\mathcal{R}(M)} = I_q$, and, consequently, $\mathcal{P}[M] = \mathcal{PR}_q(\Pi_+)$.

Lemma 4.10 *Let* $M \in \mathbb{C}^{q\times p}$ *be such that* $r := \operatorname{rank} M$ *fulfils* $r \geq 1$. *Let* $u_1, u_2, \ldots, u_r$ *be an orthonormal basis of* $\mathcal{R}(M)$ *and let* $U := (u_1, u_2, \ldots, u_r)$. *Then* $\gamma_U : \mathcal{PR}_r(\Pi_+) \to \mathcal{P}[M]$ *given by*

$$\gamma_U\left([\widetilde{\phi}; \widetilde{\psi}]\right) := [U\widetilde{\phi}U^*; U\widetilde{\psi}U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}] \tag{4.2}$$

is well defined and injective.

Proof We are going to apply an idea which was used in the proof of [20, Lem. 13.4]. Obviously, we have

$$U^*U = I_r \qquad \text{and} \qquad \mathcal{R}(U) = \mathcal{R}(M). \tag{4.3}$$

Proposition A.8 shows that

$$\mathbb{P}^2_{[\mathcal{R}(M)]^\perp} = \mathbb{P}_{[\mathcal{R}(M)]^\perp} \qquad \text{and} \qquad \mathbb{P}^*_{[\mathcal{R}(M)]^\perp} = \mathbb{P}_{[\mathcal{R}(M)]^\perp} \tag{4.4}$$

hold true. Obviously, the equations

$$\mathbb{P}_{\mathcal{R}(M)} = UU^* \qquad \text{and} \qquad \mathbb{P}_{[\mathcal{R}(M)]^\perp} = I_q - UU^* \tag{4.5}$$

are valid. Thus, (4.5) and (4.3) yield

$$\mathbb{P}_{[\mathcal{R}(M)]^\perp}U = 0_{q\times r} \qquad \text{and} \qquad U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp} = 0_{r\times q}. \tag{4.6}$$

Now we consider an arbitrary pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$. According to Definition 4.2, we know that $\widetilde{\phi}$ and $\widetilde{\psi}$ are $r \times r$ matrix-valued functions meromorphic in Π_+ and that there is a discrete subset $\widetilde{\mathcal{D}}$ of Π_+ such that $\Pi_+ \setminus \widetilde{\mathcal{D}} \subseteq \mathbb{H}_{\widetilde{\phi}} \cap \mathbb{H}_{\widetilde{\psi}}$ and that

$$\operatorname{rank} \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix} = r \qquad \text{and} \qquad \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix}^* (-\widetilde{J}_r) \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix} \in \mathbb{C}_{\geq}^{r \times r} \tag{4.7}$$

hold true for all $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$. Then the matrix-valued functions

$$\phi := U\widetilde{\phi}U^* \qquad \text{and} \qquad \psi := U\widetilde{\psi}U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} \tag{4.8}$$

are meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}$. We consider an arbitrary $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$. Then from (4.3), (4.8), (4.7), and Lemma A.16 we get $\operatorname{rank} \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix} = q$. Using (4.8), (4.4), (4.3), and (4.6), we conclude

$$\begin{aligned} [\psi(w)]^*\phi(w) &= U[\widetilde{\psi}(w)]^*U^*U\widetilde{\phi}(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}U\widetilde{\phi}(w)U^* \\ &= U[\widetilde{\psi}(w)]^*\widetilde{\phi}(w)U^*, \end{aligned}$$

and then

$$\operatorname{Im}\left([\psi(w)]^*\phi(w)\right) = \operatorname{Im}\left(U[\widetilde{\psi}(w)]^*\widetilde{\phi}(w)U^*\right) = U \operatorname{Im}\left([\widetilde{\psi}(w)]^*\widetilde{\phi}(w)\right) U^*. \tag{4.9}$$

Because of Remark 4.1, from (4.9) it follows

$$\begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix} = U \left[\begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix}^* (-\widetilde{J}_r) \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix} \right] U^*. \tag{4.10}$$

Thus, (4.7) yields that the matrix on the left-hand side of (4.10) is non-negative Hermitian. Consequently, in view of Definition 4.2, the pair $[\phi; \psi]$ belongs to $\mathcal{PR}_q(\Pi_+)$. Taking into account (4.3) and (4.8), also we get $\mathbb{P}_{\mathcal{R}(M)}\phi = \mathbb{P}_{\mathcal{R}(U)}U\widetilde{\phi}U^* = U\widetilde{\phi}U^* = \phi$. Thus, from Notation 4.7 we get $[\phi; \psi] \in \mathcal{P}[M]$.

It remains to check that the mapping γ_U is injective. Let $[\widetilde{\phi}_1; \widetilde{\psi}_1], [\widetilde{\phi}_2; \widetilde{\psi}_2] \in \mathcal{PR}_r(\Pi_+)$ be such that $\gamma_U([\widetilde{\phi}_1; \widetilde{\psi}_1]) = \gamma_U([\widetilde{\phi}_2; \widetilde{\psi}_2])$. Then $U\widetilde{\phi}_1U^* = U\widetilde{\phi}_2U^*$ and $U\widetilde{\psi}_1U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} = U\widetilde{\psi}_2U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}$ are valid. Using (4.3), immediately $\widetilde{\phi}_1 = \widetilde{\phi}_2$ and $\widetilde{\psi}_1 = \widetilde{\psi}_2$ follow. Consequently, γ_U is injective. □

Lemma 4.11 *Let $M \in \mathbb{C}^{q \times p}$ and let $r := \operatorname{rank} M$ be such that $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(M)$ and let $U := (u_1, u_2, \ldots, u_r)$. Further, let $[\phi; \psi] \in \mathcal{P}[M]$. According to Notation 4.7 and Definition 4.2, let $\mathcal{D}$ be an arbitrary discrete subset of Π_+ such that the conditions (i)–(iii) in Definition 4.2 hold true. Then the matrix-valued function $B := \psi - \mathrm{i}\phi$ is meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$ and fulfils $\det B(w) \neq 0$ for all $w \in \Pi_+ \setminus \mathcal{D}$. Moreover, the* *pair* $[\widetilde{\phi}; \widetilde{\psi}]$ *given by*

$$\widetilde{\phi} := U^* \phi B^{-1} U \qquad \textit{and} \qquad \widetilde{\psi} := U^* \psi B^{-1} U \tag{4.11}$$

belongs to $\mathcal{PR}_r(\Pi_+)$. *Furthermore, the following statements are valid:*

(iv) $\widetilde{\phi}$ *and* $\widetilde{\psi}$ *are holomorphic in* $\Pi_+ \setminus \mathcal{D}$.
(v) $\operatorname{rank} \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix} = r$ *for all* $w \in \Pi_+ \setminus \mathcal{D}$.
(vi) $\begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix}^* (-\widetilde{J}_r) \begin{pmatrix} \widetilde{\phi}(w) \\ \widetilde{\psi}(w) \end{pmatrix} \in \mathbb{C}_{\geq}^{r \times r}$ *for all* $w \in \Pi_+ \setminus \mathcal{D}$.
(vii) *The matrix-valued functions*

$$S := U \widetilde{\phi} U^* \qquad \textit{and} \qquad T := U \widetilde{\psi} U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} \tag{4.12}$$

are meromorphic in Π_+ *as well as holomorphic in* $\Pi_+ \setminus \mathcal{D}$ *and fulfil*

$$S(w) = U \widetilde{\phi}(w) U^* \quad \textit{and} \quad T(w) = U \widetilde{\psi}(w) U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} \tag{4.13}$$

as well as

$$S(w) = \phi(w)[B(w)]^{-1} \qquad \textit{and} \qquad T(w) = \psi(w)[B(w)]^{-1} \tag{4.14}$$

for all $w \in \Pi_+ \setminus \mathcal{D}$. *In particular,*

$$\operatorname{rank} \begin{pmatrix} S(w) \\ T(w) \end{pmatrix} = q \tag{4.15}$$

and

$$\det\big([S(w)]^* S(w) + [T(w)]^* T(w)\big) = \det\big([\widetilde{\phi}(w)]^* \widetilde{\phi}(w) + [\widetilde{\psi}(w)]^* \widetilde{\psi}(w)\big) \tag{4.16}$$

hold true for all $w \in \Pi_+ \setminus \mathcal{D}$.
(viii) *The pair* $[S; T]$ *belongs to* $\mathcal{P}[M]$ *and fulfils* $\langle [S; T] \rangle = \langle [\phi; \psi] \rangle$.

Proof From $[\phi; \psi] \in \mathcal{P}[M] \subseteq \mathcal{PR}_q(\Pi_+)$ and Definition 4.2 we see that B is meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$. Consider an arbitrary $w \in \Pi_+ \setminus \mathcal{D}$. According to Definition 4.2(i), we have then $w \in \mathbb{H}_\phi \cap \mathbb{H}_\psi$. Hence,

$$B(w) = \psi(w) - \mathrm{i}\phi(w). \tag{4.17}$$

From Definition 4.2(ii) we get

$$\operatorname{rank} \begin{pmatrix} \phi(w) \\ \psi(w) \end{pmatrix} = q. \tag{4.18}$$

In view of Definition 4.2(iii), Remark 4.1 yields

$$\operatorname{Im}\left([\psi(w)]^*\phi(w)\right) \in \mathbb{C}_{\geq}^{q\times q}. \tag{4.19}$$

Regarding Notation 4.7, we see

$$\mathcal{R}(\phi(w)) \subseteq \mathcal{R}(M). \tag{4.20}$$

Taking into account (4.18), (4.19), (4.20), and (4.17), from Lemma A.17 we get that $\det B(w) \neq 0$. Since $w \in \Pi_+ \setminus \mathcal{D}$ is arbitrary and $\mathcal{D}$ is a discrete subset of Π_+, we consequently can conclude that B^{-1} is a matrix-valued function meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$. By virtue of Definition 4.2(i), we then see that the following statement holds true:

(ix) $\widetilde{\phi}$ and $\widetilde{\psi}$ given by (4.11) are meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$.

In particular, (iv) is proved. Furthermore,

$$\widetilde{\phi}(w) = U^*\phi(w)[B(w)]^{-1}U \quad \text{and} \quad \widetilde{\psi}(w) = U^*\psi(w)[B(w)]^{-1}U. \tag{4.21}$$

Because of (x) the matrix-valued functions S and T given by (4.12) are meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$ and both equations in (4.13) hold true. Once more using (4.18), (4.19), (4.20), (4.17), (4.21), (4.13), and Lemma A.17, we obtain $\operatorname{rank}\begin{pmatrix}\widetilde{\phi}(w)\\ \widetilde{\psi}(w)\end{pmatrix} = r$ and

$$\left[\widetilde{\psi}(w)\right]^* \widetilde{\phi}(w) = \left([B(w)]^{-1}U\right)^* [\psi(w)]^* \phi(w) \left([B(w)]^{-1}U\right) \tag{4.22}$$

as well as (4.15), (4.16), and (4.14). Since $w \in \Pi_+ \setminus \mathcal{D}$ is arbitrary and $\mathcal{D}$ is a discrete subset of Π_+, in particular $S = \phi B^{-1}$ and $T = \psi B^{-1}$ hold true and (v) and (vii) are proved. Taking additionally into account $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$ and Remark 4.4, therefore the pair $[S; T]$ belongs to $\mathcal{PR}_q(\Pi_+)$ and fulfils $\langle [S; T]\rangle = \langle [\phi; \psi]\rangle$. Regarding $[\phi; \psi] \in \mathcal{P}[M]$, also we get $\mathbb{P}_{\mathcal{R}(M)}S = \mathbb{P}_{\mathcal{R}(M)}\phi B^{-1} = \phi B^{-1} = S$. Thus, from Notation 4.7 we see $[S; T] \in \mathcal{P}[M]$. Consequently, (viii) is proved. Now we are going to check that the pair $[\widetilde{\phi}; \widetilde{\psi}]$ given by (4.11) belongs to $\mathcal{PR}_r(\Pi_+)$. We have already observed that (x) and (v) hold true. In view of (4.22) and (4.19), we conclude

$$\begin{aligned}\operatorname{Im}\left(\left[\widetilde{\psi}(w)\right]^* \widetilde{\phi}(w)\right) &= \operatorname{Im}\left[\left([B(w)]^{-1}U\right)^* [\psi(w)]^* \phi(w) \left([B(w)]^{-1}U\right)\right] \\ &= \left([B(w)]^{-1}U\right)^* \operatorname{Im}\left([\psi(w)]^* \phi(w)\right) \left([B(w)]^{-1}U\right) \in \mathbb{C}_{\geq}^{r\times r}.\end{aligned}$$

Since $w \in \Pi_+ \setminus \mathcal{D}$ is arbitrary, by virtue of Remark 4.1 then (vi) follows. In view of (x), (v), (vi), and Definition 4.2, we succeeded in proving that the pair $[\widetilde{\phi}; \widetilde{\psi}]$ belongs to $\mathcal{PR}_r(\Pi_+)$. □

Now we turn to the main result of this section. It will be used in the proof of Theorem 9.3 which is one of the central results of this paper.

Proposition 4.12 *Let* $M \in \mathbb{C}^{q\times p}$ *and let* $r := \operatorname{rank} M$. *Then:*

(a) *If* $r = 0$*, then* $\langle \mathcal{P}[M]\rangle = \{\langle[\phi_0; \psi_0]\rangle\}$*, where* $\phi_0, \psi_0 \colon \Pi_+ \to \mathbb{C}^{q\times q}$ *are defined by* $\phi_0(w) := 0_{q\times q}$ *and* $\psi_0(w) := I_q$*, respectively.*

(b) *Suppose* $r \geq 1$. *Let* $u_1, u_2, \ldots, u_r$ *be an orthonormal basis of* $\mathcal{R}(M)$ *and let* $U := (u_1, u_2, \ldots, u_r)$. *Then the mapping* $\Gamma_U \colon \langle \mathcal{PR}_r(\Pi_+)\rangle \to \langle \mathcal{P}[M]\rangle$ *given by*

$$\Gamma_U\left(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle\right) := \langle[U\widetilde{\phi}U^*; U\widetilde{\psi}U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}]\rangle \tag{4.23}$$

is well defined and bijective.

Proof For our proof, we use the strategy used in [20, Lem. 13.7] and [28, Lem. 10.1.4] for an analogous result.

(a) In order to prove part (a), now we suppose that $r = 0$. Then $M = 0_{q\times p}$ and hence $\mathbb{P}_{\mathcal{R}(M)} = 0_{q\times q}$. Example 4.8 yields $[\phi_0; \psi_0] \in \mathcal{P}[M]$. In particular, $\{\langle[\phi_0; \psi_0]\rangle\} \subseteq \langle \mathcal{P}[M]\rangle$. Let $[\phi; \psi]$ be an arbitrary $q \times q$ Nevanlinna pair belonging to $\mathcal{P}[M]$, i.e., to $\mathcal{P}[0_{q\times p}]$. Taking into account Notation 4.7 and $\mathbb{P}_{\mathcal{R}(M)} = 0_{q\times q}$, then we have $\phi = \mathbb{P}_{\mathcal{R}(M)}\phi = \phi_0$. Because of Definition 4.2, there exists a discrete subset $\mathcal{D}$ of Π_+ such that $\Pi_+ \setminus \mathcal{D} \subseteq \mathbb{H}_\phi \cap \mathbb{H}_\psi$ and that $\operatorname{rank}\binom{\phi(w)}{\psi(w)} = q$ for all $w \in \Pi_+ \setminus \mathcal{D}$. Hence, Remark 4.3 yields $\det[\psi(w)-\mathrm{i}\phi(w)] \neq 0$ for all $w \in \Pi_+\setminus\mathcal{D}$. Thus, $\phi = \phi_0$ implies $\det\psi(w) \neq 0$ for all $w \in \Pi_+ \setminus \mathcal{D}$. Setting $g := \psi$, then we see that g is a matrix-valued function meromorphic in Π_+ with $\Pi_+ \setminus \mathcal{D} \subseteq \mathbb{H}_\phi \cap \mathbb{H}_\psi \cap \mathbb{H}_g$ such that $\phi(w) = \phi_0(w) = 0_{q\times q} = 0_{q\times q} \cdot g(w) = \phi_0(w)g(w)$ as well as $\psi(w) = I_q \cdot \psi(w) = \psi_0(w)g(w)$, and $\det g(w) = \det\psi(w) \neq 0$ for all $w \in \Pi_+ \setminus \mathcal{D}$. Remark 4.4 then yields $\langle[\phi; \psi]\rangle = \langle[\phi_0; \psi_0]\rangle$. Thus, $\langle \mathcal{P}[M]\rangle \subseteq \{\langle[\phi_0; \psi_0]\rangle\}$. Consequently, $\langle \mathcal{P}[M]\rangle = \{\langle[\phi_0; \psi_0]\rangle\}$. Part (a) is proved.

(b) Suppose $r \geq 1$. The proof of part (b) is divided into five parts.

(I) As in the proof of Lemma 4.10 we see that all the equations in (4.3), (4.4), (4.5), and (4.6) hold true. Lemma 4.10 yields that the mapping $\gamma_U \colon \mathcal{PR}_r(\Pi_+) \to \mathcal{P}[M]$ given by (4.2) is well defined and injective.

(II) Our next goal is to check that $\Gamma_U(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle)$ is independent of the choice of the particular representative $[\widetilde{\phi}; \widetilde{\psi}]$ of the equivalence class $\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle \in \langle \mathcal{PR}_r(\Pi_+)\rangle$. For this reason, we consider arbitrary pairs $[\widetilde{\phi}_1; \widetilde{\psi}_1], [\widetilde{\phi}_2; \widetilde{\psi}_2] \in \mathcal{PR}_r(\Pi_+)$ such that $\langle[\widetilde{\phi}_1; \widetilde{\psi}_1]\rangle = \langle[\widetilde{\phi}_2; \widetilde{\psi}_2]\rangle$. In view of Remark 4.4, then there are an $r \times r$ matrix-valued function $\tilde{g}$ meromorphic in Π_+ and a discrete subset $\widetilde{\mathcal{D}}$ of Π_+ such that $\widetilde{\phi}_1, \widetilde{\psi}_1, \widetilde{\phi}_2, \widetilde{\psi}_2$, and $\tilde{g}$ are holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}$ and that $\det\tilde{g}(w) \neq 0$ as well as

$$\widetilde{\phi}_2(w) = \widetilde{\phi}_1(w)\tilde{g}(w) \qquad \text{and} \qquad \widetilde{\psi}_2(w) = \widetilde{\psi}_1(w)\tilde{g}(w) \tag{4.24}$$

hold true for all $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$. For each $j \in \{1, 2\}$, we set

$$\phi_j := U\widetilde{\phi}_j U^* \qquad \text{and} \qquad \psi_j := U\widetilde{\psi}_j U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}. \tag{4.25}$$

According to (I), the pairs $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ belong to $\mathcal{P}[M]$ and thus to $\mathcal{PR}_q(\Pi_+)$. Furthermore, ϕ_1, ψ_1, ϕ_2, and ψ_2 are holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}$. Obviously, the function $g := U\tilde{g}U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}$ is meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}$. Moreover, for each $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$, from (4.25), (4.3), (4.6), (4.24), and once more, (4.25) we get

$$\begin{aligned}
\phi_1(w)g(w) &= U\widetilde{\phi}_1(w)U^* \left[U\tilde{g}(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right] \\
&= U\widetilde{\phi}_1(w)U^*U\tilde{g}(w)U^* + U\widetilde{\phi}_1(w)U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp} \\
&= U\widetilde{\phi}_1(w)\tilde{g}(w)U^* = U\widetilde{\phi}_2(w)U^* = \phi_2(w)
\end{aligned} \tag{4.26}$$

and, in view of (4.4), similarly

$$\begin{aligned}
\psi_1(w)g(w) &= \left[U\widetilde{\psi}_1(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right]\left[U\tilde{g}(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right] \\
&= U\widetilde{\psi}_1(w)U^*U\tilde{g}(w)U^* + U\widetilde{\psi}_1(w)U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp} \\
&\qquad + \mathbb{P}_{[\mathcal{R}(M)]^\perp}U\tilde{g}(w)U^* + \mathbb{P}^2_{[\mathcal{R}(M)]^\perp} \\
&= U\widetilde{\psi}_1(w)\tilde{g}(w)U^* + \mathbb{P}^2_{[\mathcal{R}(M)]^\perp} \\
&= U\widetilde{\psi}_2(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} = \psi_2(w).
\end{aligned} \tag{4.27}$$

Now we consider an arbitrary $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$. Then $\det \tilde{g}(w) \neq 0$ and, consequently, $\mathcal{N}(\tilde{g}(w)) = \{0_{r\times 1}\}$. In order to verify $\det g(w) \neq 0$, it is sufficient to prove that $\mathcal{N}(g(w)) \subseteq \{0_{q\times 1}\}$. We consider an arbitrary $x \in \mathcal{N}(g(w))$. Set $y := U^*x$. Then

$$U\tilde{g}(w)y + \mathbb{P}_{[\mathcal{R}(M)]^\perp}x = \left[U\tilde{g}(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right]x = g(w)x = 0_{q\times 1}. \tag{4.28}$$

Because of (4.3) and (4.6), from (4.28) we conclude

$$\begin{aligned}
\tilde{g}(w)y &= U^*U\tilde{g}(w)y = U^*U\tilde{g}(w)y + U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp}x \\
&= U^*\left[Ug(w)y + \mathbb{P}_{[\mathcal{R}(M)]^\perp}x\right] = U^* \cdot 0_{q\times 1} = 0_{r\times 1},
\end{aligned}$$

i. e., $y \in \mathcal{N}(\tilde{g}(w))$. Hence, $y = 0_{r\times 1}$. Taking into account (4.28), then also we obtain $\mathbb{P}_{[\mathcal{R}(M)]^\perp}x = 0_{q\times 1}$. Using (4.5), we infer $x = UU^*x = Uy = U \cdot 0_{r\times 1} = 0_{q\times 1}$. Consequently, $\mathcal{N}(g(w)) \subseteq \{0_{q\times 1}\}$, i. e., $\det g(w) \neq 0$. Thus, in view of (4.26) and (4.27), we conclude that $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$.

(III) Summarizing parts (I) and (II), the mapping Γ_U is well defined.

(IV) Now we check that the mapping Γ_U is injective. For this purpose, we consider arbitrary pairs $[\widetilde{\phi}_1; \widetilde{\psi}_1], [\widetilde{\phi}_2; \widetilde{\psi}_2] \in \mathcal{PR}_r(\Pi_+)$ such that

$$\Gamma_U\left(\langle[\widetilde{\phi}_1; \widetilde{\psi}_1]\rangle\right) = \Gamma_U\left(\langle[\widetilde{\phi}_2; \widetilde{\psi}_2]\rangle\right). \tag{4.29}$$

Part (I) of the proof shows that the pairs $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ given by (4.25) belong to $\mathcal{P}[M]$ and thus to $\mathcal{PR}_q(\Pi_+)$, whereas (4.23) and (4.29) imply $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$. According to Remark 4.4, then there are a $q \times q$ matrix-valued function g which is meromorphic in Π_+ and a discrete subset $\mathcal{D}$ of Π_+ such that ϕ_1, ψ_1, ϕ_2, ψ_2, and g are holomorphic in $\Pi_+ \setminus \mathcal{D}$ and $\det g(w) \neq 0$ as well as $\phi_2(w) = \phi_1(w)g(w)$ and $\psi_2(w) = \psi_1(w)g(w)$ hold true for all $w \in \Pi_+ \setminus \mathcal{D}$. By Definition 4.2 there exist discrete subsets $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$ of Π_+ such that $\widetilde{\phi}_1$ and $\widetilde{\psi}_1$ are holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}_1$ and $\widetilde{\phi}_2$ and $\widetilde{\psi}_2$ are holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}_2$. Let $\tilde{g} := U^* g U$ and let $\widetilde{\mathcal{D}} := \mathcal{D} \cup \widetilde{\mathcal{D}}_1 \cup \widetilde{\mathcal{D}}_2$. Then $\tilde{g}$ is an $r \times r$ matrix-valued function which is meromorphic in Π_+ and $\widetilde{\mathcal{D}}$ is a discrete subset of Π_+ such that ϕ_1, ψ_1, ϕ_2, ψ_2, and g as well as $\widetilde{\phi}_1$, $\widetilde{\psi}_1$, $\widetilde{\phi}_2$, $\widetilde{\psi}_2$, and $\tilde{g}$ are holomorphic in $\Pi_+ \setminus \widetilde{\mathcal{D}}$. Now consider an arbitrary $w \in \Pi_+ \setminus \widetilde{\mathcal{D}}$. Then $\det g(w) \neq 0$ as well as

$$\phi_2(w) = \phi_1(w)g(w) \qquad \text{and} \qquad \psi_2(w) = \psi_1(w)g(w). \tag{4.30}$$

Using (4.3), (4.25), (4.30), and, once more, (4.25) and (4.3), we have

$$\begin{aligned}\widetilde{\phi}_1(w)\tilde{g}(w) &= U^*U\widetilde{\phi}_1(w)U^*g(w)U = U^*\phi_1(w)g(w)U \\ &= U^*\phi_2(w)U = U^*U\widetilde{\phi}_2(w)U^*U = \widetilde{\phi}_2(w)\end{aligned} \tag{4.31}$$

and, in view of (4.6), similarly

$$\begin{aligned}&\widetilde{\psi}_1(w)\tilde{g}(w) = U^*U\widetilde{\psi}_1(w)U^*g(w)U = U^*\left[\psi_1(w) - \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right]g(w)U \\ &= U^*\psi_1(w)g(w)U - U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp}g(w)U = U^*\psi_2(w)U \\ &= U^*\left[U\widetilde{\psi}_2(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right]U = \widetilde{\psi}_2(w) + U^*\mathbb{P}_{[\mathcal{R}(M)]^\perp}U = \widetilde{\psi}_2(w).\end{aligned} \tag{4.32}$$

We consider an arbitrary $y \in \mathcal{N}(\tilde{g}(w))$. Setting $x := Uy$, we get

$$U^*g(w)x = U^*g(w)Uy = \tilde{g}(w)y = 0_{r\times 1}. \tag{4.33}$$

Because of (4.25), (4.6), and (4.4), for each $j \in \{1, 2\}$, we conclude

$$\begin{aligned}\mathbb{P}_{[\mathcal{R}(M)]^\perp}\psi_j(w) &= \mathbb{P}_{[\mathcal{R}(M)]^\perp}\left[U\widetilde{\psi}_j(w)U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}\right] \\ &= \mathbb{P}_{[\mathcal{R}(M)]^\perp}U\widetilde{\psi}_j(w)U^* + \mathbb{P}^2_{[\mathcal{R}(M)]^\perp} = \mathbb{P}_{[\mathcal{R}(M)]^\perp}.\end{aligned} \tag{4.34}$$

Taking into account (4.34), (4.30), once more (4.34), and (4.5), we get

$$\begin{aligned}\mathbb{P}_{[\mathcal{R}(M)]^\perp} &= \mathbb{P}_{[\mathcal{R}(M)]^\perp}\psi_2(w) = \mathbb{P}_{[\mathcal{R}(M)]^\perp}\psi_1(w)g(w)\\ &= \mathbb{P}_{[\mathcal{R}(M)]^\perp}g(w) = (I_q - UU^*)g(w) = g(w) - UU^*g(w).\end{aligned} \tag{4.35}$$

Since (4.6) implies $\mathbb{P}_{[\mathcal{R}(M)]^\perp}x = \mathbb{P}_{[\mathcal{R}(M)]^\perp}Uy = 0_{q\times r}\cdot y = 0_{q\times 1}$, from (4.35) we infer $g(w)x = \mathbb{P}_{[\mathcal{R}(M)]^\perp}x + UU^*g(w)x = UU^*g(w)x$ and, in view of (4.33), consequently, $g(w)x = 0_{q\times 1}$. By virtue of $\det g(w) \neq 0$, then $x = 0_{q\times 1}$ follows. Thus, using (4.3), we see that $y = U^*Uy = U^*x = U^*\cdot 0_{q\times 1} = 0_{r\times 1}$ is valid. Hence, $\mathcal{N}(\tilde{g}(w)) \subseteq \{0_{r\times 1}\}$. Therefore, $\det \tilde{g}(w) \neq 0$ holds true. Summarizing the last inequality, (4.31) and (4.32), we get that $\langle[\widetilde{\phi}_1; \widetilde{\psi}_1]\rangle = \langle[\widetilde{\phi}_2; \widetilde{\psi}_2]\rangle$ is valid. Consequently, the mapping Γ_U is injective.

(V) Now we are going to verify that the mapping Γ_U is surjective. For this reason, we consider an arbitrary pair $[\phi; \psi] \in \mathcal{P}[M]$. Let $\mathcal{D}$ be a discrete subset of Π_+ such that the conditions (i)–(iii) of Definition 4.2 are fulfilled. We are looking for a pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$ such that $\Gamma_U(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle) = \langle[\phi; \psi]\rangle$. According to Lemma 4.11, the pair $[\widetilde{\phi}; \widetilde{\psi}]$ given by (4.11) belongs to $\mathcal{PR}_r(\Pi_+)$, the pair $[S; T]$ defined by (4.12) belongs to $\mathcal{P}[M]$, and $\langle[S; T]\rangle = \langle[\phi; \psi]\rangle$ holds true. Taking into account (4.12), part (III) of the proof of part (b), and (4.23), then we conclude that $\Gamma_U(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle) = \langle[S; T]\rangle$. Consequently, $\Gamma_U(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle) = \langle[\phi; \psi]\rangle$ follows. Thus, the mapping Γ_U is surjective.

□

5 Some Observations on Block Hankel Matrices

In this section, we present some useful identities concerning block Hankel matrices. This material is mostly taken from [10]. We continue to use the notation introduced in Sect. 2. In the following, we are going to state a particular parametrization of special sequences of complex matrices. Therefore, for a given $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we need particular Schur complements of the matrices formed by a given sequence $(s_j)_{j=0}^{\kappa}$ of complex $p\times q$ matrices. For this reason and our further consideration, it seems to be useful to introduce some further notation.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p\times q$ matrices. For every choice of integers m and n fulfilling $0 \leq m \leq n \leq \kappa$, let

$$y_{m,n} := \begin{pmatrix} s_m \\ \vdots \\ s_n \end{pmatrix} \qquad \text{and} \qquad z_{m,n} := (s_m, \ldots, s_n).$$

Let

$$\Theta_0 := 0_{p\times q} \qquad \text{and} \qquad \Theta_n := z_{n,2n-1} H_{n-1}^{+} y_{n,2n-1} \tag{5.1}$$

for each $n \in \mathbb{N}$ such that $2n-1 \leq \kappa$. During our consideration, for each $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$, the Schur complement

$$L_n := s_{2n} - \Theta_n$$

will be of essential importance. For all $n \in \mathbb{N}_0$ fulfilling $2n+1 \leq \kappa$, we also introduce the block Hankel matrix $K_n := (s_{j+k+1})_{j,k=0}^{n}$. For every choice of $k \in \mathbb{N}$ fulfilling $2k-1 \leq \kappa$, we set

$$\Sigma_k := z_{k,2k-1} H_{k-1}^{+} K_{k-1} H_{k-1}^{+} y_{k,2k-1}.$$

For each $k \in \mathbb{N}$ fulfilling $2k \leq \kappa$, let

$$M_k := z_{k,2k-1} H_{k-1}^{+} y_{k+1,2k} \qquad \text{and} \qquad N_k := z_{k+1,2k} H_{k-1}^{+} y_{k,2k-1}.$$

Let

$$\Lambda_0 := 0_{p\times q} \qquad \text{and} \qquad \Lambda_k := M_k + N_k - \Sigma_k \tag{5.2}$$

for all $k \in \mathbb{N}$ fulfilling $2k \leq \kappa$

Now we turn our attention to sequences of complex $q \times q$ matrices which are introduced in Sect. 2 and which are defined by certain properties of block Hankel matrices built from the given sequence.

Remark 5.1 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\geq}$. Then from the definition of the set $\mathcal{H}_{q,2\kappa}^{\geq}$ and (2.1) we see that $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,2\kappa}$ and $s_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

Remark 5.2 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. It is easy to see that $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$ is valid if and only if $(s_j)_{j=0}^{k} \in \mathcal{H}_{q,k}^{\geq,\mathrm{e}}$ holds true for each $k \in \mathbb{Z}_{0,\kappa}$.

Definition 5.3 ([10, Def. 2.2]) Let $n \in \mathbb{N}_0$. A sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ is called *completely degenerate* if $L_n = 0_{q\times q}$. We will write $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$ for the set of all completely degenerate sequences $(s_j)_{j=0}^{2n}$.

Remark 5.4 ([10, Cor. 2.14]) $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}} \subseteq \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ for all $n \in \mathbb{N}_0$.

The parameters which will be introduced in the following definition play a key role in a detailed analysis of block Hankel matrices as well as in our following consideration.

Definition 5.5 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $k \in \mathbb{N}_0$ fulfilling $2k \leq \kappa$, let $\mathfrak{h}_{2k} := s_{2k} - \Theta_k$, where Θ_k is given in (5.1), and, for each $k \in \mathbb{N}_0$ fulfilling $2k+1 \leq \kappa$, let $\mathfrak{h}_{2k+1} := s_{2k+1} - \Lambda_k$, where Λ_k is given by (5.2). Then $(\mathfrak{h}_j)_{j=0}^{\kappa}$ is called the *sequence of $\mathcal{H}$-parameters of* $(s_j)_{j=0}^{\kappa}$.

Now we recall characterizations of sequences belonging to $\mathcal{H}_{q,2n}^{\geq}$, $\mathcal{H}_{q,2n}^{>}$, and $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$, respectively, by their sequences of $\mathcal{H}$-parameters.

Proposition 5.6 ([10, Prop. 2.30]) *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices with sequence of $\mathcal{H}$-parameters $(\mathfrak{h}_j)_{j=0}^{2n}$. Then:*

(a) *The sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$ if and only if the following three conditions are fulfilled:*

  (i) $\mathfrak{h}_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ *for all* $k \in \mathbb{Z}_{0,n}$.
  (ii) *If $n \geq 1$, then both $\mathfrak{h}_{2k-1}^* = \mathfrak{h}_{2k-1}$ as well as $\mathcal{R}(\mathfrak{h}_{2k-1}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ hold true for all $k \in \mathbb{Z}_{1,n}$.*
  (iii) *If $n \geq 2$, then $\mathcal{R}(\mathfrak{h}_{2k}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ for all $k \in \mathbb{Z}_{1,n-1}$.*

(b) *The sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{>}$ if and only if the following two conditions are fulfilled:*

  (iv) $\mathfrak{h}_{2k} \in \mathbb{C}_{>}^{q\times q}$ *for all* $k \in \mathbb{Z}_{0,n}$.
  (v) *If $n \geq 1$, then $\mathfrak{h}_{2k-1}^* = \mathfrak{h}_{2k-1}$ for all $k \in \mathbb{Z}_{1,n}$.*

(c) *The sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$ if and only if all the conditions (i)–(iii) as well as $\mathfrak{h}_{2n} = 0_{q\times q}$ are fulfilled.*

Observe that, in view of both the definition of the set $\mathcal{H}_{q,\infty}^{\geq}$ as well as Proposition 5.6(a), the class $\mathcal{H}_{q,\infty}^{\geq}$ can be characterized using $\mathcal{H}$-parameters as well. We omit the details. Now we characterize $\mathbb{R}$-non-negative definite extendable sequences of complex $q \times q$ matrices by their sequences of $\mathcal{H}$-parameters.

Proposition 5.7 ([10, Prop. 2.30(c)]) *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with sequence of $\mathcal{H}$-parameters $(\mathfrak{h}_j)_{j=0}^{\kappa}$. Then $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$ if and only if the following three conditions are fulfilled:*

(vi) $\mathfrak{h}_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ *for all $k \in \mathbb{N}_0$ fulfilling $2k \leq \kappa$.*
(vii) *If $\kappa \geq 1$, then $\mathfrak{h}_{2k-1}^* = \mathfrak{h}_{2k-1}$ as well as $\mathcal{R}(\mathfrak{h}_{2k-1}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ hold true for all $k \in \mathbb{N}$ fulfilling $2k-1 \leq \kappa$.*
(viii) *If $\kappa \geq 2$, then $\mathcal{R}(\mathfrak{h}_{2k}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ for all $k \in \mathbb{N}$ such that $2k \leq \kappa$.*

6 A Schur Type Algorithm for Sequences of Complex $p \times q$ Matrices

In this section, we recall some essential facts and prove some special technical results on a Schur type algorithm for sequences of complex $p \times q$ matrices which was introduced in [4] and discussed in detail in an alternative setting in [13]. The elementary step of this algorithm is based on the use of a certain reciprocal sequence of a finite or infinite sequence of complex $p \times q$ matrices. This notion plays a key role for our consideration in this section. For this reason, first we recall the definition of the reciprocal sequence.

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(s_j^{\sharp})_{j=0}^{\kappa}$ of complex $q \times p$ matrices given by $s_0^{\sharp} := s_0^{+}$ and, for all $k \in \mathbb{Z}_{1,\kappa}$, recursively by

$$s_k^{\sharp} := -s_0^{+} \sum_{j=0}^{k-1} s_{k-j} s_j^{\sharp}, \tag{6.1}$$

is called the *reciprocal sequence corresponding to* $(s_j)_{j=0}^{\kappa}$. A detailed discussion of reciprocal sequences is given in [14]. Here first we explain the elementary step of the Schur type algorithm under consideration. Let $\kappa \in \mathbb{Z}_{2,\infty} \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with reciprocal sequence $(s_j^{\sharp})_{j=0}^{\kappa}$. Then the sequence $(s_j^{(1)})_{j=0}^{\kappa-2}$ defined, for all $j \in \mathbb{Z}_{0,\kappa-2}$, by

$$s_j^{(1)} := -s_0 s_{j+2}^{\sharp} s_0 \tag{6.2}$$

is said to be the *first Schur transform of* $(s_j)_{j=0}^{\kappa}$.

Remark 6.1 ([13, Rem. 8.2]) Let $\kappa \in \mathbb{Z}_{2,\infty} \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with first Schur transform $(s_j^{(1)})_{j=0}^{\kappa-2}$. Then from (6.2) and (6.1) it is obvious that, for all $m \in \mathbb{Z}_{2,\kappa}$, the sequence $(s_j^{(1)})_{j=0}^{m-2}$ is the first Schur transform of $(s_j)_{j=0}^{m}$.

As considered in [13, Def. 9.1] already, the repeated application of the first Schur transform in a natural way generates a corresponding algorithm for (finite or infinite) sequences of complex $p \times q$ matrices:

Remark 6.2 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(s_j^{(0)})_{j=0}^{\kappa}$ given by $s_j^{(0)} := s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$, is called the 0th *Schur transform of* $(s_j)_{j=0}^{\kappa}$. If $\kappa \geq 2$, then the kth Schur transform is defined recursively: For all $k \in \mathbb{N}$ fulfilling $2k \leq \kappa$, the first Schur transform $(s_j^{(k)})_{j=0}^{\kappa-2k}$ of $(s_j^{(k-1)})_{j=0}^{\kappa-2(k-1)}$ is called the *kth Schur transform of* $(s_j)_{j=0}^{\kappa}$.

One of the central properties of the just introduced Schur type algorithm is that it preserves the $\mathbb{R}$-non-negative definite extendability of sequences of matrices.

Proposition 6.3 ([13, Propositions 9.4 and 9.5]) *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$, and let $k \in \mathbb{N}_0$ fulfilling $2k \leq \kappa$. Then the kth Schur transform $(s_j^{(k)})_{j=0}^{\kappa-2k}$ of $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{H}_{q,\kappa-2k}^{\geq,\mathrm{e}}$.*

The Schur type algorithm considered in [13] is an essential tool for the parametrization of the solution set of Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{m}, =]$ given in [15]. In order to discuss Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ in a similar manner, we need some further results concerning this Schur type algorithm. We start with a slight modification of [13, Prop. 8.23].

Lemma 6.4 *Let $\kappa \in \mathbb{Z}_{2,\infty}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $s_0^{(1)} = s_0 s_0^+ (s_2 - s_1 s_0^+ s_1) s_0^+ s_0$ and, in the case $\kappa \geq 3$, for all $j \in \mathbb{Z}_{1,\kappa-2}$, moreover,*

$$s_j^{(1)} = s_0 s_0^+ \left[(s_{j+2} - s_{j+1} s_0^+ s_1) s_0^+ s_0 - \sum_{l=0}^{j-1} s_{j-l} s_0^+ s_l^{(1)} \right].$$

Lemma 6.5 *Let $m \in \mathbb{Z}_{2,\infty}$. Further, let $(s_j)_{j=0}^{m}$ and $(t_j)_{j=0}^{m}$ be sequences of complex $p \times q$ matrices such that $s_j = t_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Then $s_{m-2}^{(1)} - t_{m-2}^{(1)} = s_0 s_0^+ (s_m - t_m) s_0^+ s_0$ and, in the case $m \geq 3$, for each $j \in \mathbb{Z}_{0,m-3}$, moreover, $s_j^{(1)} = t_j^{(1)}$.*

Proof First suppose $m \geq 3$. From Remark 6.1 we see immediately that $s_j^{(1)} = t_j^{(1)}$ holds true for all $j \in \mathbb{Z}_{0,m-3}$. Hence, applying Lemma 6.4, a straightforward calculation yields $s_{m-2}^{(1)} - t_{m-2}^{(1)} = s_0 s_0^+ (s_m - t_m) s_0^+ s_0$. If $m = 2$, then from Lemma 6.4 immediately we conclude $s_0^{(1)} - t_0^{(1)} = s_0 s_0^+ (s_2 - t_2) s_0^+ s_0$. □

Lemma 6.6 *Let $m \in \mathbb{Z}_{2,\infty}$. Let $(s_j)_{j=0}^{m}$ and $(t_j)_{j=0}^{m}$ be sequences of Hermitian complex $q \times q$ matrices such that $s_j = t_j$ for all $j \in \mathbb{Z}_{0,m-1}$ and that $t_m \leq s_m$. Then $(s_j^{(1)})^* = s_j^{(1)}$ and $(t_j^{(1)})^* = t_j^{(1)}$ for all $j \in \mathbb{Z}_{0,m-2}$ and $t_{m-2}^{(1)} \leq s_{m-2}^{(1)}$. Moreover, if $m \geq 3$, then $t_j^{(1)} = s_j^{(1)}$ for all $j \in \mathbb{Z}_{0,m-3}$.*

Proof Using (6.2) and [14, Cor. 5.17], we get

$$(s_j^{(1)})^* = (-s_0 s_{j+2}^{\sharp} s_0)^* = -s_0^* (s_{j+2}^{\sharp})^* s_0^* = -s_0 (s_{j+2}^{\sharp})^* s_0 = -s_0 s_{j+2}^{\sharp} s_0 = s_j^{(1)}$$

and, analogously, $(t_j^{(1)})^* = t_j^{(1)}$ for all $j \in \mathbb{Z}_{0,m-2}$. Lemma 6.5 yields the equation $s_{m-2}^{(1)} - t_{m-2}^{(1)} = s_0 s_0^+ (s_m - t_m) s_0^+ s_0$ and, in the case $m \geq 3$, moreover $s_j^{(1)} = t_j^{(1)}$ for all $j \in \mathbb{Z}_{0,m-3}$. We have $(s_0 s_0^+)^* = (s_0^+)^* s_0^* = (s_0^*)^+ s_0^* = s_0^+ s_0$. Thus,

$s_{m-2}^{(1)} - t_{m-2}^{(1)} = (s_0 s_0^+)(s_m - t_m)(s_0 s_0^+)^*$ and, because of the assumption $t_m \leq s_m$, then $t_{m-2}^{(1)} \leq s_{m-2}^{(1)}$ follows. □

Definition 6.7 ([13, Def. 10.1]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(t_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let A and B be complex $p \times q$ matrices. Define $t_0^{(-1,A,B)} := A$, $t_1^{(-1,A,B)} := AA^+BA^+A$, and recursively, for each $m \in \mathbb{Z}_{2,\kappa+2}$, moreover,

$$t_m^{(-1,A,B)} := \sum_{j=0}^{m-2} AA^+ t_{m-j-2} A^+ t_j^{(-1,A,B)} + AA^+BA^+ t_{m-1}^{(-1,A,B)}.$$

Then the sequence $(t_j^{(-1,A,B)})_{j=0}^{\kappa+2}$ is called the *first inverse Schur transform corresponding to* $[(t_j)_{j=0}^{\kappa}, A, B]$.

