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Berta Barquero
Ignasi Florensa
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Editors

Extended Abstracts Spring 2019

Advances in the Anthropological Theory of the Didactic

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Extended Abstracts Spring 2019

Advances in the Anthropological Theory of the Didactic

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Editors
Berta Barquero
Facultat d'Educació
Universitat de Barcelona
Barcelona, Barcelona, Spain

Ignasi Florensa
Escola Universitària Salesiana de Sarrià
Barcelona, Spain

Pedro Nicolás
Universidad de Murcia
Murcia, Spain

Noemí Ruiz-Munzón
TecnoCampus
Universitat Pompeu Fabra
Mataró, Spain

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Introduction to the Book with the "Extended Abstracts"

During June and July 2019, the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona) hosted an Intensive Research Programme focused on Advances in the Anthropological Theory of the Didactic (ATD) and their consequences in curricula and teacher education.

The programme was structured in four Advanced Courses that combined presentations of recent research results, theoretical advances and new methodologies, including participatory sessions to discuss ongoing research. The four courses were about the following topics:

- Advanced Course 1: Dialogue between theories.
- Advanced Course 2: Teacher education and the professionalisation of teaching.
- Advanced Course 3: The curriculum problem and the paradigm of questioning the world in mathematics and beyond.
- Advanced Course 4: Research in didactics at university level.

This Intensive Research Programme focused on the latest developments of the ATD and its links with other approaches in Mathematics Education. The advanced courses were structured into two parts. In the first part, invited researchers offered some lectures and workshops. In the second part, participants were invited to present ongoing research works or experiences about the course topic.

This book brings together the participants' contributions. More concretely, the book is structured in three parts, each one corresponding to one of the central topics addressed during Advanced Course 2, 3 or 4 respectively of the intensive research programme.

Part I focuses on the topic of *Mathematics teacher education and the professionalization of teaching*. It is composed by nine research articles authored by researchers from seven countries. The common core has been the study of the teaching "semiprofessions" at school and at the university, and the problem of teacher education. It is based on two main concepts, first, that of the profession (as distinguished from the notion of "semiprofession"); second, the concept of problems of a profession. The main idea here is that, however "subjective" it may seem to others, any observed "difficulty" in the exercise of the profession must be considered and reformulated in the ATD framework, to address it through a theoretically founded design.

Part II and **Part III** focus on the topic related to the study of the historical transition from the classical paradigm of *visiting works* (based on the sequential access to previously established knowledge) to an emerging didactic *paradigm of questioning the world*, in which one ideally starts from a question and tries to work out an answer to that question. In this new paradigm, in which the notions of *inquiry* (into a given question) and *study and research paths* (SRP) play a key role, the works of culture do not vanish. They are assigned a more authentically functional role, which requires studying works with the aim of providing an appropriate answer to a question or set of questions, which correlatively become part and parcel of a "new" curriculum based on the study of questions. The multiform notion of inquiry-based teaching and issues raised by the design of purportedly "inquiry-based curricula" are part of this axis.

Part II focuses more explicitly on the curriculum problem at preschool, primary and secondary school education. It presents nine research articles, which feature research work in a diversity of school levels in six countries. **Part III** focuses on the research advances in didactics at the university level. This part presents seven research articles with contributions from a large variety of university educational contexts.

We hope that the reader will find these contributions scientifically stimulating and get an inspiring insight into the latest developments of research in the framework of the ATD.

Contents

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Mathematics Teacher Education and the Professionalization of Teaching

Modifying Teachers' Didactic Praxeologies in the Perspective of the Paradigm of Questioning the World: The Case of the Technological-Theoretical Moment

Costanza Alfieri

Abstract The aim of this research is to identify some conditions and constraints of the transposition of some didactic praxeologies in teacher training. For this purpose, an inquiry (in the sense of the anthopological theory of the didactic) on the topic of functions has been implemented during the first year of secondary school in France (15-16-year-old students) by a teacher trained within the framework of the ATD. The inquiry led by the teacher is divided into two parts: the first one aims to teach the notion of function as a relation between magnitudes and its optimisation by using graphic and algorithmic techniques; the second one focuses on the study of quadratic functions and resolution of equations. This paper presents the analysis and results related to the first part. The analysis of the inquiry unveils some didactic phenomena concerning the realisation of the technological-theoretical moment that suggest ways of developing teacher training.

The *Inquiry* Implemented

In the anthropological theory of the didactic (ATD), the name *inquiry* applies to the study of a question Q by an inquiry team X (a class, a group of researchers, etc.) under the direction of Y (a teacher, a group of peers, etc.). In the case examined, X was a class of the first year of the upper secondary school in France (15-16-year-old), $y \in Y$ was the teacher and the question Q was: "How to optimise a magnitude?". Using the semi-developed *Herbartian schema* introduced in the ATD (Chevallard 2009), we can sum up the inquiry at stake as follows:

$$[S(X; y; Q) \rightarrow M] \hookrightarrow A^{\heartsuit}$$

where M is the *didactic milieu*, containing old and new resources that X could use if needed, and $A^{\heartsuit}$ is the answer looked for (which is a mathematical organisation).

C. Alfieri (✉)
Aix-Marseille University, Faculty of Mathematics, Marseille, France

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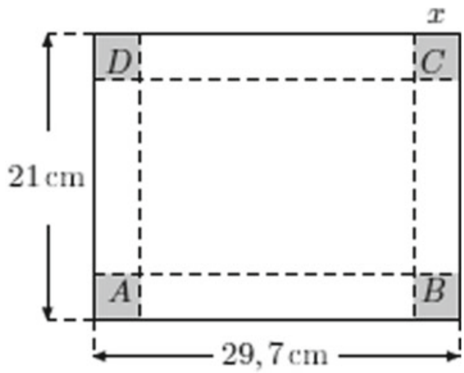

Fig. 1 The pattern of the box

The inquiry here analysed was conceived in order to study different mathematical entities: on the one hand, the notion of function as a relation between magnitudes and their optimisation by using graphic and algorithmic techniques; on the other hand, the study of quadratic functions and the resolution of 2^{nd} degree equations. In this paper, we present the results regarding the first part only.[1]

The question Q was delivered to the class X by introducing the following situation, which has been already analysed by different researchers (see e.g., Guissard et al. 2013):

> A pâtissier wants to produce some boxes having parallelepiped shape for packaging tea cookies. He wants to manufacture the boxes using an A4 paper, without cutting the paper. The pâtissier wants to make the box with the greatest volume. He has done some attempts, that we have here, without finding a solution. Can we help him find the box with the greatest volume? What are its dimensions?

The situation was introduced by giving some boxes previously constructed, according to the model shown in Fig. 1, although this model was not presented to the students. At this point, the didactic *milieu M* contains the boxes presented by the teacher, the worksheets available in the computer lab as well as some mathematical organisations previously constructed, such as linear functions and some notions of basic algebra. During the development of the inquiry, the graphic calculator that the students are becoming familiar with will replace the worksheet by using tables of values.[2]

After the end of the inquiry, the lessons have been analysed to see if the didactic organisation planned had permitted to build the mathematical organisation proposed by Artaud and Menotti (2008) and Artaud (2018), which duly happened to be the case. However, the analysis of the inquiry has unveiled some didactic phenomena

[1] This research is an extract of a master's dissertation supervised by Michèle Artaud at Aix-Marseille Université, which is is available at: https://irem.univ-amu.fr/sites/irem.univ-amu.fr/files/public/alfieri_memoire_enligne.pdf.

[2] Such a table is elaborated by a calculator for a given function: it shows the list of the x-values and the list of the corresponding $f(x)$. The x-values are established by the user, who chooses the distance Δ between two consecutives x-values.

concerning the realisation of the technological-theoretical moment, i.e. the creation of a *logos* that justifies and produces the elaborated techniques. To be specific, we focused on the teacher's role: to what extent does she *guide an inquiry,* or does she *profess some body of knowledge*?

Preliminary Analysis

We now propose a partial analysis of the 4th lesson of the inquiry, which shows an episode of the realisation of the exploratory moment, i.e. the moment during which the class explores the possibility of creating a technique to find the maximum of the volume. During this lesson, we can observe the articulation of two techniques used to optimise a function: the first one is based on the study of a table of values and the second one on the study of the graph of a function.

Determining the Box of Maximum Volume

At the beginning of lesson 4, the students have already modelled the problem with the volume function expressed as:

$$V(x) = (29,7{-}2x)(21{-}2x)x$$

The class is exploring this model by employing the tools available on the calculator, like the tables of values. During this exploration, some students encounter the graphic representation of V. This encounter leads the class to engage in the study of the following question:

Q_p: How can we determine the maximum value of the volume by reading the graph of V?

While studying Q_p, two new subquestions emerges: the first one is

Q_1: How can we determine if a curve is a straight line or not?

and the second one is

Q_2: Which values of x are acceptable in order to calculate V?

Question Q_1 arises when the students graph the curve at the calculator and observe what seems to represent a straight line (Fig. 2a).

Since the students do not know how to manipulate the settings of the screen, they are led to elaborate an algebraic technique in order to establish whether V is a linear function or not.

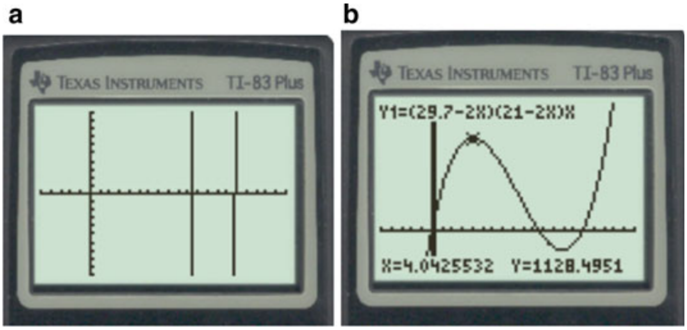

Fig. 2 **a** The straight line, **b** Volumes greater than the local maximum

Question Q_2 arises when the graph (Fig. 2b) shows that some values of the volume are greater than its local maximum (previously indicated as the maximum value of the volume). However, those values cannot be considered because the boxes can only be constructed for $x \in [0; 10.5]$, due to the measures of the A4 paper.

The Dialectic of Media and Milieus

The analysis of lesson 4 shows that the exploratory moment is led with a satisfactory *dialectic of media and milieus*, that is, a process which consists in analysing old and new resources in order to elaborate an answer A, to the question under study Q—even if the *milieu* includes few resources external to the class.

The study of Q_1, “How can we determine if a curve is a straight line or not?”, provides an example of what was previously stated. In order to answer Q_1, a student proposes to develop the expression $V(x) = (29{,}7 - 2x)(21 - 2x)x$ to check if it is of the form $f(x) = ax + b$. Since the algebraic development $V(x) = 4x^3 - 101{,}4x^2 + 623{,}7x$ does not look like a linear form, the class suggests checking if the calculations are accurate by graphing V and its algebraic development on the calculator to see if they overlap. The following extract, in which P is the teacher and S_k the student, shows a part of this episode:

Teacher: What do we observe once we have plotted the curves? What does it mean that they overlap?

Student 1: That the development [of the expression of the function] and the function are equal.

This passage highlights that the calculator is effectively used within the student’s *milieu* to check their conjectures: it is indeed the dialectic between the graph of V plotted on the calculator and the algebraic representation of the function (both in the initial form and the developed one) that permits to verify that the calculations are correct.

Globally, it appears that the dialectic of media and *milieus* is properly integrated in the class's activity within the exploratory moment. So that the inquiry takes on the form of a proof and conjecture process which allows to conduct a critical study of the question *Q*. The hypothesis that emerges during the study of a question is therefore validated by the available resources, with no intervention of the teacher. Consequently, the development of the dialectic of media and milieus during the realisation of the exploratory moment is essential to increase the student's *topos* and to install the direction of study within the paradigm of questioning the world (Chevallard 2007).

The Dialectic of the Collective and the Individual

The same evaluations can be accomplished for what concerns the *dialectic of the collective and the individual* (Kim 2015): indeed, *P* leaves enough time in autonomy to *X* to elaborate a technique and simultaneously she collects the results of all the students on the board. On the contrary, the mathematised technological justifications of the elaborated techniques, that is, mainly the definition of increasing and decreasing function (and their use), seem harder to emerge. This phenomenon is visible in the following episode:

Teacher: So, you are working on the graphical representation… We plot it and then what do we observe?
Student 2: Hem… There, where the curve goes up more.
Teacher: (rephrasing something that student 3 said) We need to change the representation and the scale in order to see where the curve goes up more

Here we see that a technological element emerges naturally: indeed, one of the students has commented that the maximum value of the function must be found "where the curve goes up more" (increasing part of the function). However, this technological element is not exploited by *P* to carry out a relevant part of the technological-theoretical moment.

The Problem with the Logos *Block*

The rest of lesson 4 is centred on elaborating a technique to determine the maximum value of the function by reading its graph; the justifications of the fact that the highest point of the curve corresponds to the maximum are not further investigated.

Throughout the inquiry, we have observed that the class does not assume the responsibility of the technological-theoretical work. On the contrary, the legitimisation of the construction of the *logos* remains almost entirely in the teacher *topos*: *P* tends to *profess* the expected mathematical knowledge, instead of *guiding the inquiry* towards a formalisation of the mathematical entities.

On the one hand, the reluctance of *P* to carry out the technological-theoretical work could be linked to a lack of didactic praxeologies. Indeed, *P* had never worked on the implementation of an inquiry in a structured way, that is, with a mathematical organisation previously conceived, along with a planned didactic organisation. On the other hand, we are aware that the students' relation to mathematics evolves slowly. In the class where the inquiry was developed, a wide change of the didactic contract was brought about, which could not reasonably be expected to be fully established after only four weeks of inquiry.

In addition, during the meetings to prepare the inquiry, *P* expressed her doubts about the achievement of the technological-theoretical moment. On one side, her lack of confidence in her praxeologies as a study director in the paradigm of questioning the world; on the other side, she was also sceptical about the class's responsiveness in this process—in fact, throughout this first part of the inquiry, the students remained somewhat passive.

Furthermore, we can remark that the milieu established during the inquiry did not contain enough technological elements. *P* could have asked to the class:

Q_θ: Why is that the greatest value?

in order to spark technological justifications. However, this question would have represented a marked change: indeed, in order to answer Q_θ, the study process should have shifted from the interaction between the system and the model (that is, the box and the volume function) to the mathematical model only. Certain professional gestures, concerning the elaboration by *X* of proper techniques and the development of their justifications, are required for *P* to fulfil this delicate passage which requires a wide negotiation of the didactic contract with the students.

Preliminary Hypothesis and Conclusions

The Dialectic of Media and Milieus Revisited

To sum up the phenomena described, we can say that the analysis conducted on the realisation of the exploratory moment shows that the dialectic of media and milieus is well conducted between the system[3] S_0, that is the box previously constructed, and its digital/algebraic model M_0, meaning the "formula" representing the volume of the box and its list of values in the worksheet or at calculator, which allows the emergence of the graph of the function. Once the notion of a graph emerges, the digital/algebraic model M_0 becomes the new system S_1, to which the graph plays

[3] In the ATD, a *system* is a piece of nature that we wish to study by questioning a *model* that should provide us information on the system.

the role of a model M_1. The dialectic between the digital/algebraic model and the graph should foster the emergence of two technological elements of the mathematical organisation at stake: the notion os graphical representation as the set of points (x, $V(x)$) and, consequently, the definition of the maximum of a function as a value $x_0 \in [0; 10.5]$ for which $V(x_0) > V(x)$ for all $x \in [0; 10.5]$. However, at this point we have observed an interruption of the dialectic of media and milieus.

The technological-theoretical moment that we have observed concerns essentially the justifications of techniques through statements poorly developed, without a meaningful dialectic of media and milieus that allow testing the statement the class has arrived at.

All this can be subsumed under what Chevallard (2007) calls an "*épistémologie autoritaire*":

> If the demonstration is supposed to guarantee the truth of the mathematical assertion that it 'demonstrates', it is in law and in fact the teacher (or the textbook) who guarantees the validity of the demonstration and thus, ultimately, of the assertion itself […] (p. 17, our translation)

Indeed, it is culturally considered that doing mathematics means proving by a deductive reasoning a statement—that is, considering only the deductive milieu without considering a broader dialectic of media and milieus to which this statement has to resist before we try to prove it deductively. As stated by the author, this way of studying mathematics seems to be fully integrated in the dominant paradigm at school, for which the truth of a mathematical assertion is guaranteed by the proof *professed* by the teacher. We therefore consider that installing a dialectic of media and milieus to test and prove mathematical assertions is an essential condition in order to realise the technological-theoretical moment within the paradigm of questioning the world.

The Teacher's Praxeological Equipment

In this perspective, it seems useful that the praxeological equipment in teacher's position would contain the idea of the relation between the system and the model as well as the relation between experimentation and deduction, given their importance in the completion of the type of tasks T_c, "checking if a result is true" (Chevallard 2007; Artaud 2019).

More precisely, the type of tasks T_c, as stated by Artaud (2019), is a didactic type of tasks for which a technique and a *logos* must be developed. In order to perform T_c, either we prove it experimentally, or we deduce it from the available theory. We can therefore highlight two subtypes of tasks, i.e., T_d, "checking if a result is deductible from the available theory" (Artaud 2019), and T_e, "checking if any form of experimentation suggests that the result could be true".

An embryo of a technique for T_c would imply first an experimentation, i.e., the performing of T_e in order to assure that the result is true in the system. Once this is confirmed, we wish to deduce the result from the available theory (therefore performing T_d). The *logos* associated to this technique therefore boils down to the following (Chevallard 2007; Artaud 2019). Let us consider a system S (magnitudes, in the case under examination) and a statement θ on S (the greatest value is "where the curve goes up more"). We wish to build a deductive theory T on S such that if θ can be deduced from T, then θ is true in S. The construction of the theory T implies that we will insert in T some assertions considered to be true in S, which will have the status of axioms in T, elicited by "questioning" the system S. This means that T will be constructed by a mechanism of induction, starting from results and definitions stated experimentally.

Ultimately, $T(S)$ is thus the outcome of a series of back-and-forth oscillations between deduction and experimentation, a process named the *dialectic of experimentation and deduction* (Kim 2015). When this *logos* block is part of the didactic praxeology about the conception and realisation of the technological-theoretical moment, it is clear that one part of the technological-theoretical moment has to be realised experimentally, by an interaction between the system and its model, and one part has to be realised in the model, in order to deduce some results (Artaud 2016).

In the case studied here, the *logos* block described is not part of the teacher position's equipment. Specific didactic techniques are required to accomplish these professional gestures, and these techniques need to be further investigated and then detailed. Simultaneously, the conditions and constraints of diffusion of this *logos* block among teachers should be examined, both for professionalising the teaching trade and changing the dominant paradigm (Artaud 2019).

The results of the study suggest that the current praxeological equipment of the teacher's position is typical of a transition yet unfulfilled towards the paradigm of questioning the world. In order to overcome the traditional stance of the *professor-who-professes* and to fully assume the position of *study director*, firmly innovative teaching praxeologies are required, which it is up to didactics to devise and design. And it is in this direction, we believe, that teacher education should be moving.

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Pre-Service Teachers' MPCK Enhancements Using the Framework of Anthropological Theory of the Didactic

Rosie Lopez-Conde, Ma. Nympha Beltran-Joaquin, and Catherine Vistro-Yu

Abstract This study aims to provide an intervention to the teacher education programs of the Philippines in response to the dismal results in TEDS-M 2008. The intervention here developed is a prototype that focuses on teaching Instrumentation in Mathematics within the anthropological theory of the didactic (ATD) by using the notions of the study and research paths (RSP) and praxeologies. The intervention is developed in the Didactics of Mathematics Course (DMC). Moreover, the DMC prototype is tested to be then replicated to help to improve the Mathematics Pedagogical Content Knowledge (MPCK) of pre-service teachers. It could also take part in the pre-service teacher education curriculum as a new course for future mathematics teachers. This intervention addressed the need to develop MPCK. We conclude that the DMC Prototype is practical and effective. Results showed that DMC enhanced the pre-service teachers' MPCK ($Z = -2.96$, $p = 0.003$).

Introduction

The Philippines' future elementary and secondary teachers ranked among the bottom compared to other countries worldwide in the three areas of Mathematics: Algebra, Geometry, and Numbers according to the Teacher Education Development Study in Mathematics (TEDS-M) of 2008. In particular, the country ranked eighth or ninth out of 10 in terms of Mathematics Content Knowledge (MCK) and Mathematics Pedagogical Content Knowledge (MPCK). These TEDS-M findings manifest the low levels of understanding of Filipino future teachers in the basic content and content pedagogic skills in Mathematics. TEDS-M 2008 is a comparative study of

R. Lopez-Conde (✉)
Philippine Normal University, Manila, Philippines
e-mail: conde.rl@pnu.edu.ph

Ma. N. Beltran-Joaquin
College of Education, University of the Philippines-Diliman, Manila, Philippines

C. Vistro-Yu
Department of Mathematics, Ateneo de Manila University, Manila, Philippines

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teacher education focusing on preparing mathematics teachers regarding provisions, opportunities to learn, and relationships on MCK and MPCK at the primary and lower secondary levels of the 17 participating countries. These countries include Botswana, Chile, Georgia, Germany, Malaysia, Norway, Oman, Philippines, Poland, Russian Federation, Singapore, Spain, Switzerland, Taiwan, Thailand, and the USA. More than 20,000 future teachers were assessed in terms of their knowledge of mathematics, mathematics pedagogy, and general pedagogical knowledge (Tatto et al. 2009, 2010).

This study aimed to provide an intervention to the teacher education programs in response to the dismal results in TEDS-M 2008. The intervention is called the Didactics of Mathematics Course (DMC). The study developed a DMC prototype which can be, in a future, replicated to help to improve the Mathematics Pedagogical Content Knowledge (MPCK) of pre-service teachers, and it could be part of the pre-service teacher education curriculum as a new course for future mathematics teachers. This intervention addressed the need for pre-service teachers to develop MPCK. Courses in MPCK are part of pre-service teacher education programs in most of the countries around the world. MPCK courses such as Didactics of Mathematics, History of Mathematics, Problem Solving, Teaching Algebra, and Teaching Geometry are staples in some countries (e.g., Korea, Germany, France, Denmark, and Spain). These courses are timely for future mathematics teachers in the Philippines as the country is implementing the new K to 12 Basic Education Curriculum. Didactics of Mathematics topics in the Philippines have been sometimes included in some education pedagogical subjects such as Strategies in Teaching Mathematics and the Teaching of Mathematics Concepts and Methods, as evident in the survey conducted from some Philippine colleges and universities. Didactics of Mathematics as a course is not yet offered in the Philippine teacher education program. The DMC Prototype was compared with the pre-existing pedagogical courses in teacher preparation programs in Philippine universities and colleges. The analysis showed that the DMC is different from the existing pedagogical courses in the Philippines due to the uniqueness of the contents and activities embedded in the course. Park (2004) stated that in Korea, being one of the top-performing countries in TIMSS and TEDS-M, PCK is embedded in pre-service teacher education programs. As a result, more colleges (Berg 2006; Silverman and Thompson 2008) of education establish courses that address PCK, such as Didactics of Mathematics, History of Mathematics, Problem Solving, Teaching Algebra, and Teaching Geometry. It is postulated that a Didactics of Mathematics Course can help strengthen pre-service mathematics teachers' PCK.

The DMC Prototype

The DMC has been designed to focus on teaching Instrumentation in Mathematics within the notions of the anthropological theory of the didactic (ATD). Two central notions in the ATD are developed in the course: on the one hand, the proposal of the study and research paths (SRP) and, on the other hand, the notion of praxeology.

Figure 1 shows some pictures of the activities implemented with pre-service teachers in the course's first try-out. For instance, Group 2 (upper right) who worked with applications on circles: "Polyboard"; Group 3 (lower right) who worked on ratios and proportions: "Tarsian Jigsaw Puzzle"; Group 4 (upper left) working on the arcs and inscribed angles: "AIC Board"; and, Group 7 (lower left) working on postulates and similarities: "Classification of Triangle."

Compared to any existing pedagogy courses in the Philippines, in DMC Prototype, the pre-service teachers took the responsibilities usually assumed by the educator. They individually formulated questions related to mathematical and didactic techniques to be able to make better designs of manipulatives and didactic activities within each group. Some of the tasks proposed during the DMC included formulation and analysis of generating questions and of the mathematical and didactic praxeologies involved, as well as the implementation of some of their designs. The enhancement of pre-service teachers' MPCK after the DMC highlighted its uniqueness compared to the existing pedagogy courses. Different from some other pedagogical courses that already exist in the Philippines, DMC started with the presentation of a generating problem for teacher education about high school students' difficulties with some basic mathematical concepts and the use of manipulative material. Thus, the analysis of the consequences of "monumentalism" (Chevallard 2015) was taken out from the beginning and during the entire study process. These manipulative materials helped address some complex mathematics concepts among high school students through the immersion into some high school teaching practices.

Fig. 1 Pictures of the group work during the implementation of the course DMC

Materials and Method

In the design of the DMC Prototype, three phases were distinguished. We call them: (1) preliminary inquiry into the needs and content analysis; (2) prototyping phase: iterative cycles of design and formative evaluation; and (3) assessment phase: semi-summative evaluation. Thus, the DMC prototype was a product of a cyclic process, as shown in Fig. 2, which corresponds to an adaptation of the instructional design proposed in Armanto (2002, p. 45).

Moreover, using quasi-experimental methods, the study also analysed the effects of DMC on pre-service teachers' MPCK. The study analysis adopted the pre-test/post-test group design, and the data were analysed using Wilcoxon Signed Rank Test at 95% level of significance. A nonparametric test was used due to the small sample size. Wilcoxon Signed Rank Test was used to determine significant effects in the post-test due to the DMC and to determine the significant effect of the scores in the post-test conducted by the pre-service teachers among high school students. The Spearman rho correlation was tested to determine the relationship between the pre-test and post-test. It also used qualitative and quantitative research methods. The participants of the study were the 28 pre-service teachers with 75% female and 25% male in a state university in Butuan City, Philippines, enrolled in the Second Semester of SY 2012–2013 and the First Semester of SY 2013–2014. There were two groups of participants: the first Try-Out Group and the second Try-Out Group. Table 1 shows the flow of the Formative Evaluation of the Didactics of Mathematics Course (DMC) prototypes and the MPCK assessment as scheduled in our study.

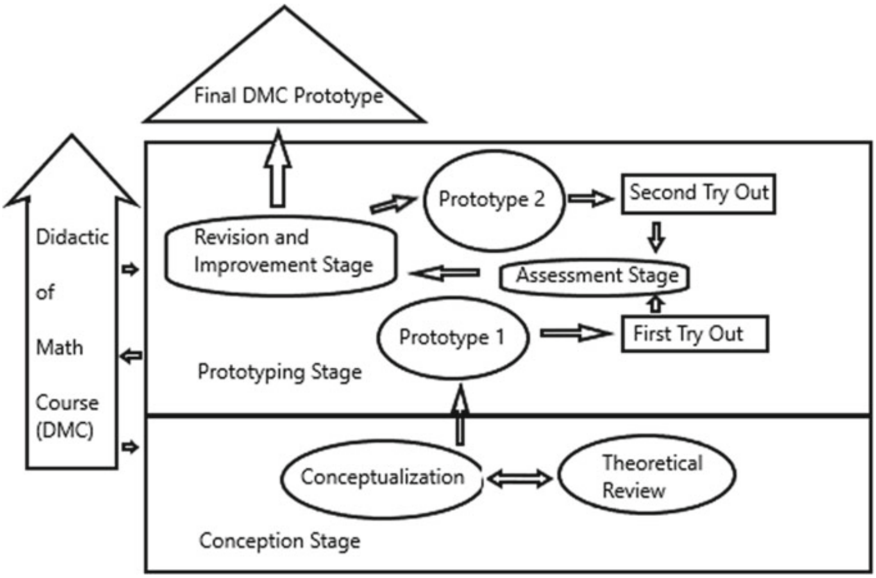

Fig. 2 The cyclic process intervening in the design of the DMC Prototype

Table 1 The flow of the formative evaluation of the DMC

	Preliminary research	Prototyping stage		Assessment stage
		Prototype 1 Stage 1	*Prototype 2 Stage 2*	
Validity	Expert appraisal	Expert apraisal		
Implementability/practicality		First try out participants	Second try out participants	DMCQ & AEQ
Effectiveness		MPCKT (Pretest) DMCQ Classroom Observation	MPCKT (Pretest) DMCQ Classroom Observation	MPCKT (Posttest) DMCQ Classroom Observation

Results and Discussion

Results showed that DMC enhanced the pre-service teachers' MPCK ($Z = -2.96$, $p = 0.003$). Integrating the Didactics of Mathematics Course (DMC) in the pre-service teacher education not only enhances their MPCK but also provided pre-service teachers with substantial research and professional opportunities to improve their skills in teaching mathematics inside the classroom. The quantitative analysis of data showed that there was a significant increase in the MPCK scores among the student-teachers participating in the DMC.

Based on the study's findings, it is important to recommend that the Didactics of Mathematics Course (DMC) take part in the curriculum of pre-service mathematics teachers. Integrating the DMC in student teaching programs can provide different kinds of experiences appropriate for the recent Philippine education setting. In doing so, the Commission on Higher Education may afford to be mindful of setting up meaningful interventions, such as the DMC here implemented. This kind of courses can provide useful and relevant tools to support pre-service teachers with their tasks as future mathematics teachers, helping them to become independent, creative, and confident in their profession.

Another important tool introduced and worked with future teachers was the praxeological analysis. The task proposed in this study was to design manipulatives and didactic games using the praxeological analysis and the notion of the study and research paths (SRP). The techniques used by the students in executing the tasks included searching on the net or into journals, and books, formulating new possible generating and derivative questions, and planning brainstorming among their peers. The presentation, discussion, and feedback sessions with the teacher, peers, and classmates provided part of the technologies used by the students. These technologies served as the learner's rationale or justifications for the chosen techniques to providing answers to the questions. That is, it helped them to justify why the techniques worked and how effective they were, compared to others. The ideas gained through self-studying, the feedback of the educator, peer, and classmates exchanged

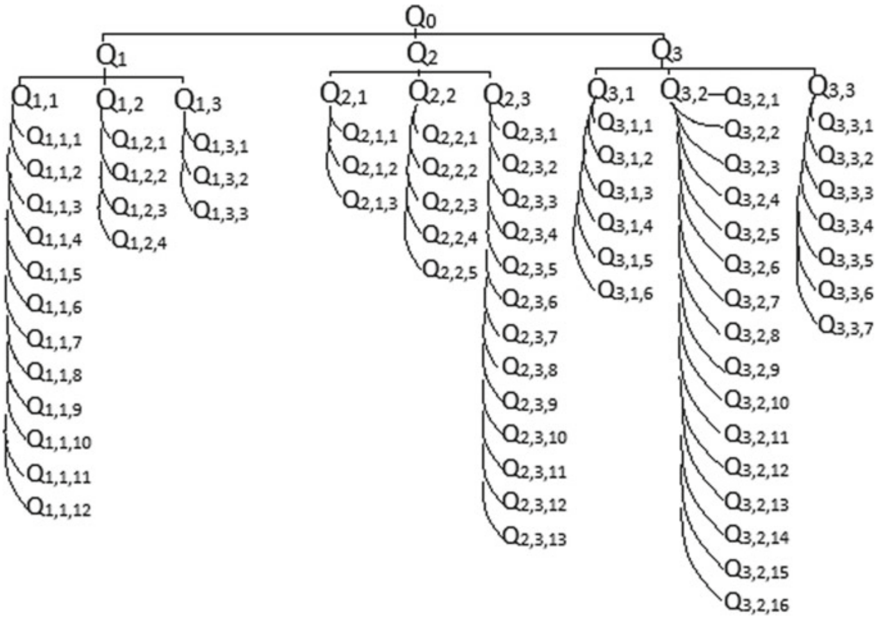

Fig. 3 Analysis of the work developed by the groups addressing the generating questions Q_0

are the theories that were used to justify the technologies. These theories are more than an abstract set of concepts and arguments arranged into a general discourse, which justifies the technology itself.

The praxeological analysis was made explicit through the analysis of the Q-A map as a result of the Study and Research Path (SRP). SRP was an important didactical tool for student-teachers in this study. An example of the resulting analysis of the experienced SRP is shown in Fig. 3. It started with a generating question Q_0 which was stated as: *What are some of the difficulties, problems, and misconceptions in this area* (*Geometry*) *that can be addressed using manipulatives?*

This generating question Q_0 was powerful enough to open many derivate questions, which are summarized and classified as:

Q_1: Given those specific areas, which of the topics is most problematic?

- $Q_{1.1}$: What are those topics?
- $Q_{1.2}$: What is its use in real life?
- $Q_{1.3}$: Why do we need to know these topics?

Q_2: What possible manipulatives can be designed on these topics?

- $Q_{2.1}$: What are those possible didactical games and activities that can be constructed using these manipulatives?
- $Q_{2.2}$: How can activity sheets be made using these manipulatives with some basic concepts in that area?

$Q_{2.3}$: What are the requisites to make?

Q_3: Does it make a difference to high school students?

$Q_{3.1}$: What are students' perceptions regarding the manipulatives with the specific areas?
$Q_{3.2}$: What are the experiences of these high school students with the created manipulatives?
$Q_{3.3}$: What are students' performance regarding the manipulatives with the specific areas?

A sample tree diagram or Q-A Map is shown in Fig. 3. This shows the relationships of the questions generated by the student teachers in the second try-out.

It was concluded, based on the results of the study, that allowing the student-teacher to experience "alive" activities allowed them to hypothesize, experiment, think, formulate more alive questions and choose and design more relevant mathematical organization and concepts. Pre-service teacher's education must bring prospective teachers far beyond their personal experiences in traditional classrooms to facilitate changes in the traditional classroom setting that still dominates nowadays in the country. It is recommended that the DMC prototype be integrated into the Filipino pre-service teacher's education since based on the results of the study, the Didactics of Mathematics Course (DMC) prototype is valid, implementable, practical, and effective.

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Evolution of a Teacher-Researcher While Developing a Co-disciplinary Study and Research Path Through Five Implementation

María Paz Gazzola, María Rita Otero, and Viviana Carolina Llanos

Abstract In this paper, we present an ethnographic study, which describes the case of a teacher-researcher (TR) who implements five times the same co-disciplinary study and research path (SRP) in mathematics and physics at courses of secondary school in Argentina. Basically, we analysed how the results evolve throughout the implementations depending on the teacher's performance in the classroom. The results among the first and the last implementation show positive differences in the development of the SRP. We aim to discuss the role of the teacher in the achieved evolution. The differences identified would not be mainly attributable to the characteristics of the study groups, but to the teacher. Regarding the five implementations, the TR seems to have modified progressively: her role in the class, the knowledge introduced by her into the didactic milieu, and the management of certain dialectics. This could be due to the acquired experience with the SRP replications.

Introduction

The implantation of the paradigm of questioning the world (PQW) (Chevallard 2013) in the current school institutions demands significant changes in the professional activity of teachers. In the more traditional paradigm, teachers assume almost completely the responsibilities of the class, they are often the "guarantor of knowledge" and, as such, responsible for communicating and evaluating it. The essential activity of students is often reduced to the reproduction of what the teacher

M. P. Gazzola (✉) · M. R. Otero · V. C. Llanos
Núcleo de Investigación en Educación en Ciencia y Tecnología (NIECyT), Tandil, Argentina
e-mail: mpgazzola@exa.unicen.edu.ar

Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN), Tandil, Argentina

Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Tandil, Argentina

M. R. Otero
e-mail: rotero@exa.unicen.edu.ar

V. C. Llanos
e-mail: vcllanos@exa.unicen.edu.ar

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explained. Thus, knowledge is "shown", in the same way that a work is exhibited in a museum, which cannot be touched or manipulated. Students "visit" knowledge, without altering, transforming or deconstructing it (Otero et al. 2013). The anthropological theory of the didactic (ATD) calls this phenomenon as the *monumentalization of knowledge.*

In contrast to this traditional and dominant paradigm, the study and research paths (SRP) are proposed to carry out functional study processes in the classrooms. The SRP are generated from a generating question Q, an "strong" question that must be answered (Chevallard 2013). The implementation of an SRP requires that the teacher's *topos* to be modified: she might be now director of the study process (Chevallard 2009), she is responsible for guiding the study and research of Q; she is possible source of information, as any other *media* that can exist (nut not the only one). She does not perform a central role in the class, nor she is considered the only "possessor" of knowledge (Parra and Otero 2018). In practice, assuming this role is very difficult for teachers, even those trained in the ATD. In this paper, we analyse the particular case of a teacher-researcher (TR) who has implemented five times the same co-disciplinary SRP at secondary school. We enquire into his performance in the classroom, the progressive modifications that she made from one implementation to another, regarding her teaching activity and how this has influenced the development of the SRP.

Focusing on it, our research questions are: *What are the main changes made by TR in its professional activity from one implementation to another? And, what influences did these changes have on the evolution of the SRP?* The answer will allow us to sketch some conclusions regarding the role of the teacher for the effective development of an SRP and the importance of incorporating this type of teaching in teacher education.

Methodology

This is an ethnographic study which describes and analyses the activity of a teacher-researcher (TR) throughout five replications of a co-disciplinary SRP in mathematics and physics. We aim to discuss the incidence of their performance in the classroom on the results of the SRP in each implementation. TR is a math teacher, with training in the ATD, who already had previous experience in this type of teaching and who had carried out a monodisciplinary SRP in secondary school for the study of rational functions (Otero et al. 2012). In this case, the starting point is the question Q_0: *Why did the Movediza Stone (MS) in Tandil fall down?* (Gazzola 2018; Gazzola et al. 2018; Otero et al. 2016). This question originates a genuine co-disciplinary SRP. This means that physics is not only a trigger that activates the study of mathematics, but that the two disciplines play a central role, being necessary to study praxeologies of both. The question Q_0 did not have a previous scientific answer. Consequently, the TR went through the experience of implementaing the SRP in person: she studied this problem in depth with a co-disciplinary team and elaborated her answer based on mathematics and physics. TR had to study physics (a discipline unknown to her) to then teach it. Considering her role of teacher and didactician, TR also developed, together with her research team, a praxeological model of reference (PMR) and analysed the scope and limitations of the SRP for its implementation at secondary school.

The implementations were carried out in mathematics courses in three different schools (with different characteristics and contexts) in Tandil, Argentina (Gazzola 2018). I_1 and I_2 were done in a state private management school, with $N_1 = 36$ and $N_2 = 32$ students. The I_3 and I_4 were carried out in a rural state school, with $N_3 = 13$ and $N_4 = 19$ students. The I_5 was developed in a sub-urban state school, with $N_5 = 16$ students. The decisions regarding the management of the SRP and the modifications that TR made after each implementation are considered to analyse the results. The teaching activity of TR is described in terms of three main focuses: (a) her role in the class (teacher's *topos*), (b) the elements she introduced into the didactic *milieu* and (c) the management of the dialectics.

The Study and Research Path

The Movediza Stone (MS), was an oscillating rock of granite of 248 tons (Peralta et al. 2008), sitting on the top of a 300 m-high hill (above sea level) in Tandil city, Argentina. Unexpectedly, on February 29, 1912, the stone fell down the cliff and the causes of it could not be explained. However, the appeal was that this stone presented very small oscillations when disturbed in a specific spot. The locals knew this property, who came to the place to perturb the stone by themselves (Rojas 1912). The SRP start with the generative question Q_0: Why did the Movediza Stone in Tandil fall down? and the objective is to provide a scientific response. Assuming that the fall can be explained by means of the Mechanical Resonance phenomenon, several questions Q_i emerged which are linked to the physical and mathematical knowledge necessary to answer Q_0. This knowledge is synthesized in Fig. 1, which represents a scheme of the praxeological model of reference (PMR) of the SRP.

If we consider that the stone was an oscillating system, the study can be carried out within the Mechanic Oscillations topic, starting from the ideal spring or the pendulum. First, simple harmonic motion is used, which is characterized by a second-order linear differential equation, and whose solution is a family of harmonic functions (sine and cosine). If friction-produced damping is considered, it provides a new term to the differential equation connected to the first derivative of the position (speed). Finally, it is possible to study systems that apart from being damped, are under the influence of an external force, and therefore called driven systems. In the case that the external force is periodic and its frequency is approximately equal, (the order of the approximation will be clarified later) to the natural (free of external forces) frequency of the oscillating system, a maximum in the oscillation amplitude is produced, generating the phenomenon known as mechanical resonance. Posteriormente, es posible complejizar el modelo y considerar un cuerpo giratorio suspendido, que conduce al estudio del torque y el momento de inercia de un cuerpo oscilante.

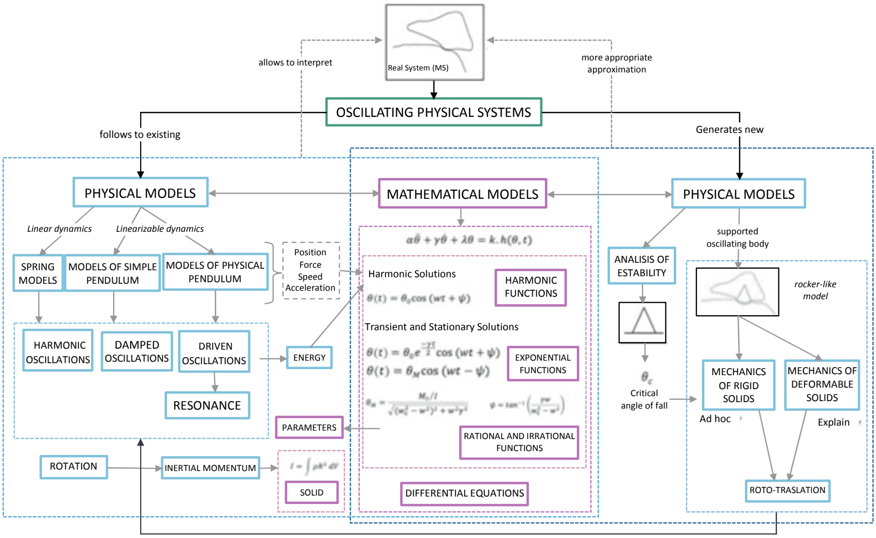

Fig. 1 Scheme of the praxeological model of reference

By increasing the complexity of the model, it is possible to consider a suspended rotating body, instead of a punctual mass. This leads to the study of the torque and the moment of inertia of an oscillating body. Here again, the linear system is for small amplitude oscillations and the damped and driven cases can be also considered, corresponding to the same mathematical model, but in which the parameters have a different physical interpretation.

However, as it refers to a suspended oscillating body, this is not a suitable physical model for the MS system. Since that, the base of the Stone was not flat, it is necessary to consider more precise models of the real situation. This leads to the mechanics of supported (and not hanging) oscillating rigid solids. In this case, we consider a rocker-like model in which the MS base is curved, and it lies on a flat surface, where the oscillation is related to a combined translational and rotational motion. In search of a more appropriate approximation of the physics model for the damping that is clearly not due to air, we consider a more sophisticated model of the stone as a deformable solid, where the contact in the support is not a point but a finite extension, along which the normal force is distributed, being larger in the motion direction and generating a rolling resistance, manifested through a torque contrary to the motion. The rolling resistance depends on the speed stone, giving a physical interpretation to the damping term. Finally, it is possible to use both models to analyse and calculate a set of possible values, compatible with the real system, that guarantee the possibility of the conjecture that explains the fall due to the fact that the oscillating system has been properly driven and consequently resonated.

Evolution of TR Throughout Implementations

Table 1 summarizes the main characteristics of the TR teaching activity in each implementation, according to the three focuses mentioned above. During I_1 and I_2, TR excessively managed the course, he had a predominant place in the class and assumed all the responsibilities of the study process. There was no internet connection or free access to the library. TR was the only media and, as such, the didactic *milieu* was completely controlled by her. She incorporated all the necessary elements and tools for the study and response of Q_0 and did not lead to students being part of its elaboration. The students constructed answers to the derivative questions from the SRP (which were proposed by TR), but mainly all decisions were the responsibility of the teacher. At that time, TR did not have adequate physical and mathematical models of the MS to enter into the milieu and, consequently, no progress was made beyond a qualitative answer of the problem (Otero et al. 2016). On the other hand, the strong monumental features of TR led her not to promote the occurrence of the gestures of a SRP. Thus, the dialectics did not work in these implementations, only some gestures (and in a very weak sense) of the dialectic of the questions and answer and the dialectic of the subject and out-of-subject (Gazzola 2018). The latter would not be related to the teacher but to the characteristics of the SRP.

In I3, TR partially assigned some responsibilities to the students: she allowed them to ask new questions and that they be responsible for disseminating the answers. Even so, TR decided what questions to study, the "acceptable" answers and at what time and how the answers would be disseminated. In this implementation, and in subsequent ones, there was free access to the library and internet connection. This allowed access to various resources and the students participated more in the development of the *milieu*, to the extent that the teacher allowed it. In I3, TR also widened the media. She incorporated more and new elements: various bibliographic materials about MS and its fall down, a mathematical model for MS considering it as a forced and driven system and also a set of data for its interpretation. This allowed some modelling activities to be carried out and to expand the study in relation to the response to reconstructed Q_0. The functioning of the dialectics was affected by the decisions of TR which is evidenced, for example, in a better functioning of the dialectic of the questions and the answers and of the dissemination and reception.

In I4, TR established a different distribution of responsibilities, more characteristic of an SRP: the students had to ask new questions but also find their answers and incorporate into the *milieu* the works they considered relevant. The teacher managed the constructed answers and the discussions that were presented in the commons moments. From the analysis of the results of the previous implementations, TR modified and proposed new activities with the objective of achieving an appropriate use of the physical and mathematical models and being able to establish links between them. Another significant change occurred in relation to the media used to study the mathematics and physics involved: in this implementation the teacher did not propose any text for the study of these knowledge, students had to search and study in books, internet, consult other teachers, etc.; and prepare their own notes together with the

Table 1 Synthesis of the teaching activity of TR in the SRP in the implementations

	I_1 and I_2	I_3	I_4	I_5
Teacher's topos	TR managed all the questions and answers studied: she determined in what order the questions would be answered, delimited the possible answers and the works. She was mainly responsible for dissemination the answers constructed in the class	She partially waned her role in the posing of the questions. But she managed which would be studied and what moment. delimited the possible answers and the works. She let the students the responsibility of disseminating the answers. She proposed the moments of sharing answers and managed the discussions that were presented	She gave more responsibility to the students: to propose questions and to decide (under TR supervision) what to study and in which order. TR allowed the students looking for answers and proposing the study of some works. She gave the students the responsibility of dissemination the answers. TR managed the answers and discussion points with the class	TR risked more: she almost completely gave up the responsibility of asking questions She allowed the students to propose new works and models
Elements introduced into the didactic milieu	TR Introduced all the questions Q_i and all the materials for the study: A book about the fall down of the MS, Study and Research Activities (SRA) oscillations, SRA trigonometry and trigonometric functions. Simulations of the website *física con ordenador*.	TR incorporated new and varied bibliographic materials. SRA oscillations, SRA trigonometry and trigonometric functions. Simulations of the website *física con ordenador* Mathematical model: The equation of the MS as a damped and driven system	TR proposed new activities: a set of situations based on an applet created by researchers. She proposed activities to study the harmonic functions and elaborated, together with the students, the study materials. She incorporated the equation of the MS as a damped and driven system.	TR introduced the questioning on mathematical and physical models. She permitted the physical modelling of the MS She proposed activities and elaborated, together with the students, the study materials. She incorporated the equation of the MS as a damped and driven system.
Management of dialectics	*questions and answers subject and out-of-subject*, (in a weakly way)	*questions and answers* (in a relatively weakly way), *subject and out-of-subject*, *diffusion and reception*	*questions and answers*, *media-milieu* (in a weakly way), *subject and out-of-subject*, *black boxes and clear boxes* (in a weakly way), *reading and writing*, *diffusion and reception*	*questions and answers*, *media-milieu*, *subject and out-of-subject*, *individual and collective*, *black boxes and clear boxes*, *reading and writing*, *diffusion and reception*

class, always under the supervision of the teacher. These decisions, allowed TR to promote and cause the occurrence of actions of certain dialectics, previously absent, such as the dialectic of the media-milieu, black boxes and clear boxes and of reading and writing, although its operation It was relatively weak.

In I5, TR assumed her *topos* of director of the study, which allows to further improve the distribution of responsibilities. Students were responsible for the knowledge to rebuild and, among other things, they decided which questions to study and which not. TR tried every time it was possible to act as a media of the class and she led the students to propose new works and models. In addition, TR enabled the questioning of physical and mathematical models and the relationships between them and the real system (MS). She also encouraged the physical modelling of the MS. This enriched the co-disciplinary nature of the answer to Q_0 developed by this study group. In this implementation, the changes in the *topos* of the teacher and the decisions related to the didactic *milieu*, allowed a more complete functioning of the dialectics mentioned above.

Discussion and Conclusions

The results among the first and the last implementation show positive differences in the development of the SRP: in relation to answer to Q_0 reconstructed by each group, product of the elaboration of an increasingly broad and rich milieu in relation to its components; also in the functioning of the dialectics. The evolution achieved would be mainly attributable to the performance of TR in the classroom, and the decisions and modifications that were made from one implementation to another. These changes could be linked to SRP replications, which allowed TR progressively: to learn more about the problem, to change and to increase her praxeological equipment and to acquire experience. This allowed her improve her anticipations about what could be happen and consequently to risk more regarding the control of the milieu.

Even so, for the teacher it has been a hard task taking and remaining his role as director of the study. On the one hand, it was difficult to resist the constant pressure and request for approval from the students, whom the *monumentalist* institution has accustomed to be "motivated" continually and to be reinforced positively. On the other hand, it was also difficult to take distance from the role of a privileged media and the only one responsible of the didactic milieu. It could be due especially to the students expected and requested continually that: the tasks to done and what and how to study be decided by the TR. This led TR intervened repeatedly in order to advance in the studying and her participation in the construction of the milieu was much greater than which, in a strict sense, it is supposed in an SRP. That happened indistinctly in all implementations. Although TR had experience in this type of teaching, she was forced to act with certain features of the monumentalism in order to make the device works in the classrooms. In any case, regarding the five implementations, TR was progressively improving the management of the milieu and the distribution of responsibilities, which led to the possibility of occurrence of some of the characteristic gestures of an SRP. Consequently, the results related to the operation of the dialectics were more favorable as the implementations occurred. In I_1 and I_2 there was a significant absence of the dialectics, which was reversed in the following implementations, in I_3 and I_4 there was a relatively weak functioning of some of the dialectics and then in I_5 a better functioning of those.

This paper shows the relevance of the role of teacher to carry out a teaching by SRP in the current school institutions and how important it is that the teacher manage the milieu in an appropriate way. This is revealed in the obtained results of this research, since it has been more favorable in the last implementation. In the latter, the management of the study was modified based on the previous results and on the experience of the teacher with the specific SRP.

Finally, we consider that making and carrying out a process of teaching based on an SRP, or any other type of teaching by *inquiry*, is difficult for any teacher. Therefore, it is essential to advance in the research and analysis of mathematical-didactic praxeological equipment that teachers need to develop a teaching by SRP and include it in their professional training.

