Spontaneous Symmetry Breaking in Nonlinear Dynamic Systems

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1 Introduction

The concept of symmetry dominates modern fundamental physics, both in quantum theory and in relativity. While symmetry plays a crucial role in modern physics, its dual concept "symmetry breaking" is also very important. Spontaneous Symmetry Breaking (SSB), in contrast to explicit symmetry breaking, is a spontaneous process through which a system governed by a symmetrical dynamic ends up in an asymmetrical state. It thus describes systems where the equations of motion or the Lagrangian function obey certain symmetries, but their lowest energy solutions do not exhibit that symmetry. So, the symmetry of the equations is not reflected by individual solutions, but it is reflected by the symmetrical coexistence of asymmetrical solutions. An actual measurement reflects only one solution, representing a breakdown in the symmetry of the underlying theory. "Hidden" is perhaps a better term than "broken" because the symmetry is always there in these equations.

2 SSB in Quantum Field Theory

Without spontaneous symmetry breaking (SSB), the local gauge principle requires the existence of a number of bosons as force carriers. However, some particles (the so-called W and Z bosons) would then be predicted to be massless. While in reality, they are observed to have mass. To overcome this conflict, spontaneous symmetry breaking is augmented by the Higgs mechanism to give these particles masses. It also suggests the presence of a new particle, the Higgs boson, reported as possibly

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identifiable with a boson detected in 2012. SSB occurs whenever a given field in a given Lagrangian has a nonzero vacuum expectation value. The Lagrangian appears symmetric under a symmetry group, but after randomly selecting a vacuum state, the system no longer behaves symmetrically.

2.1 Real Scalar Field Example

Consider the scalar Lagrangian given by

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \varphi)^2}_{\text{“kinetic term”}} + \underbrace{\frac{1}{2}\mu^2\varphi^2 - \frac{\lambda}{4!}\varphi^4}_{\text{"potential term",}} \tag{1}$$

where φ is the scalar field, μ is a sort of “mass” parameter, and λ is the coupling. Observe that there is a symmetry of $\varphi \to -\varphi$ (a discrete symmetry). One can think of the potential as being

$$V(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4!}\varphi^4, \tag{2}$$

which has extremes when its derivative is zero. Actually, there are two, given by

$$\varphi_0 = \pm v = \pm\mu\sqrt{\frac{6}{\lambda}}, \tag{3}$$

where the constant v is the “*vacuum expectation value*”.

The vacuum, that is, the lowest-energy state, is described by a randomly chosen point of these two new extremes. One can then write

$$\varphi(x) = v + \sigma(x), \tag{4}$$

and then rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4, \tag{5}$$

where the constant terms have been dropped. It turns out that the symmetry $\varphi \to -\varphi$ is no longer identifiable.

2.2 Complex Scalar Field Example

Consider a complex scalar boson φ and $\varphi^\dagger$. The Lagrangian is

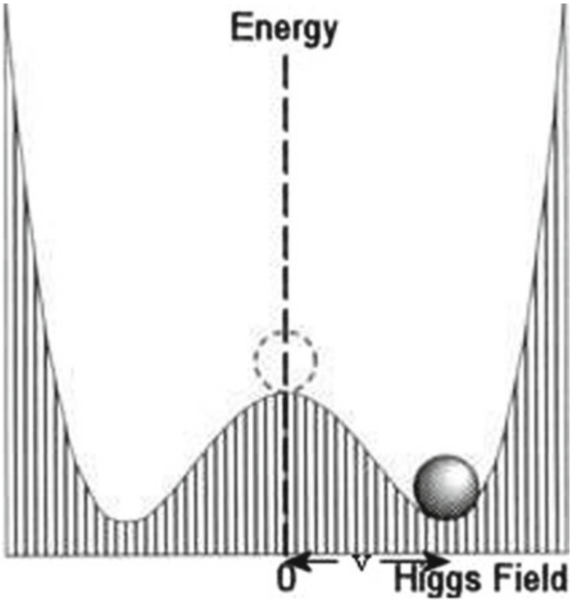

Fig. 1 Illustration of SSB in a real scalar field; figure from stick

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\varphi^{\dagger}\partial_{\mu}\varphi - \frac{1}{2}m^2\varphi^{\dagger}\varphi\,. \tag{6}$$

Naturally, one can write this as

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\varphi^{\dagger}\partial_{\mu}\varphi - V(\varphi^{\dagger},\varphi), \tag{7}$$

where

$$V(\varphi^{\dagger},\varphi) = \frac{1}{2}m^2\varphi^{\dagger}\varphi\,. \tag{8}$$

This Lagrangian has the $U(1)$ symmetry, which means that it is invariant under the transform: $\varphi \mapsto e^{i\theta}\varphi$.

