Chapter 3
On the Period Function T_1

Associated with every effective potential $U_{\text{eff}}(r, \ell) = U_Q(r) + \frac{\ell^2}{2r^2}$ is a period function $T_1(\cdot, \ell)$ that is defined for certain energies $e \in [e_{\min}(\ell), e_0]$, for which periodic solutions of $\ddot{r} = -U'_{\text{eff}}(r, \ell)$ do exist; see Appendix I, Sect. A.1, for more information. According to (A.20), this period function is given by

$$T_1(e, \ell) = 2 \int_{r_-(e,\ell)}^{r_+(e,\ell)} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r, \ell))}},$$

where $r_\pm = r_\pm(e, \ell)$ are the zeros of $0 = 2(e - U_{\text{eff}}(r, \ell))$ and satisfy $0 < r_-(e, \ell) < r_+(e, \ell)$. In addition, for every $\ell > 0$, the potential minimum $\inf \{U_{\text{eff}}(r, \ell) : r \geq 0\}$ is attained at some unique $r_0(\ell) \in]r_-(e, \ell), r_+(e, \ell)[$. The corresponding frequency function is $\omega_1(e, \ell) = \frac{2\pi}{T_1(e,\ell)}$.

3.1 Upper Boundedness of T_1

Recall that

$$D = \{(e, \beta) : \beta \in [0, \beta_*], e \in [e_{\min}(\beta), e_0]\}, \tag{3.1}$$

and

$$\mathring{D} = \{(e, \beta) : \beta \in]0, \beta_*[, e \in]e_{\min}(\beta), e_0[\}$$

is its interior. We are going to show that T_1 is bounded from above (or equivalently, ω_1 is bounded from below), uniformly in $\mathring{D}$, which is the set of relevant (e, ℓ), where T_1 is defined. As T_1 will be shown to be continuous in D (see Theorem 3.13 below), this is of course for free, but since the direct argument in Theorem 3.2 could be of

M. Kunze, *A Birman-Schwinger Principle in Galactic Dynamics*,
Progress in Mathematical Physics 77,
https://doi.org/10.1007/978-3-030-75186-9_3

general interest, we include it anyhow; the same remark applies to Theorem 3.5 on the lower boundedness of T_1.

We start with an auxiliary lemma that will be useful for the proof of Theorem 3.2 and beyond.

Lemma 3.1 *The following assertions are verified.*

(a) If $r > s > 0$, *then*

$$\frac{2\pi}{3}\rho_Q(r)(r^2 - s^2) \leq U_Q(r) - U_Q(s) \leq \frac{2\pi}{3}\rho_Q(0)(r^2 - s^2). \tag{3.2}$$

Moreover, for $r_Q \geq r > s > 0$,

$$U_Q(r) - U_Q(s) \geq \frac{\pi}{12}\,\rho_Q\Big(\frac{r_Q}{2}\Big)(r^2 - s^2). \tag{3.3}$$

(b) One has

$$\frac{\pi}{6}\,\rho_Q\Big(\frac{r_Q}{2}\Big)\,r_-^2 r_+^2 \leq \ell^2 \leq \frac{4\pi}{3}\,\rho_Q(0)\,r_-^2 r_+^2.$$

(c) One has

$$r_0 \leq \left(\frac{6}{\pi\rho_Q(\frac{r_Q}{2})}\right)^{1/4}\sqrt{\ell}.$$

Proof (a) According to (A.2), we have by changing variables $s = r\tau$, $ds = r d\tau$,

$$U_Q'(r) = \frac{4\pi}{r^2}\int_0^r s^2 \rho_Q(s)\,ds = 4\pi r\int_0^1 \tau^2\rho_Q(r\tau)\,d\tau. \tag{3.4}$$

In particular, $U_Q'(r) \geq 0$. Furthermore, for $r > s > 0$ and putting $t = \sigma/r$, $dt = d\sigma/r$,

$$\begin{aligned} U_Q(r) - U_Q(s) &= \int_s^r U_Q'(\sigma)\,d\sigma = 4\pi\int_s^r d\sigma\sigma\int_0^1 d\tau\tau^2\rho_Q(\sigma\tau) \\ &= 4\pi\int_0^1 d\tau\tau^2\int_s^r d\sigma\sigma\rho_Q(\sigma\tau) \qquad (3.5)\\ &= 4\pi r^2\int_0^1 d\tau\tau^2\int_{\frac{s}{r}}^1 dt\,t\,\rho_Q(r\tau t). \qquad (3.6) \end{aligned}$$

Due to (A.32), we have that $\rho_Q'(r) \leq 0$, i.e., ρ_Q is radially decreasing. Thus, if $\tau \in [0, 1]$ and $\sigma \in [s, r]$, then $\rho_Q(r) \leq \rho_Q(\sigma\tau) \leq \rho_Q(0)$ and (3.2) follows from (3.5). To establish (3.3), we use (3.6). To begin with, since $\rho_Q \geq 0$,

$$U_Q(r) - U_Q(s) \geq 4\pi r^2\int_0^{\frac{1}{2}} d\tau\tau^2\int_{\frac{s}{r}}^1 dt\,t\,\rho_Q(r\tau t).$$

Owing to $r \le r_Q$, $\tau \in [0, \frac{1}{2}]$, and $t \le 1$, we have $r\tau t \le \frac{r_Q}{2}$, so that $\rho_Q(r\tau t) \ge \rho_Q(\frac{r_Q}{2})$. It follows that

$$U_Q(r) - U_Q(s) \ge 4\pi r^2 \rho_Q\Big(\frac{r_Q}{2}\Big) \int_0^{\frac{1}{2}} d\tau \tau^2 \int_{\frac{s}{r}}^1 dt\, t = 4\pi r^2 \rho_Q\Big(\frac{r_Q}{2}\Big) \frac{1}{48}\Big(1 - \Big(\frac{s}{r}\Big)^2\Big),$$

which is (3.3). (b) The condition $U_{\rm eff}(r_\pm, \ell) = e$ means that $U_Q(r_\pm) + \frac{\ell^2}{2r_\pm^2} = e$, and hence

$$2r_\pm^2 U_Q(r_\pm) + \ell^2 = 2r_\pm^2 e. \tag{3.7}$$

Therefore,

$$\begin{aligned} 2(r_+^2 - r_-^2)e &= 2(r_+^2 U_Q(r_+) - r_-^2 U_Q(r_-)) \\ &= 2(r_+^2 - r_-^2) U_Q(r_+) + 2r_-^2 (U_Q(r_+) - U_Q(r_-)), \end{aligned}$$

so that

$$(r_+^2 - r_-^2)\frac{\ell^2}{r_+^2} = 2(r_+^2 - r_-^2)(e - U_Q(r_+)) = 2r_-^2 (U_Q(r_+) - U_Q(r_-)).$$

It remains to use (3.3) and the upper bound from (3.2). (c) First note that $\rho_Q(\frac{r_Q}{2}) > 0$, as otherwise supp $\rho_Q \subset [0, \frac{r_Q}{2}]$. By Lemma A.7(a), (3.4) and since ρ_Q is non-negative and radially decreasing,

$$\begin{aligned} \ell^2 &= r_0^3\, U_Q'(r_0) = 4\pi r_0^4 \int_0^1 \tau^2 \rho_Q(r_0\tau)\, d\tau \ge 4\pi r_0^4 \int_0^{\frac{1}{2}} \tau^2 \rho_Q(r_0 \tau)\, d\tau \\ &\ge 4\pi r_0^4 \int_0^{\frac{1}{2}} \tau^2 \rho_Q\Big(\frac{r_Q}{2}\Big)\, d\tau = \frac{\pi}{6}\, \rho_Q\Big(\frac{r_Q}{2}\Big)\, r_0^4. \end{aligned}$$

We will derive a more precise asymptotics of r_0 as $\ell \to 0^+$ below in (A.34). □

Now, we are in a position to derive a uniform lower bound on ω_1 or equivalently a uniform upper bound on T_1.

Theorem 3.2 *We have*

$$\delta_1 = \inf\{\omega_1(e, \ell) : (e, \ell) \in \mathring{D}\} > 0.$$

Proof Put $a_Q = \rho_Q(\frac{r_Q}{2}) > 0$. Then in particular $a_Q \le \rho_Q(0)$, so that

$$\delta_Q = 1 - \sqrt{\frac{a_Q}{16\rho_Q(0)}} \in \Big[\frac{1}{2}, 1\Big[.$$

Let $r_\pm = r_\pm(e, \ell)$ and $r_0 = r_0(\ell)$ be as before. From Lemma 3.1(c), we recall that

$$r_0 \leq \left(\frac{6}{\pi a_Q}\right)^{1/4} \sqrt{\ell}. \tag{3.8}$$

Case 1: $r_0 \geq (1-\delta_Q) r_+$. Then Lemma A.10(b) in conjunction with (3.8) implies that

$$T_1(e,\ell) \leq \pi \frac{\sqrt{r_- r_+}}{\ell}(r_- + r_+) \leq 2\pi \frac{r_+^2}{\ell} \leq \frac{2\pi}{(1-\delta_Q)^2} \frac{r_0^2}{\ell} \leq \frac{2\pi}{(1-\delta_Q)^2} \left(\frac{6}{\pi a_Q}\right)^{1/2}. \tag{3.9}$$

Case 2: $r_0 \leq (1-\delta_Q) r_+$. This is the nontrivial part of the argument. Here, we split up the integral as

$$\begin{aligned} T_1(e,\ell) &= 2\int_{r_-}^{r_+} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\ell))}} \\ &= 2\int_{r_-}^{r_0} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\ell))}} + 2\int_{r_0}^{r_+} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\ell))}} \\ &=: T_1^-(e,\ell) + T_1^+(e,\ell). \end{aligned}$$

