Chapter 6
Invariances

It is well-known that the Vlasov-Poisson system (1.5), (1.3) has many invariances, see [49, p. 427], for instance: if $f = f(t, x, v)$ is a solution, so is

$$\tilde{f}(\tilde{t}, \tilde{x}, \tilde{v}) = \frac{\mu}{\lambda^2} f\Big(\frac{\tilde{t} + t_0}{\mu\lambda}, \frac{\tilde{x} + x_0}{\lambda}, \mu\tilde{v}\Big), \tag{6.1}$$

where $\mu, \lambda > 0$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$. The associated potential and density are

$$U_{\tilde{f}}(\tilde{t}, \tilde{x}) = \frac{1}{\mu^2} U_f\Big(\frac{\tilde{t} + t_0}{\mu\lambda}, \frac{\tilde{x} + x_0}{\lambda}\Big), \quad \rho_{\tilde{f}}(\tilde{t}, \tilde{x}) = \frac{1}{\mu^2\lambda^2} \rho_f\Big(\frac{\tilde{t} + t_0}{\mu\lambda}, \frac{\tilde{x} + x_0}{\lambda}\Big). \tag{6.2}$$

It can be expected that quantities that are invariant will play a particularly important role. It is the purpose of this section to determine several such quantities.

Let $Q = Q(x, v)$ be a steady state solution. According to (6.1) and (6.2), then

$$\tilde{Q}(\tilde{x}, \tilde{v}) = \frac{\mu}{\lambda^2} Q\Big(\frac{\tilde{x}}{\lambda}, \mu\tilde{v}\Big) \tag{6.3}$$

is a steady state solution for every $\mu, \lambda > 0$. The associated potential and density are

$$U_{\tilde{Q}}(\tilde{x}) = \frac{1}{\mu^2} U_Q\Big(\frac{\tilde{x}}{\lambda}\Big), \quad \rho_{\tilde{Q}}(\tilde{x}) = \frac{1}{\mu^2\lambda^2} \rho_Q\Big(\frac{\tilde{x}}{\lambda}\Big).$$

The variables transform as $x = \frac{\tilde{x}}{\lambda}$ and $v = \mu\tilde{v}$ so that in particular $r = \frac{\tilde{r}}{\lambda}$ for $r = |x|$ and $\tilde{r} = |\tilde{x}|$.

Next let $Q = Q(e_Q)$ depend only upon $e_Q(x, v) = \frac{1}{2}|v|^2 + U_Q(x)$. Then,

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$$\begin{aligned} e_Q(x,v) &= \frac{1}{2}\,|v|^2 + U_Q(x) = \frac{1}{2}\,\mu^2|\tilde{v}|^2 + U_Q(\lambda^{-1}\tilde{x}) \\ &= \frac{1}{2}\,\mu^2|\tilde{v}|^2 + \mu^2 U_{\tilde{Q}}(\tilde{x}) = \mu^2 e_{\tilde{Q}}(\tilde{x},\tilde{v}), \end{aligned} \tag{6.4}$$

and (6.3) leads to

$$\tilde{Q}(e_{\tilde{Q}}) = \frac{\mu}{\lambda^2}\,Q(e_Q) = \frac{\mu}{\lambda^2}\,Q(\mu^2 e_{\tilde{Q}}).$$

Thus, if $Q = Q(e_Q)$ and $\tilde{Q} = \tilde{Q}(e_{\tilde{Q}})$ are understood as functions of one variable, then

$$\tilde{Q}'(e_{\tilde{Q}}) = \frac{\mu^3}{\lambda^2}\,Q'(e_Q). \tag{6.5}$$

For radial potentials and densities, we have

$$U_{\tilde{Q}}(\tilde{r}) = \frac{1}{\mu^2}\,U_Q(r), \quad \rho_{\tilde{Q}}(\tilde{r}) = \frac{1}{\lambda^2\mu^2}\,\rho_Q(r), \tag{6.6}$$

