# Chapter 4
# A Birman-Schwinger Type Operator

As has been outlined in the introduction, the eigenvalues $\lambda < \delta_1^2$ of $L = -\mathcal{T}^2 - \mathcal{K}\mathcal{T}$ from (1.16) are in one-to-one correspondence with the eigenvalues 1 of a certain Birman-Schwinger type operator $\mathcal{Q}_\lambda$ that acts on functions $\Psi = \Psi(r)$.

## 4.1 The Operator $\mathcal{Q}_z$

Let $L_r^2$ denote the $L^2$-Lebesgue space of radially symmetric functions $\Psi(x) = \Psi(r)$ on $\mathbb{R}^3$, where we take

$$\langle \Psi, \Phi \rangle = \int_{\mathbb{R}^3} \overline{\Psi(x)}\, \Phi(x)\, dx = 4\pi \int_0^\infty r^2\, \overline{\Psi(r)}\, \Phi(r)\, dr$$

as the inner product of $\Psi, \Phi \in L_r^2$. Unless otherwise stated, a generic constant (denoted by $C$) is allowed to depend only upon $Q$.

**Definition 4.1** For $z \in \Omega = \mathbb{C} \setminus [\delta_1^2, \infty[$, we introduce

$$\begin{aligned} &\mathcal{Q}_z : L_r^2 \to L_r^2, \\ &(\mathcal{Q}_z \Psi)(r) = \frac{16\pi}{r^2} \sum_{k \neq 0} \int_0^\infty d\tilde{r}\, \Psi(\tilde{r}) \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell) \leq r,\, \tilde{r} \leq r_+(e,\ell)\}} \\ &\qquad \times \frac{\omega_1(e,\ell)\, |Q'(e)|}{k^2 \omega_1^2(e,\ell) - z} \sin(k\theta(r,e,\ell)) \sin(k\theta(\tilde{r},e,\ell)), \end{aligned} \tag{4.1}$$



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where $r_\pm(e,\ell)$ and $\theta(r,e,\ell)$ are as in Appendix I, Sect. A.1, and $D$ is given by (3.1). Along with $\mathcal{Q}_z$, we also introduce the integral kernels

$$K_z(r,\tilde r) = \frac{4}{r^2\tilde r^2}\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\le r,\,\tilde r\le r_+(e,\ell)\}} \times \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-z}\,\sin(k\theta(r,e,\ell))\sin(k\theta(\tilde r,e,\ell)). \tag{4.2}$$

**Remark 4.2** (a) If $z=a+ib\in\mathbb{C}\setminus\mathbb{R}$, then $|k^2\omega_1^2(e,\ell)-z|\ge|b|>0$. More precisely,

$$|k|\ge\left[\frac{\sqrt{2|a|}}{\delta_1}\right]+1 \implies |k^2\omega_1^2(e,\ell)-z|^2 = (k^2\omega_1^2(e,\ell)-a)^2+b^2 \ge (k^2\delta_1^2-|a|)^2+b^2 \ge \frac14 k^4\delta_1^4+b^2. \tag{4.3}$$

On the other hand, if $z=\lambda\in]-\infty,\delta_1^2[$, then

$$|k^2\omega_1^2(e,\ell)-z| = k^2\omega_1^2(e,\ell)-\lambda\ge k^2\delta_1^2-\lambda\ge\delta_1^2-\lambda>0,$$

and hence

$$|k|\ge 2 \implies |k^2\omega_1^2(e,\ell)-z|\ge k^2\delta_1^2-\lambda\ge(k^2-1)\delta_1^2\ge\frac12 k^2\delta_1^2. \tag{4.4}$$

In particular, $\frac{1}{k^2\omega_1^2(e,\ell)-z}$ in (4.1) and (4.2) is well-defined for $z\in\Omega$.
(b) In the definitions, we understand the factor $|Q'(e)|$ to be zero outside of $K$, the support of $Q$, instead of carrying around another characteristic function all the time. In particular, always $r_+(e,\ell)\le r_Q$ holds, which means the following: in (4.1), $\int_0^\infty d\tilde r\,\Psi(\tilde r)$ can be replaced by $\int_0^{r_Q} d\tilde r\,\Psi(\tilde r)$; $(\mathcal{Q}_z\Psi)(r)$ can be replaced by $(\mathcal{Q}_z\Psi)(r)\,\mathbf{1}_{\{0\le r\le r_Q\}}$ and $K_z(r,\tilde r)$ can be replaced by $K_z(r,\tilde r)\,\mathbf{1}_{\{0\le r,\tilde r\le r_Q\}}$. ◇

**Lemma 4.3** *[Properties of $\mathcal{Q}_z$] The following assertions hold.*

*(a) For every $z\in\Omega$, we have $\mathcal{Q}_z\in\mathcal{B}(L_r^2)$, the space of linear and bounded operators on $L_r^2$. In addition, the map*

$$\Omega\ni z\mapsto\mathcal{Q}_z\in\mathcal{B}(L_r^2) \tag{4.5}$$

*is analytic, and for the derivatives*

$$(\mathcal{Q}_z^{(j)}\Psi)(r) = \frac{16\pi j!}{r^2} \sum_{k\neq 0} \int_0^\infty d\tilde{r}\, \Psi(\tilde{r}) \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}} \times \frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell)-z)^{j+1}} \sin(k\theta(r,e,\ell))\sin(k\theta(\tilde{r},e,\ell))$$

*for* $\Psi \in L_r^2$.

*(b) If* $z \in \Omega$, *then*

$$(\mathcal{Q}_z\Psi)(r) = \langle K_{\bar{z}}(r,\cdot), \Psi\rangle$$

*for* $\Psi \in L_r^2$. *In particular,*

$$\langle \mathcal{Q}_z\Psi, \Phi\rangle = \langle \Psi, \mathcal{Q}_{\bar{z}}\Phi\rangle$$

*for* $\Psi, \Phi \in L_r^2$, *so that* $Q_z^* = Q_{\bar{z}}$. *Thus, if* $\lambda \in ]-\infty, \delta_1^2[$, *then* $\mathcal{Q}_\lambda$ *is symmetric.*

*(c) If* $z \in \Omega$, *then* $\mathcal{Q}_z$ *is a Hilbert-Schmidt operator on* $L_r^2$.

*(d) If* $z \in \Omega$, *then*

$$\langle \mathcal{Q}_z\Psi, \Psi\rangle = 64\pi^2 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-\bar{z}} \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi(r)\sin(k\theta(r,e,\ell))\,dr \right|^2$$

*for* $\Psi \in L_r^2$. *In particular, if* $\lambda \in ]-\infty, \delta_1^2[$, *then* $\langle \mathcal{Q}_\lambda\Psi, \Psi\rangle \ge 0$ *for* $\Psi \in L_r^2$, *i.e.,* $\mathcal{Q}_\lambda$ *is positive. In addition, for the derivatives*

$$\langle \mathcal{Q}_z^{(j)}\Psi, \Psi\rangle = 64\pi^2 j! \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell)-\bar{z})^{j+1}} \times \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi(r)\sin(k\theta(r,e,\ell))\,dr \right|^2 \quad (4.6)$$

*for* $\Psi \in L_r^2$.

*(e) There is a constant* $C > 0$ *such that for* $\lambda, \tilde{\lambda} \in ]-\infty, \delta_1^2[$,

$$\|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|_{\mathrm{HS}} \le C\left(1 + \frac{1}{(\delta_1^2-\lambda)(\delta_1^2-\tilde{\lambda})}\right)|\lambda-\tilde{\lambda}|,$$

*where* $\|\cdot\|_{\mathrm{HS}}$ *denotes the Hilbert-Schmidt norm.*

*(f) If* $\lambda \in ]-\infty, \delta_1^2[$, *then the spectrum of* $\mathcal{Q}_\lambda$ *consists of* $\mu_1(\lambda) \ge \mu_2(\lambda) \ge \ldots \to 0$ *(the eigenvalues are listed according to their multiplicities). In addition,*

$$\mu_1(\lambda) = \|\mathcal{Q}_\lambda\| = \sup\{\langle \mathcal{Q}_\lambda\Psi, \Psi\rangle : \|\Psi\|_{L_r^2} \le 1\}, \quad (4.7)$$

*where* $\|\cdot\| = \|\cdot\|_{\mathcal{B}(L_r^2)}$, *and every function*

$$\mu_k(\cdot) : ]-\infty, \delta_1^2[ \to ]0, \infty[$$

*for $k \in \mathbb{N}$ is monotone increasing and locally Lipschitz continuous (and hence differentiable a.e. by Rademacher's Theorem).*

***Proof*** (a) Let $z \in \Omega$ be fixed. By Remark 4.2(a), there is $\alpha_0 > 0$ such that $|k^2\omega_1^2(e, \ell) - z| \geq \alpha_0$ for $|k| \geq 1$ and $(e, \ell) \in D$. In addition, according to (4.3) and (4.4), there is $k_0 \in \mathbb{N}$ so that $|k^2\omega_1^2(e, \ell) - z| \geq \frac{1}{2}k^2\delta_1^2$ for $|k| \geq k_0$ and $(e, \ell) \in D$; if $k_0$ is taken to be large enough, we can also make sure that $\frac{1}{2}k^2\delta_1^2 \geq k^{3/2}$. First, we observe that

$$r_-(e,\, \ell) \leq r \leq r_+(e,\, \ell) \implies \ell^2 \leq 2r^2(e_0 - U_Q(0)). \tag{4.8}$$

To establish this claim, we recall from (3.7) that $\ell^2 = 2r_-^2(e - U_Q(r_-))$ holds, where $r_\pm = r_\pm(e, \ell)$. Since $U_Q$ is increasing and $e \leq e_0$, we get $\ell^2 \leq 2r_-^2(e_0 - U_Q(0)) \leq 2r^2(e_0 - U_Q(0))$.

For $1 \leq |k| \leq k_0$ and $i \in \mathbb{N}_0$, we now apply (4.8) to $r$ and $\tilde{r}$ in order to estimate

$$s_{k,i}(r, \tilde{r}, z) = \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\, \ell) \leq r,\, \tilde{r} \leq r_+(e,\, \ell)\}} \times \frac{\omega_1(e, \ell)\, |Q'(e)|}{(k^2\omega_1^2(e, \ell) - z)^{i+1}}\, \sin(k\theta(r, e, \ell))\sin(k\theta(\tilde{r}, e, \ell)) \tag{4.9}$$

as

$$\begin{aligned} |s_{k,i}(r, \tilde{r}, z)| &\leq \alpha_0^{-(i+1)}\Delta_1\, \mathbf{1}_{\{0 \leq r, \tilde{r} \leq r_Q\}} \int_0^{l_*} d\ell\, \ell \int_{e_{\min}(\ell)}^{e_0} de \\ &\qquad \times \mathbf{1}_{\{r_-(e,\, \ell) \leq r,\, \tilde{r} \leq r_+(e,\, \ell)\}}\, |Q'(e)| \\ &\leq \alpha_0^{-(i+1)}\Delta_1\, \mathbf{1}_{\{0 \leq r, \tilde{r} \leq r_Q\}} \int_0^{l_*} d\ell\, \ell \int_{e_{\min}(\ell)}^{e_0} de \\ &\qquad \times \mathbf{1}_{\{\ell^2 \leq 2(e_0 - U_Q(0))\min\{r^2, \tilde{r}^2\}\}}\, |Q'(e)| \\ &\leq \alpha_0^{-(i+1)}\Delta_1\, \mathbf{1}_{\{0 \leq r, \tilde{r} \leq r_Q\}}\, (e_0 - U_Q(0))\left(\int_{U_Q(0)}^{e_0} |Q'(e)|\, de\right) \min\{r^2, \tilde{r}^2\}. \end{aligned}$$

Analogously, for $|k| \geq k_0$ and $i \in \mathbb{N}_0$, we deduce

$$|s_{k,i}(r, \tilde{r}, z)| \leq \frac{1}{k^{3(i+1)/2}}\, \Delta_1\, \mathbf{1}_{\{0 \leq r, \tilde{r} \leq r_Q\}}\, (e_0 - U_Q(0))\left(\int_{U_Q(0)}^{e_0} |Q'(e)|\, de\right) \min\{r^2, \tilde{r}^2\}.$$

It follows that

$$\begin{aligned} \sum_{k \neq 0} |s_{k,i}(r, \tilde{r}, z)| &\leq \sum_{|k| \leq k_0} \alpha_0^{-(i+1)}\Delta_1\, \mathbf{1}_{\{0 \leq r, \tilde{r} \leq r_Q\}}\, (e_0 - U_Q(0)) \\ &\qquad \times \left(\int_{U_Q(0)}^{e_0} |Q'(e)|\, de\right) \min\{r^2, \tilde{r}^2\} \end{aligned}$$

$$+ \sum_{|k|\ge k_0} \frac{1}{k^{3(i+1)/2}}\, \Delta_1\, \mathbf{1}_{\{0\le r,\tilde r\le r_Q\}}\, (e_0 - U_Q(0)) \times\left(\int_{U_Q(0)}^{e_0} |Q'(e)|\, de\right) \min\{r^2, \tilde r^2\}$$
$$\le C_{1,i}\, \mathbf{1}_{\{0\le r,\tilde r\le r_Q\}} \min\{r^2, \tilde r^2\} \tag{4.10}$$

for

$$C_{1,i} = \Big(2k_0\alpha_0^{-(i+1)} + 2\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\Big)\Delta_1\,(e_0 - U_Q(0))\left(\int_{U_Q(0)}^{e_0} |Q'(e)|\, de\right); \tag{4.11}$$

this constant depends upon $z$ and $Q$, but $k_0$ is independent of $i$. Therefore,

$$|(\mathcal{Q}_z\Psi)(r)| = \left|\frac{16\pi}{r^2}\sum_{k\ne 0}\int_0^\infty \Psi(\tilde r)\, s_{k,0}(r,\tilde r,z)\, d\tilde r\right|$$
$$\le \frac{16\pi C_{1,0}}{r^2}\, \mathbf{1}_{\{0\le r\le r_Q\}}\int_0^{r_Q} |\Psi(\tilde r)| \min\{r^2,\tilde r^2\}\, d\tilde r.$$

Next, note that

$$\min\{r^2, \tilde r^2\} \le r\tilde r. \tag{4.12}$$

Thus, using Hölder's inequality,

$$|(\mathcal{Q}_z\Psi)(r)|^2 \le \frac{256\pi^2 C_{1,0}^2}{r^2}\, \mathbf{1}_{\{0\le r\le r_Q\}}\left(\int_0^{r_Q} \tilde r\, |\Psi(\tilde r)|\, d\tilde r\right)^2$$
$$\le \frac{256\pi^2 C_{1,0}^2\, r_Q}{r^2}\, \mathbf{1}_{\{0\le r\le r_Q\}}\int_0^{r_Q} \tilde r^2\, |\Psi(\tilde r)|^2\, d\tilde r$$
$$\le \frac{64\pi C_{1,0}^2\, r_Q}{r^2}\, \mathbf{1}_{\{0\le r\le r_Q\}}\, \|\Psi\|_{L_r^2}^2,$$

and this in turn leads to

$$\|\mathcal{Q}_z\Psi\|_{L_r^2}^2 = 4\pi\int_0^\infty r^2\, |(\mathcal{Q}_z\Psi)(r)|^2\, dr \le 264\pi^2 C_{1,0}^2\, r_Q^2\, \|\Psi\|_{L_r^2}^2.$$

To prove the analyticity of (4.5), we recall that it suffices to show weak analyticity, in the sense that all maps $\Omega \ni z \mapsto \langle \Psi, \mathcal{Q}_z\Phi\rangle \in \mathbb{C}$ for $\Psi, \Phi \in L_r^2$ are analytic; see [85, Thm. 3.1.12]. Fix $z_0 \in \Omega$. If $|z - z_0|$ is sufficiently small, then $z \in \Omega$ and we have the series expansion

$$\frac{1}{k^2\omega_1^2(e,\ell) - z} = \sum_{i=0}^{\infty}\frac{1}{(k^2\omega_1^2(e,\ell) - z_0)^{i+1}}\,(z - z_0)^i$$

for every $k \neq 0$ and $(e, l) \in D$, which suggests that

$$\begin{aligned}\langle \Psi, Q_z \Phi \rangle &= 64\pi^2 \int_0^{r_Q} \int_0^{r_Q} dr\, d\tilde{r}\, \overline{\Psi(r)}\, \Phi(\tilde{r}) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}} \\ &\qquad \times \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell) - z} \sin(k\theta(r,e,\ell)) \sin(k\theta(\tilde{r},e,\ell)) \\ &= \sum_{i=0}^{\infty} a_i (z - z_0)^i \end{aligned} \tag{4.13}$$

for

$$\begin{aligned} a_i &= 64\pi^2 \int_0^{r_Q} \int_0^{r_Q} dr\, d\tilde{r}\, \overline{\Psi(r)}\, \Phi(\tilde{r}) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}} \\ &\qquad \times \frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell) - z_0)^{i+1}} \sin(k\theta(r,e,\ell)) \sin(k\theta(\tilde{r},e,\ell)). \end{aligned}$$

We are going to show that the series (4.13) converges near $z_0$. For this, due to (4.10) and (4.12), we deduce that

$$\begin{aligned} |a_i| &= 64\pi^2 \left| \int_0^{r_Q} \int_0^{r_Q} dr\, d\tilde{r}\, \overline{\Psi(r)}\, \Phi(\tilde{r}) \sum_{k\neq 0} s_{k,i}(r,\tilde{r},z_0) \right| \\ &\le 64\pi^2\, C_{1,i} \int_0^{r_Q} \int_0^{r_Q} dr\, d\tilde{r}\, |\Psi(r)|\, |\Phi(\tilde{r})| \min\{r^2, \tilde{r}^2\} \\ &\le 64\pi^2\, C_{1,i} \left( \int_0^{r_Q} r\, |\Psi(r)|\, dr \right) \left( \int_0^{r_Q} \tilde{r}\, |\Phi(\tilde{r})|\, d\tilde{r} \right) \\ &\le 16\pi r_Q\, C_{1,i}\, \|\Psi\|_{L^2_r} \|\Phi\|_{L^2_r}. \end{aligned}$$

If we write the constant $C_{1,i}$ from (4.11) as $C_{1,i} = \tilde{C}_1 \alpha_0^{-(i+1)} + \hat{C}_1$, with $\alpha_0$ depending only on $z_0$, then $|z - z_0| < \min\{\frac{\alpha_0}{2}, \frac{1}{2}\}$ ensures that

$$C_{1,i} |z - z_0|^i \le \tilde{C}_1 \alpha_0^{-1} 2^{-i} + \hat{C}_1 2^{-i},$$

which has a finite $\sum_{i=0}^{\infty}$. It follows that (4.13) converges for $z \in \Omega$ such that $|z - z_0| < \min\{\frac{\alpha_0}{2}, \frac{1}{2}\}$, i.e., on a sufficiently small ball about $z_0$. The formula for the derivative is gotten from $a_1$ and those for the higher order derivatives follow from this one inductively.