Using Lemma A.10(a), we can bound T_1^- as

$$\begin{aligned} T_1^-(e,\ell) &\leq 2\frac{\sqrt{r_- r_+}}{\ell} \int_{r_-}^{r_0} \frac{r\,dr}{\sqrt{(r - r_-)(r_+ - r)}} \\ &\leq 2\frac{\sqrt{r_- r_+}}{\ell} \frac{r_0}{\sqrt{r_+ - r_0}} \int_{r_-}^{r_0} \frac{dr}{\sqrt{r - r_-}} \\ &= 4\frac{\sqrt{r_- r_+}}{\ell} \frac{r_0}{\sqrt{r_+ - r_0}} \sqrt{r_0 - r_-}. \end{aligned} \tag{3.10}$$

It follows from $r_0 \leq (1-\delta_Q) r_+$ that $\sqrt{r_+} \leq \delta_Q^{-1/2} \sqrt{r_+ - r_0}$. Thus, by (3.8) and since $\delta_Q \geq 1/2$,

$$T_1^-(e,\ell) \leq 4\delta_Q^{-1/2} \frac{\sqrt{r_-}}{\ell} r_0 \sqrt{r_0 - r_-} \leq 4\delta_Q^{-1/2} \frac{r_0^2}{\ell} \leq 4\sqrt{2} \left(\frac{6}{\pi a_Q}\right)^{1/2}. \tag{3.11}$$

Regarding T_1^+, we can invoke Lemma A.7(a) to get for $r \in [r_0, r_+]$ by also using Lemma A.6(a),

$$\begin{aligned} \frac{\ell^2}{2r_+^2 r^2} &\leq \frac{\ell^2}{2r_+^2 r_0^2} = \frac{r_0\, U_Q'(r_0)}{2r_+^2} \leq \frac{r_0\, U_Q'(r_0)}{2r_0^2} (1-\delta_Q)^2 = \frac{1}{2}(1-\delta_Q)^2 A(r_0) \\ &\leq \frac{1}{2}(1-\delta_Q)^2 A(0) = \frac{2\pi}{3}(1-\delta_Q)^2 \rho_Q(0). \end{aligned} \tag{3.12}$$

We then deduce from (3.3) in Lemma 3.1(a) and (3.12) that for $r \in [r_0, r_+]$,

$$\begin{aligned} e - U_{\text{eff}}(r,\ell) &= U_{\text{eff}}(r_+,\ell) - U_{\text{eff}}(r,\ell) = U_Q(r_+) + \frac{\ell^2}{2r_+^2} - U_Q(r) - \frac{\ell^2}{2r^2} \\ &= U_Q(r_+) - U_Q(r) - \frac{\ell^2}{2r_+^2 r^2}\,(r_+^2 - r^2) \\ &\geq \left[\frac{\pi}{12}\,a_Q - \frac{2\pi}{3}\,(1-\delta_Q)^2\,\rho_Q(0)\right](r_+^2 - r^2) \\ &= \frac{\pi}{24}\,a_Q(r_+^2 - r^2), \end{aligned}$$

the latter owing to the choice of δ_Q. This in turn yields

$$\begin{aligned} T_1^+(e,\ell) &= 2\int_{r_0}^{r_+} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\ell))}} \leq \sqrt{2}\sqrt{\frac{24}{\pi a_Q}} \int_{r_0}^{r_+} \frac{dr}{\sqrt{r_+^2 - r^2}} \\ &\leq \frac{4\sqrt{3}}{\sqrt{\pi a_Q}}\,\frac{1}{\sqrt{r_+}} \int_{r_0}^{r_+} \frac{dr}{\sqrt{r_+ - r}} = \frac{8\sqrt{3}}{\sqrt{\pi a_Q}}\,\frac{1}{\sqrt{r_+}}\,\sqrt{r_+ - r_0} \leq \frac{8\sqrt{3}}{\sqrt{\pi a_Q}}. \end{aligned}$$

Adding this to (3.11), we have shown that

$$T_1(e,\ell) \leq 4\sqrt{2}\left(\frac{6}{\pi a_Q}\right)^{1/2} + \frac{8\sqrt{3}}{\sqrt{\pi a_Q}} = \frac{16\sqrt{3}}{\sqrt{\pi a_Q}}. \tag{3.13}$$

Hence, the boundedness of T_1 from above is a consequence of (3.9) and (3.13). □

Observe that in the proof of Theorem 3.2 actually no properties of the sets $\mathring{D}$ or D from (3.1) have been used, apart from the fact that $T_1(e,\ell)$ is defined for $(e,\ell) \in \mathring{D}$.

3.2 Lower Boundedness of T_1

It is the purpose of this section to verify that T_1 is bounded from below (or equivalently, ω_1 is bounded from above), uniformly in $\mathring{D}$.

In some cases, it will be convenient to be able to re-express the period function

$$T_1(e,\beta) = 2\int_{r_-(e,\beta)}^{r_+(e,\beta)} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\beta))}} \tag{3.14}$$

from (A.20), written in terms of $\beta = \ell^2$, by means of an integral with fixed limits of integration; this is more or less taken from [11, Section 2].

Lemma 3.3 *We have*

$$T_1(e,\beta) = \sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\frac{\partial h}{\partial s}(s(\sqrt{\hat{e}(\beta)}\sin\theta,\beta),\beta)},$$

where

$$h(s,\beta) = s\left(\frac{V(s,\beta)}{s^2}\right)^{1/2}, \quad h(0,\beta) = 0,$$

for

$$V(s,\beta) = U_{\mathrm{eff}}(r_0(\beta)+s,\beta) - e_{\min}(\beta).$$

Also, $\hat{e}(\beta) = e - e_{\min}(\beta)$*, and* $R \mapsto s(R,\beta) = s$ *denotes the inverse mapping to* $s \mapsto h(s,\beta) = R$*. Explicitly,*

$$\frac{\partial h}{\partial s}(s,\beta) = \frac{\mathrm{sgn}(s)}{2}\,\frac{U'_{\mathrm{eff}}(r_0(\beta)+s,\beta)}{\sqrt{U_{\mathrm{eff}}(r_0(\beta)+s,\beta)-e_{\min}(\beta)}} \geq 0, \tag{3.15}$$

so that also

$$T_1(e,\beta) = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} d\theta\, \frac{[\int_0^1 (1-\rho)\, U''_{\mathrm{eff}}(r_0(\beta)+\rho s(\sqrt{\hat{e}(\beta)}\sin\theta,\beta),\beta)\,d\rho]^{1/2}}{\int_0^1 U''_{\mathrm{eff}}(r_0(\beta)+\rho s(\sqrt{\hat{e}(\beta)}\sin\theta,\beta),\beta)\,d\rho}. \tag{3.16}$$

Proof Let $s_\pm(e,\beta) = r_\pm(e,\beta) - r_0(\beta)$. Setting $s = r - r_0(\beta)$, $ds = dr$, we obtain

$$\begin{aligned} T_1(e,\beta) &= 2\int_{s_-(e,\beta)}^{s_+(e,\beta)} \frac{ds}{\sqrt{2(e-e_{\min}(\beta)-[U_{\mathrm{eff}}(r_0(\beta)+s,\beta)-e_{\min}(\beta)])}} \\ &= 2\int_{s_-(e,\beta)}^{s_+(e,\beta)} \frac{ds}{\sqrt{2(\hat{e}(\beta)-V(s,\beta))}}. \end{aligned} \tag{3.17}$$

Note that $V(\cdot,\beta)$ is increasing in $[0, s_+(e,\beta)]$, decreasing in $[s_-(e,\beta), 0]$ and such that

$$V(s_\pm(e,\beta),\beta) = e - e_{\min}(\beta) = \hat{e}(\beta).$$

Furthermore, $V(0,\beta) = U_{\mathrm{eff}}(r_0(\beta),\beta) - e_{\min}(\beta) = 0$ by definition and $\frac{\partial V}{\partial s}(0,\beta) = U'_{\mathrm{eff}}(r_0(\beta),\beta)$ by (A.35), i.e., $V(\cdot,\beta)$ is at least quadratic about $s = 0$. The next change of variables to be applied is

$$s \mapsto R = h(s,\beta), \quad dR = \frac{\partial h}{\partial s}\,ds, \quad R^2 = V(s,\beta).$$

Then (3.17) transforms into

$$T_1(e,\beta) = 2\int_{-\sqrt{\hat{e}(\beta)}}^{\sqrt{\hat{e}(\beta)}} \frac{dR}{\frac{\partial h}{\partial s}(s(R,\beta),\beta)\sqrt{2(\hat{e}(\beta)-R^2)}}.$$

Finally, put $R = \sqrt{\hat{e}(\beta)}\sin\theta$, $dR = \sqrt{\hat{e}(\beta)}\cos\theta\, d\theta$. This yields

$$T_1(e,\beta) = \sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\frac{\partial h}{\partial s}(s(\sqrt{\hat{e}(\beta)}\sin\theta,\beta),\beta)},$$

and thus the claimed formula for T_1. The relation (3.15) is straightforward, whereas (3.16) follows from Lemma A.9. □

Corollary 3.4 *If* $s \in [r_-(e,\beta) - r_0(\beta), 0]$*, then*

$$0 \le \frac{\partial h}{\partial s}(s,\beta) \le \frac{1}{\sqrt{2B(r_Q)}}\left(3\beta\int_0^1 \frac{d\rho}{(r_0(\beta)+\rho s)^4} + \frac{28\pi}{3}\,\rho_Q(0)\right).$$

Proof Let $s_- = s_-(e,\beta) - r_0(\beta)$. If $s \in [s_-, 0]$, then

$$0 \le \frac{\partial h}{\partial s}(s,\beta) = \left|\frac{\partial h}{\partial s}(s,\beta)\right| = \frac{1}{2}\,\frac{|U'_{\text{eff}}(r_0(\beta)+s,\beta)|}{\sqrt{U_{\text{eff}}(r_0(\beta)+s,\beta) - e_{\min}(\beta)}}$$

by (3.15) in Lemma 3.3. Thus, it remains to use (A.37) and (A.38) from Lemma A.9. □