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Building Reference Paradidactic Praxeological Model for Investigating Teachers' Reflection

Ryoto Hakamata

Abstract Mathematics teachers' reflection is an interesting research topic. However, despite the great efforts already made, little is known about the role mathematical knowledge plays in reflection and how teachers' professional knowledge is developed. This problem seems to lie in the lack of appropriate models to investigate it. To approach this problem, this paper aims to construct a reference model of teachers' reflection, focusing on interactions between mathematical and didactic knowledge. For this purpose, the notion of paradidactic praxeology has been used. The reference model was built by classifying the genres of paradidactic tasks and considering model of the praxeologies. To check the effectiveness of the model, an exploratory analysis of a student-teachers' reflection was conducted. As a result, two remarkable paradidactic phenomena were found, showing the potential of the model to investigate teachers' reflection.

Introduction

This paper reports some results of an ongoing study about teachers' reflection within the anthropological theory of the didactic (ATD, hereafter). Mathematics teachers' reflection in the context of their professional development has been an interesting research topic, approached by the diversity of approaches. On the one hand, some studies have approached teachers' professional characteristics by classifying knowledge for mathematics teaching (e.g., Ball et al. 2008). On the other hand, others have addressed some essential gestures that make reflection meaningful (e.g., Jaworski 1998). However, despite these significant efforts and contributions, the interrelation among the following aspects has been little investigated: mathematical knowledge; pedagogical knowledge; teachers' activity; and development of knowledge. For example, while the widely known model of the *domains of mathematical knowledge for teaching* (Ball et al. 2008) indicates the categories of usable professional knowledge in mathematics teaching, the interrelation between the categories is not

R. Hakamata (✉)
Faculty of Education, Kochi University, Kochi, Japan
e-mail: hakamata@kochi-u.ac.jp

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shown. It is also unclear how those types of knowledge work, growth, and develop through teachers' professional activity. In particular, little is known about what role mathematical knowledge plays in teachers' reflection and how teachers' professional knowledge is developed.

This problem seems to lie in the lack of the appropriate methodologies and specific models to investigate the interaction between teachers' reflection and mathematical knowledge and how reflection contributes to the development of teachers' knowledge. The purpose of this study is to construct an analytical model of teachers' reflection focusing on the interactions between mathematical and pedagogical knowledge, and knowledge and activity.

Theoretical Tools

To construct the analytical model, the notion of *paradidactic praxeologies* (Winsløw et al. 2018) is used. To investigate teachers' professional development through a lesson, we need to consider their activity within the lesson and the activity outside the classroom, such as planning, observing, and reflecting on the lessons. These activities are not didactic practices themselves but activities *about* didactic practices. Paradidactic praxeologies refer to praxeologies developed in such activities. In other words, mathematical and didactic praxeological organisations (MO and DO respectively hereafter) are developed in *didactic systems* in lessons. In contrast, paradidactic praxeologies are developed in *paradidactic systems* both inside and outside classrooms when teachers plan a lesson, observe it and reflect about MO and DO activated. Thus, according to the levels of *didactic co-determinacy*, we can regard paradidactic praxeologies as praxeologies related to the co-determinations (Miyakawa and Winsløw 2013). Besides, we can distinguish the different paradidactic praxeologies according to whether they occur before or after a teaching practice, or simultaneously, that is, *predidactic praxeologies*, *postdidactic praxeologies*, and *observational praxeologies* (Winsløw et al. 2018). Since this paper focuses on activity and knowledge developed in teachers' reflection, we will mainly refer to postdidactic praxeologies.

In the ATD, the first step to construct an analytical model corresponds to consideration of a *reference praxeological model*. The notion of reference praxeological models was initially developed to refer to the reference models of mathematical praxeologies. However, the use of the notion can be widened since the purpose of constructing any reference praxeological model is to avoid taking the vision proposed by the educational institutions for granted (cf. Bosch and Gascón 2006; Gascón and Nicolás 2020). Therefore, we argue that researchers should consider building paradidactic praxeological models of reference as a research tool when paradidactic systems are involved.

Context and Method

In this study, we focus on post-lesson reflection carried out by student-teachers in teaching practices. The reason for conducting the study with student-teachers is that the author of this paper had been teaching at a secondary school in Japan where many student-teachers carried out teaching practices. Therefore, we could easily collect data on their activity. In Japan, students in teacher training courses engage in teaching practices for about one month when they are in the university's third or fourth year. In teaching practices, they usually write lesson plans before the actual lessons take place and hold meetings for post-lesson reflection afterwards—*hanseikai*, in Japanese—with the observers and in-service teachers in charge of the lesson. This process is quite similar to that of the *lesson study* in Japan. Hence, although the study subjects are student-teachers, we can expect that the obtained findings apply to teachers' reflection in general.

To build a reference praxeological model, first, we classified the *genres of paradidactic tasks* of teachers' activity in post-lesson reflection based on previous research about mathematics teachers' reflection in lesson study and on our experience of observing post-lesson reflections. Then, practical blocks of paradidactic praxeologies, that is, *types of paradidactic tasks* and *paradidactic techniques,* are supported by the *paradidactic technology* and *theory*, bringing to light some traits about the co-determination between MO and DO. Finally, we conducted an exploratory analysis of a post-lesson reflection carried out by Japanese student-teachers to demonstrate the effectiveness of the reference model. For this analysis, a lesson for 7th-grade students (11–12 years old) and the post-lesson reflection was recorded. The study subjects were a group of four student-teachers. During the teaching practice, they observed the lessons taught by the others and held *hanseikai* together after each lesson.

Reference Paradidactic Praxeological Model

There is a wealth of literature that describes the process of lesson study, but not many studies have classified participants' activity within the reflection phase. Warwick et al. (2016), based on the qualitative analysis of post-lesson reflection, showed that the following *key moves* promote deeper reflection: questioning, building on each other's ideas, coming to an agreement, providing evidence or reasoning, and challenging each other. In our study, based on previous research and our own experiences, we propose classifying activity in post-lesson reflection into three genres of tasks: *clarifying*, *evaluating*, and *proposing*. Clarifying includes, for example, confirming whether students could understand the teaching contents. "Questioning" and "providing evidence or reasoning", according to Warwick et al. (2016), can be included in this genre. Evaluating is the activity that judges the lessons from some perspectives.

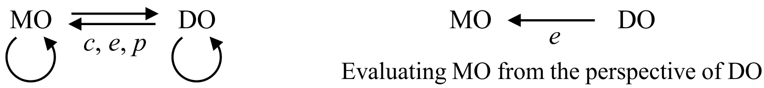

Fig. 1 Reference model of paradidactic praxeology and an example of description

For example, some may evaluate a lesson higher because they consider the mathematical tasks within the lesson as interesting ones. In contrast, others may evaluate lower the same lesson because they consider that teacher's support for the students was inappropriate. Showing agreement (see Warwick et al. 2016) is also included in this genre. In collaborative reflections, it is one of the purposes to suggest alternatives for improving the lesson. Proposing includes such kind activity, and "building on each other's idea" can be included in this genre.

Since paradidactic practices can be described as the activity involving MO and DO, we may consider the interaction and co-determination of both. For example, evaluating mathematical tasks within a lesson to assess their effectiveness for making students achieve a learning goal can be considered a paradidactic practice. This can be regarded as the paradidactic practice of "evaluating a MO by a DO." Moreover, in this example, the parts "evaluating a MO" and "by a DO" could be related to the paradidactic task and the paradidactic technique, respectively. Thus, the object of the evaluation task is a MO while the technique is performed from the perspective of a DO".

In this way, we can model any paradidactic practice by regarding the object of the activity (MO and DO, respectively) and the perspective from which this activity is tackled (from a MO or from a DO). According to this, and based on the one-to-one correspondence between a type of tasks and technique, we obtain the provisional reference model of practical blocks of paradidactic praxeologies as shown on the left side of Fig. 1. In this figure, the arrows represent a paradidactic practice, and the italic subscripts indicate the type of paradidactic tasks by its first letter (c = clarifying, e = evaluating, and p = proposing). The endpoint is the object, and the starting point is the perspective from which the task is tackled (if the task starts closer to MO or closer to DO). Since there also could happen that object and perspective are of the same nature, the looping arrows have been added. The right side of Fig. 1 shows the description of the example of evaluation mentioned above. Although this model is general and rough, it allows us to analyse in more detail by describing the components of MO and DO that appear in the lesson plans.

Result of the Exploratory Analysis

Description of the Paradidactic Practices

In this section, some distinctive remarks in the recorded *hanseikai* are introduced and briefly analysed. In the post-lesson reflection, participants focused on the value or function of the mathematical task. First, the student-teacher who performed the lesson remarked that one of the aims of the lesson was to make students realise the usefulness of graphical representation of linear functions. The mathematical task was selected to help students to achieve this goal. Then, one of the observers mentioned that the students were interested in solving the mathematical task. Besides this, another observer pointed out a mistake in the mathematical notations in the task and claimed that the mistake caused difficulty to solve it to some extent. In the *hanseikai*, there were no other remarks regarding the mathematical task.

From the perspective of the reference model, the above remarks are described as shown in Fig. 2. The first remark clarified the intention of setting the mathematical task by referring to one of the goals of the lesson. In other words, the remark explained that the student-teacher attempted to lead the students to the goal—the task of the DO—by assigning the mathematical task—the technique of the DO. In short, we can describe this remark as a clarification of the DO from the perspective of the DO (indicated as c_1).

The second remark refers to one observer comment who evaluated that the task proposed in MO by mentioning the positive attitudes of the students when solving this task. Here, we can judge that this paradidactic practice did not refer to either the MO or the DO, because her perspective was adaptable for any mathematical type of tasks. That is, the paradidactic technique to evaluate the MO was dependent on the epistemological nature of the MO. Therefore, the perspective of "attitude of students" does not belong to DO, but it is a kind of *pedagogical* perspective. In Fig. 2, the floating starting point of the arrow e_2 represents this *pedagogical* perspective.

The final remark also evaluated MO, but in this case, the evaluation was done by pointing out the internal inconsistency of the mathematical task of MO. Accordingly, we can describe the paradidactic practice as the arrow e_3.

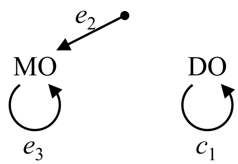

Fig. 2 Description of the paradidactic practices

Analysis of the Paradidactic Practices and Discussion

As a result of the exploratory analysis of the fragment of a post-lesson discussion, two remarkable phenomena were found. Of course, since this is not the entire description of the *hanseikai*, the generality of these phenomena is not guaranteed. However, the effectiveness of the reference model can be illustrated by the explanations of paradidactic phenomena.

Lack of Interplay Between MO and DO

The paradidactic practices certainly developed both the MO and the DO through clarifying and evaluating them. However, each praxeological organisation was clarified or evaluated from the perspective of itself; or evaluated from a pedagogical perspective. Based on the reference model, we can consider not only the existing paradidactic practices but also the non-existence of some practices. For instance, despite the explanation about the intention of setting the mathematical task, participants did not refer to its validity from the didactic perspective. This fact is reflected in our model as the absence of arrows from the DO to the MO. The description in Fig. 2 clearly shows that there was no interplay between MO and DO.

While the phenomenon of the lack of interplay between MO and DO was found in the exploratory analysis, Miyakawa and Winsløw (2013), for example, reported a case of paradidactic practices where such interplay occurred. The paper showed that the post-didactic practices developed the DO, particularly its theoretical block, through clarifying and evaluating the MO in that case. In our opinion, this interplay was promoted by the consciousness of the goal of the lesson series. In fact, the theoretical block of the DO was closely related to the goal—its practical block was justified and explained by it. If so, we can hypothesise that the phenomenon found is common to student-teachers' reflection because they rarely set the goal of the lesson series by themselves in teaching practices and, therefore, it is hard for them to conscious of the long-term goal. As it usually happens in exploratory analysis, we cannot argue the validity of the hypothesis further in this paper; however, it is certain that the proposed reference model enables us to find and compare paradidactic phenomena and to make a hypothesis and discuss them.

Existence of Unexpressed Theoretical Block of Paradidactic Praxeologies

The floating starting point of the arrow e_2 indicates that the paradidactic practice was done in terms of a pedagogical perspective. In other words, the observing student-teacher used the perspective as a paradidactic technique to solve the paradidactic task of evaluating the MO. Why was this paradidactic technique used in the post-lesson reflection? This can be explained by the theoretical block of the paradidactic praxeology, which was unexpressed at that time. For example, the observer might have had the idea that lessons should be judged by the learning attitude of students. In addition, such an idea might have been brought about by the principle of "learner-centred education" which the student-teacher learned in the teacher training course. This idea and principle can be included in the theoretical block. On the contrary, from the phenomenon mentioned above, we can argue that the student-teachers did not have the idea that MO should be evaluated in terms of DO. In this way, we can discuss theoretical blocks of paradidactic praxeologies, whether they are expressed or not, by using the reference model. This usage can suggest an approach to research on teachers' reflection. For instance, in research on lesson study, some difficulty in implementing the post-lesson reflection was pointed out (cf. Rasmussen 2016). Most of this difficulty arises from the lack of *knowledge* of what and how should be done in reflection, that is, theoretical blocks of paradidactic praxeologies. The reference model helps us to investigate and discuss which theoretical blocks are expressed, or unexpressed or spontaneously possessed and how. We thus consider the model here proposed as an effective tool for further research on teachers' reflection.

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Reflections on Teacher Professionalisation–A German Perspective

Sarah Khellaf and Reinhard Hochmuth

Abstract Mathematics teacher education in Germany faces several challenges that can be described in terms of a disconnectedness of praxeologies present in and between different institutions, for example, the so-called double discontinuity. In this contribution, we give a brief outline of how we take into account this institutional context in the design of an introductory course in didactics of mathematics. After describing general course aims, we present an example task that intends to illuminate the raison d'être of praxeologies taught in introductory mathematics lectures. We then discuss a specific difficulty that became apparent in student works. Based on an institutional interpretation and proposals from the ATD, we formulate ideas for ways to alleviate this difficulty in the example task.

Introduction: Characteristics of the Institutional Context

Mathematics teacher education in Germany faces several challenges. One of them is the so-called *double discontinuity* (see e.g. Winsløw 2017), a hot topic in the field of didactics of mathematics for some years now. The term was coined by Felix Klein as early as 1872 and denotes a disconnection between university mathematics and school mathematics discourses, encountered by becoming teachers on their way from school to university and back to school. Winsløw (2017) draws on the ATD to point out the additional relevancy of the change in teacher students' relation to these two bodies of knowledge when they move from one institution to the next. A second challenge, which is the subject of vast amounts of literature from the fields of pedagogy and educational sciences (ger. 'Erziehungswissenschaften'), is the so-called the *problem of theory and practice* (ger. 'Theorie-Praxis-Problem'; see, e.g. Hedtke 2000). Generally, this label designates a diffuse problem area connected to the

S. Khellaf (✉) · R. Hochmuth
Leibniz University, Hannover, Germany
e-mail: khellaf@idmp.uni-hannover.de

R. Hochmuth
e-mail: hochmuth@idmp.uni-hannover.de

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question, whether theory and practice (in the context of education, esp. professionalisation) are somehow related or completely different things. The associated debate can remain descriptive but, frequently, normative conclusions about the education of teachers or other professions are drawn. For example, different teacher education improvements might be proposed, depending on whether theory and practice are considered categorically different or two sides of the same coin. Arguments in this debate typically draw on philosophy, psychology or the social or educational sciences.

Connected to these two issues and a frequent subject of discussion is how teacher education has been institutionalized within the German education system. German teacher education is split into two parts, a "theoretical" part at university and a "practical" part (practical training) in schools and seminars afterwards. Arguments in favour of this model can be encountered in the debates about the problem of theory and practice and teacher professionalisation, which are mainly cultivated in pedagogy and the educational sciences but are nonetheless of high relevance to virtually all subject-specific projects aiming to improve teacher education. The split between theory and practice, which is present both in the mentioned theoretical discussions and in the institutional implementation of German teacher education, manifests as a rupture or disconnectedness between the (more "theoretical") discourses (cultures) present at (German) universities and the (more "practically oriented") discourses (cultures) among teaching staff in schools (Schrittesser and Hofer 2012). What Winsløw calls "compartmentalisation of teacher education" (2017, p. 79) adds to this general problem: many German universities' teacher education curricula are organised in a fashion that promotes disconnectedness between the different subjects taught at university. This issue may be regarded as partly didactical but is also because different subjects have very different disciplinary cultures ('Fachkulturen').

All challenges mentioned so far can be described in terms of a disconnectedness of praxeologies present in and between different institutions relevant for (German) teacher education. Similar phenomena and resulting problems have been described and analysed in ATD papers (cf. e.g. Barbé et al. 2005). For this text, we consider as relevant institutions: *school mathematics*, *university mathematics*, *didactics of mathematics* (subject at university), *practical pedagogy* (or *school pedagogy*), *theoretical pedagogy* (or *university pedagogy*), and more generally: *university* and *school*. In the following, we explain how we take the institutional context into account in the design of an introductory course in didactics of mathematics. After a brief outline of the general course aims, we present an example task from the course and characterize a difficulty, which became apparent in students' works. We then explain how this difficulty appears to be connected to the institutional context and propose a task modification.

Our Course in Didactics of Mathematics

Four years ago, in the context of a project aiming for the improvement of teacher education in Hannover (called "*Projekt Leibniz Prinzip*"[1]), Leibniz University Hannover launched a new introductory course in didactics of mathematics for first-year students of mathematics teaching. Its design and ongoing improvement draw on the *anthropological theory of the didactic (ATD)* (Chevallard 1992, 1999) and German *Critical Psychology* (Holzkamp 1985, 1995). In this course, didactics of mathematics is taught as a multi-disciplinary subject, featuring praxeologies from school and university mathematics, didactics of mathematics and theoretical pedagogy. Before the backdrop of the above described institutional context, we see the need in this course to bring together ideas from different subject areas and to illuminate through our teaching the rationale behind standard curricula of teacher education. A major concern of our activities within project *Leibniz Prinzip* has been to find means of "creating connections" between praxeologies taught in introductory mathematics lectures and praxeologies taught in our institute's didactics courses. The ATD and its terminology have helped in this endeavour by drawing attention to our institution's praxeological landscape, making it better understandable and communicable (cf. Ruge et al. 2019). In an attempt to find a local approach to dealing with the problematic institutional context, we combined the ATD with Critical Psychology to formulate the following *professionalisation aim* for teacher education, which serves us as a guideline for course development and task design (subject-scientific[2] technical terms will be avoided):

> In the first phase of teacher education, which takes place at university, explicit engagement with different discourses and views that are commonly present in institutions relevant to the teaching profession (in particular the sciences) and with their technological-theoretical blocks shall lead to an enrichment of available perspectives on questions relevant to the teaching profession, foster reflection in students and ultimately enlarge their repertoire of possible responses to profession-specific situations. Next to cognitive aspects, the development of learning environments shall take into account affective-motivational aspects and the specific nature of scientific experience (cf. Bachelard 2002). (Khellaf et al. 2021, in press, translation by the authors)

The professionalisation aim essentially proposes a general approach to the institutional context in teaching, namely discussing and comparing different praxeologies, especially their logos-blocks. The explicit mention of different discourses and the focus on their logos blocks is meant to account for German teacher education's multi-disciplinary nature. More could be said about our professionalisation aim, but due to lack of space, we discuss an example of a task.

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[1] https://www.lehrerbildung.uni-hannover.de/de/lse/projekte/qualitaetsoffensive-lehrerbildung/projekt-leibniz-prinzip/ (last visited: 27.01.2020).

[2] *Critical Psychology* can be called "*Subjektwissenschaft*" in German, hence the adjective *subject-scientific*.

Course Content and Materials: Illustrative Task and Student Reactions[3]

Our professionalisation aim inspired creating some tasks whose solution requires switching between praxeological blocks from different but related discourses. Several tasks given to students during the second half of our didactics course in summer 2018 consisted in the investigation of a subject matter question based on works studied in our course or at home. Students had to communicate the results of their investigation in the form of an essay they had to hand in two weeks after a class activity about the same topic. One such task, henceforth *fraction task*, was the following[4]:

> Describe how to represent fractions in different ways: through actions, pictograms and symbols (among them the two common representations as fraction and decimal). Explain for each type of representation whether (and how) it depicts an entire equivalence class or a single representative of such a class for the specific equivalence relation "~" from the lecture.

The *equivalence relation* ~ introduced in the lecture was related to the "crossover product":

$$\text{Two fractions } \frac{a}{b} \text{ and } \frac{c}{d} \text{ are } equivalent (\text{or } equal-valued), \text{ in symbols } \frac{a}{b} \sim \frac{c}{d}, \text{ iff } a \cdot d = c \cdot b.$$

This relation is commonly employed to divide the sets N×N or $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ into equivalence classes of equal-valued fractions. Students typically had prior knowledge about the subject matter touched in the task (for example, from university lectures or school). They were also given an assortment of learning materials to be used as required. These materials comprised excerpts from mathematics textbooks on the concept of equivalence relation and excerpts from books on the didactics of teaching fractions. Additionally, any other reasonable sources could be used (e.g. other lectures, literature, the internet). Praxeological organisations relevant for the fraction task are (I) *equivalence relation* (taught at German universities in typical introductory mathematics lectures, such as 'Analysis I' or 'Linear Algebra I'), (II) a *praxeological organization from didactics revolving around the topic of representations* (a piece of theory by Jerome Bruner called "EIS-Prinzip" in German and presented in our lecture and materials), and (III) *fractions* (a didactically reflected piece of school mathematics also presented in our lecture and materials). A good task solution requires students to understand the meaning of the symbolically given equivalence relation ~ on N×N and to translate elements of its definition into other types of representation (according to II and III). These insights can then be used to describe parallels between the generated types of representation. To produce an explanation of the findings, as demanded in the second part of the fraction task, reasonable arguments have to be formulated, drawing on organizations I, II and III.

In 2018, we received 13 task solutions in the form of essays, 3 of which did not meet our quality standards and had to be revised by their authors before they were handed in a second time. Task solutions were quite heterogeneous in terms of detecting and

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[3] This section relays information described in more detail in Khellaf et al. (2021, in press).

[4] The wording adapted for this paper from a course task presented at the CRM conference.

describing connections between organization I and fraction representations used in school (II and III). Some students struggled with the comprehension of organization I; some had trouble finding connections (some of whom made proposals which mixed the mathematical and didactical discourses in a manner that violated discourse norms); others, however, succeeded in pointing out mathematical structures in organization I which are present in school mathematics. Of interest for the remainder of this paper is an unexpected phenomenon that we observed in some student solutions: 8 out of the 13 essays include some commentary on the *legitimization of praxeological organization I* (equivalence relations), even though the task formulation did not seem to hint at this topic. 4 essays contain formulations such as "We have to begin by clarifying the mathematical background of rational numbers", hinting at a technological discourse prevalent in the institution of university mathematics. 4 others reject organization I as suitable object of learning for school students, thereby essentially questioning the *utility* of this topic for university students of mathematics teaching in their future job. Given such statements, we felt that some students had brushed over (or even discarded) equivalence relations too rashly, without having considered some important ideas. This judgement motivated us to attempt to improve our task, starting with an analysis of the students' statements encountered.

The second type of legitimization discourse we found in our students' solutions (henceforth *utility discourse*) legitimizes activities (including learning) in terms of *utility*. This is a widespread phenomenon on the level of society and an underlying notion in many justifications formulated from the university's point of view for including organization I in teacher education. A pragmatic didactical intention of using the fraction task, for example, is to illustrate the abstract concept of equivalence relation with an example from school mathematics, in the hope that some students will better understand the mathematics involved (as this would be useful for passing other university courses). Equivalence relations are furthermore an important topic in (German) didactics of mathematics and feature heavily in standard literature about subject matter didactics, so knowledge of the concept will facilitate understanding of the didactics discourse. Other justifications for including organization I in mathematics teacher education can be found in a number of already mentioned institutions, such as university mathematics (teachers must know their subject well before engaging in didactical transposition; definitions must be clarified first), German politics/educational culture (teachers of *Gymnasium* represent the science of their subject and must therefore have some amount of scientific expertise; teachers of *Gymnasium* have to prepare pupils for university education), the institution of our course (cf. our professionalisation goal: "engagement with different discourses and views").[5] Coming back to our student solutions, we noted that there was a discrepancy between the notion of utility implied by students and views on utility favoured by university institutions. Some student solutions reproduce a specific discourse often commented on in the (German) literature on teacher training, in which a considerable number of student teachers at German universities, including Hanover,

[5] This list of justifications is not meant to be exhaustive or to rule out the existence of counterarguments in the named institutions/discourses.

participates: this discourse typically revolves around ideas such as practical relevance and demands above all practical experience in teacher training while questioning the usefulness of university teacher education (or forms thereof common in Germany). (see, e.g. Schrittesser and Hofer 2012; Wenzl et al. 2018). In ATD terms, the situation could be described as a mismatch between institutional and student takes on the *raisons d'être* of the university curriculum. Somehow involved in this situation seems to be a discrepancy between the *didactical praxeologies* of certain institutions present at university (which, in the case of pedagogy or educational sciences, can be very elaborate) and the students' naïve preconceptions about the didactical praxeologies employed at university. (For lack of familiarity with the university's institutions, first-year students typically draw on knowledge from school or general society to guess the university's didactical praxeologies and, in the process, may form certain preconceptions). In the following section, we will describe how we tried to address the problem of the unclear legitimization of organization *I* in our course.

Improvement of the Fraction Task and Conclusion

One goal of task improvement was to grant students a better insight into the university's *raisons d'être* of the topic of equivalence relations, without, however, delving into any of the theoretical topics mentioned above (problem of theory and practice, professionalisation, didactic praxeologies, etc.). In more active terms, we wanted students to reflect more deeply about the legitimization of organization I (equivalence relation) and, at best, to generate by themselves some amount of arguments for or against the utility of this piece of knowledge, based on praxeologies they encountered during the investigation of works relevant for the fraction task. For example, it didn't occur to some students that organization I can be regarded as nothing more than a reformulation (in the form of mathematical symbolism) of a structure that is inherently present in the rational numbers as they are taught in school. This thought touches on ideas about the ontology of mathematical structures, which can be considered as part of the *logos* block of university mathematics. From it, arguments can be generated against the view that organization I is irrelevant for school, or discussions might be motivated about what students perceive this organization to be exactly. It is insights like this one about organization I and its relation to other pieces of knowledge that we wanted to make more accessible, more apparent to our students.

Due to institutional restrictions (e.g. examination regulations, unyielding organizational structures), we are so far not able to implement so-called SRP-TEs, in the way as Barquero et al. (2020) proposed. We decided, therefore, to consider improvements in the form of additional guidance via "control questions" (Chevallard 2015, p. 184). According to Chevallard, such questions can help in situations where the works to be studied according to the respective curriculum and educational aim would "first appear [to students], from within the cultural limits that they are precisely expected to transcend, as far removed from the matter under study"

(Chevallard 2015, p. 185). Indeed this appears to be a fitting description of the study of organization I (single work to be studied) in the context of teacher education (matter under study). According to the goal of the task improvement, such questions should aim to illuminate the *raisons d'être* of organization I. About the concept of *raison d'être* Chevallard writes:

> The study of a work *O* in the context of an inquiry into some question *Q* will depend heavily, both quantitatively and qualitatively, on the use of *O* in the making of the answer $A^{\heartsuit}$. What should be clear in such a context-bound study of *O* is that knowledge of *O* thus acquired by the investigators is functionally coherent because it is cohered by the inquiry into question *Q*, so that the raison d'être of *O* that do explain its use in the case in point are readily apparent. (2015, pp. 179–180)

O can here be interpreted as organization I, question *Q* as some general question about the teaching of fractions and $A^{\heartsuit}$ as its answer. Taking up this argument by Chevallard, we decided to search for possible *raisons d'être* of organization I in the *logos*-blocks of relevant praxeological organizations accessible to our students, i.e. which do not rely on debates or didactical praxeologies yet unfamiliar to first-year students (as for example the discussions on teacher professionalisation). In order to guide students to these *raisons d'être*, additional tasks had to be created and the original fraction task suitably modified. Following ideas surrounding the notion of SRP and the paradigm of questioning the world, we decided to employ as tasks "professional teaching question[s]" (Barquero et al. 2020, p. 191). These questions were phrased in a way that catered to the utility discourse so that students could relate to them more easily. We attempted to produce a variety of them, each connecting to different institutional discourses, in order to reach a maximally large amount of students and in order to make accessible different institutional views in accordance with our professionalisation aim. As a result, we added in winter semester 2019/20 the following "discussion questions" to the fraction task:

(1) What part of the knowledge about equivalence relations are school pupils supposed to learn?
(2) What can a teacher use this knowledge for?
(3) What can we learn from our answers to the fraction task about the teaching of fractions in general?

Question 1 aims at exploring the ontology of mathematical knowledge and can result in discussions about the challenges of and prerequisites required in the learning of mathematical symbolism. It therefore connects to *logos*-blocks pertaining to university mathematics, and possibly to related school praxeologies. Question 2 leaves space for a variety of answers: the curriculum might change (utility discourse); teachers who can understand mathematical symbolism can acquire knowledge faster (utility discourse); the mathematical knowledge is what has to be transposed into school, so a teacher needs to know it in order to do his/her job (ATD, mathematics); the teacher (as well as anyone else) can understand some specified ideas of university mathematics and thereby share an aesthetic experience (mathematics, philosophy); etc. With regard to question 3, an analysis of different representations of fractions through the application of organization II in the spirit of the fraction task can show that the only symbolic representations of fractions (present in our material and) capable of showing classes and class representatives at the same time draw on symbolism of university mathematics. This observation doesn't just illustrate the power of mathematical symbolism (*logos*-block of I), some further thoughts and arguments can lead to explanations of the specific contents of organizations III (*logos*-block of III) and provide justifications of the presence of organizations II and III in mathematics teachers' curricula (didactical *logos*-block).

In order to ensure that every student had access to the mentioned ideas, even if they did not arrive at all of them through their own thinking about the new questions, we also created new task material that explained our thoughts regarding the three questions. Starting from winter semester 19/20, we framed these thoughts as a proposal for a task solution. Presenting our ideas as the single expected answer deemed "correct" would be at odds with the general approach of our course. Students in our course exhibit freedom in the formulation of their answers, and we expect them to form their own opinions on the matters discussed in the course, even if they end up disapproving of certain views or traditions, for example the studying of organization *I*. In this sense, we take seriously Chevallard's proposal of encouraging learners to create their "proper answer (…) $A^{\heartsuit}$" (Chevallard 2015, p. 179).

Overall we conclude from the experiences we made in our introductory didactics course that it takes time for students to transition from their initial ideas about the job of a teacher to a view more in line with university discourses. This transition cannot be achieved through one single course or even within one semester, given the institutional restrictions currently at play. In this article, we have presented an attempt to guide our students through specifically designed tasks and questions to form an understanding of the rationale behind typical German university curricula in teacher education. In this sense, this contribution presents an approach to illuminating the raisons d'être of works to be studied within our specific institutional context.

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A Co-Disciplinary Study and Research Path Within Two Groups of Pre-Service Mathematics Teacher Education

Viviana Carolina Llanos, María Rita Otero, and María Paz Gazzola

Abstract We present the results of an implementation of an study and research paths (SRP) carried out with pre-service mathematics teacher in a training course at university level. A co-disciplinary SRP, starting from the generating question Q_0 about "Why did the Movediza stone in Tandil fall?" requires developing both physic and mathematics models to build a possible answer to this question. We finish by presenting some conclusions about the restrictions and relevance of introducing the SRP in pre-service teachers training courses related to modelling activity at university.

Introduction

Recent investigations emphasize that mathematics teaching cannot focus on the transmission of knowledge from the teacher to the students. Moreover, these investigations suggest to provide students tools to study and inquiry "real" phenomena, intergrading mathematical modelling in this process (Blum and Niss 1991; Blomhøj 2009; Barquero et al. 2018; Bosch et al. 2006). In order to address this aim, several changes in the mainstream paradigm are necessary. This crucial changed in the syllabus, in how teachers manage and transform knowledge and in the roles of teachers and students according to this new paradigm. The so-called paradigm of questioning the world (Chevallard 2013) called for an epistemological and didactic revolution of the teaching of mathematics (and other school disciplines), to question which knowledge should be taught based on its utility. In particular, our research works

V. C. Llanos (✉) · M. R. Otero · M. P. Gazzola
Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN), Tandil, Argentina
e-mail: vcllanos@exa.unicen.edu.ar

Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Tandil, Argentina

M. R. Otero
e-mail: rotero@exa.unicen.edu.ar

M. P. Gazzola
e-mail: mpgazzola@exa.unicen.edu.ar

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with the proposal of the study and research paths (SRPs) as epistemological and didactic model (Chevallard 2015), allowing a functional teaching of mathematics. The SRP conceive mathematical modelling as an essential tool to address the study of questions.

This work presents partial results obtained from two courses with pre-service mathematics teacher education (N = 25) working with an SRP. The aim of our work is mainly to describe and analyse how pre-service mathematics teachers deal with modelling and with physic and mathematical models. The SRP starts from the generating question Q_0 about "Why did the Movediza stone in Tandil fall?" To be able to provide an answer to Q_0–in a provisional and unfinished way – needs the use physics and mathematics jointly. The aim of this paper is to describe the teachers' activities and the difficulties they found when they experienced an SRP who needed to work with the physical and mathematical models. We lead to address the following research questions: Which mathematical and physical models did the students develop during the SRP? Which constraints did they face to manage mathematical and physical models in this SRP?

Modelling and Study and Research Paths in the ATD

The ATD defines the SRP as devices that allow the study of mathematics by means of addressing questions. According to the epistemology underlying an SRP, the starting point of mathematical knowledge are questions, called generating questions, because its study should generate new derivative questions. Teaching by means of an SRP is complex and demands profound changes in the mathematical knowledge, in the roles of the teacher and students and, more in general, in the conformation of the entire didactic system. The developed Herbartian model (Chevallard 2013) defines the SRP according to the full development of the Herbartian scheme:

$$\left[S(X; Y; Q) \to \left\{R_1, R_2, R_3, \ldots, R_n, Q_{n+1}, \ldots, Q_m, O_{m+1}, \ldots, O_p\right\}\right] \to R$$

Where Q is a certain generating question; S is a didactic system around of the study of Q. S is formed by a group of people trying to answer the question (X) and by people helping in the study (Y). In classrooms of mathematics, X represent the students and Y represent the teacher and other instruments helping in the search of answers to Q. S has to build a didactic *milieu* M to study Q, whereas M is composed by different knowledge, expressed by $R_i^{\diamond}$, Q_j and O_k. The $R_i^{\diamond}$ are any existing answer or "socially accepted answer", the Q_j are derivative questions from Q, and the O_k are any other knowledge that must be studied developing the answers. Finally, $R^{\heartsuit}$ is some possible and partial response to Q given by S. In the a priori analysis stage, the disciplinary and didactic knowledge, which could be involved within an SRP, is described through the elaboration of a praxeological reference model (PRM). The researchers analyse the potential set of questions the study of which might encompass the possible paths, together with the knowledge (mathematics and physics, in our case), necessary to answer those questions (Chevallard 2013).

Methodology

This work involves a qualitative and exploratory research that aims to carry out an SRP, as proposed by the ATD, in a course of teacher training in mathematics at the university. The SRP was implemented in a state university, in the city of Tandil, Argentina, in a discipline that is part of the didactic studies within the Mathematics Training Course. Two of the researchers are also teachers in the course. There were two implementations, in which N = 12 participated and N = 13 students in their last year (4th), with ages between 21 and 33 years. The SRP was conducted in a total of 7 weekly hours provided in two sessions per week. In both implementations, three working groups were organized with approximately 4 members each.

It is important to note that these students had not studied physics at university, but they had a relatively solid mathematical foundation. In addition, the students had been introduced to the ATD in two previous courses on didactics. However, they showed some difficulties in understanding what an SRP is and how it works. In this paper, we propose the design, implementation and analysis of a co-disciplinary SRP of physics and mathematics, adapted to the institution in which it was developed.

As we have mentioned, the SRP is truly co-disciplinary in the sense that the interaction between physics and mathematics plays a central role and requires the study of both disciplines under equal conditions. In an SRP, the generative question Q_0 is introduced by the teacher, in the first lesson. Then, the students began their inquiry in the library, selecting some texts, documents, etc. that is, looking for possible $R_i^{\diamond}$. In each session, each group presented and discussed with the teacher and the other groups their findings and possible ways to address Q_0. In this paper, we want to look at more precisely on models from physics and mathematics that appeared along their process of proposing an answer to the problem of the falling stone.

The Epistemological Model of Reference, the SRP and the Mathematical and Physical Models

As it has been introduced, the generating question is Q_0: Why did the Movediza Stone in Tandil fall down? This enormous basalt stone has remained in the city's landmark for many years, becoming a distinctive feature. Many local people and national celebrities visited the place to closely observe the natural monument. It was a 248-ton rock, sitting on the top of a 300 m-high hill (above sea level), which presented very small oscillations when disturbed in a specific spot (Figs. 1 and 2). Unexpectedly, on February 28, 1912, the stone fell down the cliff and was broken

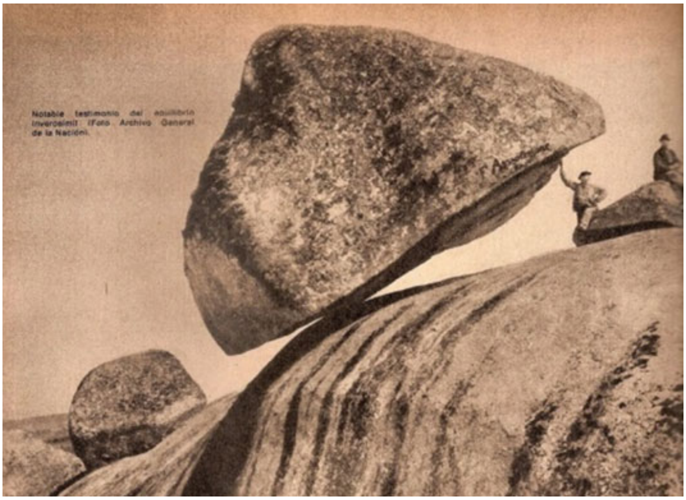


Fig. 1 Picture in the Archivo General de la Nación Argentina

Fig. 2 Picture taken from El Hage, Levy (2007) “La piedra viva”. Municipio de Tandil

into three pieces. For over 100 years, the event produced all kinds of conjectures and legends about the causes of the fall. Assuming that the fall can be explained by means of the Mechanical Resonance phenomenon, several questions Q_i emerged, linked to the physical and mathematical knowledge necessary to answer Q_0.

If we consider that the stone was an oscillating system, the study can be carried out within the Mechanic Oscillations topic, starting from the ideal spring or the pendulum. In this case, frictionless systems are used, in which the only force in action is the restoring force depending (for small amplitude oscillations) in a linear way on the deviation respect to the equilibrium position. This model is known as simple harmonic oscillator whose motion, via Newton equations, is described by a second-order linear differential equation. Progressively, the system becomes more complex. If friction-produced damping is considered. It provides a new term to the differential equation connected to the first derivative of the position (speed). Finally, it is possible to study systems that, apart from being damped, are under the influence of an external force, called driven systems. In the case that the external force is periodic, and its frequency is approximately equal (the order of the approximation will be clarified later) to the natural (free of external forces) frequency of the oscillating system, a maximum in the oscillation amplitude is produced, generating the phenomenon known as mechanical resonance. By increasing the complexity of the model, it is possible to consider a suspended rotating body, instead of a punctual mass. This leads to the study of the torque and the moment of inertia of an oscillating body. Here again, the linear system is for small amplitude oscillations and the damped and driven cases can be also considered, corresponding to the same mathematical model, but in which the parameters have a different physical interpretation.

However, as it refers to a suspended oscillating body, this is not a suitable physical model for the Movediza stone system. Since that, the base of the Stone was not flat, it is necessary to consider more precise models according to the real situation. This leads to the mechanics of supported (and not hanging) oscillating rigid solids. In this case, we consider a rocker-like model in which the Movediza stone base is curved, and it lies on a flat surface, where the oscillation is related to a combined translational and rotational motion (Otero et al. 2017). In Fig. 3, we present a brief summary of the EMR from the point of view of the models of increasing complexity that it entails. We schematized the successive extensions of the physical models and of the mathematical ones, considering the parameters mentioned before. In this case, the intrinsic co-disciplinarity in the generating question entails a succession of physical models that can be described by the same mathematical model. In Fig. 3,

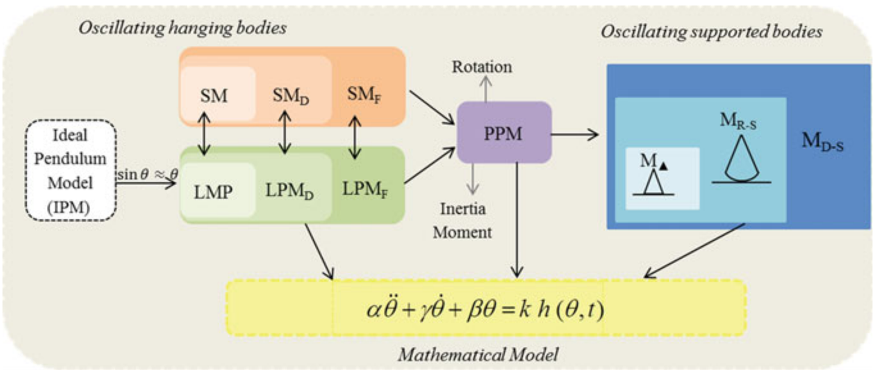


Fig. 3 Sequence of models studied in the SRP in increasing mathematical and physical complexity

they are classified between “Oscillating hanging bodies” and “Oscillating supported bodies”. In the first group they are considered the ideal pendulum model (IPM) when $sin\,\theta \approx \theta$, i.e., the lineal model pendulum (LMP) and also the spring model (SM); for the harmonic cases, damped (LMP_D and SM_D) and force (LMP_F and SM_F); and the physical pendulum model (PPM). In the second group, the flat base model ($M_\blacktriangle$), the rigid solid model ($M_{R\text{–}S}$) and the deformable solid model ($M_{D\text{–}S}$).

The application of Newton laws to the rocker model of the stone leads to a differential equation where the parameters are specific of the Movediza system: mass, geometry, inertia moments, friction at the base, external torque, etc., which is given by the following *effective* Harmonic oscillator mathematical model of the Movediza physical system, called $M_{R\text{-}S}$:

$$\ddot{\phi} + \gamma\dot{\phi} + w_0^2\phi = (M_0/I)\cos(wt) \tag{1}$$

The stationary solution to Eq. (1) is: $\phi(t) = \phi_M \cos(wt - \psi)$; being the amplitude φ_M and the phase ψ

$$\phi_M = \frac{M_0 \big/ I}{\sqrt{(w_0^2 - w^2)^2 + w^2\gamma^2}} \quad \psi = tg^{-1}\left(\frac{\gamma w}{w_0^2 - w^2}\right) \tag{2}$$

The maximum of ϕ_M is for $w_m = \sqrt{w_0^2 - \frac{\gamma^2}{2}}$. The parameters: M_0 (external torque), I (inertia moment), w_0 (natural oscillation system frequency) and γ (damping coefficient), must be estimated. Detailed data about the shape, dimensions and centre of mass position of the Movediza stone are available (Peralta et al. 2008) after a replica construction and its relocation in 2007 on the original place (although fixed to the surface and without possibility to oscillate). These data bring us the possibility to estimate some parameters in our model (such as mass, inertia moment, and the distance of 7.1 m), from which the external torque could be exerted efficiently by up to five people (according to historical chronicles) to start the small oscillation. By using these values, it is possible to study the behaviour of the $\phi_M(w)$ function for w_0 in a range of frequencies between 0,7 and 1 Hz, historically recognized (Rojas 1912) as the natural oscillation frequencies in the Movediza stone system and calculate for each case the maximum amplitude $\phi_M(w_m)$.

The Stone would fall if $\phi_c \le \phi_M(w_m)$, being $\phi_M(w_m) = M_0/w_0 I\gamma$. Note that if γ is very small (as is expected to be in this case) we can neglect it from $w_m = \sqrt{w_0^2 - \frac{\gamma^2}{2}}$, leading to $w_m \approx w_0$.By using this approximation in Eq. (2) (left) the falling condition becomes $\phi_c \le (M_0/w_0 I\gamma)$.

The value of ϕ_c can be determined by an elementary stability analysis, which according to the dimensions of the base of the stone and the centre of mass position is estimated to be approximately of 6° (Otero et al. 2016). In the present model γ is a free parameter, for which we set “ad doc” a magnitude order $\gamma \ge 10^{-2}$. With this constraint, we find several situations, comprising different torques within the mentioned frequencies interval, supporting the overcoming of the critical angle, i.e., predicting the fall. Finally, when looking for a more appropriate approximation of the physical model for the damping that is clearly not due to air, we consider a more sophisticated model of the stone as a deformable solid ($M_{D–S}$). In this case, the contact is not a point but a finite extension, along which the normal force is distributed, being larger in the motion direction and generating a rolling resistance, manifested through a torque contrary to the motion. The rolling resistance depends on the speed stone, giving a physical interpretation to the damping term. Therefore, the physics behind the damping is the same that makes a tire wheel rolling horizontally on the road come to a stop, but in the case of the stone, the deformation is much smaller. Although the deformable rocker model has extra free parameters, tabulated values of rolling resistance coefficient for stone on stone, allowed us to estimate and justify the damping values that we incorporate otherwise ad hoc in the rigid rocket Movediza model (see Otero et al. 2017).

Data Analysis

During the implementations, the students aim at answering how and why the stone fell down the cliff. Different alternatives were explored about the causes why the stone fall, accepting the hypotheses of the fall due to the physical phenomenon of Mechanical Resonance, due to the repetitive action of an external agent, possibly several people. This gave rise to the study of oscillating systems. The TTs busily searched for an "already-made" mathematical and physical model, which allowed them to solve a differential equation. Initially in both implementations, several physical and mathematical questions arose about the oscillations and resonance topics: How is an oscillation described? Which kind of oscillations are there? Which oscillation model would be the most appropriate to describe the stone? Which mathematical model should be used? The questions studied in the SRP were many. A detailed analysis of these questions can be found in the work of Otero et al. (2016). The examined group, divided into working teams, provided different answers to the problem of oscillating systems. G1 studied the harmonic oscillator (HO) (spring and simple pendulum), analysing the solutions of the motion equations with GeoGebra. G2 analysed the differences between the HO and forced and damped oscillations in the spring case considering amplitude-time graphical representations. G3 considered the HO in the context of the physical pendulum. In all cases, they had difficulties to return to the original problem. The teacher proposed filling the table showed in the Fig. 4, as an instrument of synthesis.

The teacher and students analysed the models and the meaning of the angular frequency (w), the equation of motion and its solution for each case. Regarding damped and forced oscillations for the spring, the corresponding terms of the equation and the meanings of the parameters were also considered. Among the main results we highlight those showed in Fig. 5.

Students noticed that the mathematical model were the same, but not the physical model, because the parameters represent different properties of the system. The

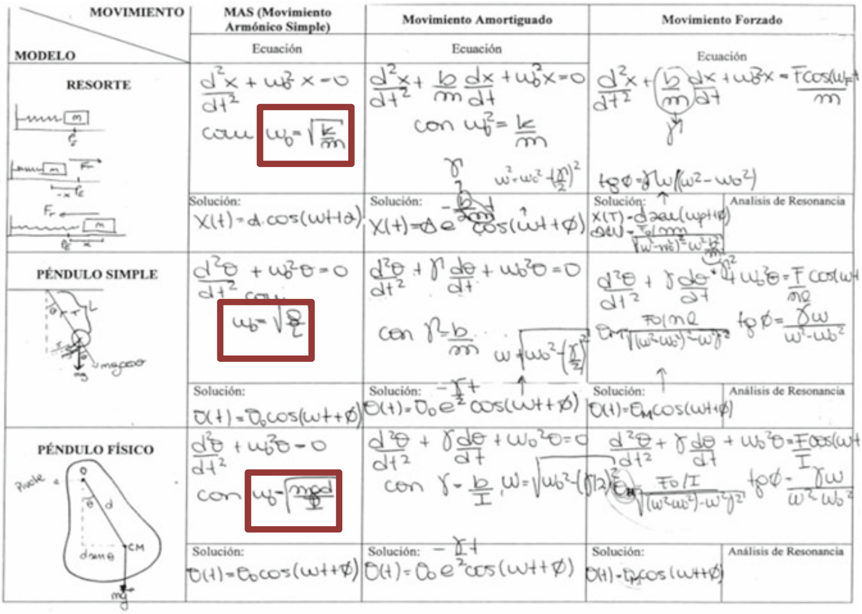
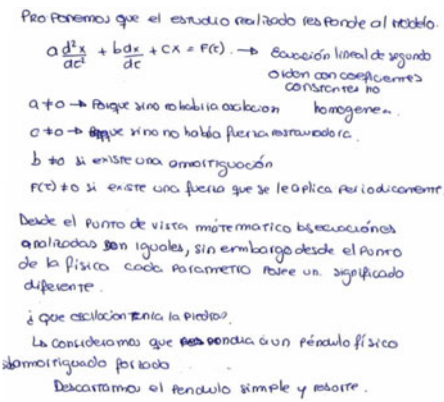


Fig. 4 Protocol of the Group 3



The study performed responds to the model:

$a \cdot \frac{d^2x}{dt^2} + b \cdot \frac{dx}{dt} + c \cdot x = f(t)$ *Second order linear nonhomogeneous differential equations*

$a \neq 0$ → *Because otherwise there would be no oscillation.*
$c \neq 0$ → *Because otherwise there would be no restoring force.*
$b \neq 0$ → *If there is damping.*
$f(t) \neq 0$ *Applying a periodically force.*
Regarding mathematics the equations analyzed are similar, but however, from the physics point of view each parameter has a different meaning. What oscillation does the stone have?

We consider that it would correspond to a forced underdamped physical pendulum.
We reject the simple pendulum and the spring

Fig. 5 Protocol of the TT9 Group 2

students verified the solutions of the DE and they arrived at the solution provided in the textbooks, helped by a text wrote by the teacher. They studied the resonance condition and analysed the amplitude function to determine the maximum. In this moment, some students presented strong objections to the possibility of using the physical pendulum model in the case of the stone, not so much in relation to a body that is supported but as an "inverted" pendulum. This led the discussion towards analysing

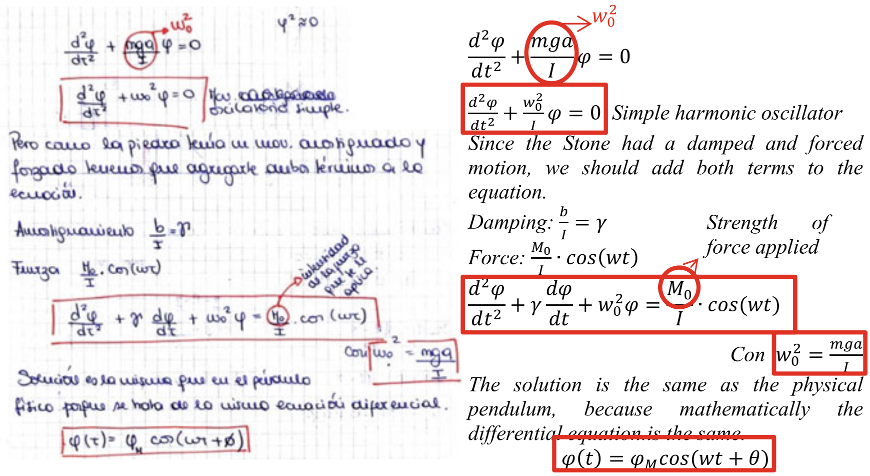


Fig. 6 Protocol of the TT11 Group 3

again the real system and the support basis, looking for models specifically related to the system to be modelled and that usually they are not present in elementary textbooks, as it is the case of the rocker model. The teacher introduced the rocker model as a rigid solid combining the oscillating and rolling motion. While they analysed the equation of motion, the students added the terms of damping and external force. Figure 6 shows a synthesis of the models analysed by the students. They considered the meaning of the angular frequency ($w_0 = \frac{mga}{I}$) concluding that the differential equation is similar same however the parameters change.