(b) By the definition of $K_z$ in (4.2), we have

$$K_z(r, \tilde{r}) = \frac{4}{r^2 \tilde{r}^2} \sum_{k\neq 0} s_{k,0}(r, \tilde{r}, z). \tag{4.14}$$

Hence,

$$(\mathcal{Q}_z\Psi)(r) = \frac{16\pi}{r^2}\sum_{k\neq 0}\int_0^\infty \Psi(\tilde r)\, s_{k,0}(r,\tilde r,z)\, d\tilde r = 4\pi\int_0^\infty \tilde r^2\, K_z(r,\tilde r)\,\Psi(\tilde r)\, d\tilde r = \langle K_{\bar z}(r,\cdot),\Psi\rangle, \tag{4.15}$$

observing that $\overline{K_z} = K_{\bar z}$. Due to $K_z(r,\tilde r) = K_z(\tilde r, r)$, we hence obtain

$$\begin{aligned}
\langle \mathcal{Q}_z\Psi,\Phi\rangle &= 4\pi\int_0^\infty r^2\,\overline{(\mathcal{Q}_z\Psi)(r)}\,\Phi(r)\,dr = 4\pi\int_0^\infty r^2\,\overline{\langle K_{\bar z}(r,\cdot),\Psi\rangle}\,\Phi(r)\,dr \\
&= 16\pi^2\int_0^\infty dr\, r^2\int_0^\infty d\tilde r\,\tilde r^2\, K_{\bar z}(r,\tilde r)\,\overline{\Psi(\tilde r)}\,\Phi(r) \\
&= 16\pi^2\int_0^\infty d\tilde r\,\tilde r^2\,\overline{\Psi(\tilde r)}\int_0^\infty dr\, r^2\, K_{\bar z}(\tilde r, r)\,\Phi(r) \\
&= 4\pi\int_0^\infty d\tilde r\,\tilde r^2\,\overline{\Psi(\tilde r)}\,\langle \overline{K_{\bar z}(\tilde r,\cdot)},\Phi\rangle = \langle \Psi, \mathcal{Q}_{\bar z}\Phi\rangle.
\end{aligned}$$

(c) According to (b), the operator $\mathcal{Q}_z$ on $L_r^2$ has the integral kernel $K_{\bar z}$. Hence, in order to verify that $\mathcal{Q}_z$ is Hilbert-Schmidt, we need to verify that

$$\begin{aligned}
\|\mathcal{Q}_z\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |K_z(x,\bar x)|^2\, dx\, d\bar x \\
&= 16\pi^2\int_0^\infty\int_0^\infty r^2\,\tilde r^2\,|K_z(r,\tilde r)|^2\, dr\, d\tilde r \\
&= 16\pi^2\int_0^{r_Q}\int_0^{r_Q} r^2\,\tilde r^2\,|K_z(r,\tilde r)|^2\, dr\, d\tilde r < \infty
\end{aligned} \tag{4.16}$$

for every $z \in \Omega$, where $K_z$ is viewed both as a function of $(x,\bar x)$ and a function of $(r,\tilde r)$ and we used Remark 4.2(b); see [35, Prop. 6.36]. From (4.14), (4.10) and (4.12), we get

$$\begin{aligned}
\int_0^{r_Q}\int_0^{r_Q} r^2\,\tilde r^2\,|K_z(r,\tilde r)|^2\, dr\, d\tilde r &\le 16\int_0^{r_Q}\int_0^{r_Q}\frac{1}{r^2\tilde r^2}\Big(\sum_{k\neq 0}|s_{k,0}(r,\tilde r,z)|\Big)^2 dr\, d\tilde r \\
&\le 16\, C_1^2\int_0^{r_Q}\int_0^{r_Q}\frac{1}{r^2\tilde r^2}\,(\min\{r^2,\tilde r^2\})^2\, dr\, d\tilde r \\
&\le 16\, C_1^2\int_0^{r_Q}\int_0^{r_Q} dr\, d\tilde r = 16\, C_1^2 r_Q^2 < \infty.
\end{aligned}$$

Note that from $\mathcal{Q}_z$ being Hilbert-Schmidt it follows that $\mathcal{Q}_z$ is bounded and $\|\mathcal{Q}_z\| \le \|\mathcal{Q}_z\|_{\mathrm{HS}}$, i.e., once again we see that (a) holds. However, since the key of the argument is (4.10) and (4.12), it needs very little additional work to derive both bounds. (d) Here, we calculate

$$\begin{aligned}\langle \mathcal{Q}_z \Psi, \Psi \rangle &= 4\pi \int_0^\infty r^2\, \overline{(\mathcal{Q}_z \Psi)(r)}\, \Psi(r)\, dr \\ &= 64\pi^2 \sum_{k\neq 0} \int_0^\infty dr\, \Psi(r) \int_0^\infty d\tilde{r}\, \overline{\Psi(\tilde{r})} \\ &\qquad \times \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}}\, \frac{\omega_1(e,\ell)\, |Q'(e)|}{k^2\omega_1^2(e,\ell) - \bar{z}} \\ &= 64\pi^2 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e,\ell)\, |Q'(e)|}{k^2\omega_1^2(e,\ell) - \bar{z}} \\ &\qquad \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi(r) \sin(k\theta(r,e,\ell))\, dr \right|^2 \ge 0.\end{aligned}$$

The proof of (4.6) is analogous. (e) For $\lambda, \tilde{\lambda} < \delta_1^2$, we have, cf. (4.16),

$$\begin{aligned}\|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|_{\mathrm{HS}}^2 &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |K_\lambda(x,\bar{x}) - K_{\tilde{\lambda}}(x,\bar{x})|^2\, dx\, d\bar{x} \\ &= 16\pi^2 \int_0^{r_Q}\int_0^{r_Q} r^2\, \bar{r}^2\, |K_\lambda(r,\bar{r}) - K_{\tilde{\lambda}}(r,\bar{r})|^2\, dr\, d\bar{r} \\ &= 256\,\pi^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \left| \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}} \right. \\ &\qquad \times \omega_1(e,\ell)\, |Q'(e)| \left[ \frac{1}{k^2\omega_1^2(e,\ell) - \lambda} - \frac{1}{k^2\omega_1^2(e,\ell) - \tilde{\lambda}} \right] \\ &\qquad \left. \times \sin(k\theta(r,e,\ell))\, \sin(k\theta(\tilde{r},e,\ell)) \right|^2 \\ &\le 512\,\pi^2 \Delta_1^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \\ &\qquad \times \Bigg( \sum_{k=1}^\infty \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde{r}\le r_+(e,\ell)\}}\, |Q'(e)| \\ &\qquad \left| \frac{1}{k^2\omega_1^2(e,\ell) - \lambda} - \frac{1}{k^2\omega_1^2(e,\ell) - \tilde{\lambda}} \right| \Bigg)^2.\end{aligned}$$

Using (4.8) and (4.12), we may continue this estimate for suitable constants $C, \hat{C} > 0$ as

$$\begin{aligned}\|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|_{\mathrm{HS}}^2 &\le 256\,\pi^2 \Delta_1^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \\ &\qquad \times \Bigg( \sum_{k=1}^\infty \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\le C\min\{r^2,\bar{r}^2\}\}}\, |Q'(e)|\end{aligned}$$

$$\left|\frac{1}{k^2\omega_1^2(e,\beta)-\lambda}-\frac{1}{k^2\omega_1^2(e,\beta)-\tilde{\lambda}}\right|\Bigg)^2 \quad (4.17)$$

$$\leq 256\,\pi^2\Delta_1^2\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2}$$

$$\times\Bigg(\sum_{k=1}^{\infty}\iint_D d\beta\,de\,\mathbf{1}_{\{\beta\leq\hat{C}r\bar{r}\}}\,|Q'(e)|$$

$$\frac{|\lambda-\tilde{\lambda}|}{(k^2\omega_1^2(e,\beta)-\lambda)(k^2\omega_1^2(e,\beta)-\tilde{\lambda})}\Bigg)^2.$$

For $k\geq 2$, we know from Remark 4.2(a) that $k^2\omega_1^2(e,\beta)-\lambda\geq k^2\delta_1^2/2$ and $k^2\omega_1^2(e,\beta)-\tilde{\lambda}\geq k^2\delta_1^2/2$ are verified. If $k=1$, then always $\omega_1^2(e,\beta)-\lambda\geq\delta_1^2-\lambda$ and $\omega_1^2(e,\beta)-\tilde{\lambda}\geq\delta_1^2-\tilde{\lambda}$ hold. Thus, we arrive at

$$\|\mathcal{Q}_\lambda-\mathcal{Q}_{\tilde{\lambda}}\|_{\mathrm{HS}}^2\leq C|\lambda-\tilde{\lambda}|^2\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2}\,r^2\tilde{r}^2$$

$$\times\left(\delta_1^{-4}\sum_{k=2}^{\infty}\frac{1}{k^4}\int_{U_Q(0)}^{e_0}|Q'(e)|\,de\right)^2$$

$$+C\,\frac{|\lambda-\tilde{\lambda}|^2}{(\delta_1^2-\lambda)^2(\delta_1^2-\tilde{\lambda})^2}\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2}\,r^2\,\tilde{r}^2$$

$$\times\left(\int_{U_Q(0)}^{e_0}|Q'(e)|\,de\right)^2$$

$$\leq C\left(1+\frac{1}{(\delta_1^2-\lambda)^2(\delta_1^2-\tilde{\lambda})^2}\right)|\lambda-\tilde{\lambda}|^2,$$

and this yields the claim. (f) According to (b–d), $\mathcal{Q}_\lambda$ is a symmetric and positive Hilbert-Schmidt operator, which is in particular compact. Thus, the assertions up to and including (4.7) are a consequence of the spectral theory for compact positive self-adjoint operators; see [35, Section 6]. Concerning the $\mu_k(\lambda)$, we have the characterization

$$\mu_k(\lambda)=\max\left\{\min_{\Psi\in S,\,\|\Psi\|_{L_r^2}=1}\langle\mathcal{Q}_\lambda\Psi,\Psi\rangle : S\subset L_r^2\text{ is a subspace of dimension }k\right\} \quad (4.18)$$

according to the Courant max-min principle. In the present situation, this follows from the spectral decomposition theorem for symmetric and compact operators. By (d), we obtain for $\tilde{\lambda}\geq\lambda$, both in $]-\infty,\delta_1^2[$ and $\Psi\in L_r^2$,

$$\langle\mathcal{Q}_{\tilde{\lambda}}\Psi,\Psi\rangle=64\pi^2\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\frac{\omega_1\,|Q'(e)|}{k^2\omega_1^2-\tilde{\lambda}}\left|\int_{r_-}^{r_+}\Psi(r)\sin(k\theta)\,dr\right|^2$$

$$\geq 64\pi^2 \sum_{k\neq 0} \iint_D d\ell\,\ell\,de\,\frac{\omega_1\,|Q'(e)|}{k^2\omega_1^2-\lambda}\left|\int_{r_-}^{r_+}\Psi(r)\sin(k\theta)\,dr\right|^2$$
$$= \langle \mathcal{Q}_\lambda\Psi, \Psi\rangle, \tag{4.19}$$

where $r_\pm = r_\pm(e,\ell)$ and $\theta = \theta(r,e,\ell)$. Hence, (4.18) implies that $\mu_k(\tilde{\lambda}) \geq \mu_k(\lambda)$ for all $k\in\mathbb{N}$. To establish the local Lipschitz continuity of $\mu_k(\cdot)$, note that

$$|\langle \mathcal{Q}_\lambda\Psi,\Psi\rangle - \langle \mathcal{Q}_{\tilde{\lambda}}\Psi,\Psi\rangle| \leq \|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|\,\|\Psi\|_{L_r^2}^2,$$

whence we deduce from (e) and $\|\cdot\| \leq \|\cdot\|_{\mathrm{HS}}$ that for $\Psi\in L_r^2$ satisfying $\|\Psi\|_{L_r^2}\leq 1$, one has

$$|\langle \mathcal{Q}_\lambda\Psi,\Psi\rangle - \langle \mathcal{Q}_{\tilde{\lambda}}\Psi,\Psi\rangle| \leq C\left(1+\frac{1}{(\delta_1^2-\lambda)(\delta_1^2-\tilde{\lambda})}\right)|\lambda-\tilde{\lambda}|.$$

Applying (4.18) once more, we arrive at

$$|\mu_k(\lambda)-\mu_k(\tilde{\lambda})| \leq C\left(1+\frac{1}{(\delta_1^2-\lambda)(\delta_1^2-\tilde{\lambda})}\right)|\lambda-\tilde{\lambda}|,$$

which completes the proof. □

In the following, we are going to derive some more specific properties of the $\mathcal{Q}_z$. See Appendix II, Sect. B.1 below for the function spaces that are being used. Once again, we understand that $|Q'(e_Q)|$ vanishes outside of $K$.