Theorem 3.5 *We have*

$$\Delta_1 = \sup\{\omega_1(e,\ell) : (e,\ell) \in \mathring{D}\} < \infty.$$

Proof As above, we write $r_\pm = r_\pm(e,\beta)$ and $r_0 = r_0(\beta)$. If $r \in [r_-, r_+]$, then by Lemma 3.1(a),

$$\begin{aligned} e - U_{\text{eff}}(r,\beta) &= U_Q(r_+) - U_Q(r) - \frac{\beta}{2r_+^2 r^2}\,(r_+^2 - r^2) \\ &\le U_Q(r_+) - U_Q(r) \le \frac{2\pi}{3}\rho_Q(0)(r_+^2 - r^2) \le \frac{4\pi}{3}\rho_Q(0) r_+(r_+ - r). \end{aligned} \tag{3.18}$$

Case 1: $r_+/2 \ge r_0$. Here (3.18) implies that

$$\begin{aligned} T_1(e,\beta) &= 2\int_{r_-}^{r_+} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\beta))}} \ge \sqrt{2}\int_{r_0}^{r_+} \frac{dr}{\sqrt{e - U_{\text{eff}}(r,\beta)}} \\ &\ge \sqrt{\frac{3}{2\pi\rho_Q(0)}}\,\frac{1}{\sqrt{r_+}}\int_{r_0}^{r_+} \frac{dr}{\sqrt{r_+ - r}} = 2\sqrt{\frac{3}{2\pi\rho_Q(0)}}\sqrt{\frac{r_+ - r_0}{r_+}} \ge \sqrt{\frac{3}{\pi\rho_Q(0)}}. \end{aligned}$$

Case 2: $r_- \le r_+/2 \le r_0$. Similarl to the first case, we obtain

$$\begin{aligned} T_1(e,\beta) &= 2\int_{r_-}^{r_+} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r,\beta))}} \ge \sqrt{2}\int_{r_+/2}^{r_+} \frac{dr}{\sqrt{e - U_{\text{eff}}(r,\beta)}} \\ &\ge \sqrt{\frac{3}{2\pi\rho_Q(0)}}\,\frac{1}{\sqrt{r_+}}\int_{r_+/2}^{r_+} \frac{dr}{\sqrt{r_+ - r}} = \sqrt{\frac{6}{\pi\rho_Q(0)}}. \end{aligned}$$

Case 3: $0 \le r_+/2 \le r_-$. Then $r_+/2 \le r_- \le r_+$ and also $r_- \le r_0 \le r_+ \le 2r_-$ as well as $r_0 \le r_+ \le 2r_- \le 2r_0$, so all of r_-, r_0 and r_+ are of comparable size. In particular, if $r \in [r_-, r_+]$, then $r_0/2 \le r \le 2r_0$. In the following, we are going to use the notation from the proof of Lemma 3.3. Let $R = \sqrt{\hat{e}(\beta)} \sin\theta$. If $\theta \in [-\pi/2, 0]$, then $R \in [-\sqrt{\hat{e}(\beta)}, 0]$ and hence $s(R, \beta) \in [s_-, 0]$. Thus, if furthermore $\rho \in [0, 1]$, then $r_0 + \rho s(R, \beta) \in r_0 + [s_-, 0] = [r_-, r_0]$, so that

$$\frac{1}{2} r_0 \le r_0 + \rho s(R, \beta) \le 2r_0. \tag{3.19}$$

Since $s(R, \beta) \in [s_-, 0]$, Corollary 3.4 and (3.19) imply that

$$\begin{aligned} 0 \le \frac{\partial h}{\partial s}(s(R, \beta), \beta) &\le \frac{1}{\sqrt{2B(r_Q)}} \left(3\beta \int_0^1 \frac{d\rho}{(r_0 + \rho s(R, \beta))^4} + \frac{28\pi}{3}\, \rho_Q(0) \right) \\ &\le \frac{1}{\sqrt{2B(r_Q)}} \left(\frac{48\beta}{r_0^4} + \frac{28\pi}{3}\, \rho_Q(0) \right) \end{aligned} \tag{3.20}$$

for $\theta \in [-\pi/2, 0]$. By (A.34) from Lemma A.7, we have

$$r_0^4 = \frac{1}{A(0)}\, \beta + \mathcal{O}(\beta^{5/4}) = \beta\Big(\frac{1}{A(0)} + \mathcal{O}(\beta^{1/4}) \Big)$$

as $\beta \to 0^+$. Hence, there is $\beta_0 \in]0, \beta_*[$ such that

$$\frac{\beta}{2A(0)} \le r_0^4 \le \frac{2\beta}{A(0)}, \quad \beta \in]0, \beta_0].$$

Accordingly, owing to Lemma A.7(a), we can find a constant $c_0 > 0$ so that $r_0 \ge c_0$ for $\beta \in [\beta_0, \beta_*]$. If we now distinguish the cases $\beta \in]0, \beta_0]$ and $\beta \in [\beta_0, \beta_*]$, by using the foregoing estimates, we deduce that in any case

$$\frac{\beta}{r_0^4} \le \max \Big\{ 2A(0), \frac{\beta_*}{c_0^4} \Big\}.$$

Upon going back to (3.20), it follows that

$$0 \le \frac{\partial h}{\partial s}(s(R, \beta), \beta) \le \frac{1}{\sqrt{2B(r_Q)}} \left(48 \max \Big\{ 2A(0), \frac{\beta_*}{c_0^4} \Big\} + \frac{28\pi}{3}\, \rho_Q(0) \right) =: C_1$$

for $\theta \in [-\pi/2, 0]$. Since generally $\frac{\partial h}{\partial s} \ge 0$, we finally get from Lemma 3.3

$$T_1(e, \beta) = \sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\frac{\partial h}{\partial s}(s(R, \beta), \beta)} \ge \sqrt{2} \int_{-\pi/2}^{0} \frac{d\theta}{\frac{\partial h}{\partial s}(s(R, \beta), \beta)} \ge \frac{\pi}{\sqrt{2}\, C_1},$$

which completes the proof, as we have found a positive lower bound on T_1 in all three cases. □

3.3 Further Properties of T_1

First, we discuss some regularity properties of T_1.

Theorem 3.6 *We have* $T_1 \in C^1(\mathring{D})$.

Proof The continuity of T_1 may be shown directly from (3.14), as we already know that $r_\pm \in C^2(\mathring{D})$ by Remark A.3; we omit the details. To prove the differentiability, we use a method that is known and that we learned from R. Ortega. It is considerably less painful than differentiating an explicit relation for T_1 like (3.14). For $(e, \beta) \in \mathring{D}$, we consider

$$\ddot{r} = -U'_{\text{eff}}(r, \beta), \quad r(0) = r_-(e, \beta), \quad \dot{r}(0) = 0,$$

where $r(t) = r(t, e, \beta)$. Defining

$$F : \mathbb{R} \times \mathring{D} \to \mathbb{R}, \quad F(t, e, \beta) = \dot{r}(t, e, \beta),$$

we have $F \in C^1(\mathbb{R} \times \mathring{D})$ by Lemma A.11(a). Next observe that $F(t, e, \beta) = 0$ exactly for

$$t = 0, \; t = \pm\frac{1}{2} T_1(e, \beta), \; t = \pm T_1(e, \beta), \; t = \pm\frac{3}{2} T_1(e, \beta), \; \ldots .$$

Fix $(\tilde{e}, \tilde{\beta}) \in \mathring{D}$ and define $\tilde{t} = T_1(\tilde{e}, \tilde{\beta})$. Then $F(\tilde{t}, \tilde{e}, \tilde{\beta}) = 0$ by the above. Furthermore,

$$\frac{\partial F}{\partial t}(t, e, \beta) = \ddot{r}(t, e, \beta) = -U'_{\text{eff}}(r(t, e, \beta), \beta)$$

and $r(\tilde{t}, \tilde{e}, \tilde{\beta}) = r(T_1(\tilde{e}, \tilde{\beta}), \tilde{e}, \tilde{\beta}) = r(0, \tilde{e}, \tilde{\beta}) = r_-(\tilde{e}, \tilde{\beta})$ in conjunction with Lemma A.4 imply that

$$\frac{\partial F}{\partial t}(\tilde{t}, \tilde{e}, \tilde{\beta}) = -U'_{\text{eff}}(r_-(\tilde{e}, \tilde{\beta}), \tilde{\beta}) > 0.$$

Hence, the implicit function theorem yields the existence of a C^1-function $t = t(e, \beta)$ that is defined for (e, β) in a neighborhood $U \subset \mathring{D}$ of $(\tilde{e}, \tilde{\beta})$, such that

$$F(t(e, \beta), e, \beta) = 0 \text{ for } (e, \beta) \in U \quad \text{and} \quad t(\tilde{e}, \tilde{\beta}) = \tilde{t} = T_1(\tilde{e}, \tilde{\beta}).$$

According to our previous remarks, for every $(e, \beta) \in U$, we must have

$$t(e,\beta) = k(e,\beta)\,\frac{1}{2}\,T_1(e,\beta)$$

for some $k(e, \beta) \in \mathbb{Z}$. Then k is continuous in U and such that $k(\tilde{e}, \tilde{\beta}) = 2$, which means that $k = 2$ throughout U. Thus, $T_1 = t$ in U shows that $T_1 \in C^1(U)$. □

Remark 3.7 If $\rho_Q \in C^k$, then $U_Q \in C^{k+2}$. As a consequence, $r_- \in C^{k+2}(\mathring{D})$ by the argument from Remark A.3. Comparing to Lemma A.11(a), this entails $F = \dot{r} \in C^{k+1}(\mathbb{R} \times \mathring{D})$, so that $t = t(e, \beta) \in C^{k+1}(U)$ for the solution function in the proof of Theorem 3.6. Hence, we get $T_1 \in C^{k+1}(\mathring{D})$ in this case. ◇

Now, we are going to show that T_1 can be extended continuously from $\mathring{D}$ to D. We start with the continuous extension to $\{(e, \beta) : \beta \in]0, \beta_*], e = e_{\min}(\beta)\}$.