Once the model was obtained, only the analysis of the parameters remained. The moment of inertia was calculated using the data provided by Peralta et al. (2008). The critical angle had already been calculated. The value $w_0 = 6{,}28$ Hz was taken from Rojas (1912). Students failed using various values for the parameters, because they conceived them as fixed and unique. The main problem was to recognize the solution as a family of functions. Based on their questions, the teacher proposed to analyse this family of functions by means of spreadsheets and graphics software, varying the different parameters. The students mainly proposed the use of GeoGebra, using sliders for M_0, γ, w_0. They emphasized the relevance of γ, determining the variation of the maximum amplitude for different values of γ. The students estimated γ to be the order of 10^{-2}, specifically between 0.01 and 0.02, depending on the people (2–5) considered. Finally, the students concluded that there is not a single set of parameters that support the fall of the stone, as originally, they thought.

Conclusions

The construction of a possible answer to the generating question Q_0 led the study and the analysis of several physical models related to oscillating systems (such as springs, single pendulum and physical pendulum, including damped and driven oscillators). However, none of these physical models were adequate to model the stone fall. By reanalysing the real system with more detail, students established that the previous models did not describe some essential aspects of the stone. One of the most important aspects was the fact that the real system is an object supported on a surface, and not a hanging, as it was assumed by the previously considered physical models. Then, in the search for a reason to make a supported physical stone model oscillate, the hypothesis that the contact surface between the stone and the base is not flat emerged. A new hypothesis that, in fact, had some historical evidences. In this way, the physical model of the rocker raised as the most appropriate to describe the oscillations of a supported object. For the pre-service teachers the most relevant obstacles were, not only concerning the necessary knowledge about physics, but the difficulties on using algebraic-functional modelling involved in the resolution and interpretation of differential equations.

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The Ecology of the Didactic Divide in Teacher Education

Koji Otaki and Yukiko Asami-Johansson

Abstract The phenomenon of didactic divide reported in Sweden expresses the disconnection between disciplinary and pedagogical knowledge in both individual teachers' professional knowledge and teacher-education programmes. Within the framework of the anthropological theory of the didactic, we firstly redefine the notion of didactic divide, and secondly, study the ecology of the phenomenon by using our new analytical model rooted in the scale of levels of didactic co-determinacy. As a result, we indicate some constraints on the didactic divide: inter-professional esoteric pact, shortage of didacticians, universality illusion on scholarly knowledge, and transparency illusion on the social reality.

Introduction: The Didactic Divide as a Teacher-Educational Problem

The *didactic divide* (Bergsten and Grevholm 2004) originally has been defined as a discontinuity between discipline and pedagogy within the teacher-education schools. In this paper, we explain this phenomenon by using several tools provided by the anthropological theory of the didactic (ATD) (cf. Chevallard 2019). We now take a quick look at an example in one Swedish mathematics teacher education programme (University of Gävle 2018). After the latest reform of the teacher-educational programme came into force in 2011, the teachers for different school grades are divided as following three levels: (1) preschool year and primary school grades 1–3; (2) primary school grades 4–6; and (3) lower secondary and upper secondary schools (grades 7–9 and *gymnasium*). The students of primary school teacher education have to have totally 240 credits, whereof 105 credits in Swedish, English, Science and

K. Otaki (✉)
Hokkaido University of Education, Hokkaido, Japan
e-mail: otaki.koji@k.hokkyodai.ac.jp

Y. Asami-Johansson
University of Gävle, Gävle, Sweden
e-mail: yuoasn@hig.se

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Mathematics, 75 credits in pedagogy, 60 credits in the school based practice (VFU) and a thesis within any subject. The students of secondary school teacher education in mathematics have totally 300 credits, whereof at least 90 in scholarly mathematics, 15 credits in mathematics education, 30 credits in writing thesis and 15 credits in VFU, 90 credits in other subject, and 60 credits in pedagogy. There are some differences between the universities, but such credits distribution is common in the Swedish teacher-education programmes in general. The disciplinary knowledge and the pedagogical knowledge are emphasized on the structure of the both courses. However, subject-matter education such as mathematics education, which holds a potential to integrate the divided bodies of knowledge, has been given low focus, instead excessively emphasizing the efficacy of the teaching practice.

Revisiting the Phenomenon of Didactic Divide from an ATD Perspective

Lectures, seminars and workshops for *prospective school teachers* $\mathfrak{X}$ by *teacher-educators* $\mathfrak{Y}$ around a *teacher-educational didactic stake* $\spadesuit$ are *didactic systems* of a special genre, denoted by $S_{\mathrm{TE}}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$. Taking a closer look at $\spadesuit$, we can notice that most of them are more or less the knowledge of the *school* didactic systems $S_{\mathrm{Sc}}(X, Y, \heartsuit)$, i.e. $\spadesuit = S_{\mathrm{Sc}}(X, Y, \heartsuit)$—there X is a set of students, Y is a set of schoolteachers, and $\heartsuit$ is a *didactic stake*. Such didactic systems of two typical types can be described in the case of mathematics teacher education as follows:

- $\mathcal{S}_{\mathrm{TE1}} = S_{\mathrm{TE}}(\mathfrak{X}, \mathfrak{Y}^{\mathcal{M}}, S_{\mathrm{Sc}}(X, Y, \heartsuit))$, where *mathematical teacher-educators* $\mathfrak{Y}^{\mathcal{M}}$ teach original or *pre-transpositive* forms of the school disciplines $\heartsuit$, ignoring the system of X and Y;
- $\mathcal{S}_{\mathrm{TE2}} = S_{\mathrm{TE}}(\mathfrak{X}, \mathfrak{Y}^{\mathcal{P}}, S_{\mathrm{Sc}}(X, Y, \heartsuit))$, where *pedagogical teacher-educators* $\mathfrak{Y}^{\mathcal{P}}$ teach the way which Y teaches mathematics to X, ignoring the didactic and epistemological study of $\heartsuit$.

By contrast, the following third type of $S_{\mathrm{TE}}(\mathfrak{X}, \mathfrak{Y}, \spadesuit)$ seems to be rare:

- $\mathcal{S}_{\mathrm{TE3}} = S_{\mathrm{TE}}(\mathfrak{X}, \mathfrak{Y}^{\mathcal{D}}, S_{\mathrm{Sc}}(X, Y, \heartsuit))$, where *didactic teacher-educators* $\mathfrak{Y}^{\mathcal{D}}$ teach knowledge of *didactic phenomena*, focusing on $S_{\mathrm{Sc}}(X, Y, \heartsuit)$ as a whole.

Using ATD, the didactic divide is described as the dominance of $\mathcal{S}_{\mathrm{TE1}}$ and $\mathcal{S}_{\mathrm{TE2}}$, and the absence of $\mathcal{S}_{\mathrm{TE3}}$ in the teacher education institution at stake. Let us emphasize here that the didactic divide—within our theory—is not a feature of prospective-teachers' knowledge and scholarship, but a property of teacher education programmes as bundles of the teacher-educational didactic systems. In other words, it is not a cognitive phenomenon but a didactic phenomenon concerning the way of organise the teacher education. Of course, didactic gestures more or less bring forth the states of cognition. However, this fact does not mean that the cognitive entity and the didactic entity belong to the same ontological category. In addition, this characterisation can clarify that the didactic divide is a specific didactic phenomenon that takes place in "didactic systems *about* didactic systems", that is, the didactic divide is a *meta-didactic* phenomenon in teacher education.

Paradidactic Ecology of the Didactic Divide

Teacher-Educational Didactic System as a Genre of the Paradidactic System

In this paper, we propose to regard the teacher-educational didactic systems as *paradidactic systems* of a special genre. The notion of paradidactic system coined by Winsløw (2012) is teachers' collaborative systems for designing and reflecting on didactic situations. In our study, such systems are denoted by a nested model $\mathfrak{S}(\mathfrak{X}, \mathfrak{Y}, S(X, Y, \heartsuit))$ with teachers as *didactic engineers* $\mathfrak{X}$ designing and reflecting on didactic systems, $\mathfrak{X}$'s *didactic mentor* $\mathfrak{Y}$, and a didactic system $S(X, Y, \heartsuit)$ as the *paradidactic stake*. Japanese lesson study is a typical example: $\mathfrak{X}$ as a group of schoolteachers; $\mathfrak{Y}$ as some teaching experts or researchers; $S(X, Y, \heartsuit)$ as a lesson. This nested structure is similar to the teacher-educational didactic system. It is the reason why we study the teacher-educational phenomenon of didactic divide from the perspective of the paradidactic system. This modelling makes us be aware of the existence of two different values inherent in every didactic system. On the one hand, the didactic system could have *didactic value* for disseminating some work of knowledge. On the other hand, it could have *cognitive value* for producing some "new" work of knowledge in some sense—at least for occupants of the system.

The Scale of Levels of Paradidactic Determinacy

The didactic system and the paradidactic system are interrelated and intimate but different systems, which are produced within different institutional ecosystems. Within ATD, ecological analysis of the didactic system can rely on a *scale of levels of didactic determinacy*. We believe, however, that there is a need for a specialised scale for the ecology of paradidactic systems. The *scale of levels of paradidactic determinacy* (Fig. 1) is thus our proposal to a new ecological tool (Otaki et al. 2020). This scale models different hierarchical categories to which conditions for paradidactic phenomena belong. It emphasizes that players $\mathfrak{X}$ and $\mathfrak{Y}$ in any paradidactic system act around some didactic system, within a certain *profession* related to paradidactic working—e.g., schoolteacher, teacher educator, mathematician, and didactician. Any profession is involved by some didactic *noosphere*, which is a fuzzy,

Fig. 1 The scale of paradidactic determinacy levels

Humankind
↓↑
Civilisations
↓↑
Societies
↓↑
Noospheres
↓↑
Professions
↓↑
Paradidactic systems

informal, even implicit institution consisting of persons in different professions—e.g., people encouraging inquiry-based mathematics education, and people following to current Spanish curriculum—for designing, observing, reflecting, and evaluating a certain school system. The noosphere here is subsumed by a *society*, a *civilisation* and a *humankind*.

Conditions for the Didactic Divide

In this section, we identify some conditions under which the didactic divide can live *well*.

Inter-professional esoteric pact at the levels of noospheres and professions: The didactic divide is an institutional systemic phenomenon, which in turn affects praxeological tendencies of *institutional positions*, and vice versa. In the case of mathematics teacher education, the systemic demarcation of the border between the scholarly discipline and the general pedagogy in its didactic systems brings about an institutional rule that compels $\mathfrak{Y}^{\mathcal{M}}$ and $\mathfrak{Y}^{\mathcal{P}}$ to teach the teacher-educational didactic stakes being strictly based on their original professions. We call the phenomenon as an *inter-professional esoteric pact*. Under this clause, $\mathfrak{Y}^{\mathcal{M}}$ must teach the scholarly mathematics without any pedagogical "distraction", and vice versa. This condition is located at the levels of noospheres and professions, because it is in an inter-institutional contract implicitly signed by different professions.

Shortage of didacticians at the levels of societies and noospheres: A condition for the didactic divide phenomenon is related to the population of didacticians who have potential to impart the knowledge about didactic phenomena as the third type of teacher-educators $\mathfrak{Y}^{\mathcal{D}}$. Didacticians are clearly fewer than other professionals like mathematicians and pedagogues. This means that the *didactic infrastructure of didactics* is relatively poor, because didacticians are in the core of this infrastructure. This is partly due to the shorter history of didactics as the science of dissemination of knowledge than the cases of pedagogy, psychology, and transposed disciplines. Such infantility of the academic profession for didactic research could bring about the similar effect to the didactic transposition of sports at school, where good players are often regarded as possible good coaches. In the same way, the belief that good teachers are good teacher-educators seems to be strong. This common-sense reduces the needs for the didacticians in the teacher-education institutions, even though we are very much aware of the fact that the former-teacher as a profession has a crucial significance for the teacher-education programmes. The shortage of didacticians is located at the levels of societies and noospheres, since it is shared by noospheres in different countries.

Universality illusion on the scholarly knowledge at the levels of humankind and civilisations: A main part of the epistemological rupture by the *didactic transposition theory*—a subtheory of ATD—is that it exposed our naive view about the scholarly knowledge. The view regards the *scholarly* disciplines as *universal entities*, and neglects the uniqueness of the *school* disciplines. The awareness of such relativity of the knowledge is a crucial condition for the emergence of didactics (cf. Bosch and Gascón 2006). Consequently, the presence of the didactic phenomena remains unrecognisable. This *universality illusion* at the levels of humankind and civilisations widely prevails in the outside of didactics.

Transparency illusion on the social reality at the levels of humankind and civilisations: We have mentioned that the young history of didactics as a realm of anthropological or social science. This fact is deeply rooted in more generic conditions. One of them is the universality illusion. Another one is a *transparency illusion on social entities* (cf. Chevallard 1992). Whether one likes it or not, we are involved in different social realities e.g., economic, political, historical, cultural, educational, religious, and *didactic*, and are possessed consciously or unconsciously by such realities. We assume that we "know about" the social facts *before studying them* in some spontaneous way, without having scientific intention. In such a situation, we easily misunderstand ourselves that we know about social entities well, and it will be difficult for us to problematise them. The notion of didactic transposition gives us a good example. We know the fact that curriculum-makers develop national mathematics curricula around the world; nevertheless we have little information about properties of each transposition and its conditions (see also, Chevallard 1989). This illusion is a condition for the didactic divide located at the levels of humankind and civilisation, since it is widely shared by most people who had not been trained on a certain kind of social science.

Epistemological Reflection: How to Confirm Identified Conditions?

We have mentioned the existence of some conditions related to the didactic divide. Let us show you here our epistemological opinions on such analysis of conditions—*ecological analysis*—in didactics. For that, we have to start with discussion about *raison d'être* of ecological analysis. Ecological analysis is analysis for answering to a problematic question: *why does a given didactic phenomenon exist as what it is?* This type of questions produces a type of tasks of explaining the *evidence of the existence* of the phenomenon. This means that the ecological analysis plays its role in a similar way of the *proving of existence theorems* in mathematics. Any didactic phenomenon is highly theoretically constructed—which never be observed by "naked eyes"—, any report of observation and identification of such a phenomenon is a *hypothesis*—or even an opinion—in itself, even though such a report applies some "strict" theories or models. In this view, the existence of the didactic phenomena has to be validated in some way. We consider that the ecological analysis is a crucial technique for realising such validation. If we can find out some conditions, which seem to be linked tightly with a certain phenomenon, then we can increase the *degree of probability of the existence* of the phenomenon. At the same time, the more such conditions are generic, the more the *degree of probability of the generality* (and also the probability of the existence) of the phenomenon rises. In our case, we found out four conditions for the didactic divide, some of which are much generic. As a consequence, this ecological analysis could confirm—not in strict meaning within mathematics but in more general scientific sense—the existence and generality of the didactic divide.

One of the ways for making the result of the ecological analysis (we can call it *didactic proof*) more plausible is connecting each identified condition with some other conditions. Let us explain by a metaphor from mathematics again. Any mathematical proof is a system of mathematical conditions, which are represented by the statements about mathematical objects and relations. An important fact is that every proof is not a *list* of statements but a *system* of them. The validity of proofs comes from well-constructed structures of the systems of mathematical conditions. In a similar way, we can presume that the plausibility of the ecological analysis can be given by finding out some *ecosystem* of the respectively linked conditions, rather than a single condition or a list of separated conditions. Of course, a didactic proof through ecological analysis is not the same thing to the mathematical proof. A main difference between them is that the condition in the case of didactics cannot be justified by a strict deductive reasoning. In other words, every didactic condition is not absolute or definite but possible or probable. However, we assume that this status of the ecological analysis is not a weakness. Let us remind you here that most fields of the sciences—not only the social but also the natural—validate scientific laws in their own fields by the constructing of a system of *probable* conditions which respectively support each other. The French sociologist and anthropologist Pierre Bourdieu and his colleagues called it the *proof by a system of convergent probabilities* (Bourdieu et al. 1991). For example, the finding of the spherical earth can be validated by many conditions, e.g., the fact that we can do a round-the-world trip. Of course, this condition can also prove the existence of another possible fact that the earth is donut-shaped. However, other facts—e.g., one cannot look at a part of ground of the earth in the sky anywhere, that is to say, no one lives inside of the donut hole—disprove the donut hypothesis. Indeed, the conditions support the findings *as a system of evidences*. Let us highlight here that each conditions can also validate each other. For example, a fact that one can observe some *circles* (projections of a sphere) of the earth from some angles in outer space makes more plausible the nonexistence of the ground in the sky—which is useful for rejecting the donut hypothesis. In our case, each of identified conditions also do not exist separately. For example, the existence of the esoteric pact can partially be explained by a probable fact that both of mathematical and pedagogical professions usually possess the universality illusion of the scholarly mathematics. In another example, the universality illusion and the transparency illusion can give circumstantial evidences to the probable fact of shortage of the didacticians, and vice versa. And, of course, there exists no contradiction within the system of such conditions. In our view, constructing a chain of the conditions allows us to make the existence of each condition more plausible.

Final Remarks: On the Possibility of Overcoming the Didactic Divide

The ecological analysis is significant for not only understanding the didactic reality but also intervening in it. If one regards the didactic divide as a teacher-educational malfunction, and tries to repair it, one has to take into account the institutional ecology of its dead or alive. Up to this point, we have indicated some *favourable* conditions, which feed the didactic divide. The result of an ecological analysis can be also reinterpreted as an identification of *restrictive* conditions for the modification of the didactic divide. We believe that knowing the conditions that bring about a difficulty or impossibility of the emergence of a given phenomenon is crucial for proposing *viable* alternatives of didactic activities. Indeed, the didactic divide can be overcome—consciously or not—in some special cases. For example, Barquero et al. (2018) reports their teacher-educational experimentation, which includes the teaching of ATD, especially the notion of *study and research path*. However, its reproducibility in other contexts seems to be quite low under the abovementioned existing conditions. The declining of the didactic divide probably goes with the growing and diffusing of didactics.

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The Gap Between Studying a Generating Question and Planning Lessons Based on It

María Rita Otero and Carolina Llanos

Abstract In this paper, we analyse how 31 in-service teachers study the question that could generate a Study and Research Path (SRP): How does a parabolic antenna work? We also consider how they design lessons based on this question, by adopting notions of the anthropological theory of the didactic (ATD). The teachers investigated the question individually and in groups during an on-line course on Mathematics Didactics. Then, the teachers were asked to organize a possible instruction oriented to an institution known to them, based on the question they analysed before. The written texts produced by the teachers in both roles, studying the question and planning lessons are analysed using qualitative techniques and the concept of SRP. The results describe the main difficulties of in-service teachers in planning instruction according to the paradigm of research and questioning the world proposed by the ATD.

Introduction

The anthropological theory of the didactic (ATD) approach advocates an epistemological and didactic revolution (Chevallard 2015) of mathematics teaching and school disciplines and calls for the dropout of the traditional paradigm. Traditional teaching has replaced the study of questions with the study of answers, enforcing a non-motivating encounter with pieces of knowledge, which have an unknown rationale. According to the study gestures as characteristic of the research pedagogy are fully experienced if the questions studied are strongly co-disciplinary, suggesting design more complex, management and implementation of teaching. According to the epistemological foundations of the ATD, the most relevant didactical and mathematical activities are referred to questioning and reorganizing the knowledge to be taught.

M. R. Otero (✉) · C. Llanos
Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN), Tandil, Argentina
e-mail: rotero@exa.unicen.edu.ar

Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Buenos Aires, Argentina

C. Llanos
e-mail: vcllanos@exa.unicen.edu.ar

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Most in-service mathematics teachers are not aware of this aspect because they tend to assume knowledge as transparent, and given, or as ATD has pointed out, as a monument. The research in the framework of the ATD (Barquero 2011; Barquero et al. 2018; Gueudet et al. 2018; Otero et al. 2016; Otero and Llanos 2019) have highlighted the relevance of disposing of relatively tested SRP for teacher education. Here, we have selected an SRP designed by the IREM of Poitiers (Bellenoué et al. 2014) driven to secondary school teachers.

We designed a course for in-service mathematics teacher education at university, based on the ATD, where we taught the fundamental ideas of the SRP's and the attitudes and gestures involved in the paradigm of questioning the world (PQW). One of the aims of the course is to train in-service mathematics teachers to conceive and design mathematics lessons involving at least some gestures of the PQW (Chevallard 2015). The generating question Q_0 of the SRP proposed in this case is "How does a parabolic antenna work?" The question will be partially responded to at the end of the process allowing teachers to live a study and research path (SRP) as a student. The main goal is to make teachers encounter an unfamiliar inquiry-based activity related to Q_0 that could exist in a normal classroom of the considered educational level. Teachers studied the question individually and in groups producing a written answer in both instances. Then, they were asked to analyze and plan a possible instruction proposal adapting the lived SRP to be hypothetically experienced in a real school. This work aims to describe the potential and difficulties of the teachers while they plan lessons according to certain gestures of the PQW. Here we are mainly interested in how teachers reorganize mathematical contents to be taught and which organization of the instruction they propose. Below we briefly present the main ATD aspects, the generating question Q_0 and the approach adopted in this work.

ATD, Monumentalism and in-Service Teachers Training

The ATD considers that any teaching situation leads to the emergence of a didactic system S (X; Y; ♥) where X represents a community of study, Y represents one or more teachers helping to study and ♥ designates the object to study. This concept of system allows introducing the notion of teaching paradigm. Regarding the so-called monumentalistic paradigm, which we will also name "traditional", the system adopts the form S (X; Y; O) where O is a theory or a work that a set of pupils X must learn helped by a unique teacher y.

Monumentalism is a metaphor proposed by the ATD, which describes a didactic phenomenon that consists of treating mathematical knowledge as a set of monuments. In general, someone is summoned to admire, visit, preserve, immortalize and even love those monuments, as if they had always been there. Consequently, the monumentalistic paradigm conceives and treats knowledge in that way. Teachers naturally invite students to visit knowledge, without altering it, transforming it or deconstructing it. When someone encounters a monument, she is supposed to discover it, at most to live an aesthetic experience with it. Monuments are rigid and non-adaptable, remaining always at the same place. In a monumentalistic epistemology, something similar happens with mathematical knowledge, which is considered immutable through time, to know him it is enough to show it, hence the ostensive treatment of the mathematical objects. Teachers living in the traditional paradigm promote monumentalistic encounter with knowledge, that is, the students meet knowledge in advance, without the need to use it. This kind of meeting is named unmotivated by the ATD. Another characteristic of monumentalism, predominant in educational systems, is to split knowledge into small units. The in-service mathematics teachers participating in this research developed their professional life in the framework of the traditional paradigm and were trained there. They are not aware of the variety of monumentalistic gestures they perform.

The ATD advocates for substituting the traditional paradigm for another yet emerging so-called paradigm of questioning the world. The theory defines the SRPs as devices allowing the study of mathematics focusing on questioning. The ATD establishes that the starting points of mathematical knowledge are questions, named generating questions because its study should generate new questions. In this case, the didactic system adopts the form $S\,(X; Y; Q_0)$, where pupils X investigate and study a question Q_0 under the direction of a teacher y or a set of teachers Y. The purpose of this kind of didactic system is to develop and provide a possible answer to Q_0, which is produced under certain constraints, but there is no universal or universally valid answer (Chevallard 2009).

During an SRP, the entire didactic system and not just the teacher produces an answer, to do this, the system uses tools, resources and works. It performs actions such as searching, analysing, describing, developing and evaluating objects, works, resources, information, etc. that is the system generates a didactic environment or *milieu* M. This *milieu* M is composed of some answers called "pre-constructed" or available answers because they are within reach of the community of study - for instance, a book, Internet, course notes, etc. It also includes questions derived from Q_0, formulated from searching for answers to Q_0. This process describes the type of epistemological activity developed in the didactic system into the paradigm of questioning the world. In other words, SRPs are didactics devices created by the ATD to face monumentalism, because they possess, among others, the following characteristics:

(1) They are developed from a so-called *generating question* Q_0 because it does not admit an immediate response. That is, it will be necessary to formulate deriving questions, and look for available answers.
(2) The didactic *milieu* M is not built a priori but from the elaboration of answers. Resources are incorporated when they are needed, at any time, under the condition that they have to be validated by the study community.
(3) The teacher directs the study process, without having a preponderant role constructing M, and their contributions may or not be incorporated into M. In an SRP, the principle of authority does not apply; there are no privileged information systems or with more authority than others, unlike what happens in the monumentalistic paradigm.

(4) The study group formulates questions, except the generating question Q_0, which is proposed by the teacher. The diffusion of the possible response to Q_0 includes proofs and it has a strongly epistemological component, unlike the narrative character of diffusion within the monumentalistic paradigm, where the teacher's role is more similar to that of a guide in the visit to a museum than to the director of a study whose path is unknown in advance.
(5) Students formulate questions, propose resources, develop responses, evaluate, disseminate, defend and critically answer other students' responses.

The SRP and an Epistemological Model of Reference

A possible epistemological model of reference to study Q_0*: How does a parabolic antenna work?* refers to the problem of the construction of the tangents to a curve from the analytical geometry. In addition to re-discovering some properties of synthetic and analytical geometry, the reflection of light on different surfaces from geometric and wave optics could be studied. In secondary school, the main mathematical know-how to deal with would be: determine the equation of a circle from its characteristic elements and determine the equation of a line, the relative position of two lines; find the canonical form of a second-degree trinomial; solve a second-degree equation; determine algebraically the coordinates of the points of intersection of two curves; show that a line is tangent to a circle, a parabola, a hyperbola and find its analytical expression. Besides, Q_0 allows studying about the historical analysis of the problem of the tangents to a curve, and the development of mathematical knowledge linked to this problem. The questions concerning the reflection of light on different surfaces, lead to the study of the conics and their tangent lines. Experiments of the reflection on different surfaces could be carried out, considering several kinds of mirrors: cylindrical, parabolic or hyperbolic and questions like "Why a surface could be considered as a mirror? What types of antennas exist? What are antennas for? Which mathematical and extra-mathematical knowledge could be necessary to study the problem?" Possible answers could include tools of the synthetic or analytical geometric framework in R^2 or R^3, among others. On the other hand, if the curves were unknown, the insufficiency of the geometric-analytical framework to determine tangent lines would require studying differential calculus. In this case, Q_0 was selected because it allows studying an important part of mathematical contents involved in the teachers training that are also relevant contents of the secondary school syllabus in Argentine.

Method

This work involves 31 in-service mathematics teachers who attended the second year of the Bachelors in Mathematics Education (BME) at a National University in Argentina. This course allows teachers of mathematics to graduate Institutes of Teacher Education, which are non-university institutions of teacher training, to complement their mathematical and didactic training. The BME curriculum consists of eight four-month courses over two years: three are Mathematics courses and the others corresponding to Didactics of Mathematics, Information and communications technologies (ICTs), Epistemology, Methodology and Cognitive Psychology. The instruction was provided completely online through the Moodle platform. The course was in charge of three teachers (one teacher per ten students). The fundamental notions of ATD were taught and some examples of various SRPs available and widely disseminated in the literature were analyzed. The last month of the course was devoted to exploring the question Q_0*: "How does a parabolic antenna work?* Teachers grouped in six teams carried out the following tasks:

- T_1: Study Q_0 as a student and prepare a possible individual written answer,
- T_2: Analyze and discuss the individual answer with the group, proposing a possible group written answer to Q_0,
- T_3: Work on the proposal of a possible instructional proposal adapting the experienced SRP (T_1 and T_2) to some hypothetical school conditions

Using the written answers given to T_1, T_2 and T_3, we seek to identify and describe the abilities, difficulties and the most relevant drawbacks found by the in-service teachers while studying Q_0 and planning lessons based on questioning, according to the paradigm of questioning the world. Teachers were not asked to test their proposal in the classroom, due they could not introduce an SRP in one week, nor could the course team help them properly. We analyzed the responses given by the six groups of teachers to the tasks T_2 and T_3 through the components of an SRP aiming to identify, describe and understand the most important difficulties and obstacles faced by teachers when they study Q_0 with the hypothetical intention of organizing a teaching under the paradigm of questioning the world. We built six tables comparing T_2 and T_3 in columns and the deriving questions, available responses, mathematical and physical knowledge linked and the possible answer elaborated by each group, in rows.

Research Questions

1. How in-service teachers transform knowledge when they work on tasks T_2 and T_3 under the role of student(s) or teacher(s), respectively?
2. What are the main difficulties students-teachers face planning the lessons according to the paradigm of questioning the world?

Results

Group A. Regarding T_2, first, they asked questions related to physics and then mathematics ones. The questioning referred to the functioning of the antennas, the parabolic antennas, the electromagnetic waves and their propagation and reflection. Finally, they adopted the geometry optics' model, they considered rays reflection and questioned the equality between the incident and reflected angle. This was justified using the Fermat principle. Then, they asked about the property of paraboloids, which would direct the reflected rays on the focus, due to their shape. Once arrived here, they defined parables in the synthetic framework and they gave a central role to the geometric proof of the existence of the tangent line to any point of a parable. Later, they used this knowledge to justify the concentration of the reflected rays on the focus. The written answer of this group was based on mathematics and physics knowledge. On the other hand, responding to T3 this group eliminated eight questions but preserved the synthetic framework of analysis. The teachers proposed to ask the students about the reflection on the parabolic antennas and they designed specific tasks to construct parables based on their geometric definition, or their elements, and to find the tangent line employing GeoGebra. They validated characteristics of ray reflection on a parabolic surface using an already made Geogebra video (https://www.geogebra.org/m/xxeRSH7H) showing the geometric construction. Thus, adopting the teacher role they removed questions and knowledge to be taught mainly focusing on parables.

Group B. In front of T_2, this group firstly addressed the mathematical question about, what is a paraboloid? and about its definition, characteristics, classification and canonical equations of quadratic surfaces and paraboloids. This group questioned the connection between antennas and electromagnetic waves (EW) as well as how signals received by satellites would be concentrated on the focus. Teachers also asked about the definition of parables, their Cartesian equations and about how to demonstrate the so-called "*reflective property of the parable*". The last question was about the reason to use a paraboloid as an antenna. Considering task T_3, the student-teachers designed traditional lessons based on exercises to teach the contents, and tend to hide all the questions. This group did not propose any transformation of knowledge to be taught, beginning with definitions followed by application exercises.

Group C. Firstly, responding to T_2, they proposed some questions from physics and later moves to more mathematical ones, such as what characteristics EW have or what types of antennas do exist? They also questioned why parabolas should be used to build satellite antennas. Group C highlighted the study of parabolas calculating their canonical equations. Regarding parabolic antennas, student-teachers analyzed the reflection and refraction on these surfaces. They used the so-called "*property of focus convergence of the parabola*" to justify that any beam parallel to the axis may be reflected passing through the focus. Regarding T_3, student-teachers eliminated many questions, and particularly questions more inside physics were almost abandoned. About the knowledge they planned to be taught, they only considered parabolas and designed lessons based on traditional exercises about polynomial functions of second degree, that are not strictly related to the generating question (as it was presented to them), which was not finally answered.

Group D. Regarding T_2, the group addressed questions about the paraboloids, parabolas and their characteristics related to the emission and detection of signals. Then, they formulated the so-called *reflective property of the parabola* using a GeoGebra's applet accessible on the website. Responding to T_3 this group changed the generating question into: "*How to build a solar kitchen?*" Group D proposed to construct a solar kitchen using cardboard guided by a video (https://youtu.be/F_fZEBw8r-c). After that, student-teachers planned to build parabolas through four manual techniques. Finally, the group formulated the answer as follows "*By means of a paraboloid and placing in the focus the object to be cooked, the solar cooker is obtained*".

Group E. Responding to T_2, Group E first asked about the satellite antennas and the wavefronts. Then, they asked how is a parabola defined? Which are their properties? and about the characteristics of paraboloids. Both parabola and paraboloids were introduced by the definitions. The so-called "*focal property of the parable*" was analytically justified through GeoGebra tools. The group formulated the response as follows: "*Any beam going out of the focus is reflected on the parabola with direction parallel to the axis. Any beam parallel to the axis is reflected passing through the focus*". Considering T_3, teachers replaced more open questions (as they came to experience) with some specific exercises to study parabolas in the analytical framework. The group concluded: "*To study a parabola is sufficient to understand the functioning of antennas*".

Group F. Regarding task T_2, the group asked more than 17 questions about physics and mathematics, in this order. For instance: How is an EW defined? What is a wavefront? What happens when an EW encounters an obstacle? How does reflection occur? What geometrical characteristics does a parabola have? How do the antennas transmit and receive EW? What is the connection between parabolas and parabolic antennas? What happens when an EW hits a parabolic surface? Physically and mathematically, what is reflection? Which equation represents a straight line? How to determine its slope? What is the equation of a parabola? What are the axis of symmetry and the focus of a parabola? The group looked at parabolas and paraboloids in the analytical approach and finally adopted the ray optics model. The student-teachers based the entire study on a single book of physics where also some mathematical justifications were proposed. Regarding T_3, the group highly reduced the questions to pose to students, which were then transformed into traditional exercises. The parabola were studied to explain that the reflected rays are concentrated on the focus if the antenna has a paraboloid shape.

Discussion and Conclusions

Regarding the questions asked by the student-teachers, many are essential questions began by: "*What is…?*" These questions promoted closed-ended answers, as definitions, and revealed that teachers tend to conceive mathematics as immutable. Furthermore, their responses are mostly coming from searching on the internet. They simply copied and pasted the answers without questioning them. The majority of the groups seem to know enough mathematics to deal with the problems raised, but they do not question what they get, in correspondence with a crystallized viewpoint of mathematical knowledge. In these cases, we can not say that is the mathematical ignorance, which hinders the reorganization of the knowledge to be taught but means a lot about how teachers conceive mathematics. There are differences between the groups that first asked physics or mathematics questions. When physics questions were asked first, the groups explored different areas and subjects on physics, to then asked mathematical questions related to the physics subjects. But when the groups started addressing more particular mathematical questions, they assumed that, first, it is necessary to know and define the mathematical contents involved in the problem to then applied them.

These groups generally conserved the same organization of knowledge in both tasks, proposing an instruction framed in the traditional paradigm. This happens with groups B and E. However, monumentalism would adopt various forms, for instance, the group D changed Q_0 by: "*How to build a solar kitchen?*" While it could considered auspicious, finally, they proposed to build a cardboard kitchen. Thus, they promoted a merely manipulating teaching activity, disregarding science and mathematics knowledge, and, in their understanding, simpler and friendlier to their students. Besides, they proposed to consider parabolas proposing four handmade activities, but without planning any questioning during the lesson. In most groups, the monumentalistic viewpoint turns out evident: only teachers control and manage the didactic environment M. Furthermore, student-teachers tend to eliminate real questions, when they moved from T_2 to T_3, they tend to eliminate the complexity of working with open questions and replaced them by traditional and directed activities. For instance, Group E proposed to the students the task of "*Observing all these paraboloids, could you say which ones have parabolas as cross-sections?*

Regarding how is the response $R^{\heartsuit}$ formulated in both tasks, it could be an epistemic (group A, B, F) or a narrative response (groups C, D, E). The activity involves communicating and discussing a possible answer with the whole class. The working groups accepted that the communication and discussion of the answer in T_2 was necessary, but not for T_3. This led the student-teacher, when they had to reorganize the knowledge to be taught to present the proposal of a teaching activity, they cannot avoid monumentalism. That is, knowledge to be taught was transparent and hard to be questioned or, at least, questioning was not necessary to be worked on the proposal. Regarding the groups starting with questions related to physics, group A is remarkable. This group adopted the ray optics model and decided on studying the parabola in the synthetic geometry framework. When they tried to justify the reflection of rays, this is the only group that noticed the relevance of demonstrating the existence of the tangent to a curve. They developed a geometric technique to find the tangent. However, when they dealt with T_3, they tend to eliminate the questions and adopted a much more conducted teaching. Group F differs from Group A because although it attached great relevance to physics, they studied only parabola from an analytical geometry perspective. Group C started by asking physics questions, but when responding to T_3 they limited the knowledge to be taught to the "quadratic function". Except for Group B, the rest of the groups limited the knowledge to be taught to the parabolas theme. Only Group A noticed and solved the problem of the tangent to the parabola. This shows the relevance of taking into account, together with student-teachers, parts of the epistemological model of reference related to the knowledge at stake as an essential didactic-mathematical tool. The in-service teachers had difficulties developing the EMR because it was alien to the traditional pedagogy. Thus, student-teachers did not consider either necessary to develop an a priori epistemological and didactic analysis of the situation or noticed that it is one of the most relevant professional practices, although during the course it was carried out several times.

To sum up, we can conclude that most teachers faced strong difficulties designing hypothetical lessons based on the paradigm of questioning the world. The reason is that the teachers avoid losing control of the didactic *milieu* and conceive themselves as the only ones responsible for managing it. The paradigm of questioning the world requires an open didactic *milieu* and non-monumentalistic encounter with the knowledge. The epistemological viewpoint of teachers would have a great influence on this fact, given that half of the teachers' groups decided to find mathematics first and then, afterwards, propose related questions, exactly in contrast to the research and questioning paradigm.

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Prospective Teachers Solving a Percentage Problem: An Analysis of the Construction of a Praxeology

Heidi Strømskag

Abstract This study focuses on a team of three primary and lower secondary school student teachers struggling with a percentage problem. The conditions and constraints under which their inquiry unfolds and the very vicissitudes of this inquiry are analyzed in the framework of the ATD. First, a praxeological reference model is constructed through a mathematical analysis. Then the model of didactic moments is used to analyze the construction of a praxeology centered on the type of tasks for which the observed students built up a technique.

From the Outside Inwards: A Centripetal Depiction

The praxeological analysis propounded in this paper will bring to light the hidden complexity of an unassuming training situation involving percentages. The general research question tackled here is the following:

> *Given a problematic question Q, what are the tools that an instance î can mobilize to solve it, and what uses can the instance î make of these tools? What are the conditions and constraints under which î is led to operate, which, given Q, determine the instance î's choice of tools and their uses?*

In what follows, I depict a set of conditions and constraints under which the observation took place. The question Q was devised by one of the teachers that appear in the observed process of study. As will become apparent, this question borders on both algebra and geometry.

H. Strømskag (✉)
Norwegian University of Science and Technology, Trondheim, Norway
e-mail: heidi.stromskag@ntnu.no

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The Institutional Setting

The research site was a department of teacher education at a university college located in Trondheim (Norway), providing four-year programs for primary and lower secondary teachers' education. There were approximately 450 students enrolled in these programs each year. Approximately 75% of the students were females.

The Class and the Curriculum

The students observed were enrolled in a program with an emphasis on mathematics and science subjects (Ministry of Education and Research 2003). The structure of this program is shown in Fig. 1. The acronym RWM stands for "Reading, Writing, and Mathematics". Didactics of subject matter and practice field experiences are integrated into all study units in the program.

The Teaching Unit

The session observed was part of an Algebra module. The topics in this teaching unit as devised by the teachers were studied chronologically in the following order:

> Number sequences; formulas (diverse modes of expressions); development of the formal mathematical language, also in a historical perspective; formulas; the concept of variable; mathematical models (project)*; mathematics in nature and society*; *mathematics in daily life*; the concept of variable*, formulas, and equations*.*

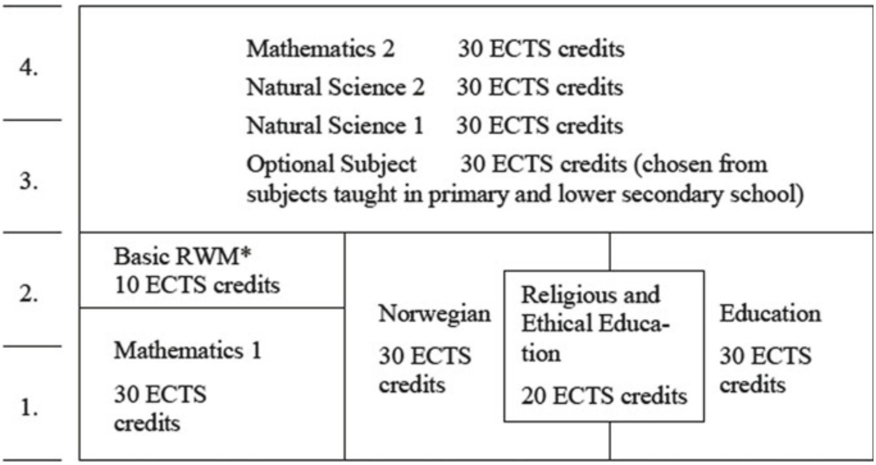

Fig. 1 Teacher education with an emphasis on mathematics and science subjects

The asterisked topics pertain to an "Algebra and Functions" unit. Those in italics were tackled during the observed class session. It has to be stressed that, while relevant to the observed situation (as we shall see), the topic "Fractions and decimal numbers", which pertains to a "Number theory" unit devised by the teachers, was studied *after* the Algebra topics. It is also worthy of note that, in Norway's prevailing mathematics curriculum for Grade 1–10, the mathematical domains include "numbers and understanding numbers, algebra, functions, geometry, statistics and probability" (Directorate for Education and Training, 2020, p. 4).

The Practical Working Conditions

The classroom was a large, horizontal floor room divided into two parts (of different sizes) by movable walls. The biggest part of the classroom had tables and chairs for all the students (the class had 66 students); this was where the introductory part of lessons took place. The observed students were placed in a small room adjacent to the classroom; it was a separate room with an entrance from the classroom. The purpose of this arrangement was to reduce the background noise on the video recordings. The observed students were informed that they could at any time go and ask for help or get suggestions from the teachers or peer students in the large classroom where the rest of the class worked in small groups on the same tasks. On their initiative, the teachers occasionally went into the room in which the observed students worked.

The Observed Didactic System

The students $X = \{x_1, x_2, x_3\}$

The didactic system working in the small room, denoted by $\mathcal{S} = S(X, Y, Q)$, consisted of a group of three students $X = \{x_1, x_2, x_3\}$, and a team of two teachers $Y = \{y_1, y_2\}$. The set X was a "practice group", i.e., a grouping decided by the faculty administration for one academic year, mainly to organize mentored in-field practice in a particular class in primary or lower secondary school. When the data were collected, X was in the second semester of the teacher education program. The 20-year-old student x_1 had attended a voluntary mathematics course (5 hrs a week) in her second year in upper secondary school, preparing for further studies in natural sciences and mathematics and achieving a low mark. The students x_2 and x_3, aged respectively 22 and 23 years, had both attended voluntary mathematics courses (5 hrs a week) in their second and third years in upper secondary school, preparing for further studies respectively in natural sciences and mathematics and in social sciences and economics, with average marks. On the basis of their marks from upper secondary school, X can be considered average strong.

The teachers $Y = \{y_1, y_2\}$

The teachers y_1 and y_2 were experienced teachers of mathematics in the department of teacher education at the university college: while y_1 was a professor of mathematics with more than ten years of practice as a mathematics teacher educator, y_2 was a senior lecturer with more than thirty years of the same practice. The two teachers played slightly distinct parts: y_1 was responsible for the lessons on algebra, including the design of tasks; y_2 had the role of a "teacher assistant" in the orchestration of the students' work and shared with y_1 the task of helping students. The information given above about the students x_1, x_2, and x_3 was not available to y_1 and y_2: they only knew that these students had a general university and college admission certification and the information they could get through teaching them mathematics.

The observer

I observed and video recorded the session for the sake of the research reported here. For the observed session to be as naturalistic as possible, I had the role of non-participant observer. The teachers Y were my colleagues at the time the data were collected.

The Question Q *Under Study*

The question Q—or more exactly the set of questions—offered for study in $\mathscr{S}$ was the following, where the questions sentences in square brackets were ignored by X, maybe for lack of time:

- Imagine that you have a square. Make a new square where the side length has increased by 50%. Q_1. How many percent has then the area increased? [$Q_1{}'$. How many percent has the perimeter increased?]
- Imagine now that the side length increases by $p\%$. Q_2. How many percent will the area consequently increase? [$Q_2{}'$. How many percent will the perimeter consequently increase?] [Q_3. If you had a cube where the side length increases by $p\%$, how many percent will the volume consequently increase?]

The tasks in the algebra module were designed by y_1, who claimed that one aim was to have students engage, in different contexts, in generalizations that would eventually be expressed in algebraic notation. Beyond that, we know about nothing concerning the choice of the question set . The document (in Norwegian) handed out to X contained a list of six "assignments", each made up of two tasks, from which every group of students had to choose one assignment. Moreover, the groups were paired so that, at the end of the class, each group in a pair was supposed to present to the other group their solution to the chosen assignment. The observed students chose Assignment 1, in which the question set was coupled with a task that consisted in devising a realistic context for a certain equation before solving that equation (in the case of X, it was: $\pi r^2 \cdot 2 + \pi r^2 \cdot 4 = 1000$). We know nothing of the reasons why X chose Assignment 1.

Conditions and Constraints

In the session observed, the students X are going to face two problematic tasks: the task t_1 consists in providing an answer to the question Q_1; the task t_2 consists in answering the question Q_2, which is clearly presented as a generalization of the question Q_1. The choice to concatenate questions Q_1 and Q_2 manifests an intention to help students build a technique to perform t_1 and then t_2. We will see that the problem statement (due to y_1) conceals other potential hints to help students tread their way to "success".

Which Mathematical Praxeology? A Reference Model

Suppose persons, not necessarily students, have to give an answer to the question Q_1. Which praxeology can they use? Let's start with the *most rudimentary praxeology*. Suppose the side length of the square is 10 (in some unit of length); the new length is therefore $10 + 5 = 15$. The original area is $10^2 = 100$ and the new area $15^2 = 225$. Here is a purely numerical rule to calculate the percentage increase (Percentage Change, n.d., To Calculate the Percentage Change section):

> First: work out the difference (increase) between the two numbers you are comparing [i.e., 225 – 100 = 125]. Then: divide the increase by the original number [125 / 100 = 1.25] and multiply the answer by 100 [1.25 × 100 = 125; the percentage increase is thus 125%].

Other, more technologically advanced, praxeologies rely on the use of letters in order to generalize the foregoing example. Let a be the original side length, a' the new side length: the rate of the side length increase is given by the ratio $\frac{a'-a}{a}$. Let A be the original area and A' the new area: the rate of the area increase is $\frac{A'-A}{A} = \frac{a'^2-a^2}{a^2}$. Given the percentage increase on a, that is, $\frac{a'-a}{a} = p\% = \frac{p}{100}$, and using some algebra, one can calculate the percentage increase on A:

$$\frac{a'^2 - a^2}{a^2} = \frac{a'-a}{a} \times \frac{a'+a}{a} = \frac{a'-a}{a} \times \frac{a'-a+2a}{a} = \left(\frac{a'-a}{a}\right)^2 + \frac{a'-a}{a} \times 2 = 2\frac{p}{100} + \left(\frac{p}{100}\right)^2$$

If $p = 10$, one arrives at: $2 \times 0.1 + 0.1^2 = 0.2 + 0.01 = 0.21 = 21\%$. A more standard way of arriving at this result consists in first establishing the following basic formula $a' = \left(1 + \frac{p}{100}\right)a = ra$, where $r = 1 + \frac{p}{100}$. We then have: $\frac{a'^2-a^2}{a^2} = \frac{(ra)^2-a^2}{a^2} = \frac{r^2a^2-a^2}{a^2} = r^2 - 1 = \left(1 + \frac{p}{100}\right)^2 - 1 = 2\frac{p}{100} + \left(\frac{p}{100}\right)^2$. Here we see clearly that the value of $\frac{a'^2-a^2}{a^2}$ is independent of a.

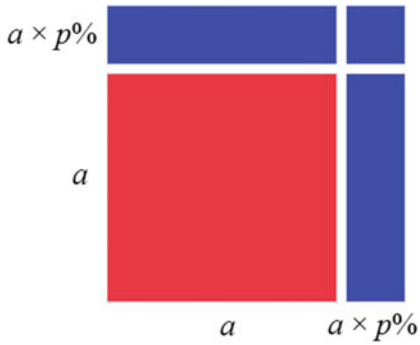

Fig. 2 A geometric model of a percentage increase

This property generalizes to any homogeneous function f with a degree of homogeneity k (if a increases by $p\% = r - 1$, then $\frac{f(a')-f(a)}{f(a)} = \frac{f(ra)-f(a)}{f(a)} = \frac{r^k f(a)-f(a)}{f(a)} = r^k - 1$) and may lead to a third, proto-algebraic praxeology, that makes us come back to the "geometric algebra" of the ancient Greeks (Høyrup 2017).

It may be supposed that the choice of squares by y_1 (instead of other similar figures, for which the same property holds) was inspired by the didactic hypothesis that this "geometric" praxeology should facilitate the students' access to a full-fledged algebraic praxeology: it would suffice to "read" Fig. 2 to conclude that the area increase is $p\% \times a \times a + (p\% \times a)^2 + p\% \times a \times a$ and, therefore, that the percentage increase of the area is $2\frac{p}{100} + \left(\frac{p}{100}\right)^2$.

Which Didactic Milieu? Some Preliminary Remarks

The more material items

The above reminds us of two generic requirements of human action, whether mathematical or not. Firstly, in order to achieve a particular result—for example in order to determine the value of a certain percentage—, we must *do something*: we are not mere guessers nor "mentalists" who can instantly, without further ado, read Nature's mind. Secondly, in order to do something, we need "something": we need ostensive and non ostensive *tools*.

What is the didactic milieu M made of that the students observed can draw on? Let us start with the more "material" elements of the milieu M. Besides their own praxeological equipment, the three students have available (1) some writing material, (2) a calculator, and (3) a mathematics didactics textbook (Selvik et al. 2007). This textbook, however, has nothing on percentages. It contains 18 chapters, the first two of which are not unrelated to the task done by the observed students. The first one is entitled "Learning of algebra" and has sections on "Why learn algebra?" and "Prealgebra". The second one is entitled "Symbols in mathematics". Its first two sections are entitled, respectively, "Symbols denoting numbers and magnitudes" and

"Symbols with which we can reckon/calculate and think". In fact, these sections both (unintentionally) point to an essential aspect of the initial deficiencies of the milieu M which the students observed had to rely upon.

Didactic Moments

I shall hereafter refer to the model of *didactic moments* (Chevallard 1999), which is at the heart of the ATD. The construction of a praxeology $\wp$ centered on a type of tasks T involves six "dimensions" called *moments* of the study process that generates the emergence of $\wp$. They are respectively (1) the *moment of the first encounter* (with T), (2) the *moment of the exploration of the type of tasks T and of the emergence of a technique* τ (of performing tasks t of the type T), (3) the *technological-theoretical moment*, when the *logos* part of the praxeology emerges, (4) the *moment of working on the praxeological organization* $\wp$ under development (in order to improve both the *praxis* block and the *logos* block), (5) the *moment of institutionalization* (of $\wp$), and (6) the *moment of the evaluation* (of both $\wp$ and of a person's relation to $\wp$). The following analyses are based on a transcript (in English) of the utterances (in Norwegian) exchanged between members of X and Y and on a description of their actions. Although the word *moment* as used in the expression "didactic moment" has no temporal meaning (it refers to a certain "dimension" of the study work), I will analyze the work session by following its time course.