It should be mentioned that in our generic application of the construction introduced in Definition 6.7 the role of the matrices A and B is played by the matrices s_0 and s_1 which are taken from a sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$.

Lemma 6.8 *Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m}$ and $(t_j)_{j=0}^{m}$ be sequences of complex $p \times q$ matrices which, in the case $m \geq 1$, fulfil $s_j = t_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Furthermore, let $A, B \in \mathbb{C}^{p\times q}$. Then $s_j^{(-1,A,B)} = t_j^{(-1,A,B)}$ for each $j \in \mathbb{Z}_{0,m+1}$ and*

$$s_{m+2}^{(-1,A,B)} - t_{m+2}^{(-1,A,B)} = AA^+(s_m - t_m)A^+A. \tag{6.3}$$

Proof First we consider the case $m = 0$. Then $s_0^{(-1,A,B)} = A = t_0^{(-1,A,B)}$ and $s_1^{(-1,A,B)} = AA^+BA^+A = t_1^{(-1,A,B)}$. Now we assume $m \geq 1$. Then from [13, Rem. 10.2] we know that $s_j^{(-1,A,B)} = t_j^{(-1,A,B)}$ holds true for all $j \in \mathbb{Z}_{0,m+1}$. Consequently, taking into account Definition 6.7, a straightforward calculation yields that the equations $s_{m+2}^{(-1,A,B)} - t_{m+2}^{(-1,A,B)} = AA^+s_mA^+s_0^{(-1,A,B)} - AA^+t_mA^+t_0^{(-1,A,B)} = AA^+(s_m - t_m)A^+A$ hold true. □

Lemma 6.9 *Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m}$ and $(t_j)_{j=0}^{m}$ be sequences of Hermitian complex $q \times q$ matrices such that $t_m \leq s_m$ and, in the case $m \geq 1$, such that $s_j = t_j$ is valid for all $j \in \mathbb{Z}_{0,m-1}$. Further, let $A, B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. For each $j \in \mathbb{Z}_{0,m+2}$, then $s_j^{(-1,A,B)}$ and $t_j^{(-1,A,B)}$ belong to $\mathbb{C}_{\mathrm{H}}^{q\times q}$ as well. Moreover, $s_j^{(-1,A,B)} = t_j^{(-1,A,B)}$ for all $j \in \mathbb{Z}_{0,m+1}$ and $t_{m+2}^{(-1,A,B)} \leq s_{m+2}^{(-1,A,B)}$ hold true.*

Proof From [13, Lem. 10.5] we know that $(s_j^{(-1,A,B)})^* = s_j^{(-1,A,B)}$ and $(t_j^{(-1,A,B)})^* = t_j^{(-1,A,B)}$ for all $j \in \mathbb{Z}_{0,m+2}$. Lemma 6.8 provides $s_j^{(-1,A,B)} = t_j^{(-1,A,B)}$ for each $j \in \mathbb{Z}_{0,m+1}$ as well as (6.3). Since $(AA^+)^* = (A^+)^*A^* =$

$(A^*)^+ A^* = A^+ A$ is valid, consequently, the assumption $t_m \leq s_m$ yields $t_{m+2}^{(-1,A,B)} \leq s_{m+2}^{(-1,A,B)}$. □

In [13, Sec. 9] and [28, Sec. 2.2], for certain classes of sequences of complex matrices, interrelations between sequences of $\mathcal{H}$-parameters and Schur transforms are worked out (see [13, Theorems 9.14 and 9.15] and [28, Satz 2.2.70 and Folgerung 2.2.71]). In the particular case of $\mathbb{R}$-non-negative definite extendable sequences of complex $q \times q$ matrices, this connection can be simplified:

Theorem 6.10 ([13, Thm. 9.15]) *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$ with sequence of $\mathcal{H}$-parameters $(\mathfrak{h}_j)_{j=0}^{\kappa}$. For each $k \in \mathbb{N}_0$ such that $2k \leq \kappa$, let $(s_j^{(k)})_{j=0}^{\kappa-2k}$ be the kth Schur transform of $(s_j)_{j=0}^{\kappa}$. Then $\mathfrak{h}_{2k} = s_0^{(k)}$ for all $k \in \mathbb{N}_0$ such that $2k \leq \kappa$ and $\mathfrak{h}_{2k+1} = s_1^{(k)}$ for all $k \in \mathbb{N}_0$ fulfilling $2k + 1 \leq \kappa$.*

Note that Theorem 6.10 is of great importance within our consideration.

Proposition 6.11 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with sequence of $\mathcal{H}$-parameters $(\mathfrak{h}_j)_{j=0}^{2n}$ and nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Then $\mathfrak{h}_{2n} = s_0^{(n)}$ and the following statements hold true:*

(a) $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$ *if and only if* $\operatorname{rank} s_0^{(n)} = q$.
(b) $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$ *if and only if* $\operatorname{rank} s_0^{(n)} = 0$.

Proof From Theorem 6.10 we know that $\mathfrak{h}_{2n} = s_0^{(n)}$ holds true. Furthermore, Proposition 5.7 shows that $\mathfrak{h}_{2k} \in \mathbb{C}_{\geq}^{q\times q}$ for all $k \in \mathbb{Z}_{0,n}$ and that, in the case $n \geq 1$, moreover, $\mathfrak{h}_{2k-1}^* = \mathfrak{h}_{2k-1}$ and $\mathcal{R}(\mathfrak{h}_{2k-1}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ as well as $\mathcal{R}(\mathfrak{h}_{2k}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ hold true for all $k \in \mathbb{Z}_{1,n}$.

(a) If $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{>}$, then $\mathfrak{h}_{2n} \in \mathbb{C}_{>}^{q\times q}$ follows from Proposition 5.6(b) which implies $\operatorname{rank} s_0^{(n)} = \operatorname{rank} \mathfrak{h}_{2n} = q$. Conversely, suppose $\operatorname{rank} s_0^{(n)} = q$. Hence, $q = \operatorname{rank} \mathfrak{h}_{2n} \leq \operatorname{rank} \mathfrak{h}_{2n-2} \leq \cdots \leq \operatorname{rank} \mathfrak{h}_0$ and, consequently, $\mathfrak{h}_{2k} \in \mathbb{C}_{>}^{q\times q}$ for all $k \in \mathbb{Z}_{0,n}$. Thus, from Proposition 5.6(b), we obtain $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$.
(b) If $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$, then $\mathfrak{h}_{2n} = s_0^{(n)}$ and Proposition 5.6(c) provide $\operatorname{rank} s_0^{(n)} = 0$. Conversely, suppose that $\operatorname{rank} \mathfrak{h}_{2n} = \operatorname{rank} s_0^{(n)} = 0$. From Proposition 5.6(c) we conclude that $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$.

□

7 Special Matrix Polynomials

In this section, we discuss special matrix polynomials, which have been used, e. g., in [4, formula (4.13)] and [15, Appendix C] already. Such matrix polynomials will be used for the description of the solution set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ of

Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \le]$ which was recognized as an equivalent reformulation of the Hamburger moment problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \le]$. More precisely, these matrix polynomials act as generating matrix-valued functions of the linear fractional transformations which establish the description of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \le]$.

Notation 7.1 Let $A, B \in \mathbb{C}^{p\times q}$. Then let $W_{A,B} \colon \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ and $V_{A,B} \colon \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ be defined by

$$W_{A,B}(z) := \begin{pmatrix} zI_p - BA^+ & A \\ -A^+ & I_q - A^+A \end{pmatrix}, \quad V_{A,B}(z) := \begin{pmatrix} 0_{p\times p} & -A \\ A^+ & zI_q - A^+B \end{pmatrix}. \tag{7.1}$$

We set $W_A := W_{A,0_{p\times q}}$ and $V_A := V_{A,0_{p\times q}}$ (see also [4, Ch. 4]).

Remark 7.2 ([15, Rem. C.1]) Let $A, B \in \mathbb{C}^{p\times q}$. Then easily one can see that $W_{A,B}$ and $V_{A,B}$ given in Notation 7.1 are matrix polynomials and, in particular, holomorphic in $\mathbb{C}$. Moreover, for all $z \in \mathbb{C}$, we have

$$V_{A,B}(z)W_{A,B}(z) = \operatorname{diag}\left(AA^+, A^+\left[A - B(I_q - A^+A)\right] + z(I_q - A^+A)\right),$$

$$W_{A,B}(z)V_{A,B}(z) = \begin{pmatrix} AA^+ & BA^+A - AA^+B \\ 0_{q\times p} & A^+A + z(I_q - A^+A) \end{pmatrix},$$

and, in particular,

$$V_A(z)W_A(z) = \operatorname{diag}\left(AA^+, A^+A + z(I_q - A^+A)\right) = W_A(z)V_A(z).$$

Lemma 7.3 *Let $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ and let $P := \mathbb{P}_{\mathcal{R}(A)}$. For all $z \in \mathbb{C}$, then the equations*

$$V_A(z)W_A(z) = \operatorname{diag}\left(P, P + z(I_q - P)\right) = W_A(z)V_A(z), \tag{7.2}$$

$$[V_A(z)]^* (-\widetilde{J}_q) V_A(z) = \left[\operatorname{diag}(P, I_q)\right]^* (-\widetilde{J}_q) \cdot \operatorname{diag}(P, I_q) + 2\operatorname{Im}(z) \cdot \operatorname{diag}(0_{q\times q}, A),$$

and

$$[W_A(z)]^* (-\widetilde{J}_q) W_A(z) = \left[\operatorname{diag}\left(P + z(I_q - P), I_q\right)\right]^* (-\widetilde{J}_q) \cdot \operatorname{diag}\left(P + z(I_q - P), I_q\right) - 2\operatorname{Im}(z) \cdot \operatorname{diag}(A^+, 0_{q\times q})$$

hold true, where the matrix $\widetilde{J}_q$ is given by (4.1).

Proof In view of $A^* = A$ we see from Remark A.13 that $P = AA^+$ and $P = A^+A$. Equation (7.2) then immediately follows from Remark 7.2. By virtue of the assumption $A^* = A$, we have $(A^+)^* = (A^*)^+ = A^+$. From Proposition A.8 we know $P^2 = P$. Thus, we get

$$
\begin{aligned}
[V_A(z)]^* (-\widetilde{J}_q) V_A(z) &= \begin{pmatrix} 0_{q\times q} & A^+ \\ -A & \overline{z} I_q \end{pmatrix} \begin{pmatrix} \mathrm{i}A^+ & \mathrm{i}zI_q \\ 0_{q\times q} & \mathrm{i}A \end{pmatrix} \\
&= \left(\begin{array}{c:c} 0_{q\times q} & \mathrm{i}A^+A \\ \hdashline -\mathrm{i}AA^+ & -\mathrm{i}(z-\overline{z})A \end{array}\right) \\
&= \left[\operatorname{diag}(P, I_q)\right]^* (-\widetilde{J}_q) \cdot \operatorname{diag}(P, I_q) + 2\operatorname{Im}(z) \cdot \operatorname{diag}(0_{q\times q}, A)
\end{aligned}
$$

for all $z \in \mathbb{C}$. Analogously, additionally using $AA^+A = A$, for every choice of $z \in \mathbb{C}$, also we have

$$
\begin{aligned}
[W_A(z)]^* (-\widetilde{J}_q) W_A(z) &= \left(\begin{array}{c:c} \overline{z} I_q & -A^+ \\ \hdashline A & I_q - AA^+ \end{array}\right) \left(\begin{array}{c:c} -\mathrm{i}A^+ & \mathrm{i}(I_q - A^+A) \\ \hdashline -\mathrm{i}zI_q & -\mathrm{i}A \end{array}\right) \\
&= \left(\begin{array}{c:c} \mathrm{i}(z-\overline{z})A^+ & \mathrm{i}P + i\overline{z}(I_q - P) \\ \hdashline -\mathrm{i}z(I_q - P) - \mathrm{i}P & 0_{q\times q} \end{array}\right) \\
&= \left[\operatorname{diag}\left(P + z(I_q - P), I_q\right)\right]^* (-\widetilde{J}_q) \cdot \operatorname{diag}\left(P + z(I_q - P), I_q\right) \\
&\quad - 2\operatorname{Im}(z) \cdot \operatorname{diag}(A^+, 0_{q\times q}).
\end{aligned}
$$

□

For the following notation, we use Notation 7.1 and the definitions given in Remark 6.2 in order to introduce matrix polynomials which also have been used in [15, Sec. 11] already.

Notation 7.4 (see [15, p. 267]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{N}_0$ such that $2m \leq \kappa$, let $(s_j^{(m)})_{j=0}^{\kappa-2m}$ be the mth Schur transform of $(s_j)_{j=0}^{\kappa}$. In view of Notation 7.1, for all $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$, let

$$
\mathfrak{V}^{((s_j)_{j=0}^{2n})} := \begin{cases} V_{s_0^{(0)}}, & \text{if } n = 0 \\ V_{s_0^{(0)}, s_1^{(0)}} V_{s_0^{(1)}, s_1^{(1)}} \cdots V_{s_0^{(n-1)}, s_1^{(n-1)}} V_{s_0^{(n)}}, & \text{if } n \geq 1 \end{cases} \tag{7.3}
$$

and, for all $n \in \mathbb{N}_0$ fulfilling $2n + 1 \leq \kappa$, let

$$
\mathfrak{V}^{((s_j)_{j=0}^{2n+1})} := V_{s_0^{(0)}, s_1^{(0)}} V_{s_0^{(1)}, s_1^{(1)}} \cdots V_{s_0^{(n)}, s_1^{(n)}}. \tag{7.4}
$$

Furthermore, for all $m \in \mathbb{Z}_{0,\kappa}$, let

$$\mathfrak{V}^{((s_j)_{j=0}^m)} = \left(\mathfrak{v}_{lk}^{((s_j)_{j=0}^m)}\right)_{l,k=1}^2 \tag{7.5}$$

be the block representation of $\mathfrak{V}^{((s_j)_{j=0}^m)}$ with $q \times q$ block $\mathfrak{v}_{11}^{((s_j)_{j=0}^m)}$.

Now we consider matrix polynomials which are used in [15, p. 268] already.

Notation 7.5 Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{N}_0$ such that $2m \leq \kappa$, let $(s_j^{(m)})_{j=0}^{\kappa-2m}$ be the mth Schur transform of $(s_j)_{j=0}^{\kappa}$. In view of Notation 7.1, for all $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$, let

$$\mathfrak{W}^{((s_j)_{j=0}^{2n})} := \begin{cases} W_{s_0^{(0)}}, & \text{if } n = 0 \\ W_{s_0^{(n)}} W_{s_0^{(n-1)}, s_1^{(n-1)}} \cdots W_{s_0^{(1)}, s_1^{(1)}} W_{s_0^{(0)}, s_1^{(0)}}, & \text{if } n \geq 1 \end{cases} \tag{7.6}$$

and, for all $n \in \mathbb{N}_0$ fulfilling $2n + 1 \leq \kappa$, let

$$\mathfrak{W}^{((s_j)_{j=0}^{2n+1})} := W_{s_0^{(n)}, s_1^{(n)}} \cdots W_{s_0^{(1)}, s_1^{(1)}} W_{s_0^{(0)}, s_1^{(0)}}. \tag{7.7}$$

Recall that, for each subspace $\mathcal{U}$ of $\mathbb{C}^q$, the notation $\mathbb{P}_{\mathcal{U}}$ refers to the orthoprojection matrix onto $\mathcal{U}$. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$ with sequence of $\mathcal{H}$-parameters $(\mathfrak{h}_j)_{j=0}^{\kappa}$. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, let

$$P_{n,-1} := I_q, \ \ P_{n,l} := \mathbb{P}_{\mathcal{R}(\mathfrak{h}_{2l})} \text{ for each } l \in \mathbb{Z}_{0,n}, \text{ and } P_{n,n+1} := 0_{q \times q}. \tag{7.8}$$

Lemma 7.6 *Let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ such that $2n + 1 \leq \kappa$, then the matrix-valued functions $\mathfrak{V}^{((s_j)_{j=0}^{2n+1})}$ and $\mathfrak{W}^{((s_j)_{j=0}^{2n+1})}$ given by Notations 7.4 and 7.5, respectively, fulfil, for all $z \in \mathbb{C}$, the equation*

$$\mathfrak{W}^{((s_j)_{j=0}^{2n+1})}(z)\mathfrak{V}^{((s_j)_{j=0}^{2n+1})}(z) = \operatorname{diag}\left(P_{n,n}, \sum_{k=0}^{n+1} z^k (P_{n,n-k} - P_{n,n-k+1})\right), \tag{7.9}$$

where, for each $l \in \mathbb{Z}_{-1,n+1}$, the matrix $P_{n,l}$ is given by (7.8).

Proof From Theorem 6.10 we obtain

$$\mathfrak{h}_{2l} = s_0^{(l)} \quad \text{and} \quad \mathfrak{h}_{2l+1} = s_1^{(l)} \quad \text{for all } l \in \mathbb{N}_0 \text{ with } 2l + 1 \leq \kappa. \tag{7.10}$$

By virtue of Proposition 5.7 we infer

$$\mathfrak{h}_j^* = \mathfrak{h}_j \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa}. \tag{7.11}$$

Thus, Remark A.2 yields

$$\mathfrak{h}_j^+\mathfrak{h}_j = \mathfrak{h}_j\mathfrak{h}_j^+ \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa}. \tag{7.12}$$

In view of (7.8) and Remark A.13 we have

$$P_{n,l} = \mathfrak{h}_{2l}\mathfrak{h}_{2l}^+ \qquad \text{for all } n \in \mathbb{N}_0 \text{ with } 2n \leq \kappa \text{ and all } l \in \mathbb{Z}_{0,n}. \tag{7.13}$$

Proposition 5.7 shows furthermore $\mathcal{R}(\mathfrak{h}_{2l+1}) \subseteq \mathcal{R}(\mathfrak{h}_{2l})$ for all $l \in \mathbb{N}_0$ with $2l+1 \leq \kappa$. Using (7.11) and Remark A.12, then $\mathcal{N}(\mathfrak{h}_{2l}) \subseteq \mathcal{N}(\mathfrak{h}_{2l+1})$ follows for all $l \in \mathbb{N}_0$ with $2l+1 \leq \kappa$. Hence, the application of Lemma A.14 yields

$$\mathfrak{h}_{2l}\mathfrak{h}_{2l}^+\mathfrak{h}_{2l+1} = \mathfrak{h}_{2l+1} \text{ and } \mathfrak{h}_{2l+1}\mathfrak{h}_{2l}^+\mathfrak{h}_{2l} = \mathfrak{h}_{2l+1} \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l+1 \leq \kappa. \tag{7.14}$$

Consider an arbitrary $z \in \mathbb{C}$. In view of (7.7), (7.4), (7.10), Remark 7.2, (7.14), (7.12), (7.13), and (7.8), then we have

$$\begin{aligned}
\mathfrak{W}^{((s_j)_{j=0}^1)}(z)\mathfrak{V}^{((s_j)_{j=0}^1)}(z) &= W_{s_0^{(0)},s_1^{(0)}}(z)V_{s_0^{(0)},s_1^{(0)}}(z) = W_{\mathfrak{h}_0,\mathfrak{h}_1}(z)V_{\mathfrak{h}_0,\mathfrak{h}_1}(z)\\
&= \begin{pmatrix} \mathfrak{h}_0\mathfrak{h}_0^+ & \mathfrak{h}_1\mathfrak{h}_0^+\mathfrak{h}_0 - \mathfrak{h}_0\mathfrak{h}_0^+\mathfrak{h}_1 \\ 0_{q\times q} & \mathfrak{h}_0^+\mathfrak{h}_0 + z(I_q - \mathfrak{h}_0^+\mathfrak{h}_0) \end{pmatrix} = \operatorname{diag}\left(\mathfrak{h}_0\mathfrak{h}_0^+, \mathfrak{h}_0\mathfrak{h}_0^+ + z(I_q - \mathfrak{h}_0\mathfrak{h}_0^+)\right)\\
&= \operatorname{diag}\left(P_{0,0}, \sum_{k=0}^{0+1} z^k(P_{0,0-k} - P_{0,0-k+1})\right).
\end{aligned}$$

In particular, the assertion is proved in the case $\kappa \leq 2$. Now assume $\kappa \geq 3$. Then we have already shown that there is an $m \in \mathbb{N}$ fulfilling $2m+1 \leq \kappa$ such that (7.9) holds true for each $n \in \mathbb{Z}_{0,m-1}$. We are going to prove that (7.9) is true in the case $n = m$ as well. Proposition 5.7 shows $\mathcal{R}(\mathfrak{h}_{2k}) \subseteq \mathcal{R}(\mathfrak{h}_{2k-2})$ for all $k \in \mathbb{N}$ with $2k \leq \kappa$. In particular, $\mathcal{R}(\mathfrak{h}_{2m}) \subseteq \mathcal{R}(\mathfrak{h}_{2l})$ for all $l \in \mathbb{Z}_{0,m-1}$. In view of (7.11), we furthermore obtain $\mathcal{R}(\mathfrak{h}_{2m}^+) = \mathcal{R}(\mathfrak{h}_{2m}^*) = \mathcal{R}(\mathfrak{h}_{2m})$. Taking additionally into account (7.8), we hence can infer

$$P_{m-1,l}\mathfrak{h}_{2m} = \mathfrak{h}_{2m} \text{ for each } l \in \mathbb{Z}_{-1,m-1} \quad \text{and} \quad P_{m-1,m}\mathfrak{h}_{2m} = 0_{q\times q}. \tag{7.15}$$

as well as $P_{m-1,l}\mathfrak{h}_{2m}^+ = \mathfrak{h}_{2m}^+$ for all $l \in \mathbb{Z}_{-1,m-1}$ and $P_{m-1,m}\mathfrak{h}_{2m}^+ = 0_{q\times q}$. Thus, we have

$$\left[\sum_{k=0}^{m} z^k(P_{m-1,m-1-k} - P_{m-1,m-k})\right]\mathfrak{h}_{2m}^+ = \mathfrak{h}_{2m}^+. \tag{7.16}$$

From (7.15) we conclude

$$P_{m-1,m-1}(-\mathfrak{h}_{2m}) = -\mathfrak{h}_{2m}. \tag{7.17}$$

Due to (7.16), we have

$$\left[\sum_{k=0}^{m} z^k (P_{m-1,m-1-k} - P_{m-1,m-k})\right](zI_q - \mathfrak{h}_{2m}^+ \mathfrak{h}_{2m+1}) = \left[\sum_{k=0}^{m} z^{k+1} (P_{m-1,m-1-k} - P_{m-1,m-k})\right] - \mathfrak{h}_{2m}^+ \mathfrak{h}_{2m+1}. \tag{7.18}$$

Combining (7.16), (7.17), and (7.18) then yields

$$\operatorname{diag}\left(P_{m-1,m-1}, \sum_{k=0}^{m} z^k (P_{m-1,m-1-k} - P_{m-1,m-k})\right) \times \begin{pmatrix} 0_{q\times q} & -\mathfrak{h}_{2m} \\ \mathfrak{h}_{2m}^+ & zI_q - \mathfrak{h}_{2m}^+ \mathfrak{h}_{2m+1} \end{pmatrix} = \begin{pmatrix} 0_{q\times q} & -\mathfrak{h}_{2m} \\ \mathfrak{h}_{2m}^+ & [\sum_{k=0}^{m} z^{k+1} (P_{m-1,m-1-k} - P_{m-1,m-k})] - \mathfrak{h}_{2m}^+ \mathfrak{h}_{2m+1} \end{pmatrix}. \tag{7.19}$$

In view of (7.14), we get

$$(zI_q - \mathfrak{h}_{2m+1}\mathfrak{h}_{2m}^+)(-\mathfrak{h}_{2m}) = -z\mathfrak{h}_{2m} + \mathfrak{h}_{2m+1}\mathfrak{h}_{2m}^+\mathfrak{h}_{2m} = -z\mathfrak{h}_{2m} + \mathfrak{h}_{2m+1}. \tag{7.20}$$

By virtue of (7.8) and Proposition A.8 we have $P_{m-1,l}^* = P_{m-1,l}$ for all $l \in \mathbb{Z}_{-1,m}$. Taking additionally into account (7.11), we can infer from (7.15) then $\mathfrak{h}_{2m} P_{m-1,l} = \mathfrak{h}_{2m}$ for all $l \in \mathbb{Z}_{-1,m-1}$ and $\mathfrak{h}_{2m} P_{m-1,m} = 0_{q\times q}$. Consequently, we obtain

$$\mathfrak{h}_{2m} \sum_{k=0}^{m} z^{k+1} (P_{m-1,m-1-k} - P_{m-1,m-k}) = z\mathfrak{h}_{2m}. \tag{7.21}$$

From (7.21) and (7.14) we get

$$\mathfrak{h}_{2m}\left(\left[\sum_{k=0}^{m} z^{k+1} (P_{m-1,m-1-k} - P_{m-1,m-k})\right] - \mathfrak{h}_{2m}^+ \mathfrak{h}_{2m+1}\right) = z\mathfrak{h}_{2m} - \mathfrak{h}_{2m+1}. \tag{7.22}$$

Moreover, the combination of (7.12), (7.13), and (7.8) provides

$$
\begin{aligned}
-\mathfrak{h}_{2m}^{+}(-\mathfrak{h}_{2m}) = \mathfrak{h}_{2m}\mathfrak{h}_{2m}^{+} = P_{m,m} &= P_{m,m} - P_{m,m+1} \\
&= z^{0}(P_{m,m-0} - P_{m,m-0+1}).
\end{aligned}
\tag{7.23}
$$

By virtue of (7.8) we see $P_{m-1,l} = P_{m,l}$ for all $l \in \mathbb{Z}_{-1,m-1}$. Consequently, additionally using (7.21), (7.11), (7.8), and (7.13), then

$$
\begin{aligned}
&(I_q - \mathfrak{h}_{2m}^{+}\mathfrak{h}_{2m})\left(\left[\sum_{k=0}^{m} z^{k+1}(P_{m-1,m-1-k} - P_{m-1,m-k})\right] - \mathfrak{h}_{2m}^{+}\mathfrak{h}_{2m+1}\right) \\
&= \sum_{k=0}^{m} z^{k+1}(P_{m-1,m-1-k} - P_{m-1,m-k}) - z\mathfrak{h}_{2m}^{+}\mathfrak{h}_{2m} \\
&= \sum_{j=1}^{m+1} z^{j}(P_{m-1,m-j} - P_{m-1,m-j+1}) - z\mathfrak{h}_{2m}\mathfrak{h}_{2m}^{+} \\
&= zP_{m,m-1} + \sum_{j=2}^{m+1} z^{j}(P_{m,m-j} - P_{m,m-j+1}) - zP_{m,m-1+1} \\
&= \sum_{k=1}^{m+1} z^{k}(P_{m,m-k} - P_{m,m-k+1})
\end{aligned}
\tag{7.24}
$$

is true. Using (7.13), (7.20), (7.22), (7.23), and (7.24) delivers

$$
\begin{aligned}
&\begin{pmatrix} zI_q - \mathfrak{h}_{2m+1}\mathfrak{h}_{2m}^{+} & \mathfrak{h}_{2m} \\ -\mathfrak{h}_{2m}^{+} & I_q - \mathfrak{h}_{2m}^{+}\mathfrak{h}_{2m} \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0_{q\times q} & -\mathfrak{h}_{2m} \\ \mathfrak{h}_{2m}^{+} & [\sum_{k=0}^{m} z^{k+1}(P_{m-1,m-1-k} - P_{m-1,m-k})] - \mathfrak{h}_{2m}^{+}\mathfrak{h}_{2m+1} \end{pmatrix} \\
&\qquad = \operatorname{diag}\left(P_{m,m}, \sum_{k=0}^{m+1} z^{k}(P_{m,m-k} - P_{m,m-k+1})\right).
\end{aligned}
\tag{7.25}
$$

Finally, due to (7.7), (7.4), (7.10), Notation 7.1, (7.9) with $n = m - 1$ as well as (7.19) and (7.25), then

$$\begin{aligned}
&\mathfrak{W}^{((s_j)_{j=0}^{2m+1})}(z)\mathfrak{V}^{((s_j)_{j=0}^{2m+1})}(z)\\
&= W_{s_0^{(m)}, s_1^{(m)}}(z)\mathfrak{W}^{((s_j)_{j=0}^{2m-1})}(z)\mathfrak{V}^{((s_j)_{j=0}^{2m-1})}(z)V_{s_0^{(m)}, s_1^{(m)}}(z)\\
&= W_{\mathfrak{h}_{2m}, \mathfrak{h}_{2m+1}}(z)\mathfrak{W}^{((s_j)_{j=0}^{2m-1})}(z)\mathfrak{V}^{((s_j)_{j=0}^{2m-1})}(z)V_{\mathfrak{h}_{2m}, \mathfrak{h}_{2m+1}}(z)\\
&= \left(\begin{array}{c|c} zI_q - \mathfrak{h}_{2m+1}\mathfrak{h}_{2m}^+ & \mathfrak{h}_{2m} \\ \hline -\mathfrak{h}_{2m}^+ & I_q - \mathfrak{h}_{2m}^+\mathfrak{h}_{2m} \end{array}\right)\\
&\quad \times \operatorname{diag}\left(P_{m-1,m-1}, \sum_{k=0}^{m} z^k (P_{m-1,m-1-k} - P_{m-1,m-k})\right)\\
&\quad\quad \times \left(\begin{array}{c|c} 0_{q\times q} & -\mathfrak{h}_{2m} \\ \hline \mathfrak{h}_{2m}^+ & zI_q - \mathfrak{h}_{2m}^+\mathfrak{h}_{2m+1} \end{array}\right)\\
&= \operatorname{diag}\left(P_{m,m}, \sum_{k=0}^{m+1} z^k (P_{m,m-k} - P_{m,m-k+1})\right),
\end{aligned}$$

which proves (7.9) in the case $n = m$. Consequently, the assertion is checked inductively. □

In order to prove a result analogous to Lemma 7.6, where an odd number of data is given, in the case of an odd number of prescribed data, we recall the following:

Remark 7.7 Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$. Furthermore, let $s_{2n+1} := \Lambda_n$, where Λ_n is given by (5.2). Regarding Definition 5.5, Proposition 5.7, and Theorem 6.10 one can check that $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ and the equations $\mathfrak{V}^{((s_j)_{j=0}^{2n})} = \mathfrak{V}^{((s_j)_{j=0}^{2n+1})}$ and $\mathfrak{W}^{((s_j)_{j=0}^{2n})} = \mathfrak{W}^{((s_j)_{j=0}^{2n+1})}$ hold true.

Lemma 7.8 *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}}$. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, then the matrix-valued functions $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ and $\mathfrak{W}^{((s_j)_{j=0}^{2n})}$ defined by Notations 7.4 and 7.5, respectively, fulfil, for all $z \in \mathbb{C}$, the equation*

$$\mathfrak{W}^{((s_j)_{j=0}^{2n})}(z)\mathfrak{V}^{((s_j)_{j=0}^{2n})}(z) = \operatorname{diag}\left(P_{n,n}, \sum_{k=0}^{n+1} z^k (P_{n,n-k} - P_{n,n-k+1})\right), \tag{7.26}$$

where, for each $l \in \mathbb{Z}_{-1,n+1}$, the matrix $P_{n,l}$ is given by (7.8).

Proof Use Remark 7.7 and Lemma 7.6. □

8 An Essential Step to a Parametrization of the Solution Set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ of the Truncated Matricial Hamburger Moment Problem

In this section, we state some results which lead to a parametrization of the solution set of the truncated matricial Hamburger moment problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$, where the parameters still depend on the given data. This parametrization is done by using a further special algorithm of Schur type. While doing so, we mainly refer to the representations given in [13, 15]. Moreover, we draw the reader's attention to Theorems 2.2 and 2.4 as well as the reformulation of Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ in Remark 3.4. In particular, we are going to use the results stated in Sect. 6.

Notation 8.1 ([15, Sec. 8]) Let $\mathcal{G}$ be a non-empty subset of $\mathbb{C}$, let $F\colon \mathcal{G} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let A and B be complex $p \times q$ matrices. Then let $F^{(+;A,B)}\colon \mathcal{G} \to \mathbb{C}^{p\times q}$ and $F^{(-;A,B)}\colon \mathcal{G} \to \mathbb{C}^{p\times q}$ be defined by $F^{(+;A,B)}(z) := -A(zI_q + [F(z)]^+A) + B$ and $F^{(-;A,B)}(z) := -A(zI_q + A^+[F(z) - B])^+$, respectively. The functions $F^{(+;A,B)}$ and $F^{(-;A,B)}$ are called the (A, B)-*Schur transform of* F and the *inverse* (A, B)-*Schur transform of* F, respectively. Abbreviating, we set $F^{(+;A)} := F^{(+;A,0_{p\times q})}$ and $F^{(-;A)} := F^{(-;A,0_{p\times q})}$.

Proposition 8.2 *Let $n \in \mathbb{N}$, let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$, and let $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$. Then $F^{(+;s_0,s_1)} \in \mathcal{R}_{0,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2(n-1)}, \leq]$, where $(s_j^{(1)})_{j=0}^{2(n-1)}$ is the first Schur transform of $(s_j)_{j=0}^{2n}$.*

Proof Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$, from Remark 5.1 we know that $s_j^* = s_j$ for all $j \in \mathbb{Z}_{0,2n}$. Since F belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, the $\mathbb{R}$-Stieltjes measure σ_F of F belongs to $\mathcal{M}_{\geq}^q[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$. In particular, $\sigma_F \in \mathcal{M}_{\geq,2n}^q(\mathbb{R})$. Setting $t_j := \int_{\mathbb{R}} x^j \sigma_F(\mathrm{d}x)$ for each $j \in \mathbb{Z}_{0,2n}$, then Remark 2.1 yields $t_j^* = t_j$ for all $j \in \mathbb{Z}_{0,2n}$, and we have $t_j = s_j$ for all $j \in \mathbb{Z}_{0,2n-1}$ and $t_{2n} \leq s_{2n}$. Lemma 6.6 provides $t_{2n-2}^{(1)} \leq s_{2n-2}^{(1)}$ and, in the case $n \geq 2$, moreover, $t_j^{(1)} = s_j^{(1)}$ for all $j \in \mathbb{Z}_{0,2n-3}$. Since σ_F belongs to $\mathcal{M}_{\geq}^q[\mathbb{R}; (t_j)_{j=0}^{2n}, =]$, Theorem 2.5 provides $(t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and the definition of the set $\mathcal{R}_{0,q}[\Pi_+; (t_j)_{j=0}^{2n}, =]$ given in Remark 3.4 shows that F belongs to $\mathcal{R}_{0,q}[\Pi_+; (t_j)_{j=0}^{2n}, =]$. Thus, from [15, Thm. 9.7] we obtain $F^{(+;s_0,s_1)} \in \mathcal{R}_{0,q}[\Pi_+; (t_j^{(1)})_{j=0}^{2(n-1)}, =]$. Thus, $F^{(+;s_0,s_1)} \in \mathcal{R}_{0,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2(n-1)}, \leq]$ follows. □

Definition 8.3 (cf. [14, Def. 4.3]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then one says that $(s_j)_{j=0}^{\kappa}$ is *first term dominated*, if $\bigcup_{j=0}^{\kappa} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$ and $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\kappa} \mathcal{N}(s_j)$ are fulfilled. The set of all first term dominated sequences $(s_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices will be denoted by $\mathcal{D}_{p\times q,\kappa}$.

Proposition 8.4 ([13, Prop. 4.24]) $\mathcal{H}_{q,\kappa}^{\geq,\mathrm{e}} \subseteq \mathcal{D}_{q\times q,\kappa}$ *for all* $\kappa \in \mathbb{N}_0 \cup \{\infty\}$.

Proposition 8.5 *Let* $n \in \mathbb{N}$, *let* $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \cap \mathcal{D}_{q\times q,2n}$, *and let* $G \in \mathcal{R}_{0,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2(n-1)}, \leq]$. *Then* $G^{(-;s_0,s_1)}$ *belongs to* $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.

Proof Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$, Remark 5.1 shows that

$$s_{2k} \in \mathbb{C}_{\geq}^{q\times q} \text{ for all } k \in \mathbb{Z}_{0,n} \qquad \text{and} \qquad s_j^* = s_j \text{ for all } j \in \mathbb{Z}_{0,2n}. \tag{8.1}$$

Since G belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2(n-1)}, \leq]$, we get that $G \in \mathcal{R}_{0,q}(\Pi_+)$ and that the $\mathbb{R}$-Stieltjes measure σ_G of G belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j^{(1)})_{j=0}^{2(n-1)}, \leq]$ and, in particular, to $\mathcal{M}_{\geq,2n-2}^{q}(\mathbb{R})$. In view of Remark 2.1, for each $j \in \mathbb{Z}_{0,2n-2}$, the matrix $t_j := \int_{\mathbb{R}} x^j \sigma_G(\mathrm{d}x)$ is Hermitian. Then σ_G belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (t_j)_{j=0}^{2n-2}, =]$ and, moreover,

$$t_{2n-2} \leq s_{2n-2}^{(1)} \qquad \text{and} \qquad t_j = s_j^{(1)} \text{ for each } j \in \mathbb{Z}_{0,2n-3}. \tag{8.2}$$

Because of $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (t_j)_{j=0}^{2n-2}, =] \neq \emptyset$, Theorem 2.5 yields $(t_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq,\mathrm{e}}$. Setting

$$r_j := t_j^{(-1,s_0,s_1)} \qquad \text{for all } j \in \mathbb{Z}_{0,2n}, \tag{8.3}$$

from (8.1) and [13, Cor. 10.8] we conclude $(r_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$. If $n \geq 2$, then, for every choice of $j \in \mathbb{Z}_{0,2n-3}$, from (8.2) and (6.2) we have

$$\mathcal{R}(t_j) = \mathcal{R}(s_j^{(1)}) = \mathcal{R}(-s_0 s_{j+2}^{\sharp} s_0) \subseteq \mathcal{R}(s_0) \tag{8.4}$$

and

$$\mathcal{N}(s_0) \subseteq \mathcal{N}(-s_0 s_{j+2}^{\sharp} s_0) = \mathcal{N}(s_j^{(1)}) = \mathcal{N}(t_j). \tag{8.5}$$

From $(t_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq,\mathrm{e}}$ and Remarks 2.3 and 5.1 we obtain $t_{2n-2} \in \mathbb{C}_{\geq}^{q\times q}$. Thus, together with (8.2) it follows $0_{q\times q} \leq t_{2n-2} \leq s_{2n-2}^{(1)}$ which, in view of Remark A.1, implies $\mathcal{R}(t_{2n-2}) \subseteq \mathcal{R}(s_{2n-2}^{(1)})$ and $\mathcal{N}(s_{2n-2}^{(1)}) \subseteq \mathcal{N}(t_{2n-2})$. Therefore, (6.2) yields

$$\mathcal{R}(t_{2n-2}) \subseteq \mathcal{R}(-s_0 s_{2n}^{\sharp} s_0) \subseteq \mathcal{R}(s_0) \text{ and } \mathcal{N}(s_0) \subseteq \mathcal{N}(-s_0 s_{2n}^{\sharp} s_0) \subseteq \mathcal{N}(t_{2n-2}). \tag{8.6}$$

Using (8.3), (8.4), (8.5), (8.6), and [13, Lem. 10.6(b)], we get $r_j^{(1)} = (t_j^{(-1,s_0,s_1)})^{(1)} = t_j$ for all $j \in \mathbb{Z}_{0,2n-2}$. Therefore, since σ_G belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (t_j)_{j=0}^{2n-2}, =]$, we see that $G \in \mathcal{R}_{0,q}[\Pi_+; (r_j^{(1)})_{j=0}^{2(n-1)}, =]$. Consequently, in view of $(r_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and [15, Thm. 10.9], we get $G^{(-;r_0,r_1)} \in$

$\mathcal{R}_{0,q}[\Pi_+; (r_j)_{j=0}^{2n}, =]$. From (8.3), Definition 6.7, the assumption $(s_j)_{j=0}^{2n} \in \mathcal{D}_{q\times q,2n}$ and Lemma A.14 we conclude $r_0 = t_0^{(-1,s_0,s_1)} = s_0$ and $r_1 = t_1^{(-1,s_0,s_1)} = s_0 s_0^+ s_1 s_0^+ s_0 = s_1$. Thus,

$$G^{(-;s_0,s_1)} \in \mathcal{R}_{0,q}[\Pi_+; (r_j)_{j=0}^{2n}, =]. \tag{8.7}$$

For each $j \in \mathbb{Z}_{0,2n-2}$, let $u_j := s_j^{(1)}$. Furthermore, let $v_j := u_j^{(-1,s_0,s_1)}$ for all $j \in \mathbb{Z}_{0,2n}$. Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$, the application of [13, Prop. 8.12] yields $(s_j^{(1)})_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq}$. In particular, Remark 5.1 then provides $u_j^* = (s_j^{(1)})^* = s_j^{(1)} = u_j$ for all $j \in \mathbb{Z}_{0,2n-2}$. Taking into account (8.3) as well as $u_j^* = u_j$ and $t_j^* = t_j$ for all $j \in \mathbb{Z}_{0,2n-2}$ as well as (8.2), the application of Lemma 6.9 to the sequences $(u_j)_{j=0}^{2n-2}$ and $(t_j)_{j=0}^{2n-2}$ yields

$$r_{2n} = t_{2n}^{(-1,s_0,s_1)} \leq u_{2n}^{(-1,s_0,s_1)} = v_{2n} \tag{8.8}$$

and $r_j = t_j^{(-1,s_0,s_1)} = u_j^{(-1,s_0,s_1)} = v_j$ for all $j \in \mathbb{Z}_{0,2n-1}$. Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{D}_{q\times q,2n}$, from [13, Thm. 10.13], for every choice of $j \in \mathbb{Z}_{0,2n}$ we conclude $v_j = u_j^{(-1,s_0,s_1)} = (s_j^{(1)})^{(-1,s_0,s_1)} = s_j$. Thus, in view of (8.7) and (8.8), the proof is complete. □

Now we are going to study Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ in the special case $n = 0$. In order to realize this goal we need some preparation.

