Prospective Teachers Solving a Percentage Problem: An Analysis of the Construction of a Praxeology

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Abstract This study focuses on a team of three primary and lower secondary school student teachers struggling with a percentage problem. The conditions and constraints under which their inquiry unfolds and the very vicissitudes of this inquiry are analyzed in the framework of the ATD. First, a praxeological reference model is constructed through a mathematical analysis. Then the model of didactic moments is used to analyze the construction of a praxeology centered on the type of tasks for which the observed students built up a technique.

From the Outside Inwards: A Centripetal Depiction

The praxeological analysis propounded in this paper will bring to light the hidden complexity of an unassuming training situation involving percentages. The general research question tackled here is the following:

> *Given a problematic question Q, what are the tools that an instance î can mobilize to solve it, and what uses can the instance î make of these tools? What are the conditions and constraints under which î is led to operate, which, given Q, determine the instance î's choice of tools and their uses?*

In what follows, I depict a set of conditions and constraints under which the observation took place. The question Q was devised by one of the teachers that appear in the observed process of study. As will become apparent, this question borders on both algebra and geometry.

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The Institutional Setting

The research site was a department of teacher education at a university college located in Trondheim (Norway), providing four-year programs for primary and lower secondary teachers' education. There were approximately 450 students enrolled in these programs each year. Approximately 75% of the students were females.

The Class and the Curriculum

The students observed were enrolled in a program with an emphasis on mathematics and science subjects (Ministry of Education and Research 2003). The structure of this program is shown in Fig. 1. The acronym RWM stands for "Reading, Writing, and Mathematics". Didactics of subject matter and practice field experiences are integrated into all study units in the program.

The Teaching Unit

The session observed was part of an Algebra module. The topics in this teaching unit as devised by the teachers were studied chronologically in the following order:

> Number sequences; formulas (diverse modes of expressions); development of the formal mathematical language, also in a historical perspective; formulas; the concept of variable; mathematical models (project)*; mathematics in nature and society*; *mathematics in daily life*; the concept of variable*, formulas, and equations**.

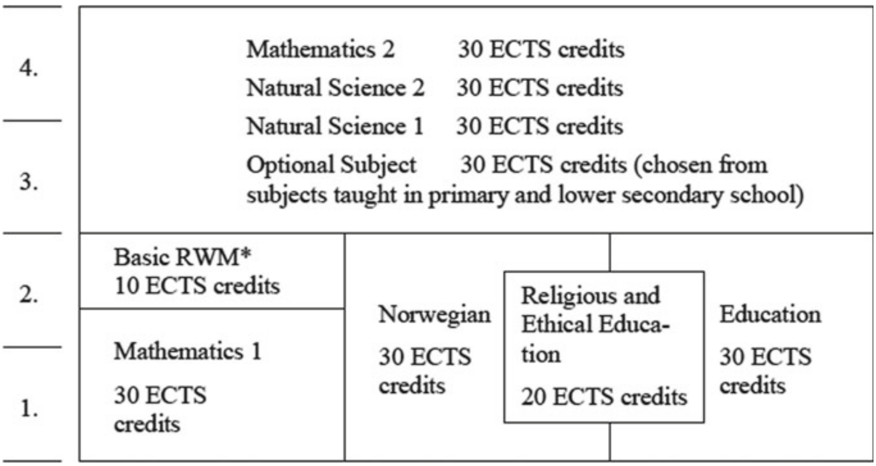

Fig. 1 Teacher education with an emphasis on mathematics and science subjects

The asterisked topics pertain to an "Algebra and Functions" unit. Those in italics were tackled during the observed class session. It has to be stressed that, while relevant to the observed situation (as we shall see), the topic "Fractions and decimal numbers", which pertains to a "Number theory" unit devised by the teachers, was studied *after* the Algebra topics. It is also worthy of note that, in Norway's prevailing mathematics curriculum for Grade 1–10, the mathematical domains include "numbers and understanding numbers, algebra, functions, geometry, statistics and probability" (Directorate for Education and Training, 2020, p. 4).

The Practical Working Conditions

The classroom was a large, horizontal floor room divided into two parts (of different sizes) by movable walls. The biggest part of the classroom had tables and chairs for all the students (the class had 66 students); this was where the introductory part of lessons took place. The observed students were placed in a small room adjacent to the classroom; it was a separate room with an entrance from the classroom. The purpose of this arrangement was to reduce the background noise on the video recordings. The observed students were informed that they could at any time go and ask for help or get suggestions from the teachers or peer students in the large classroom where the rest of the class worked in small groups on the same tasks. On their initiative, the teachers occasionally went into the room in which the observed students worked.

