Chapter 2
The Antonov Stability Estimate

The purpose of this short chapter is to provide some more details on the Antonov stability estimate, Theorem 1.2.

Proof of Theorem 1.2 We (formally) repeat the argument from [50, Prop. 4.1(ii)]; also see [30, Appendix]. Define $q(x, v) = \frac{u(x,v)}{x \cdot v}$. Then a direct calculation shows that

$$|\mathcal{T}u|^2 = (x \cdot v)^2\, |\mathcal{T}q|^2 + \mathcal{T}\Big((x \cdot v)|q|^2 \mathcal{T}(x \cdot v)\Big) - (x \cdot v)\, |q|^2 \mathcal{T}^2(x \cdot v).$$

Moreover $\mathcal{T}(x \cdot v) = |v|^2 - rU_Q'(r)$ and, using $\Delta U_Q = 4\pi\rho_Q$,

$$\mathcal{T}^2(x \cdot v) = -(x \cdot v)\Big(4\pi\rho_Q(r) + \frac{U_Q'(r)}{r}\Big). \tag{2.1}$$

Therefore,

$$|\mathcal{T}u|^2 - 4\pi\rho_Q(r)|u|^2 = (x \cdot v)^2\, |\mathcal{T}q|^2 + \mathcal{T}\Big((x \cdot v)\, |q|^2 \mathcal{T}(x \cdot v)\Big) + \frac{U_Q'(r)}{r}\, |u|^2.$$

Now integration by parts yields $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\, \mathcal{T}(\ldots) = 0$, cf. Lemma B.9(a). So what remains after integration is

$$\begin{aligned} (Lu, u)_Q &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\, |\mathcal{T}u|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\, dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\, \Big[4\pi\rho_Q(r)|u|^2 + (x \cdot v)^2\, |\mathcal{T}q|^2 + \frac{U_Q'(r)}{r}\, |u|^2\Big] \\ &\quad - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\, dx. \end{aligned}$$

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From Lemma 2.4 below, we have $U'_{\mathcal{T}u}(r) = 4\pi \int_{\mathbb{R}^3} p_r\, u\, dv$. Therefore,

$$\int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\, dx = \int_{\mathbb{R}^3} \left|\frac{x}{r}\, U'_{\mathcal{T}u}\right|^2 dx = 16\pi^2 \int_{\mathbb{R}^3} \left|\int_{\mathbb{R}^3} p_r\, u\, dv\right|^2 dx,$$

and this leads to

$$\begin{aligned}(Lu, u)_Q = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\left[(x\cdot v)^2 \left|\mathcal{T}\,\frac{u}{x\cdot v}\right|^2 + \frac{U'_Q(r)}{r}\,|u|^2\right] \\ &+ 4\pi \int_{\mathbb{R}^3} dx \left[\rho_Q(r)\int_{\mathbb{R}^3} \frac{dv}{|Q'(e_Q)|}\,|u|^2 - \left|\int_{\mathbb{R}^3} p_r\, u\, dv\right|^2\right]. \qquad (2.2)\end{aligned}$$

To obtain the lower bound, note that by Lemma 2.5 below

$$\begin{aligned}\left|\int_{\mathbb{R}^3} p_r\, u\, dv\right| &= \left|\int_{\mathbb{R}^3} p_r |Q'(e_Q)|^{1/2}\, |Q'(e_Q)|^{-1/2}\, u\, dv\right| \\ &\le \left(\int_{\mathbb{R}^3} p_r^2\, |Q'(e_Q)|\, dv\right)^{1/2}\left(\int_{\mathbb{R}^3} \frac{dv}{|Q'(e_Q)|}\,|u|^2\right)^{1/2} \\ &= \rho_Q(r)^{1/2}\left(\int_{\mathbb{R}^3} \frac{dv}{|Q'(e_Q)|}\,|u|^2\right)^{1/2}.\end{aligned}$$

Therefore, (2.2) yields

$$(Lu, u)_Q \ge \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\,\frac{U'_Q(r)}{r}\,|u|^2. \qquad (2.3)$$