which leads to

$$U'_{\tilde{Q}}(\tilde{r}) = \frac{1}{\lambda\mu^2}\,U'_Q(r). \tag{6.7}$$

The central densities are related by

$$\rho_{\tilde{Q}}(0) = \frac{1}{\lambda^2\mu^2}\,\rho_Q(0). \tag{6.8}$$

The effective potential from (7.4) is $U_{\text{eff}}(r,\ell) = U_Q(r) + \frac{\ell^2}{2r^2}$, which we also write as $U_{\text{eff}}(r,\beta) = U_Q(r) + \frac{\beta}{2r^2}$ for $\beta = \ell^2$. Let

$$\tilde{\beta} = \frac{\lambda^2}{\mu^2}\,\beta.$$

Then,

$$\tilde{U}_{\text{eff}}(\tilde{r},\tilde{\beta}) := U_{\tilde{Q}}(\tilde{r}) + \frac{\tilde{\beta}}{2\tilde{r}^2} = \frac{1}{\mu^2}\,U_Q(r) + \frac{\lambda^2}{\mu^2}\,\beta\,\frac{1}{2\lambda^2 r^2} = \frac{1}{\mu^2}\,U_{\text{eff}}(r,\beta)$$

is the corresponding transformation rule. The points $r_\pm = r_\pm(e,\beta)$ are determined by the relation $U_{\text{eff}}(r_\pm(e,\beta),\beta) = e$. Owing to

$$\tilde{U}_{\text{eff}}(\tilde{r}_\pm(\tilde{e},\tilde{\beta}),\tilde{\beta}) = \tilde{e} \quad \Longleftrightarrow \quad \frac{1}{\mu^2}\,U_{\text{eff}}(\lambda^{-1}\tilde{r}_\pm(\tilde{e},\tilde{\beta}),\beta) = \frac{1}{\mu^2}\,e$$

we obtain

$$\tilde{r}_\pm(\tilde{e}, \tilde{\beta}) = \lambda\, r_\pm(e, \beta).$$

Next, $r_0 = r_0(\beta)$ is the point where $U_{\rm eff}(\cdot, \beta)$ attains its minimum. Since $\tilde{U}_{\rm eff}(\tilde{r}, \tilde{\beta}) = \mu^{-2} U_{\rm eff}(r, \beta) = \mu^{-2} U_{\rm eff}(\lambda^{-1}\tilde{r}, \beta)$, we get

$$\tilde{U}'_{\rm eff}(\tilde{r}, \tilde{\beta}) = \lambda^{-1}\mu^{-2}\, U'_{\rm eff}(\lambda^{-1}\tilde{r}, \beta),$$

and this implies that

$$\tilde{r}_0(\tilde{\beta}) = \lambda\, r_0(\beta).$$

In terms of the variables e and β, the period function from (A.20) is

$$T_1(e, \beta) = 2 \int_{r_-(e,\beta)}^{r_+(e,\beta)} \frac{dr}{\sqrt{2(e - U_{\rm eff}(r, \beta))}}.$$

Using the transformation $\tilde{r} = \lambda r$, $d\tilde{r} = \lambda dr$, it follows that

$$\begin{aligned}
\tilde{T}_1(\tilde{e}, \tilde{\beta}) &= 2 \int_{\tilde{r}_-(\tilde{e},\tilde{\beta})}^{\tilde{r}_+(\tilde{e},\tilde{\beta})} \frac{d\tilde{r}}{\sqrt{2(\tilde{e} - \tilde{U}_{\rm eff}(\tilde{r}, \tilde{\beta}))}} \\
&= 2\lambda \int_{\lambda^{-1}\tilde{r}_-(\tilde{e},\tilde{\beta})}^{\lambda^{-1}\tilde{r}_+(\tilde{e},\tilde{\beta})} \frac{dr}{\sqrt{2(\tilde{e} - \tilde{U}_{\rm eff}(\lambda r, \tilde{\beta}))}} \\
&= 2\lambda \int_{r_-(e,\beta)}^{r_+(e,\beta)} \frac{dr}{\sqrt{2(\mu^{-2} e - \mu^{-2} U_{\rm eff}(r, \beta))}} \\
&= \lambda\mu\, T_1(e, \beta).
\end{aligned}$$