Lemma 3.8 *Let $\hat{\beta} \in]0, \beta_*]$. Then*

$$T_1(e,\beta) \to \frac{2\pi}{\sqrt{B(r_0(\hat{\beta}))}} \quad \textit{as} \quad \mathring{D} \ni (e,\beta) \to (e_{\min}(\hat{\beta}), \hat{\beta}). \tag{3.21}$$

Proof This relies on the representation (3.16) of $T_1(e, \beta)$, which we recall as

$$T_1(e,\beta) = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} d\theta\, \frac{[\int_0^1 (1-\rho)\, U_{\text{eff}}''(r_0(\beta) + \rho s(\sqrt{\hat{e}(\beta)}\sin\theta, \beta), \beta)\, d\rho]^{1/2}}{\int_0^1 U_{\text{eff}}''(r_0(\beta) + \rho s(\sqrt{\hat{e}(\beta)}\sin\theta, \beta), \beta)\, d\rho}. \tag{3.22}$$

Here, $h(s, \beta) = s(\frac{V(s,\beta)}{s^2})^{1/2}$ and $h(0, \beta) = 0$ for $V(s, \beta) = U_{\text{eff}}(r_0(\beta) + s, \beta) - e_{\min}(\beta)$. Furthermore, $\hat{e}(\beta) = e - e_{\min}(\beta)$ and $R \mapsto s(R, \beta) = s$ denotes the inverse mapping to $s \mapsto h(s, \beta) = R$. Due to $\beta \to \hat{\beta} > 0$, we can assume that $\beta \geq \hat{\beta}/2$ throughout the argument. If $r \in [r_-, r_+]$ and $\beta \in]0, \beta_*]$, then Lemma A.6(c) and (A.28) yields

$$\begin{aligned} U_{\text{eff}}'''(r,\beta) &= -\frac{12\beta}{r^5} + B'(r) - 3A'(r) = -\frac{12\beta}{r^5} + 4\pi\rho_Q'(r) - 2A'(r) \\ &= -\frac{12\beta}{r^5} + 4\pi\rho_Q'(r) - \frac{8\pi}{r^4}\int_0^r s^3\rho_Q'(s)\,ds. \end{aligned}$$

Therefore, (Q4) gives the bound

$$|U_{\text{eff}}'''(r,\beta)| \leq C\Big(1 + \frac{1}{r_-^5}\Big), \quad r \in [r_-, r_+], \quad \beta \in]0, \beta_*], \quad e \in [e_{\min}(\beta), e_0]. \tag{3.23}$$

By definition, we have $U_Q(r_-) + \frac{\beta}{2r_-} = e$. Hence, $U_Q'(r) \geq 0$ leads to

$$\frac{\beta}{2r_-^2} \leq U_Q(r_-) - U_Q(0) + \frac{\beta}{2r_-^2} = e - U_Q(0),$$

and thus

$$r_- \geq \sqrt{\frac{\beta}{2(e - U_Q(0))}} \geq \sqrt{\frac{\hat{\beta}}{4(e - U_Q(0))}}$$

for $\beta \in [\hat{\beta}/2, \beta_*]$ and $e \in [e_{\min}(\beta), e_0]$; note that we will have $e \to e_{\min}(\hat{\beta}) > U_Q(0)$. Going back to (3.23), we obtain

$$|U'''_{\text{eff}}(r, \beta)| \leq C, \quad r \in [r_-, r_+], \quad \beta \in [\hat{\beta}/2, \beta_*], \quad e \in [e_{\min}(\beta), e_0]. \tag{3.24}$$

Next, we assert that

$$\lim_{\beta \to \hat{\beta},\, e \to e_{\min}(\hat{\beta})} \sup_{\theta \in [-\pi/2, \pi/2]} |s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta)| = 0. \tag{3.25}$$

Otherwise, there would be $\varepsilon_0 > 0$ and sequences (β_j), (e_j) and (θ_j) such that $\beta_j \to \hat{\beta}$, $\theta_j \to \hat{\theta} \in [-\pi/2, \pi/2]$, $\hat{e}(\beta_j) = e_j - e_{\min}(\beta_j) \to e_{\min}(\hat{\beta}) - e_{\min}(\hat{\beta}) = 0$ as well as $|s(\sqrt{\hat{e}(\beta_j)} \sin \theta_j, \beta_j)| \geq \varepsilon_0$ for all $j \in \mathbb{N}$; here it was used that $e_{\min}(\beta) = U_{\text{eff}}(r_0(\beta), \beta)$ is continuous in $\beta \in]0, \beta_*[$, cf. Remark A.3. Thus, $\sqrt{\hat{e}(\beta_j)} \sin \theta_j \to 0$ and $s(\sqrt{\hat{e}(\beta_j)} \sin \theta_j, \beta_j) \to s(0, \hat{\beta}) = 0$, which is a contradiction. For the latter convergence, note that $s \mapsto h(s, \beta)$ for $s \in [s_-, s_+]$ is an increasing function that connects $-\sqrt{\hat{e}(\beta)}$ to $\sqrt{\hat{e}(\beta)}$. Since $\hat{e}(\beta_j) \to 0$, we must also have $s_\pm(e_j, \beta_j) \to 0$: for instance, if we had $s_+(e_j, \beta_j) \to \hat{s}_+ > 0$ (along a subsequence), then $h(s, \hat{\beta}) = 0$ for $s \in [0, \hat{s}_+]$, which is impossible. Thus, $s_\pm(e_j, \beta_j) \to 0$, and due to $|s(R, \beta)| \leq \max\{|s_-(e, \beta)|, s_+(e, \beta)\}$, we obtain $s(\sqrt{\hat{e}(\beta_j)} \sin \theta_j, \beta_j) \to 0$ as claimed.

Coming back to (3.22) and using Lemma A.7(d) and (3.24), we estimate

$$\begin{aligned}
&\left| \int_0^1 U''_{\text{eff}}(r_0(\beta) + \rho s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta), \beta)\, d\rho - B(r_0(\beta)) \right| \\
&= \left| \int_0^1 [U''_{\text{eff}}(r_0(\beta) + \rho s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta), \beta) - U''_{\text{eff}}(r_0(\beta), \beta)]\, d\rho \right| \\
&\leq C \int_0^1 |s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta)|\, d\rho \leq C S(e, \beta), \\
&\qquad S(e, \beta) = \sup_{\theta \in [-\pi/2, \pi/2]} |s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta)|.
\end{aligned} \tag{3.26}$$

Similarly,

$$\left| \int_0^1 (1 - \rho)\, U''_{\text{eff}}(r_0(\beta) + \rho s(\sqrt{\hat{e}(\beta)} \sin \theta, \beta), \beta)\, d\rho - \frac{1}{2} B(r_0(\beta)) \right| \leq C S(e, \beta). \tag{3.27}$$

From (3.26), (3.27) and (3.25), in conjunction with Lebesgue's dominated convergence theorem and $B(r_0(\hat{\beta})) > 0$, we deduce (3.21). □

Remark 3.9 Note that $T_1(e,\beta)$ is defined for $e = e_0$ and $\beta \in]0, \beta_*]$; it is the period of the orbit of $\ddot{r} = -U''_{\text{eff}}(r,\beta)$ that has the largest energy $e = e_0$. Therefore, it is straightforward that

$$T_1(e_0,\beta) = 2\int_{r_-(e_0,\beta)}^{r_+(e_0,\beta)} \frac{dr}{\sqrt{2(e_0 - U_{\text{eff}}(r,\beta))}}$$

extends T_1 continuously to $\{(e,\beta) : e = e_0, \beta \in]0, \beta_*]\}$. ◇

There is yet another way to represent T_1; see [24, Exercise 1, p. 40].

Lemma 3.10 *Define*

$$\chi(r,e,\beta) = \int_0^1 d\tau\,(1-\tau)\int_0^1 d\sigma\, U''_{\text{eff}}(\tau r_+(e,\beta) + \sigma(1-\tau)r \\ + (1-\sigma)(1-\tau)r_-(e,\beta),\beta).$$

Then

$$T_1(e,\beta) = \sqrt{2}\int_{r_-(e,\beta)}^{r_+(e,\beta)} \frac{dr}{\sqrt{(r_+(e,\beta)-r)(r-r_-(e,\beta))\,\chi(r,e,\beta)}}. \tag{3.28}$$

Proof If $r > s$, then

$$U_{\text{eff}}(r,\beta) - U_{\text{eff}}(s,\beta) = (r-s)\int_0^1 U'_{\text{eff}}(\tau r + (1-\tau)s,\beta)\,d\tau,$$

and in particular $U_{\text{eff}}(r_\pm,\beta) = e$ yields $\int_0^1 U'_{\text{eff}}(\tau r_+ + (1-\tau)r_-,\beta)\,d\tau = 0$. Therefore, we can write

$$\begin{aligned} e - U_{\text{eff}}(r,\beta) &= U_{\text{eff}}(r_+,\beta) - U_{\text{eff}}(r,\beta) \\ &= (r_+ - r)\int_0^1 U'_{\text{eff}}(\tau r_+ + (1-\tau)r,\beta)\,d\tau \\ &= (r_+ - r)\int_0^1 [U'_{\text{eff}}(\tau r_+ + (1-\tau)r,\beta) - U'_{\text{eff}}(\tau r_+ + (1-\tau)r_-,\beta)]\,d\tau \\ &= (r_+ - r)(r - r_-)\int_0^1 d\tau\,(1-\tau)\int_0^1 d\sigma\, U''_{\text{eff}}(\tau r_+ + \sigma(1-\tau)r \\ &\qquad + (1-\sigma)(1-\tau)r_-,\beta), \end{aligned}$$

which leads to (3.28). □

Lemma 3.11 *We have*

$$T_1(e,\beta) \to \frac{2\pi}{\sqrt{B(0)}} \quad as \quad \mathring{D} \ni (e,\beta) \to (U_Q(0),0). \tag{3.29}$$

Proof First, we note that, although it won't be used, $e - e_{\min}(\beta) \geq 0$ together with Lemma A.7(f) implies $e - U_Q(0) \geq e_{\min}(\beta) - U_Q(0) \sim \sqrt{U_Q''(0)}\sqrt{\beta}$ as $\beta \to 0$, which means that as $e \to U_Q(0)$, the quantity $e - U_Q(0)$ can't be too small in terms of $\beta \to 0$; due to $U_Q''(r) + \frac{2}{r}U_Q'(r) = 4\pi\rho_Q(r)$, we have $U_Q''(0) = \frac{4\pi}{3}\rho_Q(0) > 0$.