Lemma 8.6 *Let $s_0 \in \mathbb{C}_{\geq}^{q\times q}$ and let $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{0}, \leq]$. For all $z \in \Pi_+$, then $\frac{1}{\operatorname{Im}(z)} \operatorname{Im} F(z) \geq [F(z)]^* s_0^+ F(z)$.*

Proof We apply an idea used in [20, proof of Lem. 11.1]. Let $z \in \Pi_+$. Because of $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{0}, \leq]$, we have $F \in \mathcal{R}_{0,q}(\Pi_+)$ and the $\mathbb{R}$-Stieltjes measure σ_F of F belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{0}, \leq]$. In particular, $g_z \colon \mathbb{R} \to \mathbb{C}$ defined by $g_z(x) := (x - z)^{-1}$ belongs to $\mathcal{L}^1(\mathbb{R}; \mathfrak{B}_{\mathbb{R}}, \sigma_F; \mathbb{C})$ and $F(z) = \int_{\mathbb{R}} g_z \mathrm{d}\sigma_F$ holds true. Using [12, Rem. B.4], we get $\operatorname{Im} g_z \in \mathcal{L}^1(\mathbb{R}; \mathfrak{B}_{\mathbb{R}}, \sigma_F; \mathbb{C})$ and $\operatorname{Im} F(z) = \int_{\mathbb{R}} \operatorname{Im} g_z \mathrm{d}\sigma_F$. Obviously, $\operatorname{Im} g_z(x) = \operatorname{Im}(z)|g_z(x)|^2$ is valid for all $x \in \mathbb{R}$. Hence, $|g_z|^2 \in \mathcal{L}^1(\mathbb{R}; \mathfrak{B}_{\mathbb{R}}, \sigma_F; \mathbb{C})$ and $\operatorname{Im} F(z) = \operatorname{Im}(z) \int_{\mathbb{R}} |g_z|^2 \mathrm{d}\sigma_F$. From Lemma B.1 and [20, Cor. B.6] then

$$\left(\int_{\mathbb{R}} g_z \mathrm{d}\sigma_F\right)^* [\sigma_F(\mathbb{R})]^+ \left(\int_{\mathbb{R}} g_z \mathrm{d}\sigma_F\right) \leq \int_{\mathbb{R}} |g_z|^2 \mathrm{d}\sigma_F \tag{8.9}$$

follows. Both the matrices $B := \sigma_F(\mathbb{R})$ and s_0 are non-negative Hermitian and, in particular, Hermitian. Because of $\sigma_F \in \mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{0}, \leq]$, we have $s_0 \geq s_0^{(\sigma_F)} = B \geq 0_{q\times q}$. By virtue of [15, Lem. A.7], then

$$B^+ \geq B^+ B s_0^+ B B^+ \geq 0_{q\times q} \tag{8.10}$$

follows. Remark A.2 shows that $B^+B = BB^+ = (BB^+)^*$. From Lemma 3.5 we know that $\mathcal{R}(F(z)) = \mathcal{R}(B)$. Thus, Lemma A.14(a) provides $BB^+F(z) = F(z)$. Consequently, because of (8.9) and (8.10), then we have

$$\begin{aligned}\frac{1}{\operatorname{Im}(z)} \operatorname{Im} F(z) &= \int_{\mathbb{R}} |g_z|^2 \mathrm{d}\sigma_F \geq \left(\int_{\mathbb{R}} g_z \mathrm{d}\sigma_F\right)^* [\sigma_F(\mathbb{R})]^+ \left(\int_{\mathbb{R}} g_z \mathrm{d}\sigma_F\right) \\ &= [F(z)]^* B^+ F(z) \geq [F(z)]^* B^+ B s_0^+ B B^+ F(z) \\ &= \left[BB^+F(z)\right]^* s_0^+ BB^+ F(z) = [F(z)]^* s_0^+ F(z).\end{aligned}$$

□

The next two propositions deal with Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, described in Sect. 3, for the special case $n = 0$.

Proposition 8.7 *Let $s_0 \in \mathbb{C}_{\geq}^{q\times q}$ and let $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{0}, \leq]$. Let $W_{s_0} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by Notation 7.1. Then:*

(a) *The matrix-valued functions $\phi, \psi : \Pi_+ \to \mathbb{C}^{q\times q}$ defined by*

$$\phi(z) := (I_q, 0_{q\times q}) W_{s_0}(z) \begin{pmatrix} F(z) \\ I_q \end{pmatrix} \quad \text{and} \quad \psi(z) := (0_{q\times q}, I_q) W_{s_0}(z) \begin{pmatrix} F(z) \\ I_q \end{pmatrix} \tag{8.11}$$

are holomorphic in Π_+ and fulfil, for all $z \in \Pi_+$, the four conditions

$$\operatorname{rank} \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} = q, \tag{8.12}$$

$$\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \in \mathbb{C}_{\geq}^{q\times q}, \tag{8.13}$$

$\mathcal{R}(\phi(z)) \subseteq \mathcal{R}(s_0)$, and $\det[s_0^+\phi(z) + z\psi(z)] \neq 0$. In particular, $[\phi; \psi] \in \mathcal{P}[s_0]$.

(b) *For each $z \in \Pi_+$, the matrix-valued function F admits the representation*

$$F(z) = -s_0\psi(z) \left[s_0^+\phi(z) + z\psi(z)\right]^{-1}.$$

Proof Let $P := \mathbb{P}_{\mathcal{R}(s_0)}$. Since the matrix s_0 is Hermitian, we have $P = \mathbb{P}_{\mathcal{R}(s_0^*)}$. Because of $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{0}, \leq]$, the matrix-valued function F belongs to $\mathcal{R}_{0,q}(\Pi_+)$ and the $\mathbb{R}$-Stieltjes measure σ_F of F belongs to $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{0}, \leq]$. In particular,

$$0_{q\times q} \leq \sigma_F(\mathbb{R}) = \int_{\mathbb{R}} x^0 \sigma_F(\mathrm{d}x) \leq s_0. \tag{8.14}$$

Since W_{s_0} is a matrix polynomial and because F belongs to $\mathcal{R}_{0,q}(\Pi_+)$, the matrix-valued functions ϕ and ψ are holomorphic in Π_+. Consider an arbitrary $z \in \Pi_+$. By virtue of (8.11), we conclude

$$W_{s_0}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} = \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}. \tag{8.15}$$

According to Lemma 3.5, we have $\mathcal{R}(F(z)) = \mathcal{R}(\sigma_F(\mathbb{R}))$ and $\mathcal{N}(F(z)) = \mathcal{N}(\sigma_F(\mathbb{R}))$. Consequently, additionally taking into account (8.14), from Remark A.1 we get

$$\mathcal{R}(F(z)) \subseteq \mathcal{R}(s_0) \qquad \text{and} \qquad \mathcal{N}(s_0) \subseteq \mathcal{N}(F(z)).$$

Hence, Lemma A.14 as well as $P = \mathbb{P}_{\mathcal{R}(s_0^*)}$ and Remark A.13 yield

$$PF(z) = F(z) \qquad \text{and} \qquad F(z)P = F(z) \tag{8.16}$$

and, consequently,

$$\left[P + z(I_q - P)\right] F(z) = PF(z) + z\left[F(z) - PF(z)\right] = F(z). \tag{8.17}$$

Using $P^2 = P$, we conclude

$$\begin{aligned}\left[P + z(I_q - P)\right]&\left[P + \frac{1}{z}(I_q - P)\right] \\ &= P^2 + \frac{1}{z}(P - P^2) + z(P - P^2) + (I_q - P - P + P^2) = I_q.\end{aligned}$$

Consequently, we have

$$\det\left[P + z(I_q - P)\right] \neq 0 \quad \text{and} \quad \left[P + z(I_q - P)\right]^{-1} = P + \frac{1}{z}(I_q - P). \tag{8.18}$$

From Notation 7.1 and Lemma 7.3 also we obtain

$$\begin{aligned}(s_0^+, zI_q)W_{s_0}(z) &= (0_{q\times q}, I_q)V_{s_0}(z)W_{s_0}(z) \\ &= (0_{q\times q}, I_q) \cdot \operatorname{diag}\left(P, P + z(I_q - P)\right) \\ &= \left(0_{q\times q}, P + z(I_q - P)\right).\end{aligned} \tag{8.19}$$

Taking into account (8.18), (8.19), and (8.15), then

$$
\begin{aligned}
q &= \operatorname{rank}\bigl(P + z(I_q - P)\bigr) = \operatorname{rank}\left(\bigl(0_{q\times q}, P + z(I_q - P)\bigr)\begin{pmatrix} F(z) \\ I_q \end{pmatrix}\right) \\
&= \operatorname{rank}\left((s_0^+, zI_q)W_{s_0}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix}\right) = \operatorname{rank}\left((s_0^+, zI_q)\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}\right) \\
&\leq \operatorname{rank}\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \leq q
\end{aligned}
$$

and, consequently, (8.12) follow. Using (8.15), Lemma 7.3, (8.17), Remark 4.1, the assumption $\operatorname{Im}(z) \in (0, \infty)$, and Lemma 8.6, we infer

$$
\begin{aligned}
&\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} = \begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* \bigl[W_{s_0}(z)\bigr]^* (-\widetilde{J}_q) W_{s_0}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&= \begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* \Bigl\{\bigl[\operatorname{diag}\bigl(P + z(I_q - P), I_q\bigr)\bigr]^* (-\widetilde{J}_q)\cdot \operatorname{diag}\bigl(P + z(I_q - P), I_q\bigr) \\
&\qquad - 2\operatorname{Im}(z)\cdot \operatorname{diag}(s_0^+, 0_{q\times q})\Bigr\}\begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&= \begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* \bigl[\operatorname{diag}\bigl(P + z(I_q - P), I_q\bigr)\bigr]^* (-\widetilde{J}_q) \operatorname{diag}\bigl(P + z(I_q - P), I_q\bigr)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&\qquad - 2\operatorname{Im}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* \cdot \operatorname{diag}(s_0^+, 0_{q\times q})\cdot \begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&= \begin{pmatrix} [P + z(I_q - P)]F(z) \\ I_q \end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix} [P + z(I_q - P)]F(z) \\ I_q \end{pmatrix} \\
&\qquad - 2\operatorname{Im}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* \operatorname{diag}(s_0^+, 0_{q\times q})\begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&= \begin{pmatrix} F(z) \\ I_q \end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} - 2\operatorname{Im}(z)\,[F(z)]^* s_0^+ F(z) \\
&= 2\operatorname{Im}(z)\left(\frac{1}{\operatorname{Im}(z)}\operatorname{Im}[F(z)] - [F(z)]^* s_0^+ F(z)\right) \in \mathbb{C}_{\geq}^{q\times q}.
\end{aligned}
$$

Hence, (8.13) is proved. Since $z \in \Pi_+$ was arbitrary, in particular, from Definition 4.2, (8.12), and (8.13) we see that the pair $[\phi; \psi]$ belongs to $\mathcal{PR}_q(\Pi_+)$. From (8.11) and Notation 7.1 we obtain that

$$\phi(z) = (I_q, 0_{q\times q})W_{s_0}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} = (zI_q, s_0)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} = zF(z) + s_0.$$

Since $z \in \Pi_+$ was arbitrary, because of (8.16), consequently, $P\phi = \phi$. According to Notation 4.7 then $[\phi; \psi] \in \mathcal{P}[s_0]$. Using Notation 7.1, (8.15), Lemma 7.3, and (8.16), we conclude

$$\begin{aligned}
\begin{pmatrix} -s_0\psi(z) \\ s_0^+\phi(z) + z\psi(z) \end{pmatrix} &= \begin{pmatrix} 0_{q\times q} & -s_0 \\ s_0^+ & zI_q \end{pmatrix}\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} = V_{s_0}(z)\begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \\
&= V_{s_0}(z)W_{s_0}(z)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} = \operatorname{diag}\left(P, P + z(I_q - P)\right)\begin{pmatrix} F(z) \\ I_q \end{pmatrix} \\
&= \begin{pmatrix} PF(z) \\ P + z(I_q - P) \end{pmatrix} = \begin{pmatrix} F(z) \\ P + z(I_q - P) \end{pmatrix}
\end{aligned}$$

and, in particular, $-s_0\psi(z) = F(z)$ as well as $s_0^+\phi(z) + z\psi(z) = P + z(I_q - P)$. Thus, (8.18) implies $\det[s_0^+\phi(z) + z\psi(z)] \neq 0$ and $[s_0^+\phi(z) + z\psi(z)]^{-1} = P + \frac{1}{z}(I_q - P)$. In particular, the proof of (a) is complete. Consequently, in view of $-s_0\psi(z) = F(z)$, (8.16), $P^2 = P$, and once more (8.16), we get

$$\begin{aligned}
-s_0\psi(z)\left[s_0^+\phi(z) + z\psi(z)\right]^{-1} &= F(z)\left[P + \frac{1}{z}(I_q - P)\right] \\
&= F(z)P\left[P + \frac{1}{z}(I_q - P)\right] \\
&= F(z)\left[P^2 + \frac{1}{z}(P - P^2)\right] = F(z)P = F(z).
\end{aligned}$$

The proof of (b) is complete. □

Proposition 8.8 *Let* $s_0 \in \mathbb{C}_{\geq}^{q\times q}$ *and let* $[\phi; \psi] \in \mathcal{P}[s_0]$. *Regarding Notation 4.7 and Definition 4.2, let* $\mathcal{D}$ *be a discrete subset of* Π_+ *such that the conditions (i)–(iii) in Definition 4.2 are fulfilled. Let* $V_{s_0} \colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ *be given by Notation 7.1 and let* V *be the restriction of* V_{s_0} *onto* Π_+. *Furthermore, let*

$$X := (I_q, 0_{q\times q})V\begin{pmatrix} \phi \\ \psi \end{pmatrix} \qquad \text{and} \qquad Y := (0_{q\times q}, I_q)V\begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{8.20}$$

Then the following four conditions hold true:

(i) *The matrix-valued functions* X *and* Y *are meromorphic in* Π_+ *and holomorphic in* $\Pi_+ \setminus \mathcal{D}$.
(ii) $\det Y(z) \neq 0$ *for all* $z \in \Pi_+ \setminus \mathcal{D}$.
(iii) $\operatorname{rank} \begin{pmatrix} X(z) \\ Y(z) \end{pmatrix} = q$ *for all* $z \in \Pi_+ \setminus \mathcal{D}$.
(iv) $\begin{pmatrix} X(z) \\ Y(z) \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} X(z) \\ Y(z) \end{pmatrix} \in \mathbb{C}_{\geq}^{q \times q}$ *for all* $z \in \Pi_+ \setminus \mathcal{D}$.

In particular, the pair $[X; Y]$ *belongs to* $\mathcal{PR}_q(\Pi_+)$. *Moreover, the matrix-valued function* $F := XY^{-1}$ *belongs to* $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^0, \leq]$.

Proof Let $P := \mathbb{P}_{\mathcal{R}(s_0)}$. Since $[\phi; \psi]$ belongs to $\mathcal{P}[s_0]$, from Notation 4.7 we see that $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$ and $P\phi = \phi$ hold true. Thus, in view of Definition 4.2, the matrix-valued functions ϕ and ψ are meromorphic in Π_+. Regarding Definition 4.2(i), in particular

$$P\phi(z) = \phi(z) \qquad \text{for all } z \in \Pi_+ \setminus \mathcal{D}. \tag{8.21}$$

In view of (8.20), Remark 7.2 and Definition 4.2(i), we see that X and Y are well-defined functions which are meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$. We consider an arbitrary $z \in \Pi_+ \setminus \mathcal{D}$. From (8.20) immediately we get

$$\begin{pmatrix} X(z) \\ Y(z) \end{pmatrix} = V_{s_0}(z) \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}. \tag{8.22}$$

Taking into account (8.22) and Notation 7.1, then, for all $z \in \Pi_+ \setminus \mathcal{D}$, we obtain

$$X(z) = -s_0\psi(z) \qquad \text{and} \qquad Y(z) = s_0^+\phi(z) + z\psi(z). \tag{8.23}$$

Now we consider an arbitrary $v \in \mathcal{N}(Y(z))$. From (8.23) then

$$s_0^+\phi(z)v + z\psi(z)v = 0_{q \times 1} \tag{8.24}$$

follows. Using (8.21) and Remark A.13, we get

$$\phi(z)v + zs_0\psi(z)v = s_0 s_0^+\phi(z)v + zs_0\psi(z)v = s_0\left[s_0^+\phi(z) + z\psi(z)\right]v = 0_{q \times 1} \tag{8.25}$$

and, consequently,

$$v^*\left[\psi(z)\right]^*\phi(z)v + zv^*\left[\psi(z)\right]^* s_0\psi(z)v = v^*\left[\psi(z)\right]^*\left[\phi(z) + zs_0\psi(z)\right]v = 0. \tag{8.26}$$

Because of $s_0 \in \mathbb{C}_{\geq}^{q\times q}$, we obtain $v^*[\psi(z)]^* s_0 \psi(z) v \in [0,\infty) \subseteq \mathbb{R}$ and, hence,

$$\operatorname{Im}\left(zv^*\left[\psi(z)\right]^* s_0\psi(z)v\right) = \operatorname{Im}(z)\cdot v^*\left[\psi(z)\right]^* s_0\psi(z)v. \tag{8.27}$$

Thus, (8.27) and (8.26) provide

$$\begin{aligned} &v^*\operatorname{Im}\left(\left[\psi(z)\right]^*\phi(z)\right)v + \operatorname{Im}(z)\cdot v^*\left[\psi(z)\right]^* s_0\psi(z)v \\ &= v^*\left[\operatorname{Im}\left(\left[\psi(z)\right]^*\phi(z) + z\left[\psi(z)\right]^* s_0\psi(z)\right)\right]v \\ &= \operatorname{Im}\left(v^*\left[\psi(z)\right]^*\phi(z)v + zv^*\left[\psi(z)\right]^* s_0\psi(z)v\right) = 0. \end{aligned} \tag{8.28}$$

Because of Definition 4.2(iii) and Remark 4.1, we have $\operatorname{Im}([\psi(z)]^*\phi(z)) \in \mathbb{C}_{\geq}^{q\times q}$ and, consequently, $v^*[\operatorname{Im}([\psi(z)]^*\phi(z))]v \in [0,\infty)$. Due to the assumption $s_0 \in \mathbb{C}_{\geq}^{q\times q}$, from $\operatorname{Im}(z) \in (0,\infty)$ also we know that $\operatorname{Im}(z)[\psi(z)]^* s_0\psi(z) \in \mathbb{C}_{\geq}^{q\times q}$ and therefore $\operatorname{Im}(z)\cdot v^*[\psi(z)]^* s_0\psi(z)v \in [0,\infty)$ is valid. Hence, by virtue of (8.28), we get $v^*\operatorname{Im}([\psi(z)]^*\phi(z))v = 0$ and $\operatorname{Im}(z)\cdot v^*[\psi(z)]^* s_0\psi(z)v = 0$. Consequently,

$$\left[\sqrt{s_0}\psi(z)v\right]^*\left[\sqrt{s_0}\psi(z)v\right] = \frac{1}{\operatorname{Im}(z)}\operatorname{Im}(z)\cdot v^*\left[\psi(z)\right]^* s_0\psi(z)v = 0,$$

which implies $s_0\psi(z)v = \sqrt{s_0}\sqrt{s_0}\psi(z)v = 0_{q\times 1}$. Then (8.25) shows that $\phi(z)v = 0_{q\times 1}$ is fulfilled. Therefore, from (8.24) and $z \neq 0$, moreover, $\psi(z)v = 0_{q\times 1}$ follows. Thus, v belongs to $\mathcal{N}(\phi(z)) \cap \mathcal{N}(\psi(z))$, i. e., to $\mathcal{N}(\binom{\phi(z)}{\psi(z)})$. Because of Definition 4.2(ii), we can conclude that $v = 0_{q\times 1}$. This means that $\mathcal{N}(Y(z)) \subseteq \{0_{q\times 1}\}$ holds true. Consequently, $\det Y(z) \neq 0$. In particular, $\operatorname{rank}\binom{X(z)}{Y(z)} = q$ and F is well defined. Using (8.22), Lemma 7.3, (8.21), Definition 4.2(iii), and $\operatorname{Im}(z)[\psi(z)]^* s_0\psi(z) \in \mathbb{C}_{\geq}^{q\times q}$, we conclude

$$\begin{aligned} &\begin{pmatrix} X(z) \\ Y(z) \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} X(z) \\ Y(z) \end{pmatrix} = \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}^* \left[V_{s_0}(z)\right]^* (-\widetilde{J}_q) V_{s_0}(z) \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \\ &= \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}^* \Bigg\{ \left[\operatorname{diag}(P, I_q)\right]^* (-\widetilde{J}_q)\cdot \operatorname{diag}(P, I_q) \\ &\qquad\qquad + 2\operatorname{Im}(z)\cdot \operatorname{diag}(0_{q\times q}, s_0) \Bigg\} \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} \\ &= \left(\left[\phi(z)\right]^* P^*, \left[\psi(z)\right]^*\right)(-\widetilde{J}_q) \begin{pmatrix} P\phi(z) \\ \psi(z) \end{pmatrix} + 2\operatorname{Im}(z)\left[\psi(z)\right]^* s_0\psi(z) \\ &= \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} \phi(z) \\ \psi(z) \end{pmatrix} + 2\operatorname{Im}(z)\left[\psi(z)\right]^* s_0\psi(z) \in \mathbb{C}_{\geq}^{q\times q}. \end{aligned}$$

Thus, the pair $[X; Y]$ belongs to $\mathcal{PR}_q(\Pi_+)$. According to (i) and (ii), the matrix-valued function F is meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$. Using (i), (ii), Remark 4.1 as well as (8.22), Lemma 7.3, and (8.21), we get

$$\begin{aligned}
\operatorname{Im} F(z) &= \operatorname{Im}\left(X(z)\,[Y(z)]^{-1}\right) = \operatorname{Im}\left([Y(z)]^{-*}\,[Y(z)]^{*}\,X(z)\,[Y(z)]^{-1}\right)\\
&= [Y(z)]^{-*}\operatorname{Im}\left([Y(z)]^{*}\,X(z)\right)[Y(z)]^{-1}\\
&= [Y(z)]^{-*}\left(\frac{1}{2}\begin{pmatrix}X(z)\\Y(z)\end{pmatrix}^{*}(-\widetilde{J}_q)\begin{pmatrix}X(z)\\Y(z)\end{pmatrix}\right)[Y(z)]^{-1}\\
&= \frac{1}{2}[Y(z)]^{-*}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}^{*}\left[V_{s_0}(z)\right]^{*}(-\widetilde{J}_q)V_{s_0}(z)\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}[Y(z)]^{-1}\\
&= \frac{1}{2}[Y(z)]^{-*}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}^{*}\Big\{\left[\operatorname{diag}(P, I_q)\right]^{*}(-\widetilde{J}_q)\cdot\operatorname{diag}(P, I_q)\\
&\qquad\qquad + 2\operatorname{Im}(z)\cdot\operatorname{diag}(0_{q\times q}, s_0)\Big\}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}[Y(z)]^{-1}\\
&= \frac{1}{2}[Y(z)]^{-*}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}^{*}\left[\operatorname{diag}(P, I_q)\right]^{*}(-\widetilde{J}_q)\cdot\operatorname{diag}(P, I_q)\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}[Y(z)]^{-1}\\
&\qquad + \operatorname{Im}(z)\,[Y(z)]^{-*}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}^{*}\cdot\operatorname{diag}(0_{q\times q}, s_0)\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}[Y(z)]^{-1}\\
&= [Y(z)]^{-*}\left[\frac{1}{2}\begin{pmatrix}P\phi(z)\\\psi(z)\end{pmatrix}^{*}(-\widetilde{J}_q)\begin{pmatrix}P\phi(z)\\\psi(z)\end{pmatrix} + \operatorname{Im}(z)\,[\psi(z)]^{*}\,s_0\psi(z)\right][Y(z)]^{-1}\\
&= [Y(z)]^{-*}\left[\frac{1}{2}\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix}^{*}(-\widetilde{J}_q)\begin{pmatrix}\phi(z)\\\psi(z)\end{pmatrix} + \operatorname{Im}(z)\,[\psi(z)]^{*}\,s_0\psi(z)\right][Y(z)]^{-1}.
\end{aligned} \tag{8.29}$$

Thus, from Definition 4.2(iii) and $\operatorname{Im}(z)[\psi(z)]^{*}s_0\psi(z) \in \mathbb{C}_{\geq}^{q\times q}$, we see that the matrix on the right-hand side of (8.29) is non-negative Hermitian. Consequently, (8.29) yields $\operatorname{Im} F(z) \in \mathbb{C}_{\geq}^{q\times q}$. Taking into account that F is meromorphic in Π_+ and holomorphic in $\Pi_+ \setminus \mathcal{D}$, that $\mathcal{D}$ is a discrete subset of Π_+, and that $\operatorname{Im} F(z) \in \mathbb{C}_{\geq}^{q\times q}$ is fulfilled for all $z \in \Pi_+ \setminus \mathcal{D}$, then, in view of Lemma 3.2, we see that F belongs to $\mathcal{R}_q(\Pi_+)$. In particular, F is holomorphic in Π_+. Since the matrix s_0 is

Hermitian, we have $s_0^* s_0^+ s_0 = s_0 s_0^+ s_0 = s_0$. From (8.23) we then conclude

$$\begin{aligned}
&[Y(z)]^{-*}[\psi(z)]^* s_0 \psi(z)[Y(z)]^{-1} = [Y(z)]^{-*}[\psi(z)]^* s_0^* s_0^+ s_0 \psi(z)[Y(z)]^{-1} \\
&= [Y(z)]^{-*}[-s_0\psi(z)]^* s_0^+ [-s_0\psi(z)][Y(z)]^{-1} \\
&= [Y(z)]^{-*}[X(z)]^* s_0^+ X(z)[Y(z)]^{-1} = \left(X(z)[Y(z)]^{-1}\right)^* s_0^+ X(z)[Y(z)]^{-1} \\
&= [F(z)]^* s_0^+ F(z).
\end{aligned} \tag{8.30}$$

By virtue of (8.29) and (8.30), we get

$$\begin{aligned}
&\frac{F(z)-[F(z)]^*}{z-\overline{z}} - [F(z)]^* s_0^+ F(z) = \frac{1}{\operatorname{Im}(z)} \operatorname{Im} F(z) - [F(z)]^* s_0^+ F(z) \\
&= \frac{1}{\operatorname{Im}(z)}[Y(z)]^{-*}\left[\frac{1}{2}\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix} + \operatorname{Im}(z)[\psi(z)]^* s_0\psi(z)\right][Y(z)]^{-1} \\
&\quad - [Y(z)]^{-*}[\psi(z)]^* s_0\psi(z)[Y(z)]^{-1} \\
&= \frac{1}{2\operatorname{Im}(z)}[Y(z)]^{-*}\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}[Y(z)]^{-1}.
\end{aligned} \tag{8.31}$$

Because of $\operatorname{Im}(z) \in (0, \infty)$ and Definition 4.2(iii), we see that the matrix on the right-hand side of (8.31) is non-negative Hermitian. Consequently, (8.31) implies

$$\frac{F(z)-[F(z)]^*}{z-\overline{z}} - [F(z)]^* s_0^+ F(z) \in \mathbb{C}_{\geq}^{q\times q}. \tag{8.32}$$

In view of (8.23) we have

$$\mathcal{R}(F(z)) = \mathcal{R}(X(z)[Y(z)]^{-1}) \subseteq \mathcal{R}(X(z)) = \mathcal{R}(-s_0\psi(z)) \subseteq \mathcal{R}(s_0). \tag{8.33}$$

Because of the assumption $s_0 \in \mathbb{C}_{\geq}^{q\times q}$, (8.33), and (8.32), the application of Lemma A.3 shows that the block matrix

$$\begin{pmatrix} s_0 & F(z) \\ [F(z)]^* & \frac{F(z)-[F(z)]^*}{z-\overline{z}} \end{pmatrix} \tag{8.34}$$

is non-negative Hermitian. Due to $F \in \mathcal{R}_q(\Pi_+)$, the matrix-valued function F is holomorphic in Π_+. Thus, using continuity arguments, we see that the block matrix given in (8.34) is non-negative Hermitian for all $z \in \Pi_+$. Thus, [6, Lem. 8.9] shows that F belongs to $\mathcal{R}_{0,q}(\Pi_+)$ and that the $\mathbb{R}$-Stieltjes measure σ_F of F fulfils

$\int_{\mathbb{R}} x^0 \sigma_F(\mathrm{d}x) = \sigma_F(\mathbb{R}) \le s_0$. Therefore, $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{0}, \le]$ is proved as well. □

After having handled the case of a sequence $(s_j)_{j=0}^{0}$ from $\mathbb{C}_{\ge}^{q\times q} = \mathcal{H}_{q,2\cdot 0}^{\ge,\mathrm{e}}$ we turn our attention to the case of a sequence from $\mathcal{H}_{q,2n}^{\ge,\mathrm{e}}$ with arbitrary $n \in \mathbb{N}$.

Proposition 8.9 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\ge,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let*

$$\mathfrak{V}_n^{(s)} = \begin{pmatrix} \mathfrak{v}_{11;n}^{(s)} & \mathfrak{v}_{12;n}^{(s)} \\ \mathfrak{v}_{21;n}^{(s)} & \mathfrak{v}_{22;n}^{(s)} \end{pmatrix} \tag{8.35}$$

be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Let $[\phi; \psi] \in \mathcal{P}[s_0^{(n)}]$. Then there exists a discrete subset $\mathcal{D}$ of Π_+ such that the conditions (i)–(iii) in Definition 4.2 hold true and that

$$\det\left[\mathfrak{v}_{21;n}^{(s)}(z)\phi(z) + \mathfrak{v}_{22;n}^{(s)}(z)\psi(z)\right] \ne 0$$

is fulfilled for all $z \in \Pi_+ \setminus \mathcal{D}$. Furthermore, the matrix-valued function

$$(\mathfrak{v}_{11;n}^{(s)}\phi + \mathfrak{v}_{12;n}^{(s)}\psi)(\mathfrak{v}_{21;n}^{(s)}\phi + \mathfrak{v}_{22;n}^{(s)}\psi)^{-1}$$

belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \le]$.

Proof Our proof works inductively. First we consider the case $n = 0$. Since $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\ge,\mathrm{e}}$, from Remarks 2.3 and 5.1 we get $s_0 \in \mathbb{C}_{\ge}^{q\times q}$. In view of $s_0^{(0)} = s_0$, we see that the pair $[\phi; \psi]$ belongs to $\mathcal{P}[s_0]$ and, in view of (7.3), that $\mathfrak{V}^{((s_j)_{j=0}^{0})} = V_{s_0^{(0)}} = V_{s_0}$. Consequently, taking into account (8.35), the matrix-valued functions X and Y defined by (8.20) fulfil

$$X = \mathfrak{v}_{11;0}^{(s)}\phi + \mathfrak{v}_{12;0}^{(s)}\psi \qquad \text{and} \qquad Y = \mathfrak{v}_{21;0}^{(s)}\phi + \mathfrak{v}_{22;0}^{(s)}\psi.$$

Regarding Notation 4.7 and Definition 4.2, there exists a discrete subset $\mathcal{D}$ of Π_+ such that the conditions (i)–(iii) in Definition 4.2 are fulfilled for $[\phi; \psi]$. Applying Proposition 8.8, the proof is complete in the case $n = 0$.

We may now assume that there is an $m \in \mathbb{N}$ such that Proposition 8.9 is proved for all $n \in \mathbb{Z}_{0,m-1}$. We study the case $n = m$. Let $t_j := s_j^{(1)}$ for all $j \in \mathbb{Z}_{0,2m-2}$. Using the assumption $(s_j)_{j=0}^{2m} \in \mathcal{H}_{q,2m}^{\ge,\mathrm{e}}$ and Proposition 6.3, we obtain $(t_j)_{j=0}^{2(m-1)} \in \mathcal{H}_{q,2(m-1)}^{\ge,\mathrm{e}}$. In particular, also we have $t_0^{(m-1)} = s_0^{(m)} = s_0^{(n)}$. Thus, $[\phi; \psi] \in \mathcal{P}[t_0^{(m-1)}]$. Since we have assumed that Proposition 8.9 is true for

$n = m - 1$, we obtain that there exists a discrete subset $\mathcal{D}$ of Π_+ such that the conditions (i)–(iii) in Definition 4.2 as well as

$$\det\left[\mathfrak{v}_{21;m-1}^{(t)}(z)\phi(z) + \mathfrak{v}_{22;m-1}^{(t)}(z)\psi(z)\right] \neq 0 \tag{8.36}$$

for all $z \in \Pi_+ \setminus \mathcal{D}$ are valid and that

$$G := (\mathfrak{v}_{11;m-1}^{(t)}\phi + \mathfrak{v}_{12;m-1}^{(t)}\psi)(\mathfrak{v}_{21;m-1}^{(t)}\phi + \mathfrak{v}_{22;m-1}^{(t)}\psi)^{-1} \tag{8.37}$$

is a well-defined matrix-valued function such that

$$G \in \mathcal{R}_{0,q}[\Pi_+; (t_j)_{j=0}^{2(m-1)}, \leq]. \tag{8.38}$$

In particular, we get $G \in \mathcal{R}_{0,q}(\Pi_+)$ and the $\mathbb{R}$-Stieltjes measure σ_G of G fulfils

$$0_{q\times q} \leq \sigma_G(\mathbb{R}) = \int_{\mathbb{R}} x^0 \sigma_G(\mathrm{d}x) \leq t_0 \tag{8.39}$$

and, by virtue of Remark A.1, consequently, $\mathcal{N}(t_0) \subseteq \mathcal{N}(\sigma_G(\mathbb{R}))$. Because of Proposition 3.7, we have $\mathcal{R}_{0,q}(\Pi_+) \subseteq \mathcal{R}_{-1,q}(\Pi_+)$. Thus, G belongs to $\mathcal{R}_{-1,q}(\Pi_+)$. Lemma 3.8 yields $G \in \mathcal{R}_q^{[-1]}(\Pi_+)$ and $\mathcal{N}(\mu_G(\mathbb{R})) = \mathcal{N}(\sigma_G(\mathbb{R}))$, where μ_G is given via Remark 3.6. Consequently, additionally using (6.2) and $\mathcal{N}(t_0) \subseteq \mathcal{N}(\sigma_G(\mathbb{R}))$, we conclude

$$\mathcal{N}(s_0) \subseteq \mathcal{N}(-s_0 s_2^\sharp s_0) = \mathcal{N}(s_0^{(1)}) = \mathcal{N}(t_0) \subseteq \mathcal{N}(\sigma_G(\mathbb{R})) = \mathcal{N}(\mu_G(\mathbb{R})). \tag{8.40}$$

Thus, in view of (3.6), we proved that G belongs to $\mathcal{P}_q^{\mathrm{odd}}[s_0]$. In view of the assumption $(s_j)_{j=0}^{2m} \in \mathcal{H}_{q,2m}^{\geq,\mathrm{e}}$ and Remark 5.2, we know that $(s_j)_{j=0}^{1} \in \mathcal{H}_{q,1}^{\geq,\mathrm{e}}$ is valid. We consider an arbitrary $z \in \Pi_+ \setminus \mathcal{D}$. Using [15, Lem. 8.12], then we obtain $G(z) \in \mathcal{Q}_{[s_0^+, zI_q - s_0^+ s_1]}$, i. e.

$$\det\left[s_0^+ G(z) + zI_q - s_0^+ s_1\right] \neq 0, \tag{8.41}$$

and furthermore

$$G^{(-;s_0,s_1)}(z) = \mathcal{S}_{V_{s_0,s_1}(z)}^{(q,q)}(G(z)). \tag{8.42}$$

In particular, Remark C.1 yields

$$\operatorname{rank}(s_0^+, zI_q - s_0^+ s_1) = q. \tag{8.43}$$

Furthermore, applying Remark C.1, from (8.36) also we get that

$$\operatorname{rank}\left(\mathfrak{v}_{21;m-1}^{(t)}(z),\, \mathfrak{v}_{22;m-1}^{(t)}(z)\right) = q \tag{8.44}$$

holds true. Regarding (8.36) and (8.35), from (8.37) we obtain

$$G(z) = \widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{(t)}_{m-1}(z)}\left([\phi(z), \psi(z)]\right). \tag{8.45}$$

From Remark 6.2 and (7.3) (see also [15, Rem. 11.16]) we know that $\mathfrak{V}^{((s_j)_{j=0}^{2m})} = V_{s_0,s_1}\mathfrak{V}^{((t_j)_{j=0}^{2(m-1)})}$. Consequently

$$\mathfrak{V}^{(s)}_m(z) = V_{s_0,s_1}(z)\mathfrak{V}^{(t)}_{m-1}(z). \tag{8.46}$$

In view of (8.35), (7.1), (8.46), (8.44), (8.43), (8.36), (8.45), (8.41), and Proposition C.3, we conclude $\det[\mathfrak{v}^{(s)}_{21;m}(z)\phi(z) + \mathfrak{v}^{(s)}_{22;m}(z)\psi(z)] \neq 0$ and, moreover,

$$\widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{(s)}_m(z)}\left([\phi(z), \psi(z)]\right) = \mathcal{S}^{(q,q)}_{V_{s_0,s_1}(z)}\left(G(z)\right). \tag{8.47}$$

Since $(s_j)_{j=0}^{2m} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2m}$ is assumed, from Proposition 8.4 we see that $(s_j)_{j=0}^{2m}$ belongs to $\mathcal{D}_{q\times q,2m}$ as well. Furthermore, Remark 2.3 yields $(s_j)_{j=0}^{2m} \in \mathcal{H}^{\geq}_{q,2m}$. Because of (8.38) and Proposition 8.5, then we get $G^{(-;s_0,s_1)} \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2m}, \leq]$. Since from (8.42), (8.47), (8.35), and Notation C.2 we obtain that

$$\begin{aligned} G^{(-;s_0,s_1)}(z) &= \mathcal{S}^{(q,q)}_{V_{s_0,s_1}(z)}\left(G(z)\right) = \widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{(s)}_m(z)}\left([\phi(z), \psi(z)]\right) \\ &= \left[\mathfrak{v}^{(s)}_{11;m}(z)\phi(z) + \mathfrak{v}^{(s)}_{12;m}(z)\psi(z)\right]\left[\mathfrak{v}^{(s)}_{21;m}(z)\phi(z) + \mathfrak{v}^{(s)}_{22;m}(z)\psi(z)\right]^{-1} \end{aligned}$$

and since $z \in \Pi_+ \setminus \mathcal{D}$ is arbitrary and $\mathcal{D}$ is a discrete subset of Π_+, we can conclude that $F := (\mathfrak{v}^{(s)}_{11;m}\phi + \mathfrak{v}^{(s)}_{12;m}\psi)(\mathfrak{v}^{(s)}_{21;n}\phi + \mathfrak{v}^{(s)}_{22;n}\psi)^{-1}$ coincides with $G^{(-;s_0,s_1)}$. In particular, $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2m}, \leq]$. Thus, Proposition 8.9 is proved inductively. □

Proposition 8.10 *Let $n \in \mathbb{N}_0$, let $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,\mathrm{e}}_{q,2n}$ with nth Schur transform $(s^{(n)}_j)_{j=0}^{0}$, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), and let (7.5) be the $q \times q$ block representation of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$. Furthermore, let $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$. Then* *there exists a pair* $[\phi; \psi] \in \mathcal{P}[s_0^{(n)}]$ *such that the following five conditions are fulfilled:*

(i) ϕ *and* ψ *both are holomorphic in* Π_+.
(ii) $\operatorname{rank}\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix} = q$ *for all* $z \in \Pi_+$.
(iii) $\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}^* (-\widetilde{J}_q)\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix} \in \mathbb{C}_{\geq}^{q\times q}$ *for all* $z \in \Pi_+$.
(iv) *The inequality*

$$\det\left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right] \neq 0 \tag{8.48}$$

holds true for all $z \in \Pi_+$.
(v) *The matrix-valued function* F *admits, for all* $z \in \Pi_+$, *the representation*

$$F(z) = \left[\mathfrak{v}_{11}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{12}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right] \times \left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right]^{-1}. \tag{8.49}$$

Proof Our proof works inductively. From the assumption $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and Remarks 2.3 and 5.1, we get $s_0 \in \mathbb{C}_{\geq}^{q\times q}$. In view of $s_0^{(0)} = s_0$, (7.3), and Notation 7.1, then Proposition 8.7 immediately proves the assertion in the case $n = 0$. Thus, we may suppose that there is an $m \in \mathbb{N}$ such that Proposition 8.10 is proved for all $n \in \mathbb{Z}_{0,m-1}$.

We consider now the case $n = m$. Let $t_j := s_j^{(1)}$ for each $j \in \mathbb{Z}_{0,2m-2}$. According to Remark 6.2 and (7.3) (see also [15, Rem. 11.16]), we obtain

$$\mathfrak{V}^{((s_j)_{j=0}^{2m})} = V_{s_0,s_1}\mathfrak{V}^{((t_j)_{j=0}^{2(m-1)})}. \tag{8.50}$$

Using Proposition 6.3, we obtain $(t_j)_{j=0}^{2(m-1)} \in \mathcal{H}_{q,2(m-1)}^{\geq,\mathrm{e}}$. Remark 2.3 yields $(s_j)_{j=0}^{2m} \in \mathcal{H}_{q,2m}^{\geq}$. Because of the assumption $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2m}, \leq]$, from Proposition 8.2 we then see that the function $G := F^{(+;s_0,s_1)}$ fulfils (8.38). Since we have assumed that Proposition 8.10 already has been proved for $n = m-1$, then we see that there exists a pair $[\phi; \psi] \in \mathcal{P}[t_0^{(m-1)}]$ such that (i)–(iii) as well as

$$\det\left[\mathfrak{v}_{21}^{((t_j)_{j=0}^{2(m-1)})}(z)\phi(z) + \mathfrak{v}_{22}^{((t_j)_{j=0}^{2(m-1)})}(z)\psi(z)\right] \neq 0 \tag{8.51}$$

and $G(z) = [\mathfrak{v}_{11}^{((t_j)_{j=0}^{2(m-1)})}(z)\phi(z) + \mathfrak{v}_{12}^{((t_j)_{j=0}^{2(m-1)})}(z)\psi(z)][\mathfrak{v}_{21}^{((t_j)_{j=0}^{2(m-1)})}(z)\phi(z) + \mathfrak{v}_{22}^{((t_j)_{j=0}^{2(m-1)})}(z)\psi(z)]^{-1}$, i. e.,

$$G(z) = \widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{((t_j)_{j=0}^{2(m-1)})}(z)}([\phi(z), \psi(z)]), \tag{8.52}$$

hold true for all $z \in \Pi_+$. We now consider an arbitrary $z \in \Pi_+$. By virtue of Remark 6.2, we have $t_0^{(m-1)} = s_0^{(m)}$ which implies $[\phi; \psi] \in \mathcal{P}[s_0^{(m)}]$. Because of (8.51) and Remark C.1, the equation

$$\operatorname{rank}\left(\mathfrak{v}_{21}^{((t_j)_{j=0}^{2(m-1)})}(z), \mathfrak{v}_{22}^{((t_j)_{j=0}^{2(m-1)})}(z)\right) = q \tag{8.53}$$

is valid. From (8.38), in particular, we get $G \in \mathcal{R}_{0,q}(\Pi_+)$ which together with Proposition 3.7 yields $G \in \mathcal{R}_{-1,q}(\Pi_+)$. Moreover, (8.38) also provides that the $\mathbb{R}$-Stieltjes measure σ_G of G fulfils (8.39) and, in view of Remark A.1, consequently, $\mathcal{N}(t_0) \subseteq \mathcal{N}(\sigma_G(\mathbb{R}))$. Lemma 3.8 yields $G \in \mathcal{R}_q^{[-1]}(\Pi_+)$ and $\mathcal{N}(\mu_G(\mathbb{R})) = \mathcal{N}(\sigma_G(\mathbb{R}))$. In view of (6.2), then it follows (8.40). Consequently, since G belongs to $\mathcal{R}_{-1,q}(\Pi_+)$, we see from (3.6) that G belongs to $\mathcal{P}_q^{\text{odd}}[s_0]$ as well. Since $(s_j)_{j=0}^{2m}$ belongs to $\mathcal{H}_{q,2m}^{\geq,\text{e}}$, Remark 5.2 provides $(s_j)_{j=0}^{1} \in \mathcal{H}_{q,1}^{\geq,\text{e}}$. Using [15, Lem. 8.12], then we get (8.41) and (8.42). By virtue of (8.41) and Remark C.1, in particular we conclude (8.43). Regarding (7.5), (7.1), (8.50), (8.53), (8.43), (8.51), (8.52), and (8.41), the application of Proposition C.3 yields (8.48) for $n = m$ and

$$\widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{((s_j)_{j=0}^{2m})}(z)}([\phi(z), \psi(z)]) = \mathcal{S}^{(q,q)}_{V_{s_0,s_1}(z)}(G(z)). \tag{8.54}$$

Because of the assumption $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2m}, \leq]$ and $m \geq 1$, we have $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{1}, =]$. Thus, from $(s_j)_{j=0}^{1} \in \mathcal{H}_{q,1}^{\geq,\text{e}}$ and [15, Cor. 8.18(b)] we get $G^{(-;s_0,s_1)} = F$. Consequently, from (8.41), (8.42), and (8.54), it follows

$$F(z) = G^{(-;s_0,s_1)}(z) = \mathcal{S}^{(q,q)}_{V_{s_0,s_1}(z)}(G(z)) = \widetilde{\mathcal{S}}^{(q,q)}_{\mathfrak{V}^{((s_j)_{j=0}^{2m})}(z)}([\phi(z), \psi(z)])$$

and, in view of Notation C.2 and (7.5), then (8.49) for $n = m$. Proposition 8.10, thus, is proved inductively. □

Now we state the main result of this section. It is an immediate consequence of combining the Propositions 8.9 and 8.10 and gives a first parametrization of the solution set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ of Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, where, however, the parameters depend on the given data. Nevertheless, the following result can be considered as an important intermediate step on the way to the final result.