The Observed Didactic System

The students $X = \{x_1, x_2, x_3\}$

The didactic system working in the small room, denoted by $\mathcal{S} = S(X, Y, Q)$, consisted of a group of three students $X = \{x_1, x_2, x_3\}$, and a team of two teachers $Y = \{y_1, y_2\}$. The set X was a "practice group", i.e., a grouping decided by the faculty administration for one academic year, mainly to organize mentored in-field practice in a particular class in primary or lower secondary school. When the data were collected, X was in the second semester of the teacher education program. The 20-year-old student x_1 had attended a voluntary mathematics course (5 hrs a week) in her second year in upper secondary school, preparing for further studies in natural sciences and mathematics and achieving a low mark. The students x_2 and x_3, aged respectively 22 and 23 years, had both attended voluntary mathematics courses (5 hrs a week) in their second and third years in upper secondary school, preparing for further studies respectively in natural sciences and mathematics and in social sciences and economics, with average marks. On the basis of their marks from upper secondary school, X can be considered average strong.

The teachers $Y = \{y_1, y_2\}$

The teachers y_1 and y_2 were experienced teachers of mathematics in the department of teacher education at the university college: while y_1 was a professor of mathematics with more than ten years of practice as a mathematics teacher educator, y_2 was a senior lecturer with more than thirty years of the same practice. The two teachers played slightly distinct parts: y_1 was responsible for the lessons on algebra, including the design of tasks; y_2 had the role of a "teacher assistant" in the orchestration of the students' work and shared with y_1 the task of helping students. The information given above about the students x_1, x_2, and x_3 was not available to y_1 and y_2: they only knew that these students had a general university and college admission certification and the information they could get through teaching them mathematics.

The observer

I observed and video recorded the session for the sake of the research reported here. For the observed session to be as naturalistic as possible, I had the role of non-participant observer. The teachers Y were my colleagues at the time the data were collected.

The Question Q *Under Study*

The question Q—or more exactly the set of questions—offered for study in $\mathcal{S}$ was the following, where the questions sentences in square brackets were ignored by X, maybe for lack of time:

- Imagine that you have a square. Make a new square where the side length has increased by 50%. Q_1. How many percent has then the area increased? [${Q_1}'$. How many percent has the perimeter increased?]
- Imagine now that the side length increases by $p\%$. Q_2. How many percent will the area consequently increase? [${Q_2}'$. How many percent will the perimeter consequently increase?] [Q_3. If you had a cube where the side length increases by $p\%$, how many percent will the volume consequently increase?]

The tasks in the algebra module were designed by y_1, who claimed that one aim was to have students engage, in different contexts, in generalizations that would eventually be expressed in algebraic notation. Beyond that, we know about nothing concerning the choice of the question set . The document (in Norwegian) handed out to X contained a list of six "assignments", each made up of two tasks, from which every group of students had to choose one assignment. Moreover, the groups were paired so that, at the end of the class, each group in a pair was supposed to present to the other group their solution to the chosen assignment. The observed students chose Assignment 1, in which the question set was coupled with a task that consisted in devising a realistic context for a certain equation before solving that equation (in the case of X, it was: $\pi r^2 \cdot 2 + \pi r^2 \cdot 4 = 1000$). We know nothing of the reasons why X chose Assignment 1.

Conditions and Constraints

In the session observed, the students X are going to face two problematic tasks: the task t_1 consists in providing an answer to the question Q_1; the task t_2 consists in answering the question Q_2, which is clearly presented as a generalization of the question Q_1. The choice to concatenate questions Q_1 and Q_2 manifests an intention to help students build a technique to perform t_1 and then t_2. We will see that the problem statement (due to y_1) conceals other potential hints to help students tread their way to "success".

Which Mathematical Praxeology? A Reference Model

Suppose persons, not necessarily students, have to give an answer to the question Q_1. Which praxeology can they use? Let's start with the *most rudimentary praxeology*. Suppose the side length of the square is 10 (in some unit of length); the new length is therefore $10 + 5 = 15$. The original area is $10^2 = 100$ and the new area $15^2 = 225$. Here is a purely numerical rule to calculate the percentage increase (Percentage Change, n.d., To Calculate the Percentage Change section):

> First: work out the difference (increase) between the two numbers you are comparing [i.e., 225 – 100 = 125]. Then: divide the increase by the original number [125 / 100 = 1.25] and multiply the answer by 100 [1.25 × 100 = 125; the percentage increase is thus 125%].