To actually verify (3.29), we are going to write

$$T_1(e,\beta) = \sqrt{2}\int_{r_-}^{r_+} \frac{dr}{\sqrt{(r_+ - r)(r - r_-)\,\chi(r)}} \tag{3.30}$$

as in (3.28) from Lemma 3.10, where

$$\chi(r) = \chi(r,e,\beta) = \int_0^1 d\tau\,(1-\tau)\int_0^1 d\sigma\, U_{\text{eff}}''(\tau r_+ + \sigma(1-\tau)r + (1-\sigma)(1-\tau)r_-, \beta).$$

Due to Lemma A.6(c), we have $U_{\text{eff}}''(r,\beta) = \frac{3\beta}{r^4} + B(r) - 3A(r)$. By explicit integration,

$$\begin{aligned}
&3\beta\int_0^1 d\sigma\int_0^1 d\tau\,(1-\tau)\,\frac{1}{(\tau r_+ + \sigma(1-\tau)r + (1-\sigma)(1-\tau)r_-)^4}\\
&= \frac{\beta}{2r_+^2}\int_0^1 d\sigma\,\frac{2r_+ + \sigma r + (1-\sigma)r_-}{(\sigma r + (1-\sigma)r_-)^3}\\
&= \frac{\beta}{r_+}\int_0^1 d\sigma\,\frac{1}{(\sigma r + (1-\sigma)r_-)^3} + \frac{\beta}{2r_+^2}\int_0^1 d\sigma\,\frac{1}{(\sigma r + (1-\sigma)r_-)^2}\\
&= \frac{\beta}{r_+}\,\frac{r + r_-}{2r^2 r_-^2} + \frac{\beta}{2r_+^2}\,\frac{1}{r r_-}\\
&= \frac{\beta}{2}\,\frac{r(r_- + r_+) + r_- r_+}{r^2 r_-^2 r_+^2}\\
&= \frac{\beta}{2}\,\frac{(r_+ + r)(r_- + r)}{r^2 r_-^2 r_+^2} - \frac{\beta}{2}\,\frac{1}{r_-^2 r_+^2}.
\end{aligned}$$

Hence, we obtain

$$\chi(r) = \frac{\beta}{2}\,\frac{(r_+ + r)(r_- + r)}{r^2 r_-^2 r_+^2} + \chi_2(r), \tag{3.31}$$

$$\chi_2(r) = \int_0^1 d\tau\,(1-\tau)\int_0^1 d\sigma\,(B-3A)(\tau r_+ + \sigma(1-\tau)r + (1-\sigma)(1-\tau)r_-) - \frac{\beta}{2}\,\frac{1}{r_-^2 r_+^2},$$

and $\chi_2(r) = \chi_2(r, e, \beta)$. From Lemma A.6(a) and (b), we get $(B-3A)(0) = \frac{16\pi}{3}\,\rho_Q(0) - \frac{12\pi}{3}\,\rho_Q(0) = \frac{4\pi}{3}\,\rho_Q(0) = U_Q''(0)$. Since $\tau r_+ + \sigma(1-\tau)r + (1-\sigma)(1-\tau)r_- \in [r_-, r_+] \subset [0, r_+]$ for $\tau, \sigma \in [0,1]$ and $r \in [r_-, r_+]$, it follows from

$$\chi_2(r) = \int_0^1 d\tau\,(1-\tau)\int_0^1 d\sigma\,[(B-3A)(\tau r_+ + \sigma(1-\tau)r + (1-\sigma)(1-\tau)r_-) - (B-3A)(0)] + \frac{1}{2}\,U_Q''(0) - \frac{\beta}{2}\,\frac{1}{r_-^2 r_+^2}$$

and (A.26) in Lemma A.5 that

$$\sup_{r\in[r_-,r_+]} |\chi_2(r,e,\beta)| \le \frac{1}{2}\sup_{s\in[0,r_+(e,\beta)]} |(B-3A)(s) - (B-3A)(0)| + \frac{1}{2}\sup_{s\in[0,r_+(e,\beta)]} |U_Q''(s) - U_Q''(0)| \tag{3.32}$$

for $(e,\beta) \in \mathring{D}$ and $r_\pm = r_\pm(e,\beta)$.

Next, we assert that

$$r_+(e,\beta) \to 0 \quad \text{as} \quad \mathring{D} \ni (e,\beta) \to (U_Q(0), 0). \tag{3.33}$$

To establish this claim, we will use the relation

$$U_Q(r_+) - U_Q(r_0) = e - \frac{\beta}{2r_+^2} - e_{\min}(\beta) + \frac{\beta}{2r_0^2} = e - e_{\min}(\beta) + \frac{\beta}{2r_0^2 r_+^2}(r_+^2 - r_0^2).$$

Hence, (3.3) from Lemma 3.1 yields

$$\frac{\pi}{12}\,\rho_Q\Big(\frac{r_Q}{2}\Big)(r_+^2 - r_0^2) \le e - U_Q(0) + U_Q(0) - e_{\min}(\beta) + \frac{\beta}{2r_0^2 r_+^2}(r_+^2 - r_0^2). \tag{3.34}$$

Due to Lemma A.7(f), we have $|e_{\min}(\beta) - U_Q(0)| = \mathcal{O}(\beta^{1/2})$ and $r_0 = \mathcal{O}(\beta^{1/4})$ as $\beta \to 0$. Thus, if $r_+(e,\beta) \to \hat{r}_+ > 0$ as $(e,\beta) \to (U_Q(0), 0)$, and along some subsequence, then (3.34) would imply that $\frac{\pi}{12}\,\rho_Q(\frac{r_Q}{2})\,\hat{r}_+^2 \le 0$, which is a contradiction and confirms (3.33). Since both $(B-3A)(s)$ and $U''_Q(s)$ are continuous at $s=0$, (3.32) in turn shows that

$$\lim_{e\to U_Q(0),\,\beta\to 0}\ \sup_{r\in[r_-,r_+]} |\chi_2(r,e,\beta)| = 0. \tag{3.35}$$

A further preparatory step is to rewrite (3.31) as

$$\chi(r) = \frac{\beta}{2}\,\frac{(r_+ + r)(r_- + r)}{r^2 r_-^2 r_+^2}\,(1+\chi_3(r)), \tag{3.36}$$
$$\chi_3(r) = \frac{2}{\beta}\,\frac{r^2 r_-^2 r_+^2}{(r_+ + r)(r_- + r)}\,\chi_2(r),$$

for $\chi_3(r) = \chi_3(r,e,\beta)$. Owing to Lemma 3.1(b), we have

$$r_-^2 r_+^2 \le C\beta. \tag{3.37}$$

Since also $\frac{r^2}{(r_+ + r)(r_- + r)} \le 1$, it follows from (3.35) that

$$\lim_{e\to U_Q(0),\,\beta\to 0}\ \sup_{r\in[r_-,r_+]} |\chi_3(r,e,\beta)| = 0. \tag{3.38}$$

Coming back to (3.29), consider sequences $e_j \to U_Q(0)$ and $\beta_j \to 0$. Let $\varepsilon > 0$ be given. According to (3.38), there is $j_0 \in \mathbb{N}$ such that $\sup_{r\in[r_-(e_j,\beta_j),r_+(e_j,\beta_j)]} |\chi_3(r,e_j,\beta_j)| \le \varepsilon$ for $j \ge j_0$. Due to (3.36), this yields for $j \ge j_0$ and $r \in [r_-(e_j,\beta_j), r_+(e_j,\beta_j)]$

$$\frac{\beta_j}{2}\,\frac{(r_{+,j}+r)(r_{-,j}+r)}{r^2 r_{-,j}^2 r_{+,j}^2}\,(1-\varepsilon) \le \chi(r,e_j,\beta_j) \le \frac{\beta_j}{2}\,\frac{(r_{+,j}+r)(r_{-,j}+r)}{r^2 r_{-,j}^2 r_{+,j}^2}\,(1+\varepsilon),$$

where $r_{\pm,j} = r_\pm(e_j,\beta_j)$. Therefore, (3.30) leads to

$$\frac{r_{-,j}\,r_{+,j}}{\sqrt{1+\varepsilon}}\,\frac{2}{\beta_j^{1/2}}\,I_j \le T_1(e_j,\beta_j) \le \frac{r_{-,j}\,r_{+,j}}{\sqrt{1-\varepsilon}}\,\frac{2}{\beta_j^{1/2}}\,I_j$$

for $j \ge j_0$, where

$$I_j = \int_{r_{-,j}}^{r_{+,j}} \frac{r}{(r_{+,j}^2 - r^2)^{1/2}(r^2 - r_{-,j}^2)^{1/2}}\,dr.$$

Setting $s = r^2$, $ds = 2r\,dr$, this integral may be evaluated as $I_j = \pi/2$. Thus, we obtain

$$\frac{1-\varepsilon}{\pi^2}\,\frac{\beta_j}{r_{-,j}^2 r_{+,j}^2} \le \frac{1}{T_1(e_j,\beta_j)^2} \le \frac{1+\varepsilon}{\pi^2}\,\frac{\beta_j}{r_{-,j}^2 r_{+,j}^2} \tag{3.39}$$

for $j \geq j_0$. From (A.26) in Lemma A.5, we know that

$$\left|\frac{\beta_j}{r_{-,j}^2 r_{+,j}^2} - U_Q''(0)\right| \leq \sup_{r\in[0,r_{+,j}]} |U_Q''(r) - U_Q''(0)|.$$