In particular, $\tilde{\omega}_1(\tilde{e}, \tilde{\beta}) = \frac{1}{\lambda\mu}\, \omega_1(e, \beta)$ for $\tilde{\omega}_1 = \frac{2\pi}{\tilde{T}_1}$, and if we denote $\delta_1 = \inf \omega_1$, then also

$$\tilde{\delta}_1 = \frac{1}{\lambda\mu}\, \delta_1. \tag{6.9}$$

Next we consider the space $L^2_{{\rm sph}, \frac{1}{|Q'|}}(K) = X^0$ of spherically symmetric functions with the Q-dependent inner product

$$(u_1, u_2)_Q = \iint_K \frac{1}{|Q'(e_Q)|}\, \overline{u_1(x, v)}\, u_2(x, v)\, dx\, dv,$$

as in Remark B.2. Defining

$$\tilde{u}(\tilde{x}, \tilde{v}) = \frac{\mu}{\lambda^2}\, u\Big(\frac{\tilde{x}}{\lambda}, \mu\tilde{v}\Big) \tag{6.10}$$

in accordance with (6.3), we calculate, using $dx = \lambda^{-3} d\tilde{x}$ and $dv = \mu^3 d\tilde{v}$ as well as (6.5):

$$\begin{aligned} \|\tilde{u}\|_{\tilde{Q}}^2 &= \int\int \frac{d\tilde{x}\, d\tilde{v}}{|\tilde{Q}'(e_{\tilde{Q}})|} |\tilde{u}(\tilde{x}, \tilde{v})|^2 \\ &= \frac{\lambda^2}{\mu^3} \frac{\mu^2}{\lambda^4} \int\int \frac{\lambda^3 dx\, \mu^{-3} dv}{|Q'(e_Q)|} |u(x, v)|^2 = \frac{\lambda}{\mu^4} \|u\|_Q^2. \end{aligned} \tag{6.11}$$

Let the operator $(\mathcal{T} g)(x, v) = v \cdot \nabla_x g(x, v) - \nabla_v g(x, v) \cdot \nabla_x U_Q(x)$ be as in (1.11). From the above relations, it follows that

$$\begin{aligned} (\mathcal{T}\tilde{u})(\tilde{x}, \tilde{v}) &= \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{u} - \nabla_{\tilde{v}} \tilde{u} \cdot \nabla_{\tilde{x}} U_{\tilde{Q}} \\ &= \mu^{-1} \mu \lambda^{-2} \lambda^{-1}\, v \cdot \nabla_x u - \mu \lambda^{-2} \mu \mu^{-2} \lambda^{-1}\, \nabla_v u \cdot \nabla_x U_Q \\ &= \lambda^{-3} (\mathcal{T} u)(x, v). \end{aligned} \tag{6.12}$$

Alternatively, $\mathcal{T} u = \{u, e_Q\}$ can be used. From (6.4) and (6.10), we get

$$\begin{aligned} (\mathcal{T}\tilde{u})(\tilde{x}, \tilde{v}) &= \{\tilde{u}, e_{\tilde{Q}}\} = \nabla_{\tilde{x}}\, \tilde{u} \cdot \nabla_{\tilde{v}}\, e_{\tilde{Q}} - \nabla_{\tilde{x}}\, e_{\tilde{Q}} \cdot \nabla_{\tilde{v}}\, \tilde{u} \\ &= \mu \lambda^{-3} \nabla_x u \cdot \mu^{-2} \mu \nabla_v e_Q - \mu^{-2} \lambda^{-1} \nabla_x e_Q \cdot \mu^2 \lambda^{-2} \nabla_v u \\ &= \lambda^{-3} \{u, e_Q\} = \lambda^{-3} (\mathcal{T} u)(x, v). \end{aligned}$$

This in turn leads to

$$\begin{aligned} (\mathcal{T}^2 \tilde{u})(\tilde{x}, \tilde{v}) &= \{\mathcal{T}\tilde{u}, e_{\tilde{Q}}\} = \nabla_{\tilde{x}}\, (\mathcal{T}\tilde{u}) \cdot \nabla_{\tilde{v}}\, e_{\tilde{Q}} - \nabla_{\tilde{x}}\, e_{\tilde{Q}} \cdot \nabla_{\tilde{v}}\, (\mathcal{T}\tilde{u}) \\ &= \lambda^{-4} \nabla_x (\mathcal{T} u) \cdot \mu^{-2} \mu \nabla_v e_Q - \mu^{-2} \lambda^{-1} \nabla_x e_Q \cdot \lambda^{-3} \mu \nabla_v (\mathcal{T} u) \\ &= \lambda^{-4} \mu^{-1} \{\mathcal{T} u, e_Q\} = \lambda^{-4} \mu^{-1} (\mathcal{T}^2 u)(x, v). \end{aligned} \tag{6.13}$$