The function $A(r) = \frac{U'_Q(r)}{r}$, together with $B(r) = 4\pi\rho_Q(r) + A(r)$, will be considered in Lemma A.6. It turns out that this function is strictly decreasing, positive and such that $A(r_Q) = \frac{1}{r_Q^3}\|Q\|_{L^1(\mathbb{R}^6)}$. Hence (1.19) follows. □

Example 2.1 We are going to show that

$$L\left(|Q'(e_Q)|(x\cdot v)\right) = A(r)\,|Q'(e_Q)|\,(x\cdot v), \qquad (2.4)$$

and in particular $u(x, v) = |Q'(e_Q)|\,(x\cdot v)$ is not an eigenfunction for λ_* from (1.20). Regarding (2.4), (2.1) says that $\mathcal{T}^2(x\cdot v) = -(x\cdot v)(4\pi\rho_Q(r) + A(r)) = -B(r)(x\cdot v)$, cf. Lemma A.6. Hence, using $\mathcal{T}|Q'(e_Q)| = 0$ from Lemma B.9(a), also $\mathcal{T}^2 u = -B(r)u$ holds. Therefore, due to (B.37), Lemma 2.4 and Lemma 2.5,

$$\begin{aligned}\mathcal{K}\mathcal{T}u &= 4\pi\,|Q'(e_Q)|\,p_r \int_{\mathbb{R}^3} \tilde{p}_r\, u\, d\tilde{v} = 4\pi\,|Q'(e_Q)|\,p_r\, r\int_{\mathbb{R}^3} \tilde{p}_r^2\,|Q'(e_Q)|\, d\tilde{v} \\ &= 4\pi\,|Q'(e_Q)|\,(x\cdot v)\rho_Q(r) = 4\pi\rho_Q(r)u,\end{aligned}$$

where $p_r = x \cdot v/r$ and $\tilde{p}_r = x \cdot \tilde{v}/r$. As a consequence,

$$Lu = -\mathcal{T}^2 u - \mathcal{K}\mathcal{T}u = B(r)u - 4\pi\rho_Q(r)u = A(r)u,$$

so that (2.4) is established. ◇

Corollary 2.2 *We have*

$$0 < A(r_Q) \le \lambda_* < A(0) \quad \textit{and} \quad 0 < B(r_Q) \le \lambda_* < B(0).$$

Proof The lower bound $\lambda_* \ge A(r_Q)$ follows from (2.3) and $A(r) \ge A(r_Q)$. Regarding the upper bound $\lambda_* < A(0)$, consider as in Example 2.1 the function $u(x, v) = |Q'(e_Q)|(x \cdot v)$, which is odd in v. Then $Lu = A(r)u$, and thus

$$(Lu, u)_Q = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,A(r)\,|u|^2 \le A(0)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|u|^2 = A(0)\|u\|_Q^2. \tag{2.5}$$

Hence, it follows that $\lambda_* \le A(0)$, and it remains to be shown that $\lambda_* = A(0)$ is impossible. Suppose that in fact $\lambda_* = A(0)$. Then, for the same u,

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,A(0)\,|u|^2 &= A(0)\|u\|_Q^2 = \lambda_*\|u\|_Q^2 \\ &\le \left(L\,\frac{u}{\|u\|_Q}, \frac{u}{\|u\|_Q}\right)_Q \|u\|_Q^2 = (Lu, u)_Q.\end{aligned}$$

Using

$$(Lu, u)_Q = \int\int_K \frac{dx\,dv}{|Q'(e_Q)|}\,A(r)\,|u|^2$$

from (2.5), this leads to

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,(A(r) - A(0))\,|u|^2 \ge 0.$$

But this is not possible since A is strictly decreasing by Lemma A.6(a). For the estimates in terms of B, it is sufficient to note that $B(r) \ge A(r)$ and $B(r_Q) = A(r_Q)$, cf. Lemma A.6(b). □

Remark 2.3 Observe that $0 < A(r_Q) \le \lambda_* < A(0)$ and $0 < B(r_Q) \le \lambda_* < B(0)$ from Corollary 2.2 together with Lemma A.6 imply that $\lambda_* = A(r_A)$ and $\lambda_* = B(r_B)$ for certain $r_A, r_B \in]0, r_Q[$. It would be interesting to understand whether these radii r_A and r_B do have a special meaning. ◇

The following observation has been made before; see [30, p. 507] and [50, (B.1)] for instance.