In graph $V(\varphi^{\dagger},\varphi)$, plotting V vs. $|\varphi|$, one can see a "bowl" with $V_{minimum}$ at $|\varphi|^2 = 0$. The vacuum of any theory ends up being at the lowest potential point, and therefore the vacuum of this theory is at $\varphi = 0$, as one would expect.

Now, change the potential and consider

$$V(\varphi^{\dagger},\varphi) = \frac{1}{2}\lambda m^2(\varphi^{\dagger}\varphi - \Phi^2)^2, \tag{9}$$

where λ and Φ are real constants. Notice that the Lagrangian will still have the global $U(1)$ symmetry as before. But, by graphing V vs. $|\varphi|$, one gets Fig. 2.

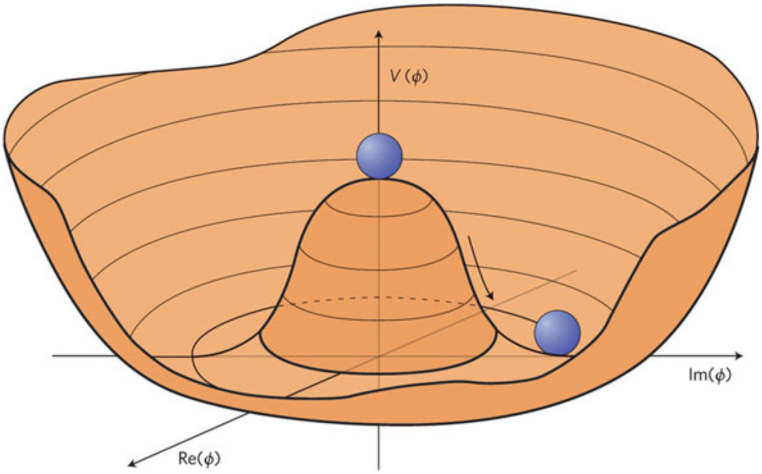

Fig. 2 Mexican hat potential function V versus Φ; figure from Eyes on a prize particle, Nature Physics 7, 2C3 (2011)

In Fig. 2, now, the vacuum $V_{minimum}$ is represented by the circle at $|\varphi| = \Phi$. In other words, there are an infinite number of vacuums in this theory. And, because the circle drawn in this figure represents a rotation through a field space, this degenerate vacuum is parameterized by $e^{i\alpha}$, the global $U(1)$. There will be a vacuum for every value of α, located at $|\varphi| = \Phi$.

In order to make sense of this theory, one must *choose* a vacuum by hand. Because the theory is completely invariant under the choice of the $U(1)$, $e^{i\alpha}$, one can choose any α and *define* that as the true vacuum. So, here, choose α to make the vacuum at $\varphi = \Phi$.

Next, it is needed to rewrite this theory in terms of the new vacuum. One can therefore expand around the constant vacuum value Φ to have a new field, as

$$\varphi \equiv \Phi + \alpha + i\beta\,, \tag{10}$$

where α and β are new real scalar fields (so, $\varphi^\dagger = \Phi + \alpha - i\beta$). One can now write out the Lagrangian as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial^\mu[\alpha - i\beta]\partial_\mu[\alpha + i\beta] - \frac{1}{2}\lambda m^2[(\Phi + \alpha - i\beta)(\Phi + \alpha + i\beta) - \Phi^2]^2 \\ &= \left[-\frac{1}{2}\partial^\mu\alpha\partial_\mu\alpha - \frac{1}{2}4\lambda m^2\Phi^2\alpha^2 - \frac{1}{2}\partial^\mu\beta\partial_\mu\beta \right] \\ &\quad -\frac{1}{2}\lambda m^2\left[4\Phi\alpha^3 + 4\Phi\alpha\beta^2 + \alpha^4 + \alpha^2\beta^2 + \beta^4\right].\end{aligned}$$