Alternatively, if we put $\hat{u}(\tilde{x}, \tilde{v}) = u(\frac{\tilde{x}}{\lambda}, \mu\tilde{v})$, then $\tilde{u} = \mu \lambda^{-2} \hat{u}$, so (6.13) may be re-expressed as

$$(\mathcal{T}^2 \hat{u})(\tilde{x}, \tilde{v}) = \lambda^{-2} \mu^{-2} (\mathcal{T}^2 u)(x, v). \tag{6.14}$$

For the density induced by $\mathcal{T}\tilde{u}$, (6.12) yields

$$\rho_{\mathcal{T}\tilde{u}}(\tilde{x}) = \int (\mathcal{T}\tilde{u})(\tilde{x}, \tilde{v})\, d\tilde{v} = \lambda^{-3} \mu^{-3} \int (\mathcal{T} u)(x, v)\, dv = \lambda^{-3} \mu^{-3} \rho_{\mathcal{T} u}(x),$$

so that

$$U_{\mathcal{T}\tilde{u}}(\tilde{x}) = \lambda^{-1} \mu^{-3}\, U_{\mathcal{T} u}(x)$$

for the potential. In particular,

$$\nabla_{\tilde{x}} U_{\mathcal{T}\tilde{u}}(\tilde{x}) = \lambda^{-2} \mu^{-3}\, \nabla_x U_{\mathcal{T} u}(x),$$

and hence

$$\int |\nabla_{\tilde{x}} U_{\mathcal{T}\tilde{u}}(\tilde{x})|^2\, d\tilde{x} = \lambda^{-4}\mu^{-6}\lambda^3 \int |\nabla_x U_{\mathcal{T}u}(x)|^2\, dx$$
$$= \lambda^{-1}\mu^{-6} \int |\nabla_x U_{\mathcal{T}u}(x)|^2\, dx. \tag{6.15}$$

For

$$(Lu, u)_Q = \int\int \frac{dx\, dv}{|Q'(e_Q)|}\, |\mathcal{T}u|^2 - \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\, dx$$

as given by (1.18), we then obtain from (6.5), (6.12) and (6.15):

$$(L\tilde{u}, \tilde{u})_{\tilde{Q}} = \int\int \frac{d\tilde{x}\, d\tilde{v}}{|\tilde{Q}'(e_{\tilde{Q}})|}\, (\mathcal{T}\tilde{u})^2 - \frac{1}{4\pi} \int |\nabla_{\tilde{x}} U_{\mathcal{T}\tilde{u}}|^2\, d\tilde{x}$$
$$= \lambda^3\mu^{-3}\lambda^2\mu^{-3}\lambda^{-6} \int\int \frac{dx\, dv}{|Q'(e_Q)|}\, (\mathcal{T}u)^2 - \frac{1}{4\pi}\, \lambda^{-1}\mu^{-6} \int |\nabla_x U_{\mathcal{T}u}|^2\, dx$$
$$= \lambda^{-1}\mu^{-6}\, (Lu, u)_Q. \tag{6.16}$$

In (1.20), the quantity

$$\lambda_* = \inf\, \{(Lu, u)_Q : u \in X^2_{\text{odd}},\, \|u\|_Q = 1\}$$

is introduced. Therefore, owing to (6.16) and (6.11),

$$\tilde{\lambda}_* = \inf\, \{(L\tilde{u}, \tilde{u})_{\tilde{Q}} : \tilde{u} \in X^2_{\text{odd}},\, \|\tilde{u}\|_{\tilde{Q}} = 1\}$$
$$= \lambda^{-1}\mu^{-6} \inf\, \{(Lu, u)_Q : u \in X^2_{\text{odd}},\, \lambda\mu^{-4}\, \|u\|_Q = 1\}$$
$$= \lambda^{-1}\mu^{-6}\lambda^{-1}\mu^4 \inf\, \{(L\hat{u}, \hat{u})_Q : \hat{u} \in X^2_{\text{odd}},\, \|\hat{u}\|_Q = 1\}$$
$$= \lambda^{-2}\mu^{-2}\lambda_*, \tag{6.17}$$

by setting $u = \lambda^{-1/2}\mu^2\, \hat{u}$; it maybe checked that $u \in X^2_{\text{odd}}$ if and only if $\tilde{u} \in X^2_{\text{odd}}$ w.r.t. the transformed variables.