Lemma 2.4 *For appropriate spherically symmetric functions g,*

$$U'_{\mathcal{T}g}(r) = 4\pi \int_{\mathbb{R}^3} p_r\, g\, dv.$$

Proof Since $\nabla_x U_Q$ is independent of v, by (1.11) the density is found to be

$$\rho_{\mathcal{T}g}(x) = \int_{\mathbb{R}^3} (v \cdot \nabla_x g - \nabla_v g \cdot \nabla_x U_Q)\, dv = \operatorname{div}_x \int_{\mathbb{R}^3} v\, g\, dv.$$

Therefore, using (A.2) below and Gauss's theorem, it follows that

$$\begin{aligned} U'_{\mathcal{T}g}(r) &= \frac{1}{r^2} \int_{|x|\le r} \rho_{\mathcal{T}g}(x)\, dx = \frac{1}{r^2} \int_{|x|\le r} dx\, \operatorname{div}_x \Big(\int_{\mathbb{R}^3} v\, g\, dv \Big) \\ &= \frac{1}{r^2} \int_{|x|=r} dS(x)\, \frac{x}{r} \cdot \Big(\int_{\mathbb{R}^3} v\, g\, dv \Big) = \frac{1}{r^2} \int_{|x|=r} dS(x) \Big(\int_{\mathbb{R}^3} p_r\, g\, dv \Big), \end{aligned}$$

where $g = g(x, v) = g(r, p_r, \ell)$. As g is spherically symmetric, the function $G(x) = \int_{\mathbb{R}^3} p_r\, g\, dv$ is invariant under rotation, i.e., $G(x) = G(|x|)$. Thus, $U'_{\mathcal{T}g}(r) = 4\pi \int_{\mathbb{R}^3} p_r\, g\, dv$ as desired. □

For the next lemma, cf. [30, p. 507] and [50, (B.3)].

Lemma 2.5 *We have*

$$\int_{\mathbb{R}^3} |Q'(e_Q)|\, p_r^2\, dv = \rho_Q(r).$$

Proof Since $\frac{d}{dp_r}[Q(e_Q)] = Q'(e_Q) p_r$ due to $e_Q = \frac{1}{2} p_r^2 + U_Q(r) + \frac{\ell^2}{2r^2}$, we have from (A.40):

$$\begin{aligned} \int_{\mathbb{R}^3} |Q'(e_Q)|\, p_r^2\, dv &= -\int_{\mathbb{R}^3} Q'(e_Q)\, p_r^2\, dv \\ &= -\frac{2\pi}{r^2} \int_{\mathbb{R}} dp_r \int_0^\infty d\ell\, \ell\, Q'(e_Q)\, p_r^2 \\ &= -\frac{2\pi}{r^2} \int_0^\infty d\ell\, \ell \int_{\mathbb{R}} dp_r\, \frac{d}{dp_r}[Q(e_Q)]\, p_r \\ &= \frac{2\pi}{r^2} \int_0^\infty d\ell\, \ell \int_{\mathbb{R}} dp_r\, Q(e_Q) \\ &= \int_{\mathbb{R}^3} Q(e_Q)\, dv = \rho_Q(r), \end{aligned}$$

where once again (A.40) has been used. Note that the boundary term vanishes in the integration by parts above. Indeed, if we use the notation from Lemma B.7, cf. (B.10), (B.11) and (B.12), then for fixed r and $\ell \in [0, \hat{l}(r)]$, we see that

$$(Q(e_Q)\, p_r)\Big|_{-\hat{p}}^{\hat{p}} = Q(e_0)\hat{p} - Q(e_0)(-\hat{p}) = 2\hat{p}\, Q(e_0) = 0,$$

since $Q(e_0) = 0$ by (Q2). □