**Lemma 4.4** *If $z\in\Omega$ and $\psi(r,p_r,\ell) = |Q'(e_Q)|\,p_r\Psi(r)$ for $\Psi\in L_r^2$, then $\psi\in X_{\mathrm{odd}}^0$,*

$$\|\psi\|_{X^0} \leq \rho_Q(0)^{1/2}\,\|\Psi\|_{L_r^2} \tag{4.20}$$

*and*

$$\mathcal{K}\mathcal{T}(-\mathcal{T}^2-z)^{-1}\psi = |Q'(e_Q)|\,p_r(\mathcal{Q}_z\Psi). \tag{4.21}$$

*In particular,*

$$\mathcal{Q}_z\Psi = U'_{\mathcal{T}(-\mathcal{T}^2-z)^{-1}\psi} = 4\pi\int_{\mathbb{R}^3} p_r\,(-\mathcal{T}^2-z)^{-1}\psi\,dv. \tag{4.22}$$

*Moreover, if also $\tilde{\psi}(r,p_r,\ell) = |Q'(e_Q)|\,p_r\tilde{\Psi}(r)$ for some $\tilde{\Psi}\in L_r^2$, then*

$$\|\psi-\tilde{\psi}\|_{X^0} \leq \rho_Q(0)^{1/2}\,\|\Psi-\tilde{\Psi}\|_{L_r^2}. \tag{4.23}$$

***Proof*** First, note that $\psi$ is odd in $v$ and has its support in $K$. Furthermore, due to Remark B.2(a), Lemma 2.5 and (A.32),

$$
\begin{aligned}
\|\psi\|_{X^0}^2 &= \|\psi\|^2_{L^2_{\text{sph},\,\frac{1}{|Q'|}}(K)} \\
&= \iint\limits_K \frac{1}{|Q'(e_Q)|}\,|\psi(x,v)|^2\,dx\,dv \\
&= \iint\limits_K |Q'(e_Q)|\,p_r^2\,|\Psi(r)|^2\,dx\,dv \\
&= \int_{|x|<r_Q} dx\,|\Psi(r)|^2 \int_{\mathbb{R}^3} dv\,|Q'(e_Q)|\,p_r^2 \\
&= \int_{|x|<r_Q} dx\,|\Psi(r)|^2\,\rho_Q(r) \\
&\le \rho_Q(0)\int_{|x|<r_Q} dx\,|\Psi(r)|^2 \le \rho_Q(0)\,\|\Psi\|^2_{L^2_r}.
\end{aligned}
$$

Thus, $\psi \in X^0_{\text{odd}} \subset X^0_0$, and accordingly Corollary B.14 yields

$$
\begin{aligned}
&\mathcal{K}\mathcal{T}(-\mathcal{T}^2 - z)^{-1}\psi \\
&= |Q'(e_Q)|\,p_r\,\frac{16\pi^2 i}{r^2}\sum_{k\neq 0}\iint\limits_D d\ell\,\ell\,de\,\mathbf{1}_{[r_-(e,\ell),\,r_+(e,\ell)]}(r)\,\frac{\sin(k\theta(r,e,\ell))}{k^2\omega_1^2(e,\ell)-z}\,\psi_k(I,\ell).
\end{aligned}
$$

On the other hand,

$$
\psi_k(I,\ell) = -\frac{i}{\pi}\,|Q'(e)|\,\omega_1(e,\ell)\int_{r_-(e,\ell)}^{r_+(e,\ell)} d\tilde r\,\Psi(\tilde r)\,\sin(k\theta(\tilde r,e,\ell)) \tag{4.24}
$$

by Lemma B.5. Therefore, we arrive at

$$
\begin{aligned}
\mathcal{K}\mathcal{T}(-\mathcal{T}^2 - z)^{-1}\psi &= |Q'(e_Q)|\,p_r\,\frac{16\pi}{r^2}\sum_{k\neq 0}\iint\limits_D d\ell\,\ell\,de\,\mathbf{1}_{[r_-(e,\ell),\,r_+(e,\ell)]}(r) \\
&\qquad\times \frac{\sin(k\theta(r,e,\ell))}{k^2\omega_1^2(e,\ell)-z}\,|Q'(e)|\,\omega_1(e,\ell) \\
&\qquad\qquad \int_{r_-(e,\ell)}^{r_+(e,\ell)} d\tilde r\,\Psi(\tilde r)\,\sin(k\theta(\tilde r,e,\ell)) \\
&= |Q'(e_Q)|\,p_r\,\frac{16\pi}{r^2}\sum_{k\neq 0}\int_0^{r_Q} d\tilde r\,\Psi(\tilde r) \\
&\qquad\qquad \iint\limits_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\le r,\,\tilde r\le r_+(e,\ell)\}}(r) \\
&\qquad\times \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-z}\,\sin(k\theta(r,e,\ell))\,\sin(k\theta(\tilde r,e,\ell)) \\
&= |Q'(e_Q)|\,p_r(\mathcal{Q}_z\Psi),
\end{aligned}
$$

and this completes the proof of (4.21), by the definition of $\mathcal{Q}_z$. Concerning (4.22), the first part follows from $\mathcal{K}g = |Q'(e_Q)|\, p_r\, U'_g(r)$, see (B.37), and for the second part, one just has to use Lemma 2.4. Lastly, (4.23) is a direct consequence of (4.20) and the fact that $(\tilde{\psi} - \psi)(r, p_r, \ell) = |Q'(e_Q)|\, p_r(\tilde{\Psi} - \Psi)(r)$. □

Now, we can make the connection from eigenvalues $\lambda < \delta_1^2$ of the self-adjoint operator

$$L = -\mathcal{T}^2 - \mathcal{K}\mathcal{T} : X^2_{\text{odd}} \to X^0_{\text{odd}},$$

cf. (1.16) and Corollary B.19, to eigenvalues 1 of $Q_\lambda$.

**Theorem 4.5** *Let $\lambda < \delta_1^2$. Then $\lambda$ is an eigenvalue of $L$ if and only if 1 is an eigenvalue of $\mathcal{Q}_\lambda$. More precisely,*

*(a) if $u \in X^2_{\text{odd}}$ is an eigenfunction of $L$ for the eigenvalue $\lambda$, then $\Psi = U'_{\mathcal{T}u} \in L^2_r$ for $r \in [0, r_Q]$ is an eigenfunction of $\mathcal{Q}_\lambda$ for the eigenvalue 1;*
*(b) if $\Psi \in L^2_r$ is an eigenfunction of $\mathcal{Q}_\lambda$ for the eigenvalue 1, then $u = (-\mathcal{T}^2 - \lambda)^{-1}(|Q'(e_Q)|\, p_r \Psi) \in X^2_{\text{odd}}$ is an eigenfunction of $L$ for the eigenvalue $\lambda$.*

***Proof*** First, suppose that $Lu = \lambda u$ for some $u \in X^2_{\text{odd}}$ and $u \neq 0$. Then $(-\mathcal{T}^2 - \lambda)u = \mathcal{K}\mathcal{T}u$. Defining $\psi = (-\mathcal{T}^2 - \lambda)u \in X^0_{\text{odd}}$, Remark B.18(a) implies that $\psi = \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi$. Since $\mathcal{K}g = |Q'(e_Q)|\, p_r\, U'_g(r)$ by (B.37), we can put

$$\Psi(r) = U'_{\mathcal{T}(-\mathcal{T}^2-\lambda)^{-1}\psi}(r) = U'_{\mathcal{T}u}(r)$$

for $r \in [0, r_Q]$ to obtain $\psi = |Q'(e_Q)|\, p_r \Psi(r)$. Then $\Psi \neq 0$, as otherwise $\psi = 0$ and $u = 0$. Next, we are going to verify that $\Psi \in L^2_r$. Using (B.40) from Lemma B.15 and Lemma B.8(c), we get

$$\begin{aligned}
\|\Psi\|^2_{L^2_r} &= \int_{\mathbb{R}^3} |U'_{\mathcal{T}(-\mathcal{T}^2-\lambda)^{-1}\psi}(r)|^2\, dx \\
&= 4\pi\Big(\mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi, (-\mathcal{T}^2 - \lambda)^{-1}\psi\Big)_{X^0} \\
&= 4\pi(\psi, (-\mathcal{T}^2 - \lambda)^{-1}\psi)_{X^0} \\
&= 4\pi((-\mathcal{T}^2 - \lambda)u, u)_{X^0} \\
&= 4\pi(\|\mathcal{T}u\|^2_{X^0} - \lambda\, \|u\|^2_{X^0}).
\end{aligned}$$

In particular, Lemma B.8(a) implies $\|\Psi\|^2_{L^2_r} \leq 4\pi\|\mathcal{T}u\|^2_{X^0} \leq 4\pi\Delta_1^2\, \|u\|^2_{X^1} < \infty$, so that indeed $\Psi \in L^2_r$. Thus, we deduce from Lemma 4.4 that

$$|Q'(e_Q)|\, p_r(\mathcal{Q}_\lambda\Psi) = \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi = \psi = |Q'(e_Q)|\, p_r\, \Psi,$$

and consequently $\mathcal{Q}_\lambda\Psi = \Psi$.

Conversely, suppose that $\mathcal{Q}_\lambda\Psi = \Psi$ is verified for some $\Psi \in L^2_r$ and $\Psi \neq 0$. According to Remark 4.2(b), $\Psi$ has its support in $[0, r_Q]$. Defining $\psi = |Q'(e_Q)$

$| p_r \Psi(r)$, we obtain $\psi \in X^0_{\rm odd}$ from Lemma 4.4. As a consequence, $u = (-\mathcal{T}^2 - \lambda)^{-1}\psi \in X^2_{\rm odd}$. Also $u \neq 0$, since otherwise $\psi = 0$ and $\Psi = 0$. From Lemma 4.4, we finally get

$$(-\mathcal{T}^2 - \lambda)u = \psi = |Q'(e_Q)|\, p_r\, \Psi = |Q'(e_Q)|\, p_r(\mathcal{Q}_\lambda \Psi)$$
$$= \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi = \mathcal{K}\mathcal{T}u,$$

so that $Lu = -\mathcal{T}^2 u - \mathcal{K}\mathcal{T}u = \lambda u$. □

**Lemma 4.6** *The following assertions hold.*

*(a) To* $\Psi \in L^2_r$ *we associate the function* $\psi(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi(r)$. *If* $z \in \Omega$, *then*

$$\langle \mathcal{Q}_z \Psi, \Psi\rangle = 64\pi^4 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, \frac{1}{k^2\omega_1^2(e,\ell) - \bar{z}}\, |\psi_k(I,\ell)|^2. \tag{4.25}$$

*(b) Let* $\Psi \in L^2_r$ *be given and suppose that* $F(r) = F(0) + \int_0^r \Psi(s)\, ds$ *for* $r \in [0, r_Q]$ *as well as* $g = -|Q'(e_Q)|(F - F_0)$, *where* $F_0$ *is the zero'th Fourier coefficient of* $F$. *Then* $\mathcal{Q}_0\Psi = U'_g$ *and furthermore*

$$\langle \mathcal{Q}_0 \Psi, \Psi\rangle = 4\pi \iint_K \frac{dx\, dv}{|Q'(e_Q)|}\, |g|^2 = 4\pi \iint_K |Q'(e_Q)|\,(F - F_0)^2\, dx\, dv. \tag{4.26}$$

*(c) Let* $\Psi \in L^2_r$ *be given and suppose that* $F(r) = F(0) + \int_0^r \Psi(s)\, ds$ *for* $r \in [0, r_Q]$. *Define* $u = -\mathcal{T}^{-1}(|Q'(e_Q)|(F - F_0))$. *Then* $u \in X^2_{\rm odd}$ *and*

$$(Lu, u)_{X^0} = \frac{1}{4\pi}\Big(\langle \mathcal{Q}_0\Psi, \Psi\rangle - \|\mathcal{Q}_0\Psi\|^2_{L^2_r}\Big). \tag{4.27}$$

***Proof*** (a) The relation (4.25) follows from Lemma 4.3(d) and (4.24).

(b) Owing to Lemma B.9, we have $g \in X^1_{\rm even}$ as well as $\mathcal{T}g = -\psi$ for $\psi$ as in (a). In addition, $g_0 = 0$ by (B.24), so that $g \in X^1_0$. Thus, Lemma B.13(c) yields $-\mathcal{T}^{-1}\psi = g - g_0 = g$.

Next, recall that $\psi$ is odd in $v$ and $\|\psi\|_{X^0} \le \rho_Q(0)^{1/2}\,\|\Psi\|_{L^2_r} < \infty$ by (4.20), which means that $\psi \in X^0_{\rm odd} \subset X^0_0$. As a consequence, $\mathcal{T}(-\mathcal{T}^2)^{-1}\psi = -\mathcal{T}^{-1}\psi = g$ by Lemma B.13(e). Hence, if we take $z = 0 \in \Omega$ in (4.22) of Lemma 4.4, then we get

$$\mathcal{Q}_0\Psi = U'_{\mathcal{T}(-\mathcal{T}^2)^{-1}\psi} = U'_g.$$

To verify (4.26), note first that $ik\omega_1 g_k = -\psi_k$ for $k \in \mathbb{Z}$. Applying (B.4) from Remark B.2(a), we obtain

$$\iint_K \frac{dx\,dv}{|Q'(e_Q)|}\,|g|^2 = 16\pi^3 \sum_{k\neq 0} \iint_D dI\,d\ell\,\ell\,\frac{1}{|Q'(e_Q)|}\,|g_k|^2$$
$$= 16\pi^3 \sum_{k\neq 0} \iint_D dI\,d\ell\,\ell\,\frac{1}{|Q'(e_Q)|}\,\frac{1}{k^2\omega_1^2}\,|\psi_k|^2$$
$$= 16\pi^3 \sum_{k\neq 0} \iint_D de\,d\ell\,\ell\,\frac{1}{|Q'(e_Q)|}\,\frac{1}{k^2\omega_1^3}\,|\psi_k|^2,$$

where we have used that $\frac{\partial e}{\partial I} = \omega_1$ owing to (A.18). Thus, the claim follow from (a) for $z = 0$. (c) We continue to use the notation and the observations from (b). Since $g \in X^1_0$, we have $u = \mathcal{T}^{-1} g \in X^2_0$. As also $g \in X^1_{\mathrm{even}}$ and $\mathcal{T}^{-1}$ reverses the parity by Remark B.18, we get $u \in X^2_{\mathrm{odd}}$. Accordingly, we deduce from (B.44) in Corollary B.19 that

$$(Lu, u)_{X^0} = \|\mathcal{T}u\|^2_{X^0} - (\mathcal{K}\mathcal{T}u, u)_{X^0}.$$

Now $\mathcal{T}u = \mathcal{T}\mathcal{T}^{-1}g = g$ due to Lemma B.13(d), so that

$$\|\mathcal{T}u\|^2_{X^0} = \|g\|^2_{X^0} = \|g\|^2_{L^2_{\mathrm{sph},\,\frac{1}{|Q'|}}(K)} = \frac{1}{4\pi}\,\langle \mathcal{Q}_0\Psi, \Psi\rangle$$

by Remark B.2(a) and (4.26). Furthermore, using (B.40) from Lemma B.15 in conjunction with (b), it follows that

$$(\mathcal{K}\mathcal{T}u, u)_{X^0} = \frac{1}{4\pi}\int_{\mathbb{R}^3} |U'_{\mathcal{T}u}|^2\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^3} |U'_g|^2\,dx$$
$$= \frac{1}{4\pi}\int_{\mathbb{R}^3} |\mathcal{Q}_0\Psi|^2\,dx = \frac{1}{4\pi}\,\|\mathcal{Q}_0\Psi\|^2_{L^2_r},$$

Altogether, this yields (4.27). □

**Lemma 4.7** *Let $\mu_1 : ]-\infty, \delta_1^2[\, \to\, ]0, \infty[$ be defined as in Lemma 4.3(f). Then*

*(a)* $0 < \mu_1(0) < 1$.

*(b)* *If $\lambda_* < \delta_1^2$ and $\lambda \in [0, \lambda_*]$, or $\lambda_* = \delta_1^2$ and $\lambda \in [0, \lambda_*[ = [0, \delta_1^2[$, then $\mu_1(\lambda) \le 1$.*

*(c)* *For $\lambda \in [0, \delta_1^2[$, let $\Psi_\lambda \in L^2_r$ denote a normalized eigenfunction of $\mathcal{Q}_\lambda$ for $\mu_1(\lambda)$. Define $\psi_\lambda(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_\lambda(r) \in X^0_{\mathrm{odd}}$ and $g_\lambda = (-\mathcal{T}^2 - \lambda)^{-1}\psi_\lambda \in X^2_{\mathrm{odd}}$. Then*

$$\mu_1(\lambda) = 4\pi\,(\psi_\lambda, g_\lambda)_{X^0}$$

*and*

$$Lg_\lambda = (1 - \mu_1(\lambda))\psi_\lambda + \lambda g_\lambda,$$

*as well as*

$$(Lg_\lambda, g_\lambda)_Q = \frac{1}{4\pi}\,\mu_1(\lambda)(1 - \mu_1(\lambda)) + \lambda\|g_\lambda\|^2_{X^0}.$$

*(d) The function* $\mu_1 : ]-\infty, \delta_1^2[ \to ]0, \infty[$ *is convex.*
*(e) We have*

$$\mu_1(\lambda) \le 16\pi\Bigg(\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\frac{d\tilde r}{\tilde r^2}\Bigg|\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\le r,\,\tilde r\le r_+(e,\ell)\}} \times\frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-\lambda}\sin(k\theta(r,e,\ell))\sin(k\theta(\tilde r,e,\ell))\Bigg|^2\Bigg)^{1/2}.$$