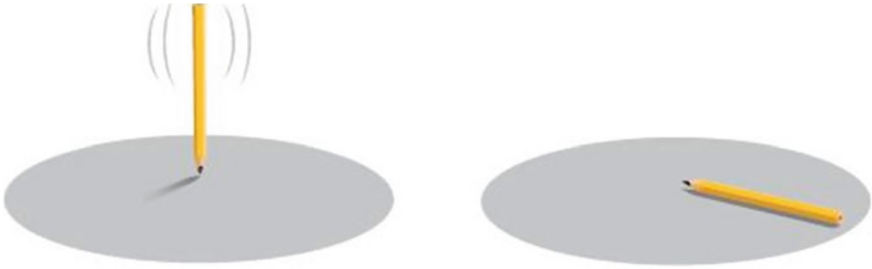

Fig. 3 Pencil illustration of SSB: Spontaneous broken symmetry. The world of this pencil is completely symmetrical. All directions are exactly equal. But this symmetry is lost when the pencil falls over. Now one direction holds. The symmetry that existed before is hidden the fallen pencil; figure from https://web.hallym.ac.kr/~physics/course/a2u/ep/ssb.htm

This is now a theory of a *massive* real scalar field α (with mass = $\sqrt{4\lambda m^2\Phi^2}$), a *massless* real scalar field β, and five different types of interactions (one allowing three α's to interact, the second allowing one α and two β's, the third allowing four α's, the fourth allowing two α's and two β's, and the last allowing four β's to do so.) In other words, there are five different types of vertices allowed in the Feynman diagrams for this theory. Thus, the $U(1)$ symmetry is no longer manifest.

3 Other Simple Illustrations

3.1 Pencil

It is an amazing idea that the symmetry of the equations could not be reflected by individual solutions, but it is reflected by the symmetrically coexistence of asymmetrical solutions.

First, consider a pencil, standing upright at the center, which is possible but unstable. The pencil tends to fall into one stable state randomly. After it falls down, the original symmetry was broken (see Fig. 3). This is exactly the illustration of the "Mexican hat" potential of a complex field, shown in Fig. 2.

3.2 Stick

Consider another daily-life example: A linear vertical stick with a compression force applied at the top and directed along its axis. The physical description is obviously invariant for all rotations around this axis. As long as the applied force is mild enough, the stick does not bend, and the equilibrium configuration (the lowest energy con-

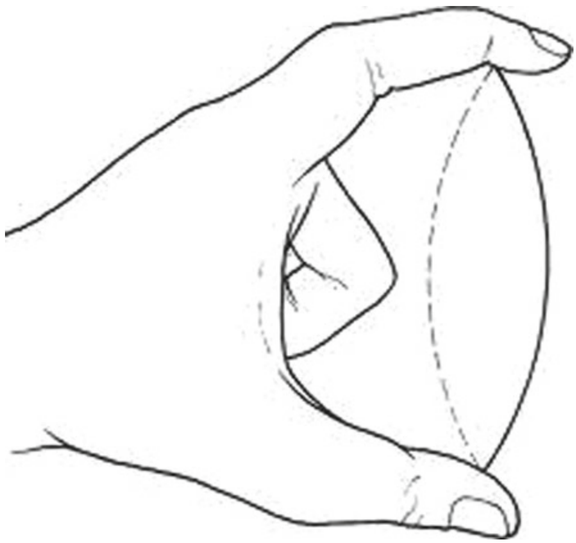

Fig. 4 Stick illustration; figure from https://web.hallym.ac.kr/~physics/course/a2u/ep/ssb.htm

figuration) is invariant under this symmetry. When the force reaches a critical value, the symmetric equilibrium configuration becomes unstable, and an infinite number of equivalent lowest energy stable states appear, which are no longer rotationally symmetric. The actual breaking of the symmetry may then easily occur due to the effect of a (however small) external asymmetric cause, and the stick bends until it reaches one of the infinite possible stable asymmetric equilibrium configurations (see Fig. 4).