Theorem 8.11 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Then:*

(a) *For each pair $[\phi; \psi] \in \mathcal{P}[s_0^{(n)}]$, the function $\det(\mathfrak{v}_{21;n}^{(s)}\phi + \mathfrak{v}_{22;n}^{(s)}\psi)$ does not vanish identically and the matrix-valued function*

$$(\mathfrak{v}_{11;n}^{(s)}\phi + \mathfrak{v}_{12;n}^{(s)}\psi)(\mathfrak{v}_{21;n}^{(s)}\phi + \mathfrak{v}_{22;n}^{(s)}\psi)^{-1} \tag{8.55}$$

belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.

(c) *For each $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, there exists a pair $[\phi; \psi] \in \mathcal{P}[s_0^{(n)}]$ such that both ϕ and ψ are holomorphic in Π_+, that (8.48) holds true for all $z \in \Pi_+$, and that F admits the representation (8.49) for all $z \in \Pi_+$.*

(c) *Let $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{P}[s_0^{(n)}]$. Then the following statements are equivalent:*

(i) $(\mathfrak{v}_{11;n}^{(s)}\phi_1 + \mathfrak{v}_{12;n}^{(s)}\psi_1)(\mathfrak{v}_{21;n}^{(s)}\phi_1 + \mathfrak{v}_{22;n}^{(s)}\psi_1)^{-1}$
$= (\mathfrak{v}_{11;n}^{(s)}\phi_2 + \mathfrak{v}_{12;n}^{(s)}\psi_2)(\mathfrak{v}_{21;n}^{(s)}\phi_2 + \mathfrak{v}_{22;n}^{(s)}\psi_2)^{-1}$.

(ii) $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$.

Proof

(a) Part (a) immediately follows from Proposition 8.9.
(b) Apply Proposition 8.10.
(c) For each $j \in \{1, 2\}$, let

$$X_j := (I_q, 0_{q\times q})\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix} \quad \text{and} \quad Y_j := (0_{q\times q}, I_q)\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix}. \tag{8.56}$$

Obviously, for each $j \in \{1, 2\}$, the matrix-valued functions ϕ_j, ψ_j, X_j, and Y_j are meromorphic in Π_+. In view of (8.35), part (a) then yields that the functions $\det Y_1$ and $\det Y_2$ do not vanish identically and that $F_1 := X_1Y_1^{-1}$ as well as $F_2 := X_2Y_2^{-1}$ and $g := Y_2^{-1}Y_1$ are well-defined matrix-valued functions meromorphic in Π_+. Moreover, $\det g$ does not vanish identically. For each $j \in \{1, 2\}$, from (8.56) we get

$$\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix}Y_j^{-1} = \begin{pmatrix}X_j\\ Y_j\end{pmatrix}Y_j^{-1} = \begin{pmatrix}X_jY_j^{-1}\\ Y_jY_j^{-1}\end{pmatrix} = \begin{pmatrix}F_j\\ I_q\end{pmatrix}. \tag{8.57}$$

Taking into account the $q \times q$ block representation (8.35) of $\mathfrak{V}_n^{(s)}$, from (8.57) we see that

$$(\mathfrak{v}_{11;n}^{(s)}\phi_j + \mathfrak{v}_{12;n}^{(s)}\psi_j)Y_j^{-1} = F_j \qquad \text{and} \qquad (\mathfrak{v}_{21;n}^{(s)}\phi_j + \mathfrak{v}_{22;n}^{(s)}\psi_j)Y_j^{-1} = I_q$$

and, consequently,

$$
\begin{aligned}
&(\mathfrak{v}_{11;n}^{(s)}\phi_j + \mathfrak{v}_{12;n}^{(s)}\psi_j)(\mathfrak{v}_{21;n}^{(s)}\phi_j + \mathfrak{v}_{22;n}^{(s)}\psi_j)^{-1} \\
&\quad = (\mathfrak{v}_{11;n}^{(s)}\phi_j + \mathfrak{v}_{12;n}^{(s)}\psi_j)Y_j^{-1}\left[(\mathfrak{v}_{21;n}^{(s)}\phi_j + \mathfrak{v}_{22;n}^{(s)}\psi_j)Y_j^{-1}\right]^{-1} = F_j I_q^{-1} = F_j
\end{aligned}
\tag{8.58}
$$

for each $j \in \{1, 2\}$. Let $(\mathfrak{h}_j)_{j=0}^{2n}$ be the sequence of $\mathcal{H}$-parameters of $(s_j)_{j=0}^{2n}$. Further, for each $l \in \mathbb{Z}_{-1,n+1}$, let $P_{n,l}$ be given by (7.8). Then Lemma 7.8 yields (7.26) for all $z \in \Pi_+$, where $\mathfrak{W}^{((s_j)_{j=0}^{2n})}$ is given by (7.6). From the assumptions that $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and that $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ belong to $\mathcal{P}[s_0^{(n)}]$ and Theorem 6.10 we get $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{P}[\mathfrak{h}_{2n}]$. Thus, regarding Notation 4.7 and Definition 4.2 we see that ϕ_1 and ϕ_2 are matrix-valued functions which are meromorphic in Π_+ and which fulfil

$$
P_{n,n}\phi_1 = \mathbb{P}_{\mathcal{R}(\mathfrak{h}_{2n})}\phi_1 = \phi_1 \qquad \text{and} \qquad P_{n,n}\phi_2 = \mathbb{P}_{\mathcal{R}(\mathfrak{h}_{2n})}\phi_2 = \phi_2. \tag{8.59}
$$

(i)⇒(ii) Let $\mathfrak{W}_n^{(s)}$ be the restriction of $\mathfrak{W}^{((s_j)_{j=0}^{2n})}$ onto Π_+ and let $Q\colon \Pi_+ \to \mathbb{C}^{q\times q}$ be defined by

$$
Q(z) := \sum_{k=0}^{n+1} z^k (P_{n,n-k} - P_{n,n-k+1}). \tag{8.60}
$$

From (7.26), then

$$
\mathfrak{W}_n^{(s)}\mathfrak{V}_n^{(s)} = \operatorname{diag}(P_{n,n}, Q) \tag{8.61}
$$

follows. Because of (i) and (8.58), we conclude $F_1 = F_2$. Thus, additionally using (8.57), we obtain

$$
\mathfrak{W}_n^{(s)}\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}Y_1^{-1} = \mathfrak{W}_n^{(s)}\begin{pmatrix}F_1\\ I_q\end{pmatrix} = \mathfrak{W}_n^{(s)}\begin{pmatrix}F_2\\ I_q\end{pmatrix} = \mathfrak{W}_n^{(s)}\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}Y_2^{-1}
$$

and, hence,

$$
\mathfrak{W}_n^{(s)}\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix} = \mathfrak{W}_n^{(s)}\mathfrak{V}_n^{(s)}\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}g. \tag{8.62}
$$

By virtue of (8.61) and (8.62), then

$$\begin{pmatrix} P_{n,n}\phi_1 \\ Q\psi_1 \end{pmatrix} = \operatorname{diag}(P_{n,n}, Q) \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = \mathfrak{W}_n^{(s)} \mathfrak{V}_n^{(s)} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}$$

$$= \mathfrak{W}_n^{(s)} \mathfrak{V}_n^{(s)} \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} g = \operatorname{diag}(P_{n,n}, Q) \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} g = \begin{pmatrix} P_{n,n}\phi_2 g \\ Q\psi_2 g \end{pmatrix} \tag{8.63}$$

follows. Using (8.59), from (8.63) we conclude

$$\phi_1 = P_{n,n}\phi_1 = P_{n,n}\phi_2 g = \phi_2 g \qquad \text{and} \qquad Q\psi_1 = Q\psi_2 g. \tag{8.64}$$

Let $\mathcal{V}_0 := \{0_{q\times 1}\}$, $\mathcal{V}_{n+2} := \mathbb{C}^q$, and, for all $l \in \mathbb{Z}_{1,n+1}$, furthermore $\mathcal{V}_l := \mathcal{R}(\mathfrak{h}_{2(n+1-l)})$. By the assumption $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ and by Proposition 5.7, then we infer

$$\{0_{q\times 1}\} = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \mathcal{V}_{n+1} \subseteq \mathcal{V}_{n+2} = \mathbb{C}^q. \tag{8.65}$$

In view of (7.8), we have $P_{n,l} = \mathbb{P}_{\mathcal{V}_{n+1-l}}$ for all $l \in \mathbb{Z}_{-1,n+1}$ and, because of (8.60), consequently,

$$Q(z) = \sum_{k=0}^{n+1} z^k (\mathbb{P}_{\mathcal{V}_{k+1}} - \mathbb{P}_{\mathcal{V}_k}) \tag{8.66}$$

for all $z \in \Pi_+$. Obviously, $z^k \neq 0$ for every choice of $k \in \mathbb{N}_0$ and $z \in \Pi_+$. Thus, taking into account (8.65) and (8.66), from Lemma A.10 we get $\det Q(z) \neq 0$ for all $z \in \Pi_+$. Hence, by virtue of (8.64), we get $\psi_1 = \psi_2 g$. Taking additionally into account the first identity in (8.64) and the fact that $\det g$ does not vanish identically, from Remark 4.4 we can conclude that (ii) holds true.

(i)⇒(ii) Since (ii) is assumed, Remark 4.4 shows that there is a matrix-valued function g meromorphic in Π_+ such that $\det g$ does not vanish identically and that $\binom{\phi_1}{\psi_1} = \binom{\phi_2}{\psi_2} g$ is fulfilled. Thus, in view of (8.56), we obtain

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \mathfrak{V}_n^{(s)} \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = \mathfrak{V}_n^{(s)} \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} g = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} g = \begin{pmatrix} X_2 g \\ Y_2 g \end{pmatrix}.$$

Consequently, since the functions $\det Y_1$ and $\det Y_2$ do not vanish identically, then

$$F_2 = X_2 Y_2^{-1} = X_2 g g^{-1} Y_2^{-1} = X_2 g (Y_2 g)^{-1} = X_1 Y_1^{-1} = F_1$$

follows which, in view of (8.58), implies (i).

□

Corollary 8.12 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Then $T: \langle \mathcal{P}[s_0^{(n)}] \rangle \to \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ given by*

$$T(\langle[\phi; \psi]\rangle) := (\mathfrak{v}_{11;n}^{(s)}\phi + \mathfrak{v}_{12;n}^{(s)}\psi)(\mathfrak{v}_{21;n}^{(s)}\phi + \mathfrak{v}_{22;n}^{(s)}\psi)^{-1} \tag{8.67}$$

is a well-defined bijective mapping.

Proof In view of parts (a) and (c) of Theorem 8.11, the mapping T is well defined. Theorem 8.11(b) shows that T is surjective and Theorem 8.11(c) yields the injectivity of T. □

9 Parametrization of the Class $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$

From Theorem 2.2 we know that the solution set $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ of the Hamburger moment problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ is non-empty if and only if the sequence $(s_j)_{j=0}^{2n}$ belongs to the class $\mathcal{H}_{q,2n}^{\geq}$. Theorem 2.4 shows that we can restrict our consideration to the case that the given sequence $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices belongs to the subclass $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ of $\mathcal{H}_{q,2n}^{\geq}$ (see also Remark 2.3). Remark 3.4 shows that Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ equivalently can be reformulated into Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ with solution set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$. In the non-degenerate case, where the given sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{>}$, I. V. Kovalishina [27, Thm. $\mathscr{H}$] obtained a parametrization of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$. In the general case, where the given sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$, V. A. Bolotnikov [2, Thm. 4.6] presents a parametrization of $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, though, the argumentation there contains a certain misstatement (see [3]). Both Kovalishina and Bolotnikov utilize the method of fundamental matrix inequalities due to V. P. Potapov. Using an approach via a Schur type algorithm, Chen and Hu [4, Thm. 3.4] achieved a parametrization of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ for a given sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$. Nevertheless, they solely give a reference for a respective proof which includes a parametrization of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, =]$. The authors of this text, nevertheless, consider a thorough explanation of the use of [4, Cor. 3.3] as necessary. (The respective detailed analysis is performed within the former sections of this text.) Moreover, it should be pointed out that the approach chosen by this time, contrasting the results in [2, 4], guarantees a representation of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ which provides, for each matrix-valued function belonging to the solution set, a representation holding for each $z \in \Pi_+$.

Now we are going to prove a parametrization of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ with parameters which are independent of the given data. As sets of independent data we work with the set of Nevanlinna pairs in Π_+ as well as (in an alternative setting) with the set of all matricial Schur functions in Π_+.

In our following discussion, we distinguish three cases all in which $(s_j^{(n)})_{j=0}^{0}$ denotes the nth Schur transform of the given sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$:

(I) $\operatorname{rank} s_0^{(n)} = q$.
(II) $1 \leq \operatorname{rank} s_0^{(n)} \leq q-1$.
(III) $\operatorname{rank} s_0^{(n)} = 0$.

Observe that from Proposition 6.11 we know that case (I) exactly is the case that the given sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{>}$ and that case (III) holds true if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$. Now we first turn our attention to the case (I).

Theorem 9.1 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Then:*

(a) *For each pair $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$, the function $\det(\mathfrak{v}_{21;n}^{(s)}\phi + \mathfrak{v}_{22;n}^{(s)}\psi)$ does not vanish identically and the matrix-valued function given in (8.55) belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.*
(b) *For each $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, there exists a pair $[\phi; \psi] \in \mathcal{PR}_q(\Pi_+)$ such that ϕ and ψ both are holomorphic in Π_+, that (8.48) holds true for all $z \in \Pi_+$, and that F admits the representation (8.49) for all $z \in \Pi_+$.*
(c) *Let $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{PR}_q(\Pi_+)$. Then*

$$(\mathfrak{v}_{11;n}^{(s)}\phi_1 + \mathfrak{v}_{12;n}^{(s)}\psi_1)(\mathfrak{v}_{21;n}^{(s)}\phi_1 + \mathfrak{v}_{22;n}^{(s)}\psi_1)^{-1} = (\mathfrak{v}_{11;n}^{(s)}\phi_2 + \mathfrak{v}_{12;n}^{(s)}\psi_2)(\mathfrak{v}_{21;n}^{(s)}\phi_2 + \mathfrak{v}_{22;n}^{(s)}\psi_2)^{-1}$$

if and only if $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$.

Proof Because of the assumption that $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{>}$, Remark 2.3 yields $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$, and from Proposition 6.11(a) we get $\operatorname{rank} s_0^{(n)} = q$, where $(s_j^{(n)})_{j=0}^{0}$ is the nth Schur transform of $(s_j)_{j=0}^{2n}$. Consequently, the combination of Theorem 8.11 and Remark 4.9 yields the assertion. □

Now we give a parametrization of the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ by aid of matricial Schur functions.

Corollary 9.2 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$. Let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let*

$$C_q := \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}I_q & \mathrm{i}I_q \\ I_q & I_q \end{pmatrix}, \tag{9.1}$$

let $\mathfrak{U}^{((s_j)_{j=0}^{2n})} := \mathfrak{V}^{((s_j)_{j=0}^{2n})} C_q$, let $\mathfrak{U}_n^{(s)}$ be the restriction of $\mathfrak{U}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let

$$\mathfrak{U}^{((s_j)_{j=0}^{2n})} = \begin{pmatrix} \mathfrak{u}_{11}^{((s_j)_{j=0}^{2n})} & \mathfrak{u}_{12}^{((s_j)_{j=0}^{2n})} \\ \mathfrak{u}_{21}^{((s_j)_{j=0}^{2n})} & \mathfrak{u}_{22}^{((s_j)_{j=0}^{2n})} \end{pmatrix} \quad \text{and} \quad \mathfrak{U}_n^{(s)} = \begin{pmatrix} \mathfrak{u}_{11;n}^{(s)} & \mathfrak{u}_{12;n}^{(s)} \\ \mathfrak{u}_{21;n}^{(s)} & \mathfrak{u}_{22;n}^{(s)} \end{pmatrix} \tag{9.2}$$

be the $q \times q$ block representations of $\mathfrak{U}^{((s_j)_{j=0}^{2n})}$ and $\mathfrak{U}_n^{(s)}$, resp. Then:

(a) *For each $S \in \mathcal{S}_{q\times q}(\Pi_+)$, the function $\det(\mathfrak{u}_{21;n}^{(s)} S + \mathfrak{u}_{22;n}^{(s)})$ does not vanish identically and the matrix-valued function*

$$(\mathfrak{u}_{11;n}^{(s)} S + \mathfrak{u}_{12;n}^{(s)})(\mathfrak{u}_{21;n}^{(s)} S + \mathfrak{u}_{22;n}^{(s)})^{-1}$$

belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.

(b) *For each $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, there exists a unique $S \in \mathcal{S}_{q\times q}(\Pi_+)$ such that the function $\det(\mathfrak{u}_{21;n}^{(s)} S + \mathfrak{u}_{22;n}^{(s)})$ does not vanish identically and that F admits the representation*

$$F = (\mathfrak{u}_{11;n}^{(s)} S + \mathfrak{u}_{12;n}^{(s)})(\mathfrak{u}_{21;n}^{(s)} S + \mathfrak{u}_{22;n}^{(s)})^{-1}.$$

Moreover,

$$\det\left[\mathfrak{u}_{21}^{((s_j)_{j=0}^{2n})}(z)S(z) + \mathfrak{u}_{22}^{((s_j)_{j=0}^{2n})}(z)\right] \neq 0$$

and

$$F(z) = \left[\mathfrak{u}_{11}^{((s_j)_{j=0}^{2n})}(z)S(z) + \mathfrak{u}_{12}^{((s_j)_{j=0}^{2n})}(z)\right]\left[\mathfrak{u}_{21}^{((s_j)_{j=0}^{2n})}(z)S(z) + \mathfrak{u}_{22}^{((s_j)_{j=0}^{2n})}(z)\right]^{-1}$$

are fulfilled for all $z \in \Pi_+$.

Proof Combine Theorem 9.1 with Lemma 4.6. □

Now (mainly) we turn our attention to the case (II), i. e., that $1 \leq \operatorname{rank} s_0^{(n)} \leq q-1$, where $(s_j^{(n)})_{j=0}^{0}$ is the nth Schur transform of the given sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$. The first result concerning this situation additionally includes the case $r = q$. Once more, we start with a parametrization using Nevanlinna pairs as parameters.

Theorem 9.3 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Suppose that $r := \operatorname{rank} s_0^{(n)}$ fulfils $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(s_0^{(n)})$ and let $U := (u_1, u_2, \ldots, u_r)$. Then, for each $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$, the function $\det[\mathfrak{v}_{21;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{22;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})]$ does not vanish identically and the mapping $\Sigma\colon \langle \mathcal{PR}_r(\Pi_+)\rangle \to \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ defined by*

$$\Sigma\left(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle\right) := \left[\mathfrak{v}_{11;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{12;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})\right] \times \left[\mathfrak{v}_{21;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{22;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})\right]^{-1} \tag{9.3}$$

is well defined and bijective.

Proof Because of the assumption $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$, from Proposition 6.3 we get $(s_j^{(n)})_{j=0}^{0} \in \mathcal{H}_{q,0}^{\geq,\mathrm{e}}$. By virtue of Remarks 2.3 and 5.1, then $(s_0^{(n)})^* = s_0^{(n)}$ follows. Consequently, Remark A.12 shows that

$$\left[\mathcal{R}(s_0^{(n)})\right]^{\perp} = \mathcal{N}(s_0^{(n)}). \tag{9.4}$$

Thus, Proposition 4.12(b) shows that $\Gamma_U\colon \langle \mathcal{PR}_r(\Pi_+)\rangle \to \langle \mathcal{P}[s_0^{(n)}]\rangle$ given by

$$\Gamma_U\left(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle\right) := \langle[U\widetilde{\phi}U^*; U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})}]\rangle \tag{9.5}$$

is a well-defined bijective mapping. From Lemma 4.10, (9.4), and Theorem 8.11(a) we see that, for each $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$, the function $\det[\mathfrak{v}_{21;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{22;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})]$ does not vanish identically. According to Corollary 8.12, the mapping $T\colon \langle \mathcal{P}[s_0^{(n)}]\rangle \to \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ defined by (8.67) also is well defined and bijective. Consequently, the composition $T \circ \Gamma_U$ of the mappings Γ_U and T is well defined and bijective as well. Taking into account (9.5), (8.67), and (9.3), easily we see that $\Sigma = T \circ \Gamma_U$ holds true. □

Now we are going to emphasize several aspects of Theorem 9.3. Guided by Theorem 9.1, first we turn our attention to the analogue of Theorem 9.1(a) in the case (II), i. e., the case $1 \leq \operatorname{rank} s_0^{(n)} \leq q-1$.

Corollary 9.4 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Suppose that $r := \operatorname{rank} s_0^{(n)}$ fulfils $1 \leq r \leq q-1$. Let $d := q - r$, let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_r$ is a basis of $\mathcal{R}(s_0^{(n)})$, and let $W := (u_1, u_2, \ldots, u_q)$. Then, for each $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$, the function $\det[\mathfrak{v}_{21;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{22;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)]$ does not vanish identically and $\Sigma \colon \langle \mathcal{PR}_r(\Pi_+)\rangle \to \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ given by (9.3) admits, for each $\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle \in \langle \mathcal{PR}_r(\Pi_+)\rangle$, the representation $\Sigma(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle) = F$, where*

$$F := \left[\mathfrak{v}_{11;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{12;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)\right] \times \left[\mathfrak{v}_{21;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{22;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)\right]^{-1}.$$

In particular, for each $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$, the matrix-valued function F belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.

Proof Similar to the proof of Theorem 9.3 we see that (9.4) holds true. Let $U := (u_1, u_2, \ldots, u_r)$. We consider an arbitrary pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$. From Remark A.15 and (9.4) we can conclude that $U\widetilde{\phi}U^* = W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) \cdot W^*$ and

$$U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})} = U\widetilde{\psi}U^* + \mathbb{P}_{[\mathcal{R}(s_0^{(n)})]^\perp} = W \cdot \operatorname{diag}(\widetilde{\psi}, I_d) \cdot W^*$$

hold true. Consequently, for each $j \in \{1, 2\}$, we have

$$\begin{aligned} &\mathfrak{v}_{j1;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{j2;n}^{(s)} (U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})}) \\ &= \mathfrak{v}_{j1;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) \cdot W^* + \mathfrak{v}_{j2;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d) \cdot W^* \\ &= \left[\mathfrak{v}_{j1;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{j2;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)\right] W^*. \end{aligned} \tag{9.6}$$

Since $W^*W = I_q$ holds true, from Theorem 9.3, (9.6), and (9.3) we see that the function $\det[\mathfrak{v}_{21;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{22;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)]$ does not vanish identically and that

$$\begin{aligned}\Sigma\left(\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle\right) &= \left[\mathfrak{v}_{11;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{12;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})\right] \\ &\quad \times \left[\mathfrak{v}_{21;n}^{(s)} U\widetilde{\phi}U^* + \mathfrak{v}_{22;n}^{(s)}(U\widetilde{\psi}U^* + \mathbb{P}_{\mathcal{N}(s_0^{(n)})})\right]^{-1} \\ &= \left[\mathfrak{v}_{11;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{12;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)\right] \\ &\quad \times \left[\mathfrak{v}_{21;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\phi}, 0_{d\times d}) + \mathfrak{v}_{22;n}^{(s)} W \cdot \operatorname{diag}(\widetilde{\psi}, I_d)\right]^{-1}\end{aligned}$$

holds true. The proof is complete. □

In the case (II), the following theorem emphasizes a certain further aspect of Theorem 9.3, namely the analogue of Theorem 9.1(b). We point out that the representation (9.8) below will be proved for each $z \in \Pi_+$ (in contrast to an existing exceptional discrete subset of Π_+).

Theorem 9.5 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3) and let (7.5) be the $q \times q$ block representation of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$. Suppose that $r := \operatorname{rank} s_0^{(n)}$ fulfils $1 \leq r \leq q-1$. Let $d := q - r$, let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_r$ is a basis of $\mathcal{R}(s_0^{(n)})$, and let $W := (u_1, u_2, \ldots, u_q)$. For each $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, then there exists a pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$ such that $\widetilde{\phi}$ and $\widetilde{\psi}$ are holomorphic in Π_+, that*

$$\det\left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z) W \phi_\square(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z) W \psi_\square(z)\right] \neq 0 \tag{9.7}$$

holds true for all $z \in \Pi_+$, and that F can be represented, for all $z \in \Pi_+$, via

$$\begin{aligned}F(z) &= \left[\mathfrak{v}_{11}^{((s_j)_{j=0}^{2n})}(z) W \phi_\square(z) + \mathfrak{v}_{12}^{((s_j)_{j=0}^{2n})}(z) W \psi_\square(z)\right] \\ &\quad \times \left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z) W \phi_\square(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z) W \psi_\square(z)\right]^{-1},\end{aligned} \tag{9.8}$$

where $\phi_\square := \operatorname{diag}(\widetilde{\phi}, 0_{d\times d})$ and $\psi_\square := \operatorname{diag}(\widetilde{\psi}, I_d)$.

Proof We consider an arbitrary $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$. By virtue of Proposition 8.10, then there is a pair $[\phi; \psi] \in \mathcal{P}[s_0^{(n)}]$ such that all the conditions (i)–(v) in Proposition 8.10 are fulfilled. In particular, with $\mathcal{D} = \emptyset$ conditions (i)–(iii) in Definition 4.2 are fulfilled, $\mathbb{H}_\phi = \mathbb{H}_\psi = \Pi_+$, the inequality (8.48) is valid for all $z \in \Pi_+$, and F admits the representation (8.49) for all $z \in \Pi_+$. Using Lemma 4.11 (with $\mathcal{D} = \emptyset$), we see that the matrix-valued function $B := \psi - \mathrm{i}\phi$ is holomorphic in Π_+ and fulfils

$$\det B(z) \neq 0 \tag{9.9}$$

for all $z \in \Pi_+$. Let $U := (u_1, u_2, \ldots, u_r)$. Moreover, Lemma 4.11 then shows that there exists a pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{PR}_r(\Pi_+)$ such that $\widetilde{\phi}$ and $\widetilde{\psi}$ are holomorphic in Π_+ and that the matrix-valued functions $S := U\widetilde{\phi}U^*$ and $T := U\widetilde{\psi}U^* + \mathbb{P}_{[\mathcal{R}(s_0^{(n)})]^\perp}$ are holomorphic in Π_+ and admit, for all $z \in \Pi_+$, the representations

$$S(z) = U\widetilde{\phi}(z)U^* \qquad \text{and} \qquad T(z) = U\widetilde{\psi}(z)U^* + \mathbb{P}_{[\mathcal{R}(s_0^{(n)})]^\perp} \tag{9.10}$$

as well as

$$S(z) = \phi(z)\,[B(z)]^{-1} \qquad \text{and} \qquad T(z) = \psi(z)\,[B(z)]^{-1}\,. \tag{9.11}$$

We now consider an arbitrary $z \in \Pi_+$. By virtue of Remark A.15, from (9.10) we get $S(z) = W\phi_\square(z)W^*$ and $T(z) = W\psi_\square(z)W^*$. Taking into account (9.11), then $\phi(z)[B(z)]^{-1} = W\phi_\square(z)W^*$ and $\psi(z)[B(z)]^{-1} = W\psi_\square(z)W^*$ follow. Consequently, $\phi(z) = W\phi_\square(z)W^*B(z)$ and $\psi(z) = W\psi_\square(z)W^*B(z)$. Hence, we conclude

$$\begin{aligned} &\mathfrak{v}_{j1}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{j2}^{((s_j)_{j=0}^{2n})}(z)\psi(z) \\ &= \mathfrak{v}_{j1}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z)W^*B(z) + \mathfrak{v}_{j2}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)W^*B(z) \\ &= \left[\mathfrak{v}_{j1}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{j2}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)\right]W^*B(z) \end{aligned} \tag{9.12}$$

for each $j \in \{1, 2\}$. Taking into account (8.48), then from (9.12), $W^*W = I_q$, and (9.9) we get the inequality (9.7) and, moreover,

$$\begin{aligned} &\left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right]^{-1} \\ &\quad = [B(z)]^{-1}\,W\left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)\right]^{-1}. \end{aligned} \tag{9.13}$$

Combining (8.48), (8.49), (9.12), (9.7), (9.13), and $W^*W = I_q$, we conclude that

$$F(z) = \left[\mathfrak{v}_{11}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{12}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right]$$
$$\times \left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)\phi(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)\psi(z)\right]^{-1}$$
$$= \left[\mathfrak{v}_{11}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{12}^{((s_j)_{j=0}^{2n})}W\psi_\square(z)\right]W^*B(z)$$
$$\times [B(z)]^{-1}\, W\left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)\right]^{-1}$$
$$= \left[\mathfrak{v}_{11}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{12}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)\right]$$
$$\times \left[\mathfrak{v}_{21}^{((s_j)_{j=0}^{2n})}(z)W\phi_\square(z) + \mathfrak{v}_{22}^{((s_j)_{j=0}^{2n})}(z)W\psi_\square(z)\right]^{-1}.$$

□

Observe that the analogue of Theorem 9.1(c) in case (II) already has been discussed in Theorem 9.3, already. Now, once more we choose matricial Schur functions in Π_+ as parameters.

Corollary 9.6 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ with nth Schur transform $(s_j^{(n)})_{j=0}^{0}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3). Suppose that $r := \operatorname{rank} s_0^{(n)}$ fulfils $1 \leq r \leq q-1$. Let $d := q - r$, let $u_1, u_2, \dots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \dots, u_r$ is a basis of $\mathcal{R}(s_0^{(n)})$, and let $W := (u_1, u_2, \dots, u_q)$. Furthermore, let $\mathfrak{U}^{((s_j)_{j=0}^{2n})} := \mathfrak{V}^{((s_j)_{j=0}^{2n})}C_q$, where C_q is given by (9.1), let $\mathfrak{U}_n^{(s)}$ be the restriction of $\mathfrak{U}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (9.2) be the $q \times q$ block representation of $\mathfrak{U}^{((s_j)_{j=0}^{2n})}$ and $\mathfrak{U}_n^{(s)}$, resp. Then:*

(a) *For each $\widetilde{S} \in \mathcal{S}_{r\times r}(\Pi_+)$, the function $\det[\mathfrak{u}_{21;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{22;n}^{(s)}W]$ does not vanish identically and the matrix-valued function*

$$F := \left[\mathfrak{u}_{11;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{12;n}^{(s)}W\right]\left[\mathfrak{u}_{21;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{22;n}^{(s)}W\right]^{-1}$$

belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$.

(b) *For each $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$, there exists a unique $\widetilde{S} \in \mathcal{S}_{r\times r}(\Pi_+)$ such that the function $\det[\mathfrak{u}_{21;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{22;n}^{(s)}W]$ does not vanish identically and that F can be represented via*

$$F = \left[\mathfrak{u}_{11;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{12;n}^{(s)}W\right]\left[\mathfrak{u}_{21;n}^{(s)}W \cdot \operatorname{diag}(\widetilde{S}, I_d) + \mathfrak{u}_{22;n}^{(s)}W\right]^{-1}.$$

Moreover, then

$$\det\left[\mathfrak{u}_{21}^{((s_j)_{j=0}^{2n})}(z)W\cdot\operatorname{diag}\left(\widetilde{S}(z),I_d\right)+\mathfrak{u}_{22}^{((s_j)_{j=0}^{2n})}(z)W\right]\neq 0$$

and

$$F(z)=\left[\mathfrak{u}_{11}^{((s_j)_{j=0}^{2n})}(z)W\cdot\operatorname{diag}\left(\widetilde{S}(z),I_d\right)+\mathfrak{u}_{12}^{((s_j)_{j=0}^{2n})}(z)W\right]\\ \times\left[\mathfrak{u}_{21}^{((s_j)_{j=0}^{2n})}(z)W\cdot\operatorname{diag}\left(\widetilde{S}(z),I_d\right)+\mathfrak{u}_{22}^{((s_j)_{j=0}^{2n})}(z)W\right]^{-1}$$

hold true for all $z\in\Pi_+$.

Proof The matrices C_q and $\mathbb{W}:=\operatorname{diag}(W,W)$ are unitary and fulfil $\mathbb{W}C_q=C_q\mathbb{W}$. In particular, $\mathfrak{U}^{((s_j)_{j=0}^{2n})}\mathbb{W}=\mathfrak{V}^{((s_j)_{j=0}^{2n})}\mathbb{W}C_q$. By virtue of Lemma 4.6(b), for all $\widetilde{S}\in\mathcal{S}_{r\times r}(\Pi_+)$, we have

$$\sqrt{2}C_q\begin{pmatrix}\widetilde{S}&0_{r\times d}\\0_{d\times r}&I_d\\\hline I_r&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix}=\begin{pmatrix}\mathrm{i}(I_r-\widetilde{S})&0_{r\times d}\\0_{d\times r}&0_{d\times d}\\\hline I_r+\widetilde{S}&0_{r\times d}\\0_{d\times r}&2I_d\end{pmatrix}\\ =\begin{pmatrix}\widetilde{\phi}&0_{r\times d}\\0_{d\times r}&0_{d\times d}\\\hline\widetilde{\psi}&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix}\begin{pmatrix}I_r&0_{r\times d}\\0_{d\times r}&2I_d\end{pmatrix},$$

where the pair $[\widetilde{\phi};\widetilde{\psi}]$ given by $\widetilde{\phi}:=\mathrm{i}(I_q-\widetilde{S})$ and $\widetilde{\psi}:=I_q+\widetilde{S}$ belongs to $\mathcal{PR}_r(\Pi_+)$. Regarding Lemma 4.6(a), for all $[\widetilde{\phi};\widetilde{\psi}]\in\mathcal{PR}_r(\Pi_+)$, we have

$$\sqrt{2}C_q^*\begin{pmatrix}\widetilde{\phi}&0_{r\times d}\\0_{d\times r}&0_{d\times d}\\\hline\widetilde{\psi}&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix}=\begin{pmatrix}\widetilde{\psi}+\mathrm{i}\widetilde{\phi}&0_{r\times d}\\0_{d\times r}&I_d\\\hline\widetilde{\psi}-\mathrm{i}\widetilde{\phi}&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix}\\ =\begin{pmatrix}\widetilde{S}&0_{r\times d}\\0_{d\times r}&I_d\\\hline I_r&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix}\begin{pmatrix}\widetilde{\psi}-\mathrm{i}\widetilde{\phi}&0_{r\times d}\\0_{d\times r}&I_d\end{pmatrix},$$

where $\widetilde{S} := (\widetilde{\psi} + i\widetilde{\phi})(\widetilde{\psi} - i\widetilde{\phi})^{-1}$ belongs to $\mathcal{S}_{r\times r}(\Pi_+)$. Using these relations and Lemma 4.6, the assertion follows from Corollary 9.4 and Theorem 9.5. We omit the details. □

Now we consider the case (III), i.e., that the given sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$. We will see that in this case, Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ has a unique solution which can be described explicitly. Once more, observe that necessary and sufficient conditions for the case that Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ has a unique solution are given in [10, Theorems 8.4 and 8.5]. According to Remark 3.4, this corresponds to the case that Problem $\mathsf{R}[\Pi_+; (s_j)_{j=0}^{2n}, \leq]$ obtains a unique solution. Furthermore, we note that, in view of Definition 5.3, Theorem 2.4, and [10, Def. 7.4 and Thm. 8.4], if $n \in \mathbb{N}$ and $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ are given, then the case (III) coincides with the situation that Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ has a unique solution.

Theorem 9.7 *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$. Further, let $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ be given by (7.3), let $\mathfrak{V}_n^{(s)}$ be the restriction of $\mathfrak{V}^{((s_j)_{j=0}^{2n})}$ onto Π_+, and let (8.35) be the $q \times q$ block representation of $\mathfrak{V}_n^{(s)}$. Then the function $\det \mathfrak{v}_{22;n}^{(s)}$ does not vanish identically and $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \leq] = \{\mathfrak{v}_{12;n}^{(s)}(\mathfrak{v}_{22;n}^{(s)})^{-1}\}$.*

Proof According to Example 4.8, the pair $[\phi_0; \psi_0]$ given by $\phi_0(z) := 0_{q\times q}$ and $\psi_0(z) := I_q$ for all $z \in \Pi_+$ belongs to $\mathcal{P}[s_0^{(n)}]$. In view of the assumption $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{cd}}$, Remark 5.4 yields $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$, whereas Proposition 6.11(b) provides $\operatorname{rank} s_0^{(n)} = 0$. Thus, Proposition 4.12(a) yields $\langle \mathcal{P}[s_0^{(n)}]\rangle = \{\langle[\phi_0; \psi_0]\rangle\}$. Applying Theorem 8.11 then completes the proof. □

Appendix A Some Particular Facts on Matrix Theory

Remark A.1 Let $A \in \mathbb{C}_{\geq}^{q\times q}$ and let $B \in \mathbb{C}_{\mathrm{H}}^{q\times q}$ be such that $B - A \in \mathbb{C}_{\geq}^{q\times q}$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.

For each $A \in \mathbb{C}^{p\times q}$, there exists a unique matrix X such that the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$ hold true. This particular matrix X is said to be the Moore–Penrose inverse of A and one writes A^+ for this matrix X. In particular, if A is a non-singular complex $q \times q$ matrix, then $A^+ = A^{-1}$. It seems to be useful stating some basic results on Moore–Penrose inverses of complex matrices.

Remark A.2 If $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, then $AA^+ = A^+A$.

Lemma A.3 (see e.g. [8, Lem. 1.1.9]) *Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ and let*

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.1}$$

be the block representation of E *with* $p \times p$ *block* a. *Then the matrix* E *is non-negative Hermitian if and only if the four conditions* $a \in \mathbb{C}_{\geq}^{p\times p}$, $\mathcal{R}(b) \subseteq \mathcal{R}(a)$, $c = b^*$, *and* $d - ca^+b \in \mathbb{C}_{\geq}^{q\times q}$ *are fulfilled.*

A complex $p \times q$ matrix K is called contractive in case $\|K\|_S \leq 1$.

Remark A.4 Let $K \in \mathbb{C}^{p\times q}$. Using Lemma A.3 (see also, e. g., [8, Thm. 1.1.2]), easily one can see that the matrix K is contractive if and only if the matrix $I_q - K^*K$ is non-negative Hermitian.

Remark A.5 Let $\widetilde{J}_q$ be given by (4.1). Let $P, Q \in \mathbb{C}^{q\times q}$ be such that

$$\begin{pmatrix} P \\ Q \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathbb{C}_{\geq}^{q\times q}. \tag{A.2}$$

In view of

$$(Q - \mathrm{i}P)^*(Q - \mathrm{i}P) = \begin{pmatrix} P \\ Q \end{pmatrix}^* \begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} P \\ Q \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} P \\ Q \end{pmatrix} \geq \begin{pmatrix} P \\ Q \end{pmatrix}^* \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathbb{C}_{\geq}^{q\times q},$$

then $\mathcal{N}(\binom{P}{Q}) = \mathcal{N}(Q - \mathrm{i}P)$. In particular, $\operatorname{rank}\binom{P}{Q} = q$ if and only if $\det(Q - \mathrm{i}P) \neq 0$.

Remark A.6 Let $P, Q \in \mathbb{C}^{q\times q}$ be such that (A.2) and $\operatorname{rank}\binom{P}{Q} = q$ hold true. In view of Remark A.5, then $\det(Q - \mathrm{i}P) \neq 0$. Because of $I_q - C^*C = 2(Q - \mathrm{i}P)^{-*}\binom{P}{Q}^*(-\widetilde{J}_q)\binom{P}{Q}(Q - \mathrm{i}P)^{-1}$ and Remark A.4, moreover, the matrix $C := (Q + \mathrm{i}P)(Q - \mathrm{i}P)^{-1}$ is contractive.

Remark A.7 Let C be a contractive complex $q \times q$ matrix. Let $P := \mathrm{i}(I_q - C)$ and $Q := I_q + C$. Because of

$$\begin{aligned} P^*P + Q^*Q = (I_q - C)^*(I_q - C) + (I_q + C)^*(I_q + C) = 2(I_q + C^*C) \\ \geq 2I_q \in \mathbb{C}_{>}^{q\times q}, \end{aligned}$$

then $\operatorname{rank}\binom{P}{Q} = \operatorname{rank}(P^*P + Q^*Q) = q$. Regarding Remark 4.1 and that C is contractive, furthermore

$$\begin{aligned} \begin{pmatrix} P \\ Q \end{pmatrix}^* (-\widetilde{J}_q) \begin{pmatrix} P \\ Q \end{pmatrix} &= \mathrm{i}(P^*Q - Q^*P) \\ &= \mathrm{i}\left[-\mathrm{i}(I_q - C)^*(I_q + C) - \mathrm{i}(I_q + C)^*(I_q - C)\right] \\ &= 2(I_q - C^*C) \in \mathbb{C}_{\geq}^{q\times q}. \end{aligned}$$

Clearly, $(Q + \mathrm{i}P)(Q - \mathrm{i}P)^{-1} = (2C)(2I_q)^{-1} = C$.