***Proof*** (a) Clearly $\mu_1(0) > 0$, since otherwise $\|\mathcal{Q}_0\| = 0$, and thus $\mathcal{Q}_0 = 0$. To show that $\mu_1(0) < 1$, let $\Psi \in L_r^2$ be given. Define $F(r) = \int_0^r \Psi(s)\,ds$ as well as $u = -\mathcal{T}^{-1}(|Q'(e_Q)|(F - F_0))$. Then $u \in X^2_{\text{odd}}$ and

$$0 \le \lambda_*\|u\|^2_{X^0} \le (Lu, u)_{X^0} = \frac{1}{4\pi}\left(\langle\mathcal{Q}_0\Psi,\Psi\rangle - \|\mathcal{Q}_0\Psi\|^2_{L_r^2}\right) \tag{4.28}$$

by (1.20) and (4.27) from Lemma 4.6. As a consequence,

$$\|\mathcal{Q}_0\Psi\|^2_{L_r^2} \le \langle\mathcal{Q}_0\Psi,\Psi\rangle \le \|\mathcal{Q}_0\Psi\|_{L_r^2}\|\Psi\|_{L_r^2}$$

implies that $\mu_1(0) = \|\mathcal{Q}_0\| \le 1$. Lastly, suppose that $\mu_1(0) = 1$. Since $\mu_1(0)$ is an eigenvalue, we have $\mathcal{Q}_0\Psi = \Psi$ for some $\Psi = \Psi(r) \neq 0$ such that $\Psi \in L_r^2$; Remark 4.2(b) implies that $\Psi$ has its support in $[0, r_Q]$. For the corresponding $u$, we deduce $u = 0$ from (4.28). Therefore, (B.24), Lemma B.13(d) and Lemma B.9(b) lead to

$$\begin{aligned}0 = \mathcal{T}^2 u &= -\mathcal{T}^2\mathcal{T}^{-1}(|Q'(e_Q)|(F - F_0))\\ &= -\mathcal{T}(|Q'(e_Q)|(F - F_0)) = -|Q'(e)|\,p_r\,\Psi,\end{aligned}$$

which is impossible. (b) Recall from Lemma 3.18 that $\lambda_* \le \delta_1^2$. Thus, if we fix $\lambda$ in one of the two cases: (i) $\lambda_* < \delta_1^2$ and $\lambda \in [0, \lambda_*]$; or (ii) $\lambda_* = \delta_1^2$ and $\lambda \in [0, \lambda_*[= [0, \delta_1^2[$, then $\lambda \in [0, \delta_1^2[$. Let $\Psi_\lambda \in L_r^2$ denote a normalized eigenfunction for $\mu_1(\lambda)$, i.e., we have $\mathcal{Q}_\lambda\Psi_\lambda = \mu_1(\lambda)\Psi_\lambda$ and $\|\Psi_\lambda\|_{L_r^2} = 1$. For $\psi_\lambda(r, p_r, \ell) = |Q'(e_Q)|\,p_r\Psi_\lambda(r)$, we get $\psi_\lambda \in X^0_{\text{odd}}$, cf. the proof of Lemma 4.6(a). Thus, $g_\lambda = (-\mathcal{T}^2 - \lambda)^{-1}\psi_\lambda \in X^2_{\text{odd}}$. Using (4.21) from Lemma 4.4, we calculate

$$\begin{aligned}\mathcal{K}\mathcal{T}g_\lambda &= \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi_\lambda = |Q'(e_Q)|\,p_r(\mathcal{Q}_\lambda\Psi_\lambda)\\ &= \mu_1(\lambda)\,|Q'(e_Q)|\,p_r\Psi_\lambda = \mu_1(\lambda)\psi_\lambda.\end{aligned}$$

In addition,

$$\mathcal{T}^2 g_\lambda = (\mathcal{T}^2 + \lambda)g_\lambda - \lambda g_\lambda = -\psi_\lambda - \lambda g_\lambda.$$

This yields

$$Lg_\lambda = -\mathcal{T}^2 g_\lambda - \mathcal{K}\mathcal{T}g_\lambda = (1 - \mu_1(\lambda))\psi_\lambda + \lambda g_\lambda \tag{4.29}$$

hence in particular

$$(Lg_\lambda, g_\lambda)_Q = (Lg_\lambda, g_\lambda)_{X^0} = (1-\mu_1(\lambda))\,(\psi_\lambda, g_\lambda)_{X^0} + \lambda\|g_\lambda\|^2_{X^0}. \tag{4.30}$$

Thus, by the Antonov stability estimate, Theorem 1.2,

$$\lambda_*\|g_\lambda\|^2_{X^0} \le (1-\mu_1(\lambda))\,(\psi_\lambda, g_\lambda)_{X^0} + \lambda\|g_\lambda\|^2_{X^0},$$

so that

$$0 \le (\lambda_* - \lambda)\|g_\lambda\|^2_{X^0} \le (1-\mu_1(\lambda))\,(\psi_\lambda, g_\lambda)_{X^0}. \tag{4.31}$$

Now, (B.26) in Corollary B.10 yields

$$\begin{aligned}(\psi_\lambda, g_\lambda)_{X^0} &= (\psi_\lambda, (-\mathcal{T}^2-\lambda)^{-1}\psi_\lambda)_{X^0} = ((-\mathcal{T}^2-\lambda)^{-1}\psi_\lambda, \psi_\lambda)_{X^0}\\ &= 16\pi^3 \sum_{k\neq 0}\iint_D dI\,d\ell\,\ell\,\frac{1}{|Q'(e)|}\,\frac{|(\psi_\lambda)_k(I,\ell)|^2}{k^2\omega_1^2(I,\ell)-\lambda},\end{aligned} \tag{4.32}$$

and in particular $(\psi_\lambda, g_\lambda)_{X^0} > 0$, as otherwise $\psi_\lambda = 0$ and consequently $\Psi_\lambda = 0$, which is impossible. Hence, (4.31) shows that $\mu_1(\lambda) \le 1$.
(c) Note that due to Lemma 4.6(a),

$$\begin{aligned}\mu_1(\lambda) &= \mu_1(\lambda)\|\Psi_\lambda\|^2_{L^2_r} = \langle\mu_1(\lambda)\Psi_\lambda, \Psi_\lambda\rangle = \langle\mathcal{Q}_\lambda\Psi_\lambda, \Psi_\lambda\rangle\\ &= 64\pi^4\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\,\frac{|(\psi_\lambda)_k(I,\ell)|^2}{k^2\omega_1^2(e,\ell)-\lambda}\\ &= 64\pi^4\sum_{k\neq 0}\iint_D d\ell\,\ell\,dI\,\frac{1}{|Q'(e)|}\,\frac{|(\psi_\lambda)_k(I,\ell)|^2}{k^2\omega_1^2(e,\ell)-\lambda},\end{aligned}$$

and therefore the first claim follows by comparing to (4.32). The other relations are due to (4.29) and (4.30). (d) If $\lambda \in ]-\infty, \delta_1^2[$ and $\Psi \in L^2_r$, then

$$\begin{aligned}\frac{d^2}{d\lambda^2}\langle\mathcal{Q}_\lambda\Psi, \Psi\rangle &= \langle\mathcal{Q}''_\lambda\Psi, \Psi\rangle\\ &= 128\pi^2\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell)-\lambda)^3}\\ &\qquad\times\left|\int_{r_-(e,\ell)}^{r_+(e,\ell)}\Psi(r)\sin(k\theta(r,e,\ell))\,dr\right|^2\\ &\ge 0\end{aligned}$$

by (4.6) from Lemma 4.3(d). Thus, every function $]-\infty, \delta_1^2[\ni\lambda\mapsto\langle\mathcal{Q}_\lambda\Psi, \Psi\rangle$ is convex. As a consequence of (4.7), also $\mu_1(\lambda) = \sup\{\langle\mathcal{Q}_\lambda\Psi, \Psi\rangle : \|\Psi\|_{L^2_r} \le 1\}$ is convex. (e) Here, we use

$$\mu_1(\lambda) = \|\mathcal{Q}_\lambda\|_{\mathcal{B}(L_r^2)} \leq \|\mathcal{Q}_\lambda\|_{\mathrm{HS}}$$

and the fact that

$$\begin{aligned}\|\mathcal{Q}_\lambda\|_{\mathrm{HS}}^2 &= 16\pi^2 \int_0^{r_Q}\int_0^{r_Q} r^2\,\tilde{r}^2\,|K_\lambda(r,\tilde{r})|^2\,dr\,d\tilde{r}\\ &= 256\pi^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\tilde{r}}{\tilde{r}^2}\Bigg|\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\leq r,\,\tilde{r}\leq r_+(e,\ell)\}}\\ &\qquad\times \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-\lambda}\,\sin(k\theta(r,e,\ell))\,\sin(k\theta(\tilde{r},e,\ell))\Bigg|^2,\end{aligned}$$

cf. [35, Prop. 6.36] and (4.16). □

According to Lemma 4.3(f), the monotone limits

$$\mu_{*,k} = \lim_{\lambda\to\delta_1^2-}\mu_k(\lambda) = \sup\{\mu_k(\lambda) : \lambda\in[0,\delta_1^2[\} \in [\mu_k(0),\infty]$$

do exist. Of particular importance to us will be the number

$$\mu_* = \mu_{*,1} = \lim_{\lambda\to\delta_1^2-}\mu_1(\lambda) = \sup\{\mu_1(\lambda) : \lambda\in[0,\delta_1^2[\} \in [\mu_1(0),\infty]. \tag{4.33}$$

**Remark 4.8** If $\lambda_* = \delta_1^2$, then $\mu_* \leq 1$. This follows from Lemma 4.7(b). ◇

The next result will use assumption ($\omega_1$-3). If $\omega_1$ attains its minimum at an interior point $(\hat{e},\hat{\beta})\in\mathring{D}$, then we are in the situation of ($\omega_1$-2), and Corollary 4.16 below applies. Otherwise, since $\omega_1$ is continuous on $D$, its minimum is attained on the boundary $\partial D$, which consists of three parts: the left side

$$\{(e,0) : e\in[U_Q(0),e_0]\},$$

the lower boundary curve

$$\{(e,\beta) : e = e_{\min}(\beta),\ \beta\in[0,\beta_*]\}$$

and the upper line

$$\{(e_0,\beta) : \beta\in[0,\beta_*]\}. \tag{4.34}$$

Corollary 3.16 shows that the minimum can only be attained on this upper line (4.34) at a point $(e_0,\hat{\beta})$, and ($\omega_1$-3) roughly concerns the case where both $\frac{\partial\omega_1}{\partial e}(e_0,\hat{\beta})\neq 0$ and $\frac{\partial\omega_1}{\partial\beta}(e_0,\hat{\beta})\neq 0$, which is reasonable to expect for a minimum on the boundary.

**Lemma 4.9** *Suppose that ($\omega_1$-3) is satisfied. Then*

$$\mathcal{Q}_{\delta_1^2} = \lim_{\lambda \to \delta_1^2 -} \mathcal{Q}_\lambda \tag{4.35}$$

*does exist in the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ of $L^2_r$. In particular, the kernel of the symmetric and positive Hilbert-Schmidt operator $\mathcal{Q}_{\delta_1^2}$ is given by*

$$K_{\delta_1^2}(r,\tilde{r}) = \frac{4}{r^2\tilde{r}^2} \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \mathbf{1}_{\{r_-(e,\ell)\leq r,\, \tilde{r}\leq r_+(e,\ell)\}} \times \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-\delta_1^2}\, \sin(k\theta(r,e,\ell))\sin(k\theta(\tilde{r},e,\ell)),$$

*and $\mu_* = \|\mathcal{Q}_{\delta_1^2}\| < \infty$. More generally, the k'th eigenvalue of $\mathcal{Q}_{\delta_1^2}$ is $\mu_{*,k}$. For $k \in \mathbb{N}$, the functions*

$$\mu_k(\cdot) : ]-\infty, \delta_1^2] \to ]0,\infty[$$

*are monotone increasing, locally Lipschitz continuous on $]-\infty,\delta_1^2[$ and continuous on $]-\infty,\delta_1^2]$, if we set $\mu_k(\delta_1^2) = \mu_{*,k}$. In particular, the $\mu_k$ are differentiable a.e. Furthermore, $\mu_1 : ]-\infty,\delta_1^2] \to ]0,\infty[$ is a convex function.*

***Proof*** We need to refine (4.17), from where we know that

$$\|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|^2_{\mathrm{HS}} \leq 256\,\pi^2\Delta_1^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \left( \sum_{k=1}^{\infty} \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\leq \hat{C}r\bar{r}\}}\,|Q'(e)| \times \left| \frac{1}{k^2\omega_1^2(e,\beta)-\lambda} - \frac{1}{k^2\omega_1^2(e,\beta)-\tilde{\lambda}} \right| \right)^2$$

for $\lambda, \tilde{\lambda} < \delta_1^2$ and a suitable constant $\hat{C} > 0$. Thus

$$\begin{aligned} &\|\mathcal{Q}_\lambda - \mathcal{Q}_{\tilde{\lambda}}\|^2_{\mathrm{HS}} \\ &\leq 512\,\pi^2\Delta_1^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \left( \sum_{k=2}^{\infty} \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\leq \hat{C}r\bar{r}\}}\,|Q'(e)| \times \frac{|\lambda-\tilde{\lambda}|}{(k^2\omega_1^2(e,\beta)-\lambda)(k^2\omega_1^2(e,\beta)-\tilde{\lambda})} \right)^2 \\ &\quad + 512\,\pi^2\Delta_1^2 \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\,\frac{d\bar{r}}{\bar{r}^2} \left( \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\leq \hat{C}r\bar{r}\}}\,|Q'(e)| \times \left| \frac{1}{\omega_1^2(e,\beta)-\lambda} - \frac{1}{\omega_1^2(e,\beta)-\tilde{\lambda}} \right| \right)^2 \end{aligned}$$

$$
\begin{aligned}
&\le 8192\,\pi^2\Delta_1^2\,\delta_1^{-8}\,|\lambda-\tilde\lambda|^2\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar r}{\bar r^2}\left(\sum_{k=2}^{\infty}\frac{1}{k^4}\iint\limits_D d\beta\,de\,\mathbf{1}_{\{\beta\le\hat C r\bar r\}}\,|Q'(e)|\right)^2\\
&\quad+1024\,\pi^2\Delta_1^2\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar r}{\bar r^2}\left(\iint\limits_D d\beta\,de\,\mathbf{1}_{\{\beta\le\hat C r\bar r\}}\,|Q'(e)|\right.\\
&\qquad\qquad\qquad\left.\times\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda}-\frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|\right)^2\\
&\quad+1024\,\pi^2\Delta_1^2\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar r}{\bar r^2}\left(\iint\limits_D d\beta\,de\,\mathbf{1}_{\{\beta\le\hat C r\bar r\}}\,|Q'(e)|\right.\\
&\qquad\qquad\qquad\left.\times\Big|\frac{1}{\omega_1^2(e,\beta)-\tilde\lambda}-\frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|\right)^2\\
&\le C|\lambda-\tilde\lambda|^2\int_0^{r_Q}\int_0^{r_Q}dr\,d\bar r\left(\int_{U_Q(0)}^{e_0}|Q'(e)|\,de\right)^2+CT(\lambda)+CT(\tilde\lambda)\\
&\le C|\lambda-\tilde\lambda|^2+CT(\lambda)+CT(\tilde\lambda),
\end{aligned}
\tag{4.36}
$$

where

$$
T(\lambda)=\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\,\frac{d\bar r}{\bar r^2}\left(\iint\limits_D d\beta\,de\,\mathbf{1}_{\{\beta\le\hat C r\bar r\}}\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda}-\frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|\right)^2.
$$

We assert that

$$
\lim_{\lambda\to\delta_1^2-}T(\lambda)=0,
\tag{4.37}
$$

and to establish this claim, we are going to use Lebesgue's dominated convergence in $\int_0^{r_Q}\int_0^{r_Q}dr\,d\bar r$ together with the condition

$$
|\omega_1(e,\beta)-\delta_1|\ge c_1|(e,\beta)-(e_0,\hat\beta)|,\quad (e,\beta)\in D,
\tag{4.38}
$$

from ($\omega_1$-3), where $(e_0,\hat\beta)\in D$ satisfies $\omega_1(e_0,\hat\beta)=\delta_1$. Let $r,\bar r>0$ be fixed and define

$$
\tau(r,\bar r)=\iint\limits_D d\beta\,de\,\mathbf{1}_{\{\beta\le\hat C r\bar r\}}\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda}-\frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|.
$$

If $(e,\beta)\in D$ are such that $\beta\le\hat C r\bar r$ and $(e,\beta)\ne(\hat e,\hat\beta)$, then $\omega_1(e,\beta)-\delta_1\ge\alpha>0$ for $\alpha=\alpha(e,\beta)$ by (4.38), and accordingly

$$
\begin{aligned}
\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda}-\frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|&=\frac{\delta_1^2-\lambda}{(\omega_1^2(e,\beta)-\lambda)(\omega_1^2(e,\beta)-\delta_1^2)}\\
&\le\delta_1^{-2}\alpha^{-2}(\delta_1^2-\lambda)\to0,\quad\lambda\to\delta_1^2-,
\end{aligned}
\tag{4.39}
$$

for this $(e, \beta)$. On the other hand,

$$\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda} - \frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big| \le 2\delta_1^{-1}\frac{1}{\omega_1(e,\beta)-\delta_1} \le 2\delta_1^{-1}c_1^{-1}\frac{1}{|(e,\beta)-(e_0,\hat{\beta})|} \tag{4.40}$$

by (4.38). Next, we are going to bound

$$I(R) = \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\le R\}} \frac{1}{|(e,\beta)-(e_0,\hat{\beta})|}, \quad R > 0. \tag{4.41}$$

Case 1: $\hat{\beta} > 0$. If $\beta \le R \le \hat{\beta}/2$, then $|(e,\beta)-(e_0,\hat{\beta})| \ge |\beta-\hat{\beta}| \ge \hat{\beta}/2$ and hence

$$I(R) \le 2\hat{\beta}^{-1}(e_0 - U_Q(0))\, R, \quad R \le \hat{\beta}/2. \tag{4.42}$$