Other, more technologically advanced, praxeologies rely on the use of letters in order to generalize the foregoing example. Let a be the original side length, a' the new side length: the rate of the side length increase is given by the ratio $\frac{a'-a}{a}$. Let A be the original area and A' the new area: the rate of the area increase is $\frac{A'-A}{A} = \frac{a'^2-a^2}{a^2}$. Given the percentage increase on a, that is, $\frac{a'-a}{a} = p\% = \frac{p}{100}$, and using some algebra, one can calculate the percentage increase on A:

$$\frac{a'^2 - a^2}{a^2} = \frac{a' - a}{a} \times \frac{a' + a}{a} = \frac{a' - a}{a} \times \frac{a' - a + 2a}{a} = \left(\frac{a' - a}{a}\right)^2 + \frac{a' - a}{a} \times 2 = 2\frac{p}{100} + \left(\frac{p}{100}\right)^2$$

If $p = 10$, one arrives at: $2 \times 0.1 + 0.1^2 = 0.2 + 0.01 = 0.21 = 21\%$. A more standard way of arriving at this result consists in first establishing the following basic formula $a' = \left(1 + \frac{p}{100}\right)a = ra$, where $r = 1 + \frac{p}{100}$. We then have: $\frac{a'^2-a^2}{a^2} = \frac{(ra)^2-a^2}{a^2} = \frac{r^2a^2-a^2}{a^2} = r^2 - 1 = \left(1 + \frac{p}{100}\right)^2 - 1 = 2\frac{p}{100} + \left(\frac{p}{100}\right)^2$. Here we see clearly that the value of $\frac{a'^2-a^2}{a^2}$ is independent of a.

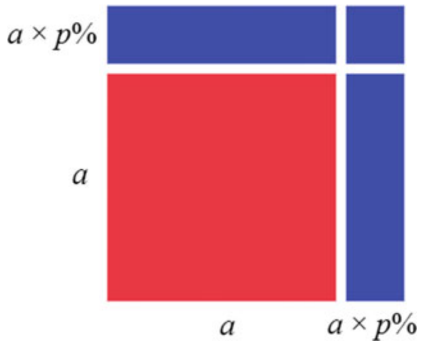

Fig. 2 A geometric model of a percentage increase

This property generalizes to any homogeneous function f with a degree of homogeneity k (if a increases by $p\% = r - 1$, then $\frac{f(a')-f(a)}{f(a)} = \frac{f(ra)-f(a)}{f(a)} = \frac{r^k f(a)-f(a)}{f(a)} = r^k - 1$) and may lead to a third, proto-algebraic praxeology, that makes us come back to the "geometric algebra" of the ancient Greeks (Høyrup 2017).

It may be supposed that the choice of squares by y_1 (instead of other similar figures, for which the same property holds) was inspired by the didactic hypothesis that this "geometric" praxeology should facilitate the students' access to a full-fledged algebraic praxeology: it would suffice to "read" Fig. 2 to conclude that the area increase is $p\% \times a \times a + (p\% \times a)^2 + p\% \times a \times a$ and, therefore, that the percentage increase of the area is $2\frac{p}{100} + \left(\frac{p}{100}\right)^2$.

Which Didactic Milieu? Some Preliminary Remarks

The more material items

The above reminds us of two generic requirements of human action, whether mathematical or not. Firstly, in order to achieve a particular result—for example in order to determine the value of a certain percentage—, we must *do something*: we are not mere guessers nor "mentalists" who can instantly, without further ado, read Nature's mind. Secondly, in order to do something, we need "something": we need ostensive and non ostensive *tools*.

What is the didactic milieu M made of that the students observed can draw on? Let us start with the more "material" elements of the milieu M. Besides their own praxeological equipment, the three students have available (1) some writing material, (2) a calculator, and (3) a mathematics didactics textbook (Selvik et al. 2007). This textbook, however, has nothing on percentages. It contains 18 chapters, the first two of which are not unrelated to the task done by the observed students. The first one is entitled "Learning of algebra" and has sections on "Why learn algebra?" and "Prealgebra". The second one is entitled "Symbols in mathematics". Its first two sections are entitled, respectively, "Symbols denoting numbers and magnitudes" and

"Symbols with which we can reckon/calculate and think". In fact, these sections both (unintentionally) point to an essential aspect of the initial deficiencies of the milieu M which the students observed had to rely upon.