3.3 Shortest Connecting Path

The above two examples are essentially the same, where all the new lowest energy solutions are asymmetric but are all related through the action of the symmetry transformation $U(1)$.

Here is yet another elegant example.

Consider the scenario of connecting four nodes located symmetrically at four vertices of a square. A simple solution one can draw by hand at once is like the one shown in Fig. 5.

This solution fully maintains the original symmetry of a foursquare, and, in fact, it is a very satisfactory solution with a total length of $2\sqrt{2} \doteq 2.828$.

However, the best solution turns out to be the one shown in Fig. 6. The total length is $1 + \sqrt{3} \doteq 2.732$, and the symmetry of the solution is less than that of the original system.

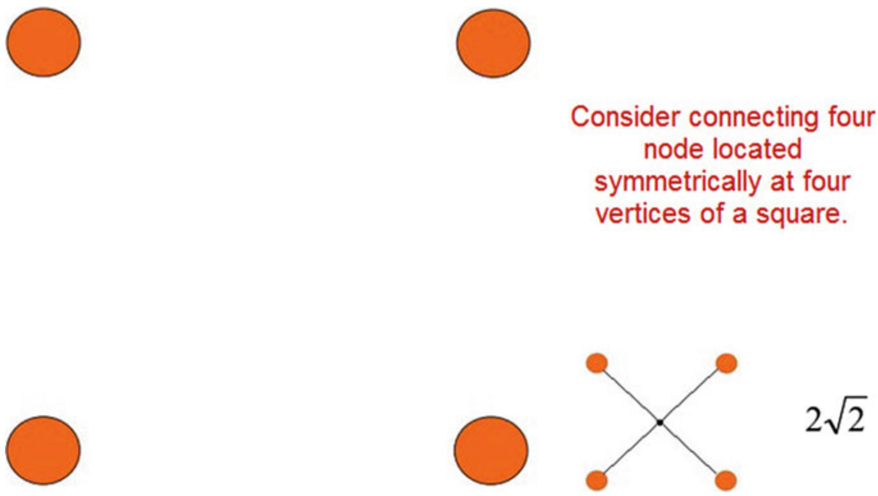

Fig. 5 Connecting four nodes located symmetrically at four vertices of a square

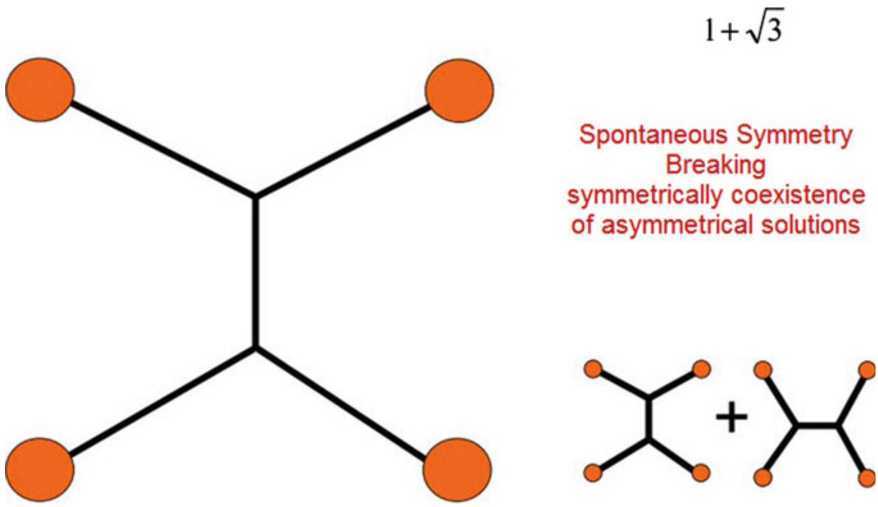

Fig. 6 Spontaneous symmetry breaking

So, the symmetry of the original problem is reflected by the symmetrical coexistence of asymmetrical solutions, each of which has less (or even no) symmetry than the original full symmetry.

4 SSB in Nonlinear Dynamic Systems

Perhaps, the best illustrations could be given by attractor(s) of nonlinear dynamic systems. Here, consider only three-dimensional autonomous systems with two quadratic terms, in a form similar to the Lorenz system.