If $R \ge \hat{\beta}/2$, then we always have

$$\begin{aligned} I(R) &\le \int_0^{\beta_*} d\beta \int_{U_Q(0)}^{e_0} de\, \frac{1}{|(e-e_0, \beta-\hat{\beta})|} \le \int_{-\hat{\beta}}^{\beta_*-\hat{\beta}} dx_2 \int_0^{e_0-U_Q(0)} dx_1\, \frac{1}{|(x_1,x_2)|} \\ &\le \int_{-\beta_*}^{\beta_*} dx_2 \int_0^{e_0-U_Q(0)} dx_1\, \frac{1}{|(x_1,x_2)|} \le C. \end{aligned} \tag{4.43}$$

Case 2: $\hat{\beta} = 0$. Then

$$\begin{aligned} I(R) &\le \int_0^R d\beta \int_{U_Q(0)}^{e_0} de\, \frac{1}{|(e-e_0,\beta)|} \le \int_0^R dx_2 \int_0^{e_0-U_Q(0)} dx_1\, \frac{1}{|(x_1,x_2)|} \\ &= \int_0^R dx_2 \ln\Big(x_1 + \sqrt{x_1^2+x_2^2}\Big)\Big|_{x_1=0}^{x_1=e_0-U_Q(0)} \\ &= \int_0^R dx_2 \Big[\ln\Big(e_0 - U_Q(0) + \sqrt{(e_0-U_Q(0))^2 + x_2^2}\Big) - \ln x_2\Big] \\ &\le CR - R(\ln R - 1) \le CR - R\ln R. \end{aligned} \tag{4.44}$$

Thus, if we summarize (4.39) and (4.42)–(4.44) for $R = \hat{C}r\bar{r}$, it follows that $\tau(r,\bar{r}) \to 0$ as $\lambda \to \delta_1^2-$ for all $r, \bar{r} > 0$. Hence, to complete the proof of (4.37), we need to obtain an integrable majorant. For, using (4.40), we can bound

$$\begin{aligned} \mathcal{I}(\lambda) &= \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\frac{d\bar{r}}{\bar{r}^2}\left(\iint_D d\beta\, de\, \mathbf{1}_{\{\beta\le \hat{C}r\bar{r}\}}\Big|\frac{1}{\omega_1^2(e,\beta)-\lambda} - \frac{1}{\omega_1^2(e,\beta)-\delta_1^2}\Big|\right)^2 \\ &= \int_0^{r_Q}\int_0^{r_Q} \frac{dr}{r^2}\frac{d\bar{r}}{\bar{r}^2}\,\tau(r,\bar{r})^2 \le C\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\frac{d\bar{r}}{\bar{r}^2}\, I(\hat{C}r\bar{r})^2. \end{aligned}$$

Case 1: $\hat{\beta} > 0$. Let $\hat{\varepsilon} = \min\{r_Q, \frac{\hat{\beta}}{2\hat{C}r_Q}\}$. If $r \le \hat{\varepsilon}$ or $\hat{r} \le \hat{\varepsilon}$, then $\hat{C}r\bar{r} \le \hat{C}\hat{\varepsilon}r_Q \le \hat{\beta}/2$, and thus we can apply (4.42) in this case, as well as (4.43) in the opposite case. Therefore, we split the integral to obtain

$$\begin{aligned}\mathcal{I}(\lambda) &\le C\int_0^{r_Q}\int_0^{r_Q} dr\, d\bar{r}\, \mathbf{1}_{\{r\le\hat{\varepsilon} \text{ or } \hat{r}\le\hat{\varepsilon}\}}\frac{1}{r^2\bar{r}^2}\, I(\hat{C}r\bar{r})^2\\ &\quad + C\int_0^{r_Q}\int_0^{r_Q} dr\, d\bar{r}\, \mathbf{1}_{\{r>\hat{\varepsilon} \text{ and } \hat{r}>\hat{\varepsilon}\}}\frac{1}{r^2\bar{r}^2}\, I(\hat{C}r\bar{r})^2\\ &\le C\int_0^{r_Q}\int_0^{r_Q} dr\, d\bar{r}\, \mathbf{1}_{\{r\le\hat{\varepsilon} \text{ or } \hat{r}\le\hat{\varepsilon}\}}\frac{1}{r^2\bar{r}^2}\, r^2\bar{r}^2\\ &\quad + C\int_0^{r_Q}\int_0^{r_Q} dr\, d\bar{r}\, \mathbf{1}_{\{r>\hat{\varepsilon} \text{ and } \hat{r}>\hat{\varepsilon}\}}\frac{1}{r^2\bar{r}^2}\\ &\le C\int_0^{r_Q}\int_0^{r_Q} dr\, d\bar{r},\end{aligned}$$

which shows that a suitably large constant provides an integrable majorant. Case 2: $\hat{\beta} = 0$. By (4.44), we get

$$\begin{aligned}\mathcal{I}(\lambda) &\le C\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\frac{d\bar{r}}{\bar{r}^2}\, I(\hat{C}r\bar{r})^2\\ &\le C\int_0^{r_Q}\int_0^{r_Q}\frac{dr}{r^2}\frac{d\bar{r}}{\bar{r}^2}\,(C\hat{C}r\bar{r} - \hat{C}r\bar{r}\ln(\hat{C}r\bar{r}))^2\\ &\le C\int_0^{r_Q}\int_0^{r_Q}(1-\ln(\hat{C}r\bar{r}))^2\, dr\, d\bar{r}\\ &\le C\int_0^{r_Q}\int_0^{r_Q}(1+|\ln r|^2 + |\ln\bar{r}|^2)\, dr\, d\bar{r}.\end{aligned}$$

Since $1 + |\ln r|^2 + |\ln\bar{r}|^2$ is integrable on $[0, r_Q]\times[0, r_Q]$, we have found an integrable majorant also in this case. Altogether, we have shown that (4.37) is verified. At the same time, this yields $\lim_{\tilde{\lambda}\to\delta_1^2-} T(\tilde{\lambda}) = 0$, and going back to (4.36), we deduce that (4.35) holds for an appropriate Hilbert-Schmidt operator $\mathcal{Q}_{\delta_1^2}$ on $L_r^2$. Since $\|\cdot\|_{\mathcal{B}(L_r^2)} \le \|\cdot\|_{\mathrm{HS}}$, (4.35) in particular implies that $\mathcal{Q}_{\delta_1^2} = \lim_{\lambda\to\delta_1^2-}\mathcal{Q}_\lambda$ in $\mathcal{B}(L_r^2)$. Recalling from (4.7) that $\mu_1(\lambda) = \|\mathcal{Q}_\lambda\|_{\mathcal{B}(L_r^2)}$, we can use (4.33) to get

$$\mu_* = \lim_{\lambda\to\delta_1^2-}\mu_1(\lambda) = \lim_{\lambda\to\delta_1^2-}\|\mathcal{Q}_\lambda\|_{\mathcal{B}(L_r^2)} = \|\mathcal{Q}_{\delta_1^2}\|_{\mathcal{B}(L_r^2)},$$

as claimed.

Let $\kappa_1 \ge \kappa_2 \ge \ldots \to 0$ denote the eigenvalues (listed according to their multiplicities) of the symmetric and positive Hilbert-Schmidt operator $\mathcal{Q}_{\delta_1^2}$. Then

$$\kappa_k = \max\left\{\min_{\Psi\in S,\, \|\Psi\|_{L_r^2}=1}\langle\mathcal{Q}_{\delta_1^2}\Psi, \Psi\rangle : S\subset L_r^2 \text{ is a subspace of dimension } k\right\}$$

by the Courant max-min principle. Passing to the limit $\lim_{\tilde{\lambda}\to\delta_1^2-}$ in (4.36), we derive

$$\|\mathcal{Q}_\lambda - \mathcal{Q}_{\delta_1^2}\|_{\mathrm{HS}} \le C|\lambda - \delta_1^2| + CT(\lambda)^{1/2},$$

where $\lim_{\lambda\to\delta_1^2-} T(\lambda) = 0$. Thus, if $\Psi \in L^2_r$ is such that $\|\Psi\|_{L^2_r} = 1$, then we have

$$|\langle \mathcal{Q}_\lambda \Psi, \Psi\rangle - \langle \mathcal{Q}_{\delta_1^2}\Psi, \Psi\rangle| \le \|\mathcal{Q}_\lambda - \mathcal{Q}_{\delta_1^2}\|_{\mathrm{HS}} \le C|\lambda - \delta_1^2| + CT(\lambda)^{1/2}.$$

Since the $\mu_k(\lambda)$ are also characterized by the Courant max-min principle, see (4.18), it follows that

$$|\mu_k(\lambda) - \kappa_k| \le C|\lambda - \delta_1^2| + CT(\lambda)^{1/2},$$

and accordingly $\mu_{*,k} = \lim_{\lambda\to\delta_1^2-} \mu_k(\lambda) = \kappa_k$.

The next assertion is due to the definition of $\mu_{*,k}$ and Lemma 4.3(f), whereas the convexity of $\mu_1$ on $]-\infty, \delta_1^2]$ is a consequence of Lemma 4.7(d). □

**Corollary 4.10** *Suppose that ($\omega_1$-3) is satisfied.*

*(a) There is a constant $C > 0$ such that for every $\lambda \in [0, \delta_1^2]$ and $r, \tilde{r} \in ]0, r_Q]$, we have*

$$|K_\lambda(r, \tilde{r})| \le \frac{C}{\tilde{r}^2}(1 + |\ln r|).$$

*(b) For $\lambda \in [0, \delta_1^2[$, let $\Psi_\lambda \in L^2_r$ denote a normalized eigenfunction of $\mathcal{Q}_\lambda$ for $\mu_1(\lambda)$. Then there is a constant $C > 0$ such that for every $\lambda \in [0, \delta_1^2[$ and $r \in ]0, r_Q]$, we have*

$$|\Psi_\lambda(r)| \le C(1 + |\ln r|)\, \|\Psi_\lambda\|_{L^2_r}.$$

*(c) For $\Psi_\lambda$ as in (b), define $\psi_\lambda(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_\lambda(r) \in X^0_{\mathrm{odd}}$. Then there is a constant $C > 0$ such that for every $\lambda \in [0, \delta_1^2[$ and $k \in \mathbb{Z}$, we have*

$$|(\psi_\lambda)_k(I, \ell)| \le C\, |Q'(e)|\, \|\Psi_\lambda\|_{L^2_r}, \quad (I, \ell) \in D,$$

*where $(\psi_\lambda)_k$ are the Fourier coefficients of $\psi_\lambda$.*

***Proof*** (a) From (4.14) and similar to the argument following (4.9), we obtain with $\min\{r^2, \tilde{r}^2\} \le r^2$ and using ($\omega_1$-3)

$$\begin{aligned}
|K_\lambda(r, \tilde{r})| &= \frac{4}{r^2\tilde{r}^2}\left|\sum_{k\ne 0} s_{k,0}(r, \tilde{r}, \lambda)\right| \\
&\le \frac{C}{r^2\tilde{r}^2}\, \mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} \sum_{k\ne 0} \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\le Cr^2\}}\, \frac{1}{k^2\omega_1^2(e,\beta) - \lambda} \\
&\le \frac{C}{r^2\tilde{r}^2}\, \mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} \sum_{|k|\ge 2} \iint_D d\beta\, de\, \mathbf{1}_{\{\beta\le Cr^2\}}\, \frac{2}{\delta_1^2 k^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} \iint\limits_D d\beta\, de\, \mathbf{1}_{\{\beta\le Cr^2\}}\, \frac{1}{\omega_1^2(e,\beta)-\lambda} \\
&\le \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}\, r^2 + \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} \iint\limits_D d\beta\, de\, \mathbf{1}_{\{\beta\le Cr^2\}}\, \frac{1}{\omega_1^2(e,\beta)-\delta_1^2} \\
&\le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} + \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} \iint\limits_D d\beta\, de\, \mathbf{1}_{\{\beta\le Cr^2\}}\, \frac{1}{|(e,\beta)-(e_0,\hat{\beta})|}.
\end{aligned}
$$

By means of the function $I$ from (4.41), this can be expressed as

$$|K_\lambda(r,\tilde{r})| \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} + \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}\, I(\hat{C}r^2)$$

for certain constants $C, \hat{C} > 0$ that only depend on $Q$. Once again, we distinguish two cases. <u>Case 1:</u> $\hat{\beta} > 0$. If $r^2 \le \frac{\hat{\beta}}{2\hat{C}}$, then we can apply (4.42) to get

$$|K_\lambda(r,\tilde{r})| \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}.$$

On the other hand, if $r^2 \ge \frac{\hat{\beta}}{2\hat{C}}$, then (4.43) leads to

$$|K_\lambda(r,\tilde{r})| \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} + \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{(\frac{\hat{\beta}}{2\hat{C}})^{1/2}\le r\le r_Q,\, 0\le\tilde{r}\le r_Q\}} \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}.$$

<u>Case 2:</u> $\hat{\beta} = 0$. Here, we invoke (4.44) to deduce that

$$|K_\lambda(r,\tilde{r})| \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}} + \frac{C}{r^2\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}\, |\hat{C}r^2\ln(\hat{C}r^2)| \le \frac{C}{\tilde{r}^2}\,\mathbf{1}_{\{0\le r,\tilde{r}\le r_Q\}}\,(1+|\ln r|).$$

Hence, in any case, we arrive at the bound

$$|K_\lambda(r,\tilde{r})| \le \frac{C}{\tilde{r}^2}\,(1+|\ln r|),$$

as desired. (b) Using (a), we obtain from (4.15) and Remark 4.2(b)

$$
\begin{aligned}
\mu_1(0)|\Psi_\lambda(r)| \le \mu_1(\lambda)|\Psi_\lambda(r)| = |(\mathcal{Q}_\lambda\Psi_\lambda)(r)| &= 4\pi\left|\int_0^{r_Q} \tilde{r}^2\, K_\lambda(r,\tilde{r})\,\Psi_\lambda(\tilde{r})\,d\tilde{r}\right| \\
&\le C(1+|\ln r|)\int_0^{r_Q} |\Psi_\lambda(\tilde{r})|\,d\tilde{r},
\end{aligned}
$$

so that

$$|\Psi_\lambda(r)| \le C_*(1+|\ln r|)\int_0^{r_Q} |\Psi_\lambda(\tilde{r})|\,d\tilde{r} \tag{4.45}$$

for a certain constant $C_* > 0$ that only depends on $Q$. Fix $a_* \in ]0, r_Q[$ such that $\int_0^{a_*} (1 + |\ln r|)\, dr \le \frac{1}{2C_*}$. Then

$$\int_0^{a_*} |\Psi_\lambda(r)|\, dr \le C_* \int_0^{a_*} (1 + |\ln r|)\, dr \int_0^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} \le \frac{1}{2} \int_0^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r}$$
$$= \frac{1}{2} \int_0^{a_*} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} + \frac{1}{2} \int_{a_*}^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r}$$

entails $\int_0^{a_*} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} \le \int_{a_*}^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r}$. Going back to (4.45), it follows by means of Hölder's inequality that

$$|\Psi_\lambda(r)| \le C_*(1 + |\ln r|) \Big[ \int_0^{a_*} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} + \int_{a_*}^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} \Big] \le 2C_*(1 + |\ln r|) \int_{a_*}^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r}$$
$$\le \frac{2C_*}{a_*} (1 + |\ln r|) \int_{a_*}^{r_Q} \tilde{r} |\Psi_\lambda(\tilde{r})|\, d\tilde{r} \le \frac{2C_* r_Q^{1/2}}{\sqrt{4\pi}\, a_*} (1 + |\ln r|)\, \|\Psi_\lambda\|_{L^2_r},$$

from where a suitable $C > 0$ can be read off. (c) Owing to (4.24), Theorem 3.5 and (b), we have

$$|(\psi_\lambda)_k(I, \ell)| = \frac{1}{\pi} |Q'(e)|\, \omega_1(e, \ell) \Big| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi_\lambda(\tilde{r}) \sin(k\theta(\tilde{r}, e, \ell))\, d\tilde{r} \Big|$$
$$\le C\, |Q'(e)| \int_0^{r_Q} |\Psi_\lambda(\tilde{r})|\, d\tilde{r}$$
$$\le C\, |Q'(e)| \Big( \int_0^{r_Q} (1 + |\ln \tilde{r}|)\, d\tilde{r} \Big) \|\Psi_\lambda\|_{L^2_r} \le C\, |Q'(e)|\, \|\Psi_\lambda\|_{L^2_r},$$

which completes the proof. □

**Corollary 4.11** *Suppose that ($\omega_1$-3) is satisfied. Let $(\lambda_j) \subset [0, \delta_1^2[$ be such that $\lim_{j\to\infty} \lambda_j = \delta_1^2$. For $j \in \mathbb{N}$, let $\Psi_j \in L^2_r$ denote a normalized eigenfunction of $Q_{\lambda_j}$ for $\mu_1(\lambda_j)$. Furthermore, define $\psi_j(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_j(r) \in X^0_{\mathrm{odd}}$. Then there is a subsequence $j' \to \infty$ so that*

$$\Psi_* = \lim_{j'\to\infty} \Psi_{j'}$$

*does exist in $L^2_r$ and*

$$\psi_* = \lim_{j'\to\infty} \psi_{j'}$$

*does exist in $X^0$, where $\psi_*(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_*(r)$. In addition, $\|\Psi_*\|_{L^2_r} = 1$ and $\mathcal{Q}_{\delta_1^2} \Psi_* = \mu_* \Psi_*$ as well as $\mu_* = \|\mathcal{Q}_{\delta_1^2}\|$.*