Didactic Moments

I shall hereafter refer to the model of *didactic moments* (Chevallard 1999), which is at the heart of the ATD. The construction of a praxeology $\wp$ centered on a type of tasks T involves six "dimensions" called *moments* of the study process that generates the emergence of $\wp$. They are respectively (1) the *moment of the first encounter* (with T), (2) the *moment of the exploration of the type of tasks T and of the emergence of a technique* τ (of performing tasks t of the type T), (3) the *technological-theoretical moment*, when the *logos* part of the praxeology emerges, (4) the *moment of working on the praxeological organization* $\wp$ under development (in order to improve both the *praxis* block and the *logos* block), (5) the *moment of institutionalization* (of $\wp$), and (6) the *moment of the evaluation* (of both $\wp$ and of a person's relation to $\wp$). The following analyses are based on a transcript (in English) of the utterances (in Norwegian) exchanged between members of X and Y and on a description of their actions. Although the word *moment* as used in the expression "didactic moment" has no temporal meaning (it refers to a certain "dimension" of the study work), I will analyze the work session by following its time course.

A bad start

The discernible aim of the work session is the building up of a praxeology $\wp$ centered on the following type of tasks T: "Determine the percentage of increase in the area of a square whose side length is increased by $p\%$." Note that a task $t \in T$ depends a priori on two parameters: the side length of the square, and the percentage $p\%$. The beginning of the work session comes under the *moment of the first encounter* with T. The student x_1 first draws a square with side length 2 cm, whose area is 4 cm^2; an increase by 50% in the side length leads x_1 to draw a square of side length 3 cm, with area 9 cm^2. The increase in area is 9 cm^2 – 4 cm^2 = 5 cm^2.

At this point, x_1 induces the trio to (erroneously) take as the rate of increase the ratio 5 / 9. The student x_2 asks why one should divide the difference by the *greatest* of the two numbers, to which x_1 rejoins assertively: "Part divided by whole. It's just a rule that I have learnt." Although x_1's assertion is vividly discussed between the students, this crucial mistake will determine the group's activity prior to the intervention of the teacher y_1. The students thus conclude that the percentage sought is equal to $5 / 9 = 0.555\ldots \approx 56\%$. All this is part of the *technological-theoretical moment*: the recipe "part divided by whole" is a *technological* self-conviction, apparently "justified" by this (implicit, unspoken) *theoretical* belief: "Any percentage is smaller than or equal to 100%."

Correcting the trajectory

Note that, until then, the milieu M contains drawings of two squares (with side lengths 2 and 3 cm), words and some arithmetic (numbers, and the division operation), but no letters or other symbols. The student x_1 tries to generalize her conclusion to any value of p using the following haphazard piece of reasoning:

> … when we increase this [the side length] by fifty percent, then we got fifty-six percent increase of the area, thus, six percent more than what is here [the side length]. So, then it will be p percent plus six.

However, x_1 is not completely sure of her conjecture and adds: "But I don't know if it will be like this." Yet the trio continues on the path opened by x_1 and goes on exploring T, taking as a new specimen a square of side length 5 cm: the new side length is therefore 7.5 cm and the new area 56.25 cm^2. At this point y_1 enters the room, while x_1 is "testing her theory", as x_3 says, arriving once again at (56.25 – 25) / 56.25 = 0.555… $\approx$ 56%. Through a fairly long and winding dialogue, y_1 corrects the trajectory of the trio: their exchange confirms that the students have first and foremost to accept that a percentage of increase can be *greater* than 100%.

Relaunching the search unencumbered

When y_1 leaves the room, he also leaves the trio in a state of bewilderment. For the first time, if we except x_1's attempt to enact a general law, the students take, however awkwardly, the parameter p into account, by trying to mimic the only calculation they now know to be correct, i.e., (9 – 4) / 4 = 1,25 = 125%—they consider for example the expressions: $2 + p - 4$; $2 \times p$; or $2 + p\%$. This comes under the *moment of the exploration of the task and of the emergence of a technique* τ. They seem to find themselves in a fog about what to do.

Considering the flow of nonproductive ideas that their activity gives rise to, the observer (myself) reminds them that they can ask one of the teachers for help—which they do one minute later. The teacher y_2 soon comes to their rescue; it seems that y_2 tries to underline the need for an appropriate study technique. But, instead of using numbers (not to mention letters), the group led by y_2 gets bogged down in considerations about the graphical and even material (using sheets of paper and scissors) modelling of the phenomenon under consideration. After an involved study of the case $p = 50$, y_2 urges the students to study the case $p = 25$.