Using (6.7), the function $A(r) = \frac{U'_Q(r)}{r}$ from (A.27) is found to scale as

$$\tilde{A}(\tilde{r}) = \frac{U'_{\tilde{Q}}(\tilde{r})}{\tilde{r}} = \lambda^{-1}\mu^{-2}\, \frac{U'_Q(r)}{\lambda r} = \lambda^{-2}\mu^{-2} A(r) \tag{6.18}$$

for $\tilde{r} \in [0, r_{\tilde{Q}}]$, with $r_{\tilde{Q}} = \lambda r_Q$ denoting the end of the support of $\rho_{\tilde{Q}}$, if r_Q denotes the end of the support of ρ_Q.

Similarly, denoting $B(r) = 4\pi\rho_Q(r) + A(r)$ as in Lemma A.7(d), owing to (6.18) and (6.6) one gets

$$\tilde{B}(\tilde{r}) = \lambda^{-2}\mu^{-2} B(r).$$

Now we turn to the operators Q_ν from Chap. 4 and their first eigenvalues $\mu_1(\nu)$ for $\nu \in]-\infty, \delta_1^2[$; note the change in notation here for the parameter of the operators, since the letter λ is already occupied from $\tilde{x} = \lambda x$, $\tilde{r} = \lambda r$. Let

$$\tilde{\nu} = \frac{1}{\lambda^2\mu^2}\,\nu.$$

If $\nu \in]-\infty, \delta_1^2[$, then $\tilde{\nu} \in]-\infty, \tilde{\delta}_1^2[$ due to (6.9). For $\Psi = \Psi(r)$ let $\tilde{\Psi}(\tilde{r}) = \Psi(\frac{\tilde{r}}{\lambda})$. Since $\tilde{p}_r = \frac{\tilde{x}\cdot\tilde{v}}{|\tilde{x}|} = \mu^{-1}\,\frac{x\cdot v}{|x|} = \mu^{-1}p_r$, we obtain from (6.5):

$$\begin{aligned}\tilde{\psi}(\tilde{r}, \tilde{p}_r, \tilde{\ell}) &= |\tilde{Q}'(e_{\tilde{Q}})|\,\tilde{p}_r\,\tilde{\Psi}(\tilde{r}) = \mu^3\lambda^{-2}|Q'(e_Q)|\,\mu^{-1}p_r\,\Psi(r)\\ &= \mu^2\lambda^{-2}\,|Q'(e_Q)|\,p_r\,\Psi(r) = \mu^2\lambda^{-2}\,\psi(r, p_r, \ell).\end{aligned} \tag{6.19}$$

First we determine the scaling of $(-\mathcal{T}^2 - z)^{-1}\psi$. Defining

$$\tilde{z} = \frac{1}{\lambda^2\mu^2}\,z,$$

we assert that

$$((-\mathcal{T}^2 - \tilde{z})^{-1}\tilde{\psi})(\tilde{x}, \tilde{v}) = \mu^4\,((-\mathcal{T}^2 - z)^{-1}\psi)(x, v). \tag{6.20}$$

To see this, let $\tilde{g} = (-\mathcal{T}^2 - \tilde{z})^{-1}\tilde{\psi}$ and $g = (-\mathcal{T}^2 - z)^{-1}\psi$. Then, (6.20) is equivalent to $\tilde{g} = \mu^4 g$, but $\tilde{g}$ and g are not necessarily related by (6.10); in fact $\tilde{g} = \mu^4\hat{g}$ or $(-\mathcal{T}^2 - \tilde{z})\tilde{g} = \mu^4(-\mathcal{T}^2 - \tilde{z})\hat{g}$ is to be shown. For, owing to (6.14) and (6.19) we have