***Proof*** Recall from (4.33) and Lemma 4.9 that $\mu_* \in [\mu_1(0), \infty[$. For $j, k \in \mathbb{N}$, we can estimate

$$\begin{aligned}
\mu_*\|\Psi_j-\Psi_k\|_{L_r^2} &\le (\mu_*-\mu_1(\lambda_j))\|\Psi_j\|_{L_r^2}+\|\mathcal{Q}_{\lambda_j}\Psi_j-\mathcal{Q}_{\lambda_k}\Psi_k\|_{L_r^2}\\
&\quad+(\mu_*-\mu_1(\lambda_k))\|\Psi_k\|_{L_r^2}\\
&\le (\mu_*-\mu_1(\lambda_j))+(\mu_*-\mu_1(\lambda_k))+\|(\mathcal{Q}_{\lambda_j}-\mathcal{Q}_{\delta_1^2})\Psi_j\|_{L_r^2}\\
&\quad+\|\mathcal{Q}_{\delta_1^2}\Psi_j-\mathcal{Q}_{\delta_1^2}\Psi_k\|_{L_r^2}+\|(\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_k})\Psi_k\|_{L_r^2}\\
&\le (\mu_*-\mu_1(\lambda_j))+(\mu_*-\mu_1(\lambda_k))+\|\mathcal{Q}_{\lambda_j}-\mathcal{Q}_{\delta_1^2}\|_{\mathcal{B}(L_r^2)}\\
&\quad+\|\mathcal{Q}_{\delta_1^2}\Psi_j-\mathcal{Q}_{\delta_1^2}\Psi_k\|_{L_r^2}+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_k}\|_{\mathcal{B}(L_r^2)}\\
&\le (\mu_*-\mu_1(\lambda_j))+(\mu_*-\mu_1(\lambda_k))+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_j}\|_{\mathrm{HS}}\\
&\quad+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_k}\|_{\mathrm{HS}}+\|\mathcal{Q}_{\delta_1^2}\Psi_j-\mathcal{Q}_{\delta_1^2}\Psi_k\|_{L_r^2}. \qquad (4.46)
\end{aligned}$$

According to Lemma 4.9, we have $\lim_{\lambda\to\delta_1^2-}\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_\lambda\|_{\mathrm{HS}}=0$ and $\mathcal{Q}_{\delta_1^2}:L_r^2\to L_r^2$ is a Hilbert-Schmidt operator, and hence compact. Thus, since $\|\Psi_j\|_{L_r^2}=1$, the set $\{\mathcal{Q}_{\delta_1^2}\Psi_j : j\in\mathbb{N}\}\subset L_r^2$ is relatively compact. Therefore, there is a subsequence $j'\to\infty$ and a function $\hat{\Psi}\in L_r^2$ so that $\lim_{j'\to\infty}\mathcal{Q}_{\delta_1^2}\Psi_{j'}=\hat{\Psi}$ in $L_r^2$. From (4.46), we deduce that along the subsequence

$$\begin{aligned}
\mu_*\|\Psi_{j'}-\Psi_{k'}\|_{L_r^2} &\le (\mu_*-\mu_1(\lambda_{j'}))+(\mu_*-\mu_1(\lambda_{k'}))\\
&\quad+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_{j'}}\|_{\mathrm{HS}}+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_{k'}}\|_{\mathrm{HS}}\\
&\quad+\|\mathcal{Q}_{\delta_1^2}\Psi_{j'}-\mathcal{Q}_{\delta_1^2}\Psi_{k'}\|_{L_r^2}\to 0,\quad j',k'\to\infty.
\end{aligned}$$

As a consequence, $\Psi_*=\lim_{j'\to\infty}\Psi_{j'}$ does exist in $L_r^2$. Since

$$\|\psi_{j'}-\psi_{k'}\|_{X^0}\le \rho_Q(0)^{1/2}\,\|\Psi_{j'}-\Psi_{k'}\|_{L_r^2}$$

by (4.23), also $\psi_*=\lim_{j'\to\infty}\psi_{j'}$ does exist in $X^0$, where $\psi_*(r,p_r,\ell)=|Q'(e_Q)|\,p_r\Psi_*(r)$ a.e. Lastly,

$$\begin{aligned}
\|\mathcal{Q}_{\delta_1^2}\Psi_*-\mu_*\Psi_*\|_{L_r^2} &\le \|\mathcal{Q}_{\delta_1^2}(\Psi_*-\Psi_{j'})\|_{L_r^2}+\|(\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_{j'}})\Psi_{j'}\|_{L_r^2}\\
&\quad+(\mu_*-\mu_1(\lambda_{j'}))\,\|\Psi_{j'}\|_{L_r^2}+\mu_*\|\Psi_{j'}-\Psi_*\|_{L_r^2}\\
&\le 2\mu_*\|\Psi_*-\Psi_{j'}\|_{L_r^2}+\|\mathcal{Q}_{\delta_1^2}-\mathcal{Q}_{\lambda_{j'}}\|_{\mathcal{B}(L_r^2)}\\
&\quad+(\mu_*-\mu_1(\lambda_{j'}))\to 0,\quad j'\to\infty,
\end{aligned}$$

implies that $\mathcal{Q}_{\delta_1^2}\Psi_*=\mu_*\Psi_*$. □

The following criterion is useful for proving that $\delta_1^2$ is an eigenvalue of $L$ in the case where $\mu_*=1$.

**Lemma 4.12** *Suppose that ($\omega_1$-3) is satisfied and that $\mu_*=1$. Let $(\lambda_j)\subset[0,\delta_1^2[$ be such that $\lim_{j\to\infty}\lambda_j=\delta_1^2$. For $j\in\mathbb{N}$, let $\Psi_j\in L_r^2$ denote a normalized eigenfunction of $Q_{\lambda_j}$ for $\mu_1(\lambda_j)$. Furthermore, define $\psi_j(r,p_r,\ell)=|Q'(e_Q)|\,p_r\Psi_j(r)\in X^0_{\mathrm{odd}}$*

*and* $g_j = (-\mathcal{T}^2 - \lambda_j)^{-1}\psi_j \in X^2_{\text{odd}}$. *If* $(g_j) \subset X^0 = L^2_{\text{sph}, \frac{1}{|Q'|}}(K)$ *is bounded, then* $\delta_1^2$ *is an eigenvalue of* $L$.

***Proof*** From (4.21), we deduce

$$\begin{aligned}\mathcal{K}\mathcal{T} g_j &= \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda_j)^{-1}\psi_j = |Q'(e_Q)|\, p_r(\mathcal{Q}_{\lambda_j}\Psi_j)\\ &= \mu_1(\lambda_j)\,|Q'(e_Q)|\, p_r\Psi_j = \mu_1(\lambda_j)\psi_j.\end{aligned} \tag{4.47}$$

Since $-\mathcal{T}^2 g_j = \psi_j + \lambda_j g_j$, using Corollary B.19, this implies that for every odd function $h \in X^{00}$, we have

$$\begin{aligned}(g_j, Lh)_{X^0} &= (Lg_j, h)_{X^0} = (-\mathcal{T}^2 g_j, h)_{X^0} - (\mathcal{K}\mathcal{T} g_j, h)_{X^0}\\ &= (\psi_j + \lambda_j g_j, h)_{X^0} - \mu_1(\lambda_j)(\psi_j, h)_{X^0}\\ &= \lambda_j (g_j, h)_{X^0} + (1 - \mu_1(\lambda_j))(\psi_j, h)_{X^0}.\end{aligned} \tag{4.48}$$

Next, from (4.20), we get $\|\psi_j\|_{X^0} \le \rho_Q(0)^{1/2}\,\|\Psi_j\|_{L^2_r} \le \rho_Q(0)^{1/2}$. Since $\lim_{j\to\infty}\mu_1(\lambda_j) = \mu_* = 1$, this yields in particular that

$$\lim_{j\to\infty}\,[(1 - \mu_1(\lambda_j))(\psi_j, h)_{X^0}] = 0. \tag{4.49}$$

By assumption, $(g_j) \subset X^0$ is bounded. Hence, passing to a subsequence (that is not relabeled), we may assume that $g_j \rightharpoonup g_*$ weakly in $X^0$ as $j \to \infty$ for some function $g_* \in X^0_{\text{odd}}$. Suppose that $g_* = 0$. Then $g_j \rightharpoonup 0$ weakly in $X^0$ implies that $\mathcal{K}\mathcal{T} g_j \rightharpoonup 0$ weakly in $X^0$ as $j \to \infty$, by Lemma B.15(d). Due to (4.47), this yields $\psi_j \rightharpoonup 0$ weakly in $X^0$ as $j \to \infty$. On the other hand, by Corollary 4.11, we may pass to a subsequence $j' \to \infty$ so that $\Psi_* = \lim_{j'\to\infty}\Psi_{j'}$ does exist in $L^2_r$ and $\psi_* = \lim_{j'\to\infty}\psi_{j'}$ does exist in $X^0$ as strong limits; the functions are linked via $\psi_*(r, p_r, \ell) = |Q'(e_Q)|\, p_r\Psi_*(r)$. But then we must have $\psi_* = 0$ and accordingly $\Psi_* = 0$, which however contradicts $\|\Psi_*\|_{L^2_r} = 1$, cf. Corollary 4.11. As a consequence, it follows that $g_* \in X^0_{\text{odd}}$ satisfies $g_* \ne 0$. Passing to the limit $j \to \infty$ in (4.48) and using (4.49), we moreover infer that $(g_*, Lh)_{X^0} = \delta_1^2 (g_*, h)_{X^0}$ for every odd function $h \in X^{00}$. From Lemma C.11, we conclude that $g_* \in X^2_{\text{odd}}$ and $Lg_* = \delta_1^2 g_*$, which completes the proof. □

## 4.2 Relating $\mu_*$ to the Fact That $\lambda_*$ is an Eigenvalue of $L$

**Theorem 4.13** *We have*

$$\mu_* > 1 \iff \lambda_* < \delta_1^2.$$

*In this case,* $\mu_1(\lambda_*) = 1$ *and* $\lambda_*$ *is an eigenvalue of* $L$.

***Proof*** If $\mu_* > 1$, then $\lambda_* = \delta_1^2$ is impossible by Remark 4.8, so that we must have $\lambda_* < \delta_1^2$. Conversely, suppose that $\lambda_* < \delta_1^2$ holds. Then, according to Theorem C.8, $\lambda_*$ is an eigenvalue of $L$. Let $u_* \in X^2_{\rm odd}$ be an eigenfunction of $L$ for the eigenvalue $\lambda_*$. Using Theorem 4.5(a), it follows that $\Psi_* = U'_{\mathcal{T}u_*} \in L^2_r$ for $r \in [0, r_Q]$ is an eigenfunction of $\mathcal{Q}_{\lambda_*}$ for the eigenvalue 1. Since $\mu_1(\lambda_*)$ is the largest eigenvalue of $\mathcal{Q}_{\lambda_*}$, we get $\mu_1(\lambda_*) \geq 1$. On the other hand, $\mu_1(\lambda_*) \leq 1$ by Lemma 4.7(b), and hence $\mu_1(\lambda_*) = 1$. It remains to show that $\mu_* > 1$. Suppose that on the contrary $\mu_* \leq 1$ is satisfied. For $\lambda \in [\lambda_*, \delta_1^2[$, the monotonicity of $\mu_1$ then yields $1 = \mu_1(\lambda_*) \leq \mu_1(\lambda) \leq \mu_* \leq 1$, which means that $\mu_1(\lambda) = 1$ is constant for $\lambda \in [\lambda_*, \delta_1^2[$. Take $\lambda_* \leq \tilde{\lambda} < \lambda < \delta_1^2$. and let $\Psi_{\tilde{\lambda}}$ denote a normalized eigenfunction for $\mu_1(\tilde{\lambda})$. Then, by (4.19) and (4.7),

$$1 = \mu_1(\tilde{\lambda}) = \langle \mathcal{Q}_{\tilde{\lambda}}\Psi_{\tilde{\lambda}}, \Psi_{\tilde{\lambda}}\rangle \leq \langle \mathcal{Q}_{\lambda}\Psi_{\tilde{\lambda}}, \Psi_{\tilde{\lambda}}\rangle \leq \|\mathcal{Q}_\lambda\| \, \|\Psi_{\tilde{\lambda}}\|^2_{L^2_r} = \mu_1(\lambda) = 1,$$

which means that $\langle \mathcal{Q}_{\lambda}\Psi_{\tilde{\lambda}}, \Psi_{\tilde{\lambda}}\rangle = 1$ for all $\lambda_* \leq \tilde{\lambda} < \lambda < \delta_1^2$. Differentiating this relation w.r. to $\lambda$ at a fixed $\lambda_0 \in ]\tilde{\lambda}, \delta_1^2[$, it follows from (4.6) that

$$\begin{aligned} 0 &= \langle \mathcal{Q}'_{\lambda_0}\Psi_{\tilde{\lambda}}, \Psi_{\tilde{\lambda}}\rangle \\ &= 64\pi^2 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell)-\lambda_0)^2} \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi_{\tilde{\lambda}}(r) \sin(k\theta(r,e,\ell))\, dr \right|^2 \end{aligned}$$

for all $\tilde{\lambda} \in [\lambda_*, \lambda_0[$. Defining $\psi_{\tilde{\lambda}}(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_{\tilde{\lambda}}(r) \in X^0_{\rm odd}$, then (4.24) implies that $(\psi_{\tilde{\lambda}})_k = 0$ for $k \in \mathbb{Z}$, so that $\psi_{\tilde{\lambda}} = 0$ and in turn $\Psi_{\tilde{\lambda}} = 0$, which however is impossible. □

**Theorem 4.14** *Suppose that ($\omega_1$-1) is satisfied. If $\mu_* < 1$, then $\lambda_* = \delta_1^2$ and this is not an eigenvalue of $L$.*

***Proof*** The approach is inspired by [20, Section 2]. Since $\lambda_* \leq \delta_1^2$ by Lemma 3.18, $\mu_* < 1$ together with Theorem 4.13 implies $\lambda_* = \delta_1^2$. Now suppose on the contrary that there is a function $u_* \in X^2_{\rm odd}$ such that $\|u_*\|_{X^0} = 1$ and $Lu_* = \delta_1^2 u_*$. If we define $\Psi_*(r) = U'_{\mathcal{T}u_*}(r)$ for $r \in [0, r_Q]$, then $\Psi_* \in L^2_r$ and (B.37) yields $\mathcal{K}\mathcal{T}u_* = |Q'(e_Q)|\, p_r\, U'_{\mathcal{T}u_*}(r) = |Q'(e_Q)|\, p_r\, \Psi_*(r)$. Hence, for $a > 0$ and $b \in \mathbb{R}$, we get

$$(-\mathcal{T}^2 - (\delta_1^2 - a + ib))u_* = \mathcal{K}\mathcal{T}u_* + (a - ib)u_*.$$

Since $z = \delta_1^2 - a + ib \in \Omega$, it follows from (4.21) that

$$\begin{aligned} |Q'(e_Q)|\, p_r(\mathcal{Q}_{\delta_1^2-a+ib}\Psi_*) &= \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - (\delta_1^2 - a + ib))^{-1}(\mathcal{K}\mathcal{T}u_*) \\ &= \mathcal{K}\mathcal{T}u_* - (a - ib)\,\mathcal{K}\mathcal{T}(-\mathcal{T}^2 - (\delta_1^2 - a + ib))^{-1}u_* \\ &= |Q'(e_Q)|\, p_r\, \Psi_* \\ &\quad -(a - ib)\,|Q'(e_Q)|\, p_r\, U'_{\mathcal{T}(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*}, \end{aligned}$$

and therefore,

$$\mathcal{Q}_{\delta_1^2-a+ib}\Psi_* = \Psi_* - (a-ib)\, U'_{\mathcal{T}(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*}. \tag{4.50}$$

We claim that if $a = a(\varepsilon) \to 0^+$ and $b = b(\varepsilon) \to 0$ as $\varepsilon \to 0$, then

$$(a-ib)\, U'_{\mathcal{T}(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*} \to 0, \quad \varepsilon \to 0^+, \tag{4.51}$$

in $L^2_r$. For, we can invoke Corollary B.16 as well as (B.25) to deduce

$$\begin{aligned}
&\|(a-ib)\, U'_{\mathcal{T}(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*}\|^2_{L^2_r}\\
&\le 16\pi^2 \rho_Q(0)\,(a^2+b^2)\,\|(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*\|^2_{X^0}\\
&= 256\pi^5 \rho_Q(0)\,(a^2+b^2) \sum_{k\neq 0} \iint_D dI\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \frac{|(u_*)_k(I,\ell)|^2}{|k^2\omega_1^2(I,\ell)-(\delta_1^2-a+ib)|^2}\\
&= 256\pi^5 \rho_Q(0)\,(a^2+b^2) \sum_{k\neq 0} \iint_D dI\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \frac{|(u_*)_k(I,\ell)|^2}{(k^2\omega_1^2(I,\ell)-\delta_1^2+a)^2+b^2}.
\end{aligned}$$