4.1 Symmetrical System with Symmetrical Attractor

The symmetry of the algebraic equations determines the symmetry of its geometric dynamics. In fact, symmetry plays an important role in generating chaos, which somehow determines the possible shape of the resulting attractor.

For example, both the Lorenz system and the Chen system have the z-axis rotational symmetry, and they both generate a two-scroll butterfly-shaped symmetrical attractor.

The Lorenz system [1] is described by

$$\begin{cases} \dot{x} = \sigma\,(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy\,, \end{cases} \tag{11}$$

which is chaotic when $\sigma = 10,\ r = 28,\ b = \frac{8}{3}$.

The Chen system[2] is described by

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = -bz + xy\,, \end{cases} \tag{12}$$

which is chaotic when $a = 35,\ b = 3,\ c = 28$.

Both Lorenz and Chen systems have z-axis rotational symmetry, in the sense that the system algebraic equations remain the same when (x, y, z) is transformed to $(-x, -y, z)$.

There are totally six possible quadratic nonlinear terms: xy, yz, xz, x^2, y^2 and z^2 in a 3D quadratic equation. Restricted by the z-axis rotational symmetry, the nonlinear terms in the second equation of the above 3D system must be either xz or yz, while in the third equation they must be either xy or z^2, or x^2, or y^2.

Based on the above observations, to maintain the z-axis rotational symmetry, the most general form of a 3D autonomous system with only linear and quadratic terms seems to be the following:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y + m_1xz + m_2yz \\ \dot{z} = a_{33}z + m_3xy + m_4x^2 + m_5y^2 + m_6z^2 + c\,, \end{cases} \tag{13}$$

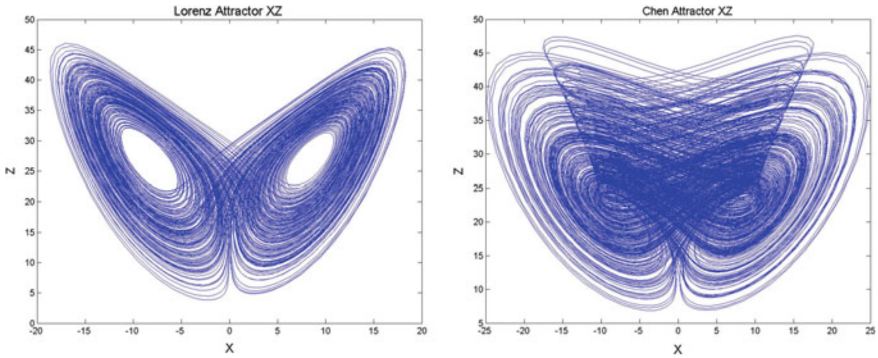

Fig. 7 Lorenz attractor and Chen attractor

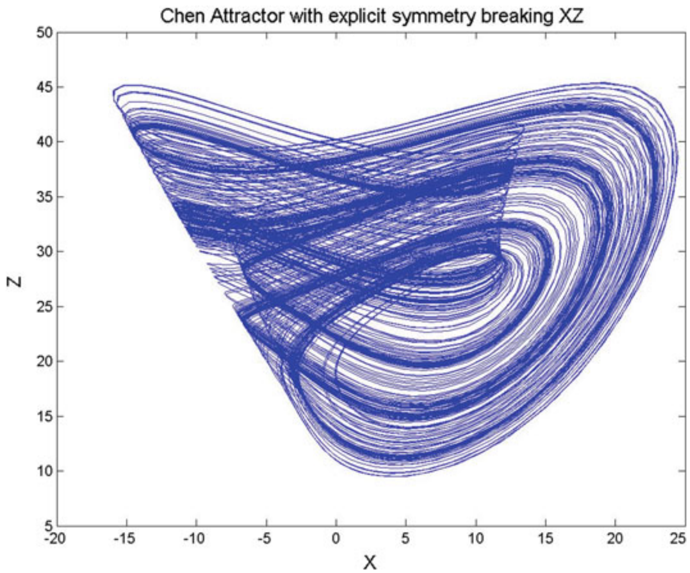

Fig. 8 Explicit symmetry breaking of the Chen system with $m = 20$

where all coefficients are real constants.