A bad start

The discernible aim of the work session is the building up of a praxeology $\wp$ centered on the following type of tasks T: "Determine the percentage of increase in the area of a square whose side length is increased by $p\%$." Note that a task $t \in T$ depends a priori on two parameters: the side length of the square, and the percentage $p\%$. The beginning of the work session comes under the *moment of the first encounter* with T. The student x_1 first draws a square with side length 2 cm, whose area is 4 cm^2; an increase by 50% in the side length leads x_1 to draw a square of side length 3 cm, with area 9 cm^2. The increase in area is 9 cm^2 – 4 cm^2 = 5 cm^2.

At this point, x_1 induces the trio to (erroneously) take as the rate of increase the ratio 5 / 9. The student x_2 asks why one should divide the difference by the *greatest* of the two numbers, to which x_1 rejoins assertively: "Part divided by whole. It's just a rule that I have learnt." Although x_1's assertion is vividly discussed between the students, this crucial mistake will determine the group's activity prior to the intervention of the teacher y_1. The students thus conclude that the percentage sought is equal to $5 / 9 = 0.555\ldots \approx 56\%$. All this is part of the *technological-theoretical moment*: the recipe "part divided by whole" is a *technological* self-conviction, apparently "justified" by this (implicit, unspoken) *theoretical* belief: "Any percentage is smaller than or equal to 100%."

Correcting the trajectory

Note that, until then, the milieu M contains drawings of two squares (with side lengths 2 and 3 cm), words and some arithmetic (numbers, and the division operation), but no letters or other symbols. The student x_1 tries to generalize her conclusion to any value of p using the following haphazard piece of reasoning:

> … when we increase this [the side length] by fifty percent, then we got fifty-six percent increase of the area, thus, six percent more than what is here [the side length]. So, then it will be p percent plus six.

However, x_1 is not completely sure of her conjecture and adds: "But I don't know if it will be like this." Yet the trio continues on the path opened by x_1 and goes on exploring T, taking as a new specimen a square of side length 5 cm: the new side length is therefore 7.5 cm and the new area 56.25 cm^2. At this point y_1 enters the room, while x_1 is "testing her theory", as x_3 says, arriving once again at (56.25 – 25) / 56.25 = 0.555… ≈ 56%. Through a fairly long and winding dialogue, y_1 corrects the trajectory of the trio: their exchange confirms that the students have first and foremost to accept that a percentage of increase can be *greater* than 100%.

Relaunching the search unencumbered

When y_1 leaves the room, he also leaves the trio in a state of bewilderment. For the first time, if we except x_1's attempt to enact a general law, the students take, however awkwardly, the parameter p into account, by trying to mimic the only calculation they now know to be correct, i.e., (9 – 4) / 4 = 1,25 = 125%—they consider for example the expressions: $2 + p - 4$; $2 \times p$; or $2 + p\%$. This comes under the *moment of the exploration of the task and of the emergence of a technique* τ. They seem to find themselves in a fog about what to do.

Considering the flow of nonproductive ideas that their activity gives rise to, the observer (myself) reminds them that they can ask one of the teachers for help—which they do one minute later. The teacher y_2 soon comes to their rescue; it seems that y_2 tries to underline the need for an appropriate study technique. But, instead of using numbers (not to mention letters), the group led by y_2 gets bogged down in considerations about the graphical and even material (using sheets of paper and scissors) modelling of the phenomenon under consideration. After an involved study of the case $p = 50$, y_2 urges the students to study the case $p = 25$.

Once again, the combined handling of paper manipulatives and percentages of areas proves arduous. After having insisted on the use of paper models, y_2 then encourages the trio to dematerialize these models, arguing as follows:

> Now it could have been interesting to do something which is manageable once more, to take ten percent for instance, and see what happens then. Because that is quite easy, you may not even need to fold paper. Perhaps you could have drawn a picture and imagined, with this [the manipulatives] as a model.

He thus surreptitiously induces the students to consider the case of a square of side length 10 cm with $p = 10$.

The teacher y_2's decisive intervention

The students strive to do it, while persistently confusing percentage points and areas of one cm^2. For his part, y_2 tries to convince himself that the process is taking shape ("Yes, now you have got on the track of something", he observes, and makes this comment directed to the observer: "I think they are on track here now"). He presses the trio to count the supplementary "small squares"—which, in the case considered, are $2 \times 10 + 1$ in number—and conclude that, therefore, the percentage of increase equals 21%. To top it off, y_2 "helps them to write down in percentage notation the arithmetic results (the areas) achieved in each of the three cases", i.e. $2 \cdot \left(\frac{50}{100} \cdot \frac{100}{100}\right) + \frac{50}{100} \cdot \frac{50}{100}$ (increase by 50%), $2 \cdot \left(\frac{25}{100} \cdot \frac{100}{100}\right) + \frac{25}{100} \cdot \frac{25}{100}$ (increase by 25%), and $2 \cdot \left(\frac{10}{100} \cdot \frac{100}{100}\right) + \frac{10}{100} \cdot \frac{10}{100}$ (increase by 10%).

This work sequence is a contribution to the *moment of working on the praxeological organization* $\wp$ being built up. All this is then abruptly expressed in algebraic form to give first $2 \cdot \left(\frac{p}{100} \cdot \frac{100}{100}\right) + \frac{p}{100} \cdot \frac{p}{100}$ then $\frac{2p}{100} + \frac{p^2}{100^2}$. This is the *moment of institutionalization.*

Learning without warning

The above did not conclude the work session. The trio had to present "their" results to another group in the class. What happened then in this very special *moment of validation*? The trio first illustrated the problem using paper manipulatives in two special cases ($p = 50$ and $p = 25$), before writing on the blackboard the expression $\frac{2p}{100} + \frac{p^2}{100^2}$ as the solution to the problem. They commented that this expression was derived from the expression $2 \cdot \left(\frac{p}{100} \cdot \frac{100}{100}\right) + \frac{p}{100} \cdot \frac{p}{100}$, which has the same structure as the concrete cases previously presented. Finally, they got positive feedback from the listening group of student teachers, who claimed to understand what was being explained. The didactic task of presenting one's own findings to a group of peers acted as a factor of learning and a revealer thereof.

Discussion

The end of the study process—when the three students present "their" results to some of their peers—sounds like a kind of miracle, given the winding and uncertain history of the making of these results, which boil down essentially to the expression $\frac{2p}{100} + \frac{p^2}{100^2}$. This is perhaps the most important discovery of the study: the role played by a pedagogical structure too rarely mobilized—the presentation to others of the results of a work—, which by contrast is so familiar to researchers in their own working communities, has a kind of positive retroactive effect on the understanding by the authors themselves of the work they present.

Having noted this, I still have to highlight two more facts revealed by the observation and analysis carried out. The first fact is twofold. On the one hand, the students observed *do* have, in their praxeological equipment, a praxeology relating to percentages (they understand the question they have to tackle and take action, however wrongly). On the other hand, there is no mechanism in the didactic system to ensure that this praxeology is adequate (in fact, it is not, as the use of a wrong ratio shows) and to update it if necessary, as if no one cared about the tools really available at the time of taking action. As we know, this will lead the observed trio to wander quite a long time in vain in search of a valid answer.

The second fact relates to the intervention of the teachers and is also twofold. On the one hand, there seems to be a (traditional) tendency to consider that geometric entities are more accessible than algebra for these students and, more precisely, that geometry is the main pathway to algebra. On the other hand, the final intervention of one of the teachers closes the inquiry by giving almost explicitly the hoped-for answer. Although this "oblique" teaching somewhat departs from the canons of modern pedagogy, we know now that its learning yield (as well as that of the work done previously during the session) was far from negligible. All this constitutes a distinctive contribution to the study of the research question made explicit at the beginning of this study and reveals, in contrast to analyses which focus on the student only, the relevance of a crucial question—Which tools are mobilized and what uses are made of them?—to which we must endlessly return.

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The Curriculum Problem and the Paradigm of Questioning the World

The Cooperative Works, a Teacher Training System Open to the Local Social Network

Salone Jean-Jacques

Abstract Since September 2017 an initial training curriculum at master's level is set up for primary school teachers in Mayotte, a French department in the southern hemisphere. It is based on a specific training system, the cooperative works, which favours the improvement of professional skills and, more specifically, a competence of contextualization. In this paper, we describe this training system with concepts from the anthropological theory of didactics such as the Herbartian schema, and we expose research results about how it allows the questioning of the Mahoran heritage and society.

Introduction

For Mayotte, a French island since 1843 and department since 2011, the professionalization of teachers is an important issue. One reason is that the School of the Republic in Mayotte educates a growing number of pupils, with a school population exceeding the symbolic hurdle of 100,000 students in 2018–2019 (Vice-Rectorat of Mayotte 2018, p. 2). It is in this social context that since September 2017, the Centre Universitaire de Formation et de Recherche (CUFR) of Mayotte, in partnership with the Vice-Rectorat of Mayotte, the University of La Reunion and the Ecole Supérieure du Professorat et de l'Education of La Reunion, has offered an initial training curriculum at master level for Primary School Teachers.

Beyond the context of higher education, Mayotte also presents a particular socio-cultural context. In the first place, the schooling language, French, is not the native language of the island. As established by Laroussi (2009), it coexists with two other vehicular languages: Shimaore, a Bantu language, belonging to the Swahili area and Kibushi, a Malagasy variety of the Malayo-Polynesian family, the only one spoken outside of Madagascar. This linguistic difference is driven forward on a more general cultural level, with a heritage that has its roots in the East African space. Furthermore, Blanchy-Dorel (1990) describes a society in which:

S. Jean-Jacques (✉)
Centre Universitaire de Formation et de Recherche de Mayotte, Mayotte, France
e-mail: jean-jacques.salone@univ-mayotte.fr

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> the worldview conveyed by the culture of Mayotte to its members is essentially a Muslim vision; but in most social circles, it is a popularized Islam that is mixed with non-Muslim elements coming as much from the pre-Islamic Arab animist traditions as from African (Bantu) traditions. (Blanchy-Dorel 1990, p. 112)

In particular, in the village where he is raised, the young child goes to the Koranic school" (Blanchy-Dorel 1990, p. 55).

One of the peculiarities of Mayotte also lies in its natural insular and tropical heritage. "Mayotte has been identified internationally as part of the 238 Ecoregions of the World Fund by nature, high places of biodiversity" (Amann et al. 2008, p. 30).

But this heritage, cultural or natural, is strongly threatened. Its preservation has become a major issue, and school is no exception. The project put forward for primary and secondary schools suggests "developing the concept of living well together: Recognizing cultural otherness and making civilization by coexisting in different cultures and languages" (Vice-Rectorat of Mayotte 2016, p. 1).

In the present paper, we describe the training system in terms of the Anthropological Theory of Didactics (ATD), and we report global results on how it enforces the students' abilities in questioning the world.

Theoretical Framework

Our focus in the graduate master courses we propose is thus placed on the *competence of contextualization*, the "ability to integrate different contextual spheres which partly drives the nature of the processes at work in the management of this professional situation of teaching" (Sauvage-Luntadi and Tupin 2012, p. 106). According to Raynaud (2005), we will not try to differentiate the notion of *context* from those of *environment* or *situation*. We prefer to adopt a *systemic approach*: "rather than causally depending or referring action to "context" or "environment", we can consider that the individual accesses information through the arcs of his social network" (Raynaud 2005, p. 17).

According to ATD (Chevallard 1985/1991; Chevallard 2007; Bosch and Gascón 2006; Winslow 2011), intentional human activity is always motivated by tasks whose achievements relies on individual or institutional knowledge. These pieces of knowledge, called *praxeologies*, are constituted by *techniques* in order to accomplish some *types of tasks* (the *practice block*) and by discourses (the *logos block*) which explain techniques, the *technologies*, or justify them, the *theories*. Thus *institutions* (a person or a socially legitimated group) have their own *praxeological equipment*. Within a *didactic systems* S based on *didactic relation*s, some of these institutions Y, teachers for examples, have some other institutions X, the students, to improve their praxeological equipements. Then professional and/or disciplinary *questions* Q are studied and collective praxeological *answers* $A^{\heartsuit}$ are provided with the help of a *milieu* M. This milieu consists in partial answers A stamped by some *referent institution* $\diamond$, among which we find at least students and teachers themselves, and other objects O (other resources for example) and derived questions Q'. The following *Herbartian* *schema* summarizes this teaching/learning process:

$$[S(X;\ Y;\ Q) \rightarrow M = \{A^{\diamondsuit},\ O,\ Q'\}\,] \rightarrow A^{\heartsuit}$$

In ATD, didactic systems are not closed systems. They are embedded in social networks which condition and constraint their underlying *disciplinary and didactic organizations*. At least four *co-determinacy levels* are identified outside classrooms: the *pedagogy*, which states teaching principles applying to all didactic systems independently from disciplines, the *school* with its internal capabilities to design and support teaching systems, the *society*, with its laws, rules or programs, and the *civilization*, an upper level beyond societies including, for example, cultural norms and traditions. Didactic systems then appear as open systems at the very core of *macro-systems* which overlap partially the non-ATD concept of *context*: in each of those upper co-determinacy levels, many a priori a-didactic institutions can be found which all have praxeological equipment and thus which all may become referent institutions. Within this frame, the competence of contextualization may be seen as a professional teaching praxeology aiming at managing the flows of praxeological resources $A^{\diamondsuit}$ or O from the available referent institutions $\diamondsuit$ into the milieu M.

In Salone (2015), three types of these *praxeological flows* are highlighted, all of them spreading through specific social networks: *scholar flows*, in which scholar knowledge is subject to external and internal didactic transpositions (Chevallard 1985/1991), *private flows* corresponding to private networks such as students' and teachers' ones, and more general *social flows* involving others referent institutions whose praxeological equipment are available directly in the nearby or indirectly through websites. These three contextual sub-systems allow teachers to make *openings* in their classes: activations of systemic relationships in order to influence the didactic systems. Three dimensions are underlined for these class openings: enrichment of the bodies of knowledge to which the class refers, diversification of the roles and *topos* of actors, widening of the didactic milieu. "In each of these three dimensions, openings can be performed not only internally, in the classroom, but also externally, by integrating non-didactical institutions" (Salone 2015, p. 332).

Our analyses in this paper adopt this systemic approach with a focus on the society co-determinacy level. More precisely, we will look at the openings allowed by the training system we have designed for Primary School Teachers-in-training (PST) with a central issue: what are the praxeological, social flows between the didactic systems PST are involved in and the Mahoran society?

Description of the Cooperative Works Training System

PST are actually involved in a specific training system: the *cooperative works*. Its goal is to equip teachers with professional praxeologies that will enable them to design courses that are both multidisciplinary and contextualized, and that abandon the paradigm of visiting works to enter into that of questioning the world. In the cooperative works system, initially the whole group X of PST is a typical didactic systems S with lectures given by the teacher trainers Y. Some generic questions Q are then asked, such as what is to be done and how, and answers $A^{\heartsuit}$ are usually produced with the help of some visited theoretical works $W^{\diamond}$. The Herbartian schema in these phases is:

$$[S(X; Y; Q) \rightarrow M = \{W^{\diamond}\}] \rightarrow A^{\heartsuit}$$

PST also freely constitute groups of 3 to 5 people to enter a project, the actual cooperative works, which includes: a *contextualized and multidisciplinary inquiry* during the first semester, a *didactic work* and an *educational booklet* during the second semester and, in semesters 3 and 4 of the second year, *teaching sequences* for their classes. The results we present in the next section concern exclusively inquiries and didactic works conducted in 2017–2018. In ATD terms, each PST group enter a two years *Study and Research Path* (SRP) with a generating question Q_0 which could be: how can we (our group) design multidisciplinary teaching situations relying on thematic resources about Mayotte's heritage?

The *contextualized and multidisciplinary inquiry* is a training and teaching technology related to the inquiry-based learning pedagogy (Dewey 1993) and to the documentary work (Gueudet and Trouche 2010). It connectsw two main types of questions: Q_1: what is known about the theme we have chosen in Mayotte's heritage or society; Q_2: what multidisciplinary bodies of academic knowledge could be related to the theme? The inquiry also looks for possible places that can serve as school field trips and for available resource persons, citizens' associations or institutional partners (derived questions Q'). When inquiring, the PST group G_i meets local actors Y_L who provides some resources A^L and who sometimes contribute to their works. It also collects resources O from the available media such books or the Internet. Each multidisciplinary survey then leads to a corpus of digital documents C^i. The Herbartian schema in these inquiries is:

$$\left[S(G_i; Y_L; Q_1, Q_2) \rightarrow M = \left\{A^L, O, Q'\right\}\right] \rightarrow C^i$$

In the second semester, the creation of a *didactic work* W^i is part of an educational approach similar to that of the "masterpieces" in the sense of Freinet (1994) or Meirieu (2015): trainees have to produce by their own a terminal work, a masterpiece, to show their professional skills and abilities. Theses masterpieces must, of course, be related to the previous inquiry theme, and they always include an aesthetic dimension that here relies mainly on digital and artistic skills. As professional productions, they also integrate a pedagogical intention directed for a general or school-related audience, with for example the transmission of a piece of knowledge or a message to bring awareness to a certain topic. The question for the PST group is then Q_3: what knowledge do we want to transmit? And the Herbartian schema is:

$$\left[S(G_i; \emptyset; Q_3) \rightarrow M = \left\{C^i\right\}\right] \rightarrow W^i$$

After a five-month production process, books, youth albums, slideshows or documentary movies are thus created. They are thereafter publicly presented and are evaluated by a jury composed of teachers' trainers, researchers and institutional partners. Finally, the best didactic works are published and disseminated in society, at least in schools, to become teaching resources.

In the context of some disciplinary teaching units such as French or Mathematics, the cooperative groups also produce an *educational booklet* whose purpose is to propose ways of using in classes the resources thus constituted, inquiry materials and didactic works. Hence these booklets generally contain examples of contextualized disciplinary organizations, with for example activities, problems or exercises statements. In the second year of the master curriculum, when every two weeks PST have their own classes, this transposition process of thematic resources into scholar ones is continued.

Methodological Framework

In this paper, we present results upon the cooperative works realized during the academic year 2017–2018. Thirty cooperative groups were formed this year. Some of their cooperatives works may be downloaded on our website (http://www.univ-mayotte.fr/fr/formation/sciences-de-l-education/travaux-des-etudiants.html), and three of them are also described in Salone (2019).

In order to answer our research question about the praxeological, social flows induced by the cooperative works training system, we examine in this paper the data coming directly from the contextualized and multidisciplinary inquiries along with the produced didactic works. Three derived questions are answered: what are the themes chosen par the PST? What academic disciplines did they involved? Which socio-economic actors have they met? Indicators are therefore used. For the first question, the indicator is the topic which appears in the titles of the didactic works. We use there a four types of categorization derived from Hartog (2005): natural heritage, tangible cultural heritage, intangible cultural heritage and societal issues. Mixed themes belonging to several categories will be admitted. For the second question, the investigated school disciplines are directly identified in the documents from the inquiries bodies. For the third question, the encountered local actors that we retain are those who are cited or referenced in both the inquiries and the didactic works and we categorize them within three types: institutions, citizens' associations and referent persons.

Results

The chosen contextualized themes are varied. In Fig. 1, we can see a cloud of the first-page photographs of the documents gathering the inquiries' data. Mostly, with

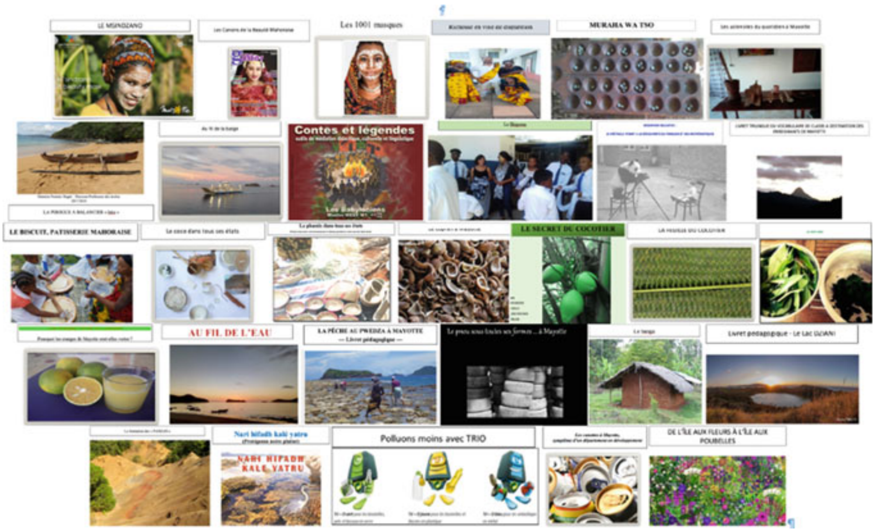

Fig. 1 Cloud of selected themes

23 works out of 30, they fall under a theme related to intangible cultural heritage, such as traditional games (for example the local *awale* type sowing game called '*mraha wa tso*'), cooking recipes (for example the '*mataba'*, a typical course from the south and central Africa made from mashed cassava leaves), arts and crafts (the countless braiding and uses of coconut leaves for example), lore (the 'msindzano' for example, a yellow, facial mask made of crushed coral, white clay and turmeric used by women) and crafts (such as the manufacturing process of the famous '*laka*', the local canoe). Twelve themes concern elements of the natural heritage, such as natural protected areas (for example the very known lagoon of Mayotte, or the volcanic lake Ziani) or endemic plant species (such as the always green oranges of Mtsamboro village) and six deal with societal issues (such as the sorting of waste, the preservation of the coral reef or the tire recycling in wall constructions). 2 themes concern the material cultural heritage (the '*banga'*, traditional tiny huts made of bamboo and clay, and sugar factories, ruined remains of the island's post-colonial industries, and the maritime transport by barge between the two main islands of Mayotte). Note that many of these themes are of course mixed (for example the 'Ziani' lake is at first a piece of Mayotte's natural heritage, but it is also a place at the very core of many legends and a highly popular tourist site) and that some themes have been addressed by several groups (such as '*msindzano'* theme which has been investigated by three groups).

With regard to the multidisciplinary, each cooperative group has clearly identified many links between its theme and almost all the academic disciplines. Coming first, Mathematics and French have been investigated by all groups. In Mathematics, 27 groups have retained pieces of knowledge from the geometry domain or from the size and measurement domain which are present in the official programs of cycles 2 and

3 of elementary school (children approximately aged from 6 to 11). In French, all the groups have considered that fluency in written or spoken language can be improved by introducing local texts into the classroom, such as stories, songs, press articles or documents produced by citizens' associations. Visual arts, History, Geography and Moral and civic education are then the next most investigated disciplines. Their scores are respectively 22 groups (out of 30), 17, 16 and 16. Then came Science and technology, Natural sciences, Sport and Music.

The local actors met by the PST and sometimes involved in the cooperative works are numerous, with an average of 5 per group including at least one institutional partner. For example, about 80% of the groups have encountered the staff or the departmental archives of Mayotte and about 60% the team of the MuMa, Mayotte's museum. Among the citizens' associations, two have often been involved: the Mahoran naturalist, an association dealing with environmental and cultural issues, and SHIME, an association of local sociolinguists. Many referent persons in Mayotte, called 'fundi', have also been met. All these partners have been involved in several ways: to provide documentary resources, to answer questionnaires or to participate in interviews, to contribute to the designing of the didactic works. More than fifty institutional, associative or private partners were thus involved by the promotion witch thus woven a real educational network between the CUFR and the Mayotte society.

Conclusion and Perspectives

The cooperative works system allows trainee teachers of schools who are engaged in our master curriculum to both strengthen their professional, praxeological equipment and acquire good knowledge about Mayotte's heritage and society. First of all, through inquiries and systemic openings on the local social networks, it fosters the emergence of contextualization skills. It also reinforces professional abilities in active pedagogies, and it improves the mastery disciplinary knowledge. Moreover, for most of the PST, questioning the local world in order to create and organize educational activities in their classes has become something possible. Another gain induced by the cooperative works system is a large amount of produced materials: the didactic works (and also the educational booklets) are disseminated in Mayotte to become teaching resources and material coming from the inquiries are shared by researchers and institutional partners.

In terms of research, our team has recently been exploring additional issues about the cooperative works training system. For example, as the vehicular languages are an important part of the cultural heritage, the questions of their place in the didactic works and their uses in the classroom are studied. More specifically, questions about the mathematical terms and idioms in these languages and their effects on the concepts' formation arise. According to an ethnomathematical approach, other questions are coming to the shore, such as the reinvestment in classes of traditional technologies or the use of handicraft artefacts. In a more sociological or ethnological approach, a main research issue also emerges: how, once tenured, will the PST involved in the cooperative works contribute to the development of Mayotte's education system?

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Didactic Praxeologies Taking into Consideration the Collective Dimension of the Question's Study Through Investigation When Implementing Mathematics' Study and Research Paths in French Secondary Schools

Karine Bernad

Abstract I will present some results from my doctoral work which concerns the study of the conditions and constraints determining teaching practices when implementing mathematics' study and research paths in French secondary schools during the 2013–2015 period (Bernad 2017). In the framework of this inquiry, I highlight elements of a teacher's praxeological equipment that are useful for the achievement of the project to implement such a study and research path (SRP). I will focus on underlining how the study of didactic praxeologies activated by a teacher has led me to make the hypothesis that he achieves to extract the collective class from the paradigm of visiting works in order to fall within a paradigm of study through research. It will also show these analyses have led me to think that the praxologies' model and the individual-collective dialectics, media-milieu dialectics, questions-answers dialectics are relevant tools for thinking about the teacher's role, and the anticipation of his decisions in the setting up of closed SRP.

My doctorate study has contributed to a study of a praxeological equipment (Chevallard 2009) which can be judged useful even indispensable, for a teacher, to achieve the project: implementation of study and research path single-discipline and closed (Chevallard 2007). To achieve this, I analyzed an evidence base composed of movies of class sessions during 3 school years (2009–2010, 2012–2013 and 2013–2014) and individual interviews conducted in 2013 concerning a teacher whom I have designated as *ý*. This professor participated in a collaborative work between mathematical didacticians and teachers, inside the IRMT[1] of Aix-Marseille; this which led to, in particular, the elaboration of a study and research path (SRP) aimed at teaching relative numbers and their operations as advocated by the French

[1] Institute for Research on Mathematics Teaching.

K. Bernad (✉)
Doctor in Science in Education from Aix-Marseille University, Marseille, France

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official curricula up to 2015. I have also studied the implementation of this SRP in the case of two other teachers, who have not participated at its conception. The data from the observations of these two professors were mainly used for the analysis of the difficulties encountered for diffusion and reception of didactical praxeologies of the professor having the plan to implement such SRP.

In this text I would like to show how the data collected from teacher *ý* has made it possible to uncover didactic praxeology elements taking into consideration the collective dimension of the study of the question by the research. The study of this questioning led me to formulate four proposals whose traces I identified when I analyzed several episodes extracted from the SRP aiming at the teaching of relative numbers. I classified them according to whether or not they mainly met the need to participate in one of the three genesis developed for the establishment of didactic contracts: mesogenesis, chronogenesis and topogenesis. In case of SRP I studied, didactic organizations are described with the model of the didactic moments (Chevallard 2002). That's why, this model is used to make proposals.

Initially, I will highlight some aspects of the mesogenetic conditions offered by the didactic praxeologies of *ý* by questioning the developments of *media-milieu* and *ostensives-not ostensives* dialectics.

Secondly, I will show how the didactic praxeologies activated by this professor cause a phenomenon related to chronogenesis.

Mesogenetic Effects of Activated Didactic Praxeologies

Bosch and Chevallard (1999) have shown that the specificity of mathematical activity lies in the deployment of an ostensive−non-ostensive dialectics, when they have studied the question: what is a given technique made of, in order to solve the type of tasks studied? What is the "implementation" of this technique? A not-ostensive object refers to ideas, concepts. For example, the relative numbers, the notion of calculation's program. Mathematical objects are not directly accessible to our senses: they are "not-ostensive" objects; we work with them through ostensive representations which can be of very diverse nature: discourse in natural language, schemas, drawings, symbolic representations, gestures, manipulations. Work with ostensive objects both shapes the development of the associated not-ostensive objects, and is shaped by the state of development of these, the not-ostensive objects. The not-ostensive objects are evoked by ostensive objects which can be perceived, manipulated, they have two valences: an instrumental valence, by which it allows to act, to work and a semiotic valence by which they allow us to suggest other object systems, ostensive and not-ostensive. The anthropological approach emphasizes the fact that mathematical objects are not directly accessible to our senses: they are "not-ostensive" objects; we work with them through ostensive representations which can be of very diverse nature: discourse in natural language, schemas, drawings, symbolic representations, gestures, manipulatives. Work with ostensive objects both shapes the development of the associated not-ostensive objects, and is shaped by the state of development of these. By examining the deployment of the *ostensives-not ostensives* dialectics, specific to the development of mathematical activity, I was able to show that this teacher brings enough ostensives to enrich collective work, to develop intermediate institutionalizations and carry out a didactic technique summarized as follows.

Proposal: The teacher deploys an *ostensives-not ostensives* dialectics that helps to explain the articulation of mathematical organizations associated with the types of tasks studied according to the didactic moment (Chevallard 2002) in which the class is mainly engaged; this presupposes that the semiotic valence of the ostensives brought by the teacher is used to nourish their instrumental valence in relation to the study of these types of tasks

In order to support this proposal, I have studied in particular the context of SRP for the teaching of relative numbers, the process of institutionalization of the concept of the calculation program conducted by the teacher *ý*. This concept bears the *raison d'être* that was chosen by the teaching of the definition of relative numbers and their calculations: relative numbers are seen as simple calculation programs of type "add or subtract". In this process of institutionalization, a first ostensive play a decisive role. This is the sentence "add… then subtract…amounts to add/subtract to this number …" then becomes "add… then subtract… amounts to subtract…", manipulated in the written and oral registers for the study of two types of tasks T_1 "to perform a calculation $a + b - c$ » and T_2, "to determine the simplest equivalent calculation program of $+ b - c$". I then describe the dialectical interplay between the "ostensives" and "not ostensives" objects of mathematical activity with an extract from a session directed by the teacher *ý*. The part that we are studying hereafter is taken from the third sitting, on the 13th of November 2009, that *ý* devotes to the implementation of SRP. During this sitting, a second ostensive is brought and designated by "simplified writing". It appears in the instruction, when *ý* asks the pupils to perform the calculation of type $a + b - c$ with $c = b + 1$: "Perform the following calculations mentally, then write the sentence and **the simplified writing**". The use of the last expression is an adjustment made by the professor: to propose the expression "simplified writing". He has decided to introduce this expression to accompany the new write ostensive « $+ \ldots - \ldots = - \ldots$ » expressing "plus… minus… equal minus" and meaning "add… subtract… amounts to…" as is shown in the following extract:

19 *ý*:	With this type of calculation we said that we could write, (*the teacher writes what he says*) add 45 then subtract 46 to a number amounts to subtract 1 to this number. And now, we are going to **write the simplified writing** as we highlighted in activity… **How can we write more simply**? Xxx?
20 Pupil:	Plus 45 minus 46 equal minus 1.
21 *ý* repeats:	Plus 45 minus 46 equal minus 1… now we are going to explain this equality. When we write plus 45 minus 46 equal minus 1, what does this mean? What is the meaning of this equality?
22 A pupil repeats the calculation:	458 + 45 – 46 is 458 – 1.

23 *ý*:	Here 458 disappears, I mean this equality.
24 Pupil:	We say that we subtract 1.
25 *ý*:	So, how do you say this?
26:	26 The pupil answers correctly.
27 *ý*:	Adding 45 and subtracting 46 amounts to subtracting 1.

In this oral exchange, recalling that the "simplified writing" ostensive refers to the instruction "to have written more simply", the professor highlights his instrumental valence. Then asking what does "it means", he emphasizes his semiotic valence. The answer proposed by a pupil is different from his expectation: the pupil proposes a calculation and the professor wants a calculation program. The professor regulates the activity by indicating that, when we see this "equality", there is something to "understand". But, at this stage of the activity, the expression "calculation program" is unknown in the pupil's vocabulary. This one is supposed to recognize "simplified writing", which means a writing "without a number in front". To articulate, in this way, the study of the two types of tasks contributes to the institutionalization of the distinction between "a calculation" and "a calculation program", this allows to come up with the definition of a relative number; this leads the teacher to define a word by indicating that the pupils needs to fill "a gap" that concerns "a vocabulary term", so he arrives at the following:

29 *ý*: Now we go back to our work. Some vocabulary. Since I introduced this simplified writing, **I am missing a vocabulary term**, when I say add 45 then abstract 46 we say that we define, a calculation program, there can be plenty of calculation programs. I can then say, I take a number, then I multiply it by 2 then I add 15 or I take a number add 50 then I subtract 10. The calculation programs, there are many kinds. **Add 45 then subtract 46 is a calculation program, from now on I will talk to you of calculation program**. OK. The calculation programs we have worked on stage 3 are all calculation programs that amount to subtracting 1. OK. Now I am going to ask you to find calculation programs that amount to subtract 2 or to subtract 3 … You are now active as is. We are now going to write stage 4 (*the teacher clears the blackboard then writes what he is saying*)… Write a calculation program that amounts to subtract 2.

As *ý* clarifies publicly to the class, the pupils have already met several examples of calculation programs that they have simplified, they practice determining "the simplified writing". A calculation program is thus defined as an extension, using examples. The teacher *ý* replaces this new ostensive: the expression calculation program into what has been worked, specifying the types of calculation programs already found and giving examples of some, not here at present, that could exist in the class. This *proposal* can be seen as a didactic technique, during the building of a mathematical organization, allowing to enrich the environment for the study. The teacher brings ostensives for which he designates their functions regarding the dimension of the mathematical activity developed ("the sentence explains a calculation technique", "the simplified writing makes it easier to write"). The teacher develops an *ostensives-not ostensives* dialectics that helps to explain the articulation of mathematical organizations associated with the types of tasks studied according to the didactic moment in which the class is mainly engaged; this presupposes that the semiotic valence of the ostensives brought by the teacher is used to nourish their instrumental valence in relation to the study of these types of tasks. The use of ostensives he embeds into the research, in which the class is engaged, allows him to set *for the time being* the mathematical activity on one of the types of tasks and relaunch the research by intermediate institutionalizations to advance the chronogenesis.

Chronogenetic Effects of Activated Didactic Praxeologies

It seems to me that the following proposal can be considered as a technologic ingredient of didactic praxeology which would guarantee the devolution of the technological dimension of mathematical organizations targeted.

Proposal: Didactic praxeologies of the teacher, by the deployment of *individual-collective dialectics* and *media-milieu dialectics*, cause a *slowing down of the didactic time* involved in a devolution of a validation's situation and produce a time proper to uncertainty which we will call ***study time by the research***.

The episodes that illustrate how the two dialectics, *individual-collective* and *media-milieu* (Chevallard 2007), develop to generate a *slowing down of the didactic time*, are the states during the mathematical activity of the class is resolutely at the technical level, while jointly developing an exploratory moment and the emergence of techniques relating to the type of tasks encountered as problems.

I rely on an episode from a session, 19th November 2009, which issue is the study of a type of tasks "add two relative numbers". This session mostly achieves the exploratory moment and of the building up of a technical embryo moment and the building up of the technologico-theorical block moment relating to the calculation of the sum of two relative numbers. At the end of an individual working time followed by a shared time, the pupils have written on the board these additions and subtractions: $+7 + (+2)$; $-7 + (+2)$; $+7 + (-2)$; $-7 + (-2)$; $+7 - (+2)$; $-7 - (+2)$; $+7 - (-2)$; $-7 - (-2)$.

A pupil wrote on the board:

$+ 7 + (-2) = 7 - 2$

This is followed by an intervention of the teacher whose main function was to situate the activity of the class at a technical level. We are going to prove how the devolution of this dimension related to the type of tasks "calculate the sum of a positive number and a negative number" as achieved by a *slowing down of the didactic time*.

78 *ý*:	Here, you don't agree? So, xxx, he writes 7 – 2. One thing is sure: + 7 that he has replace by 7, there is not a problem. **The problem is here (*ý* surrounds with his finger −2) so, wait! Here, there is a big question.** What do you propose? xxx?
Another pupil proposes:	2 – 7. *ý* write: 7 – 2 or 2 – 7.
79 *ý*:	I heard something else. Wait for the result, then we will see.
80 Pupil:	xxx
81 *ý*:	Where 0? I understand what you are now telling me. So xxx explain yourself properly.
82 Pupil:	xxx
83 *ý*:	+ 2 – 2? Why?
84 Pupil:	xxx
85 *ý*:	You have 7, you have the number 7 and you want to add −2. **Because here I don't know if your writing is correct. I don't know. Look, at present I didn't agree on anything. (*ý add "?": 7 – 2? Or 2 – 7?*). We didn't agree. I heard some other things here**.

With such a use of the didactic ostensive "?", the teacher shows all the rationale work that is required to be done. His intention is, also, to provide the development of the **tech**nological dimension related to the type of tasks "add two relative numbers". He sets the singular aspect of a pupil's proposal who consider to introduce +2 – 2; this constitutes a technical embryo to solve the task encountered as a problem. Then, when a pupil questions on the possibility to add 7 and −2, the teacher answers with a pronoun (in French "on", translated by "we") by stating that this an issue the class has set:

87 *ý*: It is what we are trying to see. **We** ("on" in French for the class) are trying to find how to add, can **we** add such numbers? How an addition works with such numbers? So, you are saying −2 – 7. Wow! **We** have got a lot of results. My "wow" is because you are finding many possible results. Results finally. **We** have still not written the results. So, finally. In all cases you have additional inputs to give me concerning these calculations. **We** have questions. **We** don't know what result we are going to find. You don't have any idea of the possible result. −5 +7 +−2 = − 5. So, listen, I think **we** are going to stay just like that.

It seems to me that such an use of the French pronoun "on", has the effect of considering the collective in the production of techniques and technological elements. I have therefore assumed that such conditions allow to make it possible to deploy an *individual-collective* dialectics by creating, in the same time, a *slowing down of the didactic time* by maintaining uncertainty to the pupils for the decisions they have to take, whether on the techniques or on the technological associated discourse.

Conclusion

Observation and analysis of didactic praxeologies activated by *ý* shows that the didactic conditions are established to extract the collective *class* from the dominant paradigm of *visiting works* (Chevallard 2002). I have made the hypothesis that this is based on one of the dimensions of its praxeological equipment. In the context of setting up a system to support teachers in the implementation of the SRP that I have studied, it seems essential to encourage teachers to question their relation to the raison d'être of the relative numbers chosen for this transposition: that is, in the case of this SRP, their relation to the notion of a "calculation program". To do this, it is necessary to make it intelligible, to justify it to them within the framework of the mathematical reference organization that has been built and that supports the mathematics taught during the SRP. The mathematical reference organization assumes the technological dimension of the mathematical organization that they will have to bring to life in the SRP. It is therefore crucial that a professor intending to take such a SRP under consideration considers the mathematical reference organization to be very relevant. Actually, I interpreted some discourse indicated by this teacher as a manifestation of *epistemological vigilance*. For example, during a meeting of researchers and teachers, on the 12th December 2013, he explains the need for the teacher, who wants to implement a SRP, to memorize the processes of mesogenesis and chronogenesis, "Why we are doing that this way, for a start, and not any other way", in order not to lose the "initial meaning", that is to say, the *raison d'être* of knowledge to be taught.

1 *ý*:	The difficulty with the SRP, is that when you want to publish them anyway is: when you start to use it, you see things that come up with the pupils. So you can change your practice and **finally there is a certain point where you can lose the meaning of certain things. You must be very careful. On the things we have worked a lot, for example the relative numbers, we are nevertheless very vigilant. Even if you do things, we question ourselves. But on other things, it's not easy**.
2 Interviewer:	Lose the meaning. What do you mean?
3 *ý*:	**The deep meaning that had been put behind. The objectives, why we did it like that, for a start and not another way. You see, when we make the pupils pass certain steps**.

One condition to develop such *epistemological vigilance* for a teacher might be to lead him to question the ecology of ostensives that instrumented didactic-mathematical praxeologies and their functions: what ostensives are needed for the progression of the class into the research for answers to the question being studied? What could promote or prevent the presence of such ostensives? To which technico-technological should or could participate the manipulation of such ostensives?

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How Long Would It Take to Open a Padlock? A Study and Research Path with Grade 10 Students

Susana Vásquez, Berta Barquero, and Marianna Bosch

Abstract We present the first implementation of a finalised study and research path (SRP) related to basic enumerative combinatorics. The SRP starts with a generating question about "How long would it take to open different kinds of padlocks?". This paper describes the main steps of the SRP as implemented with secondary school students of grade 10. We want to stress some specific features of the SRP: the role of the empirical milieu and its enrichment through the collective construction of specific terminology; the students' use of question-answer maps to organise and describe their inquiry; the experimental validation of the final answers; and the use of padlocks as models to solve combinatorics problems. We implemented the SRP in a secondary school with a long tradition in "active methodologies". It is interesting to study these particular institutional conditions and their effects in the evolution of the SRP.

Introduction

This paper presents an empirical study on the implementation of a study and research path (SRP) in secondary education. This study belongs to a research line developed within the framework of the anthropological theory of the didactic (ATD) about the conditions and constraints affecting the paradigm shift described by Chevallard (2015) from the paradigm of the visiting works towards the paradigm of questioning the world. Considered as didactic proposals, SRPs try to get as close as possible to the paradigm of questioning the world by starting study processes from open generating questions. Our research team has been working in this research line for the

S. Vásquez (✉) · B. Barquero
Universitat de Barcelona, Barcelona, Spain

B. Barquero
e-mail: bbarquero@ub.edu

M. Bosch
Universitat Ramon Llull, Barcelona, Spain
e-mail: marianna.bosch@iqs.url.edu

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past 15 years, with many experimentations at the university level, including teacher education (Bosch 2018; Florensa 2018). The implementation of SRPs in secondary education is much less studied, although there are specific investigations in Argentina, Denmark, France, and Japan (Jessen et al. 2020; Parra and Otero 2018). One possible reason would be the institutional constraints that weigh on this education level. Our study, which is part of an ongoing Ph.D. thesis, consists precisely in analysing the conditions for the implementation of SRPs in Spanish secondary education. More specifically, we are considering a school with a long innovative pedagogical tradition and an educational project based on project-based learning that, at first glance, seems supportive to the paradigm of questioning the world.

Research Questions and Methodology

The aim of the design and implementation of SRP is multiple. On the one hand, we want to analyse the conditions favouring SRPs implementation at the secondary school level. On the other hand, we want to observe if the expected secondary school institutional constraints seem weaker in this particular school and what facilitating pedagogical devices teachers and students use spontaneously.

We decided to design a finalised SRP associated with a particular mathematics theme because of the regular curriculum. To better fit the academic calendar, we chose the theme of combinatorics and reviewed the research carried out in our country (Millán 2013; Navarro et al. 1996; Roa et al. 1997). We initially thought of a generating question about password breaking. This topic very much concerns today citizens' use of technological devices: "Why do they always ask us to use capital letters, numbers and special characters in passwords?". Instead of starting with this question, the teacher-researcher proposed a more concrete one about padlocks, to create a material milieu that could provide feedback, given the facility for students to hand the padlocks and test possible combinations. The question related to the password breaking appeared later at a second stage of the SRP. In the following sections, we first present the SRP institutional context and describe its implementation and analysis. The results obtained will help us answer the following research questions: (1) What aspects of the SRP have been implemented according to its a *priori* design and which ones have not? What are the consequences for a new design of this same SRP? (2) What didactic devices facilitated the SRP's development, and how did teachers and students manage them? (3) What conditions favoured the existence and development of these devices?

Institutional Context and Conditions for the Implementation

The SRP was carried out in April-May 2019 with 58 grade 10 students at *Col·legi Natzaret,* in Esplugues de Llobregat, a town near Barcelona. In this school, there is a team of three teachers for grade 10 mathematics, one being a researcher in didactics and the first author of the paper. The other two teachers were new in the profession and without any specific training in didactics. The SRP methodology was new for the students, as well as for the teachers. Usually, in this school, the kind of work done in the mathematics classroom is based on lists of problems from an online textbook related to each thematic unit. For each unit, students must solve lists of problems within three weeks. If needed, they can attend one or two lectures—called "master classes"—where a teacher presents the unit contents and answers the students' doubts. When students think they are ready, they choose a day to perform an exam, with the possibility of taking a second-chance exam some days later. Once all the students have taken the exam, the class carry out a one-week project related to the unit.

We decided to implement the SRP during the last weeks of the course, according to the syllabus. The students participating in this implementation are good at working autonomously and in teams because they have been doing it since grade 7.

Conditions and Methodology of the Implementation

We kept the regular organisation of the mathematics grade 10 course. There were 58 students, divided into three large groups and, within the groups, in teams of three or four persons. Teachers organised the teams beforehand following this criterion. Students were sorted by their marks in the last teaching unit and were grouped into four levels; teams contained at least one student from each level. Each of the three classroom groups was guided by one teacher and teachers rotated at every session. In this way, the three teachers were able to work with all the groups. Also, in most of the sessions, one of the groups had an external observer who was a researcher in didactics.

The proposed assessment had to match with the current pedagogical organisation of the course, with some small variations for the SRP. We decided to mark the students according to the teams' workbook and daily reports (12%), their elaboration of the question-answer map (12%), a video of the experimental validation (12%), a final oral exposition (12%), the teamwork self-evaluation (12%), and an exam (40%). Furthermore, teachers were allowed to give positive and negative points for interesting contributions or wrong attitudes, directly impacting the individual mark of each student. The learning material prepared for this SRP included: a guided workbook for the students to show their investigation about the padlocks; diary templates to fill by the groups; an assessment rubric; a summary about the different combinatory formulas; a list of problems related to the combinatorial unit; two versions of a written exam; and an online questionnaire.

In Vivo Analysis of the SRP

Presentation of the Generating Question and First Answers of the Students

In the first session of the SRP, the teachers presented five padlocks, each with different features and functions. The generative question Q_0 that served as a guide for the whole project was: *How long would it take to open each padlock?*. In the workbook, students could find an image of the padlocks and an explanation. Some of them were also physically available in the classroom (Fig. 1).

The first task proposed to the group was to choose a padlock to start with (Q_1). To do this, students had a template where they had to select a padlock, begin working and explain their choice. They also had to make a first classification of the five locks and explain their criteria.

Most of the students thought they should start with padlock number 2 (the one that allows pressing different buttons at the same time). The main reason they gave was that it would depend on the number of combinations. Some groups did not know how to explain their choice, but they had some hint that it had to do with the padlock's difficulty. Teachers had previously thought that the students would choose other padlocks that seamed easier to compute the number of combinations. Curiously, padlock number 2 is one of the most challenging cases.

A critical limitation showed up in the first sessions. Teachers observed that students were using different expressions, such as "combinations", "options", "cases", "numbers", among others. They lack a common vocabulary to describe their work and to be able to understand other groups. In the second session, the team of teachers agreed to introduce the following terminology: "combination" is the password that can be entered in the padlock; "cell" is each element of the combination; "cell elements" are the values that can be inserted in a cell. The teachers gave examples to fix this common terminology, for instance: "for padlock 1, we can set different statements to exemplify these definitions: A possible combination is 1234. Padlock 2 has four cells. All boxes in padlock 2 have 1ten0 cell elements." This clarification helped the communication but was not immediately adopted. Teachers

#1 #2 #3 #4 #5

Fig. 1 Presentation of the first padlocks have to continuously remind students to use the specific vocabulary whenever they wanted to participate in a classroom discussion.

Padlock Combinations

During the following sessions, the students tried to solve the initial question for each of the padlocks. The most common resource for validation was the class discussion: students defended their answers until all groups agreed. The teacher's role was to guide the process, reminding the established vocabulary and clarifying, questioning or dismissing some of the students' contributions. Students quickly concluded that the time needed to open each padlock depends on the padlock and the person trying to open it. Therefore, the different group decided to assign an additional value for the time it took to change passwords. Even so, they had to have the same number of possible combinations for each padlock. While they were inquiring about the different padlocks, teachers led the discussions to share results and try to identify mistakes in the students' arguments. A padlock was considered resolved when all the students and the teachers were satisfied with the final reasoning. It was more important for all the teams to come to a consensual answer than to have the teacher's validation. Once a padlock was resolved, all groups expressed the reasoning on the padlock's worksheet.

Padlocks with New Restrictions

Once the first five padlocks were resolved, teachers proposed to repeat the experience including some restrictions to the padlocks: for example, cell numbers cannot be repeated in a combination; the letters of all cells are the same; etc. This second part was much more agile, as most new padlocks reproduce the first ones' conditions already solved.

Classification of Padlocks and the Elaboration of the Question-Answer Map

Once all padlocks were studied, teachers proposed to sort them into groups, depending on the method used to find the total number of combinations. There were three general groups. *Group 1* with the padlocks in which the elements' order was essential and elements can be repeated [1, 3, 4, 5]. *Group 2* with the padlocks in which the elements' order was essential, but elements cannot be repeated [1, 3, 4, 5 with restrictions]. *Group 3* with the padlocks in which the elements' order did not matter [2 and 2 with restrictions]. Padlock 4 was left without a general answer because we could not find a pattern for the letters that appeared.

Once the classification was established, students were asked to synthesise their work in a question-answer map, a diagram indicating the questions raised during the research process and the partial and final answers provided. Students were used to making mind-maps as a way to organise the contents of subject units; teachers introduce this tool by stressing the new some specificities and elements to include. In particular, the teacher stressed some common traits for the elaboration of the questions and answer maps may. In particular, it was asked that this map might:

- Highlight the questions addresses instead of defining mathematical contents.
- Aesthetically differentiate the questions, strategies, and answers.
- Relate the questions and answers one to each other.
- Show a hierarchy among the different questions by using subscripts: $Q_1 \rightarrow E_1 \rightarrow R_1$.
- Questions must be understandable without a context.
- Use of the agreed vocabulary (cells, cell elements, combination, order, repetition, etc.).
- Unanswered questions and/or incorrect answers may also be included (this should be indicated).
- Show the chronology of questions addressed and the relationship with the padlock's inquiry.
- Include all the padlock cases worked in class must be reflected.

Three sessions were devoted to this activity. In this time, teachers and researchers asked students to make many changes in their maps, and the suggestions were not sometimes well welcomed by the students, who believed their maps did comply with the instructions given initially.

Institutionalisation Moment and the Evaluation

A masterclass followed, where teachers reviewed the padlocks classification and reminded how to find the number of combinations in each case. For the first time, solutions were officially validated by teachers. When finished, the teachers proposed a new question Q_2 to the class about "How can we find a general technique for solving any padlock?". And the following derived questions and to discuss the following sub-questions:

$Q_{2.1}$: Two combinations with the same elements but arranged in different ways, count as two combinations or as one? That is, does the order of the elements influence?

$Q_{2.2}$: Can two or more elements be repeated in a combination?

$Q_{2.3}$: How many cells does each combination have?

$Q_{2.4}$: How many elements are available in each cell?