If $|k| \ge 2$, then $k^2\omega_1^2(I,\ell) - \delta_1^2 + a \ge (k^2-1)\delta_1^2 \ge 3\delta_1^2$. Thus,

$$\begin{aligned}
&\|(a-ib)\, U'_{\mathcal{T}(-\mathcal{T}^2-(\delta_1^2-a+ib))^{-1}u_*}\|^2_{L^2_r}\\
&\le 2\pi^2\delta_1^{-4}\rho_Q(0)\,(a^2+b^2)\,\|u_*\|^2_{X^0}\\
&+ 512\pi^5\rho_Q(0) \iint_D dI\, d\ell\, \ell\, \frac{a^2+b^2}{(\omega_1^2(I,\ell)-\delta_1^2+a)^2+b^2}\, \phi_1(I,\ell)
\end{aligned}$$

for $\phi_1(I,\ell) = \frac{|(u_*)_1(I,\ell)|^2}{|Q'(e)|} \in L^1(D)$. For almost all $(I,\ell) \in D$, we know from hypothesis ($\omega_1$-1) that $\omega_1(I,\ell) \neq \delta_1$, i.e., $\omega_1(I,\ell) > \delta_1$. For such an $(I,\ell)$, we have

$$\frac{a^2+b^2}{(\omega_1^2(I,\ell)-\delta_1^2+a)^2+b^2} \le \frac{a^2+b^2}{(\omega_1^2(I,\ell)-\delta_1^2)^2+b^2} \to 0, \quad \varepsilon \to 0.$$

Since always

$$\frac{a^2+b^2}{(\omega_1^2(I,\ell)-\delta_1^2+a)^2+b^2} \le 1,$$

it follows by using Lebesgue's dominated convergence theorem that indeed (4.51) is verified. Going back to (4.50), this entails that

$$\lim_{\varepsilon\to 0} \mathcal{Q}_{\delta_1^2-a+ib}\Psi_* = \Psi_* \quad \text{in} \quad L^2_r. \tag{4.52}$$

Next, we are going to compare $\mathcal{Q}_{\delta_1^2-a+ib}$ to $\mathcal{Q}_{\delta_1^2-a}$. Here, we find

$$
\begin{aligned}
|\langle \mathcal{Q}_{\delta_1^2-a+ib}\Psi_*, \Psi_*\rangle - \langle \mathcal{Q}_{\delta_1^2-a}\Psi_*, \Psi_*\rangle| &= |\langle (\mathcal{Q}_{\delta_1^2-a+ib} - \mathcal{Q}_{\delta_1^2-a})\Psi_*, \Psi_*\rangle| \\
&= 64\pi^2 \Bigg| \sum_{k\neq 0} \iint_D d\ell\, \ell\, de \\
\times \left[ \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell) - (\delta_1^2 - a - ib)} - \frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell) - (\delta_1^2 - a)} \right] \\
\times \int_0^{r_Q}\int_0^{r_Q} dr\, d\tilde r\, \Psi_*(r)\,\overline{\Psi_*(\tilde r)}\,\mathbf{1}_{\{r_-(e,\ell)\le r,\, \tilde r \le r_+(e,\ell)\}} \sin(k\theta(r,e,\ell))\sin(k\theta(\tilde r,e,\ell)) \Bigg|,
\end{aligned}
$$

cf. Lemma 4.3(d) and the definition of $\mathcal{Q}_z$. Using (4.8), (4.12) and similar arguments as in the proof of Lemma 4.3(a), we obtain

$$
\begin{aligned}
&|\langle \mathcal{Q}_{\delta_1^2-a+ib}\Psi_*, \Psi_*\rangle - \langle \mathcal{Q}_{\delta_1^2-a}\Psi_*, \Psi_*\rangle| \\
&\le C|b| \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, |Q'(e)|\, \frac{1}{|k^2\omega_1^2(e,\ell) - (\delta_1^2 - a - ib)||k^2\omega_1^2(e,\ell) - (\delta_1^2 - a)|} \\
&\qquad \times \int_0^{r_Q}\int_0^{r_Q} dr\, d\tilde r\, |\Psi_*(r)|\,|\Psi_*(\tilde r)|\,\mathbf{1}_{\{\beta \le C r\tilde r\}}.
\end{aligned}
$$

Now

$$
\begin{aligned}
|k^2\omega_1^2(e,\ell) - (\delta_1^2 - a - ib)|^2 &= (k^2\omega_1^2(e,\ell) - \delta_1^2 + a)^2 + b^2 \ge a^2, \\
|k^2\omega_1^2(e,\ell) - (\delta_1^2 - a)|^2 &= (k^2\omega_1^2(e,\ell) - \delta_1^2 + a)^2 \ge a^2,
\end{aligned}
$$

so that

$$
\begin{aligned}
&|\langle \mathcal{Q}_{\delta_1^2-a+ib}\Psi_*, \Psi_*\rangle - \langle \mathcal{Q}_{\delta_1^2-a}\Psi_*, \Psi_*\rangle| \\
&\le C\,\frac{|b|}{a^2} \sum_{k\neq 0} \int_0^{r_Q}\int_0^{r_Q} dr\, d\tilde r\, |\Psi_*(r)|\,|\Psi_*(\tilde r)|\, r\tilde r \left( \int_{U_Q(0)}^{e_0} |Q'(e)|\, de \right) \\
&\le C\,\frac{|b|}{a^2}\, \|\Psi_*\|_{L^2_r}^2.
\end{aligned}
$$

So if we take for instance $b(\varepsilon) = \varepsilon^3$ and $a(\varepsilon) = \varepsilon$, it follows that

$$
\lim_{\varepsilon\to 0} |\langle \mathcal{Q}_{\delta_1^2-\varepsilon+i\varepsilon^3}\Psi_*, \Psi_*\rangle - \langle \mathcal{Q}_{\delta_1^2-\varepsilon}\Psi_*, \Psi_*\rangle| = 0.
$$

Using also (4.52), we conclude that

$$
\lim_{\varepsilon\to 0} \langle \mathcal{Q}_{\delta_1^2-\varepsilon}\Psi_*, \Psi_*\rangle = \|\Psi_*\|_{L^2_r}^2.
$$

As a consequence,

$$\begin{aligned}\|\Psi_*\|_{L_r^2}^2 &= \lim_{\varepsilon\to 0}\langle \mathcal{Q}_{\delta_1^2-\varepsilon}\Psi_*, \Psi_*\rangle \le \limsup_{\varepsilon\to 0}\, \|\mathcal{Q}_{\delta_1^2-\varepsilon}\|\, \|\Psi_*\|_{L_r^2}^2 \\ &= \limsup_{\varepsilon\to 0}\, \mu_1(\delta_1^2-\varepsilon)\, \|\Psi_*\|_{L_r^2}^2 \le \mu_*\, \|\Psi_*\|_{L_r^2}^2.\end{aligned}$$

Since $\mu_* < 1$, this enforces $\Psi_* = 0$ and hence $\mathcal{K}\mathcal{T}u_* = 0$. Therefore, $-\mathcal{T}^2 u_* = -\mathcal{T}^2 u_* - \mathcal{K}\mathcal{T}u_* = Lu_* = \delta_1^2 u_*$, i.e., $\delta_1^2$ is an eigenvalue of $-\mathcal{T}^2$ with eigenfunction $u_*$. However, this contradicts Lemma B.12. □

The next result clarifies the case where $\mu_* = 1$.

**Theorem 4.15** *Suppose that ($\omega_1$-3) is satisfied and that $\mu_* = 1$. Then $\lambda_* = \delta_1^2$, and this is an eigenvalue of $L$ if and only if*

$$\|\mu_1'\|_{L^\infty(]-\infty,\delta_1^2[)} < \infty \tag{4.53}$$

*holds.*

***Proof*** Since $\lambda_* \le \delta_1^2$ by Lemma 3.18, $\mu_* = 1$ together with Theorem 4.13 imply $\lambda_* = \delta_1^2$. For the actual proof, recall from Lemma 4.3(f) that $\mu_1(\cdot) : ]-\infty, \delta_1^2[ \to ]0,\infty[$ is differentiable a.e., so (4.53) makes sense.

First, we consider the case where $\delta_1^2$ is an eigenvalue of $L$. Let $u_* \in X^2_{\mathrm{odd}}$ be such that $\|u_*\|_{X^0} = 1$ and $Lu_* = \delta_1^2 u_*$. If we define $\Psi_*(r) = U'_{\mathcal{T}u_*}(r)$ for $r \in [0, r_Q]$, then $\Psi_* \in L_r^2$ and (B.37) implies that $\mathcal{K}\mathcal{T}u_* = |Q'(e_Q)|\, p_r\, U'_{\mathcal{T}u_*}(r) = |Q'(e_Q)|\, p_r\, \Psi_*(r) =: \psi_* \in X^0_{\mathrm{odd}}$. For $\lambda < \delta_1^2$, we have

$$(-\mathcal{T}^2-\lambda)u_* = Lu_* + \mathcal{K}\mathcal{T}u_* - \lambda u_* = \psi_* + (\delta_1^2-\lambda)u_*, \tag{4.54}$$

and hence

$$(k^2\omega_1^2-\lambda)(u_*)_k = (\psi_*)_k + (\delta_1^2-\lambda)(u_*)_k, \quad k\in\mathbb{Z}, \tag{4.55}$$

for the Fourier coefficients. Since

$$(\psi_*, u_*)_{X^0} = (\mathcal{K}\mathcal{T}u_*, u_*)_{X^0} = \frac{1}{4\pi}\int_{\mathbb{R}^3} |U'_{\mathcal{T}u_*}(r)|^2\, dx = \frac{1}{4\pi}\, \|\Psi_*\|_{L_r^2}^2$$

by (B.40) from Lemma B.15(b), taking the inner product in $X^0$ of (4.54) with $u_*$, we deduce

$$((-\mathcal{T}^2-\lambda)u_*, u_*)_{X^0} = \frac{1}{4\pi}\, \|\Psi_*\|_{L_r^2}^2 + (\delta_1^2-\lambda)\|u_*\|_{X^0}^2. \tag{4.56}$$

Next, due to (4.25) from Lemma 4.6, we have

$$\begin{aligned}\langle \mathcal{Q}_\lambda\Psi_*, \Psi_*\rangle = 64\pi^4 \sum_{k\ne 0}\iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, \frac{1}{k^2\omega_1^2(e,\ell)-\lambda} \\ \times |(\psi_*)_k(I,\ell)|^2.\end{aligned}$$

Thus, by (B.4), (A.18), Lemma B.8(b) and (4.55) applied twice,

$$
\begin{aligned}
&((-\mathcal{T}^2-\lambda)u_*, u_*)_{X^0} \\
&= 16\pi^3 \sum_{k\neq 0} \int_0^\infty dI \int_0^\infty d\ell\, \ell \, \frac{1}{|Q'(e)|}\, \overline{[((-\mathcal{T}^2-\lambda)u_*]_k(I,\ell)}\, (u_*)_k(I,\ell) \\
&= 16\pi^3 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, (k^2\omega_1^2(e,\ell)-\lambda)\, |(u_*)_k(I,\ell)|^2 \\
&= 16\pi^3 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} (\psi_*)_k(I,\ell)\, \overline{(u_*)_k(I,\ell)} \\
&\quad + 16\pi^3(\delta_1^2-\lambda) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, |(u_*)_k(I,\ell)|^2 \\
&= 16\pi^3 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} (\psi_*)_k(I,\ell) \\
&\qquad\qquad \times \left( \frac{\overline{(\psi_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell)-\lambda} + (\delta_1^2-\lambda) \frac{\overline{(u_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell)-\lambda} \right) \\
&\quad + (\delta_1^2-\lambda)\, \|u_*\|_{X^0}^2 \\
&= \frac{1}{4\pi}\, \langle \mathcal{Q}_\lambda \Psi_*, \Psi_* \rangle \\
&\quad + 16\pi^3(\delta_1^2-\lambda) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} (\psi_*)_k(I,\ell)\, \frac{\overline{(u_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell)-\lambda} \\
&\quad + (\delta_1^2-\lambda)\, \|u_*\|_{X^0}^2.
\end{aligned}
$$

Comparing to (4.56), this yields

$$
\begin{aligned}
&\frac{1}{4\pi}\, \|\Psi_*\|_{L_r^2}^2 \\
&= \frac{1}{4\pi}\, \langle \mathcal{Q}_\lambda \Psi_*, \Psi_* \rangle \\
&\quad + 16\pi^3(\delta_1^2-\lambda) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} (\psi_*)_k(I,\ell)\, \frac{\overline{(u_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell)-\lambda}.
\end{aligned} \tag{4.57}
$$

If we had $\Psi_* = 0$, then also $\psi_* = 0$ and consequently $(k^2\omega_1^2 - \delta_1^2)(u_*)_k = 0$ in $D$ for $k \neq 0$ by (4.55). This implies that $(u_*)_k = 0$ for $|k| \geq 2$ and $(\omega_1 - \delta_1)(u_*)_1 = 0$ in $D$. Owing to ($\omega_1$-1), this enforces $(u_*)_1 = 0$ a.e. and therefore $u_* = 0$, which is a contradiction. In other words, we do know that $\Psi_* \neq 0$. Hence, by (4.7) and (4.57),

$$\begin{aligned}\mu_1(\lambda) &= \sup\{\langle \mathcal{Q}_\lambda \Psi, \Psi\rangle : \|\Psi\|_{L^2_r} \le 1\} \\ &\ge \frac{1}{\|\Psi_*\|^2_{L^2_r}} \langle \mathcal{Q}_\lambda \Psi_*, \Psi_*\rangle \\ &= 1 - \frac{64\pi^4}{\|\Psi_*\|^2_{L^2_r}} (\delta_1^2 - \lambda) \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} \\ &\qquad\qquad \times (\psi_*)_k(I,\ell)\, \frac{\overline{(u_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell) - \lambda}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1-\mu_1(\lambda)}{\delta_1^2 - \lambda} \le \frac{64\pi^4}{\|\Psi_*\|^2_{L^2_r}} \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|} \\ \times (\psi_*)_k(I,\ell)\, \frac{\overline{(u_*)_k(I,\ell)}}{k^2\omega_1^2(e,\ell) - \lambda},\end{aligned}$$

and upon using (4.55) one more time, we conclude that

$$\begin{aligned}\frac{1-\mu_1(\lambda)}{\delta_1^2 - \lambda} &\le \frac{64\pi^4}{\|\Psi_*\|^2_{L^2_r}} \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, \frac{k^2\omega_1^2(e,\ell) - \delta_1^2}{k^2\omega_1^2(e,\ell) - \lambda}\, |(u_*)_k(I,\ell)|^2 \\ &\le \frac{64\pi^4}{\|\Psi_*\|^2_{L^2_r}} \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, |(u_*)_k(I,\ell)|^2 \\ &= \frac{4\pi}{\|\Psi_*\|^2_{L^2_r}} \|u_*\|^2_{X^0} \end{aligned} \tag{4.58}$$

for all $\lambda < \delta_1^2$. Since $\mu_1$ is convex on $]-\infty, \delta_1^2]$ by Lemma 4.9(d), the difference quotients

$$\frac{\mu_1(\lambda + h) - \mu_1(\lambda)}{h}$$

for $h > 0$ are monotone increasing in $\lambda$ (and also in $h$); see [14, p. 13/14]. Let $\lambda_0 \in ]-\infty, \delta_1^2[$ be a point where $\mu_1$ is differentiable and let $h > 0$. For $\lambda_1 = \lambda_0 - h$ and $\lambda_2 = \delta_1^2 - h$, we have $\lambda_1 < \lambda_2$, whence $\mu_1(\delta_1^2) = \mu_* = 1$ in conjunction with (4.58) for $\lambda = \delta_1^2 - h$ leads to

$$\begin{aligned}\frac{\mu_1(\lambda_0) - \mu_1(\lambda_0 - h)}{h} &= \frac{\mu_1(\lambda_1 + h) - \mu_1(\lambda_1)}{h} \\ &\le \frac{\mu_1(\lambda_2 + h) - \mu_1(\lambda_2)}{h} \\ &= \frac{\mu_1(\delta_1^2) - \mu_1(\delta_1^2 - h)}{h} \le \frac{4\pi}{\|\Psi_*\|^2_{L^2_r}} \|u_*\|^2_{X^0}.\end{aligned}$$

It follows that $\|\mu_1'\|_{L^\infty(]-\infty,\delta_1^2[)} \leq \frac{4\pi}{\|\Psi_*\|^2_{L^2_r}} \|u_*\|^2_{X^0}$, which proves (4.53).