As $r_{+,j} \to 0$ by (3.33), we may assume that j_0 is already taken so large that

$$U_Q''(0) - \varepsilon \leq \frac{\beta_j}{r_{-,j}^2 r_{+,j}^2} \leq U_Q''(0) + \varepsilon$$

for $j \geq j_0$. Therefore, (3.39) implies that

$$\frac{1-\varepsilon}{\pi^2}\,(U_Q''(0) - \varepsilon) \leq \frac{1}{T_1(e_j, \beta_j)^2} \leq \frac{1+\varepsilon}{\pi^2}\,(U_Q''(0) + \varepsilon)$$

for $j \geq j_0$. Altogether, this shows that $\lim_{j\to\infty} T_1(e_j, \beta_j) = \frac{\pi}{\sqrt{U_Q''(0)}}$, and it remains to recall that $B(0) = \frac{16\pi}{3}\,\rho_Q(0) = 4U_Q''(0)$, cf. Lemma A.6(a), (b). □

Lemma 3.12 *Let $\hat{e} \in]U_Q(0), e_0]$. Then*

$$T_1(e, \beta) \to 2\int_0^{\hat{r}(\hat{e})} \frac{dr}{\sqrt{2(\hat{e} - U_Q(r))}} \quad \textit{as} \quad \mathring{D} \ni (e, \beta) \to (\hat{e}, 0), \tag{3.40}$$

where $\hat{r}(e) \in [0, r_Q]$ is the unique solution to $U_Q(\hat{r}(e)) = e$.

Proof First, we are going to show that r_+ stays away from zero in the limiting case that we are considering here. For this, we may assume that $r_+ \leq r_Q/2$. Due to (3.6), we have

$$r_+^2\varphi(r_+) + \frac{\beta}{2r_+^2} = U_Q(r_+) - U_Q(0) + \frac{\beta}{2r_+^2} = e - U_Q(0) \tag{3.41}$$

for

$$\varphi(r_+) = 4\pi \int_0^1 d\tau\tau^2 \int_0^1 dt\, t\, \rho_Q(\tau t r_+).$$

Since ρ_Q is radially decreasing and $0 \leq \tau t r_+ \leq r_Q/2$, it follows that

$$0 < c_1 = \frac{2\pi}{3}\rho_Q\left(\frac{r_Q}{2}\right) \leq \varphi(r_+) \leq \frac{2\pi}{3}\rho_Q(0) = C_1. \tag{3.42}$$

In (3.41), solving the resulting quadratic equation for r_+^2, we obtain

$$r_+^2 = \frac{e - U_Q(0) \pm \sqrt{(e - U_Q(0))^2 - 2\varphi(r_+)\beta}}{2\varphi(r_+)}. \tag{3.43}$$

Let us suppose that the sign were '−', along (a subsequence) of $e \to \hat{e}$ and $\beta \to 0$.

Then

$$r_+^2 = \frac{\beta}{e - U_Q(0) + \sqrt{(e - U_Q(0))^2 - 2\varphi(r_+)\beta}}$$

together with $\hat{e} - U_Q(0) > 0$ and (3.42) would yield $c_2\beta \le r_+^2 \le C_2\beta$ for suitable constants $C_2 > c_2 > 0$. By Lemma 3.1(b), we have the general estimate

$$c\beta \le r_-^2 r_+^2.$$

As a consequence,

$$\frac{c}{C_2} \le r_-^2.$$

However, $r_-^2 \le r_0^2 = \mathcal{O}(\beta^{1/2})$ as $\beta \to 0$ by Lemma A.7(f), which gives a contradiction. To summarize, we may suppose that the sign is '+' in (3.43). Hence,

$$r_+^2 = \frac{e - U_Q(0) + \sqrt{(e - U_Q(0))^2 - 2\varphi(r_+)\beta}}{2\varphi(r_+)} \ge \frac{1}{2C_1}(e - U_Q(0))$$

for $\beta \le \frac{1}{2C_1}(e - U_Q(0))^2$ yields the desired lower bound for r_+. Thus, in what follows, we can assume that $r_+(e, \beta) \ge \eta_0 > 0$ for an appropriate constant η_0 and $(e, \beta) \to (\hat{e}, 0)$.

Next, we are going to show that

$$T_1^-(e, \beta) = 2\int_{r_-}^{r_0} \frac{dr}{\sqrt{2(e - U_{\text{eff}}(r, \beta))}} \to 0 \quad \text{as} \quad \mathring{D} \ni (e, \beta) \to (\hat{e}, 0). \tag{3.44}$$

For, owing to (3.10), we get

$$T_1^-(e, \beta) \le 4\,\frac{\sqrt{r_- r_+}}{\sqrt{\beta}}\,\frac{r_0}{\sqrt{r_+ - r_0}}\,\sqrt{r_0 - r_-} \le 4\,\frac{\sqrt{r_- r_+}}{\sqrt{\beta}}\,\frac{r_0^{3/2}}{\sqrt{r_+ - r_0}}.$$

Since $r_-^2 r_+^2 \le C\beta$ by (3.37) and $r_0 = \mathcal{O}(\beta^{1/4})$ by Lemma A.7(f), $r_+ \ge \eta_0$ yields

$$T_1^-(e, \beta) \le C\beta^{3/8}$$

and completes the argument for (3.44).

Thus, in order to establish (3.40), we need to prove that

$$\int_{r_0}^{r_+} \frac{dr}{\sqrt{e - U_{\text{eff}}(r, \beta)}} \to \int_0^{\hat{r}(\hat{e})} \frac{dr}{\sqrt{\hat{e} - U_Q(r)}} \quad \text{as} \quad \mathring{D} \ni (e, \beta) \to (\hat{e}, 0); \tag{3.45}$$

note that $U_Q(r_+) \le U_Q(r_+) + \frac{\beta}{2r_+^2} = e = U_Q(\hat{r}(e))$ implies $r_+ \le \hat{r}(e)$. In addition, using (3.3), we obtain

$$\begin{aligned}\frac{\pi}{12}\,\rho_Q\Big(\frac{r_Q}{2}\Big)\eta_0\,(\hat{r}(e)-r_+) &\le \frac{\pi}{12}\,\rho_Q\Big(\frac{r_Q}{2}\Big)(\hat{r}(e)^2-r_+^2)\\ &\le U_Q(\hat{r}(e))-U_Q(r_+)=\frac{\beta}{2r_+^2}\le\frac{1}{2\eta_0^2}\,\beta.\end{aligned}$$

Similarly, by (3.2),

$$\begin{aligned}\frac{\beta}{2r_Q^2}\le\frac{\beta}{2r_+^2} &= U_Q(\hat{r}(e))-U_Q(r_+)\\ &\le \frac{2\pi}{3}\rho_Q(0)(\hat{r}(e)^2-r_+^2)\\ &\le \frac{4\pi}{3}\rho_Q(0)\,r_Q\,(\hat{r}(e)-r_+),\end{aligned}$$

so that $c_3\beta\le\hat{r}(e)-r_+\le C_3\beta$. To validate (3.45), we are going to show

$$\left|\int_{r_0}^{r_+}\frac{dr}{\sqrt{e-U_{\mathrm{eff}}(r,\beta)}}-\int_0^{\hat{r}(e)}\frac{dr}{\sqrt{e-U_Q(r)}}\right|\to 0, \tag{3.46}$$

$$\left|\int_0^{\hat{r}(e)}\frac{dr}{\sqrt{e-U_Q(r)}}-\int_0^{\hat{r}(\hat{e})}\frac{dr}{\sqrt{\hat{e}-U_Q(r)}}\right|\to 0, \tag{3.47}$$

both as $\mathring{D}\ni(e,\beta)\to(\hat{e},0)$; the second relation is independent of β.

To begin with,

$$\int_0^{r_0}\frac{dr}{\sqrt{e-U_Q(r)}}\to 0\quad\text{and}\quad\int_{r_+}^{\hat{r}(e)}\frac{dr}{\sqrt{e-U_Q(r)}}\to 0. \tag{3.48}$$

For the first claim, if $0\le r\le r_0=\mathcal{O}(\beta^{1/4})$ and $e\to\hat{e}>U_Q(0)$, we may suppose that $e-U_Q(r)\ge\eta_1>0$ for the e and r in question; therefore, the first claim in (3.48) follows. Regarding the second assertion, we write

$$e-U_Q(r)=U_Q(\hat{r}(e))-U_Q(r)=(\hat{r}(e)-r)\int_0^1 U_Q'(\tau\hat{r}(e)+(1-\tau)r)\,d\tau$$

for $r\in[\frac{r_+}{2},\hat{r}(e)]$. If $s\in[\frac{r_+}{2},\hat{r}(e)]$, then the fact that ρ_Q is radially decreasing yields

$$U_Q'(s)=\frac{4\pi}{s^2}\int_0^s\sigma^2\rho_Q(\sigma)\,d\sigma\ge\frac{4\pi}{r_Q^2}\int_0^{r_+/2}\sigma^2\rho_Q(\sigma)\,d\sigma\ge\frac{\pi r_+^3}{6r_Q^2}\,\rho_Q\Big(\frac{r_+}{2}\Big)\ge\eta_2>0. \tag{3.49}$$

Hence,

$$e-U_Q(r)\ge\eta_2(\hat{r}(e)-r),\quad r\in\Big[\frac{r_+}{2},\hat{r}(e)\Big], \tag{3.50}$$

and accordingly,

$$\int_{r_+}^{\hat r(e)} \frac{dr}{\sqrt{e-U_Q(r)}} \le \eta_2^{-1/2} \int_{r_+}^{\hat r(e)} \frac{dr}{\sqrt{\hat r(e)-r}} = 2\eta_2^{-1/2}\sqrt{\hat r(e)-r_+} \le C\beta^{1/2} \to 0.$$