We will write $\langle ., . \rangle_{\mathrm{E}}$ for the (left) Euclidean inner product in $\mathbb{C}^q$, i. e., for all $x, y \in \mathbb{C}^q$, let $\langle x, y \rangle_{\mathrm{E}} := y^* x$. If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^q$, then the set $\mathcal{M}^\perp$ of all $x \in \mathbb{C}^q$ which fulfil $\langle x, y \rangle_{\mathrm{E}} = 0$ for all $y \in \mathcal{M}$ is a subspace of $\mathbb{C}^q$ and is called (left) orthogonal complement of $\mathcal{M}$. If $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{C}^q$ such that $\langle u, w \rangle_{\mathrm{E}} = 0$ for every choice of u in $\mathcal{U}$ and w in $\mathcal{W}$, then $\mathcal{U} \cap \mathcal{W} = \{0_{q \times 1}\}$ and $\mathcal{U} \oplus \mathcal{W}$ is said to be the orthogonal sum of $\mathcal{U}$ and $\mathcal{W}$. If $\mathcal{U}$ is a subspace of $\mathbb{C}^q$, then there exists exactly one matrix $\mathbb{P}_{\mathcal{U}} \in \mathbb{C}^{q \times q}$ such that both $\mathbb{P}_{\mathcal{U}} x \in \mathcal{U}$ and $x - \mathbb{P}_{\mathcal{U}} x \in \mathcal{U}^\perp$ are fulfilled for each $x \in \mathbb{C}^q$. This matrix $\mathbb{P}_{\mathcal{U}}$ is called the orthoprojection matrix onto $\mathcal{U}$. In particular, $\mathbb{P}_{\mathcal{U}} u = u$ for all $u \in \mathcal{U}$. A complex $q \times q$ matrix P is said to be an orthogonal projection matrix, if there exists a subspace $\mathcal{U}$ of $\mathbb{C}^q$ such that $P = \mathbb{P}_{\mathcal{U}}$.

Proposition A.8 *Let $P \in \mathbb{C}^{q \times q}$. Then P is an orthogonal projection matrix if and only if $P^2 = P$ and $P^* = P$ hold true.*

For a detailed proof of Proposition A.8, see, e. g., [34, Satz 2.54].

Remark A.9 If $\mathcal{U}$ is a subspace of $\mathbb{C}^q$, then $\mathbb{P}_{\mathcal{U}} + \mathbb{P}_{\mathcal{U}^\perp} = I_q$.

Lemma A.10 *Let $m \in \mathbb{N}$ and let $(\mathcal{V}_j)_{j=0}^m$ be a sequence of linear subspaces of $\mathbb{C}^q$ such that*

$$\{0_{q \times 1}\} = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_{m-1} \subseteq \mathcal{V}_m = \mathbb{C}^q \tag{A.3}$$

holds true. For all $l \in \mathbb{Z}_{0,m}$, let $P_l := \mathbb{P}_{\mathcal{V}_l}$ and let $\eta_1, \eta_2, \ldots, \eta_m \in \mathbb{C} \setminus \{0\}$. Then

$$\left[\sum_{l=1}^m \eta_l (P_l - P_{l-1}) \right] \left[\sum_{l=1}^m \frac{1}{\eta_l} (P_l - P_{l-1}) \right] = I_q.$$

Proof From (A.3) and a well-known result on orthoprojection matrices (see, e. g., [34, Satz 4.31]) we get that $P_j P_k = P_{\min\{j,k\}}$ holds true for all $j, k \in \mathbb{Z}_{0,m}$. Moreover, (A.3) and a further well-known result on orthoprojection matrices (see, e. g., [34, Satz 4.30(c)]) deliver the equations $P_j - P_{j-1} = \mathbb{P}_{\mathcal{V}_j \ominus \mathcal{V}_{j-1}}$ for all $j \in \mathbb{Z}_{1,m}$ and the representation $\mathbb{C}^q = \bigoplus_{j=1}^m (\mathcal{V}_j \ominus \mathcal{V}_{j-1})$ as orthogonal sum. Therefore, using the Kronecker delta δ_{jk}, it is readily checked that $(P_j - P_{j-1})(P_k - P_{k-1}) = \delta_{jk}(P_j - P_{j-1})$ for all $j, k \in \mathbb{Z}_{1,m}$ and, consequently,

$$\begin{aligned} &\left[\sum_{l=1}^m \eta_l (P_l - P_{l-1}) \right] \left[\sum_{l=1}^m \frac{1}{\eta_l} (P_l - P_{l-1}) \right] \\ &= \sum_{j=1}^m \sum_{k=1}^m \eta_j \frac{1}{\eta_k} (P_j - P_{j-1})(P_k - P_{k-1}) = \sum_{j=1}^m \sum_{k=1}^m \eta_j \frac{1}{\eta_k} \delta_{jk} (P_j - P_{j-1}) \\ &= \sum_{l=1}^m \eta_l \frac{1}{\eta_l} (P_l - P_{l-1}) = \sum_{l=1}^m (P_l - P_{l-1}) = P_m - P_0 = I_q - 0_{q \times q} = I_q. \end{aligned}$$

□

Remark A.11 Let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$ with dimension $d := \dim \mathcal{U} \geq 1$. Let $u_1, u_2, \ldots, u_d$ be an orthonormal basis of $\mathcal{U}$ and let $U := (u_1, u_2, \ldots, u_d)$. Then $\mathbb{P}_{\mathcal{U}} = UU^*$.

Remark A.12 For each $A \in \mathbb{C}^{p\times q}$, the equations $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$ as well as $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$ hold true.

Remark A.13 If $A \in \mathbb{C}^{p\times q}$, then $AA^+ = \mathbb{P}_{\mathcal{R}(A)}$ and $A^+A = \mathbb{P}_{\mathcal{R}(A^*)}$.

Lemma A.14 *Let $A \in \mathbb{C}^{p\times q}$. Then:*

(a) *Let $B \in \mathbb{C}^{p\times r}$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ if and only if $BB^+A = A$.*
(b) *Let $B \in \mathbb{C}^{r\times q}$. Then $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ if and only if $AB^+B = A$.*

Remark A.15 Let $M \in \mathbb{C}^{q\times p}$ be such that $r := \operatorname{rank} M$ fulfils $1 \leq r \leq q-1$. Let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_r$ is a basis of $\mathcal{R}(M)$, let $U := (u_1, u_2, \ldots, u_r)$, and let $W := (u_1, u_2, \ldots, u_q)$. For every choice of $A \in \mathbb{C}^{r\times r}$, then $UAU^* = W \cdot \operatorname{diag}(A, 0_{(q-r)\times(q-r)}) \cdot W^*$ and $UAU^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp} = W \cdot \operatorname{diag}(A, I_{q-r}) \cdot W^*$.

Lemma A.16 *Let $r \in \mathbb{Z}_{1,q}$, let $U \in \mathbb{C}^{q\times r}$ be such that $U^*U = I_r$, and let $\widetilde{A}, \widetilde{B} \in \mathbb{C}^{r\times r}$. Then the matrices $A := U\widetilde{A}U^*$ and $B := U\widetilde{B}U^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp}$ fulfil $\mathcal{R}(A) \subseteq \mathcal{R}(U)$ and $\det(A^*A + B^*B) = \det(\widetilde{A}^*\widetilde{A} + \widetilde{B}^*\widetilde{B})$ as well as $B^*A = U\widetilde{B}^*\widetilde{A}U^*$. In particular, $\operatorname{rank}\binom{A}{B} = q$ if and only if $\operatorname{rank}\binom{\widetilde{A}}{\widetilde{B}} = r$.*

Proof Clearly, $\mathcal{R}(A) = \mathcal{R}(U\widetilde{A}U^*) \subseteq \mathcal{R}(U)$. Moreover, we have $A^*A = U\widetilde{A}^*U^*U\widetilde{A}U^* = U\widetilde{A}^*\widetilde{A}U^*$. Obviously, $\mathbb{P}_{[\mathcal{R}(U)]^\perp}U = 0_{q\times r}$. Taking additionally into account Proposition A.8, we consequently obtain

$$\begin{aligned} B^*B &= U\widetilde{B}^*U^*U\widetilde{B}U^* + U\widetilde{B}^*U^*\mathbb{P}_{[\mathcal{R}(U)]^\perp} \\ &\qquad + \mathbb{P}^*_{[\mathcal{R}(U)]^\perp}U\widetilde{B}U^* + \mathbb{P}^*_{[\mathcal{R}(U)]^\perp}\mathbb{P}_{[\mathcal{R}(U)]^\perp} \\ &= U\widetilde{B}^*\widetilde{B}U^* + U\widetilde{B}^*(\mathbb{P}_{[\mathcal{R}(U)]^\perp}U)^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp}U\widetilde{B}U^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp} \\ &= U\widetilde{B}^*\widetilde{B}U^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp} \end{aligned} \tag{A.4}$$

as well as

$$\begin{aligned} B^*A &= U\widetilde{B}^*U^*U\widetilde{A}U^* + \mathbb{P}^*_{[\mathcal{R}(U)]^\perp}U\widetilde{A}U^* \\ &= U\widetilde{B}^*\widetilde{A}U^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp}U\widetilde{A}U^* = U\widetilde{B}^*\widetilde{A}U^*. \end{aligned}$$

It remains to show that $\det(A^*A + B^*B) = \det(\widetilde{A}^*\widetilde{A} + \widetilde{B}^*\widetilde{B})$ is fulfilled. In view of (A.4), we have

$$A^*A + B^*B = U(\widetilde{A}^*\widetilde{A} + \widetilde{B}^*\widetilde{B})U^* + \mathbb{P}_{[\mathcal{R}(U)]^\perp}. \tag{A.5}$$

If $r = q$, then U is unitary and, therefore, $\mathbb{P}_{[\mathcal{R}(U)]^\perp} = 0_{q\times q}$ and the assertion follows from (A.5). Let $r < q$ and $d := q - r$. Then there is $V \in \mathbb{C}^{q\times d}$ such that $W := (U, V)$ is a unitary $q \times q$ matrix. In particular,

$$\begin{pmatrix} U^*U & U^*V \\ V^*U & V^*V \end{pmatrix} = W^*W = \begin{pmatrix} I_r & 0_{r\times d} \\ 0_{d\times r} & I_d \end{pmatrix} \text{ and } UU^* + VV^* = WW^* = I_q \tag{A.6}$$

hold true. Using $U^*U = I_r$ and Remarks A.11 and A.9, then $UU^* = \mathbb{P}_{\mathcal{R}(U)}$ and $VV^* = I_q - UU^* = \mathbb{P}_{[\mathcal{R}(U)]^\perp}$ follow. Additionally using (A.5) and (A.6), we obtain

$$\begin{aligned} W^*(A^*A + B^*B)W &= \begin{pmatrix} U^* \\ V^* \end{pmatrix} \left[U(\widetilde{A}^*\widetilde{A} + \widetilde{B}^*\widetilde{B})U^* + VV^*\right](U, V) \\ &= \left(\begin{array}{c:c} \widetilde{A}^*\widetilde{A} + \widetilde{B}^*\widetilde{B} & 0_{r\times d} \\ \hdashline 0_{d\times r} & I_d \end{array}\right). \end{aligned}$$

□

Lemma A.17 *Let $M \in \mathbb{C}^{q\times p}$ be such that $r := \operatorname{rank} M$ fulfils $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(M)$ and let $U := (u_1, u_2, \ldots, u_r)$. Furthermore, let P and Q be complex $q \times q$ matrices such that $\operatorname{rank}\binom{P}{Q} = q$ as well as $\operatorname{Im}(Q^*P) \in \mathbb{C}_{\geq}^{q\times q}$ and $\mathcal{R}(P) \subseteq \mathcal{R}(M)$ hold true. Then the matrix $B := Q - \mathrm{i}P$ is non-singular and the matrices $\phi := U^*PB^{-1}U$ and $\psi := U^*QB^{-1}U$ fulfil $\operatorname{rank}\binom{\phi}{\psi} = r$ and $\psi^*\phi = (B^{-1}U)^*(Q^*P)(B^{-1}U)$. Furthermore, the matrices $S := U\phi U^*$ and $T := U\psi U^* + \mathbb{P}_{[\mathcal{R}(M)]^\perp}$ fulfil the following statements:*

(a) $\operatorname{rank}\binom{S}{T} = q$ *and* $\det(S^*S + T^*T) = \det(\phi^*\phi + \psi^*\psi)$.
(b) $T^*S = B^{-*}(Q^*P)B^{-1}$.
(c) $S = PB^{-1}$ *and* $T = QB^{-1}$ *as well as* $\mathcal{R}(\binom{P}{Q}) = \mathcal{R}(\binom{S}{T})$.

Proof The idea of the proof is taken from [2, Lem. 4.3]. We only consider the case $r < q$. Let $d := q - r$. Then there is $V \in \mathbb{C}^{q\times d}$ such that $W := (U, V)$ is a unitary $q \times q$ matrix. In particular, (A.6) holds true. Using Remark 4.1, we get $\binom{P}{Q}^*(-\widetilde{J}_q)\binom{P}{Q} \in \mathbb{C}_{\geq}^{q\times q}$. Set $A := Q + \mathrm{i}P$, then Remark A.6 shows that $\det B \neq 0$ and that $C := AB^{-1}$ is contractive. Since W is unitary, then the matrix $K := W^*CW$ is contractive as well. Moreover, we have

$$K = \begin{pmatrix} U^* \\ V^* \end{pmatrix} C(U, V) = \begin{pmatrix} U^*CU & U^*CV \\ V^*CU & V^*CV \end{pmatrix}.$$

Obviously, $B - A = -2\mathrm{i}P$ and $B + A = 2Q$ are true and, consequently,

$$\frac{\mathrm{i}}{2}(I_q - C) = PB^{-1} \qquad \text{and} \qquad \frac{1}{2}(I_q + C) = QB^{-1} \tag{A.7}$$

follow. According to Remark A.11, we have $\mathbb{P}_{\mathcal{R}(M)} = UU^*$. Thus, $\mathcal{R}(P) \subseteq \mathcal{R}(M)$ yields $UU^*P = P$. Therefore, (A.7) and (A.6) imply $V^*(I_q - C) = -2\mathrm{i}V^*PB^{-1} = -2\mathrm{i}V^*UU^*PB^{-1} = 0_{d\times q}$, i. e., $V^*C = V^*$. Considering (A.6), then the lower blocks of K read $V^*CU = V^*U = 0_{d\times r}$ and $V^*CV = V^*V = I_d$ which is, in particular, unitary. Consequently, K admits the block representation

$$K = \begin{pmatrix} U^*CU & 0_{r\times d} \\ 0_{d\times r} & I_d \end{pmatrix}. \tag{A.8}$$

Using (A.6) and (A.8), we get

$$I_q = UU^* + VV^* = UU^*UU^* + VV^*,$$

$$C = WKW^* = (U, V)K \begin{pmatrix} U^* \\ V^* \end{pmatrix} = UU^*CUU^* + VV^*,$$

and, therefore,

$$I_q - C = UU^*(I_q - C)UU^* \quad \text{and} \quad I_q + C = UU^*(I_q + C)UU^* + 2VV^*.$$

Due to (A.7), we infer

$$PB^{-1} = UU^*PB^{-1}UU^* = U\phi U^*$$

and

$$QB^{-1} = UU^*QB^{-1}UU^* + VV^* = U\psi U^* + VV^*.$$

In view of (A.6) and $\mathbb{P}_{\mathcal{R}(M)} = UU^*$ and using Remark A.9, we have $VV^* = I_q - UU^* = \mathbb{P}_{[\mathcal{R}(M)]^\perp}$. Consequently, $PB^{-1} = S$ and $QB^{-1} = T$ hold true which proves (c) and $\operatorname{rank}\begin{pmatrix} S \\ T \end{pmatrix} = q$. Assertion (b) immediately follows from (c). Using $U^*U = I_r$ and $\mathcal{R}(U) = \mathcal{R}(M)$, the application of Lemma A.16 yields $\det(S^*S + T^*T) = \det(\phi^*\phi + \psi^*\psi)$ and $\operatorname{rank}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = r$. Moreover, considering that $T^*S = U(\psi^*\phi)U^*$ holds true, finally we obtain

$$\begin{aligned} \psi^*\phi = U^*U(\psi^*\phi)U^*U &= U^*T^*SU \\ &= U^*(QB^{-1})^*(PB^{-1})U = (B^{-1}U)^*(Q^*P)(B^{-1}U) \end{aligned}$$

which completes the proof. □

Appendix B Some Facts on the Integration Theory of Non-negative Hermitian Measures

In this section, we present basic facts regarding the integration theory with respect to non-negative Hermitian measures. Throughout the section, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We write $\mathfrak{B}_{\mathbb{K}}$ denoting the σ-algebra of all Borel subsets of $\mathbb{K}$. Let Ω be a non-empty set and let $\mathfrak{A}$ be a σ-algebra on Ω. Consider a measure ν on the measurable space $(\Omega, \mathfrak{A})$. We use $\mathcal{L}^1(\Omega, \mathfrak{A}, \nu; \mathbb{K})$ to denote the set of all $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{K}}$-measurable functions $f\colon \Omega \to \mathbb{K}$ such that $\int_\Omega |f| \mathrm{d}\nu < \infty$. We will write $\mathfrak{B}_{p\times q}$ for the σ-algebra of all Borel subsets of $\mathbb{C}^{p\times q}$. An $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable function $F\colon \Omega \to \mathbb{C}^{p\times q}$ is said to be integrable with respect to ν if $F = (f_{jk})_{\substack{j=1,\ldots,p\\k=1,\ldots,q}}$ belongs to $[\mathcal{L}^1(\Omega, \mathfrak{A}, \nu; \mathbb{C})]^{p\times q}$, i. e. all entries f_{jk} belong to the class $\mathcal{L}^1(\Omega, \mathfrak{A}, \nu; \mathbb{C})$. In this case, let

$$\int_\Omega F \mathrm{d}\nu := \left(\int_\Omega f_{jk} \mathrm{d}\nu \right)_{\substack{j=1,\ldots,p\\k=1,\ldots,q}}.$$

A matrix-valued function μ the domain of which is $\mathfrak{A}$ and the values of which belong to the set $\mathbb{C}^{q\times q}_{\geq}$ of all non-negative Hermitian complex $q \times q$ matrices is called non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$ if it is σ-additive, i. e., if μ fulfils $\mu(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k)$ for each sequence $(A_k)_{k=1}^\infty$ of pairwise disjoint sets belonging to $\mathfrak{A}$. By $\mathcal{M}^q_{\geq}(\Omega, \mathfrak{A})$ we denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$, i. e., the set of all σ-additive mappings $\mu\colon \mathfrak{A} \to \mathbb{C}^{q\times q}_{\geq}$. Let $\mu = (\mu_{jk})_{j,k=1}^q \in \mathcal{M}^q_{\geq}(\Omega, \mathfrak{A})$. For each $j \in \mathbb{Z}_{1,q}$ and for each $k \in \mathbb{Z}_{1,q}$, the function μ_{jk} describes a complex measure on $(\Omega, \mathfrak{A})$ and the variation ν_{jk} of μ_{jk} is a finite measure on $(\Omega, \mathfrak{A})$. Especially, $\mu_{11}, \mu_{22}, \ldots, \mu_{qq}$ and the so-called trace measure $\tau := \sum_{j=1}^q \mu_{jj}$ of μ are finite measures on $(\Omega, \mathfrak{A})$. For each function f belonging to $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K}) := \bigcap_{j,k=1}^q \mathcal{L}^1(\Omega, \mathfrak{A}, \nu_{jk}; \mathbb{K})$ we use the notation

$$\int_\Omega f \mathrm{d}\mu := \left(\int_\Omega f \mathrm{d}\mu_{jk} \right)_{j,k=1}^q.$$

For this integral, we write $\int_\Omega f(\omega)\mu(\mathrm{d}\omega)$ as well.

Lemma B.1 *Let $(\Omega, \mathfrak{A})$ be a measurable space, let $\mu = (\mu_{jk})_{j,k=1}^q \in \mathcal{M}^q_{\geq}(\Omega, \mathfrak{A})$, and let $f\colon \Omega \to \mathbb{K}$ be an $\mathfrak{A}$-$\mathfrak{B}_{\mathbb{K}}$-measurable mapping. Using standard arguments of measure and integration theory, easily one can see that the following statements are equivalent:*

(i) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{K})$.
(ii) $f \in \bigcap_{j=1}^q \mathcal{L}^1(\Omega, \mathfrak{A}, \mu_{jj}; \mathbb{K})$.
(iii) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{K})$*, where τ is the trace measure of μ.*
(iv) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, u^*\mu u; \mathbb{K})$ *for each $u \in \mathbb{C}^q$.*

Now we turn our attention to an other integral based on investigations by I. S. Kats [26] and M. Rosenberg [31]. Let $(\Omega, \mathfrak{A})$ be a measurable space and let $\mu = (\mu_{jk})_{j,k=1}^{q} \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$. Then, for every choice of j and k in $\mathbb{Z}_{1,q}$, the complex measure μ_{jk} is absolutely continuous with respect to the trace measure τ of μ. If ν describes an arbitrary measure on $(\Omega, \mathfrak{A})$ such that, for all $j, k \in \mathbb{Z}_{1,q}$, the complex measure μ_{jk} is absolutely continuous with respect to ν, we say that μ is absolutely continuous with respect to ν and the matrix-valued function $\mu'_{\nu} = (\frac{\mathrm{d}\mu_{jk}}{\mathrm{d}\nu})_{j,k=1}^{q}$ built by the corresponding Radon–Nikodym derivatives of μ_{jk} with respect to ν is said to be a version of the Radon–Nikodym derivative of μ with respect to ν and is well defined up to sets of zero ν-measure. An ordered pair $[\Phi, \Psi]$ consisting of an $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable function $\Phi\colon \Omega \to \mathbb{C}^{p\times q}$ and an $\mathfrak{A}$-$\mathfrak{B}_{r\times q}$-measurable function $\Psi\colon \Omega \to \mathbb{C}^{r\times q}$ is said to be left-integrable with respect to μ if $\Phi\mu'_{\tau}\Psi^*$ belongs to $[\mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})]^{p\times r}$. In this case the integral

$$\int_{\Omega} \Phi \mathrm{d}\mu \Psi^* := \int_{\Omega} \Phi {\mu_{\tau}}' \Psi^* \mathrm{d}\tau$$

is (well) defined and we also write $\int_{\Omega} \Phi(\omega)\mu(\mathrm{d}\omega)[\Psi(\omega)]^*$ for this integral.

Appendix C Linear Fractional Transformations of Matrices

In this appendix we summarize some basic facts on linear fractional transformations of matrices. Our considerations modify results due to V. P. Potapov [30], stated in [20].

Remark C.1 Let $c \in \mathbb{C}^{q\times p}$ and $d \in \mathbb{C}^{q\times q}$. Then easily one can check that the following statements are equivalent (see, e. g., [20, Lem. D.2]):

(i) The set $\mathcal{Q}_{[c,d]} := \{x \in \mathbb{C}^{p\times q} : \det(cx + d) \neq 0\}$ is non-empty.
(ii) The set $\widetilde{\mathcal{Q}}_{[c,d]} := \{[x, y] \in \mathbb{C}^{p\times q} \times \mathbb{C}^{q\times q} : \det(cx + dy) \neq 0\}$ is non-empty.
(iii) $\operatorname{rank}(c, d) = q$.

Notation C.2 Let $E \in \mathbb{C}^{(p+q)\times(p+q)}$ and let (A.1) be the block representation of E with $p \times p$ block a. If $\operatorname{rank}(c, d) = q$, then the linear fractional transformations $\mathcal{S}_E^{(p,q)}\colon \mathcal{Q}_{[c,d]} \to \mathbb{C}^{p\times q}$ and $\widetilde{\mathcal{S}}_E^{(p,q)}\colon \widetilde{\mathcal{Q}}_{[c,d]} \to \mathbb{C}^{p\times q}$ are defined by

$$\mathcal{S}_E^{(p,q)}(x) := (ax + b)(cx + d)^{-1} \quad \text{and} \quad \widetilde{\mathcal{S}}_E^{(p,q)}([x, y]) := (ax + by)(cx + dy)^{-1}.$$

Proposition C.3 *Let* $E_1, E_2 \in \mathbb{C}^{(p+q)\times(p+q)}$ *and let*

$$E_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \textit{and} \qquad E_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

be the block representations of E_1 *and* E_2 *with* $p \times p$ *blocks* a_1 *and* a_2. *Let* $E := E_2E_1$ *and let* (A.1) *be the block representation of* E *with* $p \times p$ *block* *a. Suppose that* $\operatorname{rank}(c_1, d_1) = q$ *and* $\operatorname{rank}(c_2, d_2) = q$ *hold true. Let* $\widetilde{\mathcal{Q}} := \{[x, y] \in \widetilde{\mathcal{Q}}_{[c_1,d_1]} : \widetilde{\mathcal{S}}_{E_1}^{(p,q)}([x, y]) \in \mathcal{Q}_{[c_2,d_2]}\}$. *Then* $\widetilde{\mathcal{Q}}_{[c,d]} \cap \widetilde{\mathcal{Q}}_{[c_1,d_1]} = \widetilde{\mathcal{Q}}$. *Furthermore, if* $\widetilde{\mathcal{Q}}_{[c,d]} \cap \widetilde{\mathcal{Q}}_{[c_1,d_1]} \neq \emptyset$, *then* $\mathcal{S}_{E_2}^{(p,q)}(\widetilde{\mathcal{S}}_{E_1}^{(p,q)}([x, y])) = \widetilde{\mathcal{S}}_{E}^{(p,q)}([x, y])$ *for all* $[x, y] \in \widetilde{\mathcal{Q}}_{[c,d]} \cap \widetilde{\mathcal{Q}}_{[c_1,d_1]}$.

A detailed proof of Proposition C.3 is given, e. g., in [20, Prop. D.4]. Note that the conditions $\widetilde{\mathcal{Q}}_{[c_1,d_1]} \neq \emptyset$ and $\mathcal{Q}_{[c_2,d_2]} \neq \emptyset$ do not imply $\widetilde{\mathcal{Q}}_{[c,d]} \cap \widetilde{\mathcal{Q}}_{[c_1,d_1]} \neq \emptyset$ (see [20, Example D.6]).

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The Wiener Algebra and Singular Integrals

E. Liflyand

Abstract Salem type conditions for trigonometric series are extended to functions from the Wiener algebra. While in the earlier one-dimensional generalization the conditions are given in terms of the Hilbert transform, for the multivariate setting all reasonable singular integrals are equally involved.

Keywords Wiener algebra · Fourier transform · Hilbert transform · Singular integral · Riesz transform · Hardy space

Mathematics Subject Classification (2000) Primary 42B10; Secondary 42B20, 42A50, 42B30

1 Introduction

Wiener's algebra has proved to be very important in various areas of analysis; for a detailed survey, see [9]. For $n = 1, 2, \dots$, we say that f belongs to $W_0(\mathbb{R}^n)$, written $f \in W_0(\mathbb{R}^n)$, if

$$f(x) = \int_{\mathbb{R}^n} g(t)e^{i(x,t)}dt, \qquad g \in L^1(\mathbb{R}^n), \tag{1.1}$$

where $(x, t) = x_1t_1 + \dots + x_nt_n$, with $\|f\|_{W_0} = \|g\|_{L^1(\mathbb{R}^n)}$. The space $W_0(\mathbb{R}^n)$ is a Banach algebra with point-wise multiplication.

In dimension one, a new necessary condition for belonging to the Wiener algebra has recently been obtained in [6]. It was represented as a non-periodic analog of Salem's necessary conditions for a trigonometric series to be the Fourier series of

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an integrable function (see [10] and Chapter II, §9 of [1], where one can find a nice discussion on this issue). Along with the proven in [8] fact that the Hilbert transform of a function from Wiener's algebra exists at every point, such necessary conditions can be given as

Theorem 1.1 *If* $f \in W_0(\mathbb{R})$, *then its Hilbert transform* $\mathcal{H}f(x)$ *exists at every* $x \in \mathbb{R}$, *is uniformly bounded and* $\lim\limits_{|x|\to\infty} \mathcal{H}f(x) = 0$.

Here the Hilbert transform $\mathcal{H}f$ is defined by

$$\mathcal{H}f(x) := \frac{1}{\pi}\ \text{(P.V.)} \int_{\mathbb{R}} f(x-u)\frac{du}{u} = \frac{1}{\pi}\ \text{(P.V.)} \int_{\mathbb{R}} \frac{f(u)}{x-u}\,du$$
$$= \frac{1}{\pi} \lim_{\delta\downarrow 0} \int_{\delta}^{\infty} \{f(x-u) - f(x+u)\}\frac{du}{u}, \qquad x \in \mathbb{R}.$$

As is well known, for $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, this limit exists for almost all x in $\mathbb{R}$. A function in Wiener's algebra need not be in any of these L^p, just continuous and vanishing at infinity, however its Hilbert transform necessarily possesses such nice properties.

In this paper, we are going to generalize Theorem 1.1 to the multivariate case. The main feature of this work is that the corresponding properties hold true not for one specific singular integral as in dimension one but for a variety of such integrals. The detailed study of singular integrals in [11, Ch.II] demonstrates that they possess the key property needed for our study. More precisely, the uniform boundedness of the truncated Fourier transform of the kernel is one of the basic features of every singular integral.

The outline of the paper is as follows. In the next section we study one specific family of singular integrals that unites the one-dimensional case and the multidimensional one. In Sect. 3, we present our approach in full generality. In the section before the last, we discuss what properties guarantee for a singular integral to be involved in this study. Finally, in the last section we present an application to radial functions.

2 Multidimensional Hilbert Transform

Among various generalizations of the Hilbert transform to the multivariate case (for example, the whole second volume of [5] is devoted to this), the closest to the one-dimensional prototype proved to be superpositions of the Hilbert transform applied to single variables. This is by no means artificial, since corresponds to one of the so called product Hardy spaces (many details are given in [7, Ch.5]).

This case will further take its place in the general scheme. However, the use of the Hilbert transform makes it, in formulation and in proof, very similar to the one-dimensional case. To show this, it is worth giving first a short proof of Theorem 1.1, moreover, in [6] it was longer and for the more restrictive in this case so-called modified Hilbert transform rather than for the usual one.

Thus, expressing f in the Hilbert transform formula by means of (1.1), we wish to justify the change of the order of integration. For each $\delta > 0$, we have

$$\int_{|x-t|>\delta} \frac{1}{x-t} \int_{\mathbb{R}} g(u)e^{itu} du\, dt = \int_{\mathbb{R}} g(u) \int_{|x-t|>\delta} e^{itu} \frac{1}{x-t}\, dt\, du.$$

In order to insert the limit as $\delta \downarrow 0$ under the sign of integration, we apply the Lebesgue dominated convergence test. For this, we prove that the integrand in u is dominated by a single integrable function independent of δ. Since g is integrable, it suffices to show that the integrals

$$\int_{|x-t|>\delta} e^{itu} \frac{1}{x-t}\, dt$$

are uniformly bounded. This is obviously the case while integrating over $|x-t| \geq 1$. Substituting $x - t = s$, we then see that the rest is equal to

$$\int_{\delta<|x-t|<1} \frac{e^{itu}}{x-t}\, dt = -ie^{ixu} \int_{\delta<|s|<1} \frac{\sin su}{s}\, ds. \tag{2.1}$$

It is well known that the integral on the right is uniformly bounded. This proves the existence everywhere of the Hilbert transform. Further, by King [5, Vol. 2, Table 1.3 (3.1)],

$$\mathcal{H}e^{iu\cdot}(x) = -i\ \operatorname{sign} u\, e^{iux}, \tag{2.2}$$

and the corresponding Fourier integral tends to zero by the Riemann-Lebesgue lemma, which completes the proof of Theorem 1.1.

To present the multidimensional result in question, we need additional notation. Let $\eta = (\eta_1, \ldots, \eta_n)$ be an n-dimensional vector with the entries either 0 or 1 only. Correspondingly, $|\eta| = \eta_1 + \ldots + \eta_n$. If only the j-th entry is one, while the rest are zeros, a natural notation for such an η is e_j. Additionally, we have special notations $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$. The inequality of vectors is meant coordinate wise. By x_η and dx_η we denote the $|\eta|$-tuple consisting only of x_j such that $\eta_j = 1$ and $\prod_{j:\eta_j=1} dx_j$, respectively. Correspondingly, by $\mathbb{R}^\eta$ we denote the $|\eta|$-dimensional Euclidean space with respect to the variables x_j such that $\eta_j = 1$. We shall need the cases where $\mathcal{H}$ is applied to some of the variables, the cases where the

Hilbert transform is applied to only one variable as well as to each of the n variables are among these. This will be denoted by $\mathcal{H}_\eta$ in general, which means applying $\mathcal{H}$ repeatedly to the j-th variables for which $\eta_j = 1$. In the case where $\mathcal{H}$ acts only at the j-th variable, we can replace $\mathcal{H}_{e_j}$ by the simpler $\mathcal{H}_j$ notation. If we apply $\mathcal{H}$ to each variable repeatedly, the notation $\mathcal{H}_{\mathbf{1}}$ automatically comes to mind.

Theorem 2.1 *If $f \in W_0(\mathbb{R}^n)$, then $\mathcal{H}_\eta f(x)$ exists at every $x \in \mathbb{R}^n$ and $\lim\limits_{|x|\to\infty} \mathcal{H}_\eta f(x) = 0$, for all $\eta \neq \mathbf{0}$.*

Proof For simplicity, it suffices to prove this for the case

$$\eta = (\underbrace{1, \dots, 1}_{\text{k times}}, 0, \dots, 0) = e_1 + \dots + e_k.$$

Just this η will be used during the following proof. The mentioned similarity to the one-dimensional case will come from the fact that the proof goes not only along the same lines applied to the indicated variable but also that we still deal with the Hilbert transform. Expressing f in $\mathcal{H}_\eta f$ by means of (1.1), we get

$$\begin{aligned}\mathcal{H}_\eta f(x) = \ &(\text{P.V.})\,\frac{1}{\pi^k}\int_{\mathbb{R}^k}\prod_{j=1}^k \frac{1}{x_j - t_j}\,dt_j \\ &\int_{\mathbb{R}^n} g(u)e^{i(t_1u_1+\dots+t_ku_k+x_{k+1}u_{k+1}+\dots+x_nu_n)}\,du \\ &= \int_{\mathbb{R}^n} g(u)e^{i(x_{k+1}u_{k+1}+\dots+x_nu_n)} \\ &(\text{P.V.})\,\frac{1}{\pi^k}\int_{\mathbb{R}^k}\prod_{j=1}^k e^{it_ju_j}\frac{1}{x_j - t_j}\,dt_j\,du. \end{aligned} \tag{2.3}$$

We must justify the change of the order of integration on the right-hand side. However, the one-dimensional argument works here in the same manner.

Let us now estimate all the summands on the right-hand side of (2.3). If we consider the product of all k Hilbert transform summands and take into account (2.2), we arrive at

$$(-i)^k\int_{\mathbb{R}^n}\left[g(u)\prod_{j=1}^k \operatorname{sign} u_j\right]e^{i(u,x)}\,du,$$

which tends to zero as $|x| \to \infty$, by the Riemann-Lebesgue lemma. This completes the proof. □

3 General Results

Proceeding to the general situation, we start with defining a general singular integral on basis of [11, Ch.II, §3.2, Th.2], where fundamental properties of singular integrals are proved. It is of the form

$$Sf(x) = \int_{\mathbb{R}^n} f(x-y)K(y)\,dy, \tag{3.1}$$

where the integral is understood as the limit, as $\varepsilon \to 0$, in that or another sense of the integrals

$$S_\varepsilon f(x) = \int_{|y|>\varepsilon} f(x-y)K(y)\,dy, \tag{3.2}$$

provided that the kernel K satisfies the conditions

$$|K(x)| \le C|x|^{-n}, \qquad |x| > 0; \tag{3.3}$$

$$\int_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le C, \qquad |y| > 0; \tag{3.4}$$

and

$$\int_{M_1<|x|<M_2} K(x)\,dx = 0, \qquad 0 < M_1 < M_2 < \infty. \tag{3.5}$$

In [11], the mentioned limit is understood in some L^p norm, with $1 < p < \infty$. However, we will see that it exists at every point provided $f \in W_0$. We have the following multidimensional extension of Theorem 1.1.

Theorem 3.1 *Let $f \in W_0(\mathbb{R}^n)$. Then $Sf(x)$ exists at every $x \in \mathbb{R}^n$ and* $\lim\limits_{|x|\to\infty} Sf(x) = 0.$

Proof In the multivariate case, our strategy will be essentially the same as that in dimension one in the previous section but applied to different singular integrals, mostly purely multidimensional. The uniform boundedness of integrals mirroring the last integral in (2.1) is the key ingredient. In fact, that integral is the truncated (sine) Fourier transform of the kernel of the Hilbert transform. The same property for general kernels is delivered by the lemma in [11, Ch.II, §3.3]; it says that if K satisfies (3.3)–(3.5) and

$$K_\delta(x) = \begin{cases} K(x), & |x| \ge \delta, \\ 0, \text{otherwise}, \end{cases}$$

then

$$\sup_{\delta} |\widehat{K_\delta}(x)| \le C,$$

where the constant C depends only on dimension n. The proof now goes along the same lines as above, where the mentioned lemma allows us to apply the Lebesgue dominated convergence theorem at an appropriate moment. □

The main important singular integrals are the Riesz transforms. In dimension n, there are n transforms. For $j = 1, 2, \dots, n$, the j-th Riesz transform $R_j f(x)$ is defined as

$$R_j f(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy,$$

with

$$c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

By this, R_j is a singular integral defined by the kernel

$$K_j(y) = \frac{y_j}{|y|^{n+1}} = \frac{\Omega_j(y)}{|y|^n},$$

with $\Omega_j(y) = \frac{y_j}{|y|}$.

It follows from Theorem 3.1 that for any $j = 1, 2, \dots, n$, we have for $f \in W_0(\mathbb{R}^n)$ that $R_j f(x)$ exists for every $x \in \mathbb{R}^n$ and $\lim\limits_{|x|\to\infty} R_j f(x) = 0$.

It is worth mentioning that while the repeated Hilbert transforms are used for defining the so-called product Hardy space $H^1(\mathbb{R} \times \dots \times \mathbb{R})$, the Riesz transforms are "responsible" for defining the real Hardy space $H^1(\mathbb{R}^n)$ (their integrability, in fact, is needed) and other related spaces. More or less detailed discussion on these can be found in [7, Ch.5]; in particular, $H^1(\mathbb{R} \times \dots \times \mathbb{R}) \subset H^1(\mathbb{R}^n)$.

4 Certain Singular Integrals

Among general singular integrals considered in Sects. 2 and 3, some play a special role in analysis. They are also studied in [11, Ch.II]. More precisely, in [11, Ch.II, §2.2, Th.1] a wider class of singular integrals is considered, for which the uniform boundedness of the Fourier transform of the L^2 kernel is assumed by definition.

However, [11, Ch.II, §3.2] is not only devoted to the proof of Theorem 2 in [11, Ch.II, §3.2] but begins with nice discussion why that wide class is unsatisfactory. More precisely, it is desirable that the class should be defined in such a way that even belonging to L^2 to be derived from the defining properties. For this, the formally smaller class of singular integrals, the one we deal with in Sect. 3, is introduced by assuming the properties (3.3)–(3.5). Then not only the belonging to L^2 is proved but the property of the Fourier transform of the kernel we are interested in is derived from the definition.

However, further restriction of the class has proved to be desirable for applications. One of the reasons is the need to deal with operators which commute not only with translations like those above but also with dilations. For this, the class which is perfectly appropriate is the one with the kernels of type

$$K(x) = \frac{\Omega(x)}{|x|^n},$$

with Ω homogeneous of degree 0, that is, $\Omega(ax) = \Omega(x)$ for any $a > 0$. By this, being constant on the rays going from zero, Ω is completely determined by its restriction to the unit sphere $\mathbb{S}^{n-1}$. Also, something should be assumed in terms of Ω in order to satisfy conditions of [11, Ch.II, §3.2, Th.2]. By (3.3) and (3.4), Ω must be bounded and thus integrable on $\mathbb{S}^{n-1}$. Further, (3.5) reduces to

$$\int_{\mathbb{S}^{n-1}} \Omega(x)\, d\sigma = 0,$$

where $d\sigma$ is the induced measure on $\mathbb{S}^{n-1}$. However, the conditions (3.3) and (3.4) are difficult to be reformulated in terms of Ω. One of the natural ways to guarantee them is to pose a Dini type condition on Ω. It can be formulated as follows. Defining one of the possible moduli of continuity of Σ in the spherical mode as

$$\omega(t) = \sup_{\substack{|x-y|\le t,\\ |x|=|y|=1}} |\Omega(x) - \Omega(y)|,$$

we assume the Dini type condition

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty.$$

Needless to say that the concrete singular integrals considered in Sects. 2 and 3 are of that last type.

It is proved in [3] that to guarantee the uniform boundedness of the Fourier transform (in the improper sense, as needed) of the kernel K, the following assumptions can be taken instead of (3.3)–(3.5):

$$\int_{|x|\ge M} |K(x)|\,dx \le CM, \qquad 0 < M < \infty; \tag{4.1}$$

$$\int_{|x|\ge 2|y|} |K(x-y) - K(x)|\,dx \le C, \qquad |y| > 0; \tag{4.2}$$

and

$$\left|\int_{M_1<|x|<M_2} K(x)\,dx\right| \le C, \qquad 0 < M_1 < M_2 < \infty. \tag{4.3}$$

All these give enough machinery for applying the obtained necessary conditions and, correspondingly, eliminate functions not belonging to the Wiener algebra.

5 Application to Radial Functions

In [1, Ch. II, §9], Salem's condition is applied to the series $\sum a_n \cos nt$. What is derived is the assertion that for this series with $\{a_n\}$ tending to zero monotonously to be a Fourier series, it is necessary that $\lim\limits_{n\to\infty}(a_n - a_{n+1})\ln n = 0$. In [6], a one-dimensional non-periodic analog of this result is the following

Theorem 5.1 *If $f_0 \in W_0(\mathbb{R}$ is even and monotone on the half-axis, then*

$$\lim_{x\to\infty}\lim_{\delta\to 0+}(f_0(x-\delta) - f_0(x+\delta))\ln\frac{x}{\delta} = 0. \tag{5.1}$$

We mention that indeed the repeated limits are taken exactly in this order.

A natural setting for a multidimensional result of such kind is radial functions. First of all, we recall the following relation.

Lemma 5.2 *For $F_0(|x|) \in W_0(\mathbb{R}^n)$, $n \ge 2$, it is necessary and sufficient that $f_0 \in W_0(\mathbb{R})$ exist such that for $t \ge 0$, there holds*

$$F_0(t) = \int_0^1 f_0(ut)(1-u^2)^{\frac{n-3}{2}}\,du. \tag{5.2}$$

Proof In its precise form, this result is due to Trigub: the functions f_0 and F_0 are assigned unambiguously by differentiation: usual for n odd and half-integer fractional for n even; see [13, 6.3.6] and [12]. However, without indicating f_0 explicitly, the result can be proved in the following simpler way.

If $f_0 \in W_0(\mathbb{R})$ in the way

$$f_0(u) = \int_0^\infty g(s) \cos us\, ds,$$

with $g \in \mathbb{R}$, then substituting the last integral for f in (5.2), we obtain

$$\begin{aligned} &\int_0^1 (1-u^2)^{\frac{n-3}{2}} \int_0^\infty g(s) \cos uts\, ds\, du \\ &= \int_0^\infty g(s) \int_0^1 (1-u^2)^{\frac{n-3}{2}} \cos uts\, du\, ds \\ &= 2^{\frac{n}{2}-2}\sqrt{\pi}\,\Gamma\Big(\frac{n-1}{2}\Big) \int_0^\infty g(s)\,(st)^{-\frac{n}{2}+1} J_{\frac{n-2}{2}}(st)\, ds, \end{aligned} \tag{5.3}$$

where J_ν is the Bessel function of first kind and the passage to the right-hand side is [2, Ch.I, 1.3(8)], with appropriate parameters. But this coincides, up to a constant multiple, with

$$(2\pi)^{\frac{n}{2}} \int_0^\infty \frac{g(s)}{s^{n-1}}\,(st)^{-\frac{n}{2}+1} J_{\frac{n-2}{2}}(st) s^{n-1}\, ds. \tag{5.4}$$

Since g is integrable on $\mathbb{R}_+$, we have $G(|x|) = \frac{g(|x|)}{|x|^{n-1}} \in L^1(\mathbb{R}^n)$, and (5.4) is exactly the multidimensional Fourier transform of $G(|\cdot|)$ (see [4, Ch.IX, §43, Thm.56]; in Remark 109 of that book, this formula is referred to Cauchy and Poisson for $n = 2, 3$, and claimed not to be new in the 50-s, the time of writing [4], for arbitrary n). Hence $F_0 \in W_0(\mathbb{R}^n)$.