To establish the converse, we assume (4.53) to hold, and we are going to verify that $\delta_1^2$ is an eigenvalue of $L$. For this, we are going to use Lemma 4.12. The operator family $Q_z$ for $z \in \Omega = \mathbb{C} \setminus [\delta_1^2, \infty[$ satisfies the assumptions of Lemma D.1 with $\lambda_0 = \delta_1^2$ and $H = L_r^2$, by Lemmas 4.3 and 4.9. Hence, there are sequences $\lambda_j \nearrow \delta_1^2$, $\varepsilon_j > 0$ and $\Phi_{j,\lambda} \in L_r^2$ for $\lambda \in ]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[$ such that $\|\Phi_{j,\lambda}\|_{L_r^2} = 1$,

$$]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[ \ni \lambda \mapsto \Phi_{j,\lambda} \in L_r^2$$

is real analytic for $j \in \mathbb{N}$, and $Q_\lambda \Phi_{j,\lambda} = \mu_1(\lambda)\Phi_{j,\lambda}$ for $j \in \mathbb{N}$ and $\lambda \in ]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[$. Furthermore, $\mu_1$ is real analytic in $]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[$ and satisfies

$$\mu_1'(\lambda) = \langle Q_\lambda' \Phi_{j,\lambda}, \Phi_{j,\lambda} \rangle \tag{4.59}$$

for $\lambda \in ]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[$. By decreasing $\varepsilon_j$ further, if necessary, we may assume that $\varepsilon_j \to 0$ as $j \to \infty$. Due to (4.53), there exists a set $N \subset ]-\infty, \delta_1^2[$ of measure zero such that $S = \sup_{\lambda \in ]-\infty,\delta_1^2[\setminus N} |\mu_1'(\lambda)| < \infty$. For each $j \in \mathbb{N}$, pick $\hat{\lambda}_j \in ]\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j[\setminus N$ and define $\Psi_j = \Phi_{j,\hat{\lambda}_j}$. It follows that $\lim_{j\to\infty} \hat{\lambda}_j = \delta_1^2$ and $\|\Psi_j\|_{L_r^2} = 1$. In addition, $Q_{\hat{\lambda}_j}\Psi_j = Q_{\hat{\lambda}_j}\Phi_{j,\hat{\lambda}_j} = \mu_1(\hat{\lambda}_j)\Phi_{j,\hat{\lambda}_j} = \mu_1(\hat{\lambda}_j)\Psi_j$, i.e., $\Psi_j$ is a normalized eigenfunction for the eigenvalue $\mu_1(\hat{\lambda}_j)$ of $Q_{\hat{\lambda}_j}$ such that

$$\sup_{j\in\mathbb{N}} \langle Q'_{\hat{\lambda}_j} \Psi_j, \Psi_j \rangle \leq S, \tag{4.60}$$

the latter due (4.59); recall that generally $\langle Q_\lambda' \Psi, \Psi \rangle \geq 0$ by (4.6). Now define $\psi_j(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_j(r) \in X^0_{\mathrm{odd}}$ and $g_j = (-\mathcal{T}^2 - \hat{\lambda}_j)^{-1}\psi_j \in X^2_{\mathrm{odd}}$. To complete the proof, we need to show that $(g_j) \subset X^0$ is bounded. From (B.4), (A.18), (B.25), (4.24) and (4.6), we obtain

$$\begin{aligned}
\|g_j\|^2_{X^0} &= 16\pi^3 \sum_{k\neq 0} \int_0^\infty dI \int_0^\infty d\ell\, \ell\, \frac{1}{|Q'(e)|}\, |(g_j)_k(I,\ell)|^2 \\
&= 16\pi^3 \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{\omega_1(e,\ell)\,|Q'(e)|}\, \frac{|(\psi_j)_k(I,\ell)|^2}{(k^2\omega_1^2(e,\ell) - \hat{\lambda}_j)^2} \\
&= 16\pi \sum_{k\neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell) - \hat{\lambda}_j)^2} \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi_j(r) \sin(k\theta(r,e,\ell))\, dr \right|^2 \\
&= \frac{1}{4\pi} \langle Q'_{\hat{\lambda}_j} \Psi_j, \Psi_j \rangle.
\end{aligned}$$

Thus, the claim follows from (4.60). □

## 4.3 Some Further Results

The following observation corresponds to the situation where $\omega_1$ is differentiable and attains its minimum at an interior point $(\hat{e}, \hat{\beta})$ of $D$; cf. assumption ($\omega_1$-2).

**Corollary 4.16** *Suppose that ($\omega_1$-2) is satisfied. Then $\mu_* = \infty$, $\lambda_* < \delta_1^2$, $\mu_1(\lambda_*) = 1$ and $\lambda_*$ is an eigenvalue of $L$.*

***Proof*** We only need to show that $\mu_* = \infty$, then the remaining assertions do follow from Theorem 4.13. The lower boundary curve $(\partial D)_3 = \{(e, \beta) \in D : e = e_{\min}(\beta)\}$ of $D$ characterizes the $(e, \beta)$ where $r_-(e, \beta) = r_0(\beta) = r_+(e, \beta)$. Since $(\hat{e}, \hat{\beta}) \in \operatorname{int} D = \{(e, \beta) : \beta \in ]0, \beta_*[, e \in ]e_{\min}(\beta), e_0[\} \subset D \setminus (\partial D)_3$ by hypothesis, we have that $r_+(\hat{e}, \hat{\beta}) - r_-(\hat{e}, \hat{\beta}) = 6\eta > 0$. The functions $r_\pm$ are known to be continuous (even $C^1$) on $\operatorname{int} D$; see [30, 50] and [88, Def./Thm. 2.4(b)]. Thus, by shrinking the neighborhood $U$ of $(\hat{e}, \hat{\beta})$ if necessary, we may assume that

$$|r_-(e, \beta) - r_-(\hat{e}, \hat{\beta})| \le \eta, \quad |r_+(e, \beta) - r_+(\hat{e}, \hat{\beta})| \le \eta, \quad (e, \beta) \in U,$$

is verified, along with

$$|\omega_1(e, \beta) - \delta_1| \le C_1 \, |(e, \beta) - (\hat{e}, \hat{\beta})|^2, \quad (e, \beta) \in U, \tag{4.61}$$

from (1.31). Next, we have $\theta(r_-(\hat{e}, \hat{\beta}), \hat{e}, \hat{\beta}) = 0$ and $\theta(r_+(\hat{e}, \hat{\beta}), \hat{e}, \hat{\beta}) = \pi$. Since $\frac{\partial \theta}{\partial r} = \frac{\omega_1}{p_r}$ due to (A.21) and $p_r > 0$ along the half-orbit, $\theta(\cdot, \hat{e}, \hat{\beta})$ is strictly increasing. In particular, we obtain

$$\sin \theta(\hat{r}_m, \hat{e}, \hat{\beta}) = 2\sigma > 0 \quad \text{for} \quad \hat{r}_m = \frac{1}{2}\,(r_-(\hat{e}, \hat{\beta}) + r_+(\hat{e}, \hat{\beta})).$$

As also

$$\theta : \{(r, e, \beta) : (e, \beta) \in \operatorname{int} D, r_-(e, \beta) < r < r_+(e, \beta)\} \to \mathbb{R}$$

is continuous, there is $\varepsilon \in ]0, \eta]$ such that $\sin \theta(r, e, \beta) \ge \sigma$ for $(e, \beta) \in U$ so that $|e - \hat{e}| \le \varepsilon$, $|\beta - \hat{\beta}| \le \varepsilon$ and $r \in [\hat{r}_m - \varepsilon, \hat{r}_m + \varepsilon] \cap ]r_-(e, \beta), r_+(e, \beta)[ = [\hat{r}_m - \varepsilon, \hat{r}_m + \varepsilon]$. If $\varepsilon > 0$ is small enough, we may assume that $[\hat{e} - \varepsilon, \hat{e} + \varepsilon] \times [\hat{\beta} - \varepsilon, \hat{\beta} + \varepsilon] \subset U \subset \operatorname{int} D$ as well as $[\hat{r}_m - \varepsilon, \hat{r}_m + \varepsilon] \subset [0, r_Q]$. Furthermore, note that in general $\sin \theta(r, e, \beta) \ge 0$ for $(e, \beta) \in \operatorname{int} D$ and $r_-(e, \beta) < r < r_+(e, \beta)$. Next, owing to $(\hat{e}, \hat{\beta}) \in \operatorname{int} D$, we have $e \in ]U_Q(0), e_0[$. Using (Q2), we can thus make sure that $\inf\{|Q'(e)| : e \in [\hat{e} - \varepsilon, \hat{e} + \varepsilon]\} = \alpha > 0$. Now, we consider the function

$$\Psi_0(r) = \gamma^{-1} \mathbf{1}_{[\hat{r}_m - \varepsilon, \hat{r}_m + \varepsilon]}(r), \quad \gamma = \Big(\frac{4\pi}{3}([\hat{r}_m + \varepsilon]^3 - [\hat{r}_m - \varepsilon]^3)\Big)^{1/2},$$

for which $\|\Psi_0\|_{L_r^2} = 1$. Hence, for $\lambda < \delta_1^2$ by Lemma 4.3(d),

$$\begin{aligned}
\mu_* &\geq \mu_1(\lambda) = \|\mathcal{Q}_\lambda\| = \sup\{\langle \mathcal{Q}_\lambda \Psi, \Psi\rangle : \|\Psi\| \leq 1\} \geq \langle \mathcal{Q}_\lambda \Psi_0, \Psi_0\rangle \\
&= 32\pi^2 \sum_{k\neq 0} \iint_D d\beta\, de\, \frac{\omega_1(e,\beta)\,|Q'(e)|}{k^2\omega_1^2(e,\beta)-\lambda} \left| \int_{r_-(e,\beta)}^{r_+(e,\beta)} \Psi_0(r)\sin(k\theta(r,e,\beta))\,dr \right|^2 \\
&\geq 32\pi^2 \iint_D d\beta\, de\, \frac{\omega_1(e,\beta)\,|Q'(e)|}{\omega_1^2(e,\beta)-\lambda} \left( \int_{r_-(e,\beta)}^{r_+(e,\beta)} \Psi_0(r)\sin(\theta(r,e,\beta))\,dr \right)^2 \\
&\geq 32\pi^2\delta_1\gamma^{-2} \int_{\hat\beta-\varepsilon}^{\hat\beta+\varepsilon} d\beta \int_{\hat e-\varepsilon}^{\hat e+\varepsilon} de\, \frac{|Q'(e)|}{\omega_1^2(e,\beta)-\lambda} \left( \int_{\hat r_m-\varepsilon}^{\hat r_m+\varepsilon} \sin(\theta(r,e,\beta))\,dr \right)^2 \\
&\geq 128\pi^2\delta_1\gamma^{-2}\alpha\,\sigma^2\varepsilon^2 \int_{\hat\beta-\varepsilon}^{\hat\beta+\varepsilon} d\beta \int_{\hat e-\varepsilon}^{\hat e+\varepsilon} de\, \frac{1}{\omega_1^2(e,\beta)-\delta_1^2+a},
\end{aligned} \tag{4.62}$$

where $a = \delta_1^2 - \lambda > 0$. From Theorem 3.5 and (4.61), we deduce that

$$\omega_1^2(e,\beta) - \delta_1^2 + a \leq 2\Delta_1 C_1\,|\xi-\hat\xi|^2 + a, \quad \xi = (e,\beta),\ \hat\xi = (\hat e,\hat\beta).$$

As a consequence,

$$\begin{aligned}
\int_{\hat\beta-\varepsilon}^{\hat\beta+\varepsilon} d\beta \int_{\hat e-\varepsilon}^{\hat e+\varepsilon} de\, \frac{1}{\omega_1^2(e,\beta)-\delta_1^2+a} &\geq \int_{|\xi-\hat\xi|\leq\varepsilon} \frac{d^2\xi}{2\Delta_1 C_1\,|\xi-\hat\xi|^2+a} \\
&= 2\pi \int_0^\varepsilon \frac{\rho}{2\Delta_1 C_1\,\rho^2+a}\,d\rho \\
&= \frac{\pi}{2\Delta_1 C_1}\ln\frac{2\Delta_1 C_1\,\varepsilon^2+a}{a} \to \infty, \quad a\to 0^+.
\end{aligned}$$

Thus, if we pass to the limit $\lambda \to \delta_1^2-$, i.e., $a \to 0^+$, in (4.62), it follows that $\mu_* = \infty$. □

Regarding Theorem 4.15, if ($\omega_1$-3) holds and if $\mu_* = 1$, then one can show that $\lambda = \delta_1^2$ is an eigenvalue of $L$, provided one is able to gain a little bit from the term $|Q'(e)|$, in the sense that $Q'(e_0) = 0$ in a controlled way, as expressed by (Q5); then the inherent logarithmic singularity can be dealt with. To simplify the presentation, we additionally assume that $\mu_*$ is simple as an eigenvalue of $Q_{\delta_1^2}$, but with some more technical efforts, this assumption could be disposed of.

**Corollary 4.17** *Suppose that ($\omega_1$-3) and (Q5) are satisfied, and assume that $\mu_* = 1$ is a simple eigenvalue of $Q_{\delta_1^2}$. Then $\lambda_* = \delta_1^2$, and this is an eigenvalue of $L$.*

***Proof*** We already know that $\lambda_* = \delta_1^2$; see the proof of Theorem 4.15. To verify that $\delta_1^2$ is an eigenvalue of $L$, we are going to use Theorem 4.15. According to Lemma D.2, there is $\varepsilon > 0$ such that $]\delta_1^2-\varepsilon, \delta_1^2[\ni \lambda \mapsto \mu_1(\lambda)$ is real analytic. In addition, there are $\Psi_\lambda \in L_r^2$ satisfying $\|\Psi_\lambda\|_{L_r^2} = 1$, $Q_\lambda\Psi_\lambda = \mu_1(\lambda)\Psi_\lambda$, and $]\delta_1^2-\varepsilon, \delta_1^2[\ni \lambda \mapsto \Psi_\lambda$

is real analytic. Also $\mu_1'(\lambda) = \langle Q_\lambda' \Psi_\lambda, \Psi_\lambda \rangle$ holds for $\lambda \in ]\delta_1^2 - \varepsilon, \delta_1^2[$. By Lemma 4.9, the function $\mu_1$ is convex, so that $\mu_1'' \geq 0$ and $\mu_1'$ is increasing. In other words,

$$\|\mu_1'\|_{L^\infty(]-\infty,\delta_1^2[)} = \lim_{\lambda \to \delta_1^2 -} \mu_1'(\lambda) =: \mu_*'$$

does exist in $]0, \infty]$, and the issue is to show that $\mu_*' < \infty$. Defining $\psi_\lambda(r, p_r, \ell) = |Q'(e_Q)|\, p_r \Psi_\lambda(r) \in X^0_{\text{odd}}$ as before, we get, from Lemma 4.3(d), (4.24) and Corollary 4.10(c),

$$\begin{aligned}
\mu_1'(\lambda) &= \langle Q_\lambda' \Psi_\lambda, \Psi_\lambda \rangle \\
&= 64\pi^2 \sum_{k \neq 0} \iint_D d\ell\, \ell\, de\, \frac{\omega_1(e, \ell)\, |Q'(e)|}{(k^2 \omega_1^2(e, \ell) - \lambda)^2} \\
&\qquad \times \left| \int_{r_-(e,\ell)}^{r_+(e,\ell)} \Psi_\lambda(r) \sin(k\theta(r, e, \ell))\, dr \right|^2 \\
&= 64\pi^4 \sum_{k \neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{(k^2 \omega_1^2(e, \ell) - \lambda)^2} \frac{|(\psi_\lambda)_k(I, \ell)|^2}{\omega_1(e, \ell)\, |Q'(e)|} \\
&\leq C \sum_{k \neq 0} \iint_D d\ell\, \ell\, de\, \frac{1}{(k^2 \omega_1^2(e, \ell) - \lambda)^2} |Q'(e)| \\
&\leq C \sum_{k=2}^{\infty} \iint_D d\ell\, \ell\, de\, \frac{4}{\delta_1^4 k^4} + C \iint_D d\ell\, \ell\, de\, \frac{|Q'(e)|}{(\omega_1(e, \ell) - \delta_1)^2}.
\end{aligned}$$

Thus, using ($\omega_1$-3) and (Q5),

$$\begin{aligned}
\mu_1'(\lambda) &\leq C + C \iint_D d\ell\, \ell\, de\, \frac{(e - e_0)^\alpha}{|(e, \beta) - (e_0, \hat{\beta})|^2} \\
&\leq C + C \int_0^{\beta_*} d\beta \int_{e_{\min}(\beta)}^{e_0} de\, \frac{1}{|(e, \beta) - (e_0, \hat{\beta})|^{2-\alpha}} \\
&\leq C + C \int_{-\hat{\beta}}^{\beta_* - \hat{\beta}} dx_2 \int_0^{e_0 - U_Q(0)} dx_1\, \frac{1}{|x|^{2-\alpha}} \leq C,
\end{aligned}$$

where $x = (x_1, x_2)$. Therefore $\mu_*' \leq C$ and the proof is complete. □