Variable *m* was proposed to name the answer the total number of combinations and variable *n* the number of elements in each cell. Then, the teachers introduced some formulas and related them to the students' proposed methods to find the number of combinations. Also, the teachers presented a list of typical combinatorial cases taken from the textbook. They then recommended linking every case to a specific padlock before trying to solve it. In the following session, students brought the list of exercises partially resolved individually. That same day, the solutions to all the problems were uploaded to the course' online platform. Finally, students took an exam containing ten questions about different situations (ice creams of different flavours, shirts of different colours, etc.) with the same structure: "In how many different ways can we…?" The next day the exam was corrected and, a week later, a second-chance exam was proposed, with the same structure.

Experimental Validation and Elaboration of the Final Answer

In this last step, students were asked to validate their answers using experimental testing. Each group was assigned one specific padlock (there were repetitions as there were only four padlocks available). Students had to check that the time they predicted matched with the experimental time and that eventually, the padlock opened. Students were asked to elaborate a video in which they had to show the strategy they used to introduce all the combinations in the padlock, a time-lapse showing all the combinations, and the comparison between the time it took and the one they had predicted.

During this experimental work, many new questions raised. In particular, students found that one padlock could be opened with three different combinations. Then they search for information on the web about this type of padlocks and the possibility of restarting it with new passwords. A second generating question was also proposed at this moment about the time needed to break a WiFi password of a fixed number of elements (to be determined in advance). Finally, students made an oral presentation in front of the three teachers and the rest of the class to show their question-answer map, their video and a proposal of WiFi password structure. They were told to find a structure that represented all the different situations studied in the previous weeks. In this exposition, the rest of the groups had to evaluate their classmates using the same rubrics the teachers used.

Some Feedback from the Students

As usual in this school, at the end of the topic students filled a questionnaire with items to be rated on a 1-4 scale and some open comments. Students valued the organisation, the duration, and the variety of activities positively. Above all, they appreciated the session with the master class and agreed that they would surely not have been able to solve the list of problems or the exam without it. They also affirmed that the timing was adjusted to the class hours, and they appreciated the flexibility and changes generated by their own needs. In general, they would be willing to exchange the classes' mechanics for this kind of projects. Regarding the more negative comments, the students considered that the question-and-answer map was redundant, even unnecessary. As they were asked to fill in the work diary daily, they considered the issue of the map to be a reiteration of the previous work. Even assuming that the general feeling was positive, some students, although a minority (12%), stated that anything that departs from class explanations and the practice of book exercises is unnecessary and preferred the traditional methodology. Also, a group of students indicated that the padlock questions resolution was not useful for the exam, yet they needed the master class.

As for the teachers, all three ended up with a positive feeling at the end of the SRP. They recognised that the manipulative work with the locks increased the master class fluidity. They were surprised by some students' intuitive reasoning (sometimes considered weak students), who had no previous combinatorics notions. Some of them arrived at correct answers and reasonings, even without having the formulas. Besides, the teachers rated it as a challenge for not being able to validate (accept or reject) the students' responses. Asking questions to get the students seeing weaknesses in the demonstrations by themselves was difficult. Besides, teachers recognised they struggled to write the daily classroom activity. The shared diary tool, where teachers had to write down what happened every session, sometimes remained empty and teachers did not find time to fill it in until some days later. Thus, as the classroom activity was very dynamic and required almost daily planning changes, finding the spaces to coordinate these changes was difficult. These tasks occurred between corridors and without the ideal reflection time. Teachers had the impression of a lack of control over the work of each group. To this was added the fact of rotating groups among the teachers. It happened that, in some sessions, students raised questions or remarks regarding their work with another teacher.

Conclusions and Perspective

We go back to the main research questions, we have introduced before, by considering the different stages of implementing the SRP about padlocks. About the a priori design, we can conclude that the generating question about the padlocks and the use of the physical padlocks in the classroom, as an initial *milieu*, had a positive effect. Usually, the field of combinatorics appears in a forced way at school, with unrealistic types of problems far from the students' concerns. This SRP introduced these tools in a functional way, for addressing the problem of the padlocks' security. A critical issue in this respect is having enough padlocks for the experimental work. This was not the case here and, as there were three groups, sometimes one of them could not have all the locks or had to wait long for them to be available. Further experiments will require more padlocks available.

However, the way the generating question was posed, and the padlocks supplied to the students, did not favour the students' search for information about the padlock's security, for instance, on the Internet. This is a very evident inquiry gesture that did not appear in the SRP. A problematic issue that was not foreseen—and is also a typical research issue—was the need to establish a common vocabulary to talk about the padlocks' combinations. This terminology problem frequently appears in the development of SRPs as far as the activities carried out do not fill into the traditional mathematical activities. The didactic transposition work provides a vocabulary to elaborate on problems statements and resolution techniques when dealing with traditional mathematics organisations, but not for the type of (non-prepared) works mobilised during SRPs.

As for the use of the question-and-answer map, we can conclude that it did not have the expected results. The students did not find it useful because teachers proposed to use it without the students' needing it. It would be preferable that teachers use it from the beginning, to share the students' contributions during debate moments, introduce the vocabulary and institutionalise the work done.

The experimental validation was also a profitable activity for the students: it helped complete counting combinations and analyse the most efficient strategies. We suggest for future implementations of the SRP to give more space to the strategy of counting the number of combinations one by one, for instance, by writing them all on a piece of paper. The strategy appeared at the end of the SRP (when preparing the videos) and was much richer than expected. Students who used it had to discover different patterns to write all the combinations without forgetting anyone, an interesting preliminary combinatorial technique.

Students found the *masterclass* very useful. It seems a pertinent way to institutionalise the knowledge built in previous sessions. It also provided students with tools to corroborate the results obtained and to count faster. Moreover, it expanded the situations that are similar to those with padlocks. Performing standard exercises and a standardised examination gave both students and teachers a sense of peace of mind regarding the standard curriculum.

About recording tools, we find that students' diaries favoured their work and their organisation, potentiating teamwork. The student who often encompasses all the reasonings and leads the research could not keep the logbook up to date at the same time. This forces the rest of the group to distribute this task and be aware of the inquiry progress. The teachers' journal helped ensure a common consensus. Even so, team meetings and talks between aisles were required to reach agreements. However, this tool meant extra work for the teachers, who sometimes could not have it in time. Lack of collaborative working hours and coordination is a significant constraint for the group of teachers.

Finally, it is necessary to mention the possibility that, in future implementations, we could change the generating question to an opener or less "finalised" one (Chevallard 2009). We might consider changing Q_0 to, for instance: "Which of these padlocks do you think it is more secure?". However, this might open the possibility for students to move their research towards more technological lines of inquiry and, consequently, overpass (o be forced to use) the combinatorial tools to be studied.

The last research question we formulated is about the favourable conditions for the SRP that emerged from the school pedagogical tradition. We can mention the students' spontaneity to work cooperatively and adopt new roles during the teamwork. Another element is their ease to analyse and refer to the learning process critically and constructively. The school's infrastructure is also to be mentioned. Students were familiar with the use of digital tools to report their activities, interact with the teachers through the online campus, and create their own digital materials.

Finally, we want to mention the importance for teachers and researchers to work more collaboratively on the SRP design. In this first experience, researchers proposed many decisions about the SRP development that teachers assumed too easily. We have already mentioned the use of questions-answers maps to present the work done, that students saw as a pure pedagogical imposition. There might be other aspects, maybe less visible, that correspond to this same situation. Time empirical validation using video recordings, assessment rubrics, planning the work to be done in each session, were aspects that had been decided beforehand by the researchers. Our future research needs to put special attention to the evolution of the didactic contract, which determines the teachers' and students' sharing of responsibilities and requires specific means to establish it. This evolution appears as a crucial aspect of the move to the new paradigm of questioning the world.

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A Praxeological Analysis of the Proposal for Teaching Probability in Brazilian Textbooks of the Compulsory Education

Janielly Taila dos Santos Verbisck and Marilena Bittar

Abstract We summarize the results of a research that aimed to characterize the proposal of probability teaching throughout compulsory education from collections of textbooks of the same authorship. To do so, we analyzed four collections of textbooks, of the same authorship, one of each level of schooling, approved by the National Program of Didactic Books of the years 2016, 2017 and 2018. Our theoretical-methodological reference is the Anthropological Theory of the Didactic, which allows us to map, model and analyze the mathematical choices and didactic choices of the author. Throughout the collections, the articulation between frequentist, geometric, axiomatic, subjective and intuitive views of probability is not predominant. Although one or other of these conceptions is sometimes proposed, the classical view is dominant in this postponement of teaching.

Introduction

Probability is fundamental to understand events and random phenomena that permeate our daily lives, (Gal, 2005). Bryant and Nunes (2012, p.9) states that:

> Our understanding of the probability of uncertain outcomes plays an extremely important part in our lives. We depend on it to decide about the medical treatment that we should follow, the insurance that we need, the car that we buy, and the precautions that we should take to protect our families and our homes. All these, and many other decisions depend on our knowledge of possible events that might happen and on our understanding of how likely these different events are.

Given this importance, in order to answer the research question—How is proposed the teaching probability throughout compulsory education[1] in collections of textbooks by the same author?—during the master progress, we seek to analyze the

[1] In Brazil, the compulsory education is composed by Elementary school—for children between six and ten years old, the Final Years for children between eleven and fourteen years old—and High school (for teenagers between fifteen and seventeen years old).

J. T. dos Santos Verbisck (✉) · M. Bittar
Federal University of Mato Grosso do Sul, Campo Grande, Brazil

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probability teaching proposal in Brazilian textbook collections Ápis (Mathematical Literacy) and Ápis (Mathematics) that make up the collection destined to the first years of elementary school, Project Teláris, destined to the final years of elementary school, and Mathematics−Contexts and Applications, concerning high school education, (Verbisck, 2019).

All these collections were adopted in National Program of Didactic Books (PNLD) the years 2016, 2017 and 2018, respectively, and all were written by Luiz Roberto Dante. In Brazil, this program is responsible to evaluate and to distribute textbooks in public schools. These collections were chosen by inferring that in textbooks of the same authorship there would be (or could be) a continuity of the proposal of the study of probability. In addition, our study of previous research has identified the absence of longitudinal analyzes on the topic of "teaching of probability", which brought more legitimacy to our study.

In order to carry out this research, we adopt as theoretical and methodological reference the Anthropological Theory of the Didactic (ATD), (Chevallard, 1998), which postulates that any human activity, including mathematical activity, can be described by a praxeological model. We also used the model developed by Bittar (2017) and the Didactic Organizations (DO) by Gascón (2003) for the analysis of textbooks, which allowed us to identify and understand the teaching proposal of the researched topic. The conceptions of praxeological organization, moments of study and phases of the methodological path of production and data analysis made it possible to identify the essential characteristics of the probability-teaching proposal in the investigated works. In this sense, we were able to respond to the specific objectives listed above: one regarding mathematical choices and the other about didactic choices for the study of probability. We present below the main results of the production and analysis of the data obtained in the research.

The Proposal of Probability Teaching in the Analyzed Textbooks

Based on investigations related to our research theme and theorists who have dedicated themselves to studies on probability, we understand that this theme is related to various concepts and ideas, such as possibilities, chance, uncertainty, among others, mobilized in situations of randomness, as well as concepts such as fraction, proportion, ratio, percentage (Santana, 2011). In the collection of textbooks intended for the initial years of elementary school we noticed that there were no specific chapters for the study of probability, but we noticed that some proposals of activities worked on notions of possibility and situations of randomness, which prepare for the study of probability.

In this collection of early years, only in the first volume did we identify activities that we believe to prepare, albeit intuitively, for the study of probability. In those textbooks, we find nine possibilities tasks involving randomness situations. These were tasks that worked to determine the possibilities of specific events in random experiments or the sample space of experiments. For this, of the four identified techniques mobilized in the tasks, two have experimental character when proposing the manipulation of ostensive six faces dice and coins or bank notes that represent the Brazilian monetary system.

A particularity of the fifth yearbook is that, in addition to possibilities tasks, there was an introduction to the concept of probability being proposed from the study of fraction and percentage. At that time, probability was institutionalized as a "measure of chance", being able to be represented by the ostensible ratio or fraction and, later, as a percentage. Of the fourteen tasks proposed in this volume, some mobilized proposed techniques for resolution and others in which there was a need for personal resolution strategies or responses that involved children's beliefs. In these, in particular, we understand that some notions of intuitive and subjective meaning probability are involved, albeit informally. The classical view of probability was the most observed in this volume.

In the collection of the final years of elementary school, we identified that in the book of the sixth year there was little exploration of the probability study, which was contemplated in only one page and had only four tasks proposed. The didactic organization of the 7th, 8th and 9th years followed a certain linearity in which, for the moments of institutionalization of concepts, the author opted for the presentation of contextualized situations and examples solved with the techniques that he proposed. After these moments of institutionalization, there were moments of work with tasks and techniques. We believe, therefore, that in this collection, a didactic organization that valued more the working moments of the presented techniques and the institutionalization of concepts for the study of probability in the elementary school predominated, to the detriment of the other moments of study.

It is extremely important to articulate the visions of probability (classical, intuitive, subjective, frequentist, axiomatic and the geometric context) throughout compulsory school, since these articulations allow the construction of a set of concepts related to probability. Although in some occasions not only the classical view of probability was contemplated, the articulation was not present.

In the collection of textbooks intended for high school, there were no chapters or topics of probability study in the first textbook. Already in the second volume, there was a chapter devoted to this theme. Frequentist, geometric, intuitive, and subjective visions were not completed in this volume. In relation to the last volume that composes the collection of high school, we identified only one topic proposed for the study of probability, related to the study of statistics. The frequentist view was also contemplated, informally, in the brief study proposed in this volume, to the detriment of the other visions of probability. It is in this collection that many of the techniques were justified, either by axioms, theorems or demonstrations of the presented theorems. Axiomatic probability was the basis of this study, and classical probability was also present throughout the praxeological organization in this collection.

We have seen in the official Brazilian curricular documents the important relation between combinatorics, statistics and probability, and that even these three make up the content block Information Processing. In some documents, such as the PNLD/2016 Guide, the combinatorial has been integrated into the Numbers and Operations block because it treats the counting principle as being a matter of numbers and operations. Despite this disagreement in some situations, including between authors, we believe that there is a close relationship between these issues. In view of this, we sought to identify whether these relationships are contemplated in the four analyzed collections. We could conclude that the relations between combinatorics and statistics in the probability study were little contemplated in the textbook collections analyzed in our research. The combinatorial (especially the concepts of arrangement, permutation and combination) is related to the study of probability in the volume relative to the second year of high school. Regarding statistics, the relation with the probability study is more valued in the volume referring to the ninth year of elementary school and in the volume referring to the third year of high school.

The contexts proposed in the tasks throughout the collections also point out the didactic choices for the study of probability. It is worth noting that we did not carry out an in-depth analysis of the contexts involved in the probability teaching proposal in the collections, since the time factor made it impossible for us. Nevertheless, it was observed that in the collection of the initial years, the fifth year presented more contexts in relation to the other volumes. This is because it is in this textbook that the probability study is introduced and that more tasks are proposed. In the collection of the final years of elementary school, seventh and eighth grade books are those that present more contexts in the proposed tasks, these are also the two volumes with more tasks in this collection. In these volumes, as well as in the volume referring to the ninth year, predominated tasks with contexts of draw. We also observed that tasks related to the contexts of throwing dices, and withdrawals from objects in containers are numerous, but it is not proposed to manipulate these ostensives. Contexts are close to daily situations that show the attempt to relate probability to reality, which corroborates with guidelines of official curricular documents that say, for example, that “applying the ideas of probability and combinatorics to natural and everyday phenomena are applications of Mathematics in real world issues” (Brazil, 2000, p.44).

Implications of Those Collections in Brazilian Education

With the research done, we identified that the four collections presented different didactic organizations. That is, in the early years an empiricist DO. In the final years, the valorization of the working moments of the presented techniques and the institutionalization of concepts for the study of probability. In high school, with a classic DO. With this, our initial hypothesis that in collections of the same authorship there would be a continuity of the probability teaching proposal appears to be flawed. However, it can be seen that even with distinct DO, there is a certain internal coherence in the volumes belonging to the same collection. On a few occasions this coherence fails, for example, in the volume referring to the sixth year of elementary school where there was a misunderstanding in treating “possibility” as synonymous with "chance". Still, it is understood that if a school chooses these collections for teachers to use as a resource, the inconsistency is small. The disarticulation is greater in relation to the conceptions of probability that, from one collection to another, is valued in some volumes, but never resumed and deepened, with the exception of the classical conception of probability.

Another point that we highlight is that our investigation allowed to characterize in the analyzed collections what Gascón (2014) denominated as being the Dominant Epistemological Model (DEM). From this DEM we have the possibility to think of other proposals of teaching of probability for Brazilian public schools. We believe that it is necessary to think about other forms of work with the subject investigated here and an alternative is the elaboration of Epistemological Reference Models (ERM) that, according to Gascón (2014), allow the emancipation of DEM as described in this article. This is one of the prospects for continuity of this research.

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Study and Research in Graph Theory: A Case of a Japanese Upper Secondary School

Ryoto Hakamata and Koji Otaki

Abstract How can we organise a teaching and learning process based on inquiry, that is, a study and research path (SRP) in school mathematics education? This big question has motived researchers under the study paradigm of questioning the world. According to this direction, we implemented an experiment for realising an SRP in a Japanese upper secondary school. The SRP developed around a theme of graph theory. For analysing it, we conducted both economic and ecological analyses of the functioning of the didactic system. We used the questions-answers map and described the occurrence of the basic gestures of inquiry for the economic analysis, and then, identified some conditions by using the scale of didactic codeterminacy levels for the ecological analysis. As a result, we pointed out some significant conditions, which can persuasively explain the economy of the SRP, at different levels. Besides that, we finally proposed a hypothesis about an obstacle that hinders the dissemination of graph theory into school mathematics curricula.

Introduction

During the past ten years, the study paradigm of *questioning the world* (Chevallard 2015) has motivated many researchers to identify favourable conditions for students' inquiry—SRP, i.e. *study and research path*. Our project follows this research direction. In the project, we have implemented a teaching experiment to realise an SRP related to some themes of discrete mathematics. To identify conditions for an SRP, we need to describe properties of it. These research tasks—identifying conditions and describing properties—are respectively included in the *ecological* and *economic* dimensions of didactic problems (cf. Gascón and Nicolás 2019). As mentioned in

R. Hakamata (✉)
Faculty of Education, Kochi University, Kochi, Japan
e-mail: hakamata@kochi-u.ac.jp

K. Otaki
Hokkaido University of Education, Hokkaido, Japan
e-mail: otaki.koji@k.hokkyodai.ac.jp

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Gascón and Nicolás (2019), “to deal with a question included in the ecological dimension [...] one seems to need certain answers to questions included in the economic dimension”. We therefore conducted economic analysis of the realised SRP at first, and ecological analysis of it after that. This paper provides the main parts of its result. From the viewpoint of *didactic engineering within ATD* (Barquero and Bosch 2015), the experimentation was implemented in a Japanese upper secondary school for about one year.

Main Theoretical Constructs and Analysing Methods

Let us explain certain methods of both analyses. For economic analysis of the SRP, we have conducted a *two-steps description*: the first was the description of the *dialectic of questions and answers* in the SRP; the second was the description of the mathematising process of the answers. To describe the dialectic of questions and answers, we have used the *questions-answers map* (Q-A map) which represents the development of derived questions raised and partial answers obtained in an SRP by a tree diagram (cf. Winsløw et al. 2013). This work enabled us to grasp the whole structure of the realised SRP and offered a base for the second step. We have used the notion of the *five basic gestures of inquiry*—observation of $A^{\diamond}$, analysis of $A^{\diamond}$, evaluation of $A^{\diamond}$, development of $A^{\heartsuit}$, and defence & illustration of $A^{\heartsuit}$ (cf. Chevallard 2019b)—to describe the mathematising process. Here, $A^{\diamond}$ denotes any *readymade answer* obtained from available resources and $A^{\heartsuit}$ denotes any *homemade answer* produced in an SRP. Some use $A^{\heartsuit}$ for representing a conclusive answer of a final form, whereas we do not restrict the use of the notation to final answers; that is, we also use $A^{\heartsuit}$ for representing all of the “temporary” homemade answers. We have expected that we can analyse the properties of the SRP by describing what kind of gestures occur.

Then, in the phase of the ecological analysis, we have identified conditions for the SRP. In ATD, a condition is anything that enables or hinders human activities. To consider various kind of conditions in deferent levels, we used the *scale of didactic codeterminacy levels*: Humankind ⇄ Civilisations ⇄ Societies ⇄ Schools ⇄ Pedagogies ⇄ Disciplines ⇄ Domains ⇄ Sectors ⇄ Themes ⇄ Subjects (cf. Chevallard with Bosch 2019). The scale shows several institutional levels which we should consider and where we can identify notable conditions.

The Setting of Our Didactic Engineering

The implementation of the SRP was developed at an upper-secondary school in Japan. The school has been one of the *super science high schools* (generally called *SSH* in Japan). The SSH is a general name of upper secondary schools that are recognised to be having high autonomy about curriculum development for inquiry-based science and mathematics education by the Japanese educational ministry. The school has had a special course of *kadai kenkyū* (project study) for inquiry-based learning. About eighty students, who have been generally interested in science and/or mathematics, took this course (there were about six hundred students in the total in the school). In the course, the students made small groups and studied their themes for about one and a half years. For every group, one or two teacher(s) supervised the inquiries. Under the intention of generating autonomous inquiry, each group was asked to pose its own initial question with suggestions of themes by its teachers.

In our case, a group of four 11th grade students inquired a question related to graph theory with the help of a teacher, who is the first author of this paper. Let us symbolise the group, the teacher, and the initial question by G, h, and Q_0 respectively. That is, the object of this study is the SRP developed by the didactic system $S(G, h, Q_0)$.

In the beginning of the SRP, the teacher h suggested some themes of graph theory. Examples of the themes included the problem of the *Seven Bridges of Königsberg* and a chessboard problem that could be solved by using graphs. For the seven bridges problem, h introduced a solution by graphs and explained that this problem connects with the theme of the *cursality* or *traversability* of graphs. Then, the group G of the students posed a question Q_0: *what kind of graphs is bicursal?* Here, a bicursal graph is any graph which can be drawn by just two strokes. In the SRP, G worked generally about two hours per week, but they could manage their time and place for working flexibly. Although most of the data collections were collected by video recording, some parts of the inquiry could not be recorded. For analysing these parts, we used the descriptions of the fieldnotes and the author's remembrance.

The Economy of the Graph-Theoretic SRP

At the beginning of the inquiry, G conjectured that "A graph is bicursal if and only if the graph has just four odd points". ($A_0^{\heartsuit}$) based on an experiment on some concrete examples of bicursal graphs. Although this conjecture is true, G could not prove it at that time. Then, h suggested G considering about the case of unicursal graphs. This led G to the question "what kind of graphs is unicursal?" (Q_1), and then, G found a proof of the theorem on Eulerian graphs in an academic textbook ($A_{1,1}^{\diamond}$; see Fig. 1, the same hereinafter). Since G had heard that the semi-Eulerian graphs are also unicursal, they began to construct its proof. Although G could not successfully complete the proof by themselves, after being given a little clue by h ($A_{1,2}^{\diamond}$), they accomplished it by applying the method of proving in the case of $A_{1,1}^{\diamond}$. Finally, applying the method in the case of $A_{1,2}^{\diamond}$, they arrived at the complete proof of $A_0^{\heartsuit}$. This was a possible final answer to Q_0. However, the SRP continued to develop: G immediately posed a new generalised question "what kind of graphs is n-cursal?" (Q_2) and conjectured "a graph is n-cursal if and only if the graph has just $2n$ odd points". In the same way, applying the method in the case of $A_0^{\heartsuit}$ again, they accomplished the proof ($A_2^{\heartsuit}$). The Q-A map shown in Fig. 1 describes the structure of the dialectic of questions and

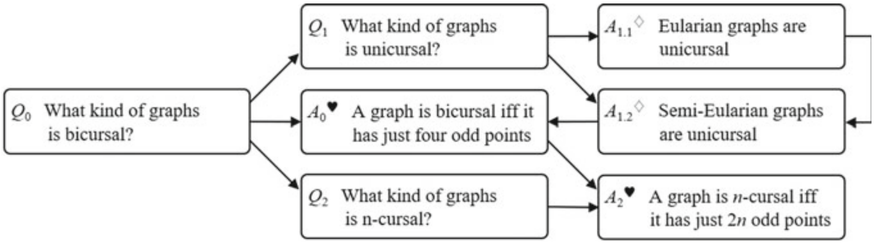


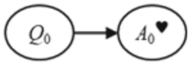
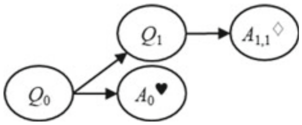
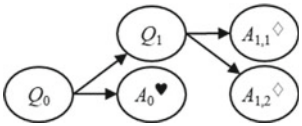
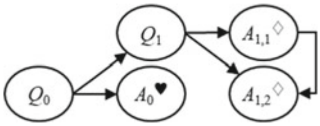
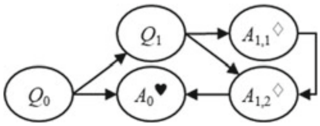
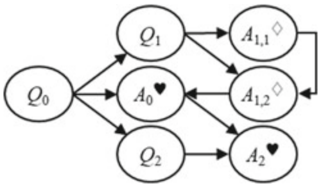
Fig. 1 Q-A map of the realised SRP

answers in the abovementioned graph-theoretic SRP. Note that the subscript numbers in the *Q-A* map do not indicate chronological order, but epistemological order.

Let us describe here the details of the realised SRP with the five basic gestures of inquiry, as the second step of economic analysis. Implementation of the five basic gestures is one of the structuring principles of any SRP (Chevallard 2019b). In this paper, we use the model of five gestures as a tool for measuring quality or soundness of the SRP, that is, a *degree of conformity* (Chevallard with Bosch 2019) between inquiry implemented by G and reference inquiry of a supposed ideal type—we assume that any occurrence of sound SRP includes all the gestures. Here, we interpreted each gesture as follows: *observation* of $A^{\diamond}$ is grasping what the information obtained from media (such as the internet, textbooks, etc.) is. In other words, the gesture is to judge whether the media offer $A^{\diamond}$ or not; *analysis* of $A^{\diamond}$ is to understand it experimentally or theoretically. This gesture includes understanding the claim of theorems by making concrete examples and the proof of them by reading some textbooks; *evaluation* of $A^{\diamond}$ is examining if the known answer is applicable to analysing another unknown answer $* A^{\diamond}$ or to developing own answer $A^{\heartsuit}$; *development* of $A^{\heartsuit}$ includes both conjecturing mathematical statement and constructing mathematical proof; *defence and illustration* of $A^{\heartsuit}$ is producing the own answer toward the outside of the didactic system, as well as answering questions and responding to criticism.

Table 1 shows the progress of the realised SRP according to the timeline together with the basic gestures occurred in it. We can see that all of the five gestures occurred in the process. In the SRP, what was remarkable is the dialectical interplay between the gesture of analysis or development and of evaluation. Regarding the analysis and evaluation, it appeared when G faced difficulty in analysing the proof in $A_{1,2}^{\diamond}$ (in scene 3 and 4). G could not understand the proof at first, however, they accomplished it by evaluating the method of proving in the case of $A_{1,1}^{\diamond}$ to apply it to $A_{1,2}^{\diamond}$. As for the development and evaluation, it occurred in constructing the proof of $A_0^{\heartsuit}$ (in scene 1 and 5). When G developed $A_0^{\heartsuit}$ as a conjecture, they could not prove it, but they eventually completed it through evaluating $A_{1,2}^{\diamond}$. The similar interplay can be seen in producing $A_2^{\heartsuit}$ (in scene 6). In this way, G developed the SRP through the dialectic between the gestures.

Table 1 Five basic gestures occurred in the realised SRP

Scene	Progress of the SRP	Basic gestures occurred in the scene
1		Development of $A_0^{\heartsuit}$ as a conjecture (without proof)
2		Observation of $A_{1,1}^{\diamond}$ from the textbook Analysis of $A_{1,1}^{\diamond}$ for understanding of the proof of the theorem of the Eulerian graphs (accomplished)
3		Analysis of $A_{1,2}^{\diamond}$ for proving of the theorem of the semi-Eulerian graphs (unaccomplished)
4		Evaluation of $A_{1,1}^{\diamond}$ for getting hints on $A_{1,2}^{\diamond}$ (this gesture helped students to apply the method in $A_{1,1}^{\diamond}$ to $A_{1,2}^{\diamond}$ and complete the proof)
5		Evaluation of $A_{1,2}^{\diamond}$ for getting hints on the proof of $A_0^{\heartsuit}$ Development of $A_0^{\heartsuit}$ as the answer (with proof) Defence and illustration of $A_0^{\heartsuit}$ in a presentation to other students and school teachers
6		Development of $A_2^{\heartsuit}$ as the final answer (without proof at first, then accomplished after evaluating) Evaluation of $A_0^{\heartsuit}$ for getting hints on the proof of $A_2^{\heartsuit}$ Defence and illustration of $A_2^{\heartsuit}$ in a presentation to other students and school teachers

The Ecology of the Graph-Theoretic SRP

This section is dedicated to an ecological analysis for identifying conditions for the implemented SRP. For the analysis, we used the scale of didactic codeterminacy levels.

The study paradigm of questioning the world at the levels of societies and schools: The SRP was implemented based on the paradigm of questioning the world, which can be regarded as an obvious but crucial condition located at the interlevel of societies and schools, that is, the level of *noospheres* which are usually-unnamed institutions for management of teaching. Let us point out here that a major effect to didactic systems comes from the paradigmatic transition from the visiting-works to the questioning-the-world. This is related to the status of *questions*, which is an implicit and susceptible entity in the scale. Under the visiting-works paradigm, questions at stake in didactic situations strongly depends on the interrelated—inter-empowered—structures of scholarly knowledge and school knowledge to be taught:

$$\rightleftarrows \text{Disciplines} \rightleftarrows \text{Domains} \rightleftarrows \text{Sectors} \rightleftarrows \text{Themes} \rightleftarrows \text{Subjects} \rightleftarrows [\text{Questions}].$$

Any didactic system of such situations S_{VM}—under the visiting works—functions with a focus on a *piece of knowledge* $\mathscr{k}$, that is to say, $S_{\mathrm{VM}} = S(X, Y, \mathscr{k})$ where X is a set of students and Y is a set of teachers. On the other hand, any didactic system under the paradigm of questioning the world S_{QW} is organised around a problematic question $\mathscr{q}$: $S_{\mathrm{QW}} = S(X, Y, \mathscr{q})$. This change about the *didactic stake* is a break or revolution by which the questions rises in the epistemological hierarchy to the top rank:

$$\rightleftarrows [\text{Questions}] \rightleftarrows \text{Disciplines} \rightleftarrows \text{Domains} \rightleftarrows \text{Sectors} \rightleftarrows \text{Themes} \rightleftarrows \text{Subjects}.$$

Compared with the didactic systems of this type, the didactic system of our case $S(G, h, Q_0)$ seems to be a compromise between the two paradigms. On the one hand, indeed, $S(G, h, Q_0)$ was organised around the initial question. However, on the other hand, the question was produced within the domain of graph theory following to the didactic contract of *kadai kenkyū*. Such an effect of the paradigmatic double-bind can be described as follows:

$$\rightleftarrows \text{Disciplines} \rightleftarrows \text{Domains} \rightleftarrows [\text{Questions}] \rightleftarrows \text{Sectors} \rightleftarrows \text{Themes} \rightleftarrows \text{Subjects}.$$

Let us underline here that this kind of compromises is not special in our experimentation. On the contrary, these are typical occurrences in researcher education like Ph.D. courses. Ph.D. students usually decide their programme, supervisor, department, and university after selecting their research discipline and domain even sector.

The SSH didactic infrastructure at the levels of schools and pedagogies: The course of *kadai kenkyū* in the SSH provided the group of students G with *didactic time* and *space* for their autonomous inquiry. Since G was allowed to manage time for the inquiry by themselves, they were able to schedule weekly activity time. For example, they sometimes shorten the time because of the school's examination, and they sometimes extended the time to prepare their research presentation. As for didactic space, for instance, they could go out the classroom for using the school library or the computer room if they wanted. The infrastructural resources of *kadai kenkyū* brought about higher responsibility of G. There is another feature of the resources. In the course, various opportunities for making research presentation were prepared. The students could make the presentation not only for other students and teachers outside the class $[G, h]$, but also for teachers in other schools as well as in universities. This feature strongly promoted the gesture of defence and illustration in the SRP.

The dialectics of inquiry at the levels of pedagogies and disciplines: In general, SRPs are conditioned by various dialectics: dialectics *of on-topic and off-topic*, *of the parachutist and the truffle hound*, *of black boxes and clear boxes*, *of conjecture and proof* (*media and milieus*), *of reading and writing* (*excribing and inscribing*), *of dissemination and reception*, *of the individual and the group* (*idionomy and synnomy*) (cf. Chevallard with Bosch 2019). In our case, two of these dialectics mainly promoted the SRP: the dialectics of conjecture and proof, and of reading and writing. After posing the initial question Q_0, the group G began their inquiry with developing the answer $A_0^{\heartsuit}$ as a conjecture. Then, the motivation to prove it activated the mathematical activity concerning the study of Q_1: Observation, evaluation, and analysis of $A_{1,1}^{\diamond}$ and $A_{1,2}^{\diamond}$. The dialectic also functioned in producing $A_2^{\heartsuit}$ in the same way. We can say that the dialectic of conjecture and proof strongly promoted the SRP as a whole. As for the dialectic of reading and writing, this refers to the process of reading some texts to understand the readymade answers $A^{\diamond}$ by reconstructing the normally hidden knowledge and of writing (re)constructed knowledge into the text in producing own answers $A^{\heartsuit}$. In the above, we have described the dialectical interplay between the basic gestures in the SRP. The dialectic of reading and writing played a significant role for conditioning this interplay.

The de-structuralisation of mathematical works at the levels of disciplines and domains: In graph theory, graphs are defined only as an ordered pair of a set of vertices and a set of edges. Here, we do not need define any algebraic operation (e.g., sum of two graphs) or topological relationship (e.g., open sets in the set of vertices). Of course, we can give those definitions and giving algebraic operations or topological relationships helps us to solve advanced problems in graph theory. However, for many basic problems, we can solve them without considering those mathematical structure. Hence, in general, mathematical activity does not include fundamental mathematical structures such as algebraic or topological ones in graph theory. While mathematical structures are essential to construct fruitful mathematical systems, their complexity—in other words, simplicity and clarity *only for experts*—can hinder students' mathematical inquiry. In the *non-structural works* in graph theory, students allow themselves to have limited elements to focus on, and therefore, they can relatively easily pose their own question or generalise the propositions.

The ostensive of graph at the levels of domains and sectors: We have already mentioned that graphs are defined as a (ordered) set and that can be favourable condition for students' inquiry. However, since sets are highly abstract and non-ostensive in themselves, students might have difficulties on dealing with the mathematical objects in graph theory. With respect to this, let us refer the graph-representation as an *ostensive register* (cf. Arzarello et al. 2008) that conditioned the development of the abstract mathematical knowledge. Any ostensive can get two values where the ostensive is activated: the *semiotic value* linked to their power to stand for non-ostensives and other ostensives, and the *instrumental value* linked to their functions as tools for accomplishing certain tasks. The *graph resister* allows us to easily and clearly represent relationships between vertices and edges in graphs, which are essentially non-ostensives. Thanks to this semiotic value of graph representation, we can understand mathematical discourse on graphs. The simplicity of graph representation leads to the ease of its operation. In fact, we can operate graphs just marking dots and drawing lines, instead of considering to add new elements into the set of vertices and corresponding elements into the set of edges. When we carry out some tasks in graph theory, writing graphs and operate them promote the activity—one of the techniques of the tasks. In the SRP, the instrumental value well functioned, therefore, students can inquire autonomously in graph theory in spite of the abstract nature of it.

The didactic media for studying the unicursal graph at the levels of sector and theme: In Japan, the topic of one stroke sketching is famous and some of secondary mathematics textbooks introduce the problem of the Eulerian graph. Furthermore, proofs of the theorem of the unicursal graph can be read in not only academic textbooks but also webpages in Japanese. Therefore, the theme of cursality of graphs has relatively rich resources for promoting mathematical inquiry. Besides this, it can also be the condition that the topic of one stroke sketching is not regarded as mathematics at first sight. In fact, many children and students in Japan know this topic through some games or children's book. Thus, inquiry on the questions in the theme can be started from *tick tasks* (Chevallard 2019b) which are well known and lead to problematic tasks.

Final Remarks: A Constraint on the Teaching of Graph Theory

Over the past decades, the domain of discrete mathematics has attracted a lot of attention in both scholarly mathematics and school mathematics. For instance, basics of number theory and the combinatorics have been included as the sectors in school mathematics in Japan. From the analysis of the favourable conditions, graph theory seems to have a potential for realising students' autonomous inquiry, and therefore, school mathematics curricula might include it as a sector. However, despite of such conditions, Japanese school mathematics curricula do not have graph theory as the teaching contents. Even if some textbooks introduce a few themes in graph theory, they are not the main teaching contents but the additional topics. Why is graph theory usually excluded from school mathematics? What is the ecology of this phenomenon, that is, the *low status of graph theory under emphasizing discrete mathematics at school*?

We conjecture that there is a constraint in higher levels of the scale, and we named it the *calculationism*. This refers to a naïve and dominant epistemology of mathematics, represented by "doing mathematics is calculating numbers." The calculationism typically appears in an English idiom *do the math* which means "calculate numbers".

Other evidences can be seen in the name of this discipline in east Asia. The word *mathematics* can be expressed in 数学or 數學in Chinese and Japanese by the character of *Kanji*, which literary mean the *discipline of the number*. *Non-math people* (cf. Chevallard 2019a) widely share this view because of different school mathematics examinations which consist of many calculation problems. Furthermore, this epistemology also seems to percolate through *math people*. From a historical point-of-view, apart from the Greek mathematics based on its special philosophy, major ancient civilisations focused on calculations and developed their own mathematics—Babylonian, Chinees, Egyptian, etc. In the case of modern mathematics, for instance, a focus on algebraic properties can be seen in the way of constructing the number system from natural to real. Usually, numbers are constructed and extended from algebraic perspective such as the closedness of operations or existence of inverse elements. Thus, most undergraduate mathematics textbooks introduce negative numbers before developing the continuity or completeness. However, if we focus not on the algebraic but topological properties, we can construct a complete positive real number system in another way as in Landau's *Foundation of analysis* (Landau 1966). We already referred to the non-structural works in graph theory as a condition for mathematical inquiry. However, according to the calculationism, the domain without algebraic structures like graph theory could not be important in school curricula. In short, the nature of graph theory could also be a restriction on didactic transposition of graph theory.

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Overview of Research on Textbooks in Brazilian Compulsory Education

Marilena Bittar

Abstract In this text I present a model for the analysis of textbooks based on the anthropological theory of the didactic. It is a model that allows highlighting the praxeologies proposed in textbooks and the importance attributed to each praxeology present in these materials. It consists of the following non-linear steps: the choice of the textbooks to be analysed; a separation of the course and the proposed activities sections; the modelling of mathematical praxeology; the modelling of didactical praxeology and the cross-checking of the data. For the presentation of this model, examples of research supervised by me over the last 15 years are brought.

Why Analyse Textbooks

There is evidence, from several research studies in different countries (Fan et al. 2013; Robitaille and Travers 1992; Sievert et al. 2019; Valverde et al. 2002; Zhu and Fan 2002) that textbooks are an essential support for teachers at any level of schooling and for any subject. It can be said that textbooks represent the official institutional knowledge that must be taught by teachers and, furthermore, that when supplemented by other sources of information, such as official documents, their analysis can allow a closer engagement with what is conveyed in the classroom. This is even more significant in the Brazilian case, since the national guidelines only provide general recommendations on the content to be worked on. In addition, considering the problems that initial teacher training faces and the difficulty of access to various materials that can help teachers in their teaching, the textbook, distributed free of charge to all students of public compulsory education, becomes the main

[1] The Grupo de Estudos em Didática da Matemática (DDMat) focuses on the study of didactic phenomena whose problematization considers mathematical knowledge as a central element. To have access to the group's productions, access http://grupoddmat.pro.br/.

M. Bittar (✉)
Universidade Federal de Mato Grosso do Sul, Campo Grande, Brazil

Researcher of Conselho Nacional de Desenvolvimento Científico e Tecnológico–CNPq, Brasília, Brazil

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material to support this teacher. Thus, one can consider that there is a great proximity between what is in the textbooks and what is transmitted in the classroom. In this sense, textbook analysis may allow us to identify the dominant praxeological model which, in turn, helps us both in the elaboration of a reference praxeological model and in the elaboration of activities to be proposed to students, aiming to contribute to their learning.

The first time I opened textbooks with the intention of understanding the teaching proposal related to a certain content was during the realization of my doctorate, whose objective was, among other aims, to study difficulties of high school students regarding the concept of vector. The analysis of textbooks was fundamental to identify and understand difficulties that the teaching proposed in these materials could generate and, from this, to propose activities to the students aiming not only to identify these difficulties, but also to overcome them. The purpose of this analysis was to identify what object was transposed to secondary education and, in this direction, I proceeded to the identification of definitions and properties present in textbooks and how these appeared in the activities solved and proposed. This was a mathematical analysis, without using theoretical tools. Years later, I returned to be interested in the analysis of textbooks, this time with the support of a theoretical and methodological referential, the anthropological theory of the didactic, which I present briefly in this paper.

In the next section I briefly discuss the National Program of Textbooks given its importance in the Brazilian scenario and its influence on the studies we carry out in the *Grupo de Estudos em Didática da Matemática* (DDMat).[1]

National Program of Textbooks

In 1929, the Ministry of Education in Brazil established the National Institute of Textbooks, with the aim of legislating on policies relating to textbooks. Since then, several modifications have taken place, which led to the creation of the National Program of Textbooks (*Programa Nacional do Livro Didático*—PNLD), which carried out in 1996 the first pedagogical evaluation of books intended for primary school. Subsequently, this evaluation system was extended and, in 2002, it reached all compulsory education.

The pedagogical assessment was coordinated by a university, chosen through a public call. Between 1996 and 2017, more than 250 teachers from different Brazilian regions participated in the evaluation process of Brazilian textbooks. The evaluators are university professors and also teachers of basic education. The evaluation considers exclusion criteria and quality criteria. The main objective is to provide students with textbooks without conceptual errors or prejudices.

The evaluation process is long, carried out by a large team and culminates in the production of a Didactic Guide made available to teachers in order to assist them in choosing the book to be purchased. The Didactic Guide contains the follows sections: Letter to the teacher; Syntheses; General considerations; Assessment criteria; Syntheses of the collections; Evaluation grid; General text. In the Syntheses of the collections, both a description of the content of each textbook and an assessment of the treatment agreed to each content are presented, as well as an overview of the textbook and some advice for the teacher who chooses this book. Thus, when we are going to analyse a textbook, the reading of the evaluation of this textbook contained in the Guide provides a first approximation with the material to be studied and has helped in the choice of textbooks to be analysed, as we will see in this article.

For more information about the *Programa Nacional do Livro Didático* (*PNLD*) I recommend reading the article of Carvalho (2018), published in ZDM.

Research Carried Out in the DDMat

Since 2008 I have been developing the investigations on mathematical and didactical choices of authors of textbooks destined for students of public schools of Brazilian compulsory education. These investigations aim at, among other objectives, a mapping of the knowledge to be taught. In this direction, the main theoretical and methodological used is the anthropological theory of the didactic–ATD (Chevallard 1998), which allows the description and, consequently, the understanding of mathematical and didactical choices of authors of textbooks. In addition, this theory allows, on the one hand, to question teaching practices and, on the other hand, to understand aspects of student learning.

Not all research done so far by the DDMat group has focused on specific mathematical concepts. This is the case of Souza's (2014) research focused on contextualization of algebra in textbooks for the 7th year of elementary school, and Costa's research (2017) on mental calculus in a collection of elementary school textbooks.

As example of research involving a mathematical object, I cite Nogueira's (2008) study on the introduction of algebra in college, and Gonçalves' (2016) research, which investigated the presentation of integers in a book for the 7th year of Middle School. In her master's research, Kaspary dos Anjos (2014) investigated the additive field in a primary school collection, which comprises 5 volumes. In her doctoral research, in development, Kaspary dos Anjos analyses textbooks intended for the initial years used in the last 20 years. Finally, I present Verbisck's (2019) research on combinatorial teaching in Brazilian textbooks intended for all 12 years of compulsory schooling.

We observe a variety of objects of study which implies in different choices of the material to be analysed, as will be seen next.

A Design for the Analysis of Textbooks

After several research carried out from the viewpoint of ATD, it was possible to systematize a way of producing the data for analysis, which I briefly present below.

Choosing the Corpus for Analysis

The textbooks to be analysed are defined according to the purpose of the research. If the intention is to know what is being done in most schools, the choice falls on the most adopted books, in case the subject to be studied is present in a single volume (Nogueira 2008), or in the most adopted collection, if the theme is presented in several volumes (Kaspary dos Anjos 2014). There is also the case of Verbisck's research (2019) that analysed the presence of combinatorics in the 12 years of compulsory schooling and for this she chose collections of the same author. This was the way of ensuring greater coherence between volumes than when the authors are different.

It should be noted that at this stage of the analysis, as I said before, the Didactic Guide is an important ally of the researcher, since it brings general information about each approved collection. For example, Nogueira (2008) investigated how algebra was proposed in its introduction (8th grade). For that, she chosed three collections of books with different didactic approaches, which was made possible thanks to the Didactic Book Guide.

Praxeological Analysis of the Course *Part*

For the analysis we divided the text of the textbook into two parts: *course* and *proposed activities*. The *course* part comprises the presentation and explanation of definitions, properties, results and solved exercises. In this part the authors of the textbook bring, sometimes implicitly, what they consider that students of that level of schooling should learn and it is in this part that students seek clues to solve what it is required. The analysis of the *course* part of a textbook allows the identification of some types of tasks that seem important in that institution. The analysis of this part, thus, allows the production of praxeological quartets, [type of tasks, technique, technology, theory], that will be tested, and perhaps modified, when studying the *proposed activity*. This first analysis allows the identification of the set of tasks to be regrouped in task types. This modeling depends, at the same time, on the institution in which the work is conducted and, of course, on the researcher doing the analysis.

It is important to note that the analysis initiated in this part is done with support from previous research and an epistemological analysis of the topic under study. Thus, when possible, we begin the praxeological analysis with a priori praxeology, as was the case of Kaspary dos Anjos's (2014) research on the additive field. This author developed a priori praxeology based mainly on the situations of the additive field defined by Vergnaud (1990) and on other situations designed by other authors who dedicated themselves to the study of this theme.

Praxeological Analysis of the Proposed Activities *Part*

Once the *course* part analysis is done, we move on to the *proposed activities* part. At that moment, it is sought to identify, with the support of the previously identified praxeologies, what is the task and what is the technique that the student is expected to use to solve the task. In order to infer what is expected of the student, we base ourselves on what is present in the teacher's Manual and, especially, what has been worked on in the *course* part. Through this analysis, we look for elements that allow us to infer how the authors of the textbook wish their users to solve their activities. Such an inference is based on the concept of didactic contract (Brousseau 1986): the student seeks, in the action of the teacher or in the textbook, to find some sign of what he is expected to do. At that moment the teacher's manual is an important ally of the researcher since, in general, it presents and justifies the choices made by the authors, as well as some resolutions or comments on proposed activities.

In this analysis the quantification of the identified tasks−consequently of the types of tasks−and of the techniques to be mobilized is carried out. It is essential to conduct a quantitative analysis for the study of mathematical praxeologies because they make it possible to infer and/or question the importance attached to several praxeological quartets with regard to the *raison d´être* of some praxeologies. Thus, every single activity is analysed.

Another factor to be considered is what the modeled data "says" or "does not say" to the researcher. Sometimes−almost always−you have to go back to the material and reformulate the model. This was the case of Freitas' research (2015), which investigated the proposal of teaching volume in high school textbooks. In the first analysis carried out, the author found 438 tasks of type T1: Calculate the volume of a solid and found that most tasks were related to known solids. To state this, as a result of research, is very vague. It was necessary to have more accurate data, to know exactly how many tasks are related to known solids and how many are not. Thus, it was possible to remodel the data around two types of subtasks, as we see in the excerpt below:

> For a more reliable analysis of the volume teaching proposal, task type T1 was divided into two subtypes: T11: Calculating the volume of a known solid and T12: Calculating the volume of an irregular solid. 408 tasks of the first subtype and 30 of the second were found, which reveals the author's choices and that it would not be possible to perceive if the modeling was done considering T1 and T2 as a single type of task. (Bittar 2017, p. 376)

Table 1 Total of non-contextualized types of tasks in each year (Kaspary dos Anjos 2014)

	T7	T8	T9	T10	T11	T12	T13	T14	T15	T16
Volume 1	29	18	40	7	54	6	14	9	0	0
Volume 2	18	22	23	35	354	17	10	28	11	1
Volume 3	24	32	5	12	212	22	4	32	26	3
Volume 4	12	27	1	7	187	19	2	84	16	26
Volume 5	8	13	0	0	192	8	2	53	14	4

Cross-Checking the Data

Once the data is produced, it is analysed. The data quantified in Table 1, for example, shows the absolute predominance of the type of tasks T11 in all the volumes of the collection, as well as four types of tasks absent or almost absent in the last volume.

These data indicate, among others, the praxeologias that one wishes to implement and possible influences of the noosphere on the choices of authors of textbooks. This is the case of Souza's (2014, p. 69) research that concluded that "The manual […] presents 435 tasks of which 153 are contextualized, which expresses a certain interest in proposing contextualized situations. Of these, most are artificial or are pseudo-contextualizations." An explanation for this great occurrence of situations that mention contexts can be found in the official guidelines when the importance of such situations is emphasized at all levels of schooling. In order for the book to be acquired by schools it is necessary to satisfy such a requirement, but the authors do so as they can or believe that it can/must be done.

Once the mathematical organization (OM) and the didactic organization (OD) are obtained, it is necessary to interpret the information obtained. Nogueira (2008) modeled the OM of the Course part (PC), of the Proposed Activities Part (PE) and also the OD of each analysed collection, which provided her with an analysis result (Ri) of each collection. Then, she compared the three results (Fig. 1) to then answer her research question. Figure 1 represents the methodological design developed by the researcher.

In the analysis of solving equations of grade 1, in three different textbooks, Nogueira (2008) modeled three techniques: $\tau 1$, $\tau 2$ and $\tau 3$. However, not all of them are present in the three books, which implies that each book adopted different praxeological evolutions. In the book of collection 1, one starts with the first technique to reach the second, $(T, \tau 1) \rightarrow (T, \tau 2)$; in the book of collection 2, $(T, \tau 1) \rightarrow (T, \tau 2) \rightarrow (T, \tau 3)$; and finally, in collection 3, we find the following praxeological evolution $(T, \tau 2) \rightarrow (T, \tau 3)$. Her data shows that the analysis of didactic praxeologies of textbooks allowed her to confirm her hypothesis when choosing textbooks: one textbook has a more technical tendency, the other presents the course interspersed with dialogues with the students and the third has a rather constructivist tendency.