$$\begin{aligned}\mu^4(-\mathcal{T}^2 - \tilde{z})\hat{g} &= \mu^4(-\lambda^{-2}\mu^{-2}\mathcal{T}^2 g - \lambda^{-2}\mu^{-2}zg) = \mu^2\lambda^{-2}(-\mathcal{T}^2 - z)g = \mu^2\lambda^{-2}\,\psi\\ &= \tilde{\psi} = (-\mathcal{T}^2 - \tilde{z})\tilde{g},\end{aligned}$$

which completes the proof of (6.20). From (4.22) together with (6.20), we obtain

$$\begin{aligned}(\widetilde{\mathcal{Q}}_{\tilde{z}}\tilde{\Psi})(\tilde{r}) &= 4\pi\int \tilde{p}_r\,((-\mathcal{T}^2 - \tilde{z})^{-1}\tilde{\psi})(\tilde{x}, \tilde{v})\,d\tilde{v}\\ &= 4\pi\mu^4\int \frac{\tilde{x}\cdot\tilde{v}}{|\tilde{x}|}\,((-\mathcal{T}^2 - z)^{-1}\psi)(\lambda^{-1}\tilde{x}, \mu\tilde{v})\,d\tilde{v}\\ &= 4\pi\int \frac{\lambda^{-1}\tilde{x}\cdot v}{|\lambda^{-1}\tilde{x}|}\,((-\mathcal{T}^2 - z)^{-1}\psi)(\lambda^{-1}\tilde{x}, v)\,dv\\ &= (\mathcal{Q}_z\Psi)(r).\end{aligned}$$

Thus, if we define

$$\tilde{\mu}_1(\tilde{\nu}) = \mu_1(\lambda^2\mu^2\tilde{\nu}), \quad \tilde{\nu} \in]-\infty, \tilde{\delta}_1^2[,$$

then $\tilde{\mu}_1(\tilde{\nu})$ is the first eigenvalue of $\widetilde{\mathcal{Q}}_{\tilde{\nu}}$, and $\tilde{\Psi} = \tilde{\Psi}(\tilde{r})$ is an associated eigenfunction if and only if $\Psi = \Psi(r)$ is an eigenfunction of $\mathcal{Q}_\nu$ for the eigenvalue $\mu_1(\nu)$. Due to (4.33) it follows that

$$\tilde{\mu}_* = \lim_{\tilde{\nu}\to\tilde{\delta}_1^2-} \tilde{\mu}_1(\tilde{\nu}) = \lim_{\nu\to\delta_1^2-} \mu_1(\nu) = \mu_*.$$

As already noted at the beginning of this chapter, it can be expected that quantities that are unaffected by the scaling do have a special relevance. Hence, μ_* is one such quantity. In addition, the condition $\lambda_* < \delta_1^2$ is invariant, as a consequence of (6.17) and (6.9). Further, we would like to mention

$$\frac{2\pi}{\sqrt{\lambda_*}}\sqrt{\rho_Q(0)},$$

cf. [59, Remark, p. 555], for which we deduce from (6.17) and (6.8):

$$\frac{2\pi}{\sqrt{\tilde{\lambda}_*}}\sqrt{\rho_{\tilde{Q}}(0)} = \frac{2\pi}{\lambda^{-1}\mu^{-1}\sqrt{\lambda_*}}\,\lambda^{-1}\mu^{-1}\sqrt{\rho_Q(0)} = \frac{2\pi}{\sqrt{\lambda_*}}\sqrt{\rho_Q(0)}.$$

This is called the Eddington-Ritter relation; also see [17, (27), p. 15] and [70, Section 4]. The relevance of the number $\frac{2\pi}{\sqrt{\lambda_*}}$ is that it is the 'linear period' of the system, in the sense that the linearized system about Q has a periodic solution of this period (if λ_* is an eigenvalue of L); recall Lemma 1.3.

Moreover, for any $r \in [0, r_Q]$ and $\tilde{r} = \lambda r$ one in fact has

$$\frac{\rho_{\tilde{Q}}(\tilde{r})}{\tilde{\lambda}_*} = \frac{\rho_Q(r)}{\lambda_*}, \quad \frac{\tilde{A}(\tilde{r})}{\tilde{\lambda}_*} = \frac{A(r)}{\lambda_*}, \quad \frac{\tilde{B}(\tilde{r})}{\tilde{\lambda}_*} = \frac{B(r)}{\lambda_*}.$$