The Lorenz attractor and Chen attractor are shown in Fig. 7.

4.2 Explicit Symmetry Breaking

The explicit symmetry breaking can be easily realized by adding terms that break the symmetry of the algebraic equations. For example, adding a constant term m into

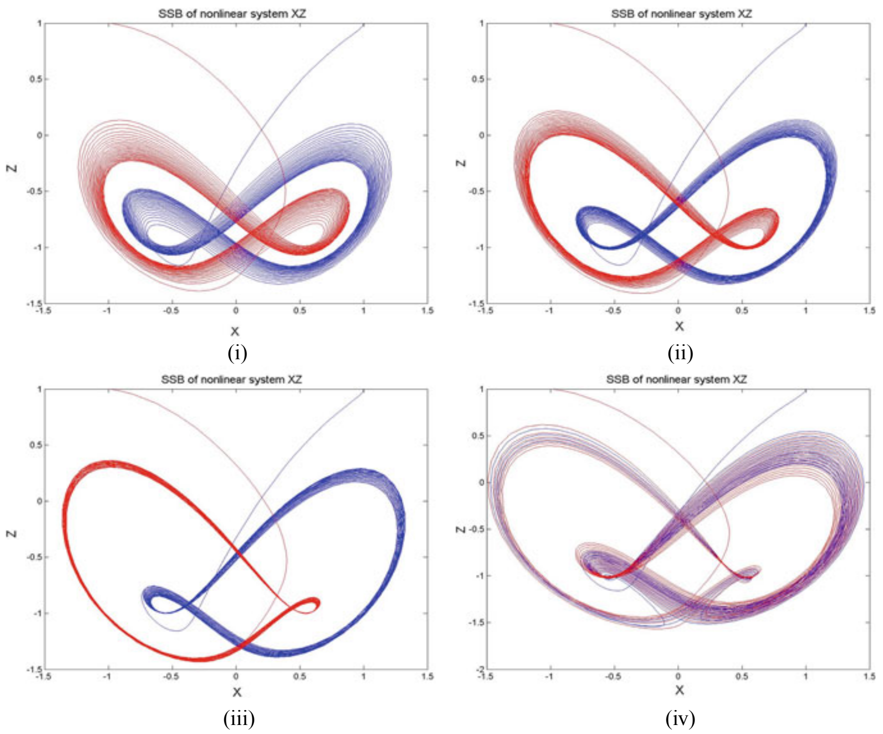

Fig. 9 Family-k of chaotic systems with two coexisting attractors: (i) $k = 0.78$, (ii) $k = 0.79$, (iii) $k = 0.8$, (iv) $k = 0.81$. The two trajectories are from two different initial conditions, for the latter case the two attractors merge into one and become symmetric

the second equation breaks the z-axis rotational symmetry:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy + m \\ \dot{z} = -bz + xy\,. \end{cases} \tag{14}$$

The resulting attractor is not symmetrical anymore due to the asymmetrical dynamic equation.

4.3 SSB of Nonlinear Systems

Constrained by the algebraical symmetry of the system equations, the above systems generate symmetrical two-scroll attractors. But this is not always the case, because this statement is based on the postulation that the dynamic system has only one attractor.

A very interesting phenomenon is that nonlinear systems may have multiple coexisting attractors. One such examples is described by Ref. [3]

5 Discussion

Constrained by the algebraically symmetry of the dynamic equations, the geometric attractors generated by such a system may or may not preserve the original symmetry. It is still puzzling to me what type of systems may break the symmetry while the others preserve it, or what type of systems may have multiple attractors.

What's more, in reality, one may not know the governing equation or may not fully understand its symmetry. What we can directly observe is the physical phenomenon, where the trajectory and the attractor may be asymmetric, thus we may overlook the underlying symmetry. If the observation (attractors) cannot reflect the underlying symmetry of the fundamental physic law, then the symmetry is considered "hidden".

Thus, SSB allows the symmetry theory to describe asymmetric reality. SSB provides a way of understanding the complexity of nature without renouncing the fundamental symmetries. So, we believe or prefer symmetric to asymmetric fundamental laws.

In one word, symmetry is simple and elegant, while symmetry breaking makes the world complex and colorful.

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