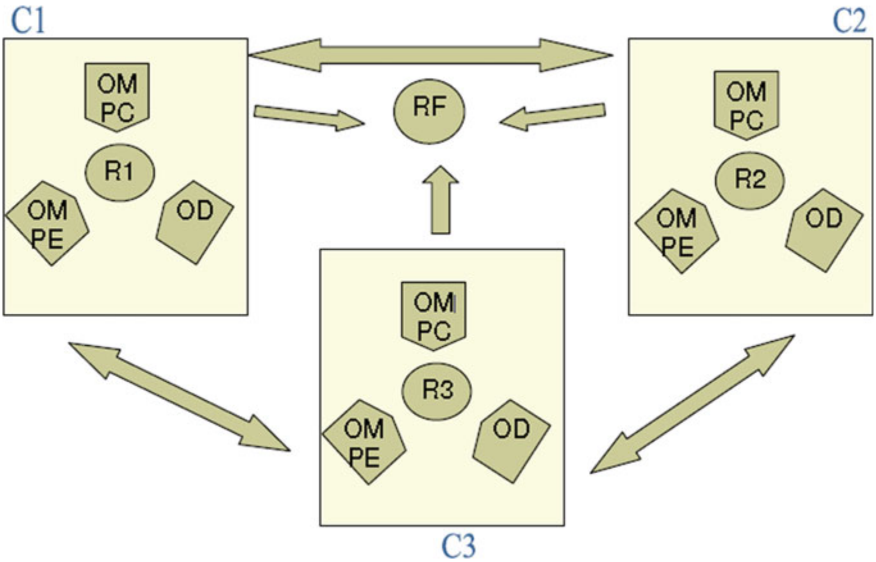

Fig. 1 Methodological design (Nogueira 2008, p. 55)

In Conclusion

Since 2008 I have supervised eleven studies that have carried out a textbook analysis and another two are in progress. Results of these investigations have been important for the development of studies on the teacher in-service training, especially for the discussion of alternative proposals for the teaching of Mathematics. Research in this field are under development in our group. Since 2019 we have been developing a project with Primary and Middle School teachers and future teachers in three cities in Mato Grosso do Sul.

In this text, I briefly presented the main steps of a model for textbook analysis. These are mandatory steps in every analysis that we carry out in our research group, DDMat, however, other theoretical and methodological elements have also been mobilized, such as levels of codetermination and the idea of ecological analysis, both of the anthropological theory of the didactic. In addition, other theories have also been used according to the need of the research, such as the theory of didactic situations (Brousseau 1986), which is from the same epistemological program as ATD, and theories that are not in the field of didactics of mathematics, as the professional knowledge of teachers developed by Shulman (2001).

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Kernels of Mathematical Thinking as Task- and Curriculum Design Tool in the Pósa Method

Dániel Katona

Abstract Some untheorised practices of the Hungarian 'guided discovery' approaches for teaching and learning mathematics are currently being theorized and redesigned from past and present practises, with long term curricular development goals, also aiming at contributing to the conceptualization of IBME. The Pósa method for talent care extracurricular mathematics education is one of the most remarkable representatives of these approaches. An a posteriori theorization of this 'intuitively developed method'– with a 'reverse didactic engineering methodology' – is being carried out within the frame of the doctoral research of the author. Main elements of this research, as well as some preliminary results and issues for further studies are presented, focusing on the Pósa method's connected task design. The concepts 'web of problem threads', 'kernel of a thread' and 'kernel of mathematical thinking' (KoMT) are presented.

The Teaching of Lajos Pósa and the Hungarian Guided Discovery Approach for IBME

There is claimed to be a rich Hungarian tradition not 'only' of doing and thinking about mathematics, but, correspondingly, a tradition of learning and teaching mathematics in a perhaps somewhat peculiar way; the analysis, description and theorization of which tradition is currently being developed, inter alia, under the term "Hungarian Guided Discovery" (Gosztonyi 2019). This tradition is partly reflected in the didactic works of Polya (1957), Dienes (1960), Péter (1961), Lakatos (1975), Halmos and Varga (1978), among others. The present paper reports on part of the research (by the author) aiming at the theorization and description of the mathematics teaching and learning method developed by Lajos Pósa in the past three decades (Győri and Juhász 2018), which also seem to represent a unique example of this supposedly unique tradition, or at least appears to be culturally highly connected to it. Further research is needed on studying the extent to which this tradition is unitary (nationally)

D. Katona (✉)
Alfréd Rényi Institute of Mathematics & ELTE - Eötvös Loránd University, Budapest, Hungary

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and the way of its peculiarity (internationally) and also whether and in what way(s) it may contribute to the on-going conceptualization of inquiry-based mathematics education (Artigue and Blomhøj 2013; Bosch and Winsløw 2016).

Lajos Pósa was a "foster nephew" of Rózsa Péter, an epsilon (student) of Pál Erdős, a classmate of László Lovász, and a young colleague of Tamás Varga, highly influenced by their teaching ideas and conceptions and their personalities. However, he, as a "charismatic leader" (Chevallard 1992, p. 220), developed his own method for "learning through discovery", as he and his colleagues (his former students) name it (Győri and Juhász 2018, p. 100), during teaching highly gifted 7–12 grade students in extracurricular weekend mathematics camps.

Theorizing and Reconstructing the Pósa Method with Reverse Didactic Engineering and Didactic Re-engineering Methodology

The Pósa-method was developed "based in teaching as craft knowledge" using the expression of Watson and Ohtani (2015, p. 5), "intuitively", lacking the conscious construction of and explicit building on any theoretical framework. In the frame of the doctoral studies of the author at the Eötvös Loránd University and the corresponding work of the MTA-Rényi Research Group on Discovery Learning in Mathematics at the Hungarian Academy of Sciences, the research aims at the (re)construction of the theoretical frame and the tools of the task- and curriculum-design behind the Pósa method. As the structure and steps of this research can be corresponded to the structure and steps of "didactic engineering" (Artigue 2014), but somewhat the other way round, the term "*reverse didactic engineering*" (RDE) was (personally) suggested for this research methodology by Angelika Bikner-Ahsbahs, on the basis of Arthur Bakker's (oral) proposal at CERME11. A more detailed description of this RDE methodology is found in a paper of the author (Katona 2020b). Subsequently to the theorization, a *didactic re-engineering* procedure, namely the re-design of the method, to be applied in public education is also targeted, through extending and restructuring the present Pósa extracurricular curriculum, along a comparative dialogue with other IBME approaches.

The main research questions of the part of the RDE research the present paper lays focus on are the following.

Q_0: How can we characterise the set of problems (or questions, tasks) used in the Pósa method?

Q_1: To what extent is the structure of the set of these problems characteristic of the method?

Q_2: How can we characterise the structure of the set of problems used in the Pósa method?

$Q_{2.1}$: Are problems connected within the set of these problems?

$Q_{2.1.1}$: To what extent are these connections characteristic of the structure of the set of these problems?
$Q_{2.1.2}$: What are the characteristic features of these connections?
$Q_{2.1.2.1}$: Are these connections connected to each other?
$Q_{2.1.2.2}$: What role(s) do these connections play in the didactic design of the Pósa method?

We try to present at least partial answers to these questions, as preliminary results of the inquiry into them (the corresponding part of the doctoral research of the author).

Web of Problem Threads (WPT)

As a preliminary result of theorizing the Pósa method, the concept *web of problem threads* (WPT) has been theorized (Katona and Szűcs 2017), based on the concept *problem thread*, which has been used by teachers of camps (Lajos Pósa and former camp students of him, becoming teacher colleagues) as a task design tool (also as a term referring to it), and by Lajos Pósa as a curriculum design tool. The term was proposed (for the research) by one of these teachers, Péter Juhász. In the problem thread theoretical construct, the focus is laid on the connections between the problems (or questions). *Threads of problems* are formed of problems that are connected in some special ways, and are posed for students sequentially, though not as "series of problems" (Gosztonyi 2019), as their order in not completely fixed (at least at the phase of their genesis), but also not as sets of problems, as there are important constraints on ordering them.

Problem threads regularly cross each other, by shared problems (belonging to more than one thread), giving birth to the whole structure called WPT. Students are concerned with several problems at a given time that belong to several threads, therefore, they walk through several connected (learning/ discovering) paths parallelly.

A Sample of a Problem Thread from the Pósa WPT — Object of Study

In the followings, some selected problems (questions) are presented from a problem thread. These questions, which were posed in Pósa camps for students grades 7–9, form part of the sample of the Pósa WPT (see Fig. 1) that was presented at the CERME11 – TWG22 poster section, though without the texts of these problems, their considered solutions and any analysis having been presented in the proceedings (Katona 2019).

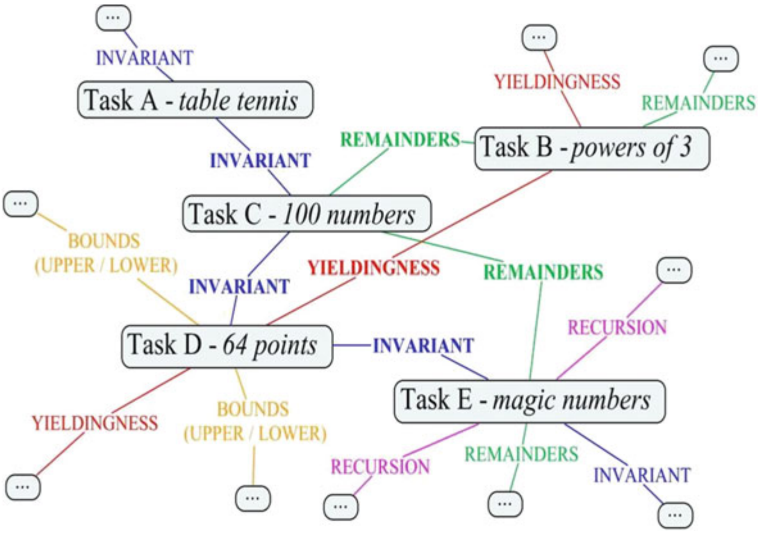

Fig. 1 A sample of the Pósa WPT with kernels (Katona 2019)

Question 1 (Task A) — Table Tennis Tournament

We are in 2345, and all the intelligent creatures in the universe have met one another. Now they are organizing a table tennis tournament for anyone who can hold a paddle with 'their' something. Due to the large number of participants, it was decided to organize a single elimination tournament, instead of a round-robin one. Now we don't know the exact number of participants. Will we be able to say the number of matches needed quite rapidly, when we know how many is the number of participants? How?

Answer 1: In each match, exactly 1 participant is eliminated. Therefore, if we start with n participants, to reach 1 participant at the end, we need n-1 matches. The sum of the number of participants not eliminated and the matches played is always n, that sum is invariant during the process.

Question 2 (Task C) — 100 Whole Numbers on Board

The first 100 positive integers are written on the board. Then we delete (any) 2 of them, and write the absolute value of their difference, and we repeat it with the new numbers again, and again (always selecting any two numbers), until we have only 1 number on the board. Can this number be 9? Can it be 10?

Answer 2: The feature that the sum of the numbers after each step is odd or even (or the remainder of the sum divided by 2) is invariant. It's even in the beginning, so it cannot be odd at the end.

Question 3 (Task D) — 64 Points with Straight Lines

The centers of each square of a (8 × 8 sized) chessboard are to be separated by straight lines (so that each 2 centers are separated). At least how many lines do we need for that?

Answer 3: You only have to look at the 30 outer squares. For their centers you need at least 14 straight lines, as each straight line can separate only two pairs of centers at the opposite sides. And 14 lines are enough for the whole board too. Invariant is the feature that every straight line separates exactly two pairs of centers at the outer squares.

Question 4 (Task E) — Magical Numbers

Anthropologists have arrived at a tribe living in a remote island, isolated from the rest of the world, and surprisingly discovered that for the aboriginals some numbers are magical. After a long observation, the scientists found out the rules what makes a number magical: if x and y are magical, then xy (x times y) and $2xy$ are also magical. Magical numbers are checked by the Council of Elders. We also know that the first magical numbers were 1, 7 and 16. (They are the only magical numbers whose magical power were not because of the aforementioned rules but for other reasons no-one knows today.) Are 123456 and 1234567 magical?

Answer 4 is based on proving the conjecture, that the magic numbers are exactly the ones which give 1 as a remainder if divided by 3. The remainder divided by 1 is invariant through creating magic numbers by the rules: If the neighbouring $3k + 1$ and $3(k + 1) + 1$ are magical, then the next, $3(k + 2) + 1$ is also magical.

Kernels of Threads Fostering the Development of "Mathematical Thinking"

Different problem threads may be built on the same kind of connection, and (a part of) a thread may be connected through more than one (type of) connection. Therefore, to separate the thing, the connection type that creates a thread from the thread itself, the concept *kernel of a thread* was introduced in (Katona and Szűcs 2017) for the types or manifestations of the connections between tasks (problem) that form a thread. Kernels in the Pósa WPT that have been detected so far are mainly "specific ways of mathematical thinking", or methods, such as *experimentation*, *yieldingness*, *invariant quantity*, *bounds* (upper and lower boundaries), *pigeonhole principle*, *recursion* (recursive thinking), *induction* (mathematical), *proof of existence*, *proof of non-existence*, *indirect proof* (proof by contradiction), *change in the representation* (e.g. of geometric configurations). There are also content-based (mathematical) kernels, such as *remainders* (within number theory). However, it seems highly questionable whether to treat the content-based ones as kernels or as belonging to a separate category.

During the implementation of the Pósa task design, these connections are revealed and discussed by the learning community. The connections are discovered, or inquired into during problem solving, and established through the process and series of discussions with posing new problems. Making these connections, the kernels "visible" to the students is a decisive part of the learning process, focusing on the development of these (and other) particular "mathematical ways of thinking", through processes of problem posing and solving, and reflecting on these processes, in which the distinction between doing (researcher) and learning (student) mathematics seem to be rather vague.

The Kernel Invariant (Quantities)

For solving the problems presented (Questions 1–4) a focus on an *invariant* parameter, invariant quantity is essential, as noted in the present texts of their (potential) answers.

In a the paper of the author (Katona 2020a), it has been shown that the type of relationship or connection that is expressed by the concept of kernel of a thread is based on the *logos blocks* of the praxeologies of the corresponding questions, more precisely, on common parts of their *technology* parts (Bosch et al. 2019).

Kernels of Mathematical Thinking (KoMT) & Curricula Based on KoMTs

The term *kernel of a problem thread* expresses, rather technically, the role of the concept within the didactic organisation (being reconstructed) of the Pósa method. However, for having a more meaningful terminology, considering the epistemological and mathematical nature of these "kernels" the term "kernel of mathematical thinking" (KoMT) is now being introduced. Further study is (definitely) needed on understanding the nature of KoMTs and collecting more, preferably all of these kernels used in the Pósa method, all the more that the development of these KoMTs observed and communicated (by the Pósa teachers) to be an essential learning goal, and the whole "Pósa curriculum" appears to be based partly but highly on them.

Based on the theorization, the re-design, that is, the "didactic re-engineering" (DRE) of the Pósa method for public education is also planned to be (hopefully) conducted further on, as a curriculum development project. The idea of a *KoMT based curriculum* for public education is planned to be elaborated on further studies. The problems (questions) in the Pósa method are created for giving birth to the kernels, and the questions are posed, by teachers and students, during a focus on the kernels, as it has been communicated by the "Pósa teachers" and as it was observed in the Pósa camps by the author. The construction and development of KoMTs may be regarded as the core curriculum development principle in the Pósa method.

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The Constitution of the « Milieu » During an Inquiry Process: An Analysis in Terms of Question-Answer and Media-Milieu Dialectics

Jana Lackova

Abstract This paper proposes to analyze the internal assessment within the standard level mathematics course in the International Baccalaureate (IB). This written and individual piece of work entitled "Mathematical Exploration" that each student must produce should involve an investigation in a field of mathematics. This analysis uses the tools of the Anthropological theory of didactics (ATD), in particular the Herbartian scheme and the dialectics of questions/answers and media/milieu. We make the hypothesis that these two dialectics play a crucial role in conducting this exploration and without these the work would be reduced to "copy-pasting" information from the media.

The Institutional Context

In our thesis, we question the place of inquiry in the context of the International Baccalaureate (IB) and focus on the place of inquiry-based education in mathematics classrooms when it becomes an institutionally recognized object, sanctioned by a summative assessment. The IB is one of the few institutions that requires a summative assessment of inquiry-specific nature in mathematics, compulsory for all students and contributing to 20% of the final grade. The internal assessment (IA) called *Mathematical exploration* is "a piece of written work that involves investigating an area of mathematics" (IBO 2012, p. 43). Its main objective is to evaluate students' performances and skills that are impossible to assess through written examinations or tests. A thorough analysis of the institutional documentation (Lackova and Dorier 2018) using the scale of didactic co-determinacy (Chevallard 2005) allowed us to identify elements pointing to the presence of inquiry-based learning (IBL) at each level of this scale. This analysis showed that the IB represents a particular institutional context that emphasizes the development of skills such as critical reflection or research skills through teaching that reflects pedagogical principles based on

J. Lackova (✉)
University of Geneva, Geneva, Switzerland
e-mail: jana.lackova@unige.ch

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inquiry as claimed in the institutional documentation (IB 2015). In the scope of this paper we will share the results of our analysis and focus on this particular form of assessment identified at the level of discipline.

The Didactic Co-determinacy Scale: The Level of Discipline

At the level of discipline, we have noticed that several choices were motivated and guided by the conditions created at the higher levels of the didactic co-determinacy scale. For example, the fact that mathematics has always been a mandatory discipline in the IB is closely linked to the "sputnik effect" and reinforced by some of the participants of the Second Sèvres Conference,[1] who found unacceptable that certain literary section of the French baccalaureate no longer required mathematics. We have also observed changes in the conception of mathematics as a discipline. The mathematics Standard Level (SL) and Higher Level (HL) syllabuses from the seventies emphasize the abstract concepts and structures which is probably the result of the new math reform in Europe. However, the mathematical studies course offered a more applied approach and included an internal assessment in the form of a personal project. This leads us to the conclusion that the form of a more applied mathematical activity was reserved to weaker students with little interests in mathematics. This conception changed though in the late nineties with the introduction of an internal assessment in the form of a Portfolio requiring completing tasks based on mathematical investigation and modelling in all mandatory mathematics subjects. This change was probably motivated by the observed decline of interest for science and mathematics (Rocard et al. 2007) and the newly published National Science Education Standards in the USA in 1996 providing a decent definition of inquiry-based education. (NRC 1996, p. 23 quoted in (Artigue and Blomhøj 2013, p. 800)) Indeed, from the late nineties until present, we have identified at the level of pedagogy a strong will to make inquiry-specific objectives more explicit which has also been reflected at the level of discipline. In 2012 the Portfolios are replaced by the Mathematical Exploration and in 2019 a totally new approach to teaching and learning mathematics has been introduced. In order to reinforce these choices and help their implementation in the classrooms the IB proceeded through four means: explicit documentation, teacher training workshop, resources and assessment. In addition, the 2019 syllabus claims to have revised and reduced content and has reserved 30 h to the development of investigative, problem-solving and modelling skills as response to the time constraints related with the implementation of inquiry-based activities. In the following part of this paper we will attempt to operationalize some tools from the ATD in order to approach and analyze this type of inquiry activity.

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[1] The objective of this conference that took place in April 1974 was to review the present state of the IB experiment in order to determine its viability, its long-term extension and consider its future developments and procedures.

The Herbartian Scheme as Model for Inquiry-Based Activities

Chevallard (2005) recalls that the dominant paradigm of school study is that of visiting works in which the teacher's role consists in making students visit the designated mathematical works to be taught. These practices gradually lead to what Chevallard calls the *monumentalisation* of the teaching of mathematics, that is to say that we study mathematical objects without meaning and without reasons for their being. The appearance of guided practical project as a baccalaureate requirement in French high schools has led to the emergence of the notion of *study and research paths* (SRP) and the development of the paradigm of questioning the world (Chevallard 2009). In this paradigm, a didactic system is formed in order to investigate the question Q in order to provide an answer. This answer $A^{\heartsuit}$ of the class is built in interaction with a learning environment—*milieu* M, which consists of *ready-made* answers validated by the institution $A^{\diamond}$, resources W, which are analysis tools for validating or rejecting partial responses. Following the type of study and research path this could be eventually completed by some derived questions Q and data collection D. The Herbartian scheme below is a condensed way that represents this reality:

$$[S(X;Y;Q) \curvearrowright \{A^{\diamond}, W_i, Q_j, D_k\}] \hookrightarrow A^{\heartsuit}$$

Bosch (2018) suggests that the dynamics of an inquiry process "is captured in terms of some *dialectics* that describe the production, validation and dissemination of $A^{\heartsuit}$"(p. 4008). In order to better understand the processes at stake behind the constitution of the milieu (mesogenesis) and its evolution (chronogenesis) we will take into account two important dialectics: media-milieu and question-answer dialectics.

Media-Milieu Dialectic

According to Chevallard (2008), the validation process is carried out thanks to the dialectic of media and milieu because:

> […] the existence of a vigorous (and rigorous) dialectic between media and milieu is a crucial condition for a process of study and research not to be reduced to the uncritical copying of scattered elements of response in the institutions of the society. (p. 345)

The fact that an inquiry-based activity can unfold will depend on the presence of these "ingredients" in the *milieu* and the existence of a dialectic between media and *milieu*.

Question-Answer Dialectic

The question-answer dialectic helps to unfold and describe the different paths followed during the inquiry process (including dead ends and abandoned attempts). When approaching the question Q at the beginning of the inquiry, one starts by searching for available answers $A^{\Diamond}$ and has to study them. This study usually generates new derived questions that help to move on with the inquiry. According to Bosch (2018) the question-answer dialectics "provides visible proof of the progress of the inquiry and contributes to what is called the *chronogenesis* of the process" (p. 4008).

The Mathematical Exploration as an Individual Study and Research Path (SRP)

The mathematical exploration is an individual, stand-alone research activity conducted on a topic chosen by a student and supervised by the mathematics teacher. The curriculum recommends dedicating 10 h of in-class time and 10 h personal work to the exploration. Naturally, the question whether the exploration could be considered as an individual SRP came up. Obviously, the conditions and constraints in this case will differ from the original idea of SRP as developed by Chevallard (2009) or the ones designed and conducted over a one semester university course in mechanical engineering (Florensa et al. 2016). The first major difference is that in a traditional SRP all students under the supervision of their teacher work on the same research question whereas during the exploration one teacher has to manage about 20 different explorations. This constraint needs to be taken into account and might represent a major obstacle to conducting a good quality inquiry process.

Square-Wheeled Bike Exploration Analysis

We propose here to present an a posteriori analysis of a part of an exploration elaborated by an IB student (17 years old) who, starting from a video showing a jump on a square-wheeled motorcycle (see Fig. 1), decided to explore this situation as part of his diploma work. This exploration is composed of three parts: a documentary research, an exploratory work using GeoGebra and an analytical solution providing a formal proof. In this paper, we propose to analyze the exploratory part of this work. The objective of the analysis is to identify the media used in this exploration by the student and determine the roles of the media-milieu and question-answer dialectics in this process. Before moving to the analysis, itself, we want to emphasize the fact that it was conducted a posteriori needs to be taken into account. We only had access

Fig. 1 Square-wheeled bike

to the final report of the student and therefore all the questions were formulated by the researcher from the answers found in the student's report in order to restore the chronogenesis of the exploration.

The Choice of Topic

At this stage the role of the teacher-supervisor is crucial. He needs to determine within a 5–10 min interview with the student whether the potential mathematics behind the suggested topic is at least at the level of the course on the one hand and whether it is accessible to the student in case it goes beyond the level of the course on the other hand. In other words, the teacher needs to make sure that the gap between the student's skills, his previous knowledge and the new mathematics is not too wide. In this case the student approached his teacher with a video showing a squared-wheeled bike. As shown on Fig. 2, the ATD allowed us to model this initial situation using the Herbartian scheme. We chose to refer to this initial situation using the indices zero as it leads to the generating question Q_0 and a partial answer $A^{\heartsuit}{}_{0}$. From now on we make the hypothesis that the inquiry process will be composed of several cycles of media-milieu dialectic and that each new partial answer $A^{\heartsuit}{}_{j}$ will be source of a new question Q_{j+1} and will thus initiate a new media-milieu dialectic cycle D_{j+1}.

Based on this information, the teacher had to evaluate whether or not Q_0 has enough of generating power and yet remains accessible to the student. The teacher's judgement and the student profile provided favorable circumstances in this case and the inquiry process could continue.

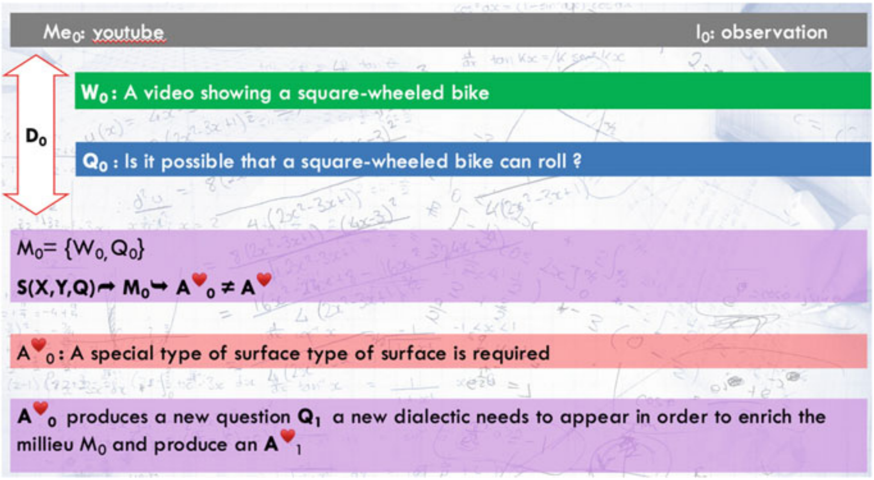


Fig. 2 Mathematical exploration—initial situation

Identifying the Ingredients of the Milieu

The bibliography refers to two sources on the Wolfram website providing information about catenaries and hyperbolic functions. Since hyperbolic functions are not part of the mathematics curriculum, the student had to search for information in these sources in order to construct the milieu. From this research, the student retains that the curve on which a square can unfold is called the catenary and he will also retrieve the hyperbolic cosine formula:

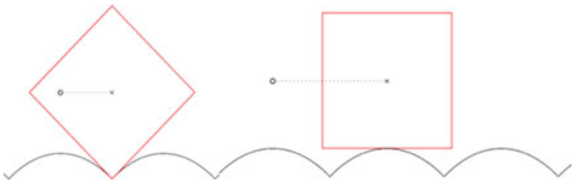
$$y = a \times cosh\ cosh\left(\frac{x}{a}\right)$$

Then the student found a GeoGeobra animation that shows the unfolding of a square on a catenary (see Fig. 3). This animation is also considered as media.

The student makes two conclusions after using this media:

(1) The square perfectly fits where two catenaries meet, which means that there has to be a 90° angle between them. (see Fig. 3)

Fig. 3 Unfolding of the square on a catenary (extract from the exploration)



(2) The side length of the square has to equal to arc length of the catenary as the square makes a complete roll.

Another important ingredient of the milieu is the previous knowledge that the student is able to mobilize in order to process the information coming for the media. Based on his previous knowledge about functions and the derivative, he makes a hypothesis that the tangents at the point of intersection of the two curves must be perpendicular with their respective slopes equal to 1 and −1. These elements lead to the construction of an experimental environment in GeoGebra which allowed the student to test and confirm his hypothesis and prepared the ground for an analytical proof.

Here is the list of the ingredients identified in the first dialectic cycle:

Dialectic D_1
Question Q_1: What type of trajectory is required for a square to roll smoothly?
Media $Me_{1.1}$: unknown source
External Answer $A^{\blacklozenge}ext_{1}$: I know that a curve on which a square can roll is called a catenary.
Question $Q_{1.1}$: What functions define catenaries?
Media $Me_{1.1.1}$: website
External Answer $A^{\blacklozenge}ext_{1.1.1}$: Catenaries are defined by hyperbolic cosine functions.
External Answer $A^{\blacklozenge}ext_{1.1.2}$: The equation is $f(x) = a \times cosh\ cosh\left(\frac{x}{a}\right)$.
External Answer $A^{\blacklozenge}ext_{1.1.3}$: a is a parameter that determines the shape of the catenary.
Previous knowledge Answer $A^{\blacklozenge}st_{1.1.1}$: The concept of a function.
Previous knowledge Answer $A^{\blacklozenge}st_{1.1.2}$: Understanding the role of a parameter
Work $W_{1.1.1}$: The graph of $f(x)$ is obtained with the slider a varying from −5 to 5
Partial Answer $A^{\heartsuit}_{1}$: The square rolls on curve defined by a hyperbolic cosine with $a < 0$

This cycle has been coded and can be easily schematized using the Herbartian scheme as shown on Fig. 4.

When looking at the second cycle, we notice that Q_2 is more specific than Q_1 and emerged from $A^{\heartsuit}_{1}$, which was considered insufficient by the didactic system. This is how the question-answer dialectic enters the scene and contributes to the chronogenesis of the inquiry process (Fig. 5).

In the experimental part of the student's exploration we identified 6 cycles of media-milieu dialectic and 3 main cycles of question-answer dialectic including some sub-cycles that resulted from searching for answers to derived questions that appeared in the main cycles. Each question-answer dialectic seems to lead to a new media-milieu dialectic that contributed progressively to the construction and enrichment of the milieu. Each dialectic allows to enrich the milieu by new external

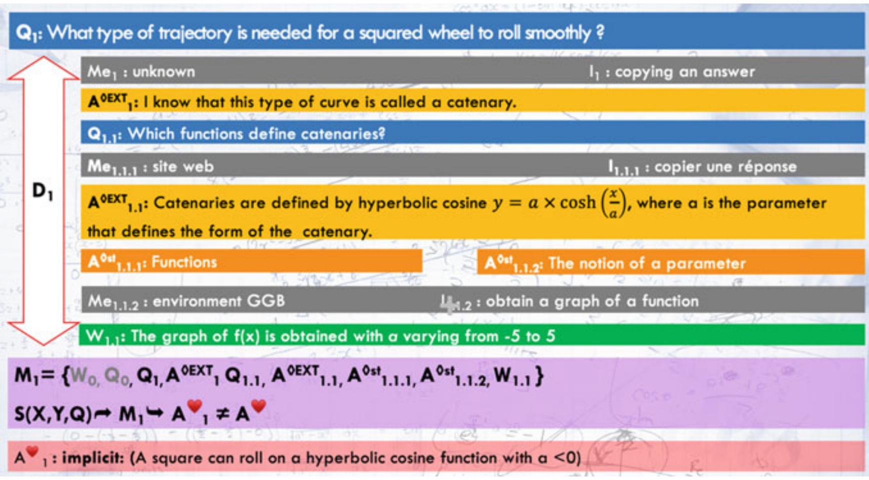


Fig. 4 Herbartian scheme for Dialectic cycle 1

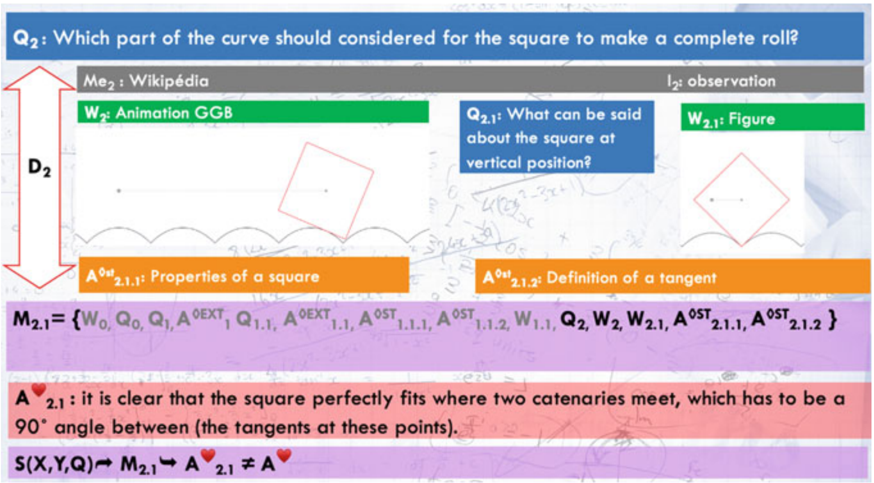


Fig. 5 Herbartian scheme for Dialectic cycle 2

answers $A^{\blacklozenge ext}$ and new works W, helps mobilize the student's previous knowledge $A^{\blacklozenge st}$ and plays the role of a bridge that connects a partial question to a partial answer $A^{\heartsuit}_{m}$. Each partial answer is then evaluated and if it the didactic system does not accept it as the final answer, it produces a new question and a new media-milieu dialectic is needed to produce a new partial answer.

Figure 6 shows the last dialectic cycle of the exploratory part of the work. The student was able to create an experimental environment in GeoGebra which enabled him to enrich the milieu by several works W and produce the answer $A^{\heartsuit}_{3}$ in form of

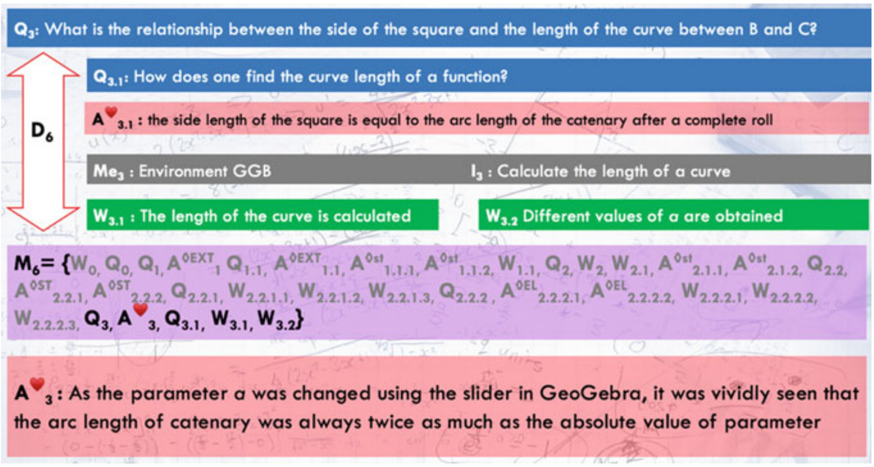


Fig. 6 Herbartian scheme for Dialectic cycle 6

a conjecture. This answer and the already rich milieu were then the departure point for the analytical solution leading to the formal proof.

Conclusion and Perspectives

This preliminary reflection and the analysis of a specific mathematical exploration show that the ATD provides powerful tools that allow to model the inquiry process within this particular setting. Within each cycle of question-answer dialectic we were able to identify a media-milieu dialectic and it seems that the quality of exploration depends on the quality of this dialectic. However, we need to take into account the main limitation of this study: the analysis was conducted a posteriori. This means that we only had access to the *success route* of the student reported in his final work which could leave the impression that the inquiry process was smooth and straightforward. It remains therefore important to gain access to all parts of the inquiry paths including the rejected media, dead ends and abandoned questions in order to further study the conditions under which the above-mentioned dialectics take place, and what determines their quality. It is nevertheless clear that the good quality of the "media-media" dialectic in the analyzed exploration played a crucial role in its success. In order to better understand and describe the conditions for a good quality dialectic to take place, more information is needed about the type of media to which students have access, how this information is treated by students, the role of prior knowledge in the process of rejecting or accepting information. In order to do so an experimental protocol has been designed in which we will follow 4 students throughout the process of the exploration. These students were given action cameras (type Go Pro) and were asked to record all their work while completing their exploration. We expect that this protocol will enable us to access and retrieve a more detailed information about the type of media and the way they are used in the inquiry process, understand the reasons why certain media are retained and others rejected, what helps to unblock certain situations and hopefully gain more insight into the mesogenesis and chronogenesis of an inquiry process.

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Pre-service Teachers' Designing of an Inquiry Task Through the Course of Study and Research Paths for Teacher Education

Tatsuya Mizoguchi and Yusuke Shinno

Purpose and Rationales of the Research

The purpose of this research is to analyse the designing process of an inquiry task by pre-service teachers.

The paradigm of questioning the world in the Anthropological Theory of the Didactic (ATD) inevitably leads to the development of the learner's ability for inquiry, and at the same time raises the issue of teacher education. A recent curriculum reform in Japan (MEXT 2017a, b) calls for changing the educational perspective from 'what to know' to 'how to learn'. However, many Japanese teachers may not be able to answer 'what is inquiry-based learning?' and 'what should it be like?'. Study and Research Paths (SRP) as a model for inquiry can be effective in the local context but conditions and constraints need to be clarified for the ecology of it. On the other hand, when focusing on teacher education, there are several problems to be elucidated: 'what kinds of difficulties do teachers experience while designing SRP?', 'what kinds of tendencies are there in their activities?' etc. In this research, we will try to address some problems mentioned above by implementing the SRP for teacher education (SRP-TE) (cf. Bosch et al. 2016) in a course for pre-service teachers in Japan.

T. Mizoguchi (✉)
Tottori University, Tottori, Japan
e-mail: mizoguci@tottori-u.ac.jp

Y. Shinno
Hiroshima University, Higashi-Hiroshima, Hiroshima, Japan
e-mail: shinno@hiroshima-u.ac.jp

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Brief Description of the Course

Three pre-service teachers—all of them university students—were engaged in this process. Two of them are majoring in middle school Social Studies, and one in high school Mathematics with special needs education. They took a course of SRP-TE titled 'Epistemology and Psychology of Mathematics Education', which consisted of fifteen sessions and was held once a week (90 min) during a semester (half a year). In the first two sessions, the lecturer (the first author) briefly explained ATD and SRP. In the third session, the lecturer posed a challenge: 'make a new initial question and its Q-A map' and the participants worked on this task in group. The rest of the sessions were conducted in the form of group presentations and discussions. The participants tried to design a new SRP and other activities based on dialectics of question–answer and media–milieu so that they would be relatively feasible to implement in an actual classroom at the high school level in Japan. As a result, the initial question finally they reached is: 'What is the water footprint (WF) of bananas imported into Japan in a year?' The Q-A map and detailed questions they posed are described below.

Theoretical Background and Research Questions

Our SRP-TE approach is part of the five general steps enumerated by Bosch et al. (2016) (more precisely, up to step3 for a priori analysis). Given that the subjects were pre-service teachers, i.e., university students, the actual didactic implementation has not yet happened. Rather, it was emphasized that they themselves experienced the type of SRP inquiry. Additionally, this helped us learn the differences they noticed between traditional learning styles and the style involved in SRP.

In this research, we therefore set up the following as the first research question: (i) *How can the ecology of SRP in Japan be characterised in the recognition of pre-service teachers from their various documentations?* We can use the scale of levels of didactic co-determinacy (Chevallard 2019) for this question. In fact, we already identified some variables for the ecology of SRP in comparative studies between Japan and Denmark (Jessen et al. 2019). The first question is on those lines and will enable us to describe the hidden consciousness of pre-service teachers. Thus, we will not discuss this first research question in this short paper.

Generally, the initial question plays a crucial role in determining SRP. It is required to produce further sub-questions, i.e., it should have the 'generative power' to lead to more knowledge. García and Ruiz-Higueras (2013) point out the following three conditions for the initial question: mathematical legitimacy, functional legitimacy, and social legitimacy. This is similar to the discussion of 'a problem' or 'a good problem' developed in traditional problem-solving learning (cf. Henderson and Pingry 1953; Krulik and Rudnick 1980).

However, as the course progressed, the pre-service teachers learnt that SRP inquiry may be different from conventional inquiry. SRP they developed was insufficient in terms of *mathematical legitimacy*—at least they felt so. Additionally, during the interview conducted after the completion of the course, they discussed encountering various water-related, i.e., environmental, issues (*social legitimacy*) and felt the possibility of further investigation (*functional legitimacy*).

Therefore, our second research question is: (ii) *How did pre-service teachers evaluate purposefully the learning of new mathematical knowledge in the course of SRP-TE and how can it be described?* In the rest of the paper, we discuss mainly the second question.

Methodology

In this paper, we approach the research question (ii) in the following manner. First, a Q-A map drawn by pre-service teachers is shown. Then, we examine each milieu designed by them in the inquiry process with the Herbartian schema: $[S(X;Y;Q) \Rightarrow \{A_m^{\Diamond}, W_n, Q_p, D_q\}] \Rightarrow A^{\heartsuit}$, that is, we come up with sub-questions and sub-answers. In other words, by investigating how each milieu networks, we will clarify what they considered (designed) about the mathematical knowledge concerned.

Results

Q-A Map by the Pre-service Teachers

The Q-A map made by pre-service teachers is shown below. However, some parts—such as the answers after the second-rank sub-questions—are omitted (Figs. 1 and 2).

Milieu-Analysis: Extracted

In the Q-A map, the direct involvement of mathematical knowledge, i.e., mathematical praxeologies, is represented by the un-shaded part in Fig. 3. Furthermore, Q111 to Q117 (excluding Q114) are sub-questions for understanding Q11. The shaded parts are related to various statistical data and the unit conversion required there. Therefore, they are not essentially mathematical praxeologies. For example, these are as follows:

Q11→Q111/M{$A^{\Diamond}$111, W111, Q1111, D111}

$A^{\Diamond}$111 Price ratio;

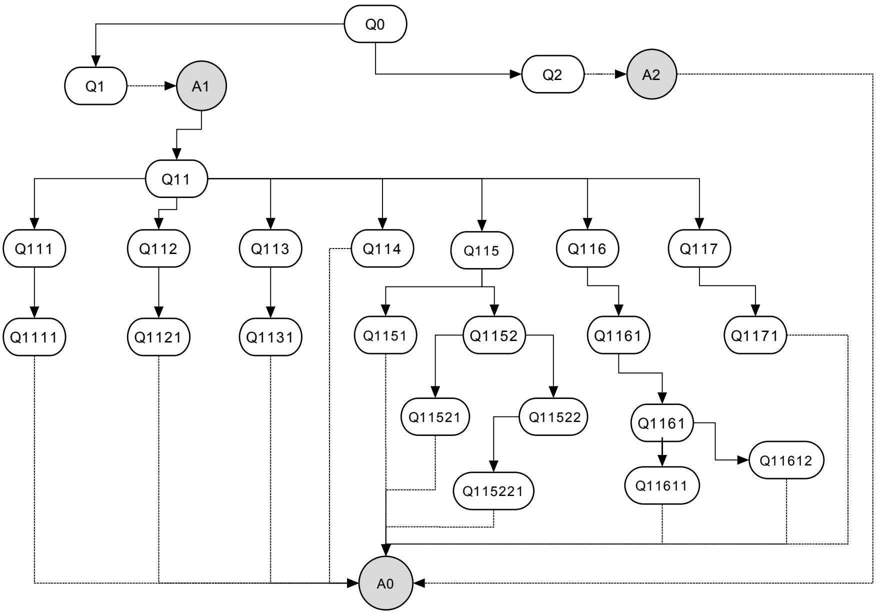

Fig. 1 Q-A map made by the pre-service teachers

Q0: What is the WF of bananas imported into Japan in a year?

Q1: What is WF?

A1: WF is the amount of water actually used in the exporting country for the imported agricultural products.

Q11: How is the WF calculated?

A11: For a certain product, the WF traded from the exporting country (e) to the importing country (i) can be obtained using the following equation;

$$WF(e,i)=\frac{pc}{r}\cdot\frac{ET(e)}{YLD(e)}TRD(e,i).$$

Q111: What is *p*?

Q1111: What is the *p* of bananas?

Q112: What is *c*?

Q1121: What is the *c* of bananas?

Q113: What is *r*?

Q1131: What is the *r* of bananas?

Q114: Japan imports bananas from which countries?

Q115: What is *ET*?

Q1151: How is *ET* calculated?

Q1152: What is *ET* of bananas in each country?

Q11521: How long does it take for bananas to grow?

Q11522: For each country, how is the transpiration amount per unit period of bananas estimated?

Q115221: How much is the amount of precipitation in each country?

Q116: What is *YLD*?

Q1161: What is the *YLD* of each country?

Q11611: What is the yield of bananas in each country?

Q116112: What is the cultivated land area of bananas in each country?

Q117: What is *TRD*?

Q1171: How much are the import amounts of bananas by country?

Q2: How much are the import amounts of bananas in Japan?

A* (A0):

$$\sum_e WF(e,Japan)=\sum_e\frac{p\cdot c}{r}\cdot\frac{ET(e)}{YLD(e)}TRD(e,Japan)=340{,}553{,}729{,}832(\text{kg})$$

Equivalent to about 945,983 of a 25m swimming pool.

Fig. 2 List of questions and answers in the Q-A map

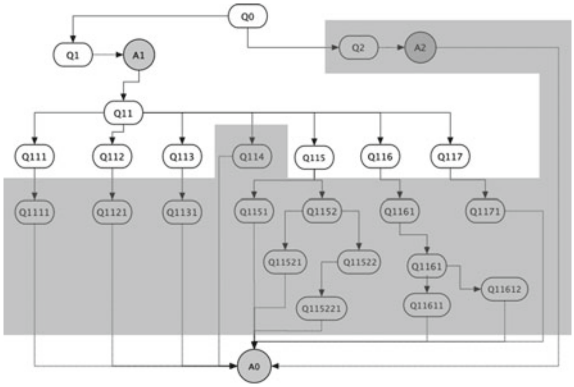

Fig. 3 Mathematical praxeologies in Q-A map

W111 Literature search (specific titles omitted);
D111 −.

$Q_{111} \to Q_{1111}/M\{A^{\diamond}{}_{1111}, W_{1111}, \_, D_{1111}\}$

$A^{\diamond}1111$ p = 1 because bananas are traded fresh;
W1111 Literature search (specific titles omitted);
D1111 →A114 (procuring data from the website, the URL is omitted).

Q116→Q1161/M{_, _, Q11611, D1161}

Q11611 (omitted);
D1161 Data shown in Table 1 is obtained from multiple websites.

Table 1 data list

Country/region	YLD (kg/ha)
Taiwan	18.928
Vietnam	16.177
Thailand	22.061
Philippines	12.765
Indonesia	50.064
Mexico	30.448
Guatemala	48.272
Costa Rica	56.813
Columbia	28.400
Ecuador	36.208
Peru	35.821
Australia	21.324

Strictly speaking, mathematical praxeology is only found in the process for Q0 (→ Q1) → Q11. That is, it requires the understanding of the mathematical formula that defines the WF (Q1 itself is about the definition of WF).

Q0→Q11/M{A$^\diamond$11, W11, Q11*n*, _}

A$^\diamond$11: The WF traded from an exporting country (e) to an importing country (i) for a certain crop is defined by the following equation:

$$WF(e,i) = \frac{p \cdot c}{r} \cdot \frac{ET(e)}{YLD(e)} TRD(e,i)$$

W11: Literature and websites search (specific titles omitted);
Q11*n*: (omitted).

Discussion and Conclusion

In order to obtain the final answer A$^\heartsuit$, the knowledge of Σ calculation is also essential: $\sum_e WF(e, Japan) = \sum_e \frac{p \cdot c}{r} \cdot \frac{ET(e)}{YLD(e)} TRD(e, Japan)$. This SRP was designed by pre-service teachers, i.e., university students. The mathematical knowledge analysed above is already known to them. Thus, they have not drawn a specific Q-A map to understand this knowledge. However, SRP was not restricted to their inquiry, but was based on a task to envision classroom implementation. Therefore, a learning of this mathematical knowledge is required, depending on which the grade is implemented. The task assigned to them during the course was very open. During the course, they focussed on environmental problems (water issues) and they set Q0. This could be because two of the three participants were aspiring social studies teachers. In fact, they state their impressions in the final report as follows (translated by authors):

- SRP will lead to the development of skills required to use data for forming one's own answers, rather than using the answers taught by teachers;
- SRP is a teaching method whose efficacy depends on the teacher's ability. In this method there is no one definite answer—it is possible for students to respond contrary to the teacher's expectations. It is, therefore, important that learners are not pushed in the direction desired by the teacher, especially by asking questions at such points where the activity stops;
- It was challenging to compose the sub-questions of Q0 because it was difficult to imagine the areas unknown to the person facing Q0. In the process of identifying Q0, the participants had acquired a lot of knowledge, which blinded them to the ignorance of other, less informed people.

In this paper, we could not sufficiently answer the research question (ii), but the following conclusion can be safely drawn: when designing SRPs in SRP-TE, it is desirable that teachers themselves have access to new mathematical knowledge.

In addition, for classroom practice, it is important to clarify the role and function of 'Y or y' in the Herbartian schema. This refers to the issue of detachment that requires the inquirers (X) to take on the role of Y. How to implement it in teacher education courses is a topic for future research.

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Research in Didactics at University Level

Deployment of *Study and Research Paths* in Mechanical Engineering

Elena Bartolomé, Ignasi Florensa, Marianna Bosch, and Josep Gascón

Abstract A Study and Research Path (SRP) is a model, grounded in the Anthropological Theory of the Didactic (ATD), which describes the inquiry processes that take place when a new, generating question (Q_0) is considered. The investigation of the question through different activities and media-milieu dialectics leads to a rooted tree sequence of derived Questions and Answers. Recently, we have investigated the potential of SRPs as project, inquiry-based, study-centered teaching formats for enriching Mechanical Engineering learning. A SRP for the subject "Strength of Materials" was designed and implemented in the last four academic years at EUSS. In every case, the six-month course derived from the study of an object-related Q_0: a "slatted-bed" (2015, 2018), or a "kart" (2016, 2017). We discuss how the choice of the Q_0, the media-milieu dialectics, adaptation to the inquiry device and inter-personal interactions influence the deployment of the SRP.

Introduction

Engineering studies need to adapt to the new challenges of industry and society. Future engineers are required to acquire not only a solid technical knowledge, but also other transversal skills, such as the ability to learn autonomously, to filter and critically validate information, the ability to adapt to technological changes, to work

E. Bartolomé (✉) · I. Florensa
Escola Universitària Salesiana de Sarrià (EUSS), Barcelona, Spain
e-mail: ebartolome@euss.es

I. Florensa
e-mail: iflorensa@euss.cat

M. Bosch
IQS School of Management, Univ. Ramón Rull, Barcelona, Spain
e-mail: marianna.bosch@iqs.url.edu

J. Gascón
Dep. Matemàtiques, Univ. Autònoma de Barcelona (UAB), Barcelona, Spain
e-mail: gascon@mat.uab.cat

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in teams, plan tasks and resolve conflicts within the group etc. This scenario makes it necessary to rethink "what, and how" to teach some subjects at university.

Study and Research Paths (SRPs), emerging from the *Anthropological Theory of the Didactic* (ATD), offer great potential as inquiry-based, study-centered teaching formats both to acquire all these competences, and systematically analyze the learning processes. The SRP begins when a new, generating question (Q_0) is considered. The search for answers through different inquiry activities, and the interaction between the student and others (*media-milieu dialectics*) leads to a rooted tree sequence of derived Questions and Answers, modelled using *Q–A maps*. As a consequence of this process, the collective constructs "own knowledge" (Chevallard 2006).

In recent years, several publications have demonstrated the great potential of SRPs to enrich the learning of different Mechanical Engineering subjects, such as "Strength of Materials" (SM) (Bartolomé et al. 2019), or "Continuous Media" (CM) (Florensa et al. 2018). An interesting issue in the frame of the ATD is to discern which common and distinct features appear in different SRPs conducted in a course. In this contribution we have tackled two main research questions:

(1) How does the deployment of a SRP depend on the choice of the generating question Q_0?
(2) For a given Q_0, how does the development of the SRP depend on the media-milieu dialectics and adaptation to the SRP device?