Conversely, let $F_0 \in W_0(\mathbb{R}^n)$. Then, by definition and radiality, it is equal to

$$(2\pi)^{\frac{n}{2}} \int_0^\infty G(s)\,(st)^{-\frac{n}{2}+1} J_{\frac{n-2}{2}}(st) s^{n-1}\, ds,$$

with some $G(|\cdot|) \in L^1(\mathbb{R}^n)$ and $t = |x|$. Furthermore, by (5.3), it is equal, again up to a constant multiple, to

$$\int_0^1 (1-u^2)^{\frac{n-3}{2}} \int_0^\infty G(s) s^{n-1} \cos uts\, ds\, du.$$

Since $G(s)s^{n-1} \in L^1(\mathbb{R}_+)$, denoting the inner integral by $f_0(ut)$, we arrive at the needed representation, with $f_0 \in W_0(\mathbb{R})$. This completes the proof. □

We are now in a position to prove the following result.

Theorem 5.3 *Let $F_0(|\cdot|) \in W_0(\mathbb{R}^n)$, $n \ge 2$, and let (5.2) hold with f_0 monotone on the half-axis. Then*

$$\lim_{|x|\to\infty} \lim_{\delta\to 0+} (F_0(|x|-\delta) - F_0(|x|+\delta)) \ln \frac{|x|}{\delta} = 0. \tag{5.5}$$

Proof By (5.2), denoting $|x| = t$, we get

$$\begin{aligned}&[F_0(t-\delta) - F_0(t+\delta)] \ln \frac{t}{\delta}\\ &= \int_0^1 [f_0(ut-u\delta) - f_0(ut+u\delta)] \ln \frac{tu}{u\delta} (1-u^2)^{\frac{n-3}{2}} du.\end{aligned}$$

Since F_0 is radial, f_0 can be considered as even. It is also bounded and vanishing at infinity by belonging to $W_0(\mathbb{R})$. While integrating on the right over $\left(\frac{1}{\sqrt{t}}, 1\right)$, we derive from (5.1) that $\lim\limits_{t\to\infty} \lim\limits_{\delta\to 0+}$ of this integral tends to zero. It remains to estimate

$$\int_0^{\frac{1}{\sqrt{t}}} [f_0(ut-u\delta) - f_0(ut+u\delta)] \ln \frac{tu}{u\delta} (1-u^2)^{\frac{n-3}{2}} du. \tag{5.6}$$

Since $\lim\limits_{\delta\to 0+}$ comes from the definition of the Hilbert transform in the principal value sense and, as we know, for a function from the Wiener algebra, it exists at every point and is uniformly bounded, we derive from the estimate

$$\int_0^{\frac{1}{\sqrt{t}}} (1-u^2)^{\frac{n-3}{2}} du = O\left(\frac{1}{\sqrt{t}}\right)$$

that (5.6) tends to zero as $\lim\limits_{\delta\to 0+}$ and then $\lim\limits_{t\to\infty}$. This completes the proof. □

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Techniques to Derive Estimates for Integral Means and Other Geometric Quantities Related to Conformal Mappings

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Abstract We will describe few methods to derive estimates for integral means and for other asymptotic expressions related to conformal mappings. One method will start from classical inequalities for conformal mappings such as the Goluzin inequalities and the exponential Goluzin inequalities. Then the simple idea of approximating integrals with the aid of their Riemann sums will serve us to obtain such estimates. A second method is to start from a certain elementary identity proved by Hardy in 1915 and use it combined with distortion theorems in S to obtain more integrals estimates. Finally, the main result in the author's Masters thesis which in fact was already known to Bendixon will give us a method to estimate the geometric distance from a point in the image of a conformal mapping to the boundary of this image. The estimate will be in terms of a rather arbitrary sequence in the domain of the definition that converges to the pre-image of the point in the image from which the distance is measured.

Keywords Conformal mappings · Schlicht functions · Goluzin inequalities · Integral means · Distance to the boundary of the image

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1 Introduction

For an $f \in H(U)$ ($U = \{z \in \mathbb{C} \mid |z| < 1\}$, and $H(U)$ is the family of the functions that are holomorphic in U) we will use the following standard notation for the p-integral mean of f on the circle $|z| = r$ for a fixed r in $0 \le r < 1$:

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

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The parameter p will usually be confined to the open positive half line, $0 < p < \infty$. We also write $M_\infty(f, r) = \max_{0 \le \theta < 2\pi} |f(re^{i\theta})|$. We also may write:

$$\log(M_0(f, r)) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta,$$

provided that the last integral converges. The motivation for the last definition comes from the following heuristics: $M_0(f, r) = \lim_{p \to 0^+} M_p(f, r)$, and

$$\log(M_0(f, r)) = \lim_{p \to 0^+} \frac{1}{p} \log\left\{\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\right\} =$$

$$= \lim_{p \to 0^+} \frac{d}{dp}\left\{\log\left\{\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\right\}\right\} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

We note that $M_0(f, r)$ is the parallel of the geometric mean of a finite sequence of numbers. We recall results of Al Baernstein II from his deep paper, [1]. A continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ is said to be convex if $\forall x, y \in \mathbb{R}$, $\Phi(\frac{1}{2}(x + y)) \le \frac{1}{2}(\Phi(x) + \Phi(y))$. It is said to be strictly convex if strict inequality holds unless $x = y$.

Theorem (Baernstein's Theorem, [1]) *Let $\Phi(x)$ be a convex non-decreasing function $\mathbb{R} \to \mathbb{R}$. Then for each $f \in S$,*

$$\int_0^{2\pi} \Phi(\log|f(re^{i\theta})|) d\theta \le \int_0^{2\pi} \Phi(\log|k(re^{i\theta})|) d\theta, \quad 0 < r < 1.$$

Here $S = \{f \in H(U) \,|\, f$ is injective $f(0) = f'(0) - 1 = 0\}$ and $k(z)$ is the Koebe function $k(z) = z/(1 - z)^2$. If Φ is strictly convex, then equality holds for some r only if f is a rotation of k.

The choice $\Phi(x) = e^{px}$, $0 < p < \infty$, gives the following conclusion:

Corollary (Al Baernstein, [1]) *For $0 < p < \infty$ and $f \in S$, $M_p(f, r) \le M_p(k, r)$, $0 < r < 1$, with equality only if f is a rotation of k.*

The proof of Baernstein's Theorem involves a certain maximal function he invented, the Baernstein star-function. Let $u(z)$ be a real-valued function defined on the annulus $r_1 < |z| < r_2$. For each r, $r_1 < r < r_2$, we suppose that $u(re^{i\theta}) \in L^1(0, 2\pi)$. The Baernstein star-function of u is:

$$u^*(re^{i\theta}) = \sup_{|E| = 2\theta} \int_E u(re^{it}) dt, \quad 0 < \theta < \pi.$$

Here $|E|$ denotes the Lebesgue measure of the measurable set $E \subseteq [-\pi, \pi]$. In [1], Baernstein showed that the star-function has the following remarkable property.

Lemma 1 ([1]) *If u is continuous and sub-harmonic in the annulus $r_1 < |z| < r_2$, then u^* is continuous in the semi-annulus $\{re^{i\theta}\} \mid r_1 < r < r_2,\ 0 \le \theta \le \pi\}$, and sub-harmonic in the interior.*

In this paper we will show how one can start from the inequalities on S, that involve finite sequence of parameters, such as the Grunsky inequalities, Goluzin and Lebedev inequalities etc... and by approximating various integral means using Riemann sums get new integral inequalities on functions of S that involve also their derivatives and other operators such as the divided quotient operator. We will use as one basic reference the book [2] of Peter Duren. Especially Chap. 4, pp 118–140 (for Grunsky inequalities, and the Goluzin Lebedev inequalities), Chap. 5, pages 142–187 (exponentiation of the Grunsky inequalities), and Chap. 7 on integral means, pages 214–231.

The natural problem of estimating sharply the integral means of derivatives of functions in S, along lines similar to the classical result of Baernstein for the functions themselves turns out to be involved. See [2, section 7.5, pages 229–231]. The first case might be to inquire if $M_p(f', r) \le M_p(k', r)$ for all $f \in S$. For $0 < p < \frac{1}{3}$, this is certainely false, for by a direct calculation $k' \in H^p(U)$ for all $0 < p < \frac{1}{3}$, so that $M_p(k', r)$ remains bounded as $r \to 1^-$. On the other hand, there exist functions $f \in S$ whose derivatives have radial limits on no set of positive measure, see [4]. In this paper it is shown that in fact f' can have much worse behavior. In particular $f' \notin H^p(U)$for any $p > 0$. For such a function f, and for each $0 < p < \frac{1}{3}$, the inequality $M_p(f', r) \le M_p(k', r)$ must fail for r near 1. Here is a quotation from [2]: "In general, the sharp bound of $M_p(f', r)$, $f \in S$ is unknown. For $p > \frac{1}{3}$ it seems a reasonable conjecture that $M_p(f', r) \le M_p(k', r)$." Duren gives in his book [2], on page 229 two kinds of evidence to support the last conjecture. First, it is asymptotically correct, at least for $p > \frac{2}{5}$. Feng and MacGregor have shown that for each $p > \frac{2}{5}$, and each positive integer $n \in \mathbb{Z}^+$, $M_p(f^{(n)}, r) = \mathrm{O}(M_p(k^{(n)}, r), r \to 1^-$, for all $f \in S$. In particular, for $p > \frac{2}{5}$, $M_p(f', r) = \mathrm{O}((1-r)^{1/p-3}), r \to 1^-$. See [3].

Second, the conjecture is true for certain sub-classes of S. It is so for the sub-class K of close-to-convex functions.

Theorem (See [2] page 229) *For $0 < p < \infty$, $M_p(f', r) \le M_p(k', r)$, $0 < r < 1$, for all $f \in K$.*

Before proceeding to the applications of Grunsky inequalities and other inequalities to the estimates of integral means in S, let us note what do the elementary theorems on S give us in that regard.

Here are some of these theorems:

A distortion theorem:

(1) For each $f \in S$,

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

For each $z \in U$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

A growth theorem:

(2) for each $f \in S$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

For each $z \in U$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

A rotation theorem:

(3) For each $f \in S$,

$$|\arg\, f'(z)| \leq \begin{cases} 4 \sin^{-1} r, & r \leq \frac{1}{\sqrt{2}}, \\ \pi + \log\left(\frac{r^2}{1-r^2}\right), & r \geq \frac{1}{\sqrt{2}}. \end{cases}$$

One more estimate:

(4) For each $f \in S$,

$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}, \quad |z| = r < 1.$$

Let us use (2) above in order to give a crude estimate of $M_p(f, r)$:

$$M_p(f, r) = \left\{\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p} \leq \frac{r}{(1-r)^2}.$$

Similarly by (1):

$$M_p(f', r) \leq \frac{1+r}{(1-r)^3}. \quad (0 < p < \infty)$$

These are relatively poor estimates. Already in 1925 Littlewood could do much better for $f \in S$ and $p = 1$. His argument was a clever geometric argument.

Theorem (Littlewood, 1925) *For each function $f \in S$,*

$$M_1(f, r) \leq \frac{r}{1-r}, \quad 0 \leq r < 1.$$

Proof Consider the square-root transform of f, $h(z) = \sqrt{f(z^2)} = \sum_{n=1}^{\infty} c_n z^n$. The growth theorem (2) gives us:

$$|h(z)| \leq \frac{z}{1-r^2}, \quad |z| = r < 1.$$

So h maps the disk $|z| < r$ conformally onto a domain D_r which lies in the disk $|w| < \frac{r}{1-r^2}$. The area A_r of D_r is therefore no greater than the area of the last disk:

$$A_r \leq \frac{\pi r^2}{(1-r^2)^2}.$$

But a calculation gives:

$$A_r = \int_0^{2\pi} \int_0^r |h'(\rho e^{i\theta})|^2 \rho d\rho d\theta = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

Consequently,

$$\sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1} \leq \frac{r}{(1-r^2)^2}, \quad 0 \leq r < 1.$$

Integration from 0 to r gives:

$$\sum_{n=1}^{\infty} |c_n|^2 r^{2n} \leq \frac{r^2}{1-r^2}, \quad \text{or}$$

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^2 d\theta \leq \frac{r^2}{1-r^2},$$

which is equivalent to:

$$M_1(f, r) \leq \frac{r}{1-r}, \quad 0 \leq r < 1.$$

One might suspect that the estimate in Littlewood's Theorem above could be improved if we use the N'th root transform $h(z) = f(z^N)^{1/N} = \sum_{n=1}^{\infty} c_n z^n$ instead of the square-root transform. The result is: □

Theorem (The N'th Root Version of Littlewood's Theorem) *For each function $f \in S$ and each integer $N \geq 2$,*

$$M_1(f, r) \leq \left(\frac{N}{N-4}\right) \times \frac{1-(1-r)^{1-4/N}}{r^{1-2/N}}, \quad 0 \leq r < 1.$$

It turns out that for $N = 2$ (as Littlewood adviced us) the upper bound is the best.

2 Inequalities on S and Σ

We recall few basic inequalities on families of conformal mappings.

(1) The Goluzin inequalities.

(a) On Σ. Let $g \in \Sigma$ and $\xi_\nu \in \Delta = \{\xi \in \mathbb{C} \,|\, |\xi| > 1\}$, $\gamma_\nu \in \mathbb{C}$ where $\nu = 1, \dots, n$, $n = 1, 2, \dots$. Then

$$\left| \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \gamma_\mu \gamma_\nu \log\left(\frac{g(\xi_\mu) - g(\xi_\nu)}{\xi_\mu - \xi_\nu} \right) \right| \leq \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \gamma_\mu \overline{\gamma}_\nu \log\left(\frac{1}{1 - (\xi_\mu \overline{\xi}_\nu)^{-1}} \right).$$

(b) On S. Let $f \in S$ and $z_\nu \in U$, $\gamma_\nu \in \mathbb{C}$ where $\nu = 1, \dots, n$, $n = 1, 2, \dots$. Then

$$\left| \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \gamma_\mu \gamma_\nu \log\left(\frac{z_\mu z_\nu}{f(z_\mu) f(z_\nu)} \cdot \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \right) \right| \leq \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \gamma_\mu \overline{\gamma}_\nu \log\left(\frac{1}{1 - z_\mu \overline{z}_\nu} \right).$$

(2) Exponentiation of the Goluzin inequalities.

The idea is to replace the inequalities in (1)(b) by similar inequalities with the logarithms removed. We refer to [2] pages 180-183. The procedure in the book is based on a theorem of Schur on the Hadamard product of two positive semi-definite matrices. We will review that but see how to obtain a slightly more general result. A real symmetric matrix $A = (a_{jk})$ is said to be positive semi-definite (denoted by $A \geq 0$) if its associated quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k$ is non-negative for all real numbers $x_1, \dots, x_n$. This will be the case if and only if all of the (real) eigenvalues of A are non-negative. The Hadamard product of two $n \times n$ matrices $A = (a_{jk})$ and $B = (b_{jk})$ is defined as the $n \times n$ matrix $(a_{jk} b_{jk})$. We will denote the Hadamard product of A and B by $A * B$ (so that it will not be confused with ordinary matrix product). This operation is commutative, i.e. $A * B = B * A$. Also if A and B are symmetric so is $A * B$. for any function f on $\mathbb{R}$ to $\mathbb{R}$ we denote by $*f(A)$ the $n \times n$ matrix $(f(a_{jk}))$. In particular $*A^m = (a_{jk}^m)$, $m = 1, 2, \dots$.

Theorem (Schur's Theorem) *If $A \geq 0$ and $B \geq 0$, then $A * B \geq 0$.*

Corollary to Schur's Theorem If $A \geq 0$ and $\phi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ is an entire function with non-negative coefficients c_ν, then $*\phi(A) \geq 0$.

We also have the following

Lemma (On Hermitian Forms) *If the real symmetric matrix $A = (a_{jk})$ is positive semi-definite, then the Hermitian form $\sum_{j=1}^{n} \sum_{k=1}^{n} \xi_j \overline{\xi}_k \geq 0$, for all complex numbers $\xi_1, \dots, \xi_n$.*

After those algebraic preparations, the book [2] on page 182 returns to the Goluzin inequalities on S, (1)(b). Choosing the parameters γ_ν to be real numbers and using the fact that for any complex number α we have $-\Re\{\alpha\} \leq |\alpha|$ we conclude that the symmetric matrix with elements:

$$a_{jk} = \left|\log\left(\frac{z_j z_k}{f(z_j)f(z_k)} \cdot \frac{f(z_j) - f(z_k)}{z_j - z_k}\right)\right|$$

is positive semi-definite. Applying the Corollary to Schur's theorem with the entire function $\phi(z) = e^{2z} - 1$ (which clearly has positive coefficients), we deduce that the matrix $C = (c_{jk}) = (e^{2a_{jk}} - 1)$ is positive semi-definite. By the Lemma (on Hermitian forms) we get, $\sum_{\mu=1}^{n}\sum_{\nu=1}^{n} c_{\mu\nu}\gamma_\mu\overline{\gamma}_\nu \geq 0$ for all complex numbers $\gamma_1, \ldots, \gamma_n$. In other words,

$$\left|\sum_{\mu=1}^{n} \gamma_\mu\right|^2 \leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \left|\frac{z_\mu z_\nu}{f(z_\mu)f(z_\nu)} \cdot \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \cdot \frac{1}{1 - z_\mu\overline{z}_\nu}\right|^2 \gamma_\mu\overline{\gamma}_\nu.$$

If the numbers γ_μ are expressed as follows:

$$\gamma_\mu = \lambda_\mu \left|\frac{f(z_\mu)}{z_\mu}\right|^2, \quad \mu = 1, \ldots, n,$$

these inequalities become:

$$\left|\sum_{\mu=1}^{n} \lambda_\mu \left|\frac{f(z_\mu)}{z_\mu}\right|^2\right|^2 \leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \lambda_\mu\overline{\lambda}_\nu \left|\frac{f(z_\mu) - f(z_\nu)}{(z_\mu - z_\nu)(1 - z_\mu\overline{z}_\nu)}\right|^2$$

for all complex numbers $\lambda_1, \ldots, \lambda_n$. These last inequalities are the exponentiation of the Goluzin inequalities.

(3) A generalization of the exponentiation of the Goluzin inequalities. The most significant thing we do, is to apply the Corollary to Schur's Theorem with the function $\phi(z) = e^{qz} - 1$ with any $0 < q$ (instead of just $q = 2$ as was the case in (2) above).Clearly the entire function $e^{qz} - 1 = \sum_{m=1}^{\infty} \frac{q^m}{m!} \cdot z^m$ has positive coefficients, as requested by the assumptions of the corollary. The result is that for $0 < q < \infty$ we have,

$$\left|\sum_{\mu=1}^{n} \gamma_\mu\right|^2 \leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \left|\frac{z_\mu z_\nu}{f(z_\mu)f(z_\nu)} \cdot \frac{f(z_\mu) - f(z_\nu)}{z_\mu - z_\nu} \cdot \frac{1}{1 - z_\mu\overline{z}_\nu}\right|^q \gamma_\mu\overline{\gamma}_\nu.$$

If the numbers γ_μ are expressed as follows:

$$\gamma_\mu = \lambda_\mu \left|\frac{f(z_\mu)}{z_\mu}\right|^q, \quad \mu = 1, \ldots, n,$$

these inequalities become:

$$\left|\sum_{\mu=1}^{n}\lambda_\mu\left|\frac{f(z_\mu)}{z_\mu}\right|^q\right|^2 \le \sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\lambda_\mu\overline{\lambda}_\nu\left|\frac{f(z_\mu)-f(z_\nu)}{(z_\mu-z_\nu)(1-z_\mu\overline{z}_\nu)}\right|^q$$

for all complex numbers $\lambda_1,\ldots,\lambda_n$.

3 Generating Integral Inequalities

The very simple idea is to turn the inequalities on S we had in Sect. 2 into inequalities between Riemann sums. We will demonstrate in the current section an example of this technique. Before stating and proving our results on integral means we will need one more inequality for a function in S that will be tailored for our needs. A major difference between this new inequality and the standard inequalities in Sect. 2 is that in the coming inequality only the left hand side contains the function f (very similar to the various exponentiation of Goluzin inequalities). The right hand side is a concrete function of the parameters.

Theorem 3.1 *Let $n \in \mathbb{Z}^+$, $z_1,\ldots,z_n \in U$, $\alpha_1,\ldots,\alpha_n \in \mathbb{C}$, $q \in \mathbb{R}^+$ and $f \in S$. Then*

$$\left|\sum_{\mu=1}^{n}\alpha_\mu\frac{1}{|f'(z_\mu)|^{q/2}}\left|\frac{f(z_\mu)}{z_\mu}\right|^q\right|^2 \le \sum_{\mu=1}^{n}\sum_{\nu=1}^{n}\alpha_\mu\overline{\alpha}_\nu\frac{(1-|z_\mu|^2)^{q/2}(1-|z_\nu|^2)^{q/2}}{(|1-\overline{z}_\mu z_\nu|-|z_\mu-z_\nu|)^{2q}}.$$

Proof We start with the following elementary, □

Lemma 3.2 *If $f \in S$, $z, w \in U$, then,*

$$\frac{1}{|f'(z)|}\cdot\frac{|f(w)-f(z)|}{|w-z||1-\overline{z}w|} \le \frac{1-|z|^2}{(|1-\overline{z}w|-|z-w|)^2}.$$

A proof of Lemma 3.2 For the variable $\xi \in U$ and for a fixed $z \in U$ we have,

$$h(\xi) = \frac{f\left(\frac{\xi+z}{1+\overline{z}\xi}\right)-f(z)}{(1-|z|^2)f'(z)} \in S.$$

Hence

$$h(\xi) \le \frac{|\xi|}{(1-|\xi|)^2}.$$

Let us denote $w = \frac{\xi+z}{1+\overline{z}\xi}$. Then $\xi = \frac{w-z}{1-\overline{z}w}$ and hence,

$$\frac{|\xi|}{(1-|\xi|)^2} = \frac{|w-z||1-\overline{z}w|}{(|1-\overline{z}w|-|z-w|)^2}.$$

By the above 3 equations and inequalities,

$$\frac{f(w)-f(z)}{(1-|z|^2)f'(z)} \leq \frac{|w-z||1-\overline{z}w|}{(|1-\overline{z}w|-|z-w|)^2}.$$

This completes the proof of Lemma 3.2.

Let us now express each λ_μ in the last inequality of Sect. 2 as follows,

$$\lambda_\mu = \alpha_\mu \frac{1}{|f'(z_\mu)|^{q/2}}, \quad \mu = 1, \ldots, n.$$

Then,

$$\left|\sum_{\mu=1}^{n} \alpha_\mu \frac{1}{|f'(z_\mu)|^{q/2}} \left|\frac{f(z_\mu)}{z_\mu}\right|^q\right|^2 \leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \alpha_\mu \overline{\alpha}_\nu \left(\frac{1}{|f'(z_\mu)|}\frac{1}{|f'(z_\nu)|}\right)^{q/2} \times$$

$$\times \left|\frac{f(z_\mu)-f(z_\nu)}{(z_\mu - z_\nu)(1-z_\mu\overline{z}_\nu)}\right|^q =$$

$$= \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \alpha_\mu \overline{\alpha}_\nu \left|\frac{1}{f'(z_\mu)} \cdot \frac{f(z_\mu)-f(z_\nu)}{z_\mu - z_\nu} \cdot \frac{1}{1-\overline{z}_\mu z_\nu}\right|^{q/2} \times$$

$$\times \left|\frac{1}{f'(z_\nu)} \cdot \frac{f(z_\nu)-f(z_\mu)}{z_\nu - z_\mu} \cdot \frac{1}{1-\overline{z}_\nu z_\mu}\right|^{q/2} \leq$$

$$\leq \sum_{\mu=1}^{n}\sum_{\nu=1}^{n} \alpha_\mu \overline{\alpha}_\nu \frac{(1-|z_\mu|^2)^{q/2}(1-|z_\nu|^2)^{q/2}}{(|1-\overline{z}_\mu z_\nu| - |z_\mu - z_\nu|)^{2q}},$$

where in the last step we used twice Lemma 3.2. □

Corollary 3.3 *Let $f \in S$ and $q \in \mathbb{R}^+$. Then:*

(1)

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|^q}{|f'(re^{i\theta})|^{q/2}} d\theta\right)^2 \leq \frac{r^{2q}(1-r^2)^q}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{dt ds}{(|1-r^2e^{i(s-t)}| - r|e^{it}-e^{is}|)^{2q}},$$

$0 \leq r < 1$.

(2)

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^q d\theta\right)^2 \le \frac{r^q}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(re^{is})-f(re^{it})}{(e^{is}-e^{it})(1-r^2e^{i(s-t)})}\right|^q dsdt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{r^2e^{is}}{1-r^2e^{is}}\right|^q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{i(s+t)})-f(re^{it})}{re^{i(s+t)}-re^{it}}\right|^q dt\right)ds,$$

$0 \le r < 1$.

Proof

(1) In Theorem 3.1 let us take $\alpha_\mu = \frac{1}{n}$, $\mu = 1, \ldots, n$, $z_\mu = re^{2\pi i\mu/n}$. Then using the corresponding Riemann sum with equal distances $\mu\frac{2\pi}{n}$, $\mu = 1, \ldots, n$, we get,

$$\left|\sum_{\mu=1}^n \alpha_\mu \frac{1}{|f'(re^{2\pi i\mu/n})|^{q/2}}\cdot\left|\frac{f(re^{2\pi i\mu/n})}{re^{2\pi i\mu/n}}\right|^q\right|^2 \to_{n\to\infty}$$

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(re^{i\theta})|^q}{|f'(re^{i\theta})|^{q/2}}\frac{d\theta}{r^q}\right)^2 = \frac{1}{r^{2q}}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(re^{i\theta})|^q}{|f'(re^{i\theta})|^{q/2}}d\theta\right)^2.$$

A similar argument applied to the right hand side gives for $n \to \infty$ the following,

$$\left(\frac{1}{2\pi}\right)^2\int_0^{2\pi}\int_0^{2\pi}\frac{(1-r^2)^q dsdt}{(|1-r^2e^{i(s-t)}|-|re^{is}-re^{it}|)^{2q}}.$$

(2) We take the data, and apply the same Riemann sums argument to the inequality at the end of Sect. 2 (A generalization of the exponentiation of Goluzin inequalities).

□

In order to express the next corollary we use the following two notations. Let $f(z)$, $g(z, w)$ be continuous functions defined on $|z| = r$ and on $|z| = |w| = r$ respectively. Then:

$$M_\infty(f(z), r) = \max_{|z|=r}|f(z)|, \quad M_\infty(g(z, w), r) = \max_{|z|=|w|=r}|g(z, w)|.$$

Corollary 3.4 *Let $f \in S$. Then:*

(1)

$$M_\infty\left(\frac{f(z)^2}{f'(z)}, r\right) \le M_\infty\left(\frac{1}{(|1-\overline{z}w| - |w-z|)^4}, r\right) \times r^4(1-r^2)^2.$$

(2)

$$M_\infty(f(z)^2, r) \le M_\infty\left(\frac{f(z)-f(w)}{(z-w)(1-\overline{w}z)}, r\right) \times r^4.$$

Proof We raise the two inequalities of Corollary 3.3 to the power $\frac{2}{q}$ and take the limits of the inequalities as $q \to \infty$. □

One can use the above technique to derive many different integral inequalities on various expressions that involve $f, f', f^{(2)}, f^{(3)}, \dots$ for arbitrary $f \in S$.

4 Using a Lemma of Hardy from 1915

Lemma (Hardy, 1915) *Let $\Phi(t)$ be a real, twice differentiable function defined on $0 \le t < \infty$. Let*

$$\Psi(t) = t\frac{d}{dt}\left\{t\Phi'(t)\right\}, \quad 0 \le t < \infty.$$

Let $f \in H(U)$ and let us denote: $M_\Phi(f, r) = \frac{1}{2\pi}\int_0^{2\pi}\Phi\left(|f(re^{i\theta})|\right)d\theta$. If $f(z) \ne 0$ for $|z| = r$ then:

$$r\frac{d}{dr}\left\{rM'_\Phi(f, r)\right\} = \frac{r^2}{2\pi}\int_0^{2\pi}\Psi(|f(re^{i\theta})|)\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta.$$

Here $M'_\Phi(f, r) = \frac{d}{dr}\{M_\Phi(f, r)\}$.

Proof *(Hardy)* Using the following 2 identities:

$$r\frac{\partial}{\partial r}\left|f(re^{i\theta})\right| = \left|f(re^{i\theta})\right|\Re\left\{(re^{i\theta})\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\},$$

$$\frac{\partial}{\partial\theta}\left|f(re^{i\theta})\right| = -\left|f(re^{i\theta})\right|\Im\left\{(re^{i\theta})\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\},$$

we obtain (here $z = re^{i\theta}$):

$$\left(r\frac{\partial}{\partial r}\right)^2 \Phi(|f(z)|) + \left(\frac{\partial}{\partial\theta}\right)^2 \Phi(|f(z)|) = \Psi(|f(z)|)\left|z\frac{f'(z)}{f(z)}\right|^2.$$

We integrate $\int_0^{2\pi} \ldots d\theta$ the last identity. The second element is 0. This gives the consequence. □

We recall that $M_\lambda(f,r)^\lambda = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta$. Let us take in Hardy's Lemma the following data: $\Phi(t) = t^\lambda$, $0 \le t < \infty$, for $\lambda \ge 2$. A computation gives: $\Psi(t) = t\frac{d}{dt}\{t\lambda t^{\lambda-1}\} = \lambda^2 t^\lambda$ and we obtain the following integral identity:

$$r\frac{d}{dr}\left\{r(M_\lambda(f,r)^\lambda\right\} = \frac{\lambda^2 r^2}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{\lambda-2}|f'(re^{i\theta})|^2 d\theta.$$

Integrating $\int_0^r \ldots dr$ gives us an identity that includes an area integral:

$$r(M_\lambda(f,r)^\lambda)' = \frac{\lambda^2}{2\pi}\iint_{|z|\le r} |f(z)|^{\lambda-2}|f'(z)|^2 d\sigma_z.$$

Theorem 4.1 *Let $f \in S$, $\lambda \ge 2$, then:*

$$\lambda^2\int_0^r \left\{\frac{1}{2\pi r}\iint_{|z|\le r} |f(z)|^{\lambda-2}|f'(z)|^2 d\sigma_z\right\} dr \le \frac{r^\lambda}{2\pi}\int_0^{2\pi} |1-re^{i\theta}|^{-2\lambda} d\theta, \quad 0 \le r < 1.$$

Equality sign holds only if $f(z)$ is a rotation of the Koebe function $k(z) = z(1-z)^{-2}$.

Proof Using the result above of Hardy with $\Phi(t) = t^\lambda$ we obtain

$$\lambda^2\int_0^r \left\{\frac{1}{2\pi r}\iint_{|z|\le r} |f(z)|^{\lambda-2}|f'(z)|^2 d\sigma_z\right\} dr = M_\lambda(f,r)^\lambda.$$

Using the celebrated theorem of Al Baernstein from [1], we have:

$$M_\lambda(f,r)^\lambda \le \frac{r^\lambda}{2\pi}\int_0^{2\pi} |1 - re^{i\theta}|^{-2\lambda} d\theta,$$

with equality only if f is a rotation of the Koebe function, $k(z)$. □

5 Estimating the Distance from $f(z_0)$ to $\partial f(U)$ Using Sequences that Converge to z_0

Theorem 5.1 *Let $f \in H(U)$ be an injective mapping on U. Then for any sequence $\{z_\nu\}_1^\infty$ of distinct points in U such that $z_0 = \lim z_\nu \in U$ we have the following estimates:*

$$\limsup_{n\to\infty} \left| \sum_{\nu=1}^{n} \frac{f(z_\nu)}{\prod_{\omega\neq\nu}(z_\omega - z_\nu)} \right|^{1/n} \leq \frac{1}{1-|z_0|}.$$

$$\limsup_{n\to\infty} \left| \sum_{\mu=1}^{n} \frac{z_\mu}{\prod_{\omega\neq\mu}(f(z_\omega) - f(z_\mu))} \right|^{1/n} \leq \frac{1}{d_{z_0}}.$$

Here $d_{z_0} = \min\{|f(z_0) - w| \mid w \in \partial f(U)\}$ the distance from $f(z_0)$ to the boundary of the image $\partial f(U)$. In particular we have an upper bound for that distance:

$$d_{z_0} \leq \liminf_{n\to\infty} \left| \sum_{\mu=1}^{n} \frac{z_\mu}{\prod_{\omega\neq\mu}(f(z_\omega) - f(z_\mu))} \right|^{-1/n}.$$

Proof We will use a theorem from the author's Masters thesis, [5]. It follows also from a result of Ivar Otto Bendixon. According to this theorem, if $g(z)$ is an analytic function in a neighborhood of z_0, and if $\{z_\mu\}_1^\infty$ is a sequence in that neighborhood and $\lim z_\mu = z_0$, then the radius of convergence of the power series of g centered at z_0, R, is given by the following formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \left| \sum_{\mu=1}^{n} \frac{g(z_\mu)}{\prod_{\omega\neq\mu}(z_\omega - z_\mu)} \right|^{1/n}.$$

In our case f is analytic in a neighborhood of z_0 with the radius of convergence about z_0, at least $1 - |z_0|$. This gives the first estimate in our theorem. Also f^{-1} is analytic in a neighborhood of $f(z_0)$ and its radius of convergence about $f(z_0)$ is at least d_{z_0}. This gives the second (and hence the third) estimate. □

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Complex Ternary Analysis and Applications

M. B. Vajiac

Abstract In this paper the author is presenting a theory of functions on complex ternary algebras. The theory developed here is a particular case of the more general case discussed in a volume the author is preparing in collaboration with A. Vajiac and a continuation of the real ternary case developed in Alpay et al. (Adv Appl Clifford Algebr 28:1–16, 2018). The complex ternary algebra has a dual nature: on one side, it is a one–dimensional (one ternary variable) theory generated by an element that cubes to ± 1, on the other it behaves as a theory of one bicomplex variable and one complex variable entangled by algebra relations.

Keywords Ternary · Complex · Bicomplex · Hypercomplex

Mathematics Subject Classification (2000) Primary 99Z99; Secondary 00A00

1 Introduction

In recent years the theory of hypercomplex analysis has taken new directions towards more exotic examples such as the ternary case, in part due to possible applications in signal processing, physics, etc. Hypercomplex numbers over $\mathbb{Q}$ have been studied by E. Artin et al. [6] in the context of analytic number theory, and recently (over $\mathbb{R}$) by physicists such as Catoni et al. [8] in the study of Minkowski space–time geometry and physics. The study of hypercomplex numbers is traced back the nineteenth century, when it formed the basis of modern group representation theory.

In this paper the author is presenting a theory of functions on complex ternary algebras, a generalization of structures that were first brought to light in [1, 14, 16]. In general, the interest in ternary algebras (as well as $n-$ary algebras) has

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been rekindled in recent years through the work of Rausch de Traubenberg and Kerner [1, 14, 16] and some interesting applications in physics [12, 13]. For more generalization from an algebraic point of view, one could see [18].

The theory developed here is a particular case of the more general case discussed in a volume the author is preparing in collaboration with A. Vajiac and a continuation of the real ternary case developed in [5]. The author is also collaborating on a Clifford ternary analysis project that would allow a cubic factorization of the Laplacian and one can obtain a Dirac-type operator in this case. Historically, Clifford analysis has been used to factor the $n-$dimensional Laplacian via the regular Dirac operator, however, new structures are needed for factorizations of different degrees. This work can be found in [9, 10].

The complex ternary algebra has a dual nature: on one side, it is a one–dimensional (one ternary variable) theory generated by an element that cubes to ± 1, on the other it behaves as a theory of one bicomplex variable and one complex variable entangled by algebra relations. In the general case, these types of algebras have known a resurgence in importance. The ternary algebras are also related to certain symmetries in high energy physics allowing one to explain, for example, the three color and three family problems of the Standard Model. Moreover, ternary algebras are known to be a strong candidate for the *algebraic confinement* model for the problem of observability of three quarks/anti–quarks in Quantum Field Theory.

The paper is structured as follows: in Sect. 2 we define the ternary complex algebra structure, then we describe the various types of norms in this case in Sect. 3. In Sect. 4 we describe the analytic function theory in this case, while in Sect. 5 we describe several applications to Cauchy–Kowalevskaya type Theorem in several cases. We conclude with a small Sect. 6 of conclusions and future planned work.

2 Complex Ternary Algebra

The *ternary algebra* is a particular case of a *hypercomplex algebra* $\mathbb{HF}_3$ of dimension three (see [6]), initially constructed from a generic unit (i.e. not in the field F) that cubes to ± 1 and extended linearly over a field $\mathbb{F}$ (in our case $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$), as follows. Consider three units

$$\mathbf{e}_1 = \mathbf{1}, \quad \mathbf{e}_2, \quad \mathbf{e}_3 = \mathbf{e}_2^2, \tag{2.1}$$

where $\mathbf{e}_2^3 = 1$, having the commutative multiplication table:

$\cdot$	$\mathbf{1}$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{1}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{1}$	$\mathbf{e}_2$

(2.2)

The case when $\mathbf{e}_2^3 = -1$ is very similar and has been introduced in [5]. Following [6], the structure constants matrices are:

$$\Gamma_1 = (\Gamma_{1j}^{\ell})_{\ell j} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_2 = (\Gamma_{2j}^{\ell})_{\ell j} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\Gamma_3 = (\Gamma_{3j}^{\ell})_{\ell j} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

Notice that $\Gamma_{km}^{\ell} = \Gamma_{mk}^{\ell}$, equivalent to the fact that $\mathbb{HF}_3$ is a commutative algebra. Moreover, $\Gamma_2 = (\Gamma_3)^t$. A *ternary number* is written as

$$\mathbf{z} = z_1\mathbf{1} + z_2\mathbf{e}_2 + z_3\mathbf{e}_3, \tag{2.4}$$

where $z_\ell \in \mathbb{F}$. Following [6], the *associated matrix* of the ternary number is given by the circulant matrix $S(\mathbf{z})$:

$$S(\mathbf{z}) = \begin{bmatrix} z_1 & z_3 & z_2 \\ z_2 & z_1 & z_3 \\ z_3 & z_2 & z_1 \end{bmatrix} \in \mathcal{M}_3(\mathbb{F}). \tag{2.5}$$

Remark 2.1 In this case the transpose of the structure matrix is given by:

$$S(\mathbf{z})^t = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_3 & z_1 & z_2 \\ z_2 & z_3 & z_1 \end{bmatrix}, \tag{2.6}$$

which corresponds to the ternary number $\mathbf{z}^t = z_1\mathbf{1} + z_3\mathbf{e}_2 + z_2\mathbf{e}_3$, and we have:

$$\begin{aligned} \mathbf{z}\mathbf{z}^t &= (z_1\mathbf{1} + z_2\mathbf{e}_2 + z_3\mathbf{e}_3)(z_1\mathbf{1} + z_3\mathbf{e}_2 + z_2\mathbf{e}_3) \\ &= (z_1^2 + z_2^2 + z_3^2)\mathbf{1} + (z_1z_2 + z_2z_3 + z_3z_1)\mathbf{e}_2 + (z_1z_2 + z_2z_3 + z_3z_1)\mathbf{e}_3. \end{aligned}$$

Note also that $\mathbf{e}_2^t = \mathbf{e}_3$, thus $\mathbf{e}_3^t = \mathbf{e}_2$, and $\mathbf{e}_2\mathbf{e}_2^t = \mathbf{e}_3\mathbf{e}_3^t = 1$.

Remark 2.2 In the case of the mirror ternary algebra given by $\mathbf{e}_2^3 = -1$ one obtains, as expected:

$$S(\mathbf{z}) = \begin{bmatrix} z_1 & z_3 & z_2 \\ -z_2 & z_1 & z_3 \\ -z_3 & -z_2 & z_1 \end{bmatrix} \in \mathcal{M}_3(\mathbb{F}). \tag{2.7}$$

Here the matrix is called *anticirculant*, as studied in [20].