Thus, both relations in (3.48) hold, and therefore (3.46) comes down to proving that

$$\left| \int_{r_0}^{r_+} \frac{dr}{\sqrt{e-U_{\rm eff}(r,\beta)}} - \int_{r_0}^{r_+} \frac{dr}{\sqrt{e-U_Q(r)}} \right| \to 0 \quad \text{as} \quad \mathring{D} \ni (e,\beta) \to (\hat e, 0).$$

If $r \in [r_0, (1-\beta^{1/4})r_+]$, then $\frac{\beta}{2r^2} \le \frac{\beta}{2r_0^2} \le C\beta^{1/2}$, as $r_0 = \mathcal{O}(\beta^{1/4})$. Therefore, (3.3) yields

$$\begin{aligned} e - U_Q(r) \ge e - U_{\rm eff}(r,\beta) &= e - U_Q(r) - \frac{\beta}{2r^2} \\ &\ge U_Q(\hat r(e)) - U_Q((1-\beta^{1/4})r_+) - C\beta^{1/2} \\ &\ge \frac{\pi}{12}\rho_Q\left(\frac{r_Q}{2}\right)(\hat r(e)^2 - (1-\beta^{1/4})^2 r_+^2) - C\beta^{1/2} \\ &\ge \frac{\pi}{12}\rho_Q\left(\frac{r_Q}{2}\right)\eta_0\,(\hat r(e) - r_+ + \beta^{1/4} r_+) - C\beta^{1/2} \\ &\ge c_4\beta + c_5\beta^{1/4} - C\beta^{1/2} \\ &\ge c_6\beta^{1/4}. \end{aligned}$$

From the estimate $\frac{1}{\sqrt a} - \frac{1}{\sqrt b} \le \frac{b-a}{a\sqrt b}$ for $b \ge a > 0$, we hence infer

$$\begin{aligned} &\left| \int_{r_0}^{(1-\beta^{1/4})r_+} \frac{dr}{\sqrt{e-U_{\rm eff}(r,\beta)}} - \int_{r_0}^{(1-\beta^{1/4})r_+} \frac{dr}{\sqrt{e-U_Q(r)}} \right| \\ &\le \frac{\beta}{2}\int_{r_0}^{(1-\beta^{1/4})r_+} \frac{1}{(e-U_{\rm eff}(r,\beta))\sqrt{e-U_Q(r)}}\,\frac{dr}{r^2} \\ &\le \frac{\beta}{2r_0^2}\int_{r_0}^{(1-\beta^{1/4})r_+} \frac{1}{c_6^{3/2}\beta^{3/8}}\,dr \le C\beta^{1/8} \to 0. \end{aligned} \tag{3.51}$$

For the remaining part, $r \in [(1-\beta^{1/4})r_+, r_+]$, we note that for such r, by (3.50),

$$e - U_Q(r) \ge \eta_2(\hat r(e) - r) \ge \eta_2(\hat r(e) - r_+) \ge \eta_2 c_3\,\beta.$$

In addition,

$$\begin{aligned} e - U_{\rm eff}(r,\beta) &= U_{\rm eff}(r_+,\beta) - U_{\rm eff}(r,\beta) \\ &= (r_+ - r)\int_0^1 U'_{\rm eff}(\tau r_+ + (1-\tau)r, \beta)\,d\tau. \end{aligned} \tag{3.52}$$

If $r \in [(1-\beta^{1/4})r_+, r_+]$, then $s = \tau r_+ + (1-\tau) r \in [(1-\beta^{1/4})r_+, r_+] \subset [\frac{r_+}{2}, \hat{r}(e)]$ for instance, and

$$U'_{\rm eff}(s,\beta) = U'_Q(s) - \frac{\beta}{s^3} \geq \eta_2 - \frac{8\beta}{r_+^3} \geq \eta_2 - \frac{8\beta}{\eta_0^3} \geq \frac{1}{2}\,\eta_2$$

by (3.49), if β is small enough. Thus, (3.52) leads to

$$e - U_{\rm eff}(r,\beta) \geq \frac{1}{2}\,\eta_2\,(r_+ - r), \quad r \in [(1-\beta^{1/4})r_+, r_+].$$

If we now use that $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \leq \frac{b-a}{\sqrt{ab}}$ for $b \geq a > 0$, we obtain the bound

$$\begin{aligned} &\left| \int_{(1-\beta^{1/4})r_+}^{r_+} \frac{dr}{\sqrt{e - U_{\rm eff}(r,\beta)}} - \int_{(1-\beta^{1/4})r_+}^{r_+} \frac{dr}{\sqrt{e - U_Q(r)}} \right| \\ &\leq \frac{\beta}{2} \int_{(1-\beta^{1/4})r_+}^{r_+} \frac{1}{\sqrt{e - U_{\rm eff}(r,\beta)(e - U_Q(r))}}\,\frac{dr}{r^2} \\ &\leq \frac{\beta}{2}\,\frac{4}{\eta_0^2}\,\frac{1}{\eta_2 c_3\,\beta}\sqrt{\frac{2}{\eta_2}} \int_{(1-\beta^{1/4})r_+}^{r_+} \frac{1}{\sqrt{r_+ - r}}\,dr \leq C\beta^{1/8} \to 0. \end{aligned} \tag{3.53}$$

By (3.51) and (3.53), the proof of (3.46) is complete.

Therefore, it remains to check that (3.47) is satisfied. This is not worked out, since it is just the continuity of the standard period function in the potential $V(x) = U_Q(x) - U_Q(0)$ for $x \geq 0$ and $V(x) = U_Q(-x) - U_Q(0)$ for $x \leq 0$, for energies $\hat{e} = e - U_Q(0) \in]0, e_0 - U_Q(0)]$. □

If we now summarize Lemma 3.8, Remark 3.9 and Lemmas 3.11 and 3.12, then we have shown the following result (note that $e_{\min}(0) = U_Q(0)$ and $r_0(0) = 0$).

Theorem 3.13 *We have $T_1 \in C(D)$. The extensions to ∂D are given by*

$$T_1(e,\beta) = \begin{cases} \frac{2\pi}{\sqrt{B(r_0(\beta))}} & : \; e = e_{\min}(\beta),\ \beta \in [0,\beta_*] \\ 2\int_{r_-(e_0,\beta)}^{r_+(e_0,\beta)} \frac{dr}{\sqrt{2(e_0 - U_{\rm eff}(r,\beta))}} & : \; e = e_0,\ \beta \in]0,\beta_*] \\ 2\int_0^{\hat{r}(e)} \frac{dr}{\sqrt{2(e - U_Q(r))}} & : \; e \in]U_Q(0), e_0],\ \beta = 0 \end{cases},$$

where $\hat{r}(e) \in [0, r_Q]$ is the unique solution to $U_Q(\hat{r}(e)) = e$.

In the remaining part of this section, we will discuss some monotonicity properties of T_1.

Lemma 3.14 *The function $[0,\beta_*] \ni \beta \mapsto T_1(e_{\min}(\beta),\beta)$ is strictly increasing.*

Proof We know from Lemma A.7(e) that $\beta \mapsto r_0(\beta)$ is strictly increasing, and furthermore $r \mapsto B(r)$ is strictly decreasing by Lemma A.6(b). Hence, the claim follows from $T_1(e_{\min}(\beta), \beta) = \frac{2\pi}{\sqrt{B(r_0(\beta))}}$. □

Lemma 3.15 *The function $[U_Q(0), e_0] \ni e \mapsto T_1(e, 0)$ is strictly increasing.*

Proof The argument is analogous to the fact that for a one degree of freedom oscillator $\ddot{x} = -V'(x)$ about a stable center, where $V(0) = V'(0) = 0$ and $V(-x) = V(x)$ for simplicity, the condition $V'(x) > 0$ and $V''(x) > V'(x)/x$ for $x > 0$ guarantees that the period function of the periodic orbits about $x = 0$ is decreasing in the energy $e = \frac{1}{2}\dot{x}^2 + V(x)$. The first reference to point this out seems to be [64] (which we basically follow); related papers are [11, 78, 79]. To see the connection, first observe that, by (1.13), Remark A.1 and (A.32),

$$U_Q''(r) - \frac{U_Q'(r)}{r} = 4\pi\rho_Q(r) - \frac{3}{r}\,U_Q'(r) = \frac{4\pi}{r^3}\int_0^r s^3\rho_Q'(s)\,ds < 0, \quad r > 0.$$

Thus, $(U_Q'(r)/r)' = (rU_Q''(r) - U_Q'(r))/r^2 < 0$ for $r > 0$, and it follows that

$$\frac{1}{p}\,U_Q'(pr) < U_Q'(r), \quad p > 1, \quad r > 0. \tag{3.54}$$

The function $\hat{r}$ is strictly increasing, due to $1 = U_Q'(\hat{r}(e))\,\hat{r}'(e)$ and $U_Q'(r) > 0$ for $r > 0$. Therefore, its inverse $[0, r_Q] \ni \hat{r} \mapsto e(\hat{r}) \in [U_Q(0), e_0]$ is well-defined and strictly increasing too; note that $\hat{r}(U_Q(0)) = 0$ and $\hat{r}(e_0) = r_Q$. Let

$$\hat{T}(\hat{r}) = 2\int_0^{\hat{r}} \frac{dr}{\sqrt{2(U_Q(\hat{r}) - U_Q(r))}}.$$

Then

$$T_1(e, 0) = 2\int_0^{\hat{r}(e)} \frac{dr}{\sqrt{2(U_Q(\hat{r}(e)) - U_Q(r)}} = \hat{T}(\hat{r}(e)),$$

which implies that $e \mapsto T_1(e, 0)$ is increasing if and only if $\hat{r} \mapsto \hat{T}(\hat{r})$ is increasing. If $p > 1$ and $s \in [0, \hat{r}]$, then by (3.54), one has

$$U_Q(p\hat{r}) - U_Q(ps) = p\int_s^{\hat{r}} U_Q'(p\tau)\,d\tau \le p^2\int_s^{\hat{r}} U_Q'(\tau)\,d\tau = p^2(U_Q(\hat{r}) - U_Q(s)).$$