Methodology

For the study, we considered four SRPs conducted in *Strength of Materials*, a six-month compulsory subject of the Mechanical Engineering degree at EUSS university (Barcelona). The SRP course was designed, on the basis of *Didacting Engineering*, to overcome certain problematic *didactic phenomena* detected in the traditional course (Bartolomé et al. 2019). In every case, the course derived from the resistance study of a real, object-related Q_0: a "slatted-bed" (2015, 2018), or a competition "kart" (2016, 2017). Each SRP spanned over a semester (17 weeks, 4 h/week). The average cohort was 25 students. The SRP schedule typically followed the next cyclic structure:

1. The class considers a (new) Q_0 question. Through brainstorming and discussion, a Q–A map emerges containing different sub-questions (Q_{0i}) relevant for the study. The class is divided in different groups, each tackling a different Q_{0i}.
2. The students discuss the strategy to tackle the question, and perform different inquiry/study activities, which may include looking information (in textbooks, Internet, videos…), making calculations or simulations, experimenting, consulting experts (other teachers, more experimented students, professionals), visiting companies etc. During the work session, the teacher moves from one group to the other observing the group progress and dynamics, offering help, suggesting questions, moderating debates etc.

3. The students summarize the result of their findings into an "Output", and present it to the rest of the class orally. This is the occasion to discuss the validity of results, and propose new questions.
4. After each session, the teacher writes in a "class diary" the impressions over the march of the class, the students' implication and participation, teamwork, teacher-student's interactions etc.

The analysis of the research questions was based on the comparison of the developed Q–A maps, the observed student–teacher, student–student interactions and *media-milieu* dialectics noted in the class diary and the student's perceptions, collected in a questionnaire passed at the end of each course.

Results

Q-A Maps: Influence of the Q_0

The comparison of the Q–A maps (see e.g. Fig. 1) allowed identifying common and different features in the four lived SRPs. The main *common characteristics* which were observed to appear in all SRPs are the following:

- Both in the "slatted-bed" and the "kart" SRPs, only a certain limited number of parts could be studied under the hypothesis of the theory of SM (essentially, prismatic bars submitted to axial, shear, torsion and flexure load). These elements constitute the "core" of the subject, as traditionally considered in reference textbooks (Timoshenko 1983; Beer 2015).
- In all the SRPs, the students discovered at a certain moment the limitations of the hypothesis of the SM theory, and the needed to employ Elasticity theory and Finite Element Method (FEM) simulations to tackle the resistance and stiffness study of complex-shaped parts. The students were eager to learn FEM simulations because the project required it, even without previous knowledge on the software.
- Other times, the teacher was obliged to propose certain Q_{0i}'s to pass through compulsory issues included in the subject curriculum, which would not appear naturally from the study of the considered object (i.e. different cross-sections, and not only tubes, in the "kart" SRP). In this sense, the teacher has a large influence on the deployment and openness of the SRP, by deciding how much freedom and time the students have to investigate certain "branches" of the Q-A map. It is also noted that there is some "hysteresis" between SRPs, i.e. the Q-A evolution depends on the previous history and SRPs lived by the teacher, who may direct the SRP towards successful "branches". Moreover, sometimes the teacher felt a "theoretical pill" was needed to help the project progress. These moments arrived at certain typical points (e.g. when learning the determination of Ty, Mz diagrams, the calculation of elastic curves, and application of Navier and Colignon's laws).

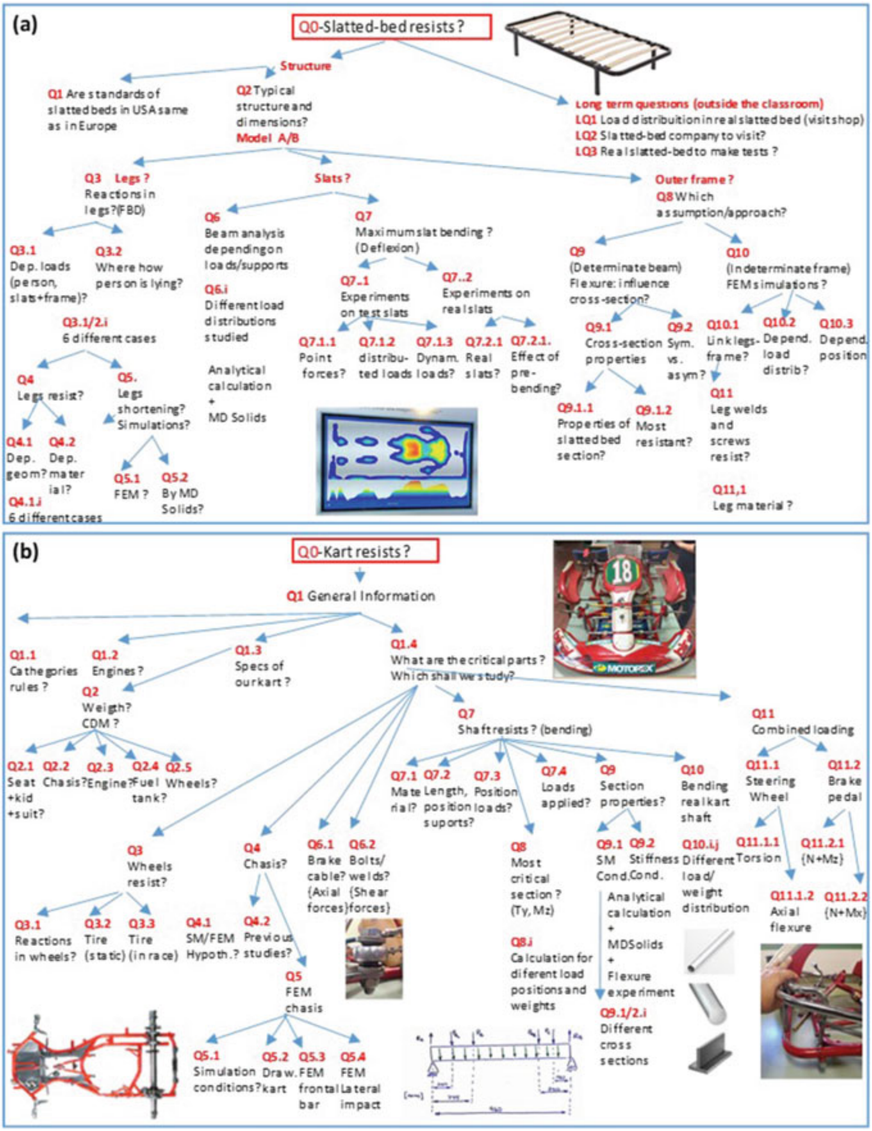

Fig. 19.1 Q-A map of SRP, based on (a) Q_0 -slatted-bed (2015), (b) Q_0-kart (2016)

Different than in traditional classes, these theoretical concepts appeared naturally, when needed in the project.

- The need to pass through certain milestones of the curriculum, and the fact that available resources (simulation software, lab set-ups) were always the same, had as consequence that certain questions appeared and were tackled similarly in all the SRPs. E.g. in every case, the study of the influence of the cross-section on bending was done by making flexure experiments of Al profiles in a universal testing machine available in the lab.

- Through the SRP, not all the groups tackled the same sub-questions. Two modes of inquiry were typically developed: (a) "*Parallel inquiry*": in which all students faced the same "conceptual question", but each group would study a different "variation on the theme"; (b) "*Distinct inquiry*": in which every group would be working on a completely different question.

On the other hand, the "kart" and "slatted-bed" SRPs presented certain *different characteristics*:

- In all the SRPs, the students spent the initial weeks looking up general information about the Q_0 and adapting to the SRP dynamics. In the case of the "kart", the time devoted to study "collateral issues" (e.g. racing rules, tires…), not directly related to SM but interesting for the students was larger than for the "slatted-bed" (the SM-questions were more "hidden" in the former).
- The "slatted-bed" Q_0 is simpler to study; it contains elements of study which can be easily studied under SM hypothesis (e.g. the slats under flexure, the legs under compression…). In order to have sufficient Q_{0i} for all the groups, the teacher has to introduce "academic variations" (e.g. studying legs with different geometries). In comparison, the "kart" Q_0 is more complex; it contains complex-shaped parts, which require CM treatment, or simplifying assumptions. In advantage, the kart includes many different elements, allowing to tackle the study of all types of loads (axial, shear, bending, torsion and combined loading), and permitting all the groups to consider different pieces.

Media-Milieu Dialectics

Several common features regarding the *media-milieu* were observed in all SRPs, briefly summarized here. Students broadly used Internet and video tutorials as source of information, in detriment of typical text books and written sources. In spite of that, many lacked efficient strategies for searching and validating information. Through the SRP, the teacher encouraged students to be critical with the information found on the Internet, and provided strategies to contrast results through comparison with different sources. Also, at the beginning of the SRP many students did not plan their search activity and directly turned to the Internet as the only possible way to find answers. As the SRP progressed, the students learned to plan and organize their search using Q–A maps, and diversified the strategies to find answers. Notably, they progressively gained confidence in experimenting and proposing imaginative solutions. Students showed large initiative to contact experts outside the classroom to obtain professional information, and also consulted teachers of subjects collaterally related to SM (such as Material Sciences, Continuous Media, CAE etc.).

Students' Adaptation to the SRP Inquiry Device

The questionnaires evidence the student's general satisfaction with the SRP in every edition. They find this project-based methodology motivating and enjoy practical work on an Engineering project. They learn presentation skills, and get fluent in English. They declare to have adapted well to an open-ended project, where the scope was not predefined, ideas were generated by the students themselves, and questions were tackled in several ways. They state to feel "learning and not only passing exams". They enjoy working collaboratively and recognize the necessity to learn this competence, although notably, most of the reported incidences in the course have to do with "problems to work in group". In every case, the success of the SRP depends upon the adaptation of the students to the new *didactic contract*. While traditionally the teacher is the only "owner and validator of knowledge", the SRP encourages the use of all possible means to find answers. At the beginning of the SRPs, all students turn to the teacher to validate their findings. However, a change of mindset is achieved, with variations between the SRPs depending on the particular class composition. This becomes particularly evident when comparing the results of SRPs staring with a similar Q_0.

Generally, the students needed only about 4 sessions to get used to the SRP dynamics. They engaged enthusiastically in the project, proposed original ideas and there was a lot of participation in the brainstorming sessions. In particular, the presence of charismatic "question posers" in the class fueled sessions, and in that case, heated debates took place. In one of the courses (kart 2016) the adaptation to the SRP was somewhat more difficult: there was less participation in the brainstorming sessions, so the teacher was forced to guide more the "branches" of the SRP, and some students continued considering the teacher as the only knowledge-owner at the end of the semester; these declared in the questionnaires to miss more "theoretical sessions".

Conclusions

SRP is an enriching format for learning Engineering subjects such as Strength of Materials: theory and practice are integrated and have a "*raison d'être*", associated to the question Q_0, specific competences, related to the analysis of prismatic elements, as well as transversal competences are worked naturally through the course, and students are motivated by this project-based, collaborative learning methodology. The comparison of the Q–A maps, surveys and class notes of four SM-SRPs implemented at EUSS have allowed us to establish which common and distinct features appear in SRPs, depending on the choice of the Q_0, the media-milieu dialectics and the adaptation of the community of study to the SRP.

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Didactic Potentials of a Task Sequence: Teaching the Notion of a Space Curve

Matija Bašić and Željka Milin Šipuš

Abstract In this paper we analyze our own practice with the focus on the noticed disparity in students' use of different representations of a curve in a space and their theoretical underpinnings given by the Implicit Function Theorem. We designed two tasks with the aim to scaffold students' learning in the way that supports linking of procedures and theory, and thus possibly leads to formation of a more comprehensive and coherent students' knowledge. Tasks that required students' use of different representations of curves in space referred to determining the tangent line of a curve given by the implicit equation(s). In the design and analysis of students' interaction with the tasks, we used the adaptation of Theory of Didactic Situations to university mathematics and analysis of features of a task regarding its adidactic, linking and deepening potential. In this paper we discuss the affordances of the designed task sequence and also observed students' difficulties related to the notion of curve in space that could be related to similar difficulties observed with linear objects - straight lines and planes in space.

Introduction

Different representations of curves and surfaces and their flexible conversions are essential for various mathematical courses that involve multivariable calculus. Many students' difficulties can be described and explained by the lack of coordination of different representations (Duval 1993). Students' difficulties appear when using the algebraic representation of straight objects in space, and vice-versa, in recognizing them from their equations. In particular, there is clear evidence of these difficulties in the conversion between parametric and implicit forms (Alves-Dias 1998; Nihoul 2016). In our recent study (Bašić and Milin Šipuš 2019) we investigated

M. Bašić (✉) · Ž. Milin Šipuš
Faculty of Science, Department of Mathematics, University of Zagreb, Zagreb, Croatia
e-mail: mbasic@math.hr

Ž. Milin Šipuš
e-mail: milin@math.hr

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students' understanding of representations of curves and surfaces in space when they were presented by their implicit equation(s) and their subsequent conversion to a representation by parametric equations.

The tangent line is a notion that students encounter at many levels of education and in different contexts. In geometry, it is mostly studied related to a circle and its properties, whereas in single-variable calculus, the tangent line of a function graph is often used as motivation for introduction of the notion of derivative. More generally, the tangent line of a parametrized curve is a line passing through a given point of a curve with a direction determined by the derivative of a curve parametrization. Many studies report on students' difficulties with this concept in single-variable calculus (e.g. Biza 2017), while many efforts are also made to support students' meaningful development of the concept (e.g. Biza 2017; Bos et al. 2019).

Mathematical Context

Implicit Function Theorem (IFT) roughly provides sufficient conditions under which a system of m equations in R^{m+n} defines a functional relationship among variables by expressing local dependence of m variables on the remaining n variables. If the equations are given as zeros of a smooth implicit function in $m + n$ variables, then the resulting vector function with m coordinate functions and n variables is also smooth, and its differential may be calculated by using the chain rule. Formulated in algebraic setting, the theorem is further used in a geometric setting in order to provide the existence of a parametric representation of an n-dimensional surface in R^{m+n} given implicitly. The IFT also provides that an n-surface is locally described as a function graph, and its parametrization is obtained by introducing variables as the parameters. In particular, for a plane curve (that is, a 1-surface) given by an implicit relationship $F(x,y) = 0$, IFT provides conditions under which the same set of points (x, y) of a curve in plane is locally described as e.g. the graph of a function $y = y(x)$. Then the local parametrization of the curve is given by $(t, y(t))$. Similarly, a space curve given by $F(x,y,z) = 0$, $G(x,y,z) = 0$, that is, as intersection of two 2-surfaces, is locally described as the graph of a vector function with coordinate functions $y = y(x)$, $z = z(x)$, with the local parametrization of the curve given by $(t, y(t), z(t))$. The same description is used for 2-surfaces in 3-space, and more generally, in abstract differential geometry, for n-submanifolds of an $(m + n)$-manifold. Further functional value of the IFT is that it provides a way how to calculate values of a differential or a directional derivative, without explicitly obtaining a parametrization, which is not always a straightforward task. However, both, the statement of IFT and its proof, seem to be overwhelming for most of the students, as it is quite often evidenced in exams and during tutorial hours. On the other hand, requirement to deal with different representations of curves and surfaces forces students to learn certain procedures without linking them to the theory that justifies them, and thus does not provide opportunity that engages students in work that could possibly lead to more complete understanding.

University mathematics education is characterized by the high speed of introduction of new objects, and requests for efficiency and economy in teaching (Artigue 2016; Kondratieva and Winsløw 2018), leaving to students' responsibility to cope with many gaps concerning taught mathematical knowledge. As instructors in the course *Introduction to differential geometry*, we focused on the often-overlooked issue concerning the representations of a curve described earlier and decided to set up a design of new tasks that could be used by students autonomously or during the exercise classes. To enhance students' learning of the topic, our goal was to design tasks with certain *adidactic, linking and deepening potential,* as developed in Gravesen et al. (2017).

Theoretical Framework and Research Questions

We base our investigation on two important tenets from the Theory of Didactic Situations (Brousseau 1997), the notion of adidacticity and, to lesser extent, the didactic contract. Adidactic potential (Hersant and Perrin-Glorian 2005) describes a characteristic of the milieu of the teaching situation, that is, a specially designed environment, to provide feedback to the students. It is called a potential, as it might not be realized or may be insufficient due to various constraints. The existence of a problem does not ensure the possibility of building a didactic situation and it is very subtle to achieve learning in an adidactic way. That relies on the mutual expectations set between the teacher and the students, collectively referred to as the didactic contract.

Hersant and Perrin-Glorian (2005) describe various didactic contracts based on four dimensions: mathematical field/domain, status of knowledge at stake, nature of the didactic situation and the distribution of responsibility. The status of knowledge is divided in three stages: completely new, in development and old knowledge. Contract may vary from being very weak (meaning that the students are mostly accountable for the learning) to very strong (where the instructor leads the situation). Adaptation of TDS for university mathematics is given by González-Martín et al. (2014), and it emphasizes increased transfer of responsibility to the students. In this adaptation, adidactic situations also include student work with exercises at times when they cannot interact directly with the teacher and the teacher cannot modify the milieu (Gravesen et al. 2017). Ideas that support this type of design are to put hints and divide the problem in smaller parts.

Along with the adidactic potential, Gravesen et al. (2017) discuss three more types of potential that a task may have: research, linking and deepening. While the research potential revolves around certain types of activities in which students mimic the work of a researcher, the deepening potential refers to the study of specific details and overall structure (deepening the knowledge in development) and the linking potential describes opportunities to relate theorems to a problem (building new knowledge on top of old). For each potential one can associate its theoretical value determined in the a priori analysis and the observed value determined in the a posteriori analysis. We also find the framework of the four potentials useful to use as design guidelines in a slightly different (but related) domain of representations of curves.

We were interested if it was possible to design a sequence of tasks with adidactic, linking and deepening potential for the topic of curves in 3D space to support students' building of more coherent understanding of the notion. We have decided not to discuss the research potential of the designed tasks as our intention was not to engage students in open inquiry. In the next section we present the tasks and the a priori analysis of the mathematical aspects of the tasks and the theoretical values of the adidactical, linking and deepening potential. We formulate the following research questions:

(1) To what extent did we observe assumed theoretical didactical potentials of the proposed task sequence in students' work? Does the observed students' work point to a purposeful design?
(2) To what extent does the students' lack of fluency in using different representations of curve in plane and space hinder students' reasoning and passage from 2 to 3D problems?

Context and Methodology

The participants in our study were six undergraduate students of mathematics education programme at the department of mathematics in Croatia, enrolled in the elective course *Introduction to differential geometry,* who volunteered in solving the prepared tasks in a tutorial meeting given by the authors. The course *Introduction to differential geometry* is given during the third year of the study programme, after the standard courses in linear algebra, and single-variable and multi-variable calculus. In academic year 2018/2019, the course was given by the authors. For the reason that teaching assistants change from year to year, authors' intention was to design tasks with certain didactical potentials that would be feasible to assistants. At the very beginning of the course, the notion of curve is recalled as given by implicit equation(s) or by parametrization, as students have already met in many examples during their previous mathematics education. In differential geometry, a curve is further usually treated as parametrized, however, for many purposes it is still given in its implicit form. As already stated, this requires either explicit conversion to parametric form, or by using IFT, implicit conversion to parametric form, since the theorem provides conditions under which the local conversion is possible.

Before the meeting with the students who volunteered in the tutorial meeting, students worked in the usual exercise class given for all students with the task that invoked use of different differentiation techniques, parametric and implicit, for curves in plane. For the tutorial meeting, students were presented with a task dealing with curves in space.

Task for the exercise class:

1. The ellipse is given by the implicit Eq. $2x^2 + y^2 = 6$.
   (a) Determine its tangent line at the point (1, 2) using implicit differentiation.
   (b) Parametrize the ellipse.
   (c) Determine the tangent line at the point (1, 2) using parametric differentiation.
   (d) Which theoretical results connects two ways of calculating the derivatives?

Task for the tutorial meeting:

2. A curve is given as the intersection of the elliptic paraboloid $x^2 + y^2 = 3\text{z}$ and the plane $4x + 4y + 3z = 1$.
   (a) Is the curve planar? Explain.
   (b) What does the Eq. $4\text{x} + 4y + x^2 + y^2 = 1$ represent?
   (c) What is the curve at stake?
   (d) Determine the tangent line to the curve at the point (−2, 1, 5/3).

At the tutorial meeting, students were also instructed to invoke the previous task for curves in the plane and different possible techniques how to solve it, as possible inspiration for them how to proceed in three dimensions. The main idea of the design of the tasks was to explicitly refer to the connection to the theoretical background of the procedures that are standardly used in this context. Hence our analysis is in particular focused on questions (1d) and (2d). In (1d) we have expected students to show understanding of the chain rule by supposing that one variable can locally be expressed in terms of the other (IFT). In the subtask (2d), if students reparametrized the curve, they were instructed to proceed by using implicit description of a curve as well. We were interested if students may find on their own or provide *the analogue* of the chain rule for functions of several variables by recognizing that two variables can locally be expressed in terms of the third variable. In this task, analogous reasoning is possible only to certain extent, and we were interested to see which elements would emerge.

Apart from this main design idea behind the tasks, by the subquestion (2b) we also addressed already observed students' difficulties in the passage from 2 to 3D (Bašić and Milin Šipuš 2019). The equation in the subquestion is obtained by simple substitution of $3z$ of the first equation into the second equation. This question addresses the following two issues: the first one, that manipulation of equations is considered as a process of solving of (non-linear) system of equations and the newly obtained equation in no matter what step is seen as a solution; and the second one, that an equation in three unknowns (variables) x, y, z but without z-variable is considered as a planar geometrical object in xy-coordinate plane, in this case as a circle. This difficulty is observed also for the case of equations without z-variable of the straight lines (Nihoul 2016). Furthermore, by subquestion (2a) we addressed this issue of identifying geometrical properties like "being planar" from the given equation which is an equation of a plane in space. This question therefore strongly underlines conversion between different representations —geometrical objects are given by and are meant to be recognized from algebraic equations.

The theoretical potentials of these two tasks were conceived as the following:

Adidactical potential could be realized in situations when students work alone or in groups, without teacher's direct instructions towards solution. Important aspect of adidacticity is that the milieu provides enough feedback to students to validate their answers. It was assumed by authors that the task is not out of the reach of students, leaving them without opportunities how to begin. We considered the task in 2D as a preparation or a hint for the richer task in 3D where students can reason by analogy. We also considered the structure of the 3D task, which consists of many smaller subquestions, as an offered scaffolding towards the solution. Besides, in the task in 3D, students might validate their answers by working using one form of a curve equation (parametric) and comparing the result to the required solution in the implicit form.

Linking potential of the tasks is embedded by the mathematical context involved, the questions require from students to reason about algebraic equations geometrically, that is, by using different representations. Moreover, the same geometrical object can be presented by two different forms of equations, implicit and parametric, and the conversion between them was under our direct examination. Therefore, once the students determine the tangent line of a curve by using its parametrization, the teaching assistant would encourage them to obtain it in "another representation as well".

Deepening potential is involved in task's requirement to deal with various specific details, while the last question (in particular 1d) and its 3D-analogue in (2d) asks for reflection and formulation of the theoretical underpinning. "Hidden" deepening potential was assumed to be found in possibility of analogous reasoning in passage from 2 to 3D for interpretation of functional dependence of variables that are given in implicit viewpoint of a curve, and in examining the existence of such a dependence.

The course of the teaching was organized as follows: for the exercise class, the task for a plane curve has been given on a separate piece of paper to each student and their solutions were gathered. The solutions and the discussion that the teaching assistants led with the students during the exercise class showed that the students struggled with the procedural part of the task; they sporadically solved the questions (1a)–(1c) on their own, and hence almost none of the students were able to answer (1d). This data has been characterized as uninformative for the study and has not been further analyzed. The teaching assistant has organized a formulation of both approaches (solutions to (1a)–(1c)) and institutionalized the link given by the chain rule.

In the subsequent tutorial meeting, students were divided in three pairs. Students were solving the second task for 90 min, during which they were very much engaged. Data was collected in the form of students' written productions. Tutors were supposed to answer questions only when students were considerably held up, in order to analyse how far students could go in possible construction of their answers. Tutors (the authors) have occasionally asked the students to explain their thinking, without leading them to ways how to solve tasks. In particular, when students would phrase some of their reasoning, tutors would ask them to write down their explanations. More than to expectations, students have very clearly expressed their difficulties and misconceptions so that the evidence of their reasoning was collected.

Data and Findings

In the tutorial meeting, three groups have worked with different pace and used different approaches. The first group struggled with the first question having a severe conflict in understanding why the equation in (2b) is not the required intersection since it is obtained by "solving the system", while on the other hand geometrically it was obvious that the intersection did not lie in the *xy*-coordinate plane (Fig. 1). Further students' interpretation of the obtained algebraic equation without *z*-variable (interpreted as a circle instead a circular cylinder) points to the difficulty in passage 2D to 3D. This students' difficulty was also observed in the previous study by the authors (Bašić and Milin Šipuš 2019).

The second group, and the third similarly, has understood the geometrical situation and managed to find the parametrization of the intersection. As planned, then they were encouraged to recall and repeat the procedure needed in the task 1 which was given in the exercise class. They have confronted difficulties in setting up the situation for implicit differentiation which would be meaningful in 3D, and finally managed to provide it (Fig. 2).

With respect to theoretical potentials of the tasks, we have observed that adidactical potential was mostly achieved, students were engaged in solving the tasks on their own, although they usually do not meet such situations in exercise classes. However, students heavily relied on what the tutors said about the correctness of the

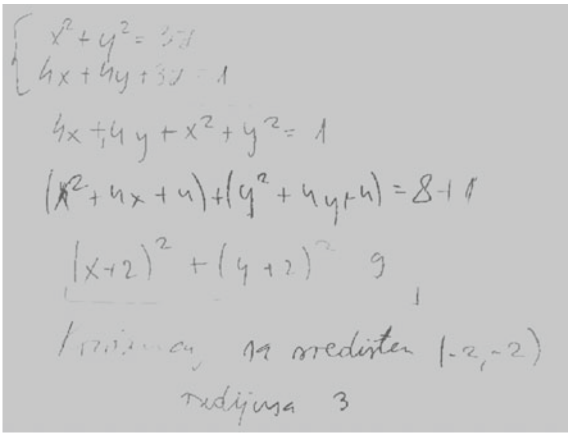

Fig. 1 Equation in the question (2b) is a required intersection and it represents a *circle*

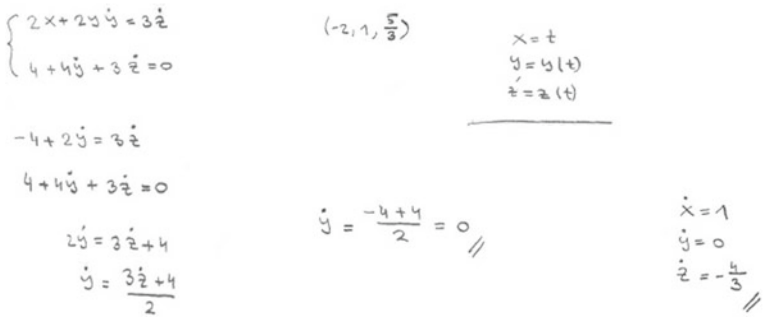

Fig. 2 Students' implicit differentiation in the task (2b)

answer. This indicates a part of the didactic contract regarding the students' responsibility for the solution, but it is also connected to more global constraints valid for the whole study programme. Moreover, there was indication that the milieu in our study did not provide enough feedback to students to validate their answers, as they did not use the possibility of correct reasoning by analogy and obtaining answers in two different ways. Realization of the linking potential is obtained from the students' answers that in the end show that they have tried to interpret connections between the two representations, algebraic and geometrical, and their different forms of equations, implicit and parametric. Hence the tasks provided the opportunity for students to engage in making conversions and to become more conscious of them. Concerning the deepening potential, students indeed used reasoning by analogy, however they struggled with the formulation of the 3D implicit differentiation and showed many difficulties in passage from 2D. The proposed task sequence provided them with an opportunity to discuss these misunderstandings. Students found challenging to formulate their reasoning, which suggests that work on the tasks supported their mathematical discourse. Since the groups used different techniques for different parts of the task, they also managed to discuss more details on different representations of the curve. The IFT did not appear in the students' explanations or conclusion. The students did not write (or speak) about the existence of the parametrization nor about the theory that justified the procedure. On the other hand, their conclusion is formulated in the form of a chain rule (Fig. 3), so we could say they used only the second part of the theorem. It might be also a consequence of the superficial mentioning of IFT in the previous course.

Conclusion and Perspectives

In this study we intended to analyze students' interaction with two tasks aimed to scaffold students' learning on the notion of a curve in space. The notion requires students' use of different representations, algebraic and geometric, and different forms of equations, parametric and implicit with their fluent conversion, either explicitly by direct

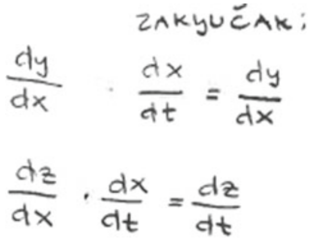

Fig. 3 Students' explanation (conclusion) of the link between the two approaches

search for e.g. parametrization, or implicitly, which is provided theoretically by the Implicit Function Theorem. We designed two tasks in which it was required to calculate the tangent line of a curve given by the implicit equation(s). For each of the tasks, certain theoretical values of the didactical potentials (adidactic, linking and deepening) were assumed. Concerning the practice, we could say that some values of the didactical potentials were observed, although not all that were theoretically assumed. Regarding students' difficulties, we also observed those related to the passage from 2 to 3D, like hasty manipulation and interpretation of newly obtained equation as a required solution, and false interpretation of equations without z-variable. Interestingly, these students' misunderstandings of curve representations did not hinder the attempts to calculate the tangent line to the extent that no student work was attempted. Additionally, which was not expected, students made extensive use of the graphical register by trying to identify surfaces given by implicit equations, sketch them, and sketch the required intersection curve. This prevented some of them to make hasty conclusions in interpretation of equations. Therefore, this research raises new questions on understanding the differences between theoretical and observed values of didactical potentials of tasks, using of the graphical register, and generally of redesigning the proposed tasks in desirable directions.

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A Reference Epistemological Model for the Case of Conics

Ricardo Nicasso Benito and Maria José Ferreira da Silva

Abstract Conics have been the subject of several pieces of research in mathematics education, with most of them covering problems that involve content teaching or learning, at the basic and higher levels of education, in addition to continuing teacher development, that focuses mainly on the analytic geometry context. Nevertheless, the development of the tools used in these researches, such as teaching sequences or textbook criteria analysis, were not build upon an epistemological study of the mathematical object that would provide a distinct view from the predominant model. This bibliographic study aims to present a reference epistemological model on conics and some of its geometries, i.e., synthetic, analytic and linear, that has sought to identify both the evolution of the construction process of this model and the insufficiency of each of them individually, in addition to the problems that allow the complementarity of the three.

Introduction

This paper presents a portion of Benito's doctoral thesis (2019) with the aim of presenting a reference epistemological model (REM) for the teaching of conics (parabola, ellipse and hyperbola) which allowed to identify mathematical praxeologies and it was used to determine the dominant model (DM), both of them used in a study and research paths for teacher education (SRP-TE), developed and experimented in the thesis.

By means of an investigation in assignments that dealt with the teaching of conics, we have found researches that confirm the fact that these curves are predominantly inserted in analytic geometry, pointing to possible consequences, both in teacher training and in the knowledge construction of elementary school students.

R. N. Benito (✉)
Federal University of Sergipe, Aracaju, Brazil
e-mail: nicasso@mat.ufs.br

M. J. F. da Silva
Pontifical Catholic University of São Paulo, São Paulo, Brazil
e-mail: zeze@pucsp.br

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Almouloud, Koné, and Sangaré (2014) observed, through an analysis in the copybooks in the third grade of the Secondary School (exact sciences and biology) of the Republic of Mali "a discrepancy between the knowledge to be taught and the knowledge actually taught about conics" (Almouloud et al. 2014, p. 3), besides that the geometric aspects are practically ignored when compared to what is taught in the analytical context. These authors analyzed two series of teaching materials, used in the Republic of Mali, and noticed that although the conics are introduced in synthetic geometry through the notion of locus, this approach is quickly replaced by algebraic study and the study of this object is limited to the study of its equation. They also claim that this replacement can cause epistemological difficulties in the construction of meaning of conics by the students, particularly for the meaning of focus, directrix, eccentricity and elements of symmetry, as with this replacement, the geometrical meanings of these elements are reduced to the nomenclatures assigned to them in the analytic geometry.

Regarding the teaching of conics in Brazil, Paques and Sebastiani Ferreira (2011) analyzed teaching materials that have been used for more than one century in Brazilian high school and they found that this was marked by a rotation between the following approaches such as: graph of a function, section of a cone, notion of locus and analytic geometry depending on the time. We note that these approaches move between analytical and synthetic geometry independently, with no connection between the two contexts and, in that direction, Gascón (2002) states that synthetic and analytic geometries must be worked in a continuous and complementary way as from synthetic geometry problems that will only be solved with the creation of analytical/Cartesian techniques. This author also clarifies that such problems can be produced by an evolution based on small changes in the statements of initial problems, that is, those that until then were solved only by means of ruler and compass.

In addition to synthetic and analytic geometry, we found conics as quadratic forms in books used in university higher education in a context that is based on analytic geometry using mathematical objects of linear algebra such as matrices, vectors, determinants, eigenvalue, eigenvector, diagonalization, etc. Geometry that deals with the conics in this context is called linear geometry.

Thus, we saw that conics are inserted, or better, they live in at least three geometries and it was in the face of this scenario, with the desire to seek solutions for teaching conics, that we focused our efforts on an epistemological study of the object that considers the problems of each of these geometries and helps us in the analysis of what is put in the current textbooks.

In the scope of the Anthropological Theory of the Didactic (ATD), this epistemological reference is currently presented as a reference epistemological model (REM), which according to Licera (2017), is not a model in the normative sense of the term, but rather a technical-experimental work tool capable of being used and expanded according to the need for research.

The function of this model, according to Barquero et al. (2013), is to lead us to know more deeply what this object of study is, for what reason it is studied and serve as a reference to analyze a dominant model (DM)—that is, a text that shows us how the mathematical object is found in books (textbooks or teaching materials) used in educational institutions. In ATD, the REM is also fundamental for building and conducting both the study and research paths (SRP) and the study and research for teacher education, providing the mathematical praxeologies that should be considered during the development of these devices. Finally, this model must be developed in such a way that, once it is finished, it can always be modified by the addition of new information to become more and more complete.

Given the above, we built a reference epistemological model on what we call "conics geometries", in which we seek to characterize them through their praxeologies, showing what role conics play in each one and in what way they can be connected so that we have a more complete mathematical praxeology proposal than those presented in current textbooks. Below, we present some elements of this REM.

A Reference Epistemological Model on Conics Geometries

This REM was built based on the work of Ruiz-Olarría (2015) and Licera (2017) and is divided into three geometry models (synthetic, analytic and linear). The explanation of the study of these models was carried out utilizing questions that made possible to notice the insufficiency of each one and the complementarity between the three, in the sense that together these models form a regional mathematical praxeology to be used in the teaching of conics in a more complete way than that found in texts and textbooks. These questions were divided into two categories: theoretical and technical. The theoretical questions guided the construction of the REM and the search for their answers made it possible to understand the role of conics in each of these geometries and how they are related. The technical questions contemplated the mathematical particularities present in each of these models and their answers helped to answer the theoretical ones. Each technical question represents a task for the praxeology in question.

We started with the synthetic geometry model, which had as a theoretical question.

> It is possible to develop a praxeology for the study of conics in synthetic geometry that favors the passage from the plane to the space context and vice versa, showing continuity and complementarity between them? What elements should be considered in this praxeology for this to occur? (Benito 2019, p. 56)

The technical questions that make up this model are found in the rectangles of Fig. 1. The arrows go through the ideas or concepts used to build the solutions and point to the next question.

According to Benito (2019), synthetic geometry can be considered.

> as an epistemological model for the conics because it allows to work with the definitions for these curves by locus and to identify its elements such as directrix, focus, axes of symmetry, eccentricity, etc., both in space and in plane. (Benito 2019, p. 56)

In particular, the last question of this model, Q_10MS, brings the Pappus's problem that, according to Roque (2012), it was during the resolution of this problem that

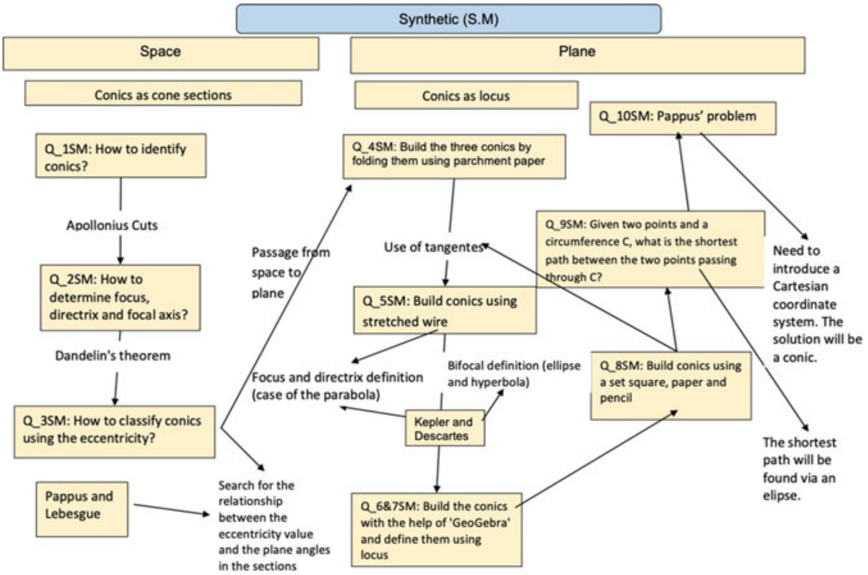

Fig. 1 Model for synthetic geometry

René Descartes noticed the need to use lines as reference axes to locate points, thus emerging the context of analytic geometry.

The theoretical question used in the analytic geometry model developed by Benito (2019, p. 80) was "what are conics in analytic geometry? What are the reasons that allow making of your study by integrating the synthetic geometry model with the analytic geometry model?".

Still according to Benito (2019, p. 79), this model "can be seen as a first algebraization of the synthetic geometry model" and conics are defined as curves associated with second-degree polynomial equations. For the author, the reason for being of this object in the analytic context is to solve problems involving sums and/or differences in distances, as is the case of the problem stated in the question Q_1MA (Fig. 2) which proposes the study of how the LORAN (Long-Range Navigation) system locates a ship from two stations emitting radio wave.

Analytic geometry also allows working with the reduced equation of conics, with the graph of quadratic functions and of functions used to relate inversely proportional quantities, but it does not make it possible to identify and justify which conic is represented by a second-degree polynomial equation in two variables of the type $ax^2 + by^2 + cxy + dx + ey + f = 0$, hence the need for linear geometry arises naturally.

In the linear geometry model the conics are treated as quadratic forms and we can work with equations using matrixes, diagonalization, determinants, characteristic equation, eigenvalue and eigenvector not only to study the conic identification given

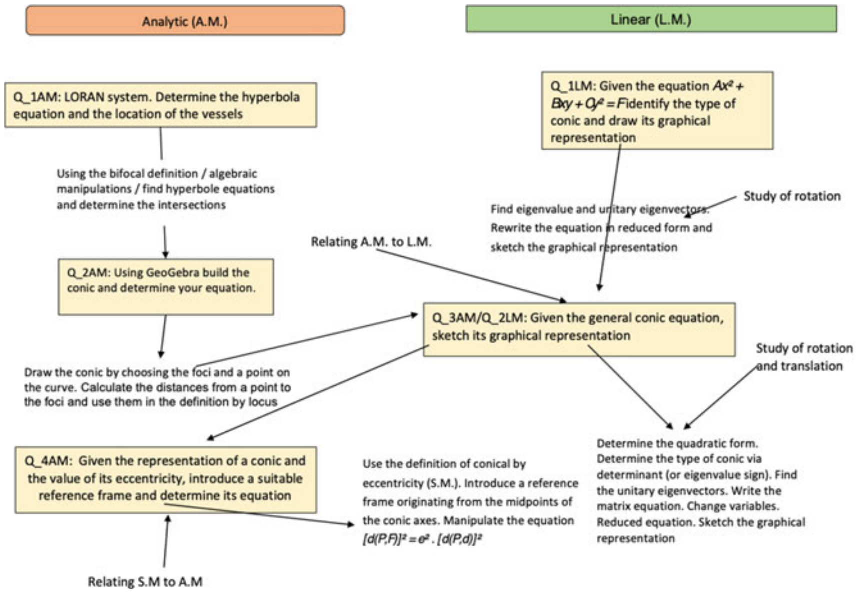

Fig. 2 Models for analytic geometry and linear geometry

by a complete quadratic equation in two variables, but also the characteristics of the matrices responsible for the rotational and translational movements of these curves.

The theoretical question that guided the discussions of this model, enunciated by Benito (2019, p. 88), was "how could we extend the complementarity between the models of synthetic geometry and analytic geometry to the model of linear geometry?".

In Fig. 2 we find in the rectangles the technical questions that compose both models of analytic geometry and linear geometry. In this figure, some arrows go through the ideas of solutions for each question and point to the next and in addition, there are also arrows that relate the questions that allowed the transition between the two models: Q_3MA/Q_2ML and Q_4MA.

An Analysis of the Dominant Model Under the Lens of the REM

With the reference epistemological model built, we sought to analyze what and how conics are presented in textbooks used in basic education and initial training courses for mathematics teachers, in the states of Sergipe and São Paulo (Brazil).

The textbook used as support material by teachers in the state education network of São Paulo is called Cadernos do Professor (Teacher's Notebook) and was used between the years 2014–2017 in all schools in the state of São Paulo. With four volumes for each year/grade—one volume for every two months[1]—this material was developed for all four years of middle school (students between 11 and 14 years old) and the three grades of high school (between 15 and 17 years old).

All 28 volumes were analyzed to verify which contexts (synthetic, analytic or linear) are worked on for the teaching of conics, which choices are made by the authors and how this content is managed.

In middle school, we found the hyperbola and the parabola in the material for the last year of this cycle (14-year-old students), both in the analytic context. In São Paulo (2014b) the hyperbola is defined as the points (x, y) of the plane that satisfy an equation of the type $xy = k$, for $x, y, k \in R$, $k \neq 0$, so that the teachers work on inversely proportional quantities. The parabola, on the other hand, appears as the graphical representation of a quadratic function of a real variable, so that it can be used in the search for the maximum or minimum points of the function. The ellipse was not found in the material used in this cycle, and there are also no discussions of conics in the contexts of synthetic geometry or linear geometry.

In high school, we found conics in the materials used for both the first year (15 years old students) and the third year (17 years old), with analytic geometry being the predominant context in both years. For the first year, the hyperbola is presented as a graph of a function, but the number of proposed tasks is insignificant when compared to the tasks for the parabola.

The context of synthetic geometry only appears in the third year, in the study of the ellipse. In that year, the three conics are introduced as sections of a cone, but this context is immediately replaced by the analytic one, with no reason to justify the introduction of a coordinate system and studies are started to obtain the equations for the three curves. The authors of this material chose to define the ellipse as a "flattened circumference", that is, "it is the curve obtained when we reduce (or enlarge) all the strings perpendicular to a given diameter in the same proportion" (São Paulo 2014a, p. 47) and its equation is determined based on that idea.

Finally, despite the predominance of the analytic context, the authors do not propose the study of conics as graphical representations of the second-degree polynomial equation in two variables of the type $ax^2 + by^2 + cxy + dx + ey + f = 0$, which would allow the introduction of linear geometry.

We also analyzed two collections of textbooks used in state schools in the state of Sergipe and found few differences concerning the São Paulo material. In middle school, the parabola is presented as a graphical representation of a quadratic function and the rest of the conics are not mentioned in any of the books analyzed for this cycle.

The novelties were on account of the volumes used in high school, in which the parabola is used as a tool for solving second-degree polynomial inequalities and there is a step by step for the construction of the three conics using cardboard, nails and a stretched wire. For these constructions, the definitions of parabola, ellipse, and hyperbola by geometrical place are explored and elements such as foci, center,

[1] In Brazil, the school year in basic education is divided into 4 periods (2 months each).

axes, and eccentricity are determined. Right after these constructions, without any discussion or motivation, the transition from synthetic to analytic geometry occurs.

As for university education, analyzing the syllabus of undergraduate courses in mathematics, we find that the conics are worked on the three models of our REM. At these universities, conics are worked on analytic geometry and linear geometry, both contexts inserted in the same disciplines, while synthetic geometry is found isolated in the syllabus of disciplines such as geometry and geometric design.

Final Considerations

Among the results, we highlight the fact that the each of the models (synthetic, analytic and linear) presented in the REM can be considered as a local mathematical praxeology and the three together could constitute a regional mathematical praxeology for the teaching of conics in Elementary education considering that each question represents a task. On the other hand, this regional mathematical praxeology must be studied in the initial formation of teachers. In addition, the option to develop the study looking at different geometries in which the conics live can be an inspiration for the construction of models for the teaching of other mathematical contents.

We believe that this strategy of using questions for the development of the REM made it possible to visualize the insufficiency of each model in producing techniques for solving questions that deal with conics in all their scope. On the other hand, this strategy made it possible to identify tasks that show a complementarity between these three geometry models.

The REM built was used as a reference for analysis of textbooks from a collection used in the state network of Sergipe and Cadernos do Professor (Teacher's Notebook) in the state of São Paulo, in addition to the syllabus of disciplines that deal with conics in undergraduate mathematics courses at some Brazilian universities. The results found show that, in basic education, conics are treated in the analytic model and with discrete approaches in the synthetic model, without any deepening that makes it possible to work with their elements. Nothing was found regarding the linear geometry model. In addition, we observed that there are few extra-mathematical discussions in the analyzed books and the reason for these curves is only mathematics, the books show no other reason to study this content other than simply learning the math presented.

Regarding the initial training of mathematics teachers, the results showed that the analytic geometry model also prevails over the other two, but most universities work with the three models, although the linear geometry model is not disconnected from the analytic context.

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Two Tasks to Teach Randomness and Probability Reasoning

Debora Coimbra

Abstract Using the Anthropological Theory of Didactics framework, we have studied how several groups of teachers implemented two tasks: a competitive two player 20-faced dice game and a collaborative coin game. The goal of the first one is, within an ensemble of a physical system, to guess a state secretly chosen from a given set, in the fewest number of measurements using a 20-sided dice. In the second one, given a set of identical coins, the group must toss the coins and select heads or tails; using only the quantity with chosen face from the previous play, toss them again and select the same kind; repeat until finishing the coins. In both, teachers obtained different results each time the experiment was carried out and the experiment cannot be reversed. When analyzing the tasks and techniques involved, we demonstrate randomness, probabilistic nature of individual measurement, to distinguish between theoretical expectation values and outcomes of measurements.

Introduction

Probability and statistics are important parts of mathematics curricula for primary and secondary school classes in many countries. To develop skills, institutional knowledge demanded within this branch is the sample space of equiprobable events, using tree of possibilities and the multiplicative principle to estimate the probability of success of one of the events. Also, to perform sample survey, interpreting measures of central tendency, solving problems and to communicate the results obtained through reports, including adequate graphical representations (Brasil 2017) are important aspects of this domain. According to Batanero et al. beyond mathematical reasoning.

> The reasons to include probability and statistics teaching have been repeatedly highlighted over the past 20 years [...] and include the usefulness of statistics and probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions (2004, p. 1).

D. Coimbra (✉)
Federal University of Uberlândia, Uberlândia, Brazil
e-mail: debora.coimbra@ufu.br

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Fig. 1 Milieu—20-sided dice or a random number generator (RNG) mobile app

In Brazil, an undergraduate degree is sufficient to become physics and mathematics teacher in basic education. More factual information about management, career and regional contexts could be find in Marcondes et al. (2017). In all five regions of country, some more than others, the professional context provides little stimulation to improvement. In last decades, master's courses in specific teaching has been a successful initiative to offer in-service teacher training certified by CAPES.[1] We have studied how several groups of in-service teachers perform two tasks and how they stablished relations with their practical knowledge.

Tossing coins and rolling dices are useful milieus to teach basic statistical contents. Considering the Anthropological Theory of Didactics (Rodriguez et al. 2008), mathematical activities consist in mobilizing praxiologies, including tasks (game implementation as a didactical situation), techniques to carry them out and the technology and theory—probability and statistical approach to solve problems, to explain and justify through formal arguments.

Rolling Dice as a Milieu

In this task, a two-player competitive game goal is, within an ensemble of a physical system, specially a quantum one, to guess a state secretly chosen from a given set, in the fewest number of measurements using a 20-sided die or a random number generator—RNG (Fig. 1). This game aims to reveal the probabilistic intrinsic nature of individual quantum measurements. In addition other aspects are relevant such ashow the statistical means of measurements approach the expectation values in the

[1] Coordination of Superior Level Staff Improvement, a foundation subordinated to Brazilian Education Department.

Table 1 Game measurement eigenstate

State	P_x (+1)	P_y (+1)	P_z (+1)
A	100% 1–20	50% 1–10	50% 1–10
B	0% –	50% 1–10	50% 1–10
C	50% 1–10	100% 1–20	50% 1–10
D	50% 1–10	0% –	50% 1–10
E	50% 1–10	50% 1–10	100% 1–20
F	50% 1–10	50% 1–10	0% –

limit of many measurements, to relate and to distinguish between the theoretical expectation values and outcomes of individual measurements, eigenstates and non-eigenstates. In exceptional cases, game may provide a model system for practicing statistical estimation techniques and as a last but equally valid objective, it is possible to have fun learning quantum mechanics (Corcovilos 2018).

One player is named Scientist and the other, Experiment. This last one chooses the secret state, conducts the measurements, and reports the results. Both gamers must use a game table (as Table 1, part of game script) showing a predetermined set of possible state Bloch vectors **s** and probabilities of measuring +1 from measuring the spin in the x, y, or z directions. The range of values under percentage means the outcome of die (+1 measurement). Any dice rolls greater than this value yields a −1 measurement.

The protocol outlined in Fig. 2 and summarized in Game Script is: Experiment chooses secretly the state from the Table 1 (obtained from Corcovilos 2018, p. 514); Scientist asks for a measurement of spin in either the x, y, or z direction; Experiment secretly rolls the 20-sided die and consults the table entry corresponding to the row of the secret state and the column of the requested measurement direction; Scientist may try to guess the secret state or may request a new measurement in any direction. If the guess is incorrect, the Scientist receives a penalty of 5 points. The Scientist may either guess again or ask for a new measurement. If the Scientist guesses correctly, the round ends and the Scientist's score is the number of measurements performed plus any penalties for bad guesses. Gamers must swap roles and win the game who makes the fewest score.

Table 2 presents class situation, organized according phases of Brousseau's Theory of Didactical Situations, adapted from Jessen and Winslow (2017). It takes about 100 min. In the devolution phase, students are divided in pairs, receiving a die and copies of Game Script. Teacher explains briefly the didactical contract, introducing the game. According Jessen and Winslow teacher and students have mutual expectations about each other's roles and responsibilities in classroom, and it is a crucial challenge not to correct or guide students in the direction of the best strategy (2017, p. 36).