In this paper we study the case when the scalar field $\mathbb{F} = \mathbb{C}$ and we denote the complex ternary algebra by $\mathbb{TC}$. We now prove a primary decomposition theorem in this case:

Theorem 2.3 *The algebra* $\mathbb{TC}$ *decomposes as a direct sum of three ideals:*

$$\mathbb{TC} = \langle \epsilon_1 \rangle \oplus \langle \epsilon_2 \rangle \oplus \langle \epsilon_3 \rangle,$$

where $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ *is an* idempotent basis *of* $\mathbb{TC}$*, given by:*

$$\epsilon_1 = \frac{1}{3}(\mathbf{1} + \mathbf{e}_2 + \mathbf{e}_3), \qquad \epsilon_2 = \frac{1}{3}(\mathbf{1} + \psi \mathbf{e}_2 + \psi^2 \mathbf{e}_3), \qquad \epsilon_3 = \frac{1}{3}(\mathbf{1} + \psi^2 \mathbf{e}_2 + \psi \mathbf{e}_3), \tag{2.8}$$

where ψ *is the complex primary root of unity, with the usual properties. Therefore, any complex ternary number* $\mathbf{z} \in \mathbb{TC}$ *has the following idempotent representation:*

$$\mathbf{z} = \lambda_1(\mathbf{z})\epsilon_1 + \lambda_2(\mathbf{z})\epsilon_2 + \lambda_3(\mathbf{z})\epsilon_3, \tag{2.9}$$

where $\lambda_\ell(\mathbf{z}) \in \mathbb{C}$ *are the* idempotent coordinates *of* $\mathbf{z}$*.*

Proof As the complex ternary algebra $\mathbb{TC}$ is commutative, for simplicity we compute the eigenvalues and eigenvectors of the matrix associated to the units $\mathbf{e}_2$, then $\mathbf{e}_3$ and then we use linearity for determining the eigenvalues of a general complex ternary number $\mathbf{z}$. We have:

$$S(\mathbf{e}_2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad S(\mathbf{e}_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

The two characteristic polynomials are the same:

$$\begin{aligned} \sigma_{\mathbf{e}_2}(\lambda) = \sigma_{\mathbf{e}_3}(\lambda) &= \det(\lambda I - S(\mathbf{e}_2)) = \det(\lambda I - S(\mathbf{e}_3)) \\ &= \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1), \end{aligned}$$

thus there are three simple complex eigenvalues in each case. For $\ell = 2$, we denote:

$$\lambda_1(\mathbf{e}_2) = 1, \quad \lambda_2(\mathbf{e}_2) = \overline{\psi} = \frac{1}{2}(-1 - \sqrt{3}\mathbf{i}) \in \mathbb{C}, \quad \lambda_3(\mathbf{e}_2) = \psi = \frac{1}{2}(-1 + \sqrt{3}\mathbf{i}) \in \mathbb{C},$$

where $\psi, \overline{\psi} \in \mathbb{C}$ are the usual non–trivial complex roots of 1. Since $\mathbf{e}_3 = \mathbf{e}_2^2$, and $\mathbf{e}_2$ has simple eigenvalues and obviously commutes with $\mathbf{e}_3$, we obtain: $\lambda_k(\mathbf{e}_3) = \lambda_k(\mathbf{e}_2)^2$, for $1 \le k \le 3$, and we have the corresponding eigenvalues for $\mathbf{e}_3$:

$$\lambda_1(\mathbf{e}_3) = 1, \qquad \lambda_2(\mathbf{e}_3) = \overline{\psi}^2 = \psi, \qquad \lambda_3(\mathbf{e}_3) = \psi^2 = \overline{\psi}.$$

Three corresponding linear independent eigenvectors of $S(\mathbf{e}_2)$ and $S(\mathbf{e}_3)$ are:

$$\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \overrightarrow{u_2} = \begin{bmatrix} 1 \\ \psi \\ \psi^2 \end{bmatrix}, \quad \overrightarrow{u_3} = \begin{bmatrix} 1 \\ \psi^2 \\ \psi \end{bmatrix},$$

thus their corresponding complex ternary numbers are:

$$\mathbf{u}_1 = \mathbf{1} + \mathbf{e}_2 + \mathbf{e}_3, \qquad \mathbf{u}_2 = \mathbf{1} + \psi\mathbf{e}_2 + \psi^2\mathbf{e}_3, \qquad \mathbf{u}_3 = \mathbf{1} + \psi^2\mathbf{e}_2 + \psi\mathbf{e}_3.$$

Therefore, for a generic $\mathbf{z} = \sum_{k=1}^{3} z_k\mathbf{e}_k \in \mathbb{TC}$, the eigenvalue problem of $S(\mathbf{z})$ yields the following expressions for the eigenvalues of a generic complex ternary number:

$$\begin{aligned}
\lambda_1(\mathbf{z}) &= z_1 + z_2\lambda_1(\mathbf{e}_2) + z_3\lambda_1(\mathbf{e}_3) = z_1 + z_2 + z_3, \\
\lambda_2(\mathbf{z}) &= z_1 + z_2\lambda_2(\mathbf{e}_2) + z_3\lambda_2(\mathbf{e}_3) = z_1 + \psi^2 z_2 + \psi z_3, \\
\lambda_3(\mathbf{z}) &= z_1 + z_2\lambda_3(\mathbf{e}_2) + z_3\lambda_3(\mathbf{e}_3) = z_1 + \psi z_2 + \psi^2 z_3.
\end{aligned}$$

The characteristic polynomial of $\mathbf{z}$ is given by:

$$\sigma_{\mathbf{z}}(X) = X^3 - 3z_1X^2 + (3z_1^2 - 3z_1z_2z_3)X - \det S(\mathbf{z}),$$

where

$$\begin{aligned}
\lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z}) &= \text{trace}(S(\mathbf{z})) = 3z_1, \\
\lambda_1(\mathbf{z})\lambda_2(\mathbf{z}) + \lambda_1(\mathbf{z})\lambda_2(\mathbf{z}) + \lambda_2(\mathbf{z})\lambda_3(\mathbf{z}) &= 3z_1^2 - 3z_1z_2z_3, \\
\lambda_1(\mathbf{z})\lambda_2(\mathbf{z})\lambda_3(\mathbf{z}) &= \det(S(\mathbf{z})) = z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3.
\end{aligned}$$

It follows that any complex ternary number $\mathbf{z} = \sum_{\ell=1}^{3} z_\ell\mathbf{e}_\ell \in \mathbb{TC}$ is written in both *idempotent* and *standard* bases as:

$$\begin{aligned}
\mathbf{z} &= \lambda_1(\mathbf{z})\epsilon_1 + \lambda_2(\mathbf{z})\epsilon_2 + \lambda_3(\mathbf{z})\epsilon_3 \\
&= \frac{1}{3}\left(\lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z})\right)\mathbf{1} + \frac{1}{3}(\lambda_1(\mathbf{z}) + \psi\lambda_2(\mathbf{z}) + \psi^2\lambda_3(\mathbf{z}))\mathbf{e}_2 \\
&\quad + \frac{1}{3}(\lambda_1(\mathbf{z}) + \psi^2\lambda_2(\mathbf{z}) + \psi\lambda_3(\mathbf{z}))\mathbf{e}_3.
\end{aligned}$$

This gives the inverse formulas for the change of basis:

$$z_1 = \frac{1}{3}\left(\lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z})\right),$$

$$z_2 = \frac{1}{3}(\lambda_1(\mathbf{z}) + \psi\lambda_2(\mathbf{z}) + \psi^2\lambda_3(\mathbf{z})),$$

$$z_3 = \frac{1}{3}(\lambda_1(\mathbf{z}) + \psi^2\lambda_2(\mathbf{z}) + \psi\lambda_3(\mathbf{z})),$$

and the decomposition of $\mathbb{TC}$ follows immediately. □

Remark 2.4 We chose to include the computational details of the previous theorem, as it gives a solid foundation to the seemingly "magical" choices found in literature in [1, 14].

2.1 Conjugates and the Idempotent Representation

As in the other commutative theory of bicomplex and multicomplex algebras [17, 19], using the idempotent representation we can build two associated conjugates to each ternary number:

Definition 2.5 For the ternary number $z = \sum_{k=1}^{3} \lambda_k(z)\epsilon_k$ we define:

$$z^\dagger = \sum_{k=1}^{3} \lambda_{[k+1]}(z)\epsilon_k,$$

$$z^* = \sum_{k=1}^{3} \lambda_{[k+2]}(z)\epsilon_k,$$

where the square brackets represent permutations of the λ's. (E.g. $\lambda_{[2+2]} = \lambda_1$.)

Example For the ternary number $\mathbf{e}_2$, written in this idempotent representation as $\mathbf{e}_2 = \sum_{k=1}^{3} \lambda_k(\mathbf{e}_2)\epsilon_k = \epsilon_1 + \psi^2\epsilon_2 + \psi\epsilon_3$, we have these two conjugates:

$$\mathbf{e}_2^\dagger = \sum_{k=1}^{3} \lambda_{[k+1]}(\mathbf{e}_2)\epsilon_k = \psi\epsilon_1 + \epsilon_2 + \psi^2\epsilon_3 = \psi\mathbf{e}_2,$$

$$\mathbf{e}_2^* = \sum_{k=1}^{3} \lambda_{[k+2]}(\mathbf{e}_2)\epsilon_k = \psi^2\epsilon_1 + \psi\epsilon_2 + \epsilon_3 = \psi^2\mathbf{e}_2.$$

Similarly, since $\mathbf{e}_3 = (\mathbf{e}_2)^2$, we have:

$$\mathbf{e}_3 = \epsilon_1 + \psi\epsilon_2 + \psi^2\epsilon_3,$$
$$\mathbf{e}_3^\dagger = \psi^2\epsilon_1 + \epsilon_2 + \psi\epsilon_3 = \psi^2\mathbf{e}_3,$$
$$\mathbf{e}_3^* = \psi\epsilon_1 + \psi^2\epsilon_2 + \epsilon_3 = \psi\mathbf{e}_3.$$

Note that

$$\mathbf{e}_2\mathbf{e}_2^\dagger = \psi\mathbf{e}_2^2 = \psi\mathbf{e}_3, \qquad \mathbf{e}_2\mathbf{e}_2^* = \psi^2\mathbf{e}_2^2 = \psi^2\mathbf{e}_3, \qquad \mathbf{e}_2^\dagger\mathbf{e}_2^* = \mathbf{e}_2^2 = \mathbf{e}_3,$$

and we have the following:

Remark 2.6 A simple consequence of the fact that the characteristic polynomial $\mathbf{e}_2$ is equal to $S(\mathbf{e}_2) = \lambda^3 - 1$ is:

$$\mathbf{e}_2 + \mathbf{e}_2^\dagger + \mathbf{e}_2^* = \mathbf{e}_2\mathbf{e}_2^\dagger + \mathbf{e}_2\mathbf{e}_2^* + \mathbf{e}_2^\dagger\mathbf{e}_2^* = 0$$
$$\mathbf{e}_2\mathbf{e}_2^\dagger\mathbf{e}_2^* = \mathbf{1} = \det(S(\mathbf{e}_2)).$$

Remark 2.7 The two conjugations are not involutive, as:

$$\epsilon_1^\dagger = \epsilon_2, \qquad \epsilon_2^\dagger = \epsilon_3, \qquad \epsilon_3^\dagger = \epsilon_1,$$

and

$$\epsilon_1^* = \epsilon_3, \qquad \epsilon_2^* = \epsilon_1, \qquad \epsilon_3^* = \epsilon_2.$$

Lemma 2.8 *For a generic* $\mathbf{z} = \sum_{k=1}^{3} z_k\mathbf{e}_k = z_1\mathbf{1} + z_2\mathbf{e}_2 + z_3\mathbf{e}_3, \in \mathbb{TC}$, *the two conjugates of* $\mathbf{z}$ *are given by:*

$$\mathbf{z}^\dagger = z_1\mathbf{1} + \psi z_2\mathbf{e}_2 + \psi^2 z_3\mathbf{e}_3,$$
$$\mathbf{z}^* = z_1\mathbf{1} + \psi^2 z_2\mathbf{e}_2 + \psi z_3\mathbf{e}_3.$$

Proof To show the intricacies of the conjugations, we write the following computation which yields the proof:

$$\begin{aligned}\mathbf{z}^\dagger &= (\lambda_1(\mathbf{z})\epsilon_1 + \lambda_2(\mathbf{z})\epsilon_2 + \lambda_3(\mathbf{z})\epsilon_3)^\dagger = \lambda_3(\mathbf{z})\epsilon_1 + \lambda_1(\mathbf{z})\epsilon_2 + \lambda_2(\mathbf{z})\epsilon_3 \\ &= \frac{1}{3}(\lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z}))\mathbf{1} + \frac{1}{3}\psi(\lambda_1(\mathbf{z}) + \psi\lambda_2(\mathbf{z}) + \psi^2\lambda_3(\mathbf{z}))\mathbf{e}_2 \\ &\quad + \frac{1}{3}\psi^2(\lambda_1(\mathbf{z}) + \psi^2\lambda_2(\mathbf{z}) + \psi\lambda_3(\mathbf{z}))\mathbf{e}_3 \\ &= z_1\mathbf{1} + \psi z_2\mathbf{e}_2 + \psi^2 z_3\mathbf{e}_3,\end{aligned}$$

and

$$\begin{aligned}
\mathbf{z}^* &= (\lambda_1(\mathbf{z})\epsilon_1 + \lambda_2(\mathbf{z})\epsilon_2 + \lambda_3(\mathbf{z})\epsilon_3)^* \\
&= \lambda_2(\mathbf{z})\epsilon_1 + \lambda_3(\mathbf{z})\epsilon_2 + \lambda_1(\mathbf{z})\epsilon_3 \\
&= \frac{1}{3}(\lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z}))\mathbf{1} + \frac{1}{3}\psi^2(\lambda_1(\mathbf{z}) + \psi\lambda_2(\mathbf{z}) + \psi^2\lambda_3(\mathbf{z}))\mathbf{e}_2 \\
&\quad + \frac{1}{3}\psi(\lambda_1(\mathbf{z}) + \psi^2\lambda_2(\mathbf{z}) + \psi\lambda_3(\mathbf{z}))\mathbf{e}_3 \\
&= z_1\mathbf{1} + \psi^2 z_2\mathbf{e}_2 + \psi z_3\mathbf{e}_3.
\end{aligned}$$

□

It is also true that both conjugations are linear operators, so they can be computed directly from the conjugations of the standard basis elements:

$$\begin{aligned}
\mathbf{z}^\dagger &= z_1\mathbf{1} + z_2\mathbf{e}_2^\dagger + z_3\mathbf{e}_3^\dagger, \\
\mathbf{z}^* &= z_1\mathbf{1} + z_2\mathbf{e}_2^* + z_3\mathbf{e}_3^*.
\end{aligned}$$

Remark 2.9 We have the following useful relations:

$$\begin{aligned}
\mathbf{z} + \mathbf{z}^\dagger + \mathbf{z}^* &= \lambda_1(\mathbf{z}) + \lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z}) = 3z_1\mathbf{1}, \\
\mathbf{z}\mathbf{z}^\dagger\mathbf{z}^* &= \lambda_1(\mathbf{z})\lambda_2(\mathbf{z})\lambda_3(\mathbf{z}) \cdot \mathbf{1} = \det(S(\mathbf{z}))\mathbf{1} = (z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3)\mathbf{1}, \\
(\mathbf{z}^\dagger)^\dagger &= \mathbf{z}^*, \quad (\mathbf{z}^\dagger)^* = \mathbf{z}, \quad (\mathbf{z}^*)^* = \mathbf{z}^\dagger.
\end{aligned}$$

Since neither of these conjugates is an involution, we will give them the moniker of *simple conjugates* instead.

Definition 2.10 We define the *total conjugate* of $\mathbf{z}$ to be the product of the simple ones, i.e. $\overline{\mathbf{z}} := (\mathbf{z}^\dagger)(z^*)$.

Remark 2.11 This conjugate is also not involutive, however, since $\mathbf{z}\overline{\mathbf{z}} = \det(S(\mathbf{z}))\mathbf{1}$ we can use it to define an inverse for the ternary complex number.

2.2 An Alternative Basis

In an algebra of this type there are important elements that help define additional analytic structures; these are obtained from the usual basis through a linear transformation. We now mention a useful alternative basis that simplifies calculations and relates the ternary complex algebra with other known examples. This new basis turns out to be very useful also in the study of the real ternary algebra $\mathbb{TR}$.

Definition 2.12 The *symmetric basis* for $\mathbb{TC}$ is defined as follows:

$$\kappa_1 := \epsilon_1 = \frac{1}{3}(\mathbf{1} + \mathbf{e}_2 + \mathbf{e}_3), \qquad \kappa_2 := \epsilon_2 + \epsilon_3 = \frac{1}{3}(2 - \mathbf{e}_2 - \mathbf{e}_3),$$
$$\kappa_3 := -\mathbf{i}(\epsilon_2 - \epsilon_3) = \frac{1}{\sqrt{3}}(\mathbf{e}_2 - \mathbf{e}_3).$$

It is obvious that $\{\kappa_1, \kappa_2, \kappa_3\}$ is a basis of $\mathbb{TC}$ and we have the following properties for the symmetric basis elements:

$$(\kappa_1)^2 = \epsilon_1^2 = \epsilon_1 = \kappa_1, \qquad (\kappa_2)^2 = \epsilon_2^2 + \epsilon_3^2 = \epsilon_2 + \epsilon_3 = \kappa_2,$$
$$(\kappa_3)^2 = -(\epsilon_2^2 + \epsilon_3^2) = -(\epsilon_2 + \epsilon_3) = -\kappa_2,$$
$$\kappa_2\kappa_3 = -\mathbf{i}(\epsilon_2^2 - \epsilon_3^2) = -\mathbf{i}(\epsilon_2 - \epsilon_3) = \kappa_3, \qquad \kappa_1\kappa_2 = \kappa_1\kappa_3 = 0, \qquad \kappa_1 + \kappa_2 = \mathbf{1}. \tag{2.11}$$

The inverse change of base formulae from the symmetric to the idempotent ones are given by:

$$\epsilon_1 = \kappa_1, \qquad \epsilon_2 = \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3), \qquad \epsilon_3 = \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3).$$

Considering an arbitrary complex ternary number $\mathbf{z}$, we can now write it in all three standard, idempotent and symmetric bases as follows:

$$\mathbf{z} = \sum_{\ell=1}^{3} z_\ell \mathbf{e}_\ell = \sum_{k=1}^{3} \lambda_k(\mathbf{z})\epsilon_k = \sum_{j=1}^{3} w_j(\mathbf{z})\kappa_j,$$

where the symmetric coordinates $w_j(\mathbf{z})$ are given by:

$$\begin{aligned}\mathbf{z} &= \lambda_1(\mathbf{z})\kappa_1 + \frac{1}{2}(\lambda_2(\mathbf{z}) + \lambda_3(\mathbf{z}))\kappa_2 + \frac{\mathbf{i}}{2}(\lambda_2(\mathbf{z}) - \lambda_3(\mathbf{z}))\kappa_3 \\ &= (z_1 + z_2 + z_3)\kappa_1 + \frac{1}{2}(2z_1 - z_2 - z_3)\kappa_2 + \frac{\sqrt{3}}{2}(z_2 - z_3)\kappa_3 \\ &=: w_1(\mathbf{z})\kappa_1 + w_2(\mathbf{z})\kappa_2 + w_3(\mathbf{z})\kappa_3. \end{aligned} \tag{2.12}$$

For simplicity of notation we will omit $(\mathbf{z})$ from now on, in all coordinates corresponding to the three types of basis.

The simple conjugations of the symmetric basis elements in $\mathbb{TC}$ are as follows:

$$\kappa_1^\dagger = \epsilon_1^\dagger = \epsilon_2 = \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3),$$

$$\kappa_2^\dagger = \epsilon_2^\dagger + \epsilon_3^\dagger = \epsilon_3 + \epsilon_1 = \kappa_1 + \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3),$$

$$\kappa_3^\dagger = -\mathbf{i}(\epsilon_2^\dagger - \epsilon_3^\dagger) = \mathbf{i}(\epsilon_1 - \epsilon_3) = \mathbf{i}\kappa_1 - \frac{\mathbf{i}}{2}(\kappa_2 - \mathbf{i}\kappa_3).$$

For an arbitrary ternary number $\mathbf{z} = w_1\kappa_1 + w_2\kappa_2 + w_3\kappa_3 \in \mathbb{TC}$, we obtain:

$$\begin{aligned}\mathbf{z}^\dagger &= w_1\kappa_1^\dagger + w_2\kappa_2^\dagger + w_3\kappa_3^\dagger \\ &= (w_2 + \mathbf{i}w_3)\kappa_1 + \frac{1}{2}(w_1 + w_2 - \mathbf{i}w_3)\kappa_2 + \frac{\mathbf{i}}{2}(w_1 - w_2 + \mathbf{i}w_3)\kappa_3 \end{aligned} \tag{2.13}$$

Similarly, the $*$–conjugates of κ_ℓ are:

$$\kappa_1^* = \epsilon_1^* = \epsilon_3 = \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3),$$

$$\kappa_2^* = \epsilon_2^* + \epsilon_3^* = \epsilon_1 + \epsilon_2 = \kappa_1 + \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3),$$

$$\kappa_3^* = -\mathbf{i}(\epsilon_2^* - \epsilon_3^*) = -\mathbf{i}(\epsilon_1 - \epsilon_2) = -\mathbf{i}\kappa_1 + \frac{\mathbf{i}}{2}(\kappa_2 + \mathbf{i}\kappa_3),$$

therefore we have:

$$\begin{aligned}\mathbf{z}^* &= w_1\kappa_1^* + w_2\kappa_2^* + w_3\kappa_3^* \\ &= (w_2 - \mathbf{i}w_3)\kappa_1 + \frac{1}{2}(w_1 + w_2 + \mathbf{i}w_3)\kappa_2 + \frac{\mathbf{i}}{2}(-w_1 + w_2 + \mathbf{i}w_3)\kappa_3. \end{aligned} \tag{2.14}$$

The total conjugate is given by:

$$\begin{aligned}\overline{\mathbf{z}} = \mathbf{z}^\dagger\mathbf{z}^* &= (z_1^2 - z_2z_3)\mathbf{1} + (z_3^2 - z_1z_2)\mathbf{e}_2 + (z_2^2 - z_1z_3)\mathbf{e}_3 \\ &= \lambda_2(\mathbf{z})\lambda_3(\mathbf{z})\epsilon_1 + \lambda_1(\mathbf{z})\lambda_3(\mathbf{z})\epsilon_2 + \lambda_1(\mathbf{z})\lambda_2(\mathbf{z})\epsilon_3 \\ &= (w_2^2 + w_3^2)\kappa_1 + (w_1w_2)\kappa_2 - (w_1w_3)\kappa_3, \end{aligned}$$

and the determinant of $S(\mathbf{z})$ is given by:

$$\det(S(\mathbf{z})) = \mathbf{z}\mathbf{z}^\dagger\mathbf{z}^* = w_1(w_2^2 + w_3^2) \in \mathbb{C}.$$

2.3 The Real Ternary Algebra

An interesting application of the complex case becomes the real ternary case, as studies in [5] In this section we restrict a complex ternary number to real coordinates, i.e. $\mathbf{z} =: \mathbf{x} = \sum_{\ell=1}^{3} x_\ell \mathbf{e}_\ell$, with $x_\ell \in \mathbb{R}$. We denote the real ternary algebra by $\mathbb{TR}$.

Remark 2.13 The matrix $S(\mathbf{x})$ has the same form (2.5) only that it has real entries. However, its eigenvalues and eigenvectors take $\mathbb{C}(\mathbf{i})$–complex valued coefficients:

$$\lambda_1(\mathbf{x}) = x_1 + x_2 + x_3, \qquad \lambda_2(\mathbf{x}) = x_1 + \psi^2 x_2 + \psi x_3, \qquad \lambda_3(\mathbf{x}) = x_1 + \psi x_2 + \psi^2 x_3.$$

Nevertheless, the sum and difference of $\lambda_2(\mathbf{x})$ and $\lambda_3(\mathbf{x})$ have either real or purely imaginary coefficients, which will allow the coefficients of $\mathbf{x}$ in the symmetric base to remain real:

$$\lambda_2(\mathbf{x}) + \lambda_3(\mathbf{x}) = 2x_1 + (\psi^2 + \psi)x_2 + (\psi + \psi^2)x_3 = 2x_1 - x_2 - x_3,$$
$$\lambda_2(\mathbf{x}) - \lambda_3(\mathbf{x}) = (\psi^2 - \psi)x_2 + (\psi - \psi^2)x_3 = -\sqrt{3}\mathbf{i}(x_2 - x_3).$$

In this view, the symmetric basis comes to the rescue, as it becomes the only valuable way to intrinsically describe $\mathbb{TR}$. We noticed that the elements of the complex symmetric basis $\{\kappa_1, \kappa_2, \kappa_3\}$ are already in $\mathbb{TR}$. Adapting the writings (2.12) for a real ternary number $\mathbf{x}$, we obtain:

$$\begin{aligned}\mathbf{x} &= \lambda_1(\mathbf{x})\kappa_1 + \frac{1}{2}(\lambda_2(\mathbf{x}) + \lambda_3(\mathbf{x}))\kappa_2 + \frac{\mathbf{i}}{2}(\lambda_2(\mathbf{x}) - \lambda_3(\mathbf{x}))\kappa_3 \\ &= (x_1 + x_2 + x_3)\kappa_1 + \frac{1}{2}(2x_1 - x_2 - x_3)\kappa_2 + \frac{\sqrt{3}}{2}(x_2 - x_3)\kappa_3 \\ &= w_1(\mathbf{x})\kappa_1 + w_2(\mathbf{x})\kappa_2 + w_3(\mathbf{x})\kappa_3, \end{aligned} \tag{2.15}$$

with $w_\ell(\mathbf{x}) \in \mathbb{R}$. The change of base formulas from the standard basis to the real symmetric basis are:

$$w_1(\mathbf{x}) = x_1 + x_2 + x_3, \quad w_2(\mathbf{x}) = \frac{1}{2}(2x_1 - x_2 - x_3), \quad w_3(\mathbf{x}) = \frac{\sqrt{3}}{2}(x_2 - x_3),$$
$$x_1 = \frac{1}{3}(w_1(\mathbf{x}) + 2w_2(\mathbf{x})), \quad x_2 = \frac{1}{3}(w_1(\mathbf{x}) - w_2(\mathbf{x}) + \sqrt{3}w_3(\mathbf{x})),$$
$$x_3 = \frac{1}{3}(w_1(\mathbf{x}) - w_2(\mathbf{x}) - \sqrt{3}w_3(\mathbf{x})).$$

Remark 2.14 A real ternary number can be viewed, roughly, as a split in a real number $w_1(\mathbf{x}) \in \mathbb{R}$ and a complex–type number $\zeta(\mathbf{x}) \in \mathbb{C}[\kappa_2; \kappa_3]$, as $\kappa_3^2 = -\kappa_2$:

$$\mathbf{x} = w_1(\mathbf{x})\kappa_1 + \zeta(\mathbf{x})\kappa_2 := w_1(\mathbf{x})\kappa_1 + (w_2(\mathbf{x})\kappa_2 + w_3(\mathbf{x})\kappa_3)\,\kappa_2, \tag{2.16}$$

with κ_1 and κ_2 being a partition of the unity: $\kappa_1 + \kappa_2 = \mathbf{1}$.

This fact does not imply that $\mathbb{TR}$ is merely a product $\mathbb{C} \times \mathbb{R}$, the same as $\mathbb{R} \times \mathbb{R}$ is not the same as $\mathbb{C}$. The algebra allowing multiplication of "vectors" in $\mathbb{TR}$ plays a very important role, as it does in the traditional complex case.

Multiplication in the writing (2.16) is realized component–wise, as $\kappa_1\kappa_2 = 0$:

$$\mathbf{xy} = w_1(\mathbf{x})w_1(\mathbf{y})\kappa_1 + \zeta(\mathbf{x})\zeta(\mathbf{y})\kappa_2,$$

where $\zeta(\mathbf{x})\zeta(\mathbf{y})$ is the complex product in $\mathbb{C}[\kappa_2; \kappa_3]$.

For $\mathbf{x} \in \mathbb{TR}$, its conjugates $\mathbf{x}^\dagger$ and $\mathbf{x}^*$ are not in $\mathbb{TR}$, this would happen only when $x_2 = x_3 = 0$, i.e. $\mathbf{x} \in \mathbb{R}$. Nevertheless, the total conjugate of a non–singular real ternary number $\mathbf{x}$ stays in $\mathbb{TR}$ (omitting $(\mathbf{x})$ from notation):

$$\begin{aligned}\overline{\mathbf{x}} = \mathbf{x}^\dagger\mathbf{x}^* &= (x_1^2 - x_2x_3)\mathbf{1} + (x_3^2 - x_1x_2)\mathbf{e}_2 + (x_2^2 - x_1x_3)\mathbf{e}_3 \\ &= (w_2^2 + w_3^2)\kappa_1 + (w_1w_2)\kappa_2 - (w_1w_3)\kappa_3 \in \mathbb{TR}.\end{aligned}$$

We denote by $\tilde{\zeta}(\mathbf{x}) := w_2(\mathbf{x})\kappa_2 - w_3(\mathbf{x})\kappa_3$ the $\mathbb{C}[\kappa_2; \kappa_3]$–complex conjugate of $\zeta(\mathbf{x})$ and its $\mathbb{C}[\kappa_2; \kappa_3]$–*complex norm* by

$$|\zeta(\mathbf{x})|^2\kappa_2 := \zeta(\mathbf{x})\tilde{\zeta}(\mathbf{x}) = \left(w_2(\mathbf{x})^2 + w_3(\mathbf{x})^2\right)\kappa_2,$$

so $|\zeta(\mathbf{x})|^2 := w_2(\mathbf{x})^2 + w_3(\mathbf{x})^2 \in \mathbb{R}_+$. Then the total conjugate of $\mathbf{x}$ becomes:

$$\overline{\mathbf{x}} = |\zeta(\mathbf{x})|^2\kappa_1 + \tilde{\zeta}(\mathbf{x})w_1(\mathbf{x})\kappa_2.$$

The determinant of $S(\mathbf{x})$ is a real number:

$$\det(S(\mathbf{x})) = \mathbf{x}\overline{\mathbf{x}} = w_1(\mathbf{x})(w_2(\mathbf{x})^2 + w_3(\mathbf{x})^2) = w_1(\mathbf{x})|\zeta(\mathbf{x})|^2 \in \mathbb{R}.$$

Note that, as expected:

$$\det(S(\overline{\mathbf{x}})) = (w_2^2 + w_3^2)((w_1w_2)^2 + (w_1w_3)^2) = w_1^2(w_2^2 + w_3^2)^2 = \det(S(\mathbf{x}))^2.$$

Moreover, computing

$$\begin{aligned}\overline{\overline{\mathbf{x}}} = \overline{\mathbf{x}^\dagger\mathbf{x}^*} &= (\mathbf{x}^\dagger\mathbf{x}^*)^\dagger(\mathbf{x}^\dagger\mathbf{x}^*)^* \\ &= \mathbf{x}^*\mathbf{x}^2\mathbf{x}^\dagger = \det(S(\mathbf{x}))\mathbf{x},\end{aligned}$$

we obtain that the total conjugation is not an involution, but a dilation with a factor of $\det(S(\mathbf{x}))$. When $\mathbf{x}$ is a zero–divisor, i.e. $\det(S(\mathbf{x})) = 0$, then $\overline{\overline{\mathbf{x}}} = 0$. A real ternary number $\mathbf{x}$ is a zero–divisor if and only if:

$$w_1(w_2^2 + w_3^2) = 0 \iff w_1 = 0 \text{ or } \{w_2 = w_3 = 0\}.$$

Using the notation (2.16), $\mathbf{x}$ is a zero–divisor if and only if $w_1(\mathbf{x}) = 0$ or $\zeta(\mathbf{x}) = 0$. In case $\mathbf{x}$ is invertible in $\mathbb{TR}$, its inverse is:

$$\begin{aligned}\frac{1}{\mathbf{x}} = \frac{\overline{\mathbf{x}}}{\det(S(\mathbf{x}))} &= \frac{1}{w_1}\kappa_1 + \frac{w_2}{(w_2^2 + w_3^2)}\kappa_2 - \frac{w_3}{(w_2^2 + w_3^2)}\kappa_3 \\ &= \frac{1}{w_1(\mathbf{x})}\kappa_1 + \frac{\tilde{\zeta}(\mathbf{x})}{|\zeta(\mathbf{x})|^2}\kappa_2.\end{aligned}$$

2.4 From Complex Ternary to Bicomplex Algebras

Using relations (2.11) among the basis ternary complex numbers κ_ℓ, we write a generic complex ternary number $\mathbf{z}$ as follows:

$$\mathbf{z} = w_1(\mathbf{z})\kappa_1 + (w_2(\mathbf{z})\kappa_2 + w_3(\mathbf{z})\kappa_3)\,\kappa_2 =: w_1(\mathbf{z})\kappa_1 + Z(\mathbf{z})\kappa_2, \tag{2.17}$$

where $Z(\mathbf{z}) = w_2(\mathbf{z})\kappa_2 + w_3(\mathbf{z})\kappa_3$ is a bicomplex number in the algebra over the scalar field $\mathbb{C}(\mathbf{i})$ generated by the "imaginary" unit κ_3. This is a two–dimensional unital hypercomplex algebra, with multiplicative unit κ_2 and relation $\kappa_3^2 = -\kappa_2$.

Remark 2.15 This bicomplex algebra is a subalgebra of $\mathbb{TC}$, but not a unital one, as its multiplicative unit, κ_2, is not equal to the ternary multiplicative unit $\mathbf{1} \in \mathbb{TC}$. This fact implies that there are three bicomplex–type subalgebras of $\mathbb{TC}$, thus the complex ternary numbers inherit many of the properties of bicomplex numbers. This is in fact part of a general fact of hypercomplex algebras which have elements with simple (complex) eigenvalues.

Note though that this is not a unital subalgebra of $\mathbb{TC}$, κ_2 is still part of a partition of $\mathbf{1} = \kappa_1 + \kappa_2$. Moreover, note that

$$\lambda_2(\mathbf{z})\kappa_2 = (w_2(\mathbf{z}) - \mathbf{i}w_3(\mathbf{z}))\kappa_2, \qquad \lambda_3(\mathbf{z})\kappa_2 = (w_2(\mathbf{z}) + \mathbf{i}w_3(\mathbf{z}))\kappa_2$$

are the bicomplex $\mathbb{C}(\mathbf{i})$–idempotent coordinates of $Z(\mathbf{z})$ (see [15]), over idempotent basis generated by:

$$\epsilon_2 = \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3), \qquad \epsilon_3 = \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3),$$

having the properties:

$$\epsilon_2\epsilon_3 = 0, \quad \epsilon_2 + \epsilon_3 = \kappa_2, \quad (\epsilon_2)^2 = \epsilon_2, \quad (\epsilon_3)^2 = \epsilon_3.$$

Definition 2.16 We denote this bicomplex algebra by $\mathbb{BC}_2[\kappa_2; \kappa_3]$, which is a hypercomplex algebra over the field $\mathbb{C}(\mathbf{i})$ generated by units κ_2 and κ_3.

Remark 2.17 In this notation, over the field $\mathbb{C}(\mathbf{i})$, we view the κ_2 before the semicolon as the main unit, and κ_3 commuting with $\mathbf{i}$.

Multiplication of complex ternary numbers can also be achieved using the bicomplex algebra multiplication and the fact that $\kappa_1\kappa_2 = 0$:

$$\mathbf{zw} = w_1(\mathbf{z})w_1(\mathbf{w})\kappa_1 + Z(\mathbf{z})Z(\mathbf{w})\kappa_2.$$

A complex ternary number $\mathbf{z}$ is a zero–divisor if and only if $\det(S(\mathbf{z})) = 0$, equivalent to:

$$w_1(w_2^2 + w_3^2) = 0 \iff w_1 = 0 \text{ or } \{w_2 = w_3 = 0\}.$$

Using the notation (2.17), $\mathbf{z}$ is a zero–divisor if and only if $w_1(\mathbf{z}) = 0$ or $Z(\mathbf{z})$ is a bicomplex zero–divisor. In case $\mathbf{z}$ is invertible in $\mathbb{TC}$, its inverse is:

$$\begin{aligned}\frac{1}{\mathbf{z}} := \frac{\overline{\mathbf{z}}}{\mathbf{z}\overline{\mathbf{z}}} = \frac{\overline{\mathbf{z}}}{\det(S(\mathbf{z}))} &= \frac{w_2^2 + w_3^2}{w_1(w_2^2 + w_3^2)}\kappa_1 + \frac{w_1 w_2}{w_1(w_2^2 + w_3^2)}\kappa_2 - \frac{w_1 w_3}{w_1(w_2^2 + w_3^2)}\kappa_3 \\ &= \frac{1}{w_1}\kappa_1 + \frac{w_2}{(w_2^2 + w_3^2)}\kappa_2 - \frac{w_3}{(w_2^2 + w_3^2)}\kappa_3 \\ &= \frac{1}{w_1(\mathbf{z})}\kappa_1 + \frac{Z^\dagger(\mathbf{z})}{|Z(\mathbf{z})|_{\mathbf{i}}^2}\kappa_2.\end{aligned}$$

Note that in the writing (2.17), bicomplex conjugations do not have any input into ternary simple conjugations. This is because the latter ones are permuting all three idempotent variables, not conserving the bicomplex writing. The (second order) total conjugation can be written in bicomplex terms, involving the $\mathbb{C}(\mathbf{i})$–complex modulus and the †–conjugation (see [15] for details):

$$\overline{\mathbf{z}} = |Z(\mathbf{z})|_{\mathbf{i}}\kappa_1 + w_1(\mathbf{z})Z^\dagger(\mathbf{z})\kappa_2.$$

3 Ternary Norms

There are several norms of interest for the algebra of ternary numbers. The first one worth mentioning is the analog of a field norm (which is not the usual notion of a vector space norm):

Definition 3.1 The *field–type norm* of a ternary number is defined by:

$$\begin{aligned} N(\mathbf{z}) &:= \mathbf{z}\overline{\overline{\mathbf{z}}} = \det(S(\mathbf{z})) = z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 \\ &= \lambda_1(\mathbf{z})\lambda_2(\mathbf{z})\lambda_3(\mathbf{z}) = w_1(w_2^2 + w_3^2). \end{aligned}$$

We defer the study of this field–type norm for the moment. The most useful norm for the analysis of ternary numbers is the following ternary–valued norm (see [4] for a similar theory of hyperbolic–valued norms).

Definition 3.2 The *complex–valued ternary norm* is defined by:

$$\begin{aligned} \|\mathbf{z}\|_{\mathbb{TC}} &:= |\lambda_1(\mathbf{z})|\epsilon_1 + |\lambda_2(\mathbf{z})|\epsilon_2 + |\lambda_3(\mathbf{z})|\epsilon_3 \\ &= |\lambda_1(\mathbf{z})|\kappa_1 + \frac{1}{2}(|\lambda_2(\mathbf{z})| + |\lambda_3(\mathbf{z})|)\kappa_2 + \frac{\mathbf{i}}{2}(|\lambda_2(\mathbf{z})| - |\lambda_3(\mathbf{z})|)\kappa_3. \end{aligned}$$

As ϵ_2 and ϵ_3 do not have real coefficients, note that $\|\mathbf{z}\|_{\mathbb{TC}} \in \mathbb{TC}$. Equivalently, in the symmetric basis, the coefficient in front of κ_3 is purely imaginary in $\mathbf{i}$. Moreover, as mentioned above,

$$\epsilon_2 = \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3), \qquad \epsilon_3 = \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3),$$

form the idempotent basis of $\mathbb{BC}[\kappa_2; \mathbf{i}, \kappa_3]$. Note that $\mathbf{k} := \mathbf{i}\kappa_3$ is an element that squares to κ_2, the multiplicative unit of $\mathbb{BC}[\kappa_2; \mathbf{i}, \kappa_3]$, therefore it is a *hyperbolic–type* unit. We write:

$$\|\mathbf{z}\|_{\mathbb{TC}} = |\lambda_1(\mathbf{z})|\kappa_1 + |Z(\mathbf{z})|_{\mathbf{k}}\,\kappa_2,$$

where

$$|Z(\mathbf{z})|_{\mathbf{k}} := |\lambda_2(\mathbf{z})|\epsilon_2 + |\lambda_3(\mathbf{z})|\epsilon_3$$

is a *non–negative hyperbolic–type* number (see [4] for more details).

Next, we define a (partial) order relation on the set $\mathbb{TC}\big|_{\mathbb{R}}$ (which is *not* the same as $\mathbb{TR}$ described later here) that will lead to subadditivity of the complex–valued ternary norm:

Definition 3.3 A complex ternary number with real idempotent coefficients is *non–negative*, denoted by $\mathbf{z} \succeq 0$, if and only if $\lambda_\ell(\mathbf{z}) \in \mathbb{R}_+$. Moreover, two ternary numbers $\mathbf{z}, \mathbf{w} \in \mathbb{TC}\big|_{\mathbb{R}}$ are related, $\mathbf{z} \preceq \mathbf{w}$ if and only if $\mathbf{w} - \mathbf{z} \succeq 0$, i.e. their (real) idempotent coordinates satisfy $\lambda_\ell(\mathbf{z}) \leq \lambda_\ell(\mathbf{w})$, for all $\ell = 1..3$.

Using the bicomplex description for $\mathbf{z} \in \mathbb{TC}\big|_{\mathbb{R}}$, note that $Z(\mathbf{z})$ is a hyperbolic–type number, therefore

$$\mathbf{z} = w_1(\mathbf{z})\kappa_1 + Z(\mathbf{z})\kappa_2 \succeq 0$$

if and only if $w_1(\mathbf{z}) \geq 0$, and $Z(\mathbf{z})$ is a non–negative hyperbolic-type number. It is easy to check that the relation above is indeed a partial order relation on $\mathbb{TC}\big|_{\mathbb{R}}$.

Note that indeed $\|\mathbf{z}\|_{\mathbb{TC}}$ and has the usual properties of a norm, including being multiplicative:

$$\begin{aligned}
\|\mathbf{z}\|_{\mathbb{TC}} &= \mathbf{0} \iff \mathbf{z} = \mathbf{0} \\
\|\mu\mathbf{z}\|_{\mathbb{TC}} &= |\mu|\,\|\mathbf{z}\|_{\mathbb{TC}}, \qquad \forall \mu \in \mathbb{C}(\mathbf{i}), \\
\|\mathbf{z}\cdot\mathbf{w}\|_{\mathbb{TC}} &= \|\mathbf{z}\|_{\mathbb{TC}} \cdot \|\mathbf{w}\|_{\mathbb{TC}},
\end{aligned}$$

and because of the subadditivity of the usual complex norm, we get:

$$\begin{aligned}
\|\mathbf{z}+\mathbf{w}\|_{\mathbb{TC}} &= |\lambda_1(\mathbf{z}) + \lambda_1(\mathbf{w})|\epsilon_1 + |\lambda_2(\mathbf{z}) + \lambda_1(\mathbf{w})|\epsilon_2 + |\lambda_3(\mathbf{z}) + \lambda_1(\mathbf{w})|\epsilon_3 \\
&\preceq (|\lambda_1(\mathbf{z})| + |\lambda_1(\mathbf{w})|)\epsilon_1 + (|\lambda_2(\mathbf{z})| + |\lambda_1(\mathbf{w})|)\epsilon_2 + (|\lambda_3(\mathbf{z})| + |\lambda_1(\mathbf{w})|)\epsilon_3 \\
&\preceq \|\mathbf{z}\|_{\mathbb{TC}} + \|\mathbf{w}\|_{\mathbb{TC}}.
\end{aligned}$$

For $\mathbf{z} = \mathbf{x} \in \mathbb{TR}$, we note that:

$$\begin{aligned}
|\lambda_2(\mathbf{x})|^2 &= |x_1 + \psi^2 x_2 + \psi x_3|^2 = \left| x_1 + \frac{1}{2}(-x_2 - x_3) - \frac{\sqrt{3}\mathbf{i}}{2}(x_2 - x_3)\right|^2 \\
&= (x_1 + \frac{1}{2}(-x_2 - x_3))^2 - \frac{3}{4}(x_2 - x_3)^2 \\
|\lambda_3(\mathbf{x})|^2 &= |x_1 + \psi x_2 + \psi^2 x_3|^2 = \left| x_1 + \frac{1}{2}(-x_2 - x_3) + \frac{\sqrt{3}\mathbf{i}}{2}(x_2 - x_3)\right|^2 \\
&= (x_1 + \frac{1}{2}(-x_2 - x_3))^2 - \frac{3}{4}(x_2 - x_3)^2 = |\lambda_2(\mathbf{x})|^2,
\end{aligned}$$

therefore $|\lambda_2(\mathbf{x})| = |\lambda_3(\mathbf{x})|$. It follows that the ternary–valued norm for real ternary numbers is a real–ternary number:

$$\begin{aligned}
\|\mathbf{x}\|_{\mathbb{TC}} &= |\lambda_1(\mathbf{x})|\kappa_1 + |\lambda_3(\mathbf{x})|\kappa_2 = |w_1(\mathbf{x})|\kappa_1 + |w_2(\mathbf{x}) + \mathbf{i}w_3(\mathbf{x})|\kappa_2 \\
&= |w_1(\mathbf{x})|\kappa_1 + |\zeta(\mathbf{x})|\kappa_2, \qquad (3.1)
\end{aligned}$$

where we note that

$$|\zeta(\mathbf{x})| = |w_2(\mathbf{x}) + \mathbf{i}w_3(\mathbf{x})| = |w_2(\mathbf{x})\kappa_2 + w_3(\mathbf{x})\kappa_3| = \sqrt{w_2(\mathbf{x})^2 + w_3(\mathbf{x})^2} \in \mathbb{R}_+.$$

Next, for a real ternary number $\mathbf{x} \in \mathbb{TR}$, $\lambda_1(\mathbf{x})$ is always real, but $\lambda_2(\mathbf{x})$ and $\lambda_3(\mathbf{x})$ are real if and only if $x_2 = x_3$, equivalent to $\lambda_2(\mathbf{x}) = \lambda_3(\mathbf{x})$ and to $w_3(\mathbf{x}) = 0$. Moreover, in this case $w_2(\mathbf{x}) = \lambda_2(\mathbf{x}) = x_1 - x_2$. Therefore, for a real ternary number:

$$\mathbf{x} \succeq 0 \iff w_1(\mathbf{x}) \geq 0, w_2(\mathbf{x}) \geq 0, w_3(\mathbf{x}) = 0. \tag{3.2}$$

In conclusion, a non–negative real ternary number is of the form:

$$\begin{aligned}
\mathbf{x} &= x_1\mathbf{1} + x_2\mathbf{e}_2 + x_2\mathbf{e}_3 \\
&= \lambda_1(\mathbf{x})\epsilon_1 + \lambda_2(\mathbf{x})\epsilon_2 + \lambda_2(\mathbf{x})\epsilon_3 \\
&= w_1(\mathbf{x})\kappa_1 + w_2(\mathbf{x})\kappa_2,
\end{aligned}$$

where $\lambda_\ell(\mathbf{x}) \geq 0$, $w_\ell(\mathbf{x}) \geq 0$, for $\ell = 1, 2$, equivalent to $x_1 + 2x_2 \geq 0$ and $x_1 \geq x_2$. This implies that $x_1 \geq 0$, and if $x_2 \geq 0$ then $x_1 \geq x_2 \geq 0$, or if $x_2 \leq 0$ then $x_1 \geq -2x_2$.