As a consequence,

$$\begin{aligned}\hat{T}(p\hat{r}) &= 2\int_0^{p\hat{r}} \frac{dr}{\sqrt{2(U_Q(p\hat{r}) - U_Q(r))}} = 2p\int_0^{\hat{r}} \frac{ds}{\sqrt{2(U_Q(p\hat{r}) - U_Q(ps))}} \\ &\ge 2\int_0^{\hat{r}} \frac{ds}{\sqrt{2(U_Q(\hat{r}) - U_Q(s))}} = \hat{T}(\hat{r}),\end{aligned}$$

which completes the proof. □

Corollary 3.16 *Suppose that* $\delta_1 = \inf_{\mathring{D}} \omega_1 = \min_D \omega_1$ *is attained at some point* $(\hat{e}, \hat{\beta}) \in \partial D$. *Then* $(\hat{e}, \hat{\beta})$ *lies on the 'upper line'* $\{(e, \beta) : e = e_0, \beta \in [0, \beta_*]\}$ *of the boundary.*

Proof This follows from $\omega_1 = \frac{2\pi}{T_1}$ together with Lemmas 3.14 and 3.15. □

Remark 3.17 It will certainly be important to gain a better understanding of the monotonicity properties of ω_1 (or, equivalently, T_1) in D. In particular, we expect that it should be significant to locate those points in D, where ω_1 attains its minimum δ_1. Some relations for $\frac{\partial T_1}{\partial e}$ and $\frac{\partial T_1}{\partial \beta}$ are stated in Lemma A.12(b), (c). For instance, we have

$$\frac{\partial T_1}{\partial \beta}(e, \beta) = -\frac{1}{2} \frac{\partial}{\partial e} \int_0^{T_1(e,\beta)} \frac{ds}{r(s)^2} = -\frac{\partial}{\partial e} \int_{r_-(e,\beta)}^{r_+(e,\beta)} \frac{dr}{r^2 p_r}, \tag{3.55}$$

which could provide a way to approach the monotonicity of T_1 in β. To see this, we apply the transformation $\rho = \sqrt{\beta}\, r^{-1}$, $d\rho = -\sqrt{\beta}\, r^{-2}\, dr$, like for the 'apsidal angle' [77]. Defining

$$\tilde{U}(\rho, \beta) = \frac{1}{2}\rho^2 + U_Q\Big(\frac{\sqrt{\beta}}{\rho}\Big), \quad \rho_{\mp}(e, \beta) = \frac{\sqrt{\beta}}{r_{\pm}(e, \beta)},$$

and recalling that $p_r = \sqrt{2(e - U_Q(r) - \frac{\beta}{2r^2})}$, we get

$$\frac{\partial T_1}{\partial \beta}(e, \beta) = -\frac{1}{\sqrt{\beta}} \frac{\partial}{\partial e} \int_{\rho_-(e,\beta)}^{\rho_+(e,\beta)} \frac{d\rho}{\sqrt{2(e - \tilde{U}(\rho, \beta))}}.$$

At fixed β, this has turned the integral on the right-hand side of (3.55) into the period function

$$\tilde{T}(e) = \int_{\rho_-(e,\beta)}^{\rho_+(e,\beta)} \frac{d\rho}{\sqrt{2(e - \tilde{U}(\rho, \beta))}}$$

for the transformed potential $\tilde{U}$; note that $0 < \rho_- < \rho_+$ and $\tilde{U}(\rho_{\pm}, \beta) = e$. One could study the monotonicity of $\tilde{T}(e)$ in the energy e by checking the criteria that have been listed in the papers we mentioned in the proof of Lemma 3.15 or which can be found in similar works. Let us state a remarkable relation that could be useful in this respect. Writing $\tilde{U}(\rho) = \tilde{U}(\rho, \beta)$, it is calculated that

$$\tilde{U}'(\rho) = -\frac{\sqrt{\beta}}{\rho^2} U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big) + \rho, \quad \tilde{U}''(\rho) = \frac{\beta}{\rho^4} U_Q''\Big(\frac{\sqrt{\beta}}{\rho}\Big) + \frac{2\sqrt{\beta}}{\rho^3} U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big) + 1,$$

and using (1.13) this yields

$$\begin{aligned}\tilde{U}''(\rho) - \frac{\tilde{U}'(\rho)}{\rho} &= \frac{\beta}{\rho^4}\, U_Q''\Big(\frac{\sqrt{\beta}}{\rho}\Big) + \frac{3\sqrt{\beta}}{\rho^3}\, U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big) \\ &= \frac{\beta}{\rho^4}\left[4\pi\rho_Q\Big(\frac{\sqrt{\beta}}{\rho}\Big) - \frac{2\rho}{\sqrt{\beta}}\, U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big)\right] + \frac{3\sqrt{\beta}}{\rho^3}\, U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big) \\ &= \frac{\beta}{\rho^4}\left[4\pi\rho_Q\Big(\frac{\sqrt{\beta}}{\rho}\Big) + \frac{\rho}{\sqrt{\beta}}\, U_Q'\Big(\frac{\sqrt{\beta}}{\rho}\Big)\right] \\ &= \frac{\beta}{\rho^4}\, B\Big(\frac{\sqrt{\beta}}{\rho}\Big).\end{aligned}$$

In other words,

$$\Big(\frac{\tilde{U}'(\rho)}{\rho}\Big)' = \frac{\beta}{\rho^5}\, B\Big(\frac{\sqrt{\beta}}{\rho}\Big),$$

and the function B is strictly positive. Comparing to the reasoning in Lemma 3.15, this looks promising for proving that $\tilde{T}(e)$ is increasing in e, i.e., that $\frac{\partial T_1}{\partial \beta} < 0$. However, the argument does not seem to work properly, since the integral defining $\tilde{T}(e)$ is on $[\rho_-, \rho_+]$, instead of it beginning at zero, as is the case in Lemma 3.15. ◇

3.4 $\lambda_* \le \delta_1^2$

From (1.20), recall the definition of λ_*.

Lemma 3.18 *We have* $\lambda_* \le \delta_1^2$.

Proof From (1.18), cf. Corollary B.19 and Lemma B.8(c), we deduce that, for $u \in X^2_{\mathrm{odd}}$,

$$\begin{aligned}(Lu, u)_Q &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|\mathcal{T}u|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\,dx \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|\mathcal{T}u|^2 = \|\mathcal{T}u\|^2_{X^0} = 16\pi^3 \sum_{k\neq 0} k^2 \|\omega_1 u_k\|^2_{L^2_{\frac{1}{|Q'|}}(D)}.\end{aligned}$$

Since $u_{-k} = -u_k$ by Lemma B.3(b), this yields

$$\lambda_* \le 32\pi^3 \sum_{k=1}^{\infty} k^2 \|\omega_1 u_k\|^2_{L^2_{\frac{1}{|Q'|}}(D)} \tag{3.56}$$

for all $u \in X^2_{\mathrm{odd}}$ such that $\|u\|_{X^0} = \|u\|_Q = 1$. Now we specialize (3.56) to $u \cong (\ldots, 0, u_{-1}, 0, u_1, 0, \ldots) = (\ldots, 0, -u_1, 0, u_1, 0, \ldots)$ to find that

$$\lambda_* \le 32\pi^3 \|\omega_1 u_1\|^2_{L^2_{\frac{1}{|Q'|}}(D)} = 32\pi^3 \iint_D dI\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \omega_1^2(I,\ell)\, |u_1(I,\ell)|^2$$
$$= 32\pi^3 \iint_D de\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \omega_1(e,\ell)\, |u_1(e,\ell)|^2 \tag{3.57}$$

for all $u_1 = u_1(I,\ell) = u_1(e,\ell) \in L^2_{\frac{1}{|Q'|}}(D)$ satisfying

$$1 = 32\pi^3 \iint_D dI\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, |u_1(I,\ell)|^2$$
$$= 32\pi^3 \iint_D de\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \frac{1}{\omega_1(e,\ell)}\, |u_1(e,\ell)|^2;$$

see Definition B.1 and cf. (A.18). Let $\varepsilon > 0$. Since $\delta_1 = \inf_{\mathring{D}} \omega_1$, there is $(\hat{e}, \hat{l}) \in \mathring{D}$ such that $\omega_1(\hat{e}, \hat{l}) < \delta_1 + \varepsilon/2$. As ω_1 is continuous in $\mathring{D}$ by Theorem 3.6, there is an open neighborhood $U \subset \mathring{D}$ of $(\hat{e}, \hat{l})$ with the property that $\omega_1(e,l) < \delta_1 + \varepsilon$ for $(e,l) \in U$; then $\iint_U de\, d\ell\, \ell > 0$. Define

$$\chi(e,\ell) = \begin{cases} 1 & : \ (e,\ell) \in U \\ 0 & : \ (e,\ell) \in D \setminus U \end{cases}$$

and $u_1(e,\ell) = a\, |Q'(e)|^{1/2} \omega_1(e,\ell)^{1/2}\, \chi(e,\ell)$ for $a = (32\pi^3 \iint_U de\, d\ell\, \ell)^{-1/2}$. It follows that

$$32\pi^3 \iint_D de\, d\ell\, \ell\, \frac{1}{|Q'(e)|\omega_1}\, |u_1|^2 = 32\pi^3 a^2 \iint_U de\, d\ell\, \ell = 1.$$

Thus, by (3.57),

$$\lambda_* \le 32\pi^3 \iint_D de\, d\ell\, \ell\, \frac{\omega_1}{|Q'|}\, |u_1|^2 = 32\pi^3 a^2 \iint_U de\, d\ell\, \ell\, \omega_1^2$$
$$\le 32\pi^3 a^2\, (\delta_1 + \varepsilon)^2 \iint_U de\, d\ell\, \ell = (\delta_1 + \varepsilon)^2.$$

As $\varepsilon \to 0^+$, we get $\lambda_* \le \delta_1^2$. □