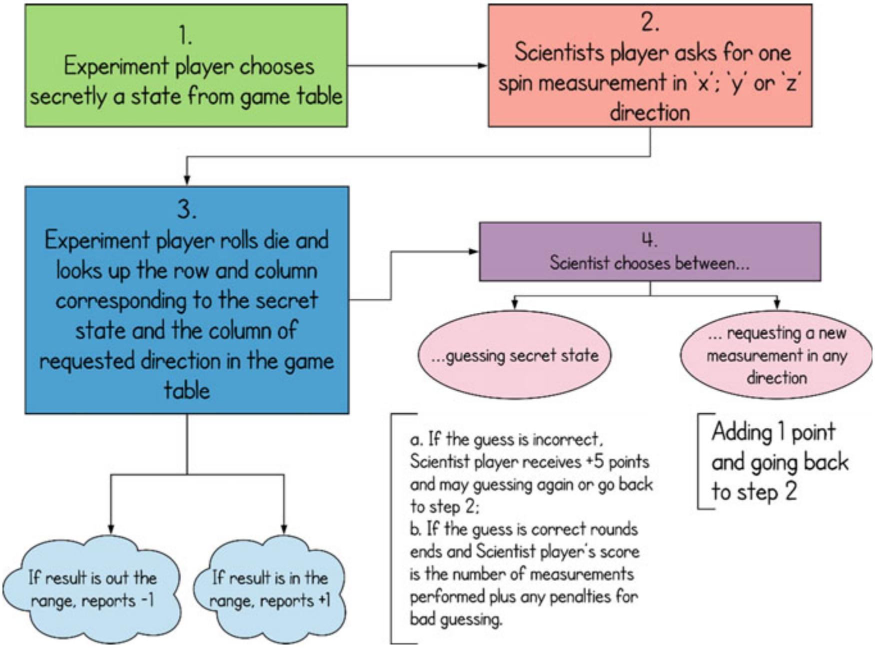

Fig. 2 Diagram representing game steps

In the action phase, students start to play, usually using a "trial and error" strategy. This phase could overlap with formulation one, because students demand help to teacher to explain or interpret again the Game Script. After, thevalidation phase consists to ask students to present their strategy for winning the game. To verbalize their actions helps students to become explicit about their somewhat imprecise ideas and hypotheses from the inquiry process. A last match could test the optimal strategy (state elimination) or it could be validated so to speak.

To approach the institutionalized knowledge, during the last phase, teacher highlights probabilistic intrinsic nature of individual measurement; distinguishes between the theoretical expectation values and the outcomes of individual measurements and provide a model for practicing statistical techniques.

Tossing Coins

To these collective activities, in-service teachers must attend: (1) How many coins do I have (Fig. 3)? (2) Toss the coins and select or heads (H) or tails (T); using only the quantity with chosen face (H or T) from the previous play, toss them again and select the same kind; store your data in a table and repeat until finish the coins. Repeating the whole procedure at least three times, the resulting frequencies and mean values must be represented in a table and graphics.

Table 2 Phases of Rolling Die didactical situation indicating actions of participants

Phase	Role of teacher	Role of students	Milieu
Devolution (5 min)	Hands over the milieu and explains briefly the rules of game using Fig. 2	Receive and try to take on the game	Die or RNG Game script
Action (20 min)	Observes and reflect	Work based on "trial and error" Progressively, realize state elimination strategy	Die or RNG Game script
Formulation (25 min)	Inquiry each student pairs, help them identify strategies	Formulate as specifically as possible strategies identified	Guided discussion
Validation (20 min)	Inquiry about the solution way to whole group; Gather ideas, organize, listen and evaluate if needed	Share individual strategies Listen, argue and consider others' arguments	Open and guided discussion
Institutiona- lisation (30 min)	Sum up main points of shared strategies and present it as an optimal one Presents and explain institutional knowledge, articulating with problem solved	Listen, associate and reflect	Institutionalised knowledge

Fig. 3 Ensemble of Brazilian coins

The first task is a counting problem and group must decide quickly the best strategy to obtain the total. In the devolution phase, the milieu is handed over to the participants. In the action phase, they engaged autonomously to implement counting procedures. It seems easy, but they need cooperate and check the final result, establishing an overlap between validation and formulation phases. The institutionalization phase consists to register the value and sum up the best strategy.

The second task is a procedure. As showed in Table 3, in the devolution phase, teacher explain the steps of this procedure and proposes a table and a graph. In the action phase, tacit rules must be negotiated to work division, as who counts heads, who counts tails andwho registers. Before the second turn, teacher asks about a comparison of obtained result with personal knowledge, their previsions. Validation phase relies on the teacher who approves or rejects the participants' accurate description of procedure implemented. In Table 4, partial results of an implementation with in-service teacher training group are presented and in Fig. 4 the average of tails measuring for about 330 coins are represented.

The institutionalized knowledge established sampling frequency, probability theory and its axiomatization. The instructor explanation must distinguish between the theoretical values and the outcomes of individual measurements; provide a model system for practicing statistical estimation techniques (Dunn 2005); highlight Markov chains. Considering learning theories, mainly Vygotsky's one, teachers could realize difficulties to perform collective work, because work division.

Table 3 Phases of tossing coins didactical situation indicating actions of participants

Phase	Role of teacher	Role of students	Milieu
Devolution	Hands over the milieu and explains briefly task procedure	Receive and try to take on	Coins
Action	Observes and reflect	Work division Counting Register Check results	Coins and Group Interlocution
Formulation	Ask about personal knowledge in comparison with obtained result	Formulate as specifically as possible strategies	Guided discussion
Validation	Inquiry about the solution way to whole group; Gather ideas, organize, listen and evaluate if needed	Share individual strategies Listen, argue and take into account others' arguments	Open and guided discussion
Institutiona- lisation	Sum up main points of shared strategies and present it as an optimal one Presents and explain institutional knowledge, articulating with problem solved	Listen, associate and reflect	Institutionalised knowledge

Table 4 An example of measurements in tossing coins procedure

Trial 1		Trial 2		Trial 3		Average	
H	T	H	T	H	T	H	T
161	169	170	160	165	165	165,33	164,66
76	93	81	89	83	82	80,00	88,00
38	38	37	44	48	35	41,00	39,00
19	19	16	21	16	32	17,00	24,00
7	12	12	4	4	12	7,66	9,33
5	2	6	6	2	2	4,33	3,33
2	3	2	4	2	0		
0	2	2	0	1	1		

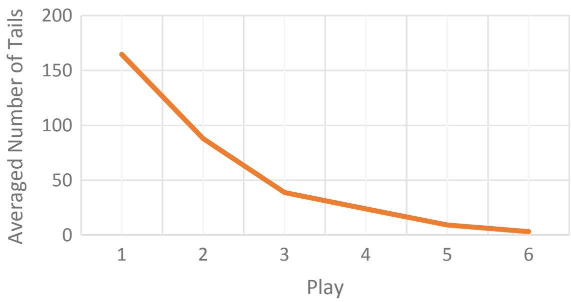

Fig. 4 Graphical representation of averaged values in tails measurement

Final Remarks

We were able to investigate randomness, probabilistic nature of individual measurement, and other concepts using inquiry based solving problems, organized according Brousseau's Theory of Didactical Situations. The local mathematical organization were unified by the technology related to solving statistical problems. Both Tossing Coins tasks were applied in groups of teachers within in-service training, from 2015 to 2018, in Federal University of Uberlândia, Brazil. The Rolling Die was validated in preservice teacher training contexts.

An important point was the epistemological reflection, which can help them to understand the role of concepts within statistics and other areas of Natural Sciences, its importance in students' learning and students' conceptual difficulties in problem solving. Comparable results were found in teacher's strategies to win the first game. Considering randomness, the first one starts out from a data table as if the experiment has already been carried out, and randomness conception would be enlarged by implementing game procedure, rolling die.

The second task emphasizes the role of experimenting in probability which leads to the conclusion that didactic training of teachers must show them how to carry out didactic analysis of content, not only in theoretical as well as procedural perspectives.

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The Problem-Based Learning Through the E-portfolio as a Methodological Basis for the Acquisition of the Transversal Competence of Solidarity and Social Commitment in Mathematics Subjects

M. I. García-Planas, J. Taberna, and S. Domínguez-García

Abstract Problem-based learning (PBL) is a learning methodology based on constructivist theories and focused on learning processes that is gaining strength at all levels of education and on which there are already experiences in several European and American universities. The PBL methodology together with the tool and portfolio have been the basis to implement the competence "sustainability and social commitment" in the subject of linear algebra in the ETSEIB UPC, in order that students understand and become aware of global issues, not only in its social, economic and environmental aspects, but also from the point of view of mathematics. Results of the experience are presented regarding students' self-perception of the competences studied and the academic results.

Introduction

The notion of sustainable development became the basis of the United Nations Conference on Environment and Development held in Rio de Janeiro in 1992. This summit marked the first international attempt to develop action plans and strategies to move towards a more sustainable development model. Specifically, at the Earth Summit in Rio de Janeiro in 1992 an appeal was made to education professionals of all levels and disciplines to help citizens acquire an adequate vision of the challenges and problems that afflict humanity and can thus participate in the necessary informed decision making (Vilches and Gil 2003).

Concretely, the sustainable development concept is defined as development that meets the needs of the current generation without compromising the ability of future generations. However, the lack of understanding and even misinterpretation

M. I. García-Planas (✉) · J. Taberna
Universitat Politècnica de Catalunya, Barcelona, Spain
e-mail: maria.isabel.garcia@upc.edu

S. Domínguez-García
Escola Els Llorers, Barcelona, Spain

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of this definition has delayed the implementation of sustainable practices within the framework of Higher Education.

It is usual to associate the term "education" with what happens in the classroom. Nevertheless, "learning" in Education for sustainable development takes into account a wide variety of social contexts. It includes what happens in the formal education system but also extends to daily and professional life (Tilbury 2011).

Universities must act as agents of change by promoting the principles of sustainable development within their institutions and in society through education and the development of competencies that will contribute to a more sustainable future (Barth and Reickman 2012).

Problem-based learning (PBL) is a learning methodology based on constructivist theories and focused on learning processes that is gaining strength at all levels of education and on which there are already experiences in several European and American universities.

The PBL methodology seeks the development of active learning through the resolution of complex situations close to reality, on which they must also assess the solution according to the context (see Domínguez et al. 2014, for more details).

The digital portfolio is becoming a basic, plural and global tool in the acquisition of transversal competences, establishing a methodology of continuous evaluation through digital rubrics incorporated into the portfolio. The e-portfolio tool, which includes evaluation rubrics, allows students to collect, select, reflectively interpret, and present the results obtained in the resolution of the proposed problem by self-evaluating during its elaboration through the rubrics. Remember that a rubric is an evaluation tool that lists the criteria and standards for evaluating work related to learning objectives. Depending on what you want to evaluate, the rubrics can be holistic in which you do not separate the parts of a job evaluating the whole of the work, or analytics that evaluate each part of an activity or a set of activities. The rubrics are applicable to all levels and educational fields, from stages of initiation to reading to the evaluation, for example, of doctoral theses or innovation projects. In particular, in university degrees the rubrics allow to evaluate both specific and generic competences, so they become a good tool for the evaluation of competence in sustainability and social commitment, see Brookhart (1999), Domínguez-García (2014) and Koh (2013), for example.

In the search for alternatives to improve the teaching and learning of linear algebra, different experiments have been designed and implemented, concluding that to improve the teaching and learning of linear algebra can be useful to include alternatives such as creation of e-portfolios, implementing projects both individually and in groups, and the use of new technologies (Domínguez et al. 2016).

The PBL methodology together with the tool and portfolio have been the basis to implement the competence "sustainability and social commitment" in the subject of linear algebra in the ETSEIB UPC, in order that students understand and become aware of global issues, not only in its social, economic and environmental aspects, but also from the point of view of mathematics.

It is well known that linear algebra is fundamental in different areas of science due to the multiple problems that can be modeled by linear systems where linear algebra becomes essential to obtain and discuss the solution. However, one of the main difficulties for first-year university students who have enrolled in scientific or technical degrees other than mathematics is that they do not see the importance of mathematics, in general, and linear algebra, in particular, may have in their fields of interest. The implementation of the competence "sustainability and social commitment" in the subject of linear algebra gives a motivation to students and working problems students understand the concepts of linear algebra and see its applicability so they themselves create the need to learn such matter.

Acquisition of Sustainability Competence Through the Mathematics Subject

The competence about sustainable development and social commitment is understood as the ability to know and understand the complexity of social and economic phenomena that are typical of the welfare state.

With the projects, it is possible to work the different levels of achievement. The first level of achievement corresponds a systematic analysis and critical global situation, considering the sustainability of interdisciplinary and sustainable human development, and recognizing the social and environmental implications of the professional activity of the same field. The other two levels of achievement consist of applying sustainability criteria and professional codes of conduct in the design and assessment of technological solutions for level two, and taking social, economic and environmental factors into account in the application of solutions, undertaking projects that tie in with human development and sustainability for level three.

Many people are convinced that Sustainable Development is completely disconnected with mathematical reasoning. However, they are very connected, and they need each other, and it is essential not only to show this relationship to the students but must work both competencies simultaneously. In this sense, we are committed to the use of the PBL methodology since this allows some guarantee that.

(a) An integrative concept of the various areas of knowledge is created.
(b) An awareness of respect for other cultures, languages and people is promoted.
(c) Empathy develops towards others.
(d) Work relationships are developed with people of various kinds.
(e) Disciplinary work is promoted.
(f) Research capacity is promoted.
(g) It provides a tool and a methodology to learn new things effectively.

When preparing the project, the questions that the teacher should ask to ensure its success are:

(1) Is it provided at the beginning, a frame of reference before the activities, which helps to locate the terms and evaluate the acquisition of competition?

(2) Does the work proposed to students induce?

- To reflection both from an environmental, social, economic, technological and temporal point of view
- To the identification of technical needs from an environmental, social and economic point of view
- To the description of the environmental, social and economic impact as a result of the decisions taken?

(3) Does the proposed work allow describing the links of the problem posed with other social realities?

If the answers are not all the positive that one expects, the work proposed should be reviewed until such positivity is achieved.

Once the project has been completed by the students, the teacher should ask himself the following question: have the expectations been met? If the response is not satisfactory enough, the proposal should be reviewed to detect the deficiency and be able to correct it for future projects.

In this work is studying how you can relate first-year college-level math with the acquisition of sustainability competencies.

Case Study

The participants in this study are the first-year students of a Bachelor's degreee in Industrial Technology Engineering at ETSEIB during the academic years: 2014–15, 2015–16, 2016–17.

With the aim of addressing the Sustainable Development Goals in the subject of linear algebra, we have prepared a series of contextualized projects that refer to the different Sustainable Development Goals for students to develop throughout of the linear algebra course.

The different project activities carried out by the students are made visible through an e-portfolio that each student creates for that purpose. The teacher follows up on this activity, feeds back and evaluates through previously established e-rubrics. The management of the tasks is done through a learning management system (LMS), the UPC has developed its own virtual environment that has been named "Atenea" using the Moodle open software platform as a technological base.

The greatest difficulty in carrying out this project is the choice of problems and their presentation. The same problem can be introduced from different points of view and here it is a matter of selecting in an appropriate way the point of view of sustainable development. Thus, for example, the study of the control of an elevator with variable load can be considered so that it follows a periodic trajectory. Thus raised is a common exercise, but if it is stated that a school must install an elevator to be accessible for all students with reduced mobility, the problem happens to have a component more to reflect from the point of view of SDGs, since it becomes important to think how long the elevator should be on each floor so that a student with reduced mobility can enter without being closed the door.

All proposed projects are based on real problems related to the specific objective of the SDGs to be addressed. Some of them have been simplified so that they can be solved with the knowledge acquired in a linear algebra course, it is clear that the statements can be readjusted based on the level of knowledge of the specific course that will be put into practice.

The list of prepared projects has been compiled in (García-Planas et al. 2016, 2017 and 2018), each of these projects is related to one of the 17 Objectives of the SDGs.

There are no results with Impact on students regard due to the withdrawal of the evaluable competence of "Sustainability and Social Commitment" of the teaching guide of the subject of Linear Algebra by the University. We are struggling to recover transversal competencies as soon as possible.

There are very positive results on sustainability during the course that the University allowed to evaluate the competence, and for the type of proposed projects it is possible to think that if they were implemented, they would also be very positive.

Example of Project Proposal

This project has been designed for a context of engineering school where a course of linear algebra is taught.

The proposal responds to the need to introduce to the linear algebra matter a competence that adds criteria and values consistent with sustainability and answer for the economic, social, cultural and environmental aspects of human development. Concretely, the topic considered is water because it is a critical resource that cannot be substituted. Having potable water is a universal human right and is also a key factor for public health. The way we maintain and expand this critical good is a fundamental problem for building an environmentally and socially sustainable world. Certainly, the organized task for PBL is precisely aligned with one of the Sustainable Development Objectives 2030, more concretely with the goal number six "Ensure access to water and sanitation for all". To achieve this aim is essential training to increase the efficiency of water resources in all sectors and ensure the sustainability of freshwater harvesting and water supply to address water scarcity by reducing the number of people suffering from water shortages.

The formulation project presented to students is as follows:

In a particular country, it is proposed to build a reservoir to regulate the basin of one of its rivers to satisfy the needs of water for irrigation.

For the realization of the project, the students are given the following data:

- Maximum reservoir capacity,
- Quantities required for irrigation.

– Volume to be left to maintain water quality standards for other uses, provided that the water level of the dam plus the weekly contribution by the water of the river, does not reach a minimum that does not allow the exit of water;

The primary objective to be achieved with this project is: "Study the viability of the reservoir" by analyzing.

1. The stability of the reservoir under the given conditions.
2. The sustainability of the reservoir under the same conditions.
3. The stability of the reservoir imposing the variation in the time of the river's contribution to climate change.
4. The sustainability of the reservoir by imposing the previous condition.

It is also intended that the following awareness-raising objectives be achieved:

1. Make an assessment of the social benefit by counterbalancing the benefits to be obtained by irrigation in the face of social conflict caused by expropriations and the resulting displacement of the inhabitants of the area.
2. Make an assessment of the problem by extrapolating the case to a large dam.

To solve the project, the student must:

1. Describe using a matrix equation of the type $p(k + 1) = Ap(k)$, the weekly transition of the probable water units.
2. Express $p(k)$ as a function of $p(0)$
3. Starting from a particular amount of water impounded:
   (i) Analyze the probability that in two weeks the reservoir will be below minimums.
   (ii) Critical assessment of the result within the context of the work.
4. Find out how Linear Algebra will give us a solution to the matrix equation proposed
5. Apply it to the case at hand
6. Use the calculations to analyze the situation of the reservoir at the week $k = 10$.
7. Study and analysis of stability and sustainability.

With this project not only introduces and evaluates the competence on sustainability but also the social commitment because they must also value the social cost that involves the displacement of people with the uprooting that this entails.

Conclusions

Sustainability is often emphasized as an essential objective of higher education, but more as a principle that at a practical level. With this work, it is shown how it is possible to carry out the implementation of sustainability competence in an evaluable form, in a subject of higher education mathematics.

Using the e-portfolio the improvement of the autonomy of the students is obtained. They have the opportunity of working on individual tasks but also in group projects, so they can also to cooperate and to work together solving the different tasks.

Is important to create the need to study mathematics to the students, in this sense, the decision to do projects related to real life has been taken, this fact motive the students to ask questions and understand that the subject they were studying had a practical application. We should not look for real problems to present as exercises in the application of mathematics, but we should ask: *What mathematics can help solve a real problem?*

The use of technological tools, the e-portfolio, implies that the professor has to prepare all the material very carefully including templates, and also a subject planning with all the structure of the course.

With the implementation of this methodology, an improvement of the qualifications of the students has been observed. What is more, students have achieved the concrete and general skills of Linear Algebra and about the competence of sustainability and social commitment.

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On a Vector Calculus Task that Led to Formation of a Praxeology: A Lucky Accident or an Implicit Fine Design?

Margo Kondratieva

Abstract Acknowledging a complex relation between different teaching practises at the university level, this paper considers an episode from a lecture based course on vector calculus, where students were successful in discovering and justifying a method for constructing an osculating plane to a given curve. While this learning situation was not explicitly designed in the first place, we reconstruct a hypothetical questions and answers sequence that could warrant an emergence of this result in similar settings.

Introduction

At the university level oftentimes a significant portion of mathematical knowledge is given to students in already refined form (e.g., theorems, proofs, algorithms) and students are expected to learn and apply them in appropriate situations. This teaching approach is contrasted with the inquiry based pedagogical methods (Artigue and Blomhøj 2013), such as problem solving and project-based learning, where students are led to discovery of mathematical ideas. There are endorsement and criticism of both teaching strategies. On the one hand, transmitted knowledge is systematic, but being memorized without much understanding by student, it is often inflexible and quick to forget. On the other hand, inquiry-based learning gives students more meaningful experience, but it requires more time and recourses; since advanced mathematical knowledge had been accumulated during centuries, it is unrealistic for students to rediscover them in a short period of one semester or so. It has been of practical interest for a long time to find a way of combining the two teaching approaches so that students can benefit from their advantages respectively.

One may start from a *micro* scale, by looking at how students can at least partly rediscover some knowledge, admitting that this process is necessary for students' proper comprehension of the transmitted material and its meaningful applications. One such case is discussed in (Kondratieva 2019) using an excerpt from a university

M. Kondratieva (✉)
Memorial University, St. John's, NL, Canada
e-mail: mkondra@mun.ca

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course where a certain method was presented to students in a lecture, but they accidentally discovered an alternative one while working on a routine exercise. To complete their experience this discovery was followed by the task to justify their actions. The new method (praxis) together with its justification given by the students (logos) signified the emergence of a new praxeology (as the term is defined in Chevallard (1999)). Note that this experience was not initiated by a specifically planned *adidactic* situation (Brousseau 1997), but rather began from an exercise where students were supposed to use given formulas. My analysis (Kondratieva 2019) using six steps of construction a praxeology (Chevallard 1999) suggested that the first two steps (first encounter and exploration of an unfamiliar situation) were missing in this case. Instead, a technique alternative to the one given in lectures was in the focus of students' inquiry. This example is informative for design of tasks that "aim directly to develop and refine knowledge in progress" (Gravesen et al. 2017) using two types of resources: media, intending to provide an information, and milieu, affording a support for further development of knowledge.

Here I extend my previous analysis using tools offered by the Anthropological Theory of Didactic (ATD) that allow revealing richer media-milieu dialectic of a study process. My research question is: Given a positive outcome (formation of a new praxeology) in what appears to be a spontaneous development, how could we start in a systematic way to design an activity with a similar result?

Theoretical Tools

ATD is a theoretical framework suitable for analysing a study process. Praxeology is one of the central concepts within ATD. A praxeology comprises a type of tasks, a technique to accomplish it, a discourse (technology) explaining this technique and a theory supporting this discourse. Various praxeologies constitute the content of mathematical courses and it is of interest how they could be taught. ATD promotes a shift from the teaching paradigm of "visiting monuments" to one of "questioning the world", where open-ended questions generate so called Study and Research Paths (SRP). Following an SRP the learners find, explore and adapt information that could be helpful for answering these questions, and as a by-product they build related praxeologies. The hypothetical process of obtaining sequences of questions and answers that result from the open (generating) question can be modelled using the question-answer (Q-A) maps (Winsløw et al. 2013). Barquero et al. (2016) point out, "on some occasions, SRPs have been wrongly identified as 'inquiry-based proposals', as if the transmission of knowledge was not related to their internal functioning". They further argue that SRPs "arise to dialectically combine 'inquiry' with 'transmission'", and demonstrate that SRPs provide "a productive framework to analyze the necessary connections between 'study' activities consisting of making available a given pieces of knowledge and 'research' activities that consist of raising questions and searching, de-constructing and re-constructing answers."

Further developing this idea, Jessen (2018) advocates for the critical role of students' ability to pose questions for knowledge acquisition. In particular, "autonomous questioning refers to situations where students pose questions to an answer they have studied or been presented. In this context, an answer can be a piece of knowledge, a technique or a theorem. Autonomous questioning does not cover questions such as "will you please repeat?" or "please, explain". The goal is students raising questions, which address a mathematical notion, technique or phenomenon, thus it generates a confirmed mathematical study process for the students posing the question."

One may distinguish two types of questions, "how" and "why"; respective answers constitute "praxis" and "logos" components of mathematical knowledge. Further both questions and answers could be custom made (given by teacher) or home made (proposed by student). In ATD, the former are marked by the sign of diamond, and the latter by the sign of heart. (These notations are used in Fig. 1 below.)

This Socratic approach allows emphasizing the epistemological dimension while analyzing the learning situation in hand. A useful ATD means to perform such an analysis is didactic engineering methodology. It consists of four phases: (1) preliminary analysis of the knowledge to be taught in a given institution, (2) design and a priori analysis of the teaching and learning situations, (3) in vivo implementation and experimentation, and (4) a posteriori analysis and validation (Barquero and Bosch 2015). An unusual treatment resides in my reliance on the data for the last two phases as described in (Kondratieva 2019), while attempting to reconstruct and make implicit the first two phases so that they would match the rest. Thus, in this paper, I first identify and formulate the didactic phenomena that need to be addressed.

Fig. 1 In vivo questions and answers map

$Q^{\diamondsuit}_{how}(0)$: How to find OP for curve $\vec{r}(t)$ at point P?
$A^{\diamondsuit}_{how}(0)$: Frenet-Serret frame $\vec{T}, \vec{N}, \vec{B} = \vec{T} \times \vec{N}$.
$A^{\diamondsuit}_{why}(0)$: By def OP $=\text{span}\{\vec{T}, \vec{N}\}$ and $\vec{B} \perp$ OP.
$Q^{\diamondsuit}_{how}(1)$: Find OP for $\vec{r}(t) = (\cos t, \sin t, t)$ at ...
$A^{\diamondsuit}_{how}(1)$: $\vec{B} = k\ \vec{r}(t)' \times \vec{r}(t)''$
$A^{\diamondsuit}_{why}(1)$: Use formulas given in the lecture.
$Q^{\diamondsuit}_{how}(2)$: Find OP for $\vec{r}(t) = (\cos t, \sin t, \cos t)$
$A^{\heartsuit}_{how}(2a)$: Notice $x(t) = z(t)$ thus $x = z$ is OP
$A^{\heartsuit}_{why}(2a)$: because our curve is planar
$A^{\heartsuit}_{how}(2b)$: the normal to OP is $\vec{r}(t)' \times \vec{r}(t)''$
$A^{\heartsuit}_{why}(2b)$: because $\vec{T} = k_1\vec{r}', \vec{N} = k_2\vec{r}''$ (mistake!)
$Q^{\diamondsuit}_{why}(3)$: Why $A^{\heartsuit}_{how}(2b)$ is equivalent to $A^{\diamondsuit}_{how}(0)$?
$A^{\heartsuit}_{why}(3)$: OP $=\text{span}\{\vec{T}, \vec{N}\}=\text{span}\{\vec{r}', \vec{r}''\}$

Second, I state possible questions that provide a motivation for the study of this mathematical topic. I follow the idea that "even if praxeologies can be considered on their own and in a decontextualized way, they always originally appear as the result of the inquiry of some questions arising in institutional settings" (Kidron et al. 2014, p. 157) and therefore a didactic system is formed around these questions, even if they are initially unseen by the participants. Considering chains of related questions and answers partly produced by students and partly offered to them by the instructor, textbook and other resources, one can describe hypothetical classroom scenarios for study of a given topic. In doing that, one addresses the issue of dis-balanced media-milieu dialectic "where students overuse a small number of media (the teacher and the textbook, for instance) without feeling the need to test the validity of the information they get from them; while other activities (like a session of practical exercises, for instance) seem not to allow access to any extra media (other students' or other people's answers)" (ibid p. 159).

The First Two Phases of Analysis

The object of our attention is a topic within a vector calculus course that, similarly to the prerequisite courses, focuses mostly on computational techniques and obtaining numerical answers. We refer the reader to Kondratieva (2019) for more details about the instructional setting and learning episodes. Looking back, we identify the didactic phenomenon that was unsatisfactory: students were explained formulas and methods in the lectures, but when working on their own they often applied them formally without any critical reflection. Sometimes alternative (more efficient) methods could be found and compared with the original ones. Students needed such experiences of flexible use of the given material as opposed to rigid memorization that largely took place. We attribute students' actions to the existing didactical contract of the instruction, that is, "specific habits of the teacher expected by the student and the behavior of the student expected by the teacher" (Brousseau 1997, p. 225), and argue for the need of changing the existing didactical contract.

For the purpose of our discussion we will concentrate on the notion of the osculating plane (OP). Specifically, we consider the task of constructing the OP at a given point for the curve $r(t) = (\cos t,\ \sin t,\ \cos t)$. The problem has three different solutions, respectively based on:

i. the idea that OP is the plane to which the curve locally belongs. Since the entire curve lies in the intersection of a cylinder and the plane $x = z$, the answer is obvious;
ii. the description of the OP in terms of the Frenet-Serret moving frame (T, N, B) of unit (tangent, normal and binormal) vectors, namely the OP at r(t) by definition is OP = span{T(t), N(t)}, and thus OP is orthogonal to B(t);
iii. the fact from physics that both the velocity v(t) and acceleration a(t) vectors lie in the OP of a curve at point r(t).

In the described case students could use the definition and calculate B(t), which is however computationally challenging. The original design aimed at finding the first solution, which is way more efficient. In the in vivo phase some students, by accident of making a mistake, discovered the third method (but without attribution of their result to the domain of physics). Then as an ad hoc task they investigated algebraic relationships between methods (ii) and (iii) proving the validity of the last one. This process is depicted in Fig. 1. Stage 0 corresponds to the question and answers given by instructor in the lecture. Stage 1 shows an example given by instructor in a tutorial. Stage 2 shows students' exercise. Answer (2a) corresponds to method (i). Answer (2b) gives a correct equation for OP, however, is based on a wrong explanation. Stage 3 is related to students' investigation and discovery. Answer (3) finally validates the alternative method.

Now we are interested in explicitly formulating more questions and possible answers that provide a motivation for establishing the connection between different methods of finding the OP. These questions are broad enough to start a discussion in which all students can try to contribute before an answer is formed. We call it "the extended Q-A sequence", which could be also reworked into a Q-A map by showing additional links between presented ideas.

Q: *How can we describe a straight line and a circle?*

A: We can either draw a picture, or write an equation, or refer to linear or circular motion. We can think of the intersection of two planes (for line) or a plane and a sphere (for circle).

Q: *How can we describe a curve in general?*

A (how): Either using physics or geometry or algebra.

A (why): Because a curve could refer to a trajectory of a moving object as well as to the underlying path (road).

Q: *How can we describe a curve geometrically?*

A: We can draw an image of a curve in 2D, make a 3D model (e.g. spiral).

Q: *How can we describe a curve algebraically?*

A: We can use (a system of) algebraic equations or parametric equations. The former option requires a choice of a system of coordinates (Cartesian, polar, etc.,) while the latter–also a choice of a parameterization.

Q: *How can we describe a curve in physics?*

A: We can use the position, and also velocity and acceleration of a moving object.

Q: *How curves are different from other geometrical objects?*

A: Traveling along a curve one has only one degree of freedom (moving forward or backward). These could be contrasted with e.g. surfaces in 3D, where one has two degrees of freedom.

Q: *How different curves can be compared?*

A: We can first look at discrete curves that consist of several linear segments attached to each other. Each linear segment has a space orientation and length. Two adjacent segments lie in a (osculating) plane in which an angle is formed. If a chain of three segments does not lie in the same plane, then another angle is formed by the adjacent osculating planes. (The measures of the angles between the adjacent segments and the adjacent planes of discrete curves motivates the notions of curvature and torsion in a more general case.) This type of description that uses the lengths and the angles is handy e.g. in Robotics in order to program a particular motion. Discrete curves are used as an approximation for more general trajectories related to the motion of cars, planes, etc.

Q: *How a curve could be parameterized*?

A: The time (of a journey along a road) could serve as a metaphor for a parameter along a curve. The idea that the same road could be passed at different speeds gives rise to different parametric descriptions of the same curve. The constant speed of 1 'unit of length' per 'unit of time' gives the special parameterization by the distance measured along the curve.

Q: *How can we connect geometric, algebraic and physical characteristics of a curve?*

A: Take for example a geometric description "curve without corners". If we zoom in at any point of such a curve we will locally see a *small arc of a circle* that measures the curvature at the point. Zooming in even further will give us a *straight line*, which is tangent to the curve at the point. Suppose the curve has a parametric description r(t). The direction of the tangent line is given by the vector r'(t) whose physical meaning is the velocity v(t) and the norm ||r'(t)|| is the speed of motion. Note that r'(t) = ||r'(t)|| T(t) and therefore the acceleration vector a(t) = r''(t) is a linear combination of unit vectors T(t) and N(t). (Here we use T' = ||T'|| N.) This establishes a connection between the acceleration and the OP.

Q: *How can we describe the osculating plane in geometry and physics?* (for a curve without corners.)

A: In geometry, the OP contains the osculating circle (of the same curvature as the curve at the point), for which T is tangent and N points towards the centre. In physics, if a moving particle is allowed to move with the constant acceleration $a(t^*)$, the trajectory remains in the OP for $t \geq t^*$. This illustrates the formally derived equation showing that the acceleration $a(t^*)$ is a linear combination of vectors $T(t^*)$ and $N(t^*)$. This fact supports the validity of the third method above, confirming that at each point of a curve the osculating plane OP = span{T, N} = span{r', r''}

By presenting this example of Q-A sequence we accomplish our goal of embedding the actual episode (Fig. 1) in a wider frame of questions and answers.

Conclusion

As it has been shows within the ATD, transmission and discovery of knowledge are processes related respectively to study and research, the two components that are vital for successful learning of mathematics at the university level. The ability to critically analyze already existing answers, notions and facts is an important step towards finding new good questions for which the answers may or may not be known. This principle is important for students' learning; however, it also applies in the case of professor teaching the course.

In his editorial Winsløw (2011), discussing the relation between teaching and research in mathematics, argues that even if "creating new mathematics is not exactly the same as finding a good situation (explanations, problems and so on) for students to learn the law of sines … it is also not something entirely different". He supports his proposition by the observation that "80% of mathematics research consists in reorganizing, reformulating, and "problematizing" mathematics that has already been "done", by the researcher himself or by others" (Brousseau 1999), and this is exactly what is required in preliminary epistemological analysis of a mathematical topic to teach. Hypothetical Q-A sequences and maps result form such instructor's preparatory activity.

In this paper we attempted at developing of a given classroom episode into a method of task design. Comparing the hypothetical possibilities of the extended Q-A sequence (leading to geometrical, algebraic and physical descriptions of a curve and its OP at a point) with what had happen in vivo we conclude that the latter is a part the former.

The in vivo episode (Fig. 1) included questions and answers at four levels: (0) lecturing, (1) tutorial, (2) exercise and (3) mini-research. It illustrates that "through working on exercises using given information and techniques [in lecture and tutorial], the groundwork for developing alternative techniques and raising new questions is established." (Kondratieva 2019) Indeed, students had found two bases in the OP as if they were partly following the extended Q-A sequence. A version of the extended Q-A sequence seems to exist in the "head of the instructor" and it was triggered by an accidental mistake made by students. This observation suggests that the process of instructional planning similar to the first two phases of didactical engineering may occur implicitly without conscious effort of experienced instructors, for whom this skill could be a part of their tacit professional knowledge. However, it is not practical to always rely on that. Making the design process explicit by all instructors as well as taking into the account "students' misconceptions or preferred strategies might even add more paths" (Jessen 2018) to the Q-A map and the a priori analysis. This would enrich the lessons and allow fuller control over the situation in the classroom, not only for the junior colleagues.

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On the Didactic Transposition of the Concept of Ideal at the Bachelor Level

Julie Candy

Abstract The concept of ideal is a central concept in abstract algebra, in particular in Ring theory. This concept involves issues regarding teaching and learning at university. This article presents the results of a crossed praxeological analysis and interview analysis to study the didactic transposition of the concept of ideal of a ring at Bachelor level in France.

Introduction

In order to study the teaching of the concept of ideal we have chosen to use the theoretical framework of didactic transposition (Chevallard 1991). Indeed, the concept of ideal is taught in several institutions in France (competitive preparation to access engineering schools, second year of Bachelor, third year of Bachelor). However, the theoretical framework of the didactic transposition allows us to model the teaching of the concept of ideal while taking into account the institutional dependencies.[1] This will lead us to answer the following question: are there different didactic transposition choices among the institutions? Moreover, the didactic transposition choices of a professor are of course always subject to constraints. However, some are clear, such as the curriculum, and others are to be identified. In this article we will focus on the study of the internal didactic transposition process by providing some answers to the question: what are the constraints and their influence on the internal didactic transposition process?

First, we will present the theoretical frameworks involved in our study. Then, we will give the methodology and a description of the institutions involved. Finally, we will provide answers to our research questions through the results of our analysis.

[1] An ideal of a ring $(A, +, \cdot)$ is a subset of A endowed with the following properties: I is a subgroup of the additive group (A,+) and if $a \in A$ and $x \in I$, then $a \cdot x \in I$.

J. Candy (✉)
High Pedagogical School of Wallis, St-Maurice, Switzerland
e-mail: julie.candy@hepvs.ch

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Theoretical Framework

Mathematical Organizations

In this ongoing study, the main framework is the anthropological theory of the didactic (Chevallard 1998). This theory assumes that every human activity may be described by quadruples [T, τ, θ, Θ], called *praxeologies* or *mathematical organizations*. In those praxeologies, [T, τ] is the praxis block (a type of tasks T and a set of techniques τ to solve T) and [θ, Θ] the logos block. The logos block includes two levels of description and justification of the praxis: the technology θ and the theory Θ.

Ecology

The focal point of the ecology of the knowledge is that the mathematical concepts does not live in isolation but are connected as in a biological ecosystem (Artaud 1997). Therefore, as in a biological ecosystem, in the didactic ecosystem it is necessary to study the interrelations between the mathematical organizations. This will notably lead us to consider the trophic needs of mathematical organizations. That is the same principle as in the food chain: a mathematical organization A may need to have a mathematical organization B built beforehand in order to build itself. For example, this is the case for the mathematical organization where the concept of minimal polynomial lives. To define the minimal polynomial it will be necessary, among other things, to use the principality of K[X]. So, the mathematical organization where the principality of K[X] lives meets the trophic needs of the mathematical organization where the minimal polynomial lives.

We chose to use the co-determination scale (Chevallard 2002) to organize our ecological model: Subject ↔ Themes ↔ Sector ↔ Domain ↔ Discipline ↔ Pedagogy ↔ School ↔ Society↔ Civilization.

Methodology

In this study, we will conduct a cross-analysis of abstract algebra teacher teaching material and their interviews.

First, the interview consisted of three parts. The first one is about the personal relationship of the teacher to the knowledge to be taught. The second one relates to the internal transposition process. The third one deals with the external transposition process and the product of this process as a constraint on teachers. An a priori analysis enabled to approve the questions of the interviews in view of the study goals. We interviewed separately abstract algebra teachers.

Secondly, the ecological analysis of the curriculum (or the syllabus) gave us an a priori ecology of the concept of ideal. As a product of the external didactic transposition process, this a priori ecology acts as a constraint on the internal didactic transposition process.

Finally, we carried out a praxeological analysis on the teaching material (course and exercises) of the teachers involved. This praxeological study enabled the building of the ecological models of the institutions involved. Thanks to a precise description of each of the praxeologies at stake (technology and theory included), we were able to model the ecosystem of the concept of ideal in the institutions by grouping via the sectors and themes of the co- determination scale.

These ecological analyses will permit to compare didactic transposition choices between the institutions. Furthermore, thanks to the cross-analysis of the interviews and the ecological analyses we will also be able to describe the constraints on the internal didactic transposition process within the institutions described in the following section.

Presentation of the Institutions

The first institution involved in this study is the CPGE. That is a two years competitive preparation to access engineering schools in France. The entry to these engineering schools is by competitive examination. In the institution CPGE, there are two separate cursus MP and MP*. MP* is the cursus where the best students go at the end of the first year to prepare for the admission to prestigious engineering schools. Regardless they come from MP or MP*, students can all take the same competitive examination. Moreover, there is one official curriculum not differentiated between MP and MP*. With this in mind, we chose to consider CPGE as a unique institution. Two teachers are part of this study: MP1 who is a MP* teacher and MP2 a MP teacher.

The second institution is the second year of Bachelor in France (L2). L2 students are not yet specialized in mathematics, they will choose in the third year. In this research, there are two teachers involved, from two separate universities. The first one, EC2, teaches in a course called “Algebra 3” and the second one, EC9, in a course called “Polynomial and linear algebra” in the autumn semester and “Endomorphism theory” in the spring semester.

In addition to the fact that these two institutions bring the concept of ideal to life in parallel during the second year, the interest of this study also lies in the fact that the students will, for some of them, find themselves all together in the third year of a Bachelor’s degree in mathematics.

Ecological Analysis

Institution CPGE

The ecological description of the curriculum of the CPGE is the following (Table 1).

First, we have to notice that the chapters of the curriculum have not the same status in the co- determination scale. Indeed the chapter "usual algebraic structures" cannot be seen as a sector in our study. This would lead to a level of granularity that is too broad and does not enable the detailed study of the choices of didactic transposition and the constraints weighing on the process of didactic transposition. Inversely the chapter "reduction of endomorphisms and square matrix" is a sector in our study. This explains the methodological choice presented above to start from the praxeological analysis to identify the sectors and themes present in our study.

In the chapter "usual algebraic structures", there is mention of the divisibility relation formulated in term of ideals, the Bézout theorem and the Gauss Lemma. Therefore, an a priori function of the concept of ideal is to be a tool for the definition of the GCD and the LCM and to prove Bézout theorem and Gauss lemma. The curriculum is contextualized in $\mathbb{Z}$ and **K**[X]. There is no mention of the concept of principality. In the chapter "endomorphisms and square matrix theory" the curriculum mentions that "the kernel of the morphism $P \rightarrow P(u)$ is an ideal" and gives the definition of the minimal polynomial of an endomorphism *u*. In this sector, an a priori function of the concept of ideal is to be a tool for the definition of the minimal polynomial.

The praxeological analysis led us to the ecological models below (Table 2).

"Principality of euclidean domains" is not a theme explicitly suggested by the curriculum. The praxeological analysis showed that MP1 develops a praxeological background around the principality of several Euclidean rings without proving that a Euclidean domain is a principal ideal domain. MP2 on the other hand focuses on the shape of the ideals of $\mathbb{Z}$ and K[X]without defining the concept of principal ideal. For this theme, MP2 is very close to the prescriptions of the curriculum.

MP1 chose to introduce all algebraic structures in a theoretical chapter "Usual algebraic structures". In the interview, MP1 explained that two factors led to this introduction. The first factor is the constraint of the economy of the didactic time: "I

Table 1 Ecological description of the CPGE curriculum

Chapter	Contents	Details
Usual structures algebraic	Ideals of domains integral	Ideal of an integral domain, the kernel of a ring morphism is an ideal, divisibility relation in an integral domain, ideals of $\mathbb{Z}$
	Polynomial rings (one indeterminate)	Ideal of K[X], gcd of two polynomials
Reduction of endomorphisms and square matrix	Polynomial of an endomorphism	Minimal polynomial of an endomorphism

Table 2 Ecological model of the CPGE institution

CPGE ecological models			
Sector	Theme	MP1	MP2
Ring theory	Ideals of integral domains	✓	✓
	Principality of Euclidean domains	✓	✓
	Divisibility relation on $\mathbb{Z}$ and K[X]	✓	✓
Endomorphisms theory	Minimal polynomial of endomorphisms	✓	✓
Arithmetic of $\mathbb{Z}$	Gcd and lcm in $\mathbb{Z}$		✓
Arithmetic of K[X]	Gcd and lcm in K[X]	✓	✓
Algebraic numbers theory	Minimal polynomial of algebraic numbers	✓	

have to introduce the theoretical notions quickly in order to apply it when I need". The second factor is the constraint of the public: "I did not make a non-traumatic introduction to the concept of ideal […] because I do not have the time and I have the public that understands". Moreover, the presence of the chapter "usual algebraic structures" in the curriculum may have reinforced this choice.

The "algebraic numbers theory" sector is present only in the MP1 course. The presence of this sector is due to the constraint imposed by other institutions. Indeed, as MP1 explained: "when you talk to a CPGE teacher, he is really constrained by the curriculum and the engineering schools competitive examinations".

We note that the ecology of mathematical organizations of MP2 does not include any sectors or themes that are not suggested by the curriculum. MP2 confirmed that the constraint of the public strongly constrains the choices of didactic transposition: "algebra, I would not have taught it the same way depending on whether I have an MP or an MP*". Moreover, the constraint imposed by other institutions is also felt. MP2 explained that he does not do general algebra exercises because at the main entrance examination to engineering school (CCP) "there are rarely any general algebra exercises".

To conclude, as expected, the curriculum is a constraint on the process of internal didactic transposition. It gives ecological injunctions that teachers follow. However, it does not prevent the development of a wider ecosystem. Rather, it is the other constraints such as the public or the competitive examinations to be prepared that push or hinder the development of the ecosystem in this institution.

Institution L2

At the University, there is no official program that would be identified for every second year of Bachelor's degree. All universities have their own syllabus. The table below presents the ecological description of the syllabus of the EC2 course: "Algebra 3" (Table 3).

Table 3 Ecological description of the syllabus of EC2

Chapter	Detail
Polynomials	Divisibility, gcd and lcm, Gauss and Bézout theorems
Endomorphisms theory	Minimal polynomial

Table 4 Ecological description of the syllabus of EC9

Chapter	Detail
Polynomials	Divisibility, ideals and gcd
Endomorphisms theory	Minimal polynomial

In the EC2 syllabus, there is no explicit mention of the concept of ideal. However, the above analysis leads us to identify a priori two functions of the concept of ideal. The first would be to introduce the GCD and the LCM and to be a tool to prove the Bézout and Gauss theorems. The second would be to introduce the notion of minimal polynomial in the endomorphism theory.

EC9 teaches in two separate courses in the second year but to the same students. The courses are "polynomial and linear algebra" (given in the first semester) and "endomorphisms theory" (given in the second semester). For this reason, we consider the two courses as permitting the construction of a single ecosystem around the concept of ideal. The table below takes in account the a priori ecological organization of the two courses (Table 4).

We notice that the external didactic transposition choices around the concept of ideal are the same between the two institutions. Bosch et al. (2019) underlined this homogeneity more generally. The difference between the two syllabi lies in the fact that the EC9 syllabus clearly mentions the concept of ideal in relation with the gcd.

The praxeological analysis led us to the ecological models below:

L2 ecological models			
Sector	Theme	EC2	EC9
Ring theory	Ideals of integral domains	✓	
	Principality of Euclidean domains	✓	
	Divisibility relation on $\mathbb{Z}$ and K[X]	✓	
Endomorphisms theory	Minimal polynomial of endomorphisms	✓	✓
Arithmetic of K[X]	Gcd and lcm in K[X]		✓

Contrary to what we assumed, EC2 did not use the concept of ideal to define gcd. He only did a reformulation of the concept of gcd in term of ideals without mention of the simplification aspect. EC9 defined the gcd via the concept of ideal. According to us, the syllabus constraint is responsible of this difference. The EC9 syllabus makes explicit reference to the concept of ideal in relation to the gcd.

In the corpus of EC2, there is only three different type of tasks (in three different themes) where the concept of ideal is not mobilized only in the theory of the praxeologies. There is one where the concept is mobilized only in the theory of the praxeology. EC9 uses the concept of ideal only to define the concept of gcd and minimal polynomial. This is the reason why, in his corpus, the concept is always mobilized in the theory of the praxeologies in tasks using gcd or minimal polynomial. Thus, in the L2 institution, the concept of ideal does not live in enough different subjects to constitute themes around this concept. The only themes in which the concept of ideal is included are those where it is part of the theory of the praxeologies. As a result, the topos of the students around this concept is strongly reduced and possibly almost non-existent. In the interviews, EC2 and EC9 explained that the appropriation of the concept of ideal is not a learning objective in their courses. The first reason for this is that the concept is present to satisfy the trophic needs of other mathematical organizations present in the syllabus. As a result, the trophic needs of other mathematical organizations act as constraints on the ecological development of concepts. In addition, the teachers pointed out the lack of time to deal with this concept more depth.

Conclusion

The table below resume the different constraints appeared in the ecological analysis led in this article (Table 5).

First of all, the economy of the didactic time is an obvious constraint imposed by the Society. Both institutions underlined this constraint as a factor of didactic transposition choices.

Then, we can notice that, in the CPGE, the constraints that weigh on the didactic transposition process are all at the level of the society. On the contrary, in the L2, the syllabus is a constraint that belongs to the School level. Indeed, the external didactic transposition process takes place within the mathematics teaching department of universities. We saw that the syllabi are very concise, this may be due to the fact that teachers are included in the external transposition processes and therefore the

Table 5 Summary of the constraints

Constraint	Institution	Level on the codetermination scale
Economy of the didactic time	CPGE, L2	Society
Public	CPGE	Society
Curriculum	CPGE	Society
Syllabus	L2	School
Competitive examination	CPGE	Society
Trophic needs of other mathematical organizations	L2	Sector

institution does not feel the need to go into more detail. It could also come from the desire to maintain a high degree of academic freedom. In any case, this very different choice from the CPGE's very detailed one leads to great differences in the way the concept is taught in both institutions. In the CPGE, the concept of ideal is developed in several sectors and themes, whereas in the L2 it is punctually developed at the subject level. Therefore, the curriculum and the syllabus play two similar role in the institutions. The teachers develop at least mathematical organisations mentioned in the curriculum but the other constraints promotes the development of other mathematical organisations.

The competitive examinations could also be the reason for the development of themes around the concept of the ideal in the CPGE. Indeed, the teachers preparing their students for should allow them to manipulate, certain concepts, during the competitive examinations dealing with a specific theme of algebra. For example, in the case of a competitive examinations subject on the reduction of endomorphisms, students could be led to redefine the minimal polynomial. Teachers must therefore ensure that this reconstruction will be possible. Whereas in L2, if the learning objective is not focused on the concept of ideal then this reconstruction will never be asked to the students. Moreover, we saw that competitive examination can promote the

development of mathematical organisation not present in the curriculum as for example the one dealing with algebraic numbers.

Regarding the constraint of the trophic needs, we can explain that it appears only in the L2. As we saw, in this institution the product of the external didactic transposition process is reduce, teachers themselves must rebuild the mathematical organizations they need to achieve their learning objectives. At the contrary, in the CPGE curriculum, the *noosphère* already took at her charge a pre-construction of the mathematical organisations. For example, we saw that MP2 was able to construct his course only with the mathematical organizations suggested in the curriculum.

To conclude, the concept of ideal in CPGE and L2 plays the same role: it is a tool for introducing new concepts. It therefore satisfies the trophic needs of mathematical organizations. However, the different constraints of the different institutions have led to different ecological developments. Yet, these students may find themselves together in the third year of the Bachelor's degree (L3), so it will be important to study how the L3 manage this transition with audiences with different ecological and praxeological backgrounds.

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