The next valuable norm on $\mathbb{TC}$ is the Euclidean–type, still real ternary–valued:

Definition 3.4 The *Euclidean–type ternary–valued norm* of a complex ternary number $\mathbf{z} \in \mathbb{TC}$ is defined by:

$$\begin{aligned}
\|\mathbf{z}\|_{\mathbb{E}} &:= |\lambda_1(\mathbf{z})|\kappa_1 + \left|\frac{1}{2}(|\lambda_2(\mathbf{z})| + |\lambda_3(\mathbf{z})|)\kappa_2 + \frac{\mathbf{i}}{2}(|\lambda_2(\mathbf{z})| - |\lambda_3(\mathbf{z})|)\kappa_3\right|\kappa_2 \\
&= |\lambda_1(\mathbf{z})|\kappa_1 + \frac{1}{2}\sqrt{(|\lambda_2(\mathbf{z})| + |\lambda_3(\mathbf{z})|)^2 + (|\lambda_2(\mathbf{z})| - |\lambda_3(\mathbf{z})|)^2}\kappa_2 \\
&= |\lambda_1(\mathbf{z})|\kappa_1 + \frac{1}{\sqrt{2}}\sqrt{|\lambda_2(\mathbf{z})|^2 + |\lambda_3(\mathbf{z})|^2}\kappa_2 \\
&= |w_1(\mathbf{z})|\kappa_1 + |Z(\mathbf{z})|\kappa_2,
\end{aligned} \tag{3.3}$$

where $|\cdot|$ denotes the usual norm of complex numbers, with the exception of $|Z(\mathbf{z})|$ which is the Euclidean norm of the bicomplex number $Z(\mathbf{z})$ written in its idempotent representation.

It follows easily that $\|\mathbf{z}\|_{\mathbb{E}} \succeq 0$ and, using the subadditivity of the Euclidean norms, the ternary–valued norm is also subadditive but not multiplicative, as the bicomplex Euclidean norm is not.

Using the complex ternary–valued norm, we define ternary disks and domains as follows.

Definition 3.5 A *ternary open disk* $\mathbb{D}(\mathbf{z}_0, \rho)$ centered in $\mathbf{z}_0 = \sum_{\ell=1}^{3} \lambda_\ell(\mathbf{z}_0)\epsilon_\ell$ and *radius* $\rho \in \mathbb{TC}\big|_{\mathbb{R}}$, $\rho \succeq 0$, is defined by $\|\mathbf{z} - \mathbf{z}_0\|_{\mathbb{TC}} \prec \rho$, i.e. the relations:

$$|\lambda_\ell(\mathbf{z}) - \lambda_\ell(\mathbf{z}_0)| < \lambda_\ell(\rho), \quad \ell = 1..3.$$

In case we replace $\prec$ by $\preceq$ above, we obtain a ternary *closed* disk.

Definition 3.6 A complex *ternary domain* is an connected open subset $\Omega \subseteq \mathbb{TC}$, i.e. for each point $\mathbf{z}_0 \in \Omega$ there is a ternary open disk $\mathbb{D}(\mathbf{z}_0, \rho) \subset \Omega$. If written in the projection form:

$$\Omega = \Omega \cdot \epsilon_1 + \Omega \cdot \epsilon_2 + \Omega \cdot \epsilon_3,$$

the projection sets $\Omega \cdot \epsilon_\ell$ are one–variable complex domains (disks) in $\mathbb{C}(\mathbf{i})$, for $\ell = 1..3$.

In the symmetric basis, a ternary domain Ω decomposes as follows:

$$\Omega = \Omega \cdot \kappa_1 + \left(\Omega \cdot \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3) + \Omega \cdot \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3)\right)\kappa_2 = \Omega \cdot \kappa_1 + \Omega \cdot \kappa_2,$$

where we have used the bicomplex domain decomposition in the bicomplex idempotents $\frac{1}{2}(\kappa_2 \pm \mathbf{i}\kappa_3)$ (see [15] for details):

$$\Omega \cdot \kappa_2 = \left(\Omega \cdot \frac{1}{2}(\kappa_2 + \mathbf{i}\kappa_3)_2 + \Omega \cdot \frac{1}{2}(\kappa_2 - \mathbf{i}\kappa_3)\right)\kappa_2$$

When necessary, we work also with Euclidean–type ternary disks, defined in a similarly way:

Definition 3.7 An *Euclidean ternary open disk* centered in $\mathbf{z}_0 = |w_1(\mathbf{z}_0)|\kappa_1 + |Z(\mathbf{z}_0)|\kappa_2$ and *radius* $\gamma \in \mathbb{TR}_+$, $\gamma \succeq 0$, is defined by $\|\mathbf{z} - \mathbf{z}_0\|_{\mathbb{E}} \prec \gamma$, i.e. the relations:

$$|w_1(\mathbf{z}) - w_1(\mathbf{z}_0)| < w_1(\gamma), \quad |Z(\mathbf{z}) - Z(\mathbf{z}_0)| < |Z(\gamma)|.$$

4 Complex Ternary Regularity

Consider a complex ternary number written in all three standard, idempotent and symmetric bases:

$$\mathbf{z} = \sum_{k=1}^{3} z_k \mathbf{e}_k = \sum_{k=1}^{3} \lambda_k(\mathbf{z})\epsilon_k = \sum_{\ell=1}^{3} w_\ell(\mathbf{z})\kappa_\ell = w_1(\mathbf{z})\kappa_1 + Z(\mathbf{z})\kappa_2. \tag{4.1}$$

Following the same set-up, a complex ternary function $\mathbf{F} : \Omega \subseteq \mathbb{TC} \to \mathbb{TC}$ can also be written in three ways:

$$F(\mathbf{z}) = \sum_{k=1}^{3} f_k(\mathbf{z})\mathbf{e}_k = \sum_{k=1}^{3} \varphi_k(\mathbf{z})\epsilon_k = \sum_{\ell=1}^{3} W_\ell(\mathbf{z})\kappa_\ell = W_1(\mathbf{z})\kappa_1 + G(\mathbf{z})\kappa_2, , \tag{4.2}$$

where $\Omega \subseteq \mathbb{TC}$ is a complex ternary domain and:

1. $f_k : \Omega \subset \mathbb{C}(\mathbf{i})^3 \to \mathbb{C}(\mathbf{i})$, $f(\mathbf{z}) = f_k(z_1, z_2, z_3)$ are $\mathbb{C}(\mathbf{i})$–valued functions of three complex variables z_k,
2. $\varphi_k : \prod_{\ell=1}^{3} \Omega \cdot \epsilon_\ell \subset \mathbb{C}(\mathbf{i})^3 \to \mathbb{C}(\mathbf{i})$, $\varphi_k(\mathbf{z}) = \varphi_k(\lambda_1(\mathbf{z}), \lambda_2(\mathbf{z}), \lambda_3(\mathbf{z}))$ are $\mathbb{C}(\mathbf{i})$–valued functions of three complex variables $\lambda_k(\mathbf{z})$,
3. $W_k : \prod_{\ell=1}^{3} \Omega \cdot \kappa_\ell \subset \mathbb{C}(\mathbf{i})^3 \to \mathbb{C}(\mathbf{i})$, $W_k(\mathbf{z}) = W_k(w_1(\mathbf{z}), w_2(\mathbf{z}), w_3(\mathbf{z}))$ are $\mathbb{C}(\mathbf{i})$–valued functions of three complex variables $w_k(\mathbf{z})$.
4. In this context, the function

$$G : \Omega \cdot \kappa_1 \times \Omega \cdot \kappa_2 \subset \mathbb{C}(\mathbf{i}) \times \mathbb{BC}(\kappa_2; \kappa_3) \to \mathbb{BC}(\kappa_2; \kappa_3), \tag{4.3}$$

where $G(\mathbf{z}) = G(w_1(\mathbf{z}), Z(\mathbf{z}))$, becomes a bicomplex–valued function of one complex variable $w_1(\mathbf{z})$ and one bicomplex variable $Z(\mathbf{z})$.

The relations among the coordinate functions for F are:

$$\varphi_1(\mathbf{z}) = f_1(\mathbf{z}) + f_2(\mathbf{z}) + f_3(\mathbf{z}), \qquad \varphi_2(\mathbf{z}) = f_1(\mathbf{z}) + \psi^2 f_2(\mathbf{z}) + \psi f_3(\mathbf{z}),$$
$$\varphi_3(\mathbf{z}) = f_1(\mathbf{z}) + \psi f_2(\mathbf{z}) + \psi^2 f_3(\mathbf{z}), \tag{4.4}$$

and the inverse transformations are:

$$f_1(\mathbf{z}) = \frac{1}{3}\left(\varphi_1(\mathbf{z}) + \varphi_2(\mathbf{z}) + \varphi_3(\mathbf{z})\right), \qquad f_2(\mathbf{z}) = \frac{1}{3}(\varphi_1(\mathbf{z}) + \psi\varphi_2(\mathbf{z}) + \psi^2\varphi_3(\mathbf{z})),$$
$$f_3(\mathbf{z}) = \frac{1}{3}(\varphi_1(\mathbf{z}) + \psi^2\varphi_2(\mathbf{z}) + \psi\varphi_3(\mathbf{z})). \tag{4.5}$$

In the symmetric basis, we have:

$$W_1(\mathbf{z}) = \varphi_1(\mathbf{z}) = f_1(\mathbf{z}) + f_2(\mathbf{z}) + f_3(\mathbf{z}),$$
$$W_2(\mathbf{z}) = \frac{1}{2}(\varphi_2(\mathbf{z}) + \varphi_3(\mathbf{z})) = \frac{1}{2}(2f_1(\mathbf{z}) - f_2(\mathbf{z}) - f_3(\mathbf{z}))$$
$$W_3(\mathbf{z}) = \frac{\mathbf{i}}{2}(\varphi_2(\mathbf{z}) - \varphi_3(\mathbf{z})) = \frac{\sqrt{3}}{2}(f_2(\mathbf{z}) - f_3(\mathbf{z})), \tag{4.6}$$

and the inverse transformations:

$$\varphi_1(\mathbf{z}) = W_1(\mathbf{z}), \qquad \varphi_2(\mathbf{z}) = W_2(\mathbf{z}) - \mathbf{i}W_3(\mathbf{z}), \qquad \varphi_3(\mathbf{z}) = W_2(\mathbf{z}) + \mathbf{i}W_3(\mathbf{z}), \tag{4.7}$$

and

$$f_1(\mathbf{z}) = \frac{1}{3}(W_1(\mathbf{z}) + 2W_2(\mathbf{z})), \qquad f_2(\mathbf{z}) = \frac{1}{3}(W_1(\mathbf{z}) - W_2(\mathbf{z}) + \sqrt{3}W_3(\mathbf{z})),$$
$$f_3(\mathbf{z}) = -\frac{1}{3}(-W_1(\mathbf{z}) + W_2(\mathbf{z}) + \sqrt{3}W_3(\mathbf{z})). \tag{4.8}$$

Note that all relations among the functions $f_k(\mathbf{z})$, $\varphi_k(\mathbf{z})$, $W_k(\mathbf{z})$ and $G(\mathbf{z})$ are *linear transformations*.

Definition 4.1 A ternary function $\mathbf{F} : \Omega \subseteq \mathbb{TC} \to \mathbb{TC}$ is *complex differentiable* if the coordinate functions $f_k(\mathbf{z})$ (and implicitly $\varphi_k(\mathbf{z})$, $W_k(\mathbf{z})$ and $G(\mathbf{z})$) have continous partial derivatives

$$\frac{\partial f_k}{\partial z_\ell}, \quad \frac{\partial \varphi_k}{\partial \lambda_\ell(\mathbf{z})}, \quad \frac{\partial W_k}{\partial w_\ell(\mathbf{z})}, \qquad \frac{\partial G}{\partial w_2(\mathbf{z})}, \quad \frac{\partial G}{\partial w_3(\mathbf{z})},$$

for all $k, \ell = 1..3$, i.e. the coordinate functions are $\mathbb{C}(\mathbf{i})$–complex holomorphic functions in variables z_ℓ (respectively $\lambda_\ell(\mathbf{z})$ and $w_\ell(\mathbf{z})$).

From now on we will omit $(\mathbf{z})$ from the notation of $\lambda_k(\mathbf{z})$ et al. In terms of the usual complex conjugate differential operators $\dfrac{\partial}{\partial \overline{z}_\ell}$ in $\mathbb{C}(\mathbf{i})^3$, a function $\mathbf{F}$ is complex differentiable if it is in the kernel of all three operators:

$$\frac{\partial f_k}{\partial \overline{z}_1} = \frac{\partial f_k}{\partial \overline{z}_2} = \frac{\partial f_k}{\partial \overline{z}_3} = 0, \qquad k = 1..3.$$

Equivalently, in the idempotent and symmetric bases:

$$\frac{\partial \varphi_k}{\partial \overline{\lambda}_1} = \frac{\partial \varphi_k}{\partial \overline{\lambda}_2} = \frac{\partial \varphi_k}{\partial \overline{\lambda}_3} = 0 \quad \Longleftrightarrow \quad \frac{\partial W_k}{\partial \overline{w}_1} = \frac{\partial W_k}{\partial \overline{w}_2} = \frac{\partial W_k}{\partial \overline{w}_3} = 0.$$

In terms of the bicomplex function $G(\mathbf{z})$, we have the equivalent description of complex differentiability of ternary functions:

$$\frac{\partial G}{\partial \overline{w}_1} = \frac{\partial G}{\partial \overline{Z}} = \frac{\partial F}{\partial Z^*} = 0.$$

In the last equality, $\overline{Z}$ and Z^* denote the bicomplex conjugates (see [15] for details).

Definition 4.2 The *ternary conjugate operators* acting on complex differentiable ternary functions $\mathbf{F}(\mathbf{z})$ are defined in terms of the idempotent representation:

$$\frac{d}{d\mathbf{z}^\dagger} := \frac{\partial}{\partial\lambda_3}\epsilon_1 + \frac{\partial}{\partial\lambda_1}\epsilon_2 + \frac{\partial}{\partial\lambda_2}\epsilon_3, \qquad \frac{d}{d\mathbf{z}^*} := \frac{\partial}{\partial\lambda_2}\epsilon_1 + \frac{\partial}{\partial\lambda_3}\epsilon_2 + \frac{\partial}{\partial\lambda_1}\epsilon_3.$$

Because $\epsilon_j\epsilon_k = 0$ for $j \neq k$ and $\epsilon_k^2 = \epsilon_k$, for all $k, \ell = 1..3$ we obtain:

$$\frac{\partial \mathbf{F}}{\partial \lambda_\ell}\epsilon_k = \frac{\partial\left(\sum_{j=1}^{3}\varphi_j \cdot \epsilon_j\right)}{\partial\lambda_\ell}\epsilon_k = \frac{\partial\varphi_k}{\partial\lambda_\ell}\epsilon_k.$$

Then the explicit action of the conjugate operators is given by:

$$\frac{d\mathbf{F}}{d\mathbf{z}^\dagger} = \frac{\partial\mathbf{F}}{\partial\lambda_3}\epsilon_1 + \frac{\partial\mathbf{F}}{\partial\lambda_1}\epsilon_2 + \frac{\partial\mathbf{F}}{\partial\lambda_2}\epsilon_3 = \frac{\partial\varphi_1}{\partial\lambda_3}\epsilon_1 + \frac{\partial\varphi_2}{\partial\lambda_1}\epsilon_2 + \frac{\partial\varphi_3}{\partial\lambda_2}\epsilon_3,$$

and similarly

$$\frac{d\mathbf{F}}{d\mathbf{z}^*} = \frac{\partial\varphi_1}{\partial\lambda_2}\epsilon_1 + \frac{\partial\varphi_2}{\partial\lambda_3}\epsilon_2 + \frac{\partial\varphi_3}{\partial\lambda_1}\epsilon_3.$$

In terms of the ternary conjugate differential operators, we define the notion of regularity as follows:

Definition 4.3 A complex differentiable ternary function $\mathbf{F} : \Omega \to \mathbb{TC}$ is *ternary regular* on Ω if and only if $\mathbf{F}$ is in the kernel of the two ternary conjugate operators:

$$\frac{d\mathbf{F}}{d\mathbf{z}^\dagger}(\mathbf{z}) = \frac{d\mathbf{F}}{d\mathbf{z}^*}(\mathbf{z}) = 0,$$

for all $\mathbf{z} \in \Omega$.

The following theorem characterizes ternary complex regular functions in the idempotent writing.

Theorem 4.4 *A complex differentiable ternary function* $\mathbf{F}$ *is regular on* Ω *if and only if the functions* $\varphi_k(\mathbf{z}) = \varphi_k(\lambda_k(\mathbf{z}))$*, i.e. they depend only on one variable, thus are* $\mathbb{C}(\mathbf{i})$*–holomorphic functions in the variable* $\lambda_k(\mathbf{z})$*, for* $k = 1..3$*, on their projection complex domains* $\Omega \cdot \epsilon_k$*.*

Proof The proof is very similar to the proof in the bicomplex case [15] and a direct implication from the differential operators above. □

In the symmetric basis we obtain the equivalent statement:

Theorem 4.5 *A complex differentiable ternary functions* $\mathbf{F}$ *is regular on* Ω *if and only if both of the following hold:*

1. *the function* $W_1(\mathbf{z}) = W_1(w_1)$*, i.e. it depends only on one symmetric variable* w_1 *and thus it is* $\mathbb{C}(\mathbf{i})$*–holomorphic on* $\Omega \cdot \kappa_1$*,*
2. *the function* $G(\mathbf{z}) = G(Z)$*, i.e. it depends only on the bicomplex variable* Z *and it is bicomplex holomorphic on* $\Omega \cdot \kappa_2$*.*

Proof This is a consequence of the theorem above and the equivalence with bicomplex derivability. □

Remark 4.6 Coming from general commutative hypercomplex theory, one can define the notion of *derivability* for ternary functions, i.e. in terms of limits of the ternary rate of change quotients. It turns out that a complex ternary function $\mathbf{F} : \Omega \to \mathbb{TC}$ is derivable on Ω if and only if it is ternary regular on Ω. In terms of the *ternary derivative operator*, defined by:

$$\frac{d}{d\mathbf{z}} = \sum_{k=1}^{3} \frac{\partial}{\partial \lambda_k(\mathbf{z})} \epsilon_k,$$

if $\mathbf{F}$ is derivable then its derivative is given by:

$$\mathbf{F}'(\mathbf{z}) = \frac{d\mathbf{F}}{d\mathbf{z}} = \varphi_1'(\lambda_1(\mathbf{z}))\epsilon_1 + \varphi_2'(\lambda_2(\mathbf{z}))\epsilon_2 + \varphi_3'(\lambda_3(\mathbf{z}))\epsilon_3.$$

In symmetric coordinates, the derivative of $\mathbf{F}$ is given by:

$$\mathbf{F}'(\mathbf{z}) = W_1'(w_1(\mathbf{z}))\kappa_1 + G'(Z(\mathbf{z}))\kappa_2.$$

The Generalized Cauchy–Riemann (GCR) conditions give an equivalent description of ternary regularity. Starting with a ternary function $\mathbf{F} : \Omega \to \mathbb{TC}$ which is holormophic as a function of three complex variables, the GCR system is:

$$\frac{\partial \mathbf{F}}{\partial z_k}\mathbf{e}_\ell = \frac{\partial \mathbf{F}}{\partial z_\ell}\mathbf{e}_k, \qquad k, \ell = 1..3.$$

Explicit computations yield the following system:

$$\frac{\partial f_3}{\partial z_1}\mathbf{1} + \frac{\partial f_1}{\partial z_1}\mathbf{e}_2 + \frac{\partial f_2}{\partial z_1}\mathbf{e}_3 = \frac{\partial f_1}{\partial z_2}\mathbf{1} + \frac{\partial f_2}{\partial z_2}\mathbf{e}_2 + \frac{\partial f_3}{\partial z_2}\mathbf{e}_3,$$
$$\frac{\partial f_2}{\partial z_2}\mathbf{1} + \frac{\partial f_3}{\partial z_2}\mathbf{e}_2 + \frac{\partial f_1}{\partial z_2}\mathbf{e}_3 = \frac{\partial f_3}{\partial z_3}\mathbf{1} + \frac{\partial f_1}{\partial z_3}\mathbf{e}_2 + \frac{\partial f_2}{\partial z_3}\mathbf{e}_3,$$
$$\frac{\partial f_1}{\partial z_3}\mathbf{1} + \frac{\partial f_2}{\partial z_3}\mathbf{e}_2 + \frac{\partial f_3}{\partial z_3}\mathbf{e}_3 = \frac{\partial f_2}{\partial z_1}\mathbf{1} + \frac{\partial f_3}{\partial z_1}\mathbf{e}_2 + \frac{\partial f_1}{\partial z_1}\mathbf{e}_3.$$

which is equivalent to the following system of differential equations in terms of the partial derivatives in standard coordinates z_ℓ:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} = \frac{\partial f_3}{\partial z_3},$$
$$\frac{\partial f_1}{\partial z_2} = \frac{\partial f_2}{\partial z_3} = \frac{\partial f_3}{\partial z_1},$$
$$\frac{\partial f_1}{\partial z_3} = \frac{\partial f_2}{\partial z_1} = \frac{\partial f_3}{\partial z_2}. \tag{4.9}$$

In the symmetric basis the GCR system is equivalent to the following:

$$\frac{\partial W_1}{\partial w_1} = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_3},$$

a complex Cauchy–Riemann system:

$$\frac{\partial W_2}{\partial w_2} = \frac{\partial W_3}{\partial w_3} = \frac{1}{2}\left(2\frac{\partial f_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2} - \frac{\partial f_1}{\partial z_3}\right),$$
$$\frac{\partial W_2}{\partial w_3} = -\frac{\partial W_3}{\partial w_2} = \frac{\sqrt{3}}{2}\left(\frac{\partial f_1}{\partial z_2} - \frac{\partial f_1}{\partial z_3}\right),$$

which is equivalent to $\frac{\partial G}{\partial Z^\dagger} = 0$, and the projector system:

$$\frac{\partial W_2}{\partial w_1} = \frac{\partial W_3}{\partial w_1} = 0, \qquad \frac{\partial W_1}{\partial w_2} = \frac{\partial W_1}{\partial w_3} = 0.$$

5 Applications of Complex Ternary Regularity

5.1 Ternary Power Series and Cauchy–Kovalevskaya Extensions

A complex ternary regular function $\mathbf{F} : \Omega \to \mathbb{TC}$ inherits a power series expansion from its complex holomorphic idempotent coordinate functions:

$$F(\mathbf{z}) := \varphi_1(\lambda_1(\mathbf{z}))\epsilon_1 + \varphi_2(\lambda_2(\mathbf{z}))\epsilon_2 + \varphi_3(\lambda_3(\mathbf{z}))\epsilon_3.$$

Omitting ($\mathbf{z}$) from notation, consider the following power series expansions of φ_ℓ on complex disks $\mathbb{D}_\ell(\lambda_{\ell 0}) \subseteq \Omega \cdot \epsilon_\ell$

$$\varphi_\ell(\lambda_\ell) := \sum_{n=0}^{\infty} a_{\ell n}(\lambda_\ell - \lambda_{\ell 0})^n, \qquad \ell = 1..3.$$

Then

$$\begin{aligned}
\mathbf{F}(\mathbf{z}) &:= \sum_{n=0}^{\infty} a_{1n}(\lambda_1 - \lambda_{10})^n \epsilon_1 + \sum_{n=0}^{\infty} a_{2n}(\lambda_2 - \lambda_{20})^n \epsilon_2 + \sum_{n=0}^{\infty} a_{3n}(\lambda_3 - \lambda_{30})^n \epsilon_3 \\
&= \sum_{n=0}^{\infty} (a_{1n}\epsilon_1 + a_{2n}\epsilon_2 + a_{3n}\epsilon_3) \cdot \big((\lambda_1 - \lambda_{10})^n \epsilon_1 + (\lambda_2 - \lambda_{20})^n \epsilon_2 + (\lambda_3 - \lambda_{30})^n \epsilon_3\big) \\
&= \sum_{n=0}^{\infty} (a_{1n}\epsilon_1 + a_{2n}\epsilon_2 + a_{3n}\epsilon_3) \cdot ((\lambda_1 - \lambda_{10})\epsilon_1 + (\lambda_2 - \lambda_{20})\epsilon_2 + (\lambda_3 - \lambda_{30})\epsilon_3)^n \\
&=: \sum_{n=0}^{\infty} A_n(\mathbf{z} - \mathbf{z}_0)^n,
\end{aligned}$$

where $A_n = a_{1n}\epsilon_1 + a_{2n}\epsilon_2 + a_{3n}\epsilon_3$ are ternary coefficients. This proves the theorem:

Theorem 5.1 *A complex ternary function $F : \Omega \to \mathbb{TC}$ of class $\mathcal{C}^1(\Omega)$ is* holomorphic *on Ω if and only if it is derivable on Ω.*

Just like in the bicomplex case [11], we obtain a first Cauchy-Kovalevskaya Theorem for complex ternary functions:

Theorem 5.2 *A real analytic function $f : U \subseteq \mathbb{R} \to \mathbb{TC}$*

$$\begin{aligned}
f(x) &= f_1(x)\mathbf{1} + f_2(x)\mathbf{e}_2 + f_3(x)\mathbf{e}_3 \\
&= \varphi_1(x)\epsilon_1 + \varphi_2(x)\epsilon_2 + \varphi_3(x)\epsilon_3
\end{aligned}$$

(equivalent to $f_\ell(x)$ are real analytic on U), extends uniquely to a complex ternary regular function $\mathbf{F} : \Omega \subseteq \mathbb{TC} \to \mathbb{TC}$, where Ω is an open neighborhood of U in $\mathbb{TC}$.

Proof Recall that

$$\begin{aligned}
\varphi_1(x) &= f_1(x) + f_2(x) + f_3(x), \\
\varphi_2(x) &= f_1(x) + \psi^2 f_2(x) + \psi f_3(x), \\
\varphi_3(x) &= f_1(x) + \psi f_2(x) + \psi^2 f_3(x).
\end{aligned}$$

First we extend f to a holomorphic function on $\tilde{U} \subseteq \mathbb{C}$, where $\tilde{U}$ is a neighborhood of U in $\mathbb{C}$ by $f(z) := f_1(z)\mathbf{1} + f_2(z)\mathbf{e}_2 + f_3(z)\mathbf{e}_3$, for $z = x + \mathbf{i}y$. This is equivalent to use the regular CK extension for the real analytics functions $f_\ell(x)$ to holomorphic $f_\ell(z)$, for $\ell = 1..3$. It follows that the extensions $\varphi_\ell(z)$ are complex holomorphic, as they are $\mathbb{C}(\mathbf{i})$–linear combinations of $f_\ell(z)$.

Now consider the ternary variable

$$\mathbf{z} = z_1\mathbf{1} + z_2\mathbf{e}_2 + z_3\mathbf{e}_3 = \lambda_1(\mathbf{z})\epsilon_1 + \lambda_2(\mathbf{z})\epsilon_2 + \lambda_3(\mathbf{z})\epsilon_3$$

and define

$$\mathbf{F}(\mathbf{z}) := \varphi_1(\lambda_1(\mathbf{z}))\epsilon_1 + \varphi_2(\lambda_2(\mathbf{z}))\epsilon_2 + \varphi_3(\lambda_3(\mathbf{z}))\epsilon_3.$$

Note that $\mathbf{F}\big|_{\mathbb{R}} = f$ and $\mathbf{F}$ is ternary regular. The uniqueness of $\mathbf{F}$ follows from the uniqueness of the extensions $f_\ell(x)$ to $f_\ell(z)$. □

5.2 *Analysis of Real Ternary Functions*

A real ternary number $\mathbf{x} \in \mathbb{TR}$ has the following representations:

$$\mathbf{x} = \sum_{k=1}^{3} x_k\mathbf{e}_k = \sum_{k=1}^{3} \lambda_k(\mathbf{x})\mathbf{e}_k = \sum_{\ell=1}^{3} w_\ell(\mathbf{x})\kappa_\ell = w_1(\mathbf{x})\kappa_1 + \zeta(\mathbf{x})\kappa_2,$$

where $\lambda_1(\mathbf{x}) \in \mathbb{R}$, $x_k, w_k(\mathbf{x}) \in \mathbb{R}$ for $k = 1..3$, $\lambda_2(\mathbf{x}), \lambda_3(\mathbf{x}) \in \mathbb{C}(\mathbf{i})$, and $\zeta(\mathbf{x}) \in \mathbb{C}(\kappa_2; \kappa_3)$.

A real ternary function $F : \Omega \to \mathbb{TR}$ is also written in the following ways:

$$F(\mathbf{x}) = \sum_{k=1}^{3} f_k(\mathbf{x})\mathbf{e}_k = \sum_{k=1}^{3} \varphi_k(\mathbf{x})\mathbf{e}_k = \sum_{\ell=1}^{3} W_\ell(\mathbf{x})\kappa_\ell = W_1(\mathbf{x})\kappa_1 + G(\mathbf{x})\kappa_2,$$

where $\Omega \subseteq \mathbb{TR}$ is a real ternary domain and the coordinate functions are as in the complex case, with a few exceptions:

1. $f_k : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = f_k(x_1, x_2, x_3)$ are real–valued functions of three real variables x_k.
2. $\varphi_k : \prod_{\ell=1}^{3} \Omega \cdot \epsilon_\ell \subset \mathbb{C}(\mathbf{i})^3 \to \mathbb{R}$, $\varphi_k(\mathbf{z}) = \varphi_1(\lambda_1(\mathbf{z}), \lambda_2(\mathbf{z}), \lambda_3(\mathbf{z}))$ are functions of one real variable $\lambda_1(\mathbf{z})$ and two complex variables $\lambda_2(\mathbf{z})$ and $\lambda_3(\mathbf{z})$, i.e. $\Omega \cdot \epsilon_1 \subset \mathbb{R}^3$, while $\Omega \cdot \epsilon_k \subset \mathbb{C}(\mathbf{i})^3$ for $k = 2, 3$. The difference is that φ_1 is real–valued while φ_2 and φ_3 are complex–valued functions.

3. $W_k : \prod_{\ell=1}^{3} \Omega \cdot \kappa_\ell \subset \mathbb{C}(\mathbf{i})^3 \to \mathbb{C}(\mathbf{i})$, $W_k(\mathbf{z}) = W_k(w_1(\mathbf{z}), w_2(\mathbf{z}), w_3(\mathbf{z}))$ are $\mathbb{C}(\mathbf{i})$–valued functions of three complex variables $w_k(\mathbf{z})$. The function $G : \Omega \cdot \kappa_1 \times \Omega \cdot \kappa_2 \subset \mathbb{BC}(\kappa_2; \mathbf{i}, \kappa_3)$, $G(\mathbf{z}) = G(w_1(\mathbf{z}), Z(\mathbf{z}))$ is a bicomplex–valued function of one complex variable $w_1(\mathbf{z})$ and one bicomplex variable $Z(\mathbf{z})$.

Theorem 5.3 **F** *is regular on* Ω *if and only if:*

1. *The real function* $W_1(\mathbf{x}) = W_1(w_1(\mathbf{x}))$*, i.e. it depends only on one real symmetric variable, and the complex function* $g(\mathbf{x}) = g(\zeta(\mathbf{x}))$*, i.e. it depends only on one complex variable.*
2. *The real function* $W_1(w_1(\mathbf{x}))$ *is of class* $\mathcal{C}^1(\Omega \cdot \kappa_1)$ *and the complex function* $G(\zeta(\mathbf{x}))$ *is complex derivable (i.e. holomorphic) in the complex holomorphic on* $\Omega \cdot \kappa_2$.

Moreover, if the derivative of F exists, it is given by:

$$F'(\mathbf{x}) = \frac{dF}{d\mathbf{x}} = W_1'(w_1(\mathbf{x}))\kappa_1 + g'(\zeta(\mathbf{x}))\kappa_2.$$

The equivalent description is the GCR system (4.9), replacing the variables z_ℓ by x_ℓ. In the symmetric basis, we get the similar formulation of real ternary regularity in terms of differential operators:

$$\frac{\partial W_1}{\partial w_1} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial x_3},$$

a CR system:

$$\frac{\partial W_2}{\partial w_2} = \frac{\partial W_3}{\partial w_3} = \frac{1}{2}\left(2\frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial x_3}\right),$$
$$\frac{\partial W_2}{\partial w_3} = -\frac{\partial W_3}{\partial w_2} = \frac{\sqrt{3}}{2}\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial x_3}\right),$$

which is equivalent to $\frac{\partial g}{\partial \overline{\zeta}} = 0$, and the projection properties:

$$\frac{\partial W_2}{\partial w_1} = \frac{\partial W_3}{\partial w_1} = 0, \qquad \frac{\partial W_1}{\partial w_2} = \frac{\partial W_1}{\partial w_3} = 0.$$

The derivative real ternary differential operator is given by:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{x}} &= \frac{\partial}{\partial w_1}\kappa_1 + \frac{\partial}{\partial w_2}\kappa_2 + \frac{\partial}{\partial w_3}\kappa_3 \\ &= \frac{1}{3}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right)\kappa_1 + \frac{1}{3}\left(2\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right)\kappa_2 + \frac{1}{\sqrt{3}}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right)\kappa_3.\end{aligned}$$

5.3 Cauchy-Kovalevskaya Theorem for Real Ternary Regular Functions

In the same way as the previous case, we have the following Cauchy-Kovalevskaya theorem for real ternary regular functions:

Theorem 5.4 *Consider a real ternary regular function* $\mathbf{F} : \Omega \subseteq \mathbb{TR} \to \mathbb{TC}$

$$\mathbf{F}(\mathbf{x}) = W_1(w_1(\mathbf{x}))\kappa_1 + G(\zeta(\mathbf{x}))\kappa_2$$

with the additional condition that $W_1(\mathbf{x})$ *is real analytic on* $\Omega \cdot \kappa_\ell$. *Then* $\mathbf{F}$ *extends uniquely to a complex ternary regular function* $\tilde{\mathbf{F}} : \tilde{\Omega} \subseteq \mathbb{TC} \to \mathbb{TC}$, *where* $\tilde{\Omega}$ *is an open neighborhood of* Ω *in* $\mathbb{TC}$.

Proof Recall that $\mathbf{F}$ is ternary regular implies that $G(\zeta(\mathbf{x}))$ is holomorphic in $\zeta(\mathbf{x}) = w_2(\mathbf{x})\kappa_2 + w_3(\mathbf{x})\kappa_3$ on $\Omega \cdot \kappa_2$. First we extend $W_1(w_1(\mathbf{x}))$ to a holomorphic function on $\tilde{U}_1 \subseteq \mathbb{C}$, where $\tilde{U}_1$ is a neighborhood of U in $\mathbb{C}$ by:

$$W_1(z) := U_1(z) + \mathbf{i}V_1(z)$$

for $z_1 = w_1(x) + \mathbf{i}y_1$. Now we extend the complex holomorphic function $G(\zeta(\mathbf{x}))$ from $\mathbb{C}(\kappa_3)$ to a bicomplex holomorphic function $G(Z)$ on $\mathbb{BC}[\kappa_2; \kappa_3]$.

Now consider the ternary variable

$$\mathbf{z} = w_1(\mathbf{z})\kappa_1 + Z(\mathbf{z})\kappa_3$$

and define

$$\tilde{\mathbf{F}}(\mathbf{z}) := W_1(w_1(\mathbf{z}))\kappa_1 + G(Z(\mathbf{z}))\kappa_2$$

Note that $\tilde{\mathbf{F}}\big|_{\mathbb{TR}} = F$ and $v\tilde{F}$ is complex ternary holomorphic. The uniqueness of $\mathbf{F}$ follows from the uniqueness of the complex and respectively bicomplex extensions. □

5.4 Other Applications, Extensions to the Tri-complex Space $\mathbb{BC}_3$

We will first write a generalization of the tri-complex space $\mathbb{BC}_3$, i.e. given by three commuting complex units as in [17, 19], as follows:

Remark 5.5 Consider the tricomplex space $\mathbb{BC}_3[\mathbf{1}; \mathbf{i}_2, \mathbf{i}_3]$ (where before the semi-colon we have the main unit and $\mathbf{i}_2, \mathbf{i}_3$ commute with $\mathbf{i}$, i.e. this is a commutative algebra over $\mathbb{C}(\mathbf{i})$) written in both standard and idempotent representations:

$$\mathcal{Z}_3 = Z_{21} + \mathbf{i}_3 Z_{22} = (Z_{21} - \mathbf{i}_2 Z_{22})\mathbf{e}_{23} + (Z_{21} + \mathbf{i}_2 Z_{22})\mathbf{e}_{23}^{\dagger}$$

where Z_{21} and Z_{22} are bicomplex numbers in $\mathbb{BC}_2[\mathbf{1}; \mathbf{i}_2]$, and:

$$\begin{aligned} \gamma_1 &:= \mathbf{e}_{23} := \frac{1}{2}(1 + \mathbf{i}_2\mathbf{i}_3) & \gamma_2 &:= \mathbf{e}_{23}^{\dagger} := \frac{1}{2}(1 - \mathbf{i}_2\mathbf{i}_3), \\ \gamma_4 &:= \mathbf{i}_2\mathbf{e}_{23} = -\mathbf{i}_3\mathbf{e}_{23}, & \gamma_3 &:= \mathbf{i}_2\mathbf{e}_{23}^{\dagger} = \mathbf{i}_3\mathbf{e}_{23}^{\dagger} \\ \gamma_4^2 = (\mathbf{i}_2\mathbf{e}_{23})^2 &= (-\mathbf{i}_3\mathbf{e}_{23})^2 = -\mathbf{e}_{23} = -\gamma_1, & \gamma_3^2 &= (\mathbf{i}_2\mathbf{e}_{23}^{\dagger})^2 = (\mathbf{i}_3\mathbf{e}_{23}^{\dagger})^2 = -\mathbf{e}_{23}^{\dagger} = -\gamma_2. \end{aligned} \tag{5.1}$$

Using the properties of tricomplex numbers, we can write $\mathcal{Z}_3$ also as follows:

$$\begin{aligned} \mathcal{Z}_3 &= (Z_{21} - \mathbf{i}_2 Z_{22})\mathbf{e}_{23} + (Z_{21} + \mathbf{i}_2 Z_{22})\mathbf{e}_{23}^{\dagger} \\ &= (Z_{21}\mathbf{e}_{23} - (\mathbf{i}_2\mathbf{e}_{23})Z_{22})\,\mathbf{e}_{23} + \left(Z_{21}\mathbf{e}_{23}^{\dagger} + (\mathbf{i}_2\mathbf{e}_{23}^{\dagger})Z_{22}\right)\mathbf{e}_{23}^{\dagger} \\ &= (Z_{21}\mathbf{e}_{23} + (\mathbf{i}_3\mathbf{e}_{23})Z_{22})\,\mathbf{e}_{23} + \left(Z_{21}\mathbf{e}_{23}^{\dagger} + (\mathbf{i}_3\mathbf{e}_{23}^{\dagger})Z_{22}\right)\mathbf{e}_{23}^{\dagger} \\ &= (Z_{21}\gamma_1 + Z_{22}\gamma_4)\,\gamma_1 + (Z_{21}\gamma_2 + Z_{22}\gamma_3)\,\gamma_2 \\ &=: Z_1\gamma_1 + Z_2\gamma_2, \end{aligned}$$

where, up to isomorphism of bicomplex algebras, $Z_1 \in \mathbb{BC}_2[\gamma_1; \gamma_4]$ and $Z_2 \in \mathbb{BC}_2[\gamma_2; \gamma_3]$ are treated as independent bicomplex variables. We denote the tricomplex algebra also by $\mathbb{BC}_3[\mathbf{1}; \gamma_1, \gamma_4; \gamma_2, \gamma_3]$

Relations (5.1) among the elements γ_1, γ_2 and γ_3 (note also that $\gamma_1 + \gamma_2 = \mathbf{1}$) allow us to identify a complex ternary number of the form

$$\mathbf{z} = w_1(\mathbf{z})\gamma_1 + Z(\mathbf{z})\gamma_2 = (w_1(\mathbf{z})\gamma_1 + 0\gamma_4)\gamma_1 + Z(\mathbf{z})\gamma_2$$

is also an element of the tricomplex algebra above, as the ternary symmetric basis elements κ_1, κ_2 and κ_3 have the same properties (see (2.11)). All these *identifications* are obtained by algebra isomorphisms, which are easier to understand from the context rather than complicating the notations further.

Using a similar proof as in the two theorems above, one can prove the following:

Theorem 5.6 *Consider a complex ternary regular function* $\mathbf{F} : \Omega \subseteq \mathbb{TC} \to \mathbb{BC}_3[\mathbf{1}; \kappa_1, \kappa_4; \kappa_2, \kappa_3]$ *(denoted simply by* $\mathbb{BC}_3$*)*

$$\mathbf{F}(\mathbf{z}) = W_1(w_1(\mathbf{z}))\kappa_1 + G(Z(\mathbf{z}))\kappa_2,$$

i.e. $W_1(w_1(\mathbf{z}))$ *is holomorphic in* $\Omega \cdot \kappa_1 \in \mathbb{C}(\mathbf{i}) \cdot \kappa_1$ *and* $G(Z(\mathbf{z}))$ *bicomplex holomorphic in* $\Omega \cdot \kappa_2 \in \mathbb{BC}_2[\kappa_2; \mathbf{i}, \kappa_3]$. *Then* $\mathbf{F}$ *extends uniquely to a tricomplex* *holomorphic function* $\tilde{\mathbf{F}} : \tilde{\Omega} \subseteq \mathbb{BC}_3 \to \mathbb{BC}_3$*, where* $\tilde{\Omega}$ *is an open neighborhood of* Ω *in* $\mathbb{BC}_3$.

We leave the proof to the reader.

6 Comparison, Conclusions, and Future Work

The analysis of spaces of regular functions have been studied in [5] where the authors solved the Gleason's problem in the ternary case, using the method described in [2] for several complex variables and in [3] for the quaternionic case. In this work, in a way parallel to the quaternionic case, in solving the Gleason's problem, the authors obtained Fueter–type variables (as discussed, for example, in [7]), as the primary ternary variable is not regular in a usual sense. These Fueter–type variables are commutative and this allowed to define the counterpart of the Arveson space and solve the Gleason's problem defined in the ternary case. The Fueter variables themselves behave as a complex variable and a "light–cone"–type one which reduce to a two real variable theory when restricted to a plane. The fact that these variables only show up in a non–commutative case before is interesting in itself as well. This paper outlines a comparison between complex ternary analysis and the real ternary one, clarifying the notions found in [5]. For [5] we have used the theory generated by an element that cubes to -1, however all results translate with small and careful changes of sign. The analytic theory also carries over smoothly.

The author is now working on a collaborative project on solving the Gleason problem in the complex ternary case, using one of the ternary norms described in Sect. 3.

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