Once again, the combined handling of paper manipulatives and percentages of areas proves arduous. After having insisted on the use of paper models, y_2 then encourages the trio to dematerialize these models, arguing as follows:

> Now it could have been interesting to do something which is manageable once more, to take ten percent for instance, and see what happens then. Because that is quite easy, you may not even need to fold paper. Perhaps you could have drawn a picture and imagined, with this [the manipulatives] as a model.

He thus surreptitiously induces the students to consider the case of a square of side length 10 cm with $p = 10$.

The teacher y_2's decisive intervention

The students strive to do it, while persistently confusing percentage points and areas of one cm^2. For his part, y_2 tries to convince himself that the process is taking shape ("Yes, now you have got on the track of something", he observes, and makes this comment directed to the observer: "I think they are on track here now"). He presses the trio to count the supplementary "small squares"—which, in the case considered, are $2 \times 10 + 1$ in number—and conclude that, therefore, the percentage of increase equals 21%. To top it off, y_2 "helps them to write down in percentage notation the arithmetic results (the areas) achieved in each of the three cases", i.e. $2 \cdot \left(\frac{50}{100} \cdot \frac{100}{100}\right) + \frac{50}{100} \cdot \frac{50}{100}$ (increase by 50%), $2 \cdot \left(\frac{25}{100} \cdot \frac{100}{100}\right) + \frac{25}{100} \cdot \frac{25}{100}$ (increase by 25%), and $2 \cdot \left(\frac{10}{100} \cdot \frac{100}{100}\right) + \frac{10}{100} \cdot \frac{10}{100}$ (increase by 10%).

This work sequence is a contribution to the *moment of working on the praxeological organization* $\wp$ being built up. All this is then abruptly expressed in algebraic form to give first $2 \cdot \left(\frac{p}{100} \cdot \frac{100}{100}\right) + \frac{p}{100} \cdot \frac{p}{100}$ then $\frac{2p}{100} + \frac{p^2}{100^2}$. This is the *moment of institutionalization.*

Learning without warning

The above did not conclude the work session. The trio had to present "their" results to another group in the class. What happened then in this very special *moment of validation*? The trio first illustrated the problem using paper manipulatives in two special cases ($p = 50$ and $p = 25$), before writing on the blackboard the expression $\frac{2p}{100} + \frac{p^2}{100^2}$ as the solution to the problem. They commented that this expression was derived from the expression $2 \cdot \left(\frac{p}{100} \cdot \frac{100}{100}\right) + \frac{p}{100} \cdot \frac{p}{100}$, which has the same structure as the concrete cases previously presented. Finally, they got positive feedback from the listening group of student teachers, who claimed to understand what was being explained. The didactic task of presenting one's own findings to a group of peers acted as a factor of learning and a revealer thereof.

Discussion

The end of the study process—when the three students present "their" results to some of their peers—sounds like a kind of miracle, given the winding and uncertain history of the making of these results, which boil down essentially to the expression $\frac{2p}{100} + \frac{p^2}{100^2}$. This is perhaps the most important discovery of the study: the role played by a pedagogical structure too rarely mobilized—the presentation to others of the results of a work—, which by contrast is so familiar to researchers in their own working communities, has a kind of positive retroactive effect on the understanding by the authors themselves of the work they present.

Having noted this, I still have to highlight two more facts revealed by the observation and analysis carried out. The first fact is twofold. On the one hand, the students observed *do* have, in their praxeological equipment, a praxeology relating to percentages (they understand the question they have to tackle and take action, however wrongly). On the other hand, there is no mechanism in the didactic system to ensure that this praxeology is adequate (in fact, it is not, as the use of a wrong ratio shows) and to update it if necessary, as if no one cared about the tools really available at the time of taking action. As we know, this will lead the observed trio to wander quite a long time in vain in search of a valid answer.

The second fact relates to the intervention of the teachers and is also twofold. On the one hand, there seems to be a (traditional) tendency to consider that geometric entities are more accessible than algebra for these students and, more precisely, that geometry is the main pathway to algebra. On the other hand, the final intervention of one of the teachers closes the inquiry by giving almost explicitly the hoped-for answer. Although this "oblique" teaching somewhat departs from the canons of modern pedagogy, we know now that its learning yield (as well as that of the work done previously during the session) was far from negligible. All this constitutes a distinctive contribution to the study of the research question made explicit at the beginning of this study and reveals, in contrast to analyses which focus on the student only, the relevance of a crucial question—Which tools are mobilized and what uses are made of them?—to which we must endlessly return.

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